Lecture 14, Thursday, October/26/2023

Outline

Also Known As

- Subtitution method (5.5, 5.6, AKA change of variable)
- Even and odd functions (5.6)

FTC2 states that in order to compute $\int_a^b f(x)dx$, it suffices to find a computable antiderivative F of f. In the second half of the course, we are going to introduce a few techniques for finding F, the first being the substitution method.

Substitution Method (Change of Variable)

e.g. What is an antiderivative of $f(x) = \frac{1}{\sqrt{x}} \cos\sqrt{x}$?

$2(\sqrt{x})'$ $\qquad$ $\sin'$

- $f(x) = 2(\sqrt{x})' \sin'(\sqrt{x})$
- $f(x) = 2\frac{d}{dx}(\sin\sqrt{x}) = \frac{d}{dx}(2\sin\sqrt{x})$
- $F(x) = 2\sin\sqrt{x}$, $\quad F'(x) = f(x)$.

i.e., $\int \frac{1}{\sqrt{x}} \cos x \, dx = 2\sin\sqrt{x} + C$.

In general, the substitution method seeks to reverse the chain rule. It is usually more convenient to write in the form of indefinite integrals.

THEOREM ~~6~~ 5.5.6—The Substitution Rule If $\underline{u = g(x)}$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int \underbrace{f(g(x))}_{f(u)}\underbrace{g'(x)\,dx}_{du} = \int f(u)\,du.$$

It means LHS = RHS when both are expressed in terms of x.

Before proving it, let us demonstrate with one example.

e.g. 1
$$\int \sin^3 x\,dx = \int (1-\cos^2 x)\underbrace{\sin x\,dx}_{du??}$$

$u = -\cos x$

$\frac{du}{dx} = \sin x$

$du = \sin x\,dx$

$$= \int (1-u^2)\,du$$

$$= u - \frac{1}{3}u^3 + C$$

$$= -\cos x + \frac{1}{3}\cos^3 x + C.$$

exercise: check that this is correct using differentiation.

Proof of Substitution Rule:

- Since f is cts (on I), it has an antiderivative F (on I), by FTC I.

- Then for all x ($\in D$),

$$f(g(x))g'(x) = F'(g(x))g'(x) = (F\circ g)'(x),$$

So

$$\int f(g(x))g'(x)\,dx = (F\circ g)(x) + C = F(g(x)) + C.$$

- On the other hand,

$$\int f(u)\,du = F(u) + C = F(g(x)) + C,$$

So LHS = RHS.

$\square$

e.g. 2 $\int x\sqrt{2x+1}\,dx =$ ______

$$= \frac{1}{4}\left(\frac{2}{5}(2x+1)^{5/2} - \frac{2}{3}(2x+1)^{3/2}\right) + C.$$

e.g. 3 If f satisfies the condition in the substitution rule, then

$$\int f(Ax+B)\,dx = \int f(u)\frac{1}{A}\,du = \frac{1}{A}F(u) + C \quad \begin{pmatrix} u = Ax+B \\ du = A\,dx \end{pmatrix}$$

$$= \frac{1}{A}F(Ax+B) + C.$$

e.g. 4
- $\int \sec^2(5x+1)\,dx = \frac{1}{5}\tan(5x+1) + C.$
- $\int \cos(7\theta+3)\,dx = \frac{1}{7}\sin(7\theta+3) + C.$

After getting used to the notation, one may write $d(g(x))$ instead of du if $u = g(x)$.

e.g. 5 $\int \sin^3(x)\,dx = \int (1-\cos^2 x)\sin x\,dx = \int (\cos^2 x - 1)(-\sin x\,dx)$

$$= \int [(\cos x)^2 - 1]\,d(\cos x) = \frac{1}{3}(\cos x)^3 - \cos x + C$$

The substitution rule above can help in computing $\int_a^b f(x)\,dx$.

e.g.6 $\displaystyle\int_{-1}^{1} \underbrace{3x^2\sqrt{x^3+1}}_{f}\,dx = ?$

- Find an antiderivative of f first.

- $\displaystyle\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{x^3+1}\,d(x^3+1) = \underbrace{\frac{2}{3}(x^3+1)^{3/2}}_{F(x)} + C$

$u = x^3+1$, $du = 3x^2\,dx$

- Hence, $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx \overset{FTC2}{=} F(1) - F(-1) = \frac{2}{3}\left(2^{3/2} - 0\right) = \frac{4\sqrt{2}}{3}$.

Alternatively, one may do the following: $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = ?$

- Let $u = x^3+1$, $du = 3x^2\,dx$.
- When $x=-1$, $u=0$; when $x=1$, $u=2$.
- $\displaystyle\int_{-1}^{1} 3x^3\sqrt{x^3+1}\,dx = \int_0^2 \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big|_{u=0}^{2} = \frac{2}{3}\cdot 2\sqrt{2}$.

This second method in e.g.6 is summarized below.

THEOREM ~~7~~ 5.6.7—Substitution in Definite Integrals If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

<u>Proof</u>: • Let F be any antiderivative of f on range(g).

• Then $\int_{g(a)}^{g(b)} f(u)du = F(g(b)) - F(g(a)) = (F\circ g)(x)\Big|_a^b$. ①

f is cts.

• Since $g'(x)$ is continuous, so is $f(g(x))g'(x)$.

• Since $(F\circ g)'(x) = F'(g(x))g'(x) = f(g(x))g'(x)$, we have

$$\int_a^b f(g(x))g'(x)dx \overset{\text{FTC2}}{=\!=} (F\circ g)(x)\Big|_a^b. \quad ②$$

• By ① & ②, we are done. $\square$

<u>e.g. 7</u> $\int_0^{\pi/4} (\sin 2x - 2\sin^2 x \sin 2x)\, dx$

$$= \int_0^{\pi/4} (1 - 2\sin^2 x)\sin(2x)\, dx$$

$$= \int_0^{\pi/4} \cos(2x)\sin(2x)\, dx$$

$$= \frac{1}{2}\int_0^1 u\, du$$

$$= \frac{1}{2}\cdot\frac{1}{2}(1^2 - 0^2) = \frac{1}{4}.$$

$u = \sin 2x$
$du = 2\cos(2x)dx$
$x = 0 \Rightarrow u = 0$
$x = \pi/4 \Rightarrow u = 1$

In-Class Discussion

e.g. 8 Compute $\int \sin x \cos x \, dx$ using:

1. Double-angle formula.
2. Substitution with $u = \sin x$.
3. Substitution with $u = \cos x$.

Which method is correct? Which one is not?

Message ______________________.

Even and Odd Functions

Def: A function $f: D \to \mathbb{R}$ is called

This implies D is symmetric about $x=0$.

- an even function, if $f(x) = f(-x)$ for all $x \in D$;
- an odd function, if $f(x) = -f(-x)$ for all $x \in D$.

Theorem (Integrals of Symmetric functions)

Let $f : [-a, a] \to \mathbb{R}$ be an integrable function.

- *If f is an even function, then* $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.
- *If f is an odd function, then* $\int_{-a}^{a} f(x)\,dx = 0$.

Before proving the theorem, let us see two examples.

e.g. 9 $$\int_{-\sqrt{2}}^{\sqrt{2}} (15x^4 - 4x^3 + 6x^2 + 7x)\, dx = 32\sqrt{2}.$$

Sol: Call the integral I. Then

$$I = \int_{-\sqrt{2}}^{\sqrt{2}} \underbrace{(15x^4+6x^2)}_{\text{even}}\, dx + \underbrace{\int_{-\sqrt{2}}^{\sqrt{2}} \underbrace{(7x-4x^3)}_{\text{odd}}\, dx}_{=0}$$

$$= 2\int_0^{\sqrt{2}} (15x^4+6x^2)\,dx = 2\left(\left[3x^5 + 2x^3\right]\Big|_0^{\sqrt{2}}\right)$$

$$= 2(3\cdot 4\sqrt{2} + 4\sqrt{2}) = 32\sqrt{2}.$$

e.g. 10 $$\underbrace{\int_{-1}^{3} (x+1)^2(x-3)^2\, dx}_{I} = \frac{512}{15}.$$

Sol: · Observe the symmetry of integrand about $x=2$.

· Let $u = x-1$. Then $du = dx$.

· $$I = \int_{-2}^{2} (u+2)^2(u-2)^2\, du = \int_{-2}^{2} \underbrace{(u^2-4)^2}_{\text{Even}}\, du$$

$$= 2\int_0^2 (u^2-4)^2\, du = 2\int_0^2 (u^4 - 8u^2 + 16)\,du$$

$$= 2\left(\frac{1}{5}2^5 - \frac{8}{3}2^3 + 16\cdot 2\right) = \frac{512}{15}.$$

Proof: $\int_{-a}^{a} f(x)dx = \underbrace{\int_0^a f(x)dx}_{I_1} + \underbrace{\int_{-a}^{0} f(x)dx}_{I_2}$.

(1) If f is even, then

$$I_2 \underset{\text{f is even}}{=} \int_{-a}^{0} f(-x)dx \underset{u=-x,\ du=-dx}{=} \int_a^0 f(u)(-du) = \int_0^a f(u)du = I_1$$

Hence $\int_{-a}^{a} f(x)dx = 2I_1 = 2\int_0^a f(x)dx$.

(2) If f is odd, then

$$I_2 \underset{\text{f is odd}}{=} \int_{-a}^{0} -f(-x)dx = -\int_{-a}^{0} f(-x)dx \underset{\text{by (1)}}{=} -I_1$$

Hence $\int_{-a}^{a} f(x)dx = I_1 - I_1 = 0$. $\square$