

Lecture 23, Tuesday , Nov /28/2023

Outline

- Integration by partial fractions (8.5)
 - ↳ Finding undetermined coefficients
 - ↳ Basic forms
- Numerical integration (8.7)
 - ↳ Overview
 - ↳ Trapezoidal rule
 - ↳ Simpson's rule
 - ↳ Error bounds

Integration by Partial Fractions

Finding Undetermined Coefficients : Differentiating

This technique will be useful for function of the form $\frac{f(x)}{(x-r)^n}$.

e.g. $\frac{f(x)}{(x-r)^4} = \frac{A}{(x-r)^4} + \frac{B}{(x-r)^3} + \frac{C}{(x-r)^2} + \frac{D}{x-r}$

1. $f(x) = A + B(x-r) + C(x-r)^2 + D(x-r)^3 \quad \textcircled{1}$

$$\Rightarrow \underbrace{f(r)}_{f^{(0)}(r)} = A$$

2. $\textcircled{1} \Rightarrow f'(x) = B + 2C(x-r) + 3D(x-r)^2 \quad \textcircled{2}$

$$\Rightarrow \underbrace{f'(r)}_{f'(0)} = B$$

3. $\textcircled{2} \Rightarrow f''(x) = 2C + 6D(x-r) \quad \textcircled{3}$

$$\Rightarrow f''(r) = 2C \Rightarrow \underbrace{C = f''(r)/2}_{\text{in green}}$$

4. $\textcircled{3} \Rightarrow f'''(x) = 6D \Rightarrow \underbrace{D = f'''(r)/6}_{\text{in green}}$

General form : _____ ?

e.g. Find the partial fraction decomposition for $\frac{3x^2-16x+21}{(x-1)^2(x+3)}$.

Sol: 1. General decomposition : $\frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+3}$.

2. Use Heaviside's method to get C :

$$C = \left. \frac{3x^2 - 16x + 21}{(x-1)^2(x+3)} \right|_{x=-3} = \frac{27 + 48 + 21}{16} = 6.$$

$$3. \frac{A}{(x-1)^2} + \frac{B}{x-1} = \frac{3x^2 - 16x + 21}{(x-1)^2(x+3)} - \frac{6}{x+3}$$

$$= \frac{3x^2 - 16x + 21 - 6(x^2 - 2x + 1)}{(x-1)^2(x+3)}$$

$$= \frac{-3x^2 - 4x + 15}{(x-1)^2(x+3)}$$

$$= \frac{-3x+5}{(x-1)^2} \leftarrow =: f(x)$$

$$\begin{array}{r} -3x+5 \\ x+3 \sqrt{-3x^2 - 4x + 15} \\ \hline -3x^2 - 9x \\ \hline 5x+15 \\ 5x+15 \\ \hline 0 \end{array}$$

4. Differentiation method yields

$$A = f(1) = 2, \quad B = \frac{f'(1)}{1} = -3.$$

Integrating the Basic Forms

After partial fraction decomposition, we need to know how to compute two types of integrals:

$$(I) \int \frac{A}{(ax+b)^n} dx$$

$$(II) \int \frac{Ax+B}{(ax^2+bx+c)^n} dx$$

$$(I) \int \frac{A}{ax+b} dx = \frac{A}{a} \ln|ax+b| + K$$

$$\int \frac{A}{(ax+b)^n} dx = A \int (ax+b)^{-n} dx = \frac{A}{a} \frac{1}{-n+1} (ax+b)^{-n+1} + K$$

if $n \neq 1$.

For (II), we demonstrate with an example.

e.g. Compute $\int \frac{x^2+1}{(x^2-2x+2)^2} dx$ ($=: I$).

Sol:

$$1. \text{ Decompose } \frac{x^2+1}{(x^2-2x+2)^2} \text{ into } \frac{A_1x+B_1}{x^2-2x+2} + \frac{A_2x+B_2}{(x^2-2x+2)^2}.$$

$$2. \frac{x^2+1}{(x^2-2x+2)^2} = \frac{(x^2-2x+2) \cdot (1+2x-1)}{(x^2-2x+2)^2} = \frac{1}{x^2-2x+2} + \frac{2x-1}{(x^2-2x+2)^2}.$$

Long division could be effective

$$3. \int \frac{x^2+1}{(x^2-2x+2)^2} dx = \underbrace{\int \frac{dx}{x^2-2x+2}}_{I_1} + \underbrace{\int \frac{2x-1}{(x^2-2x+2)^2} dx}_{I_2}.$$

$$4. I_1 = \int \frac{dx}{(x-1)^2 + 1} = \arctan(x-1) (+C_1).$$

$$5. I_2 = \underbrace{\int \frac{2x-2}{(x^2-2x+2)^2} dx}_J + \underbrace{\int \frac{1}{(x^2-2x+2)^2} dx}_K.$$

6. Reason of considering J is $d(x^2 - 2x + 2) = (2x - 2)dx$.

$$J = - (x^2 - 2x + 2)^{-1} = \frac{-1}{x^2 - 2x + 2} (+ C_2)$$

7. To find K , use trigonometric substitution after completing the square.

$$K = \int \frac{dx}{((x-1)^2 + 1)^2}$$

Let $x-1 = \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$
 $dx = \sec^2 \theta d\theta$

$$\begin{aligned} &= \int \frac{\sec^2 \theta d\theta}{(\sec^2 \theta)^2} = \int \cos^2 \theta d\theta \\ &= \int \frac{1 + \cos 2\theta}{2} d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C_3 \\ &= \frac{1}{4}(2\theta + \sin 2\theta) + C_3 = \frac{1}{4}(2\theta + 2\sin \theta \cos \theta) + C_3 \end{aligned}$$

$$= \frac{1}{2}(\theta + \tan \theta \cos \theta \cos \theta) + C_3$$

$$= \frac{1}{2}(\theta + \frac{\tan \theta}{1 + \tan^2 \theta}) + C_3$$

$$= \frac{1}{2}(\arctan(x-1) + \frac{x-1}{1 + (x-1)^2}) + C_3$$

$$= \frac{1}{2} \left(\frac{x-1}{x^2 - 2x + 2} + \arctan(x-1) \right) + C_3$$

For (II)

- Completing square of the denominator
- Arctan
- U-substitution
- Trig substitution

8. $I = I_1 + J + K$. (Can write C instead of $C_1 + C_2 + C_3$.)

Numerical Integration

- ▶ Although the Fundamental Theorem of Calculus gives us the ability to compute $\int_a^b f(x) dx$ by finding $F(x)$, it is not always possible to find a closed form of $F(x)$.
- ▶ In particular, some elementary functions, such as

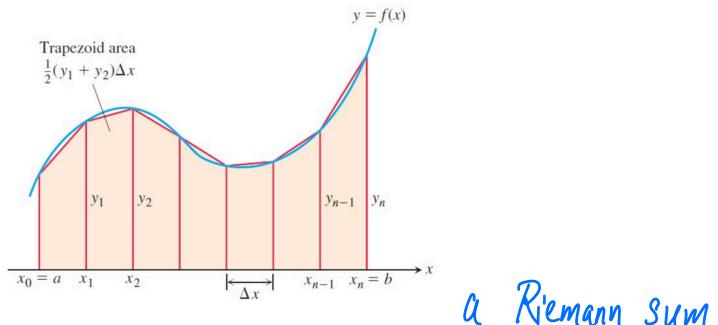
$$f(x) = \frac{\sin x}{x} \quad \text{and} \quad f(x) = e^{x^2},$$

do not have an elementary antiderivative.

- In practice, approximations are often used.
- For integration approximations, the following approach could be taken:
 - ↪ Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$, $\Delta x_k = \frac{b-a}{n}$
 - ↪ Then $\int_a^b f(x) dx = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(x) dx$.
 - ↪ Approximate each $\int_{x_{k-1}}^{x_k} f(x) dx$.
 - ↪ In a Riemann sum approximation, we pick some $c_k \in [x_{k-1}, x_k]$ and take $\int_{x_{k-1}}^{x_k} f(x) dx \approx f(c_k) \Delta x_k$.
- Commonly used Riemann sum approximations include:
 - ↪ Left-hand rule
 - ↪ Right-hand rule
 - ↪ Midpoint rule
- We will introduce two other numerical integration methods.

Trapezoidal Rule

The trapezoidal rule approximates each $\int_{x_{k-1}}^{x_k} f(x) dx$ with the "area" of a trapezoid.



a Riemann sum

More specifically, instead of $f(c_k)\Delta x_k$ in the Midpoint Rule, we take

$$T_k := \frac{(y_{k-1} + y_k)\Delta x}{2},$$

where $y_j := f(x_j)$. (Note that T_k is the area of a trapezoid if f is nonnegative.)

Note: f does not have to be nonnegative.

$$\int_a^b f(x) dx \approx \sum_{k=1}^n T_k = \frac{\Delta x}{2} (y_0 + 2y_1 + 2y_2 + \dots + 2y_{n-1} + y_n),$$

$\underbrace{(y_0+y_1)+\underbrace{(y_1+y_2)}+\dots+\underbrace{(y_{n-1}+y_n)}$

where $\Delta x := (b - a)/n$.

How good are these approximation methods? Let us define

$$\text{error} := E := \text{approximated value} - \underbrace{\int_a^b f(x) dx}_{\text{exact value}}.$$

We want to look at $|E|$.

e.g. Consider $I := \int_1^{1.6} \frac{1}{x} dx$. With $\Delta x = 0.1$, use different rules to approximate I .

- Left-hand rule : $0.489260\dots$
 - Midpoint rule : $0.469750\dots$
 - $I = \ln 1.6 = 0.470003\dots$
- $\begin{aligned} & \text{Right-hand rule : } 0.451760\dots \\ & \text{Trapezoidal rule : } 0.470510\dots \\ & = \frac{0.1}{2} \left(\frac{2}{1.1} + \frac{2}{1.2} + \frac{2}{1.3} + \frac{2}{1.4} + \frac{2}{1.5} + \frac{2}{1.6} \right) \end{aligned}$

- In the example above, if we consider the errors E_L , E_R , E_m and E_T , then

$$|E_L| = 0.019203, |E_R| = 0.018243, |E_m| = 0.000253, |E_T| = 0.000507$$

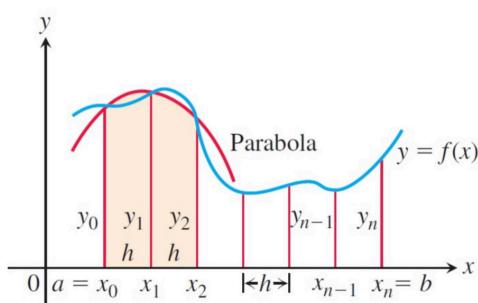
$$|E_m| < |E_T| < |E_R| < |E_L|.$$

We will see that these are consistent with the error bounds to be given soon.

Simpson's Rule

any Riemann sum approximation

- In the Midpoint Rule, f is approximated by a **constant** $f(c_k)$ over the interval $[x_{k-1}, x_k]$.
- In the **Trapezoidal Rule**, over $[x_{k-1}, x_k]$, the function f is approximated by a **linear function** $y = Ax + B$ whose graph passes through the points (x_{k-1}, y_{k-1}) and (x_k, y_k) .
- We can take this approach a step further by approximating f with **quadratic polynomials** $Ax^2 + Bx + C$.



Fact :

Any three non-collinear points on the xy -plane that have distinct x -coordinates determine a quadratic curve uniquely.

- ▶ Consider an evenly spaced partition $P := \{x_0, x_1, \dots, x_n\}$ of $[a, b]$, where n is even.
- ▶ For $k \in \{1, 3, 5, \dots, n-1\}$, over the interval $[x_{k-1}, x_{k+1}]$, consider approximating f by the ~~quadratic~~ function $p_k(x) := A_k x^2 + B_k x + C_k$ whose graph passes through the three points (x_{k-1}, y_{k-1}) , (x_k, y_k) and (x_{k+1}, y_{k+1}) .
- ▶ This approximation is called **Simpson's Rule**, which can be stated as

$$\int_a^b f(x) dx \approx \sum_{k \in \{1, 3, \dots, n-1\}} \int_{x_{k-1}}^{x_{k+1}} p_k(x) dx.$$

- ▶ We will derive an explicit formula for this approximation.

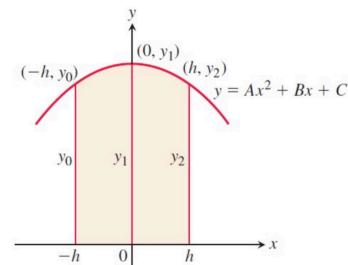
We first look at $\int_{x_0}^{x_2} p_1(x) dx$. To compute it, we may shift the graph so that $x_0 = -\Delta x$, $x_1 = 0$ and $x_2 = \Delta x$, then compute the integral. This is O.K., since horizontal shifts do not change the value of an integral :

$$\int_c^d f(x) dx = \int_{c-s}^{d-s} f(x+s) dx. \quad (u=x+s, du=dx)$$

Let $h := \Delta x$, and write

$$q_1(x) = Ax^2 + Bx + C = p_1(x+h+a).$$

(the shifted polynomial)



Then

$$\begin{aligned} \int_{x_0}^{x_2} p_1(x) dx &= \int_a^{a+2h} p_1(x) dx = \int_{-h}^h q_1(x) dx \\ &= \int_{-h}^h (Ax^2 + Bx + C) dx = 2 \int_0^h (Ax^2 + C) dx \end{aligned}$$

$$= 2\left(\frac{A}{3}h^3 + Ch\right) = \frac{h}{3}(2Ah^2 + 6C). \quad \textcircled{1}$$

Since $(-h, y_0)$, $(0, y_1)$ and (h, y_2) are all on the graph of $y = Ax^2 + Bx + C$, we have

$$\begin{cases} y_0 = Ah^2 - Bh + C \\ y_1 = C \\ y_2 = Ah^2 + Bh + C \end{cases}$$

$$\Rightarrow \begin{cases} C = y_1 \\ y_0 + y_2 = 2Ah^2 + 2C = 2Ah^2 + 2y_1 \end{cases} \Rightarrow \begin{cases} 6C = 6y_1 \\ 2Ah^2 = y_0 + y_2 - 2y_1 \end{cases}.$$

By ①, we have

$$\int_{x_0}^{x_2} p_1(x) dx = \frac{h}{3}(2Ah^2 + 6C) = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

The same argument also leads to

$$\int_{x_{k-1}}^{x_{k+1}} p_k(x) dx = \frac{h}{3}(y_{k-1} + 4y_k + y_{k+1}).$$

Therefore,

$$\begin{aligned} \int_a^b f(x) dx &\approx \sum_{k \in \{1, 3, \dots, n-1\}} \int_{x_{k-1}}^{x_{k+1}} p_k(x) dx \\ &= \int_{x_0}^{x_2} p_1(x) dx + \int_{x_2}^{x_4} p_3(x) dx + \dots + \int_{x_{n-2}}^{x_n} p_{n-1}(x) dx \\ &= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \dots + 2y_{n-2} + 4y_{n-1} + y_n). \end{aligned}$$

Hence, an explicit formula for [Simpson's Rule](#) is given by

$$\int_a^b f(x) dx \approx \frac{h}{3} (y_0 + 4y_1 + 2y_2 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n),$$

where n is even, and the coefficients are 1, 4, 2, 4, 2, 4, 2, ..., 4, 2, 4, and 1.

e.g. Use Simpson's rule to approximate $I := \int_1^{1.6} \frac{1}{x} dx$, with $\Delta x = 0.1$, and compare its error with those of other rules.

- $I \approx \frac{1}{30} \left(1 + \frac{4}{1.1} + \frac{2}{1.2} + \frac{4}{1.3} + \frac{2}{1.4} + \frac{4}{1.5} + \frac{1}{1.6} \right) = 0.470006\dots$
- "Exact": $I = \ln 1.6 = 0.470003\dots$
- $|E_s| = 0.000003$
- $|E_s| < |E_m| (< |E_T| < |E_R| < |E_L|)$.

Error Bounds

Let f be a function that is integrable on $[a, b]$. Let E_s, E_T, E_m, E_L and E_R be the error of Simpson's rule, trapezoidal rule, midpoint rule, left-hand rule and right-hand rule for approximating $\int_a^b f(x) dx$, respectively. Let

$$\max |f^{(i)}| := \max_{x \in [a, b]} |f^{(i)}(x)|.$$

Then

$$\begin{aligned}|E_L| &\leq \frac{(b-a)^2}{2n} \max|f'|, & |E_R| &\leq \frac{(b-a)^2}{2n} \max|f'|, \\ |E_T| &\leq \frac{(b-a)^3}{12n^2} \max|f''|, & |E_M| &\leq \frac{(b-a)^3}{24n^2} \max|f''|, \\ |E_S| &\leq \frac{(b-a)^5}{180n^4} \max|f^{(4)}|.\end{aligned}$$

no
need to
memorize

Generally speaking, Simpson's rule is the most accurate, followed by the midpoint rule, then trapezoidal rule, and lastly, left- and right-hand rule. (This applies to bigger values of n .)

e.g. If we want to approximate $\int_0^1 e^{x^2} dx$ with error having absolute value less than 10^{-5} , how many sub-intervals do we need at least? Give the answer for Simpson's rule and the trapezoidal rule.

Sol: Let $f(x) = e^{x^2}$.

$$f'(x) = e^{x^2} \cdot 2x,$$

$$f''(x) = 2e^{x^2} + 2x(e^{x^2} \cdot 2x) = (4x^2+2)e^{x^2}, \quad \max|f''|=6e$$

$$f'''(x) = 8x \cdot e^{x^2} + (4x^2+2)e^{x^2} \cdot 2x$$

$$= e^{x^2}(8x+8x^3+4x) = (8x^3+12x)e^{x^2},$$

$$f''''(x) = (24x^2+12)e^{x^2} + (8x^3+12x)e^{x^2} \cdot 2x$$

$$\begin{aligned}
 &= e^{x^2} (24x^2 + 12 + 16x^4 + 24x^2) \\
 &= e^{x^2} (16x^4 + 48x^2 + 12), \quad \max|f^{(4)}| = 76e
 \end{aligned}$$

For Simpson's rule, we want $\frac{1}{180n^4} 76e < 10^{-5}$,

$$\Leftrightarrow 180n^4 > 76 \cdot 10^5 e \Leftrightarrow n \geq 19.$$

Since n is even, we need n to be at least 20.

For trapezoidal rule, we want $\frac{1}{12n^2} 6e < 10^{-5}$,

$$\Leftrightarrow 12n^2 > 6 \cdot 10^5 e \Leftrightarrow \underline{n \geq 369}.$$

Some (optional) proofs regarding the error bounds can be found on Blackboard.