Lecture 9, Tuesday, Oct/10/2023

Outline

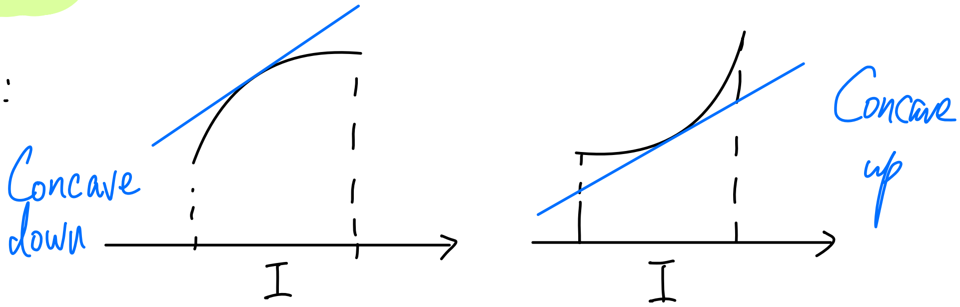
- Concavity (4.4)
- Second derivative test (4.4)
- Concavity, secants and tangents (4.4 extended, not in book)

Need to know the facts (theorems); proofs are optional.

Concavity

The graph of a function can bend up or down on a given interval.

Intuitively:



Formally:

DEFINITION The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I;

(b) **concave down** on an open interval I if f' is decreasing on I.

The property of being concave up or concave down is called concavity.

Remarks

- We may say that "f is concave up" instead of "the graph of $y=f(x)$ is concave up", for simplicity.
- One may extend the definition to say that f is concave up (or down) on $[a,b)$, $(a,b]$, or $[a,b]$, with f'_+ and

f'_- replacing f' for left and right endpoints, respectively.

Second Derivatives and Concavity

Suppose $f''(x) > 0$ for all x in (a,b) and f' is continuous on $[a,b]$. By Corollary 4.3.3, f' is increasing on $[a,b]$, so f is concave up on $[a,b]$. A similar statement can be made to the case where $f''(x) < 0$.

Theorem

Let f be a function where f' is continuous on $[a,b]$.

- If $f''(x) > 0$ for all $x \in (a,b)$, then f is concave up on $[a,b]$.
- If $f''(x) < 0$ for all $x \in (a,b)$, then f is concave down on $[a,b]$.

Remarks

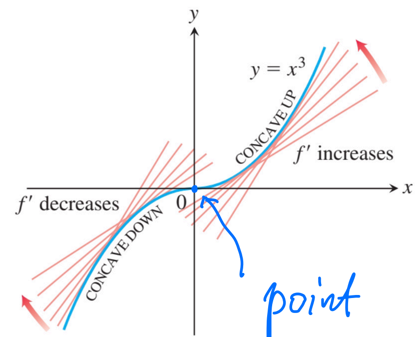
- The theorem can be extended to cover $[a,\infty)$, $(-\infty,b]$, and $(-\infty,\infty)$.
- This theorem is **NOT** the definition of concavity.

e.g. 1 Consider $y = x^3$. $f(x) = x^3$, $f'(x) = 3x^2$,

$$f''(x) = 6x \begin{cases} > 0, & \text{if } x > 0 \\ < 0, & \text{if } x < 0 \end{cases}.$$

So f is concave up on $[0, \infty)$

and concave down on $(-\infty, 0]$



point of inflection

Def: A function f is said to have a point of inflection (or an inflection point) at $(c, f(c))$ if:

- f has a tangent line (or a vertical tangent) at $x = c$.
- $\exists\, a > 0$ such that the concavity of f on $(c-a, c)$ is different from that on $(c, c+a)$.

Remarks

- We can also say that f has an inflection point at $x = c$.
- A continuous curve is said to have a vertical tangent at $x = c$ if $\displaystyle\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \infty$ or $= -\infty$;

  e.g., $y = x^{1/3}$ has a vertical tangent at $x = 0$, and

$y = x^{2/3}$ does not.

Q: How do we find inflection points?

> Fact: If $(c, f(c))$ is an inflection point of f, then either $f''(c)$ does not exist or $f''(c) = 0$.

$\hookrightarrow$ The proof of this requires the so-called "intermediate value property of derivative functions", which we will not discuss in this course. We omit the proof of the fact.

e.g. 2 Determine all points of inflection for the curves:

(a) $y = x^{1/3}$, $D = \mathbb{R}$.

(b) $y = x^4$, $D = \mathbb{R}$.

(c) $y = x^{4/3} - 4x^{1/3}$, $D = \mathbb{R}$.

Ans: (a) $x = 0$. (b) None. (c) $x = -2, 0$.

Discussion: Is $f(x) = x^4$ concave up on $\mathbb{R}$?

Second Derivative Test

THEOREM ~~5~~ 4.4.5—Second Derivative Test for Local Extrema Suppose f'' is continuous on an open interval that contains $x = c$. ← Continuity is not essential: only need f'' exists.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

Proof ← (Optional): 2. • Suppose $f'(c)=0$ and $f''(c)>0$.

• Then $0 < f''(c) = \lim_{x\to c} \frac{f'(x)-f'(c)}{x-c} = \lim_{x\to c}\frac{f'(x)}{x-c}$.

• Let $L = f''(c)$. For $\varepsilon = L/2$, $\exists \delta > 0$ such that

$\frac{f'(x)}{x-c} \in (L-\varepsilon, L+\varepsilon) = (\frac{L}{2}, \frac{3L}{2})$ for all $x \in (c-\delta, c+\delta)\setminus\{c\}$.

• This means that $\frac{f'(x)}{x-c} > \frac{L}{2} > 0$ for all $x \in (c-\delta, c+\delta)\setminus\{c\}$,

so $f'(x)$ has the same sign as $x-c$:

$$f'(x) < 0 \text{ for } x \in (c-\delta, c) \text{ and}$$
$$f'(x) > 0 \text{ for } x \in (c, c+\delta).$$

• By the first derivative test, f has a local minimum at c.

The proof of 1 is similar. How do you prove 3? ________. $\square$

e.g.3 $f(x) = x^4 - 4x^3 + 10$. Find all local extrema.

Sol. $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$ $(D = \mathbb{R})$

- $f'(x) = 0 \iff \underline{x = 0 \text{ or } x = 3}$. critical pts

$f''(x) = 12x^2 - 24x$.

- $f''(0) = 0$, second derivative test gives no info.

- $f''(3) = 12 \cdot 9 - 72 > 0$, $x=3$ gives a local minimum $f(3) = 81 - 4 \cdot 27 + 10 = -17$.
- For $x=0$, note that $f'(-1) < 0$ and $f'(1) < 0$, so it gives no local extrema by first derivative test.

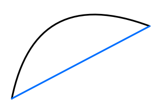
Concavity vs Secant & Tangent Lines

Here, we prove two geometric facts about concavity formally and see the power of the MVT.

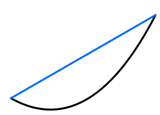
Theorem (Concavity and Secant Lines)

Let f be continuous on $[a,b]$ and differentiable on (a,b).

(i) If f is concave down on (a,b), then the graph of f lies above the secant line joining $(a, f(a))$ and $(b, f(b))$ on (a,b).



(ii) If f is concave up on (a,b), then the graph of f lies below the secant line joining $(a, f(a))$ and $(b, f(b))$ on (a,b).



(Optional)

Proof (i) The secant line has graph

$$y = g(x) = f(a) + \left(\frac{f(b)-f(a)}{b-a}\right)(x-a). \qquad (1)$$

Will show that $f(x) > g(x)$ for all $x \in (a,b)$. Fix any $x_0 \in (a,b)$. By the MVT,

$$f(x_0) = f(a) + f'(c_1)(x_0 - a) \text{ for some } c_1 \in (a, x_0). \qquad (2)$$

If we can show that $\frac{f(b)-f(a)}{b-a} < f'(c_1)$ ③, then by ① & ②,

$$f(x_0) - g(x_0) = \left(f'(c_1) - \frac{f(b)-f(a)}{b-a}\right)(x_0 - a) > 0$$

and we are done. It remains to show ③. Now

$$\begin{aligned} f(b) - f(a) &= f(b) - f(x_0) + (f(x_0) - f(a)) \\ &= f'(c_2)(b - x_0) + f'(c_1)(x_0 - a) && \left(\text{MVT, for some } c_2 \in (x_0, b)\right) \\ &< f'(c_1)(b - x_0) + f'(c_1)(x_0 - a) && (f' \text{ is decreasing by concavity}) \\ &= f'(c_1)(b-a), \end{aligned}$$

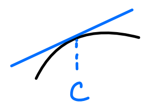
So $f(b) - f(a) < f'(c_1)(b-a)$, proving ③.

(ii) Similar. $\square$

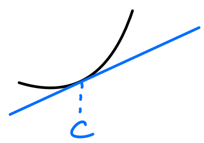
Theorem (Concavity and Tangent Lines)

Let f be continuous on $[a,b]$ and differentiable on (a,b).

(i) If f is concave down on (a,b), then for any $c \in (a,b)$, the tangent line to $y=f(x)$ at c lies above the graph of $y=f(x)$.



(ii) If f is concave up on (a,b), then for any $c \in (a,b)$, the tangent line to $y=f(x)$ at c lies below the graph of $y=f(x)$.



(Optional)

Proof: (i) The tangent line at c has graph

$$y = g(x) = g_c(x) = f(c) + f'(c)(x-c).$$

Will show that $g(x) > f(x)$ for all $x \in [a,b] \setminus \{c\}$.

Fix any $x_0 \in [a,b] \setminus \{c\}$. Note that

$$\begin{aligned}
g(x_0) > f(x_0) &\iff f(c) + f'(c)(x_0 - c) > f(x_0) \\
&\iff f'(c)(x_0 - c) > f(x_0) - f(c) \\
&\iff f'(c) \begin{cases} > \dfrac{f(x_0)-f(c)}{x_0-c}, & \text{if } x_0 > c; \\[2ex] < \dfrac{f(x_0)-f(c)}{x_0-c}, & \text{if } x_0 < c, \end{cases}
\end{aligned}$$

So it suffices to show that

$$f'(c) \begin{cases} > \dfrac{f(x_0)-f(c)}{x_0-c}, & \text{if } x_0 \in (c,b]; \quad ① \\ < \dfrac{f(x_0)-f(c)}{x_0-c}, & \text{if } x_0 \in [a,c). \quad ② \end{cases}$$

Assume that $x_0 \in [a,c)$. By the MVT,

$$f'(c_1) = \frac{f(x_0)-f(c)}{x_0-c} \text{ for some } c_1 \in (x_0, c). \quad (*)$$

Since f is concave down on (a,b), $f'(c_1) > f'(c)$, which, together with $(*)$, shows ②.

① can be proven similarly. This proved (i).

(ii) Similar. $\square$