Lecture 15, Tuesday, October/31/2023

Outline

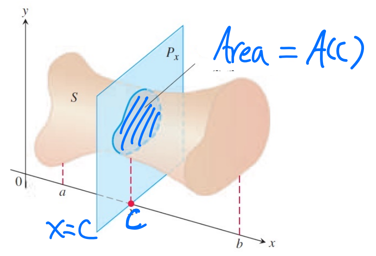
- Volumes using cross-sections (6.1)
- Solid of Revolution (6.1)
- Cylindrical shells (6.2)
- Arc length (6.3)
- Areas of surfaces of revolution (6.4)

How to compute its volume?

Volumes Using Cross-Sections

Let S be a solid in the three dimensional Euclidean space (the xyz-space), lying between the planes $x = a$ and $x = b$. How do we compute the volume of the solid?

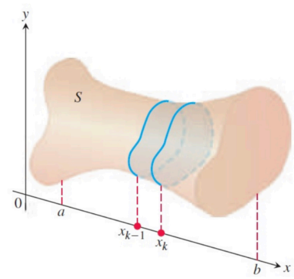
For $c \in [a, b]$, let $A(c)$ be the area of the cross-section obtained by intersecting S with the plane $x = c$.

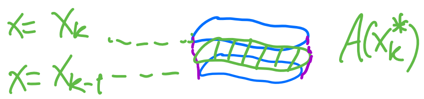


- Consider a partition $P := \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$.
- When Δx_k is small, the volume of the solid lying between $x = x_{k-1}$ and $x = x_k$ is approximately $A(x_k^*)\Delta x_k$.
- When $\|P\|$ is small, the volume of S is approximately

$x_k^* \in [x_{k-1}, x_k]$

$$\sum_{k=1}^{n} A(x_k^*)\Delta x_k.$$





Definition

Let S be a solid that lies between the planes $x = a$ and $x = b$. The volume V of S is defined by

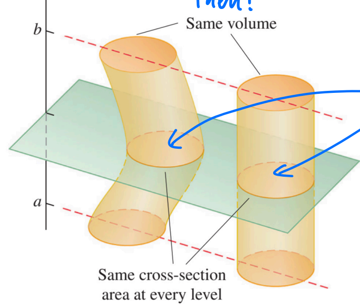
$$V := \int_a^b A(x)\,dx,$$

on $[a,b]$

provided that the cross-section area function $A(x)$ is integrable.

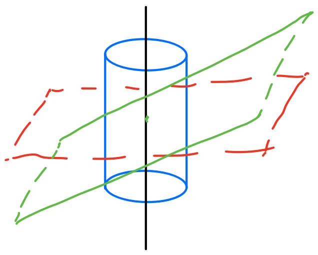
Hence, to find V, we can find a formula for the cross section area first.

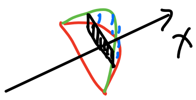
Then:



$A_1(x) = A_2(x)$, $\forall x \in [a,b]$

EXAMPLE ~~2~~ 6.1.2 A curved wedge is cut from a circular cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.



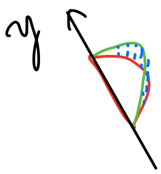


Sol: $V = \int_0^3 A(x)\,dx$

$= \int_0^3 2\sqrt{9-x^2}\,x\,dx = \int_9^0 \sqrt{u}\,(-du)$

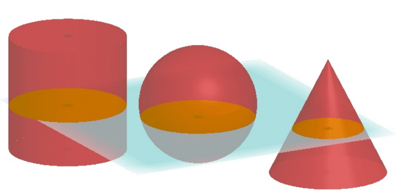
$= \int_0^9 u^{\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{3}{2}}\Big|_{u=0}^{9} = \frac{2}{3}(27) = 18.$

Alternatively:



$V = \boxed{}$

Hence, to analyze the volume of a 3D-solid, it suffices to analyze its cross sections. Many well-known solids have circular cross sections:



Solid of Revolution

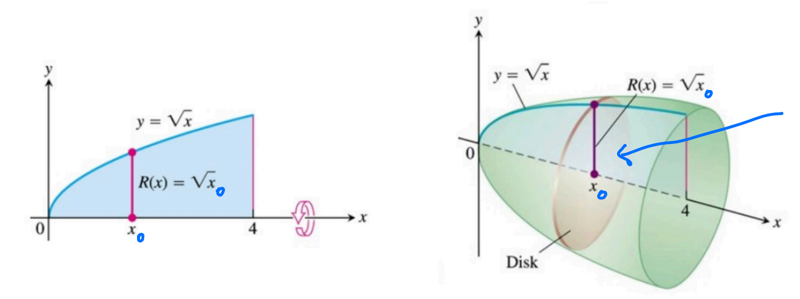
If the solid S is generated by rotating the region

$$\{(x, y) : 0 \leq y \leq R(x),\ a \leq x \leq b\}$$

around the x-axis, then the cross-sections of S are discs with radii $R(x)$. Consequently, $A(x) = \pi R(x)^2$, and

= "solid circles"

$$V = \int_a^b \pi R(x)^2 \, dx.$$



$A(x_0) = \pi R(x_0)^2$

e.g. Derive the volume formula of a sphere with radius r.

Sol. · Such a sphere can be obtained by revolving

________________________________ .

around the x-axis.

· $V =$ ________________________________

________________________ $= \frac{4}{3}\pi r^3$.

Exercise: Verify using the cross-sectional method that the volume of a cone is indeed $\frac{1}{3}\pi r^2 h$.

EXAMPLE ~~6~~ 6.1.6 Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines ~~y~~ $x = 1$, $x = 4$ about the line $y = 1$.

Sol. $V = \dfrac{}{} = \dfrac{7\pi}{6}$.

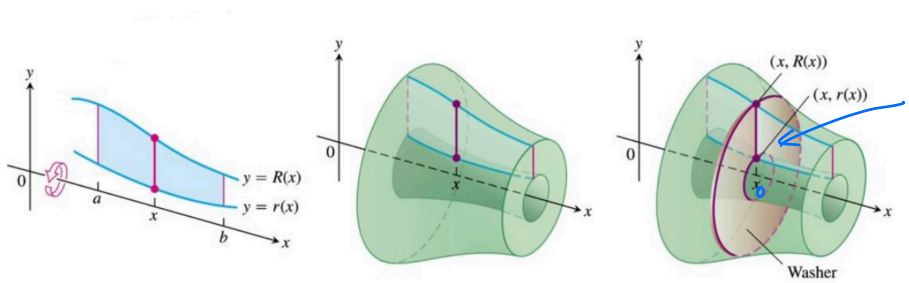
If the solid S is generated by rotating the region

$$\{(x, y) : 0 \le r(x) \le y \le R(x),\ a \le x \le b\}$$

around the x-axis, then similarly,

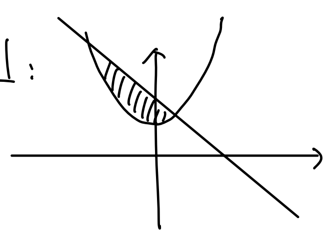
$$V = \int_a^b \pi\left(R(x)^2 - r(x)^2\right)\, dx.$$

$A(x_0) = \pi\left[R^2(x_0) - r^2(x_0)\right]$



EXAMPLE ~~9~~ 6.1.9 The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Sol:

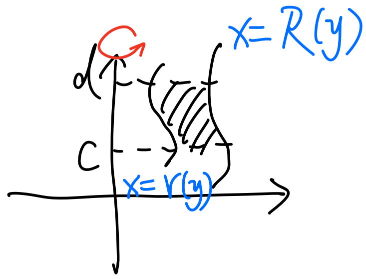


- Only one bounded region
- $x^2 + 1 = -x + 3 \iff x = -2$ or $x = 1$.

$$\cdot\ V=\int_{-2}^{1}\pi\left[(-x+3)^2-(x^2+1)^2\right]dx=\int_{-2}^{1}\pi\left(x^2-6x+9-x^4-2x^2-1\right)dx$$

$$=\pi\int_{-2}^{1}\left(-x^4-x^2-6x+8\right)dx=\cdots=\frac{117\pi}{5}.$$

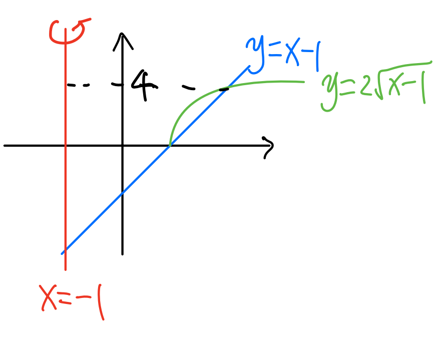
The ideas above can be applied similarly to a solid obtained by revolving a region around the y-axis, or around lines of the form $y=K$ or $x=K$ in general.



$$V=\int_c^d \pi\left[R(y)^2-r(y)^2\right]dy$$

$0\le r(y)\le R(y)$
$\forall\, y\in[c,d]$

e.g. Find the volume of the solid obtained by rotating the region bounded by $y=2\sqrt{x-1}$ and $y=x-1$ about the line $x=-1$.



Sol: $\cdot\ y=x-1 \iff x=y+1$

$\cdot\ y=2\sqrt{x-1} \iff y^2=4(x-1)$
$\iff x=\frac{y^2}{4}+1$

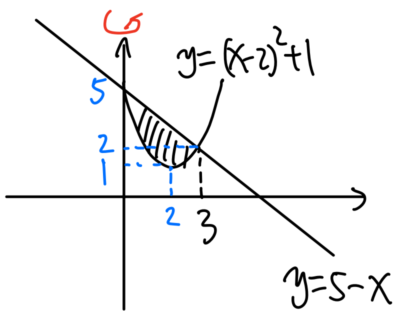
$\cdot\ y+1=\frac{y^2}{4}+1 \iff y=0$ or $y=4$.

- Let $R(y) = (y+1)+1$ and $r(y) = \left(\frac{y^2}{4}+1\right)+1$.
- Then volume is the same as $\int_0^4 \pi\left(R^2(y) - r^2(y)\right)\,dy$
- $V = \pi \int_0^4 \left[(y+2)^2 - \left(\frac{y^2+8}{4}\right)^2\right] dy$

$$= \pi \int_0^4 \left(4y - \frac{1}{16}y^4\right) dy$$

$$= \pi \left(2y^2 - \frac{1}{80}y^5\right)\Big|_0^4 = \pi\left(32 - \frac{1}{5\cdot 16}4^5\right) = \frac{96\pi}{5}.$$

<u>e.g.</u> A region is bounded by $y = (x-2)^2+1$ and $y = 5-x$. Find the volume of the solid generated by revolving the region around the y-axis.



<u>Sol</u>
- Basic algebra shows that the points of intersection of the two curves are

$$(x,y) = (0,5) \text{ and } (3,2)$$

- $y = 5-x \iff x = 5-y$

$y = (x-2)^2+1 \iff x = \pm\sqrt{y-1}+2$

- The method of cross-section is rather complicated here:

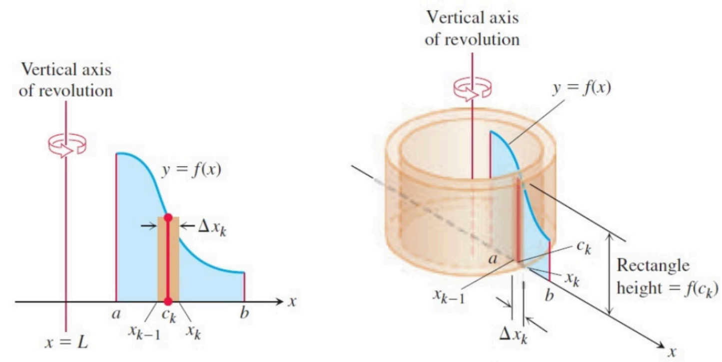
$$V = \int_1^2 \pi\left((\sqrt{y-1}+2)^2 - (-\sqrt{y-1}+2)^2\right) dy + \int_2^5 \pi\left((5-y)^2 - (2-\sqrt{y-1})^2\right) dy$$

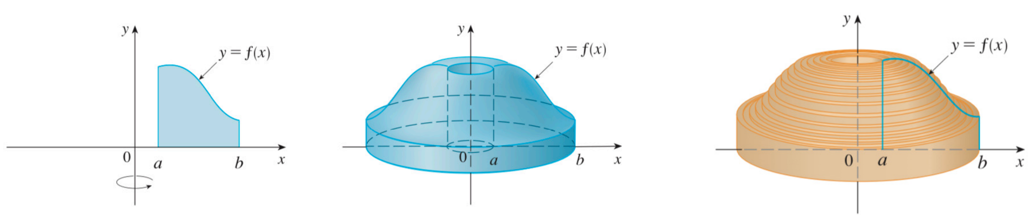
$$= \text{UGLY COMPUTATION} \ldots = \frac{27\pi}{2}.$$

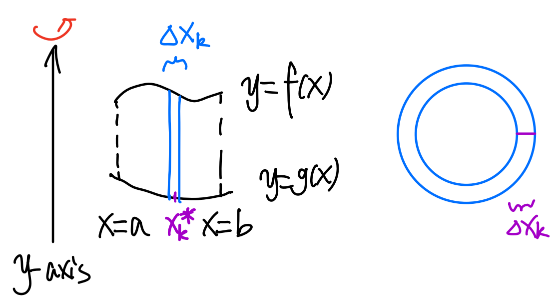
Volumes Using Cylindrical Shells

Another way to compute the volume of a solid generated by rotations around a coordinate axis is to use cylindrical shells.

Consider revolving the following region in blue about the y-axis to generate a solid. Its volume can be computed by adding the volumes of all the "cylindrical shells", one of which is displayed in orange in the figure.







Let $h(x) := f(x) - g(x)$.
Take $h(x_k^*)$ as height,
where $x_k^* \in [x_{k-1}, x_k]$.

- Volume of "thin cylindrical shell" $\approx$ (circumference) $\cdot$ (height) $\cdot$ (thickness)

  $= 2\pi x_k^* \cdot h(x_k^*) \cdot \Delta x_k$

- Riemann sum for volume $= \sum_{k=1}^{n} 2\pi x_k^* h(x_k^*) \Delta x_k$.

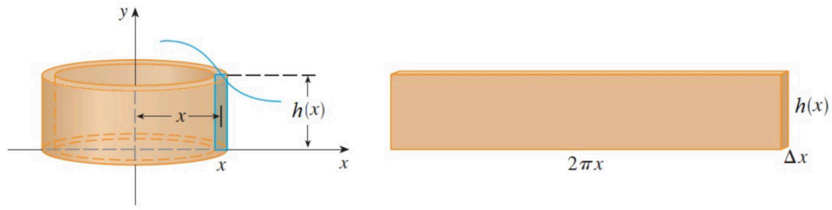
- Exact volume = limit of Riemann sums

In general, given the solid S generated by revolving the region

$$\{(x, y) : g(x) \le y \le f(x),\ a \le x \le b\}$$

about the y-axis, let $h(x) := f(x) - g(x)$ be the height of the region at x. Then the volume V of S can be computed by

$$\boxed{V = \int_a^b 2\pi x h(x)\, dx.}$$



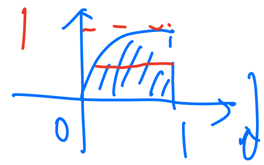
e.g. A region is bounded by $y=(x-2)^2+1$ and $y=5-x$. Find the volume of the solid generated by revolving the region around the y-axis.

Sol. Following the shape of the region given on Page 7, using cylindrical shells,

$$V = \boxed{} = \frac{27\pi}{2}.$$

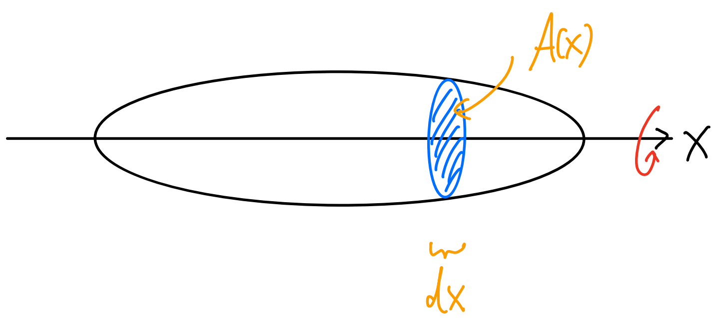
Easier than the method of revolution for this example.

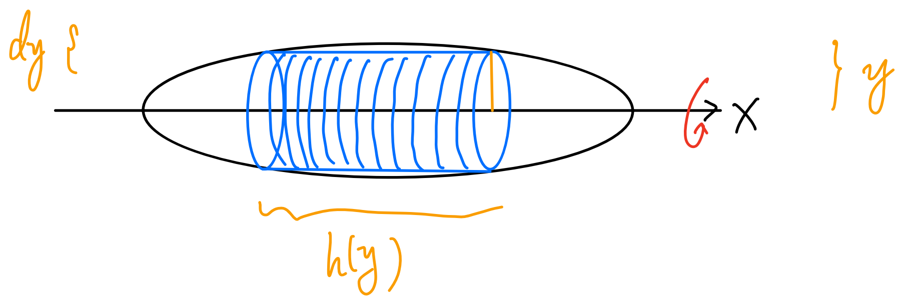
e.g.: Find the volume of the solid generated by revolving the region between $y=\sqrt{x}$ and $y=0$, from 0 to 1, around the x-axis, in two different ways.



Ans: $\pi/2$.

Moral of story

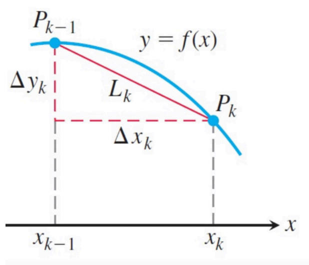




- Choose a method that makes the computation convenient.

Arc Length

Consider a curve given by a continuous function $y = f(x)$ defined on the interval $[a, b]$, and let P be a partition of $[a, b]$.



If $y_k := f(x_k)$ and $\Delta y_k = y_k - y_{k-1}$, then the length of the curve between the points (x_{k-1}, y_{k-1}) and (x_k, y_k) is approximately

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\,\Delta x_k$$

when Δx_k is small. The definition of arc length is obtained by taking limit of $\sum_{k=1}^{n} L_k$ as $\|P\| \to 0$.

$\frac{\Delta y_k}{\Delta x_k} = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(c_k)$ for some $c_k \in (x_{k-1}, x_k)$

$\Rightarrow L_k = \sqrt{1+[f'(c_k)]^2}\,\Delta x_k \Rightarrow \sum_{k=1}^{n} L_k$ is a Riemann sum

DEFINITION If f' is continuous on $[a, b]$, then the **length (arc length)** of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{3}$$

Example

Compute the length of the curve given by $y = x^{3/2}$, $0 \le x \le 3$.

Solution

The length is $\frac{8}{27}\left(\left(\frac{31}{4}\right)^{3/2} - 1\right)$.

$\left(L = \int_0^3 \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\,dx = \int_0^3 \sqrt{1+\frac{9}{4}x}\,dx = \frac{4}{9}\cdot\left(\frac{2}{3}\left(1+\frac{9}{4}x\right)^{3/2}\right)\Big|_0^3\right)$

Example

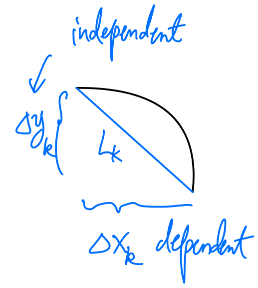
Compute the length of the curve given by $y = (x/2)^{2/3}$, $0 \le x \le 2$.

$$L = \int_0^2 \sqrt{1 + \left(\frac{2}{3}\left(\frac{x}{2}\right)^{-1/3} \cdot \frac{1}{2}\right)^2}\, dx = \int_0^2 \underbrace{\sqrt{1 + \frac{1}{9}\left(\frac{x}{2}\right)^{-2/3}}\, dx}_{\text{not very friendly}}$$

Try to write $x = g(y)$.

If the curve is given by $x = g(y)$, $c \leq y \leq d$, and g' is continuous, then the arc length can be computed by

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + (g'(y))^2}\, dy.$$



For the example above,

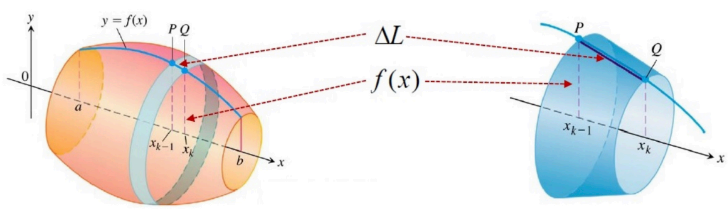
$$y = \left(\frac{x}{2}\right)^{2/3}, \quad 0 \leq x \leq 2 \iff \underbrace{2y^{3/2}}_{g(y)} = x, \quad 0 \leq y \leq 1,$$

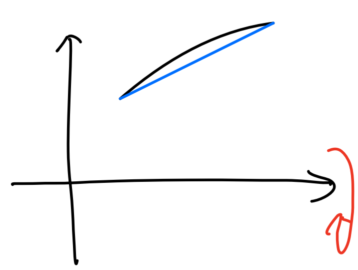
So $$L = \int_0^1 \sqrt{1 + [g'(y)]^2}\, dy = \int_0^1 \sqrt{1 + (3y^{1/2})^2}\, dy$$

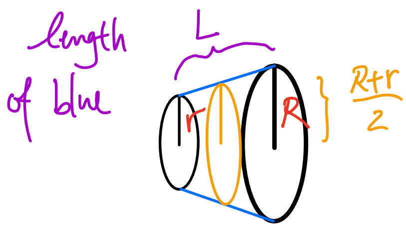
$$= \int_0^1 \sqrt{1 + 9y}\, dy = \frac{1}{9}\left(\frac{2}{3}(1 + 9y)^{\frac{3}{2}}\right)\Big|_{y=0}^{1} = \frac{2}{27}\left(10^{\frac{3}{2}} - 1\right).$$

Areas of Surfaces of Revolution

Consider a surface generated by revolving about the x-axis a curve $y = f(x)$, where f is ~~positive~~ nonnegative, for $x \in [a, b]$.







"Conical frustum"

Surface area of the conical frustum is

$$2\pi\left(\frac{R+r}{2}\right)L \quad (= \pi(R+r)L)$$

(See additional note on BB regarding frustum's surface area)

For area of a surface of revolution ($y=f(x)$, about x-axis, for $a \le x \le b$):

- Partition $[a,b]$ using $x_0, x_1, \ldots, x_n$.
- The k^{th} portion of curve has length $\approx \overbrace{\sqrt{1+[f'(c_k)]^2}\,\Delta x_k}^{L_k}$.
- k^{th} portion of surface $\approx \underbrace{\pi\left(f(x_{k-1})+f(x_k)\right)\sqrt{1+[f'(c_k)]^2}\,\Delta x_k}_{\text{Conical frustum}}$

When $\|P\|$ is small.

$$\approx \pi\left(f(c_k)+f(c_k)\right)\sqrt{1+[f'(c_k)]^2}\,\Delta x_k.$$

$c_k \in [x_{k-1}, x_k]$

This motivates the following definition.

Definition
Let f be a ~~positive~~ nonnegative function such that f' is continuous on $[a,b]$. The area S of the surface generated by revolving about the x-axis the curve $y=f(x)$, $a \le x \le b$, is defined by

$$S = \int_a^b 2\pi f(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b 2\pi f(x)\underbrace{\sqrt{1+(f'(x))^2}\,dx}_{\text{"arc length differential"}}.$$

Similarly, if $x = g(y)$ (g nonnegative, g' continuous on $[c,d]$) is revolved about the y-axis for $c \le y \le d$, then

$$S = \int_c^d 2\pi g(y)\sqrt{1+(g'(y))^2}\,dy.$$

e.g. The curve $y = x^2$ from $(1,1)$ to $(2,4)$ is revoled about the y-axis to obtain a surface S. Find Area(S).

Sol: · Curve is the same as $x = \sqrt{y}$, $1 \le y \le 4$.

· $$\text{Area}(S) = \int_1^4 2\pi\sqrt{y}\left(1+\left(\frac{dx}{dy}\right)^2\right)^{\frac{1}{2}} dy = 2\pi\int_1^4 \sqrt{y}\sqrt{1+\left(\frac{1}{2}y^{-\frac{1}{2}}\right)^2}\,dy$$

$$= 2\pi\int_1^4 \sqrt{y}\sqrt{1+\frac{1}{4y}}\,dy = 2\pi\int_1^4 \sqrt{y+\frac{1}{4}}\,dy$$

$$= 2\pi\left[\frac{2}{3}\left(y+\frac{1}{4}\right)^{3/2}\Big|_{y=1}^{4}\right] = \frac{4\pi}{3}\left[\left(\frac{17}{4}\right)^{3/2} - \left(\frac{5}{4}\right)^{3/2}\right]$$

$$= \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right).$$

Alternatively,

Area(S) =