Lecture 21, Tuesday, November/21/2023

Outline

- Integration by parts (8.2)
  - ↳ Indefinite integrals
  - ↳ Reduction formulae
  - ↳ Definite integrals
  - ↳ Tabular integration
- Trigonometric Integrals (8.3)
  - ↳ $\int \sin^m x \cos^n x \, dx$
  - ↳ $\int \tan^m x \sec^n x \, dx$

Integration by Parts

Indefinite Integrals

Integration by parts is a technique that simplifies integrals. If f and g are differentiable at x, then

$$\left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x),$$

So

$$\int \left(f'(x)g(x) + f(x)g'(x)\right) dx = f(x)g(x) + C.$$

Formula (Integration by Parts)

The "+C" above is absorbed by this

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

Example

To find $\int x\cos x\,dx$, consider $f(x) := x$ and $g'(x) := \cos x$. We would like to integrate $f(x)g'(x)$. Note that $f'(x) = 1$, and $g(x) = \sin x$ is one antiderivative of g'. Now $f'(x)g(x) = \sin x$ is easy to integrate, and

(a) $$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

Alternative notation:

If we set $u = f(x)$ and $v = g(x)$, then we have the following notation:

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

Example

(b) $\displaystyle\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C.$

(c) $\displaystyle\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C.$

(d) $\displaystyle\int \ln x \, dx = x \ln x - x + C.$

Reduction Formulae

Consider finding $I = \int \sin^n x \, dx$, where $n \neq 0$.

↳ Set $u = \sin^{n-1} x$ and $dv = \sin x \, dx$. Then $I = \int u \, dv$.

↳ Then $du = (n-1) \sin^{n-2} x \, (\cos x) \, dx$, $v = -\cos x$, so

$$I = \int \sin^n x \, dx = -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x \cdot \overset{1-\sin^2 x}{\cancel{\cos^2 x}} \, dx$$

$$= -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x \cdot dx - (n-1) I$$

↳ $$I = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx.$$

This gives the following reduction formula.

$$\boxed{\int \sin^n x \, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \, dx.}$$

Similarly, we have

$$\boxed{\int \cos^n x \, dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x \, dx.}$$

e.g. (e) $\int \sin^3 x = -\frac{1}{3}\cos x \sin^2 x + \frac{2}{3}\int \sin x \, dx$

$= -\frac{1}{3}\cos x \sin^2 x - \frac{2}{3}\cos x + C$

Definite Integrals

If $f(x)g'(x) + f'(x)g(x)$ is continuous on $[a,b]$, then by FTC2,

$$\int_a^b \left(f(x)g'(x) + f'(x)g(x)\right) dx = f(x)g(x)\Big|_a^b .$$

Hence,

$$\boxed{\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx .}$$

e.g. (f) Evaluate $\int_0^1 \arctan x \, dx$.

Sol.

Ans: $\frac{\pi}{4} - \frac{1}{2}\ln 2$.

Tabular Integration

Consider finding $\int x^{10}\cos x\,dx$. You may apply integration by parts 10 times ——— although you probably would rather spend your time on something else. Tabular integration can simplify the notation.

Differentiation $\left(f_i'(x) = f_{i+1}(x)\right)$

$f_0(x)$

Suppose we have
$$\begin{cases} f(x) \to f_1(x) \to f_2(x) \to \cdots \to f_m(x) = 0 \\ g(x) \to g_1(x) \to g_2(x) \to \cdots \to g_m(x) \end{cases}$$

$g_0(x)$

Integration
$\left(\quad g_i'(x) = g_{i-1}(x)\right)$

Then
$$\int f_0(x)g_0(x)\,dx = f_0(x)g_1(x) - \int f_1(x)g_1(x)\,dx$$
$$= f_0(x)g_1(x) - f_1(x)g_2(x) + \int f_2(x)g_2(x)\,dx$$
$$= \cdots$$
$$= f_0(x)g_1(x) - f_1(x)g_2(x) + f_2(x)g_3(x) - \cdots \pm \underbrace{\int f_m(x)g_m(x)\,dx}_{0}$$

This method is particular effective for $\int f(x)g(x)\,dx$ if f is a polynomial and integrating g will not give a more complicated integrand.

e.g. (g) Evalute $\int x^3 3^x dx$

$f(x)$ and derivatives		$g(x)$ and antiderivatives
x^3	+	3^x
$3x^2$	−	$(\ln 3)^{-1} 3^x$
$6x$	+	$(\ln 3)^{-2} 3^x$
6	−	$(\ln 3)^{-3} 3^x$
0		$(\ln 3)^{-4} 3^x$

$$\text{Ans} = x^3 \frac{3^x}{\ln 3} - 3x^2 \frac{3^x}{(\ln 3)^2} + 6x \frac{3^x}{(\ln 3)^3} - 6 \frac{3^x}{(\ln 3)^4} + C.$$

Trigonometric Integrals

In this section, we will see how various trigonometric identities may help with integration.

$\int \sin^m x \cos^n x \, dx$ $\quad (m, n \in \mathbb{N} := \{0, 1, 2, 3, \ldots\}.)$

1. If m or n is odd, consider an odd power, and take out all the even powers. Use $\sin^2 x + \cos^2 x = 1$.

If $m = 2k+1$, then $\sin^m x = (\sin^2 x)^k \sin x$

e.g. $\int \sin^5 x \cos^7 x \, dx$

$$= \int (\sin^2 x)^2 \sin x \cos^7 x \, dx = \int (1 - \cos^2 x)^2 \sin x \cos^7 x \, dx$$

$$\overset{u=\cos x}{=} \int (1-u^2)^2 u^7 (-du) = -\int u^7(1-2u^2+u^4)\,du$$

$$= \ldots\ldots$$

can use reduction formula, but longer.

e.g. $\int \cos^7 x\,dx = \int \cos^6 x \cdot \cos x\,dx = \int (1-\sin^2 x)^3 \cos x\,dx$

$$= \int (1-u^2)^3\,du = \int (1-3u^2+3u^4-u^6)\,du = \ldots$$

2. If both m and n are even, then use half-angle identities:

$$\sin^2 x = \frac{1-\cos 2x}{2}, \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

e.g. $\int \sin^2 x \cos^4 x\,dx = \int \sin^2 x\,(\cos^2 x)^2\,dx$

$$= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{8}\int (1+\cos 2x-\cos^2 2x-\cos^3 2x)\,dx$$

- $\int (1+\cos 2x)\,dx = x+\frac{1}{2}\sin 2x + C_1.$
- $\int \cos^2 2x = \int \frac{1+\cos 4x}{2} = \frac{1}{2}x+\frac{1}{8}\sin 4x + C_2.$
- $\int \cos^3 2x = \int (1-\sin^2 2x)\cos 2x\,dx$

$$= \int (1-u^2)\frac{1}{2}\,du = \frac{1}{2}\left(u-\frac{1}{3}u^3\right)+C_3$$

$$= \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right) + C_3$$

Combining and simplifying, we have

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{8}\left(\frac{1}{2}x - \frac{1}{8}\sin 4x + \frac{1}{6}\sin^3 2x\right) + C.$$

$\int \tan^m x \, dx$ $\quad (m \in \mathbb{N})$.

- Recall that $\int \tan x \, dx = \ln|\sec x| + C$ $\left(= -\ln|\cos x| + C\right)$.
- Use $\tan^2 x + 1 = \sec^2 x$ to obtain a reduction formula:
  - ↳ If $m \in \{0, 1\}$: ✓.
  - ↳ Assume $m \geq 2$. Then

$$\begin{aligned}
\int \tan^m x \, dx &= \int \tan^{m-2} x \cdot \tan^2 x \, dx \\
&= \int \tan^{m-2} x \left(\sec^2 x - 1\right) dx \\
&= \int \tan^{m-2} x \sec^2 x \, dx - \int \tan^{m-2} x \, dx \\
&= \int \tan^{m-2} x \, d(\tan x) - \int \tan^{m-2} x \, dx \\
&= \frac{\tan^{m-1} x}{m-1} - \int \tan^{m-2} x \, dx.
\end{aligned}$$

Numerical example: 8.3, example 5.

$\int \sec^n x \, dx \quad (n \in \mathbb{N})$

- Recall that $\int \sec x \, dx = \ln|\sec x + \tan x| + C$
- Use integration by parts to obtain a reduction formula:
  - ↳ If $n \in \{0, 1, 2\}$: ✓
  - ↳ Assume $n \geq 3$.

    Let $u = \sec^{n-2} x$, $\quad dv = \sec^2 x \, dx$.

    $du = (n-2) \sec^{n-2} x \tan x \, dx$, $\quad v = \tan x$.

Then $\int \sec^n x \, dx = \int u \, dv = uv - \int v \, du$ $\quad \sec^2 x - 1$

$$= \sec^{n-2} x \cdot \tan x - (n-2) \int \sec^{n-2} x \tan^2 x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^n x \, dx + (n-2) \int \sec^{n-2} x \, dx$$

Isolating $\int \sec^n x \, dx$ yields

$$\boxed{\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx.}$$

Note that for even powers of sec, using $\sec^2 x = 1 + \tan^2 x$ and making substitution $u = \tan x$ could be easier, as item 1 on the next page outlines.

$\int \tan^m x \sec^n x \, dx \qquad (m, n \in \mathbb{Z}_+ := \{1, 2, 3, \ldots\})$

1. If n is even, then take out a copy of $\sec^2 x$ and express everything in terms of $\tan x$ (with $d(\tan x) = \sec^2 x \, dx$ and $\sec^2 x = \tan^2 x + 1$). This works even if $m = 0$:

$$\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x \sec^{2k-2} x \sec^2 x \, dx$$
$$= \int \tan^m x (1 + \tan^2 x)^{k-1} \, d(\tan x)$$

Numerical example: 8.4, example 7.

2. If m is odd, then take out a copy of $\tan x \sec x$, and express the rest in terms of $\sec x$.

$$\int \tan^{2k+1} x \sec^n x \, dx = \int \tan^{2k} x \cdot \sec^{n-1} x \, (\tan x \sec x) \, dx$$
$$= \int (\sec^2 x - 1)^k \cdot (\sec^{n-1} x) \, d(\sec x)$$

3. If m is even and n is odd, then use $\tan^2 x = \sec^2 x - 1$ to reduce the problem to the case involving only $\int \sec^i x \, dx$.

$$\int \tan^{2k} x \sec^n x \, dx = \int (\sec^2 x - 1)^k \sec^n x \, dx$$

Can use reduction formula for $\int \sec^i x \, dx$.