Extended Exercise

(a) Show that $\int_a^\infty e^{-kt}\,dt$ is convergent for all $a \in \mathbb{R}$ and all $k > 0$.

(b) The gamma function is the function $\Gamma : (0,\infty) \to \mathbb{R}$ defined by

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\,dt = \int_0^\infty \frac{t^{x-1}}{e^t}\,dt.$$

This is an extention of the factorial function to the domain $(0,\infty)$: one can check that $\Gamma(n+1) = n!$ for $n \in \mathbb{N} := \{0,1,2,\dots\}$ (see Chp 8, additional and advanced exercises Q51). Prove that $\int_0^\infty t^{x-1} e^{-t}\,dt$ converges for each $x > 0$ (so the gamma function is well-defined).

Sol: (a) $\displaystyle\int_a^b e^{-kt} = -\frac{1}{k} e^{-kt}\Big|_a^b = \frac{1}{k}\left(e^{-ka} - e^{-kb}\right) =: I.$

As $b \to \infty$, $I \to \frac{1}{k} e^{-ka}$ (since $k > 0$).

(b) Fix $x > 0$. Then $\Gamma(x) = \underbrace{\int_0^1 t^{x-1} e^{-t}\,dt}_{I_1} + \underbrace{\int_1^\infty t^{x-1} e^{-t}\,dt}_{I_2}.$

It suffices to show that both I_1 and I_2 are finite.

- $\underline{\underline{I_1}}$: If $x \geq 1$, I_1 is finite (as a definite integral).

If $0 < x < 1$, then $-1 < x-1 < 0$, so $\int_0^1 t^{x-1} dt$ converges.

Since $t^{x-1} > \frac{t^{x-1}}{e^t} > 0$ $\forall t \in (0,1]$, by comparison test,

I_1 converges.

- $\underline{\underline{I_2}}$: Limit-Compare with $\int_1^\infty e^{-\frac{1}{2}t} dt$, which is convergent by (a).

Since

$$\lim_{t \to \infty} \frac{e^{-t} t^{x-1}}{e^{-1/2 t}} = \lim_{t \to \infty} \frac{t^{x-1}}{e^{1/2 t}} = 0,$$

Clearly 0 if $0 < x \leq 1$; if $x > 1$, use the fact that positive power functions grow slower than exponential functions.

I_2 is convergent by limit comparison test.