Lecture 24, Thursday, November/30/2023

Outline

- Improper integrals (8.8)
  - ↳ Type I : unbounded intervals
  - ↳ Type II : discontinuous integrands (including unbounded ones)
  - ↳ Convergence tests

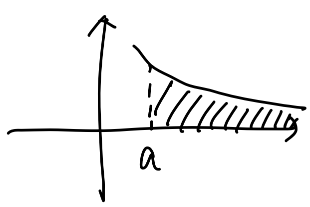
Improper Integrals

Definite (Riemann) integrals $\int_a^b f(x)dx$ assume two conditions:

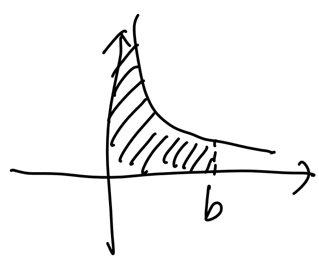
- Interval $[a,b]$ is bounded, and;
- f is bounded on $[a,b]$ (otherwise not Riemann integrable).

Hence, Riemann integrals cannot handle functions over unbounded intervals or those that have unbounded ranges.

We introduce improper integrals, which are limits of Riemann integrals. These will cover unbounded domains or unbounded integrands.



or



In statistics, these are very fundamental; e.g., a probability density function is a certain nonnegative function f that satisfies

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

Type I (Unbounded intervals)

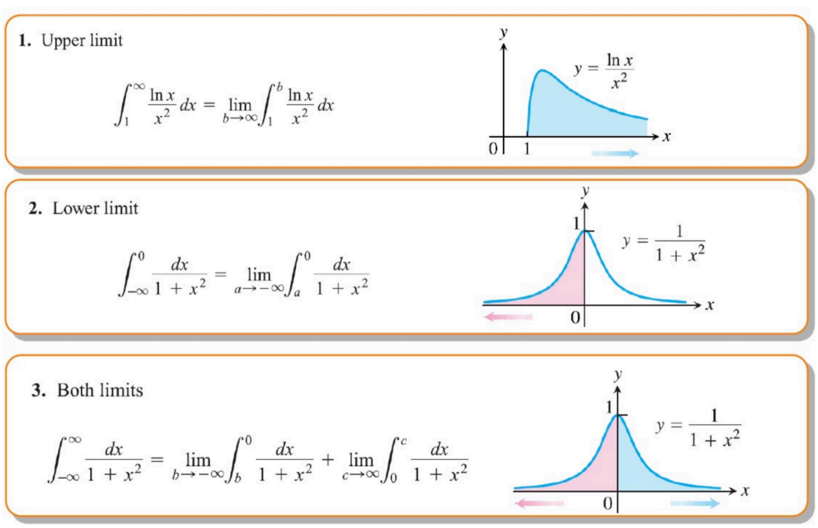
Definition

- Let $a \in \mathbb{R}$. If f is integrable on $[a, b]$ for every $b \in [a, \infty)$, then
$$\int_a^\infty f(x)\,dx := \lim_{b\to\infty} \int_a^b f(x)\,dx.$$
- Let $b \in \mathbb{R}$. If f is integrable on $[a, b]$ for every $a \in (-\infty, b]$, then
$$\int_{-\infty}^b f(x)\,dx := \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

Definition

- An improper integral is said to be convergent if the corresponding limit exists, and is said to be divergent if the limit does not exist (as a real number).

  $\pm\infty \Rightarrow$ divergent
- We define
$$\int_{-\infty}^\infty f(x)\,dx := \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$
whenever both improper integrals on the right converge. We may use any real number c in this definition.



Remark: Our definition of $\int_{-\infty}^{\infty}$ is NOT the same as $\lim_{a\to\infty}\int_{-a}^{a}$.

(Why?) ____________________________.

Extra learning: Search "Cauchy principal value".

e.g. (a) $\int_1^\infty \frac{1}{x}\,dx$.

$$\int_1^b \frac{1}{x}\,dx = \ln b - \ln 1 = \ln b \longrightarrow \infty \quad \text{as } b\to\infty,$$

So $\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \infty$. (Divergent)

(b) $\int_1^\infty x^\alpha\,dx$, $\alpha \neq -1$.

$$\int_1^b x^\alpha\,dx = \frac{1}{\alpha+1}\left(b^{\alpha+1}-1\right) =: F(b).$$

$$\int_1^\infty x^\alpha\,dx = \lim_{b\to\infty} F(b) = \begin{cases} \infty, & \text{if } \alpha>-1 \\ \frac{-1}{\alpha+1}, & \text{if } \alpha<-1 \end{cases}.$$

Considering (a), we have $\int_1^\infty x^\alpha\,dx = \begin{cases} \infty, & \text{if } \alpha\geq-1 \\ \frac{-1}{\alpha+1}, & \text{if } \alpha<-1 \end{cases}$.

An equivalent form is given by

Set $p:=-\alpha$

$$\boxed{\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1}, & \text{if } p>1 \quad \text{(Convergent)} \\ \infty, & \text{if } p\leq 1 \quad \text{(Divergent)} \end{cases}.}$$

(c) $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

$$\int_0^b \frac{1}{1+x^2}\,dx = \arctan b - \arctan 0 = \arctan b$$

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \arctan b = \frac{\pi}{2}. =: B$$

$$\int_a^0 \frac{1}{1+x^2}\,dx = -\arctan a$$

$$\int_{-\infty}^0 \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty} -\arctan a = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2} =: A$$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = A+B = \pi.$$

Type II (Discontinous/unbounded integrands)

The second type of improper integrals involves discontinuous integrands. In particular, it covers cases like $f(x) = \frac{1}{x}$ which approaches $\pm\infty$ as $x \to 0$.

Definition

- If f is continuous on $[a, b)$ (~~and is~~ but may be discontinuous at b), then
$$\int_a^b f(x)\,dx := \lim_{c\to b^-} \int_a^c f(x)\,dx.$$
- If f is continuous on $(a, b]$ (~~and~~ but may be is discontinuous at a), then
$$\int_a^b f(x)\,dx := \lim_{c\to a^+} \int_c^b f(x)\,dx.$$

Definition

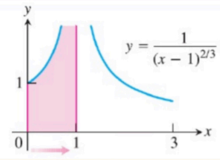
- If f is discontinuous at c, where $a < c < b$, and is continuous on $[a, b] \setminus \{c\}$, then

$$\int_a^b f(x)\,dx := \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

provided that both improper integrals on the right converge.

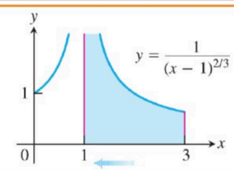
4. Upper endpoint

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}}$$



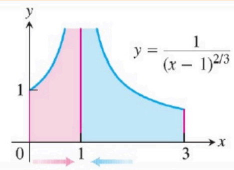
5. Lower endpoint

$$\int_1^3 \frac{dx}{(x-1)^{2/3}} = \lim_{d\to 1^+} \int_d^3 \frac{dx}{(x-1)^{2/3}}$$



6. Interior point

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$



Remarks:

- If f is continuous on $[a,b)$ but $\lim_{x\to b^-} f(x) \in \{\infty, -\infty\}$, then $\int_a^b f(x)\,dx$ is not a Riemann integral (since unbounded function is not Riemann integrable) although the notation looks the same.

- If f is cts on $[a,b)$ and f has a jump/removable left-discontinuity at b, then it can be shown that the improper integral "$\int_a^b f(x)dx$" has the same value as the Riemann integral "$\int_a^b f(x)dx$". (Proof omitted.)

e.g. Suppose $p>0$. $\int_0^1 \frac{1}{x^p}\,dx = ?$

- $\lim_{x\to 0^+} \frac{1}{x^p} = \infty$, essential discontinuity at $x=0$. Continuous on $(0,1]$. (if you include 0 in the domain by giving it a value)
- $\int_a^1 \frac{1}{x^p}\,dx \overset{\text{If } p\neq 1}{=} \frac{1}{-p+1} x^{-p+1} \Big|_a^1 = \frac{1}{1-p}\left(1-\frac{1}{a^{p-1}}\right)$
- $\int_0^1 \frac{1}{x^p}\,dx = \lim_{a\to 0^+} \frac{1}{1-p}\left(1-\frac{1}{a^{p-1}}\right) = \begin{cases} \infty, & \text{if } p>1; \\ \frac{1}{1-p}, & \text{if } 0<p<1. \end{cases}$
- If $p=1$, $\int_0^1 \frac{1}{x}\,dx = \lim_{a\to 0^+} (\ln 1 - \ln a) = \infty$
- Hence, $\int_0^1 \frac{1}{x^p}\,dx \begin{cases} \text{Converges to } \frac{1}{1-p}, & \text{if } 0<p<1; \\ \text{diverges (to } \infty\text{)}, & \text{if } p\geq 1. \end{cases}$

Combined with Type-I improper integrals, this means that:

- If $0<p<1$: $\int_0^1 \frac{1}{x^p}\,dx$ converges, $\int_1^\infty \frac{1}{x^p}\,dx$ diverges.
- If $p>1$: $\int_0^1 \frac{1}{x^p}\,dx$ diverges, $\int_1^\infty \frac{1}{x^p}\,dx$ converges.
- If $p=1$: Both $\int_0^1 \frac{1}{x^p}\,dx$ and $\int_1^\infty \frac{1}{x^p}\,dx$ diverge.

Convergence Tests

Consider $\int_0^\infty e^{-x^2}\,dx$. $f(x) = e^{-x^2}$ has no elementary antiderivative, so numerical method is required. Before approximating its value, we may want to know if it converges at all. Convergence tests may be used for such purposes.

> **Theorem (Direct Comparison Test)**
> *Suppose that $a \in \mathbb{R}$, and suppose that f and g are continuous functions on $[a, \infty)$. If there exists $c \in [a, \infty)$ such that $0 \le f(x) \le g(x)$ for all $x \in [c, \infty)$, then the following statements hold:*
>
> (i) *If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.*
>
> (ii) *If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.*

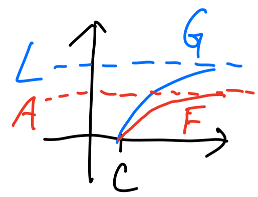
"integrable on $[a, b]$ $\forall b \ge a$" is enough

Idea (Not a complete proof).

- Note that (i) and (ii) are equivalent. We outline (i).
- Since $\int_a^\infty g(x)dx = \underbrace{\int_a^c g(x)dx}_{\text{definite integral, finite}} + \int_c^\infty g(x)dx$, $\int_a^\infty g(x)dx$ converges if and only if $\int_c^\infty g(x)dx$ converges.
- Similarly, $\int_a^\infty f(x)dx$ converges if and only if $\int_c^\infty f(x)dx$.

- Hence, it suffices to show that if $\int_c^\infty g(x)\,dx$ converges then $\int_c^\infty f(x)\,dx$ converges.
- Let $F(t) := \int_c^t f(x)dx$ and $G(t) := \int_c^t g(x)dx$, for $t \geq c$.
- Since $0 \leq f(x) \leq g(x)$ for all $x \in [c, \infty)$, we have

$$0 \leq F(t) \leq G(t), \quad \forall t \in [c, \infty).$$



- Assume that $\int_c^\infty g(x)dx$ converges. Then $\lim_{t\to\infty} G(t) = L$ for some $L \in \mathbb{R}$.
- Then $F(t)$ is bounded above by L and has a "least upper bound", say A. (This makes use of the "least upper bound property" of $\mathbb{R}$, which states that any nonempty set that is bounded above must have a least upper bound.)
- Then $\lim_{t\to\infty} F(t) = A$, so $\int_c^\infty f(x)dx = A$ (convergent).

e.g. Determine whether $\int_0^\infty e^{-x^2}dx$ converges or not.

Sol: - Since $e^{x^2} \geq e^x > 0$ for all $x \in [1, \infty)$, we have

$$0 < e^{-x^2} \leq e^{-x}, \quad \forall x \in [1, \infty).$$

- $\int_0^\infty e^{-x}dx = \lim_{b\to\infty} (-e^{-b} + e^{-0}) = 1$, so $\int_0^\infty e^{-x}dx$ converges.

- By comparison test, $\int_0^{\infty} e^{-x^2}dx$ also converges.

e.g. $\int_2^{\infty} \frac{\sqrt[3]{x^7+2}}{x^3 \ln x}dx$ converges or not?

Sol:

Remark: The direct comparison test also holds for Type-II improper integrals.

The following "comparison test" makes use of relative growth rate.

Theorem (Limit Comparison Test)

Suppose that $a \in \mathbb{R}$, and suppose that f and g are positive continuous functions on $[a, \infty)$. If

"integrable on $[a,b]$ $\forall b \geq a$" is enough

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

f & g grow at the same rate.

for some $L \in \mathbb{R}_+ := (0, \infty)$, then

$$\int_a^{\infty} f(x)\, dx \quad \text{and} \quad \int_a^{\infty} g(x)\, dx$$

both converge or both diverge.

<u>Proof</u>:

- Since $\lim_{x\to\infty}\frac{f(x)}{g(x)} = L$, $\exists N \in \mathbb{R}$ $(N \geq a)$ s.t. $\forall x \in [N, \infty)$ $(L>0)$

$$\frac{L}{2} \leqslant \frac{f(x)}{g(x)} \leqslant \frac{3L}{2}.$$

$$\left(\Rightarrow \quad 0 < \frac{L}{2} g(x) \leqslant f(x) \leqslant \frac{3L}{2} g(x). \quad (*) \right)$$

- If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty \frac{L}{2} g(x)\,dx$ converges by $(*)$, so

$$\int_a^\infty g(x)\,dx = \frac{2}{L}\int_a^\infty \frac{L}{2} g(x)\,dx \text{ converges.}$$

- If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty \frac{3L}{2} g(x)$ diverges by $(*)$, so

$$\int_a^\infty g(x)\,dx \text{ diverges.}$$

$\square$

The limit comparison test may be more convenient sometimes.

<u>e.g.</u> $\int_1^\infty \frac{1-e^{-x}}{x}\,dx$.

- $0 < \frac{1-e^{-x}}{x} < \frac{1}{x}$; $\int_1^\infty \frac{1}{x}\,dx$ diverges, but no conclusion can be drawn by direct comparison.

- $\lim_{x\to\infty} \frac{(1-e^{-x})/x}{1/x} = \lim_{x\to\infty}(1-e^{-x}) = 1.$

- By limit comparison test (with $\int_1^\infty \frac{1}{x}\,dx$), $\int_1^\infty \frac{1-e^{-x}}{x}\,dx$ diverges.

Remarks (1) The limit comparison test can be extended. If f and g are positive and continuous on $[a,\infty)$, and

"integrable on $[a,b]$ $\forall b\geq a$" is enough.

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = L,$$

then:

- if $L=0$ and $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.
- if $L=\infty$ and $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

(2) The test also applies to Type-II improper integrals.

Extended Exercise

(a) Show that $\int_a^\infty e^{-kt}\,dt$ is convergent for all $a\in\mathbb{R}$ and all $k>0$.

(b) The gamma function is the function $\Gamma:(0,\infty)\to\mathbb{R}$ defined by

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\,dt = \int_0^\infty \frac{t^{x-1}}{e^t}\,dt.$$

This is an extention of the factorial function to the domain $(0,\infty)$: one can check that $\Gamma(n+1)=n!$ for $n\in\mathbb{N}:=\{0,1,2,\dots\}$ (see Chp 8, additional and advanced exercises Q51). Prove that $\int_0^\infty t^{x-1}e^{-t}\,dt$ converges for each $x>0$ (so the gamma function is well-defined).