Lecture 22, Thursday, November / 23 / 2023

Outline

- Trigonometric integrals (8.3)
  - ↳ Integrands with $\sqrt{1 \pm \cos 2\theta}$
  - ↳ $\int \sin mx \sin nx \, dx$, $\int \sin mx \cos nx \, dx$, $\int \cos mx \cos nx \, dx$
- Trigonometric substitutions (8.4)
- Integration by partial fractions (8.5)
  - ↳ Partial fraction decomposition
  - ↳ Finding undetermined coefficients

Trigonometric Integrals

Integrands with $\sqrt{1 \pm \cos 2\theta}$

Trigonometric identities involving squares, such as

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \quad \text{and} \quad \sin^2\theta = \frac{1-\cos 2\theta}{2},$$

may help evaluate integrals.

e.g. $\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx =$ ______________________

______________________ $= \frac{\sqrt{2}}{2}$.

$\int \sin mx \sin nx \, dx$, $\int \sin mx \cos nx \, dx$, $\int \cos mx \cos nx \, dx$

These integrals can be computed by using integration by parts twice. Alternatively, one may memorize the following identities:

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x],$$
$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x],$$
$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x].$$

e.g. $\int \sin 4x \sin 8x \, dx = \frac{1}{2}\int (\cos 4x - \cos 12x)\, dx$

$\rightarrow$ Try it again using integration by parts.

Trigonometric Substitution

We start by considering $\int \frac{dx}{\sqrt{9+x^2}}$, with integrand defined on $\mathbb{R}$.

If we consider $x = 3\tan\theta$, then all x will be covered by considering all $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Hence, we may make the following substitution

$$\int \frac{dx}{\sqrt{9+x^2}} = \int \frac{3\sec^2\theta \, d\theta}{\sqrt{9+9\tan^2\theta}}$$

Let $x = 3\tan\theta$, $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$

$dx = 3\sec^2\theta \, d\theta$

$$= 3\int \frac{\sec^2\theta \, d\theta}{3\sqrt{\sec^2\theta}} = \int \sec\theta \, d\theta$$

$\sec\theta > 0$ for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C$$

$\tan\theta = x/3$,

$\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\frac{x^2}{9}}$

$= \frac{\sqrt{9+x^2}}{3}$

The technique of trigonometric substitutions may be useful when integrating functions involving $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$ or $\sqrt{a^2 + x^2}$. The key idea is to make use the following trigonometric identities:

- $1 - \sin^2\theta = \cos^2\theta$.
- $\sec^2\theta - 1 = \tan^2\theta$.
- $1 + \tan^2\theta = \sec^2\theta$.

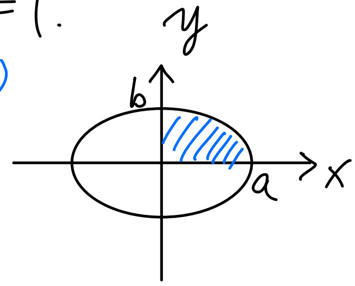
Depending on whether the integrand involves $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$ or $\sqrt{a^2 + x^2}$, we can make the substitution $x = a\sin\theta$, $x = a\sec\theta$ or $x = a\tan\theta$ accordingly.

Remark: For the substitution $x=f(\theta)$, the domain of f is chosen so that f covers all the possible values of x.

e.g. Compute $\int_{-4}^{-3} \frac{1}{x^2\sqrt{x^2-4}}\,dx =: I$.

- $I = \frac{1}{4}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{5}}{3}\right)$.

e.g. Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
($a>0$, $b>0$)



Sol: • Area $A = 4A_1$, where A_1 is the shaded area.

• $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \iff y^2 = b^2\left(1-\frac{x^2}{a^2}\right) = \frac{b^2}{a^2}(a^2-x^2)$

• $A_1 = \int_0^a y\,dx = \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$

$=$

• $A = 4A_1 = ab\pi$.

Integration by Partial Fractions

We now introduce a method that works well for integrating rational functions $P(x)/Q(x)$. Consider finding

i.e., P & Q are polynomials

$$\int \frac{5x-3}{x^2-2x-3}\,dx.$$

This becomes easy if we realize that

$$\frac{5x-3}{x^2-2x-3} = \frac{2}{x+1} + \frac{3}{x-3},$$

since the right-hand side is easy to integrate. How do we split $P(x)/Q(x)$ into rational functions that are easier to integrate?

Def If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ with $a_n \neq 0$, then $\deg(P(x)) = n$. Here $\deg(P(x))$ is called the degree of $P(x)$. We do not consider the degree of a zero polynomial here.

e.g. $\deg(7x^3 - 2x^2 + \pi x + e) = 3$; $\deg(a_0) = 0$ (for $a_0 \neq 0$).

Fact: If $P(x)$ and $Q(x)$ are polynomials with $Q(x)$ nonzero, then there exist polynomials $S(x)$ (quotient) and $R(x)$ (remainder) such that

- $P(x) = S(x)Q(x) + R(x)$, and;
- $R(x) \equiv 0$ or $\deg(R(x)) < \deg(Q(x))$.

identically equals

This means that $\dfrac{P(x)}{Q(x)} = S(x) + \dfrac{R(x)}{Q(x)}$, so

$$\int \frac{P(x)}{Q(x)}\,dx = \int \overset{①}{S(x)}\,dx + \int \overset{②}{\frac{R(x)}{Q(x)}}\,dx.$$

Since ① is easy to integrate, we focus on ②. Hence, we may assume that $\deg(P(x)) < \deg(Q(x))$.

<u>e.g.</u> (Long division of polynomials.)

Since $$\frac{x^3-4x^2+2x-3}{x+2} = x^2-6x+14 - \frac{31}{x+2},$$

$$\int \frac{x^3-4x^2+2x-3}{x+2}\,dx = \underline{\int (x^2-6x+14)\,dx} - \underline{31\int \frac{1}{x+2}\,dx}$$

Easy

Long division:

$$\begin{array}{r|l}
 & x^2-6x+14 \quad \leftarrow \text{Quotient} \\
\hline
x+2 & x^3-4x^2+2x-3 \quad \leftarrow \text{Dividend} \\
 & x^3+2x^2 \\
\hline
 & -6x^2+2x \\
 & -6x^2-12x \\
\hline
 & 14x-3 \\
 & 14x+28 \\
\hline
 & -31 \quad \leftarrow \text{Remainder}
\end{array}$$

Divisor

To integrate a rational function $P(x)/Q(x)$:

1. Apply long division to reduce the problem to one that has $$\deg(P(x)) < \deg(Q(x)).$$
2. Decompose $P(x)/Q(x)$ into partial fractions.
3. Integrate each partial fraction.

Factors of Polynomials

It is a fact from algebra that every polynomial factors into linear or irreducible quadratic polynomials over the real numbers. (A quadratic polynomial $ax^2 + bx + c$ is called irreducible if it has no real root, i.e., if $b^2 - 4ac < 0$.)

That is, every nonzero polynomial $Q(x)$ can be written as

$$Q_1(x)Q_2(x)\cdots Q_k(x),$$

where each $Q_i(x)$ has one of the following forms:

- $a_i x + b_i$, $a_i \neq 0$; (linear)
- $a_i x^2 + b_i x + c_i$, $a_i \neq 0$, $b_i^2 - 4a_i c_i < 0$. (irreducible quadratic)

e.g.

- $x^2 - 1 = (x+1)(x-1)$
- x^2+1 is irreducible over $\mathbb{R}$. (It is reducible over $\mathbb{C}$, the set of complex numbers, since $x^2+1 = (x+i)(x-i)$, where $i := \sqrt{-1}$.)
- You can check that $x^4+1 = (x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$.

↳ This is for demonstration of the fact only. Do not worry about how to factor a general polynomial.

Now we look at $\int \frac{P(x)}{Q(x)} dx$, where $\deg(P(x)) < \deg(Q(x))$.

Case 1: $Q(x)$ **is a product of distinct linear factors**:

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\dots(a_kx + b_k).$$

In this case, there exist constants $A_1, A_2, \dots, A_k$ such that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \dots + \frac{A_k}{a_kx + b_k}. \tag{1}$$

Example

Find $\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx$.

Answer: $\frac{1}{2} \ln|x| + \frac{1}{10} \ln|2x - 1| - \frac{1}{10} \ln|x + 2| + C$.

Sol: • $2x^3+3x^2-2x = x(2x^2+3x-2) = x(2x-1)(x+2)$

• $\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$

$= \frac{(2A+B+2C)x^2+(3A+2B-C)x-2A}{x(2x-1)(x+2)}$

A, B, C are "undetermined coefficients".

• $\begin{cases} 2A+B+2C=1 \\ 3A+2B-C=2 \\ -2A=-1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{2} \\ B=\frac{1}{5} \\ C=-\frac{1}{10} \end{cases}$

• $\int \frac{P(x)}{Q(x)} dx = \frac{1}{2}\int \frac{1}{x}dx + \frac{1}{5}\int \frac{1}{2x-1}dx - \frac{1}{10}\int \frac{1}{x+2}dx$

$= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C.$

What happens if we have a linear factor $a_i x + b_i$ repeating?

e.g. $\frac{x^2+x+1}{(x+1)^3}$

- Note that $x^2+x+1 = x(x+1)+1$

$$\Rightarrow \frac{x^2+x+1}{(x+1)^3} = \frac{x(x+1)+1}{(x+1)^3} = \frac{x}{(x+1)^2} + \frac{1}{(x+1)^3} = \frac{x+1-1}{(x+1)^2} + \frac{1}{(x+1)^3}$$

$$= \frac{1}{x+1} - \frac{1}{(x+1)^2} + \frac{1}{(x+1)^3}$$

The general principle is summarized below.

Case 2: $Q(x)$ is a product of linear factors, some of which are repeated.

Suppose that a linear factor $(a_1x + b_1)$ is repeated r times; that is, $(a_1x + b_1)^r$ appears in the factorization of $Q(x)$. Then instead of the single term $A_1/(a_1x + b_1)$ in Equation (1) on the previous slide, we would use

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}. \qquad (2)$$

Example

Find $\displaystyle\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx. =: I$

Sol: • Apply long division and get

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1} \qquad \frac{P(x)}{Q(x)}$$

• $x^3-x^2-x+1 = x^2(x-1)-(x-1) = (x^2-1)(x-1) = (x+1)(x-1)^2$

• $\dfrac{P(x)}{Q(x)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+1} = \dfrac{(A+C)x^2+(B-2C)x+(-A+B+C)}{(x+1)(x-1)^2}$

• $\begin{cases} A+C=0 \\ B-2C=4 \\ -A+B+C=0 \end{cases} \Longrightarrow \begin{cases} A=1 \\ B=2 \\ C=-1 \end{cases}$

• $\displaystyle\frac{P(x)}{Q(x)} = \int \frac{1}{x-1}dx + 2\int\frac{1}{(x-1)^2}dx - \int\frac{1}{x+1}dx$

$= \ln|x-1| + (-2)(x-1)^{-1} - \ln|x+1| + C$

$= \ln\left|\dfrac{x-1}{x+1}\right| - \dfrac{2}{x-1} + C$

• $I = \frac{1}{2}x^2 + x + \ln\left|\dfrac{x-1}{x+1}\right| - \dfrac{2}{x-1} + C$

Case 3: $Q(x)$ contains irreducible quadratic factors, none of which is repeated.

If $Q(x)$ has a factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then, in addition to the partial fractions in Equations (1) and (2), the expression for $P(x)/Q(x)$ will have a term of the form

$$\frac{Ax + B}{ax^2 + bx + c}. \tag{3}$$

For example, there exist constants A, B, C, D and E such that

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{x^2+4}.$$

e.g. $\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx \quad (=: I)$

Sol: • $\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$

• $-2x+4 = \underbrace{(A+C)}_{0}x^3 + \underbrace{(-2A+B-C+D)}_{0}x^2 + \underbrace{(A-2B+C)}_{-2}x + \underbrace{(B-C+D)}_{4}$

• Solving linear system gives $A=2$, $B=1$, $C=-2$, $D=1$.

• $I = \int \frac{2x+1}{x^2+1}dx - 2\int \frac{1}{x-1}dx + \int \frac{1}{(x-1)^2}dx$

$\int \frac{2x}{x^2+1}dx + \int \frac{dx}{x^2+1}$

$= \ln(x^2+1) + \arctan x - 2\ln|x-1| - \frac{1}{x-1} + C.$

Case 4: $Q(x)$ **contains a repeated irreducible quadratic factor**.

If $Q(x)$ has a factor $(ax^2+bx+c)^r$, where $b^2-4ac<0$, then instead of a single term (3) on the previous slide, the sum

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

appears in the partial fraction decomposition of $P(x)/Q(x)$. For example, there exist constants A, B, ..., I and J such that

$$\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$
$$=\frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+x+1}+\frac{Ex+F}{x^2+1}+\frac{Gx+H}{(x^2+1)^2}+\frac{Ix+J}{(x^2+1)^3}$$

Finding Undetermined Coefficients

Heaviside "Cover-up" Method

In case 1, where

$$\frac{f(x)}{g(x)}=\frac{f(x)}{(x-r_1)\dots(x-r_n)}=\frac{A_1}{x-r_1}+\dots+\frac{A_n}{x-r_n},$$

there is a quickway for finding $A_1,\dots,A_n$. Indeed, by multiplying both sides by $x-r_1$, we have

$$\frac{f(x)}{(x-r_2)\dots(x-r_n)}=A_1+(x-r_1)\left(\frac{A_2}{x-r_2}+\dots+\frac{A_n}{x-r_n}\right).$$

Substituting $x=r_1$ yields

$$A_1=\frac{f(r_1)}{(r_1-r_2)\dots(r_1-r_n)}.$$

(Cover up $(x-r_1)$ in $\frac{f(x)}{g(x)}$ then sub in $x=r_1$ gives you A_1: hence the name.)

The similar approach works for all the A_i's: remove $x-r_i$ from the bottom of $f(x)/g(x)$, then sub in $x=r_i$ to get A_i

e.g. In one of our previous examples, we have

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{f(x)}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}.$$

- $A = \frac{f(0)}{(2\cdot 0-1)(0+2)} = \frac{-1}{(-1)2} = \frac{1}{2}.$
- $B = \frac{f(\frac{1}{2})}{\frac{1}{2}(\frac{1}{2}+2)} = \frac{1/4}{5/4} = \frac{1}{5}.$
- $C = \frac{f(-2)}{-2(2(-2)-1)} = \frac{-1}{-2(-5)} = -\frac{1}{10}.$

This approach may be modified slightly to handle other cases.

e.g. $$\frac{3x^2-16x+21}{(x-1)^2(x+3)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+3}$$

- To get A, cover up $(x-1)^2$ on LHS and evaluate at $x=1$:

$$A = \frac{3-16+21}{1+3} = \frac{8}{4} = 2.$$

· Move $\frac{2}{(x-1)^2}$ to LHS :

$$\frac{3x^2-16x+21-2(x+3)}{(x-1)^2(x+3)} = \frac{B}{x-1} + \frac{C}{x+3}$$

$$\Rightarrow \quad \frac{3x^2-18x+15}{(x-1)^2(x+3)} = \frac{B}{x-1} + \frac{C}{x+3}$$

$$\begin{array}{r}
3x-15 \\
x-1 \overline{\big) 3x^2-18x+15} \\
\underline{3x^2-3x } \\
-15x+15 \\
\underline{-15x+15} \\
0
\end{array}$$

$$\Rightarrow \quad \frac{3x-15}{(x-1)(x+3)} = \frac{B}{x-1} + \frac{C}{x+3} .$$

Now B & C can be found by the cover-up method again.