Lecture 7, Tuesday, September/26/2023

Outline

- Proof of the chain rule (3.9)
- Related rates (3.8)
- Extreme values of functions (4.1)
- Rolle's theorem (4.2)

Proof of Chain Rule

Want to show $(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$.

- Let $y = f(x)$, $z = g(y) = g(f(x))$.
- Consider a change of x-value from x_0 by Δx.
- Let Δy and Δz be corresponding changes in y- and z-values, respectively.
- Then $\Delta y = f'(x_0)\Delta x + \varepsilon_1 \Delta x$ for some ε_1 with $\lim\limits_{\Delta x \to 0} \varepsilon_1 = 0$.
- Also, $\Delta z = g'(y_0)\Delta y + \varepsilon_2 \Delta y$ for some ε_2 with $\lim\limits_{\Delta y \to 0} \varepsilon_2 = 0$.

  $f(x_0)$

- Now $\frac{\Delta z}{\Delta x} = (g'(y_0) + \varepsilon_2)\frac{\Delta y}{\Delta x} = (g'(y_0) + \varepsilon_2)(f'(x_0) + \varepsilon_1)$, where $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $\Delta x \to 0$.

  since $\Delta y \to 0$ as $\Delta x \to 0$.

- Therefore, $(g \circ f)'(x_0) = \left.\frac{dz}{dx}\right|_{x = x_0} = \lim\limits_{x \to 0} \frac{\Delta z}{\Delta x}$

$$= (g'(y_0) + 0)(f'(x_0) + 0) = g'(f(x_0))f'(x_0).$$

$\square$

Related Rates

EXAMPLE ~~1~~ 3.8.1 Water runs into a conical tank at the rate of $0.25 \text{ m}^3/\text{min}$. The tank stands point down and has a height of 3 m and a base radius of 1.5 m. How fast is the water level rising when the water is 1.8 m deep?

Ans: ________________ ... $= \frac{25}{81\pi}$ m/min

(≈ 0.098)

Example 2 A rocket is launched so that it rises vertically. A camera is positioned 5000 feet away from the launch pad on a flat ground, and it always stays focus on the bottom of the rocket. When the rocket is 1000 feet above the launch pad, its velocity is 600 feet/sec. Find the rate of change of the angle made by the ground and the camera with respect to time.

Ans: ________________ ... $= \frac{3}{26}$ radian/sec

(≈ 0.115)

Extreme Values of Functions

DEFINITIONS Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

- Absolute maximum/minimum are also called global maximum/minimum.
- Extremum means maximum or minimum.
- Plural forms: maxima, minima, extrema.

e.g. $f: D \to \mathbb{R}$, $f(x) = x^2$.

- $D = \mathbb{R}$: no absolute max;
  absolute min at $x=0$, value $= 0$.
- $D = (0,3]$: no absolute min;
  absolute max at $x=3$, value $= 3^2 = 9$.
- $D = [0,3]$: absolute min at $x=0$, value $= 0$;
  absolute max at $x=3$, value $= 9$.

THEOREM ~~1~~ 4.1.1—The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \le f(x) \le M$ for every other x in $[a, b]$.

$a \le b$. finite.

May fail if
- f is not continuous, or;
- interval is not of the form $[a,b]$

Prove: Omitted.
(Optional reading: Wikipedia.)

Def: Let $f: D \to \mathbb{R}$.

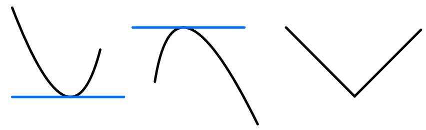
- The function f is said to have a local maximum at c if there exists $a > 0$ such that

$$f(x) \le f(c) \quad \text{for all } x \in (c-a, c+a) \cap D.$$

- The function f is said to have a local minimum at c if there exists $a > 0$ such that

$$f(x) \ge f(c) \quad \text{for all } x \in (c-a, c+a) \cap D.$$

Q: How to find local/global extrema?



Def: Let $f: D \to \mathbb{R}$ and let c be an interior point of D. Then c is a critical point of f is

(i) $f'(c) = 0$, or; (ii) $f'(c)$ does not exist. (in $\mathbb{R}$)

e.g. 3 What are all the critical points of the function

$$f(x) = \begin{cases} |x|, & \text{if } x < 1; \\ 1, & \text{if } x \geqslant 1. \end{cases} \quad ?$$

What about local extrema?

THEOREM ~~2~~ 4.1.2—The First Derivative Theorem for Local Extreme Values If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

The theorem above can be rephrased as follows.

Let c be an interior point of D. If a function $f : D \to \mathbb{R}$ has a local extremum at c, then c is a critical point of f.

Before proving this, recall "limits preserve order".

2.2.5

THEOREM ~~5~~ If $f(x) \leq g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

Proof of Theorem 4.1.2 :

- Suppose that f has a local maximum at c; the case where c gives a minimum is similar.
- Then $\exists\, a$ with $a > 0$ such that $f(c) \geq f(x)$ for all $x \in (c-a, c+a)$. (Recall that c is an interior point.)
- Let $g(x) = \dfrac{f(x) - f(c)}{x - c}$, for $x \in (c-a, c+a) \setminus \{c\}$
- For $x \in (c-a, c)$, $f(x) - f(c) \leq 0$ and $x - c < 0$, so $g(x) \geq 0$. By Theorem 2.2.5, we have

$$\lim_{x \to c^-} g(x) \geq \lim_{x \to c^-} 0 = 0.$$

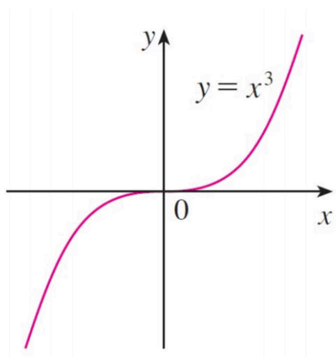
- Similarly, for $x \in (c, c+a)$, $f(x) - f(c) \leq 0$ and $x - c > 0$, so $g(x) \leq 0$. Hence, $\lim_{x \to c^+} g(x) \leq 0$.
- Now

Since f is differentiable at c.

$$0 \leq \lim_{x \to c^-} g(x) = f'_-(c) = f'(c) = f'_+(c) = \lim_{x \to c^+} g(x) \leq 0,$$

So $f'(c) = 0$. $\square$

The converse of the theorem is not true; having a critical point at c does not imply that a local extremum must occur at c — The function $f(x) := x^3$ is a counterexample.



Theorem 4.1.1 and 4.1.2 give the following strategy for finding all the absolute extrema for a continuous function on $[a,b]$.

How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval $[a,b]$

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

Think about the logic behind this method.

e.g.4 Find all absolute extrema (with values and positions) of:

(a) $f: [-2,4] \to \mathbb{R}$, $f(x) = 2x^3 - 3x^2 - 12x + 15$.

(b) $f: [-2,3] \to \mathbb{R}$, $f(x) = x^{2/3}$.

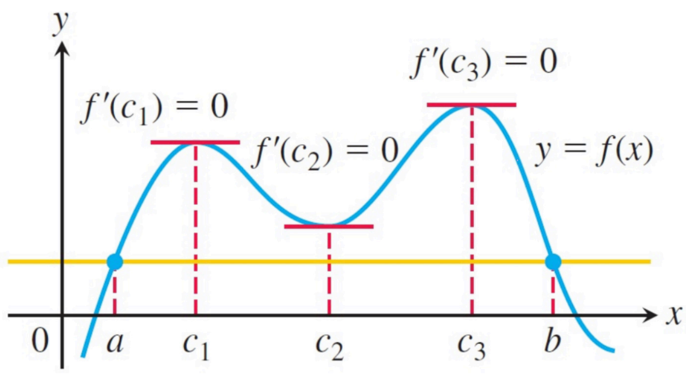
Ans: (a) ________. Absolute maximum $= 47$, at $x = 4$;
" minimum $= -5$, at $x = 2$.

(b) ______. Abs. max $= \sqrt[3]{9}$, at $x = 3$; abs min $= 0$, at $x = 0$.

Rolle's Theorem

Theorem (Rolle's theorem)

Suppose that a function f is continuous on $[a, b]$ and differentiable on (a, b), and it satisfies $f(a) = f(b)$. Then there exists c in (a, b) such that $f'(c) = 0$.



Remark: A physical consequence of Rolle's theorem is that if an object starts and stops an movement at the same position, then at some point in between, it must stop moving.

Proof of Rolle's theorem

- By the Extreme Value Theorem, $\exists x_m, x_M \in [a,b]$ such that $f(x_m)=m$ and $f(x_M)=M$, where m and M are absolute minimum and maximum, respectively.
- Then m and M are local extrema as well.
- If $m=M$, then $f(x)=M$ for all $x\in[a,b]$, so $f'(x)=0$ for any x in (a,b), done. Can choose any $c\in(a,b)$
- Suppose $m<M$. Then

  $f(a)=f(b)\neq m$ or $f(a)=f(b)\neq M$ (or both).

  Suppose $f(a)=f(b)\neq m$.
- Since $f(x_m)=m$, we have $x_m\in(a,b)$. Since $f'(x_m)$ exists, by Thm 4.1.2, $f'(x_m)=0$. Let $c:=x_m$.

· If $f(a)=f(b)\neq M$, similar: take $c=x_M$. $\square$

e.g.5 Using calculus theory discussed up to this point, prove that if $a>0$ and $b>0$, then

$$ax^3+bx+d=0$$

has exactly one solution.

Proof: ________________________.