Lecture 5, Tuesday, September/19/2023

Outline

- Differentiability vs. continuity (3.2)
- f' vs Δx and Δy
- Differentiation rules (3.3)
  - ↳ Linearity and power rule
  - ↳ Product and quotient rules
- Higher-order derivatives (3.3)
- Trigonometric functions (3.5)
- Linear motion and cost/production (3.4)

Differentiability Implies Continuity

Theorem 3.2.1 If f is differentiable at c, then f is continuous at c.

Proof of theorem:

- It suffices to show that $\lim_{x\to c} f(x) = f(c)$.
- Note that

$$\left(\lim_{x\to c} f(x)\right) - f(c) = \lim_{x\to c} f(x) - \lim_{x\to c} f(c) = \lim_{x\to c}\left(f(x) - f(c)\right)$$

$$= \lim_{x\to c} \frac{f(x)-f(c)}{x-c}(x-c)$$

$$= \left(\lim_{x\to c} \frac{f(x)-f(c)}{x-c}\right)\left(\lim_{x\to c}(x-c)\right)$$

$$= f'(c)\cdot 0 = 0$$

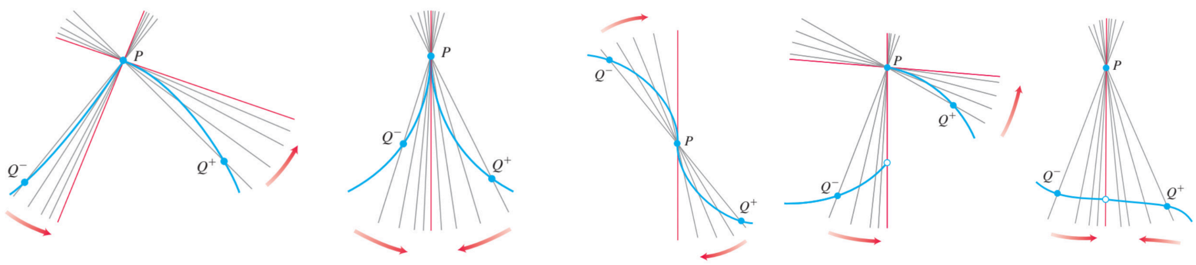
So $\lim_{x\to c} f(x) = f(c)$. $\square$

Note that the converse is false:

f continuous at c $\not\Rightarrow$ f differentiable at c.

We can see this in e.g. 7 and e.g. 8 in Lecture 4's notes.

Not Differentiable at a Point: Common Cases



$f'_-(c) \neq f'_+(c)$ vertical tangent   discontinuous at c

A Useful Notation

Let $y = f(x)$. Fix x_0, and let Δy be the change of y-value if x-value is changed by Δx:

$$\Delta y := f(x_0 + \Delta x) - f(x_0).$$

Then

$$\boxed{\left.\frac{dy}{dx}\right|_{x=x_0} = f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.}$$

Differentiation Rules

We do not want to use the definition every time when we compute $f'(x)$. In this lecture, we derive some computational rules for computing $f'(x)$.

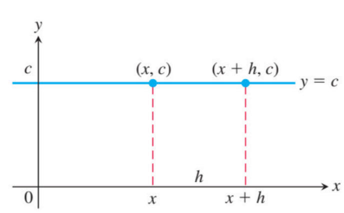
↳ show with steps

- Constant Function

If $f(x) = c$ is a constant function, then

$(f(x) \equiv c)$

$$f'(x) = 0, \quad \forall x \in \mathbb{R}.$$



$\left(\text{For any fixed } x_0, \; f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} 0 = 0. \right)$

- Linearity

For proof, see P. 134–135 of textbook.

Theorem (Linearity of Differentiation)

For any constants $\alpha, \beta \in \mathbb{R}$,

$$(\alpha f(x) + \beta g(x))' = \alpha f'(x) + \beta g'(x).$$

In particular, the linearity implies the following rules:

- $(\alpha f(x))' = \alpha f'(x)$. (Scalar multiplication rule)
- $(f(x) + g(x))' = f'(x) + g'(x)$. (Sum rule)
- $(f(x) - g(x))' = (f(x) + (-1)g(x))' = f'(x) + (-1)g'(x) = f'(x) - g'(x)$.

(Difference Rule)

- Power Rule

Let $f(x) = x^\alpha$, where $\alpha \in \mathbb{R}$ is a constant. Then

$$f'(x) = \alpha x^{\alpha-1}$$

for all x where x^α and $x^{\alpha-1}$ are both defined.

e.g.

$$\frac{d}{dx}\left[\pi x^3 - \frac{2}{3}x^{\sqrt{2}} + \overset{5x^{\frac{1}{2}}}{\cancel{5\sqrt{x}}} + 48 - \overset{x^{-\frac{2}{3}}}{\cancel{\left(\frac{1}{\sqrt[3]{x}}\right)^2}}\right]$$

$$= 3\pi x^2 - \frac{2}{3}\sqrt{2}\,x^{\sqrt{2}-1} + \frac{5}{2}x^{-\frac{1}{2}} + 0 + \frac{2}{3}x^{-\frac{5}{3}}.$$

Let us first check the power rule when $\alpha = n$ is a nonnegative integer: let $f(x) = x^n$, $n \in \mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}$.

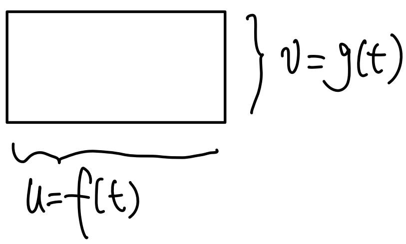
- If $n = 0$, then $f(x_0) = x_0^0 = 1$ (defined for $x_0 \neq 0$).
  By the constant rule, $f'(x_0) = 0 = 0 x_0^{0-1}$. ✓
- If $n \geq 1$, then

$$f'(x_0) = \lim_{x \to x_0} \frac{x^n - x_0^n}{x - x_0} = \begin{cases} \lim_{x \to x_0} \left(x^{n-1} + x^{n-2}x_0 + \ldots + x x_0^{n-2} + x_0^{n-1}\right), & \text{if } n \geq 2 \\ 1, & \text{if } n = 1 \end{cases}$$

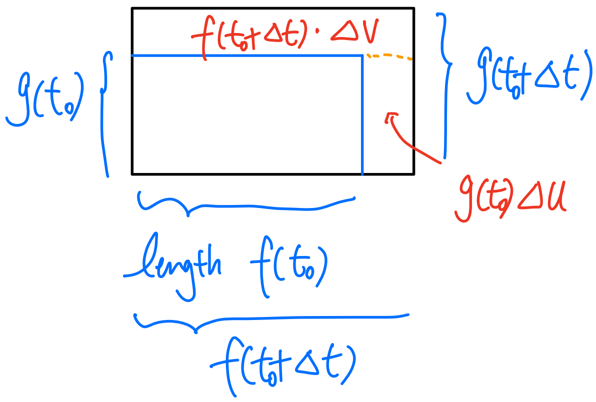
$$= \begin{cases} n x_0^{n-1}, & \text{if } n \geq 2 \\ 1 \cdot x_0^{1-1}, & \text{if } n = 1. \end{cases}$$ ✓

- For $\alpha \in \mathbb{Z}$, $\alpha < 0$, we will prove the rule after the quotient rule.
- For other values of α, we will prove the rule in Chapter 7.

- Product Rule



Q: How fast is area changing with respect to time?



Consider moment $t = t_0$

Suppose f and g are both differentiable at t_0.

- $$\left.\frac{d(uv)}{dt}\right|_{t=t_0} = \lim_{\Delta t\to 0}\frac{\Delta(uv)}{\Delta t}.$$

- $$\begin{aligned}\Delta(uv) &= f(t_0+\Delta t)g(t_0+\Delta t) - f(t_0)g(t_0)\\ &= \underline{f(t_0+\Delta t)g(t_0+\Delta t)} - \underline{f(t_0)g(t_0)} - \underline{f(t_0+\Delta t)g(t_0)} \\ &\quad + \underline{f(t_0+\Delta t)g(t_0)}\\ &= \underline{f(t_0+\Delta t)\Delta V} + \underline{g(t_0)\Delta u}\end{aligned}$$

- $$\lim_{\Delta t\to 0}\frac{\Delta(uv)}{\Delta t} = \lim_{\Delta t\to 0}\frac{f(t_0+\Delta t)\Delta V + g(t_0)\Delta u}{\Delta t}$$

$$= \underbrace{\lim_{\Delta t \to 0} f(t_0+\Delta t)}_{f(t_0),\text{ since } f \text{ is differentiable and hence continuous at } t_0} \underbrace{\lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}}_{\left.\frac{dv}{dt}\right|_{t=t_0} = g'(t_0)} + g(t_0) \underbrace{\lim_{\Delta t \to 0} \frac{\Delta u}{\Delta t}}_{\left.\frac{du}{dt}\right|_{t=t_0} = f'(t_0)}$$

$$= f(t_0)g'(t_0) + g(t_0)f'(t_0).$$

This is known as the product rule.

<u>Product Rule</u> If f and g are differentiable at x, then

$$\big(f(x)g(x)\big)' = f'(x)g(x) + f(x)g'(x).$$

<u>e.g.</u>

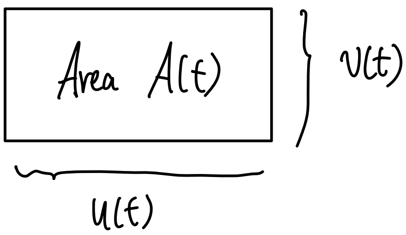
$$y = (x^2+1)(x^3+1) \quad (= x^5 + x^3 + x^2 + 1)$$

$$y' = (x^2+1)(x^3+1)' + (x^2+1)'(x^3+1)$$

$$= (x^2+1)\,3x^2 + 2x(x^3+1)$$

$$= 3x^4 + 3x^2 + 2x^4 + 2x$$

$$= 5x^4 + 3x^2 + 2x.$$



Q: How if we know $\frac{dA}{dt}$ and $\frac{du}{dt}$? What is $\frac{dv}{dt}$?

· Quotient Rule

> **Quotient Rule** If f and g are differentiable at x, and $g(x) \neq 0$, then
> $$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

e.g. $y = \dfrac{\overbrace{t^2-1}^{f(t)}}{\underbrace{t^3+1}_{g(t)}}$. Then

$$\frac{dy}{dt} = \frac{(t^3+1)(t^2-1)' - (t^2-1)(t^3+1)'}{(t^3+1)^2} = \frac{(t^3+1)2t - (t^2-1)3t^2}{(t^3+1)^2}$$

$$= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3+1)^2} = \frac{-t^4 + 3t^2 + 2t}{(t^3+1)^2}.$$

For a proof of the quotient rule, see Chapter 3.3.

Alternatively, $f'(x) = \left(\frac{f(x)}{g(x)} g(x)\right)' = \left(\frac{f}{g}\right)'(x)\, g(x) + \frac{f(x)}{g(x)} g'(x)$

$$\Rightarrow \quad \left(\frac{f}{g}\right)'(x) = \frac{f'(x) - \frac{f(x)}{g(x)} g'(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

For $n \in \mathbb{Z}$, $n=-m$, $m>0$, we can use the quotient rule to show that $(x^n)' = nx^{n-1}$. Indeed,

$$(x^n)' = \left(\frac{1}{x^m}\right)' = \frac{x^m \cdot 1' - (x^m)' \cdot 1}{x^{2m}} \quad \text{(quotient rule)}$$

$$= \frac{-mx^{m-1}}{x^{2m}} \quad \text{(power rule for positive integer powers)}$$

$$= -m\frac{1}{x^{m+1}} = -mx^{-m-1} = nx^{n-1}.$$

Higher-Order Derivatives

- $f^{(2)} := f'' := (f')'$ Second derivative of f
- In general, for $n \in \mathbb{Z}$, $n \geq 0$:
  - ↳ $f^{(0)} := f$.
  - ↳ $f^{(n)} := \left(f^{(n-1)}\right)'$, if $n \geq 1$.

    n^{th} derivative of f.
- Alternative notations: if $y = f(x)$:
  - ↳ $f''(x)$: y'', $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, $\frac{d^2}{dx^2}y$
  - ↳ $f^{(n)}(x)$: $y^{(n)}$, $\frac{d^n}{dx^n}y$.

e.g. $y = x^3 - 3x^2 + 2$

↳ $y' = 3x^2 - 6x$, $y'' = 6x - 6$, $y''' = 6$, $y^{(n)} \equiv 0$, $\forall n \geq 4$.

Derivatives of Trigonometric Functions

We derive some derivative formulae for trigonometric functions here. Since

$$\sin(A+B) = \sin A \cos B + \sin B \cos A,$$

$$\sin'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{\sin x \cosh + \sin h \cos x - \sin x}{h}$$

$$= \sin x \underbrace{\lim_{h\to 0} \frac{\cos h - 1}{h}}_{=0,\ \text{see e.g.5, lec3}} + \cos x \lim_{h\to 0} \frac{\sin h}{h}$$

$$= \cos x.$$

Hence,

$$\boxed{\sin' x = \cos x, \quad \forall x \in \mathbb{R}.}$$

Similarly, since $\cos(A+B) = \cos A \cos B - \sin A \sin B$, we have

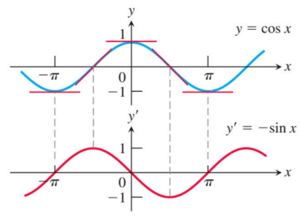
$$\cos'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \cos x \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h}$$

$$= -\sin x.$$

Hence,

$$\boxed{\cos' x = -\sin x, \quad \forall x \in \mathbb{R}.}$$



For $\tan(x)$ and $\sec(x)$, we have

$$\tan' x = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \sin'(x) - \sin x \cos'(x)}{\cos^2 x}$$ Quotient rule

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

$$\sec'(x) = \left(\frac{1}{\cos x}\right)' = \frac{\cos x \cdot 1' - 1 \cdot \cos' x}{\cos^2 x} = \frac{\sin x}{\cos^2 x}$$

$$= \tan x \cdot \sec x.$$

Similarly, we can find $\cot' x$ and $\csc' x$.

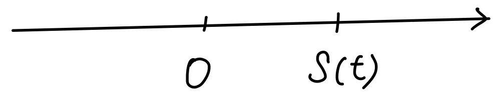
The derivatives of the other trigonometric functions:

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Derivatives: Motion along a Line

Suppose an object is moving along a line, whose position is $s = S(t)$ at time t.



Note that s can be negative.

Def.: • The displacement of the object over a time interval $[a, b]$ is $S(b) - S(a)$. $(b \geq a)$.

• The velocity of the object at time t is

$$v(t) := \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{S(t+\Delta t) - S(t)}{\Delta t}.$$

• The speed of the object at time t is $|v(t)|$.

• The acceleration of the object at time t is

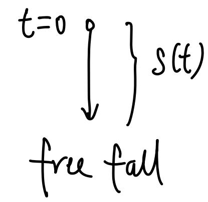
$$a(t) := \frac{dv}{dt} = \frac{d^2}{dt^2} s.$$

• The jerk of the object at time t is

$$j(t) := \frac{da}{dt} = \frac{d^3}{dt^3} s.$$

e.g. $s(t) = 4.9t^2$, $v(t) = 9.8t$,

$a(t) = 9.8$, $j(t) = 0$.



Derivatives: Cost and Production

Suppose that the cost for producing x (product) thousand units of goods is $C(x)$ (in thousand dollars \$).

total cost function

What do we know if $C(x) = 0.001x^2 + 2x + 1500$?

- Fixed cost is $C(0) = 1500$ (K\$).
- Average cost is $\frac{C(x)}{x} = 0.001x + 2 + \frac{1500}{x}$.
- Marginal cost is $C'(x) = 0.002x + 2$. This means that at the product level x, the additional cost for producing one additional unit is approximately $0.002x + 2$ dollars.

Def

$$\text{marginal cost at } x_0 \text{ units of production} := C'(x_0) = \lim_{h \to 0} \frac{C(x_0+h) - C(x_0)}{h}.$$