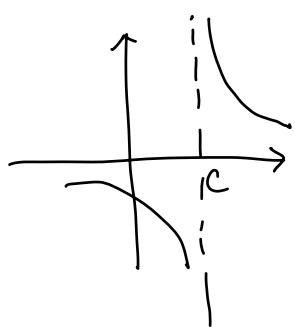


Lecture 4, Thursday, September 14/2023

Outline

- Infinite limits (2.6)
- Oblique asymptotes (2.6)
- The derivative at a point (3.1)
- Derivative functions (3.2)

Infinite Limits

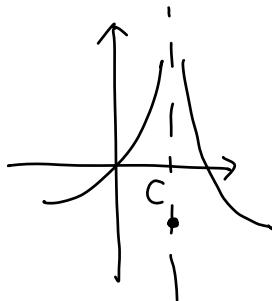


$$\lim_{x \rightarrow c^-} f(x) = -\infty$$

$$\lim_{x \rightarrow c^+} f(x) = \infty$$

$$\lim_{x \rightarrow c} f(x) \text{ D.N.E.}$$

(not even as an infinite limit)



$$\lim_{x \rightarrow c} f(x) = \infty$$

(Again, ∞ means $+\infty$, not $\pm\infty$)

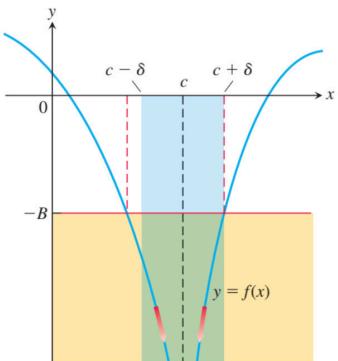


FIGURE 2.62 For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies below the line $y = -B$.

DEFINITIONS

1. We say that $f(x)$ approaches infinity as x approaches c , and write

$$\lim_{x \rightarrow c} f(x) = \infty,$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \implies f(x) > B.$$

2. We say that $f(x)$ approaches minus infinity as x approaches c , and write

$$\lim_{x \rightarrow c} f(x) = -\infty,$$

if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \implies f(x) < -B.$$

Q: What is $\lim_{x \rightarrow 0^+} \frac{1}{x} (\sin \frac{1}{x} + 1)$?

A: _____

Def: If $\lim_{x \rightarrow a^-} f(x) = \infty$ or $\lim_{x \rightarrow a^-} f(x) = -\infty$ or $\lim_{x \rightarrow a^+} f(x) = \infty$ or $\lim_{x \rightarrow a^+} f(x) = -\infty$, then the line $y=a$ is called a **vertical asymptote** of $y=f(x)$.

Computation

It can be shown formally (using definitions) that the familiar limit properties still hold when dealing with infinite limits.

We use the following Convention:

- $\infty + \infty = \infty$; $a \pm \infty = \pm \infty$, $\forall a \in \mathbb{R}$.
- $a \times (\pm \infty) = \pm \infty$ ($\mp \infty$), $\frac{\pm \infty}{a} = \pm \infty$ ($\mp \infty$), if $a > 0$ ($a < 0$).
- $\infty \times (\pm \infty) = \pm \infty$, $-\infty \times (\pm \infty) = \mp \infty$.
- $\frac{a}{\pm \infty} = 0$, $\forall a \in \mathbb{R}$, $\frac{\pm \infty}{\pm 0} = \pm \infty$ ($\mp \infty$) if $a > 0$ ($a < 0$).
- $\sqrt[n]{\infty} = \infty$.

Remarks

1. The "rules" above are not real number operations. They serve as a shortcut for you to use the limit laws.
2. $\infty - \infty$, $0 \times \infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$ are "undefined".
(indeterminate forms)

e.g.1 $\lim_{x \rightarrow -\infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} = \underline{\quad ? \quad} = \frac{5}{3}.$

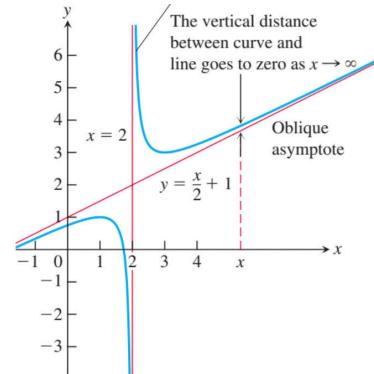
e.g.2 $\lim_{x \rightarrow \infty} (\sqrt{x^2 + 1} - x) = \underline{\quad ? \quad} = 0$

e.g.3 $\lim_{x \rightarrow 2} \frac{x-3}{x^2-4} = ?$

e.g.4 $\lim_{x \rightarrow -\infty} \frac{2x^5 + 6x^4 + 1}{3x^5 + x - 7} = ?$

Obligee Asymptotes

More generally, an asymptote does not have to be horizontal or vertical.



Definition

The line given by $y = Ax + B$ is called an **oblique asymptote** of the graph of a function $y = f(x)$ if

$$\lim_{x \rightarrow \infty} [f(x) - Ax - B] = 0 \quad \text{or} \quad \lim_{x \rightarrow -\infty} [f(x) - Ax - B] = 0.$$

Suppose that $y = f(x)$ has an oblique asymptote $y = Ax + B$ as $x \rightarrow \infty$. Then $f(x) - Ax - B \rightarrow 0$ as $x \rightarrow \infty$, so

$$\lim_{x \rightarrow \infty} \frac{f(x) - Ax - B}{x} = 0.$$

From this we can deduce that

$$A = \lim_{x \rightarrow \infty} \frac{f(x)}{x} \quad \text{and} \quad B = \lim_{x \rightarrow \infty} [f(x) - Ax].$$

This reasoning is also valid if $x \rightarrow \infty$ is replaced by $x \rightarrow -\infty$.

- ▶ The reasoning above gives a strategy of finding the oblique asymptotes.
- ▶ The limits above may not exist. If one of the limits does not exist, then the graph does not have an oblique asymptotes. (For example, consider $f(x) := x + \sqrt{x}$.)

Why are the formulae above correct?

(a plural form of formula)

$$0 = \lim_{x \rightarrow \infty} \frac{f(x) - Ax - B}{x} = \underbrace{\lim_{x \rightarrow \infty} \frac{f(x)}{x}}_A - \underbrace{\lim_{x \rightarrow \infty} \frac{Ax}{x}}_0 - \underbrace{\lim_{x \rightarrow \infty} \frac{B}{x}}_0$$

$$\Rightarrow A = \lim_{x \rightarrow \infty} \frac{f(x)}{x}$$

$$0 = \lim_{x \rightarrow \infty} (f(x) - Ax - B) = \lim_{x \rightarrow \infty} (f(x) - Ax) - \cancel{\lim_{x \rightarrow \infty} B}$$

$$\Rightarrow B = \lim_{x \rightarrow \infty} (f(x) - Ax)$$

e.g.5 Find the oblique asymptotes of the graph of

$$y = \frac{x^2 - 3}{2x - 4}.$$

Sol. $A = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{x^2 - 3}{2x^2 - 4x} = \lim_{x \rightarrow \infty} \frac{1 - \frac{3}{x^2}}{2 - \frac{4}{x}} = \frac{1}{2}$

$$B = \lim_{x \rightarrow \infty} (f(x) - Ax) = \lim_{x \rightarrow \infty} \frac{x^2 - 3 - x^2 + 2x}{2x - 4} = \lim_{x \rightarrow \infty} \frac{2x - 3}{2x - 4} = 1$$

$\therefore y = \frac{1}{2}x + 1$ is the oblique asymptote as $x \rightarrow \infty$.

By taking $\lim_{x \rightarrow -\infty}$, we found the same asymptote as $x \rightarrow -\infty$.

e.g.6 Do the same for $y = x + \sqrt{x}$.

Sol. $A = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{\sqrt{x}}\right) = 1.$

But $\lim_{x \rightarrow \infty} (f(x) - Ax) = \lim_{x \rightarrow \infty} \sqrt{x} = \infty$
does not exist in \mathbb{R} .

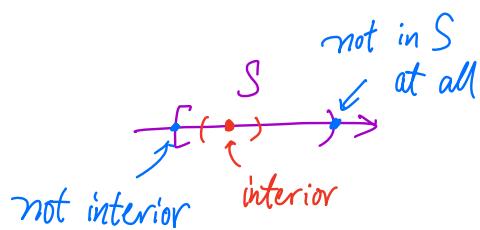
No oblique asymptote as $x \rightarrow \infty$.

Derivatives

Def: Let $S \subseteq \mathbb{R}$. A point $c \in S$ is called an **interior point** of S if there exists $\alpha > 0$ such that $(c-\alpha, c+\alpha) \subseteq S$.

Remark Every point in an open interval I

is an interior point of I .



Def: Let $f: D \rightarrow \mathbb{R}$ be a function and let x_0 be an interior point of D . Then the **derivative of f at x_0** , denoted by $f'(x_0)$, is defined by

$$f'(x_0) := \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

provided that the limit exists (as a real number).

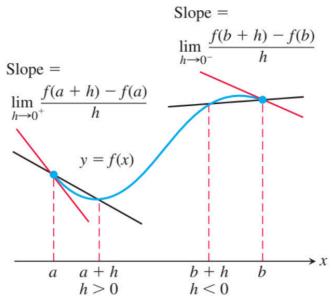
Note: f' is pronounced as "f prime".

As discussed in Lecture 1, for $y = f(x)$, $f'(x_0)$ is:

- Slope of tangent line of graph at $x = x_0$.
- Instantaneous rate of change of y w.r.t. X at $x = x_0$.

We can define one-sided derivative

Similarly.



Def.: Let x_0 be in the domain of f .

- The right-hand derivative of f at x_0 is $f'_+(x_0)$, where

$$f'_+(x_0) := \lim_{h \rightarrow 0^+} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{x \rightarrow x_0^+} \frac{f(x) - f(x_0)}{x - x_0}.$$

- The left-hand derivative of f at x_0 is $f'_-(x_0)$, where

$$f'_-(x_0) := \lim_{h \rightarrow 0^-} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{x \rightarrow x_0^-} \frac{f(x) - f(x_0)}{x - x_0}.$$

e.g.7 $f(x) = |x|.$

$$= \begin{cases} 1, & \text{if } h > 0 \\ -1, & \text{if } h < 0 \end{cases}$$

Since $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{|h|}{h}$ does not exist,
 $f'(0)$ does not exist.

e.g.8 $f(x) = \begin{cases} x \sin(\frac{1}{x}), & \text{if } x \neq 0 \\ 0, & \text{if } x=0. \end{cases}$ What is $f'(0)$?

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h \sin \frac{1}{h}}{h} = \lim_{h \rightarrow 0} \sin \frac{1}{h},$$

which does not exist. Note that f is cts at 0.

e.g.9 $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & \text{if } x \neq 0 \\ 0, & \text{if } x=0. \end{cases}$ What is $f'(0)$?

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h^2 \sin(\frac{1}{h})}{h}$$

$$= \lim_{\substack{h \rightarrow 0 \\ \text{w w bdd}}} h \sin(\frac{1}{h}) = 0 \quad \leftarrow \begin{array}{l} \text{Horizontal} \\ \text{tangent line in} \\ \text{graph.} \end{array}$$

e.g.10 $f(x) = \sqrt{x}$. What is $f'_+(0)$?

$$f'_+(0) = \lim_{h \rightarrow 0^+} \frac{\sqrt{0+h} - \sqrt{0}}{h} = \lim_{h \rightarrow 0^+} \frac{1}{\sqrt{h}} = \infty,$$

So $f'_+(0)$ D.N.E. (as a real number).

Def.: Let f be a function.

- If $f'(x_0)$ exists, then f is said to be **differentiable at x_0** .
- If $f'_+(x_0)$ exists, then f is **right differentiable at x_0** .
- If $f'_-(x_0)$ exists, then f is **left differentiable at x_0** .
- f is said to be **differentiable on D** if f is differentiable at every interior point of D , and f is one-sided differentiable at every endpoint. (Note that open intervals contain no endpoint.)

Derivative Functions

Note that f' itself defines a function: if $f: D \rightarrow \mathbb{R}$, then

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h},$$

with domain of f' being the set of all $x \in D$ at which f is differentiable.

e.g. $f(x) = |x| \Rightarrow f'(x) = \begin{cases} -1, & \text{if } x < 0 \\ 1, & \text{if } x > 0 \end{cases}$
 $D = \mathbb{R} \setminus \{0\}.$

e.g. $f(x) = x^2 \Rightarrow f'(x) = 2x$
 $D = \mathbb{R}$

$$f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{h^2 + 2xh}{h} = \lim_{h \rightarrow 0} (h + 2x) = 2x.$$

Alternative notation for f'

If $y = f(x)$, then:

Leibniz's notation

- f' is the same as y' , $\frac{dy}{dx}$, $\frac{df}{dx}$, $\frac{d}{dx} f$. (Derivative functions)
- $f'(x_0)$ is the same as $y'(x_0)$, $\left. \frac{dy}{dx} \right|_{x=x_0}$, $\left. \frac{d}{dx} f \right|_{x=x_0}$. (Derivative at a point)