Lecture 11, Tuesday, October/17/2023

Outline

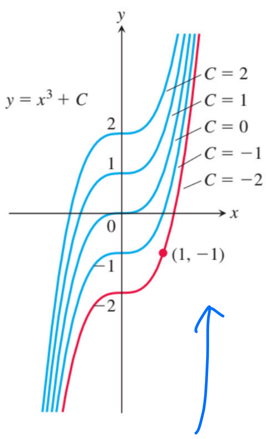
- Antiderivatives and indefinite integrals (4.7)
- Estimation with finite sums (5.1)
- Riemann sums (5.1)
- Finite sums (5.2)
- Definite integrals: definitions (5.3)

Antiderivatives

Def: If $F'(x) = f(x)$ for all x in an interval I, then F is said to be an antiderivative of f on I.

By a corollary of the MVT, we have:

Theorem 4.7.8 If F is an antiderivative of f on an interval I, then all the antiderivatives of f on I are $F(x)+C$, where C is an arbitrary constant.



- All antiderivatives are "parallel".
- Given a fixed point (x_0, y_0), there is only one antiderivative F that satisfies $F(x_0) = y_0$.

Some common formulae.

	Function	General antiderivative
1.	x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$
2.	$\sin kx$	$-\frac{1}{k}\cos kx + C$
3.	$\cos kx$	$\frac{1}{k}\sin kx + C$
4.	$\sec^2 kx$	$\frac{1}{k}\tan kx + C$

e.g. Find all antiderivatives of $f(x) = 3\sqrt{x} + \sin 2x$ on $[0, \infty)$.

Sol: Since $\left(2x^{3/2} - \frac{1}{2}\cos 2x\right)' = f(x)$, all antiderivatives are

$$F(x) = 2x^{3/2} - \frac{1}{2}\cos 2x + C,$$

where C is a constant.

EXAMPLE ~~5~~ 4.7.5 A hot-air balloon ascending at the rate of 3.6 m/s is at a height 24.5 m above the ground when a package is dropped. How long does it take the package to reach the ground?

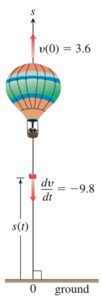


FIGURE 4.52 A package dropped from a rising hot-air balloon (Example 5).

Sol: ________________ .

Ans: $t = \dfrac{-3.6 + \sqrt{493.16}}{-9.8} \approx 2.63$ (sec).

Indefinite Integrals

> Def: The set of all antiderivates of f is called the **indefinite integral of f**. If F is one antiderivate of f, we write
>
> $$\int f(x)\,dx = F(x) + C.$$
>
> Here, we mean C is any constant.

→ Here, we assume the domain is an interval I, i.e., $F' = f$ on I.

$$\int f(x)\,dx$$

Indefinite integral of f with respect to x

$f(x)$: integrand dx: variable of integration

e.g. $\int (x^2-2x+5)\,dx = \frac{1}{3}x^3 - x^2 + 5x + C.$

Remark: Indefinite integrals satisfy linearity; that is,

$$\int (\alpha f(x) + \beta g(x))\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx.$$

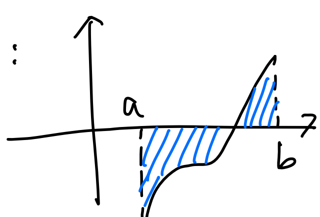
α, β constants.

- $\int (x + x^2)\,dx = \frac{1}{2}x^2 + \frac{1}{3}x^3 + C.$
- $\int x\,dx + \int x^2\,dx = \left(\frac{1}{2}x^2 + C_1\right) + \left(\frac{1}{3}x^3 + C_2\right) = \frac{1}{2}x^2 + \frac{1}{3}x^3 + \underbrace{(C_1 + C_2)}.$

This constant is arbitrary; in practice, can just write C instead.

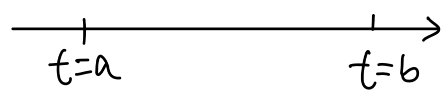
Integrals

Q1:



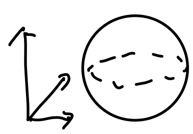
What is shaded area?

Q2: Given the velocity function $v(t)$



What is distance traveled from $t=a$ to $t=b$?

Q3:

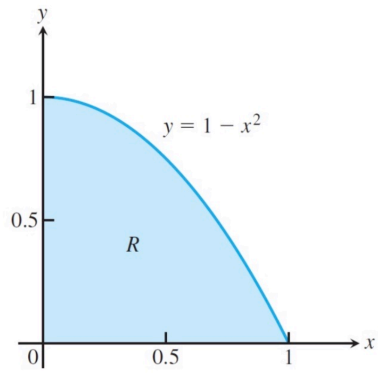


density at (x,y,z) is $\rho(x,y,z)$. What is total mass of solid ball?

MAT1002

Finding Area

Consider finding the area R under the graph of the function $y = 1 - x^2$, above the x-axis, between the vertical lines $x = 0$ and $x = 1$.

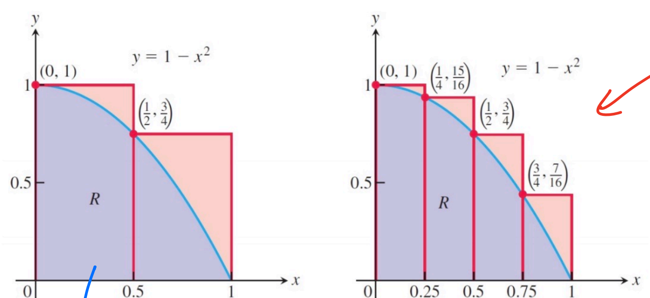


Note that $1 - x^2 \geq 0$ for all $x \in [0, 1]$.

Approximating Area by Rectangles

We may approximate R by summing areas of rectangles:

- Divide $[0, 1]$ into subintervals with equal length, and construct rectangles using the function values of the left endpoints, as demonstrated in the figure below.
- The sum of the areas of these rectangles is an approximation of R.

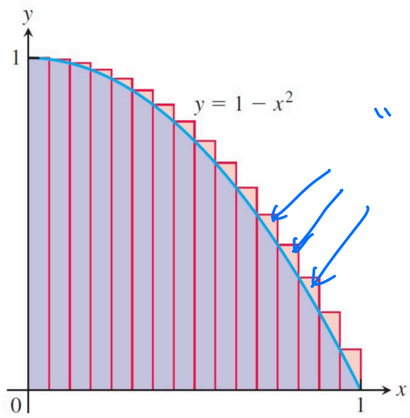


Note that this approximation over-estimate the area.

$$\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{3}{4} = \frac{7}{8} = 0.875$$

$$\frac{1}{4}\left(1 + \frac{15}{16} + \frac{3}{4} + \frac{7}{16}\right) = \frac{25}{32} = 0.78125$$

Intuitively, this approximation becomes better if we divide $[0,1]$ into more subintervals.



" Smaller errors "

In the above example:

- We divide $[0,1]$ into n subintervals of length $\Delta x = \frac{1}{n}$, corresponding to points $x_0, x_1, \dots, x_n$, where

  $$x_0 = 0, \quad x_1 = \frac{1}{n}, \quad x_2 = \frac{2}{n}, \quad \dots, \quad x_n = \frac{n}{n} = 1.$$

- $[a,b]$
- Δx

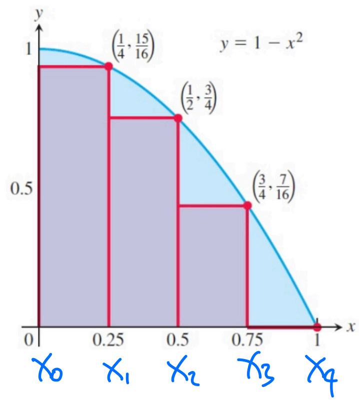
- The approximated area is

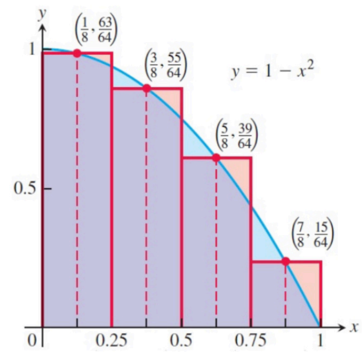
$$\frac{1}{n}\left(f(c_1) + f(c_2) + \dots + f(c_n)\right) = \frac{1}{n}\sum_{i=1}^{n} f(c_i),$$

where each $c_i \in [x_{i-1}, x_i]$ is chosen to be the left endpoint x_{i-1} (here the sum is called the left-hand sum).

- In this example, the function is decreasing, so $f(c_i) = f(x_{i-1})$ is always the maximum of f over $[x_{i-1}, x_i]$; when c_i is chosen to give maximum the sum $\frac{1}{n}\sum_{i=1}^{n} f(c_i)$ is called an upper sum.

Instead of left endpoints, one may also approximate by taking midpoints or right endpoints of the subintervals:





Right-hand sum

$$= \frac{1}{n}\sum_{i=1}^{n} f(c_i), \quad c_i = x_i,$$

= Lower sum

Midpoint sum

$$= \frac{1}{n}\sum_{i=1}^{n} f(c_i), \quad c_i = \frac{x_{i-1}+x_i}{2}$$

TABLE 5.1 Finite approximations for the area of R

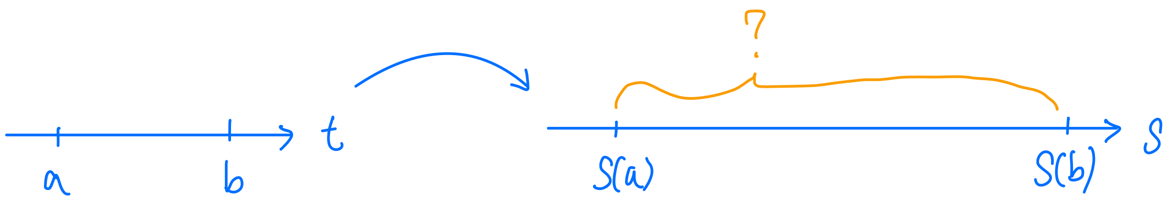
Number of subintervals (n)	Right-endpoint sum	Midpoint sum	Left-endpoint sum
2	0.375	0.6875	0.875
4	0.53125	0.671875	0.78125
16	0.634765625	0.6669921875	0.697265625
50	0.6566	0.6667	0.6766
100	0.66165	0.666675	0.67165
1000	0.6661665	0.66666675	0.6671665

lower sum; under-estimate

Exact area must be in between

upper sum; over-estimate

Q: If $v(t)$ is the velocity of an object at time t and $v(t) \geq 0$ for all $t \in [a,b]$, how can we approximate the total distance travelled from $t=a$ to $t=b$?



Suppose that you are sitting in a car. How can you tell how far you have travelled without checking the signs on the road? Check your speed meter often!

Suppose that the speed of a car is observed every two seconds, and is recorded in the following table.

Time (sec)	0	2	4	6	8	10
Velocity (ft/sec)	20	30	38	44	48	50

If we estimate using the starting speed in each interval, then the total distance travelled is

$$20 \cdot 2 + 30 \cdot 2 + 38 \cdot 2 + 44 \cdot 2 + 48 \cdot 2 = 360 \text{ feet.}$$

If we estimate using the ending speed in each interval, then the total distance travelled is

$$30 \cdot 2 + 38 \cdot 2 + 44 \cdot 2 + 48 \cdot 2 + 50 \cdot 2 = 420 \text{ feet.}$$

How if the speed is observed every second?

Time (sec)	0	1	2	3	4	5	6	7	8	9	10
Velocity (ft/sec)	20	26	30	35	38	42	44	46	48	49	50

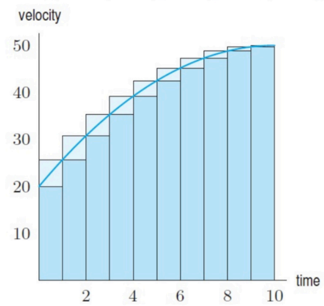
Again using the starting speed and ending speed to estimate, the total distance travelled is approximately:

$$20 + 26 + 30 + 35 + 38 + 42 + 44 + 46 + 48 + 49 = 378 \text{ feet}$$

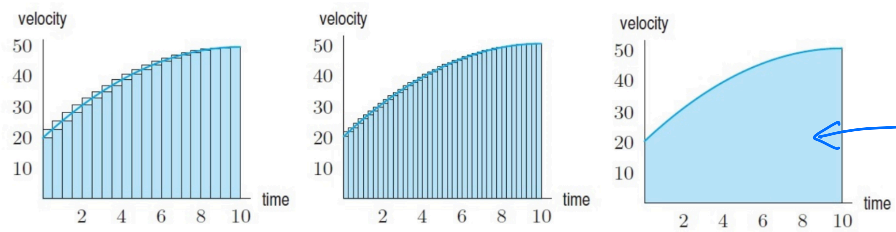
and

$$26 + 30 + 35 + 38 + 42 + 44 + 46 + 48 + 49 + 50 = 408 \text{ feet.}$$

Time (sec)	0	1	2	3	4	5	6	7	8	9	10
Velocity (ft/sec)	20	26	30	35	38	42	44	46	48	49	50



Dark blue = using starting speed; light blue = using ending speed.



Area = Total distance travelled.

Riemann Sums

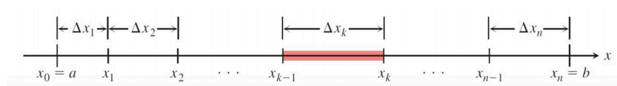
Def: A partition of the interval $[a,b]$ is a set $P=\{x_0, x_1, \dots, x_{n-1}, x_n\}$ such that

$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b.$$

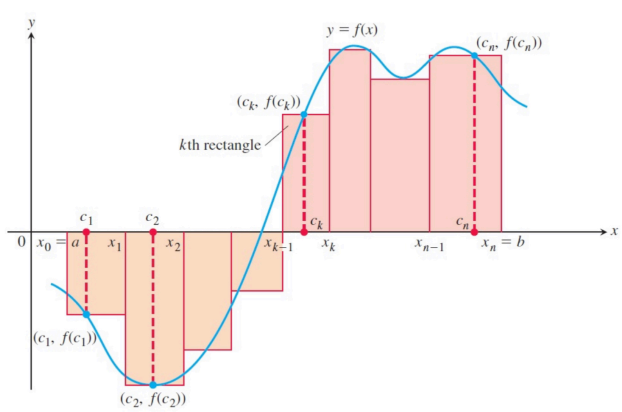
Given a function $f:[a,b]\to\mathbb{R}$ with a partition P of $[a,b]$, a Riemann sum of f (with respect to P) is a sum of the form

$$S_P = S_P(f) = \sum_{k=1}^{n} f(c_k)\Delta x_k = f(c_1)\Delta x_1 + \dots + f(c_n)\Delta x_n,$$

where $c_k \in [x_{k-1}, x_k]$ and $\Delta x_k = x_k - x_{k-1}$, for each $k \in \{1, \dots, n\}$.



Note that there are many Riemann sums for a function f: a Riemann sum depends on the partition P and the points c_k chosen from the subintervals.



Special cases:

- Left-hand sum
- Right-hand sum
- Midpoint sum
- Upper sum
- Lower sum

Note that in general, the values of the Δx_k may be different.

In-class discussion

Suppose that $y=f(x)$ is continuous, concave down, and positive on $[a,b]$. If we approximate the area between the curve and the x-axis, for $a \le x \le b$, using a midpoint sum S, which of the following is true?

(a) S over-estimates the area always.

(b) S under-estimates the area always.

(c) Both can happen depending on the choice of f and the partition of $[a,b]$.

Finite Sums

- Sigma notation : $\sum_{i=1}^{n} a_i := \sum_{i \in \{1,2,\dots,n\}} a_i := a_1 + a_2 + \dots + a_n$.
- Linearity of $\sum$:

$$\sum_{i=1}^{n}(ka_i + tb_i) = k\sum_{i=1}^{n} a_i + t\sum_{i=1}^{n} b_i .$$

$\hookrightarrow$ $(ka_1+tb_1)+(ka_2+tb_2)+\dots+(ka_n+tb_n)$

$= k(a_1+a_2+\dots+a_n) + t(b_1+b_2+\dots+b_n)$.

- $\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j$

Choice of symbols is not important : Such a variable is called a dummy variable.

The following are some useful identities:

- $\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \frac{1}{2} n(n+1).$ ①
- $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{1}{6} n(n+1)(2n+1).$ ②
- $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \frac{1}{4} n^2(n+1)^2.$ ③

① is the famous identity observed by Gauss.

Q: Is there a strategy for finding $\sum_{k=1}^{n} k^m$ for a fixed m?

A: Start with n^{m+1}, and use formulae derived for smaller m.

e.g. Find $\sum_{k=1}^{n} k$ (again).

↳ Start with n^2. Let $S_n = \sum_{k=1}^{n} k$.

↳ $n^2 = (n^2 - (n-1)^2) + ((n-1)^2 - (n-2)^2) + \ldots + (2^2 - 1^2) + (1^2 - 0^2)$

$$= \sum_{k=1}^{n} (k^2 - (k-1)^2) = \sum_{k=1}^{n} (\cancel{k^2 - k^2} - 1 + 2k)$$

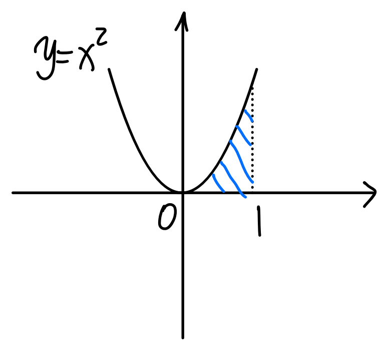
$$= 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = 2S_n - n$$

$$\Rightarrow \quad S_n = \frac{n^2+n}{2} = \frac{n(n+1)}{2}. \qquad \checkmark$$

<u>e.g.</u> Prove formula ② on the previous page.

<u>Sol</u>: ________________ .

<u>e.g.</u> By using right-hand sum approximation and limits, determine the shaded area.



<u>Sol</u>: ________________ $= \frac{1}{3}$.

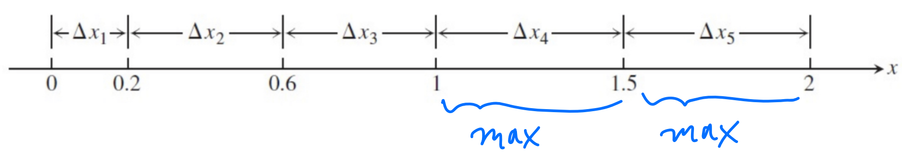
Definite Integrals (Riemann Integrals)

Definition
Let $P := \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. The norm of P, denoted by $\|P\|$, is defined by

$$\|P\| := \max_{k: 1 \le k \le n} \Delta x_k.$$

That is, $\|P\|$ is the length of the largest subinterval given by P.

Example
The partition P of $[0, 2]$ represented by the following figure has norm $\|P\| = 0.5$.



DEFINITION Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral of f over $[a, b]$** and that J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - J \right| < \epsilon.$$

Plain English: J is the limit of the Riemann sums of f if all Riemann sums of f are arbitrarily close to J as long as the partition is sufficiently fine.

Remark: With the definition above:

- We write $\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = J$.
- If the limit J exist, we say that f is (Riemann) integrable on $[a,b]$, and write the limit J as (or over $[a,b]$)

$$\int_a^b f(x)\,dx.$$

This symbol is called the definite integral or the Riemann integral of f over $[a,b]$ (or from a to b).