Lecture 1, Tuesday, September/05/2023

Outline

- What is calculus?
- Functions (notation $f : D \to Y$)
- Rates of change (2.1)
- Limits of functions (2.2)

What is Calculus?

Calculus is the study of continuous changes and limits.

Calculus = "calculus of infinitesimal" (old name)

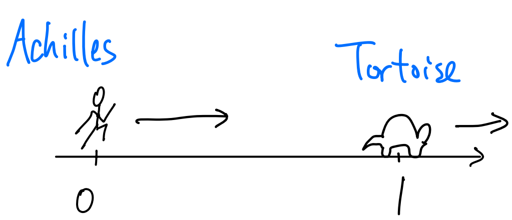
calculus: a formal system of symbolic expression

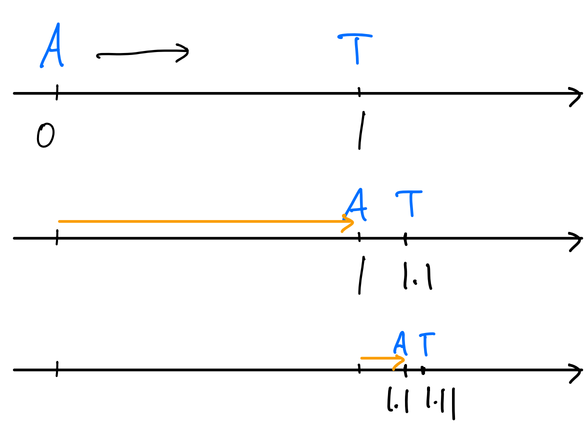
infinitesimal: "An amount that gets arbitrarily close to 0"

$$\boxed{\text{Calculus} < \begin{array}{l}\text{differential calculus}\\ \text{integral calculus}\end{array}}$$

Motivating Example 1: Achilles and the tortoise / Zeno's paradox

(Around 500– 400 B.C.)



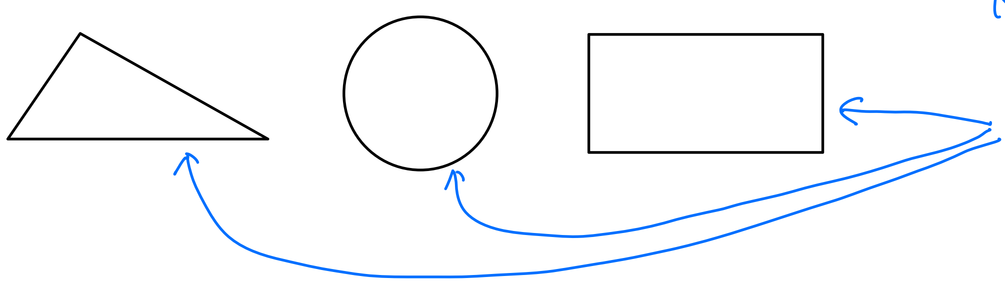


Every time Achilles reaches the tortoise' (original) position, the tortoise will be ahead again.

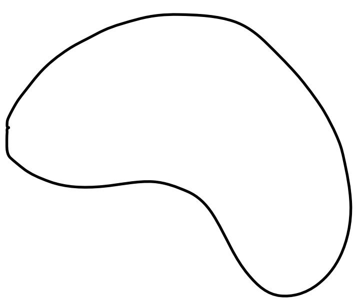
Therefore, Achilles will never be able to catch up with the tortoise.

This is obviously false, and calculus can explain this.

Motivating Example 2 : area of shapes



Know how to find areas

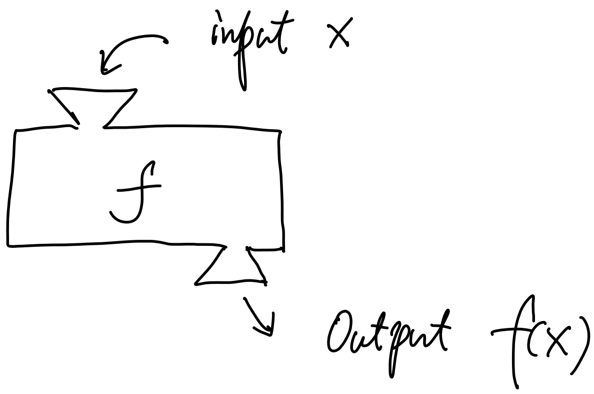


What is its area? Even more importantly, what would be a reasonable definition of its area?

Calculus can be used to formalize this.

Functions

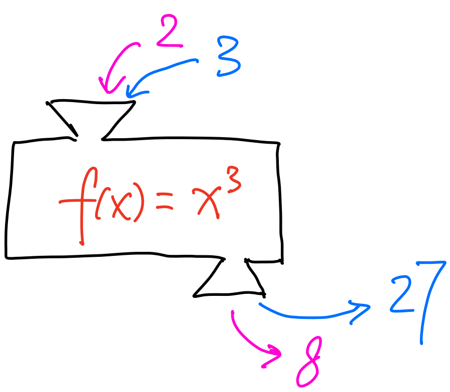
Intuition:



x in Domain D

$f(x)$ in Codomain Y

e.g.



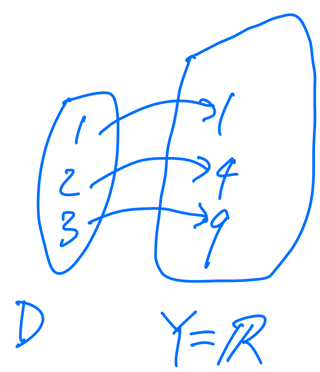
Def: A function consists of three objects:

- a nonempty set D, called the domain; Set of all inputs
- a nonempty set Y, called the codomain; A set that can contain all outputs.
- a rule f such that, for each element $x_0 \in D$, f assigns exactly one element in Y to x_0; this element is denoted by $f(x_0)$.

Notation: $f: D \to Y$

domain codomain

e.g. $f: \{1,2,3\} \to \mathbb{R}$, $f(x) = x^2$.



- $\mathbb{R}$ denotes the set of all real numbers.
- All outputs by f are real numbers.
- Not all real numbers appear in the outputs.
- The range of f is $\{1, 4, 9\}$.

Def: The range of a function $f: D \to Y$ is the set

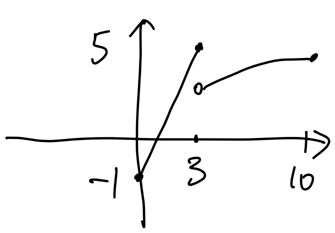
$$\text{range}(f) := \{ f(x) : x \in D \}.$$

This symbol $:=$ means "is defined to be" or "equals by definition".

Remarks

- Given $f: D \to Y$, we have $\text{range}(f) \subseteq Y$.
- In general, $\text{range}(f) \neq Y$.
- Y is a pre-specified "container" that all the outputs must lie in.

e.g. $f: [0, 10) \to \mathbb{R}$, $f(x) = \begin{cases} 2x-1, & 0 \le x \le 3 \\ \sqrt{x+6}, & 3 < x < 10 \end{cases}$.



Piecewise-defined function.

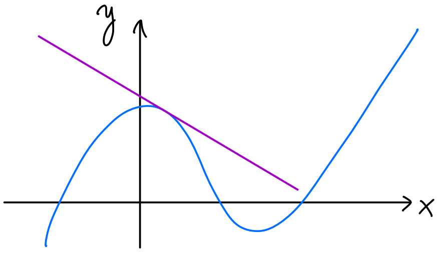
$\text{range}(f) = [-1, 5]$.

In many occasions where the domain and codomain are clear from the context or not important, one may just use rule f to denote the function.

Remarks If x and y are variables related by $y=f(x)$, then x is called the independent variable and y is call the dependent variable (e.g., time vs position; production vs cost).

Rates of Change

Motivations:
- Movement speed
- Tangent lines



How do we define these concepts in a formal and reasonable way?

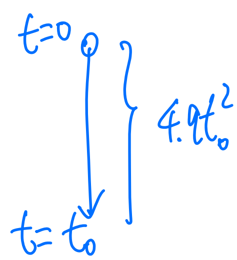
- Average rates of change
- Instantaneous rates of change

Average Rates of Change

Def: Let $y = f(x)$ and suppose $[a,b] \subseteq D$. The average
domain
rate of change of y with respect to x over $[a,b]$ is
(w. r. t.)

$$\frac{f(b)-f(a)}{b-a}.$$

e.g. Free fall (movement speed)



y: distance fallen (meters)

t: time (seconds)

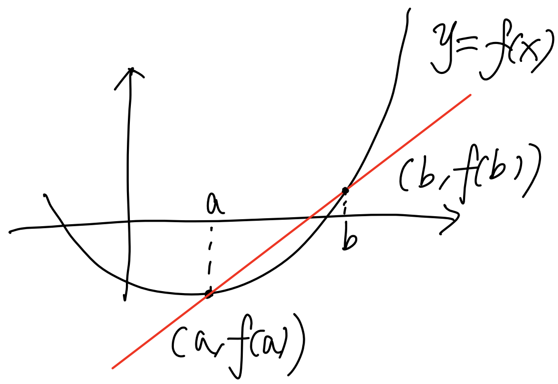
$y = f(t) = 4.9t^2$.

Average speed between the first and third second is

$$\frac{f(3)-f(1)}{3-1} = \frac{(4.9)8}{2} = 19.6 \quad (m/s)$$

e.g. Secant lines

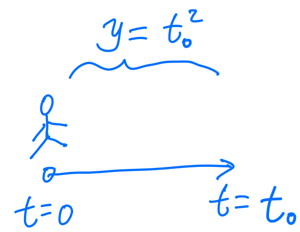
Def: A secant line to a curve is a line joining two distinct points of the curve.



slope of the secant line to the graph of $y=f(x)$ through $(a, f(a))$ and $(b, f(b))$ = Average rate of change of y w.r.t. x over $[a, b]$.

Instantaneous Rates of Change

Tracy is moving to the right,
what is her speed at $t=1$?



Idea: Use average speed from $t=1$ to $t=b$ to approximate, but take b "very close" to 1.

$$y=f(t)=t^2.$$

b	0.9	0.99	0.999	1.001	1.01	1.1
$\frac{f(b)-f(1)}{b-1}$	1.9	1.99	1.999	2.001	2.01	2.1

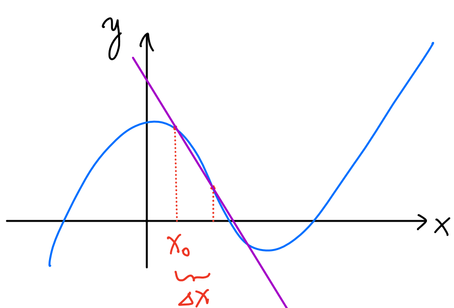
- It seems that the average speed is getting closer to 2 as b approaches 1. We would say that the (instantaneous) speed of Tracy at $t=1$ is 2.
- Speed at $t=a$ is "limit of average speed between $t=a$ and $t=b$ as b approaches a".

Let $y=f(x)$ and fix $x_0 \in D$. domain

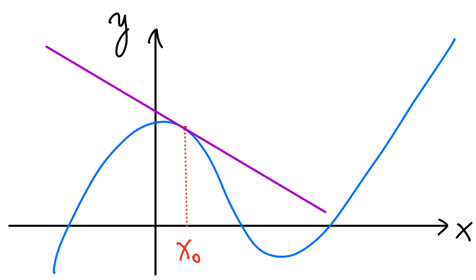
- Consider a change in x-value by $\Delta x \, (\neq 0)$.
- Let $\Delta y := f(x_0+\Delta x) - f(x_0)$, the change in y-value.
- Then $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_0+\Delta x) - f(x_0)}{x_0+\Delta x - x_0}$ is the average rate of change in y value between $x=x_0$ and $x=x_0+\Delta x$.
- The instantaneous rate of change of y at $x=x_0$ is

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

Geometry : Tangent Lines



As $\Delta x \to 0$



Secant line :

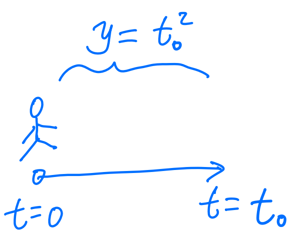
Slope = Average rate of change between $x=x_0$ & $x=x_0+\Delta x$

Tangent line :

Slope = Instantaneous rate of change at $x=x_0$.

Def The tangent line to the graph of $y=f(x)$ at a point $P=(x_0, f(x_0))$ on the graph is the line through P with slope being the instananeous rate of change of y with respect to x at $x=x_0$.

e.g. Speed.



What is tracy speed at $t=1$?

fixed

speed at $t=1$ is

$$\lim_{\Delta t\to 0}\frac{(1+\Delta t)^2-1^2}{\Delta t}=\lim_{\Delta t\to 0}\frac{\Delta t^2+2\Delta t}{\Delta t}=\lim_{\Delta t\to 0}\Delta t+2=2.$$

(Try it yourself using Desmos!)

e.g. Consider the graph of $y = x^2$. Consider fixing $x = 1$.

- One can check that the secant line between $(1,1)$ and $(1+\Delta x, (1+\Delta x)^2)$ is

$$y = (\Delta x + 2)x - \Delta x - 1$$

- Analyze the secant lines as $\Delta x \to 0$.

- Since the limit of the secant slopes is 2, the tangent line to the graph at $(1,1)$ is

$$y = 2x - 1.$$

Discussion summary

Average rate of change $\rightleftharpoons$ slope of secant line

Instantaneous rate of change $\rightleftharpoons$ slop of tangent line

Limits

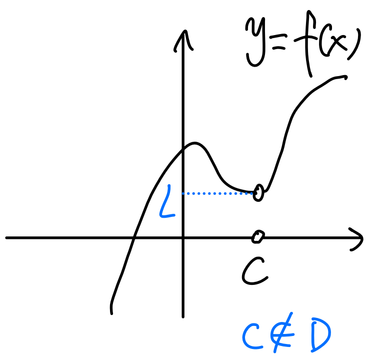
Def: Let A and B be sets. The set A minus B is the set $A \setminus B := \{x \in A : x \notin B\}$.

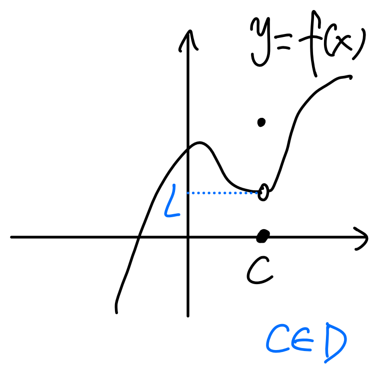
When we discuss $\lim_{x \to c} f(x)$, we assume that $f : D \to \mathbb{R}$ is defined near c (except possibly at c); that is, there exists $a > 0$ such that $(c-a, c+a) \setminus \{c\} \subseteq D$.

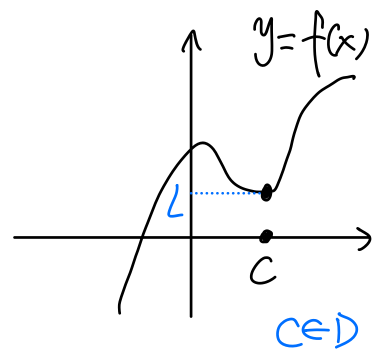
Notation (finite limit)

Let $L \in \mathbb{R}$ (so $L \neq \pm\infty$). The symbol $\lim_{x \to c} f(x) = L$ is read as "the limit of $f(x)$ as x approaches c is L".

e.g. For all of the following graphs, the functions f all satisfy $\lim_{x \to c} f(x) = L$.



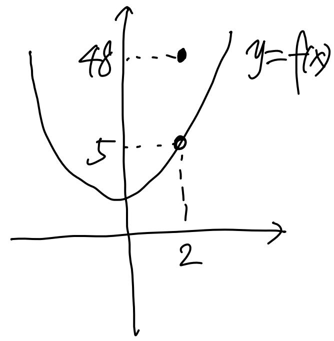




Remark

$\lim_{x\to c} f(x)$ measures the tendency of f as x approaches c; in general, $\lim_{x\to c} f(x) \neq f(c)$.

e.g. $f(x) = \begin{cases} x^2+1, & \text{if } x\neq 2; \\ 48, & \text{if } x=2. \end{cases}$

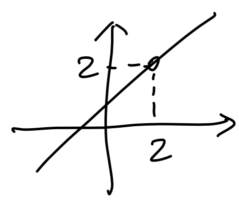


Then $\lim_{x\to 2} f(x) = 5$

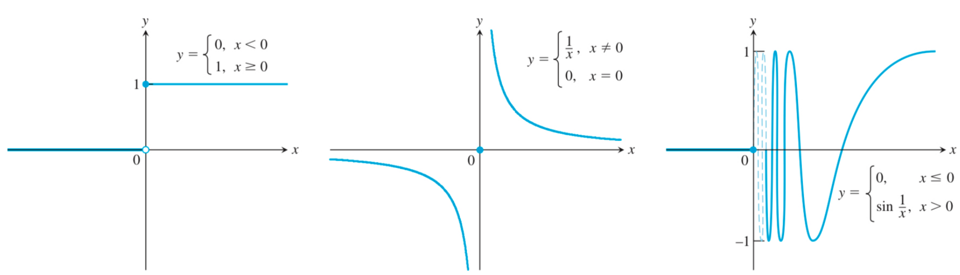
e.g. $\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)(x-1)}{x-1}$

Not defined at $x=1$, but is defined for all other real numbers

$= \lim_{x\to 1} (x+1) = 2$



e.g. $\lim_{x\to 0} f(x)$ does not exist for the following functions.



(a) Unit step function $U(x)$ (b) $g(x)$ (c) $f(x)$

Reason (informal):

(a) Approaching different numbers from different sides:

$f(x) \to 0$ as $x \to 0$ from left, and

$f(x) \to 1$ as $x \to 0$ from right:

not approaching the same number.

(b) $f(x)$ can get arbitrarily large no matter how close x gets to 0.

(c) Oscillating from the right between two numbers:

no matter how close x gets to 0 from the right, there exist two values $x = x_0$ and $x = x_1$ such that

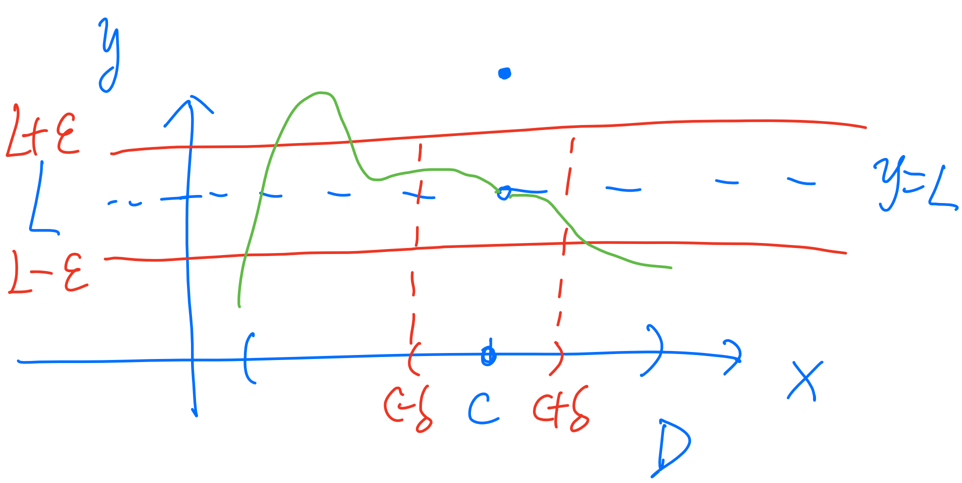
$$\sin\left(\frac{1}{x_0}\right) = -1 \quad \text{and} \quad \sin\left(\frac{1}{x_1}\right) = 1,$$

So $\sin\frac{1}{x}$ does not approach any single value L as $x \to 0$.

Lecture 2, Thursday, September/07/2023

Outline

- Formal definition of limits (2.3)
- Continuity (2.5)
- Limit properties / limit laws (2.2)
- Composition of continuous functions (2.5)
- Limit computation (basic)



Limits

$(c-a, c+a) \setminus \{c\}$
$\subseteq D$ for some $a>0$

Def: Let $f: D \to \mathbb{R}$ be a function defined on an open interval containing c, except possibly at c itself. Let $L \in \mathbb{R}$ (so $L \neq \pm\infty$). Then we write

$$\lim_{x \to c} f(x) = L$$

if, for all $\varepsilon > 0$, there exists a $\delta > 0$ such
(epsilon) (delta)

that, for all $x \in D$ with $0 < |x - c| < \delta$, we have
(for all x in domain close enough to c that are $\neq c$)

$$|f(x) - L| < \varepsilon.$$

($f(x)$ is close to L)

Alternative notation We could write $f(x) \to L$ as $x \to c$ to mean $\lim_{x \to c} f(x) = L$.

e.g. 1 Prove that $\lim_{x\to 5}(4x-6) = 14$. (Optional)

Proof: Let $\varepsilon > 0$ be fixed. We want $|4x-6-\overset{L}{\underline{14}}| < \varepsilon$ whenever $0 < |x-\underset{c}{\underline{5}}| < \delta$, and we want to choose such a δ. Note that

$$|4x-6-14| < \varepsilon \iff |4x-20| < \varepsilon \iff |x-5| < \frac{\varepsilon}{4}. \quad (*)$$

Set $\delta := \frac{\varepsilon}{4}$. By $(*)$, whenever x satisfies $0 < |x-5| < \delta$, we have $|4x-6-14| < \varepsilon$. By definition of limits,

$$\lim_{x\to 5}(4x-6) = 14.$$

$\square$

e.g. 2 Prove that $\lim_{x\to c} x^2 = c^2$. (Optional)

Proof: Let $\varepsilon > 0$ (be arbitrary, but fixed).

Q: How should we choose δ?

- We want to achieve $|x^2 - c^2| < \varepsilon$, i.e.,

$$|x+c||x-c| < \varepsilon.$$

- Narrow the search first: Consider x with $|x-c| < 1$.
- Now, $|x+c| = |x-c+2c| \underset{\text{triangle inequality}}{\leq} |x-c| + 2|c| < 1 + 2|c|$

- If $|x-c|<\delta$, then $|x+c||x-c| < |x+c|\delta < (1+2|c|)\delta$.

  still trying to find

- Hence, if we set $\delta := \frac{\varepsilon}{1+2|c|}$, then $|x+c||x-c| < \varepsilon$.

Pick $\delta := \min\left(1, \frac{\varepsilon}{1+2|c|}\right)$. Then, if $|x-c|<\delta$, then $|x-c|<1$ and $|x-c|<\frac{\varepsilon}{1+2|c|}$. By the argument above,

$$|x^2-c^2| = |x+c||x-c| < (1+|c|)\frac{\varepsilon}{1+|c|} = \varepsilon.$$

By definition, $\lim_{x\to c} x^2 = c^2$. $\square$

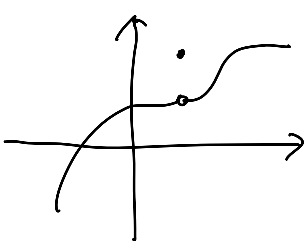
From e.g. 2 above, we see that even a very basic function may have a relatively complicated proof using δ-ε definition.

For this course, we will not test on δ-ε proofs.

Continuity

We have seen that in general, $\lim_{x\to c} f(x) \neq f(c)$.

e.g.



Def: Let $f: D \to \mathbb{R}$ be a function defined on an open interval containing c. We say that f is continuous at c if

This implies $c \in D$.

$$\lim_{x \to c} f(x) = f(c).$$

For this to be true, $\lim_{x \to c} f(x)$ must exist first.

A function defined on an open interval D is said to be continuous if it is continuous at every point in D.

The following functions are continuous (on domain D):

- Constant function: $f(x) = K$, where K constant; $D = \mathbb{R}$.
- Identity function: $f(x) = x$, $D = \mathbb{R}$.
- Absolute value function: $f(x) = |x|$, $D = \mathbb{R}$.
- Natural exponential function: $f(x) = e^x$, $D = \mathbb{R}$.
- Natural logarithmic function: $f(x) = \ln x$, $D = (0, \infty)$.

(Continued)

- Basic trigonometric functions: $f(x)=\sin x$, $g(x)=\cos x$, $D=\mathbb{R}$ for both.

All of these can be proven using δ-ε definition, which we omit.

By definition, limit of a continuous function can be computed by direct substitutions for any point in the domain: $\lim_{x\to 1} \sin x = \sin 1$.

Limit Properties

The following limit properties may make computations more convenient.

All of these can be proven using δ-ε definition, which we omit.

THEOREM ~~1~~ 2.2.1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to c}(f(x)+g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to c}(f(x)-g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x\to c}(k\cdot f(x)) = k\cdot L$
4. *Product Rule:* $\lim_{x\to c}(f(x)\cdot g(x)) = L\cdot M$
5. *Quotient Rule:* $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x\to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x\to c} f(x) = L > 0$.)

For example, the first and fourth law states that

$$\lim_{x\to c}\left(f(x)+g(x)\right)=\lim_{x\to c}f(x)+\lim_{x\to c}g(x)$$

and

$$\left(\lim_{x\to c}f(x)g(x)\right)=\left(\lim_{x\to c}f(x)\right)\left(\lim_{x\to c}g(x)\right),$$

given that both $\lim_{x\to c}f(x)$ and $\lim_{x\to c}g(x)$ exist as real numbers.

By limit properties, limits of algebraic combinations $(+,-,\times,\div,(\cdot)^n,\sqrt[n]{\cdot})$ of continuous function can also be computed by substitutions (for points in the domains).

<u>e.g.</u> $$\lim_{x\to 2}\frac{\sqrt{\ln x}}{x^2-3}=\frac{\sqrt{\ln 2}}{4-3}=\sqrt{\ln 2}.$$

<u>Reason</u>: $$\lim_{x\to 2}\frac{\sqrt{\ln x}}{x^2-3}=\frac{\lim_{x\to 2}\sqrt{\ln x}}{\lim_{x\to 2}(x^2-3)}$$ (limit law 5)

$$=\frac{\sqrt{\lim_{x\to 2}\ln x}}{\lim_{x\to 2}x^2-\lim_{x\to 2}3}$$ (limit law 7 and 2)

$$=\frac{\sqrt{\lim_{x\to 2}\ln x}}{\left(\lim_{x\to 2}x\right)^2-\lim_{x\to 2}3}$$ (limit law 6)

$$= \frac{\sqrt{\ln 2}}{2^2 - 3} \qquad \text{(continuity)}$$

$$= \sqrt{\ln 2}.$$

<u>Def</u>. A polynomial is a function of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$$

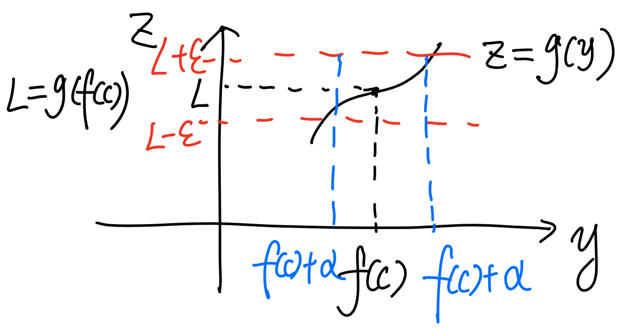
where $a_0, a_1, \dots, a_n$ are constants. A rational function is a function of the form $p(x)/q(x)$, where $p(x)$ and $q(x)$ are both polynomials (and $q(x)$ is not the zero function).

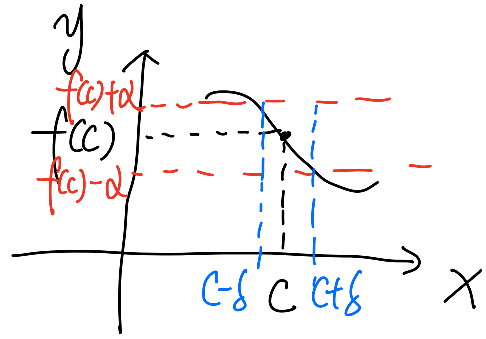
<u>Remark</u>: Since polynomials and rational functions are finite algebraic combinations of continuous functions $f(x) = K$ and $g(x) = x$, by limit laws, their limits (for points in the domain) can be found by direct substitutions. Hence they are continuous (in their domains).

Composition and continuity

> **Theorem** (2.5.9) If f is continuous at c and g is continuous at $f(c)$, then $g \circ f$ is continuous at c.

Proof: (Optional)





We show that $\lim_{x \to c} (g \circ f)(x) = (g \circ f)(c)$.

Let $z = g(y)$, $y = f(x)$, so $z = (g \circ f)(x)$.

Let $\varepsilon > 0$. Since g is continuous at $f(c)$, $\exists\, \alpha > 0$ "there exists"
such that $\forall y$ (in the domain of g), ($\forall$ means "for all")

$$|y - f(c)| < \alpha \Rightarrow |g(y) - g(f(c))| < \varepsilon. \quad (1)$$

Since f is continuous at c, $\exists\, \delta > 0$ such that $\forall x$ (in the domain of f),

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \alpha. \quad (2)$$

Now ①, ② $\Rightarrow$ $\forall x$ (in the domain of f),

$$|x-c|<\delta \Rightarrow |g(f(x))-g(f(c))|<\varepsilon.$$

Hence, $\lim_{x\to c}(g\circ f)(x) = (g\circ f)(c)$. $\square$

In fact, a more general result holds.

Theorem (2.5.10) If g is continuous at b, and $b=\lim_{x\to c} f(x)$, then

$$\lim_{x\to c} g(f(x)) = g(b) = g\left(\lim_{x\to c} f(x)\right).$$

Its proof is similar. A consequence of Thm 2.5.10 is that you may "move" the "$\lim_{x\to c}$" inside whenever the outside function is continuous (provided that the limit inside exists).

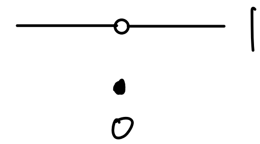
e.g. $g(x) = \sin x$, $f(x) = \begin{cases} 1, & \text{if } x=0 \\ x, & \text{if } x\neq 0 \end{cases}$

By Theorem 2.5.10, since g is continuous,

$$\lim_{x\to 0} g(f(x)) = g\left(\lim_{x\to 0} f(x)\right) = g(0) = \sin 0 = 0.$$

Note that you cannot do the same in general if the "outside" function is not continuous.

<u>e.g.</u> Let $f(x)=x$ and $g(x)=\begin{cases}0, & \text{if } x=0\\ 1, & \text{if } x\neq 0\end{cases}$



Then $g(f(x))=g(x)$, so $\lim\limits_{x\to 0} g(f(x))=1$.

But $g\left(\lim\limits_{x\to 0} f(x)\right)=g(0)=0\neq 1=\lim\limits_{x\to 0} g(f(x))$.

Here the outside function g is not continuous.

Limit Computation

How to compute limit if we cannot make direct substitution?

- Eliminate zero denominator:

e.g. $$\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}=\lim_{x\to 0}\frac{(\sqrt{x^2+100}-10)(\sqrt{x^2+100}+10)}{x^2(\sqrt{x^2+100}+10)}$$

$$=\lim_{x\to 0}\frac{x^2}{x^2(\sqrt{x^2+100}+10)}=\lim_{x\to 0}\frac{1}{\sqrt{x^2+100}+10}=\frac{1}{20}.$$

- Squeeze theorem (Sandwich theorem)

THEOREM ~~4~~ 2.2.4—The Sandwich Theorem Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

e.g. $\lim_{x \to 0} x^2 \sin\frac{1}{x} = ?$

As a final property of limit in this lecture, we have the following theorem.

THEOREM ~~5~~ 2.2.5 If $f(x) \leq g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$