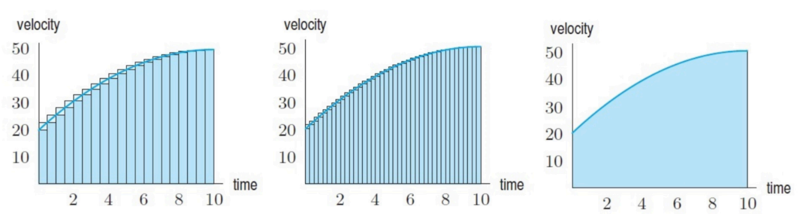
Lecture 12, Thursday, October/19/2023

- Definite integrals (5.3)
  - ↳ Nonintegrable functions
  - ↳ Integrabable functions
  - ↳ Computation using Riemann sums
  - ↳ Properties of definite integrals
  - ↳ Average value of a function



Definite Integrals (Riemann Integrals)

Q: • Which functions are integrable? ← This lecture's goal.

• If f is integrable, then how to compute $\int_a^b f(x)dx$?

Given f defined on $[a,b]$ and a partition $P=\{x_0,x_1,\dots,x_n\}$ of $[a,b]$, define

$$M_k := \max_{x\in[x_{k-1},x_k]} f(x) \text{ and } m_k := \min_{x\in[x_{k-1},x_k]} f(x) \text{ for } k=1,2,\dots,n.$$

Then $U_P(f) := \sum_{k=1}^{n} M_k \Delta x_k$ is an upper sum of f and

$L_P(f) := \sum_{k=1}^{n} m_k \Delta x_k$ is a lower sum of f.

Observe that $U_P(f)$ can only decrease as P gets finer (i.e., more points are added to P) and $L_P(f)$ can only increase as P gets finer.

Intuition A function is integrable (on $[a,b]$) if $U_P(f)$ and $L_P(f)$ are finite, and the gap between $U_P(f)$ and $L_P(f)$ goes to 0 as $\|P\|\to 0$. The reason is that all the Riemann sums would be squeezed by $L_P(f)$ and $U_P(f)$ as $\|P\|\to 0$, and the squeezed number is the limit J.

e.g. 1 Consider the Dirichlet function

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}.$$

(Here $\mathbb{Q} = \left\{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{Z}, n \neq 0 \right\}$ is the set of rational numbers.)

It is not integrable on any $[a,b]$.

- Any interval contains a rational AND an irrational number.
- Hence, for any portion P, in any sub-interval $[x_{k-1}, x_k]$, $m_k = 0$ and $M_k = 1$.
- This means $U_P(f) = \sum_{k=1}^{n} M_k \Delta x_k = \sum_{k=1}^{n} \Delta x_k = b-a > 0$

  and $L_P(f) = \sum_{k=1}^{n} m_k \Delta x_k = 0$.

No matter how small $\|P\|$ is, there is a Riemann sum with value $b-a$ and one with value 0. So the Riemann sums have no limit, and f is not integrable.

You may think of this as "there is no well-defined area under the graph of $y = f(x)$".

On the other hand, unbounded functions are always non-integrable, as their Riemann sums are not bounded even if $\|P\|$ gets very small.

<u>Theorem A</u> If f is unbounded on $[a,b]$, then it is not integrable on $[a,b]$.

<u>Idea</u>:

- Suppose f is not bounded above (idea is similar if f is not bounded below).
- Take $f(x) := \begin{cases} \frac{1}{x}, & x \in (0,1] \\ 0, & x = 0 \end{cases}$ as an example.
- For any partion P of $[0,1]$, consider a Riemann sum
$$\sum_{k=1}^{n} f(c_k) \Delta x_k .$$
- Since f is unbounded above on $[x_0, x_1]$, one may let
$$f(c_1)\underbrace{\Delta x_1}$$
could be small, but FIXED !

  be arbitrarily big by choosing c_1 suitably.
- Keeping all other c_k's unchanged, we see that there is

an arbitrarily big Riemann sum.

- Since this is true for each P (no matter how small $\|P\|$ is), the Riemann sums have no limit, so f is not integrable on $[0,1]$.

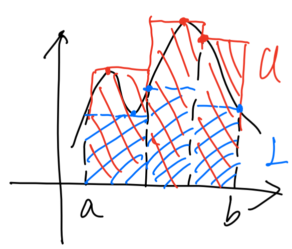
* In Chapter 8, we will introduce another type of integrals that handle unbounded integrands.

<u>Remark</u>: When we talk about definite integrals on $[a,b]$, we may assume that f is bounded on $[a,b]$.

> <u>Theorem B</u> If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

<u>Idea</u>:

- Define $U_P(f)$ and $L_P(f)$ as before, and show that
  $U_P(f) - L_P(f) \to 0$ as $\|P\| \to 0$.



- By continuity, as long as we choose $\|P\|$ to be small enough, $M_k - m_k$ can be arbitrarily small $\forall k$, and $U_P - L_P$ can be arbitrarily small.

  ↳ Formally, for any given $\varepsilon > 0$, choose $\|P\|$ small enough such that $M_k - m_k < \frac{\varepsilon}{b-a}$, $\forall k$.

↳ Then

$$U_P(f) - L_P(f) = \sum_{k=1}^{n} (M_k - m_k)\Delta x_k < \frac{\varepsilon}{b-a} \sum_{k=1}^{n} \Delta x_k$$
$$= \frac{\varepsilon}{b-a}(b-a) = \varepsilon .$$

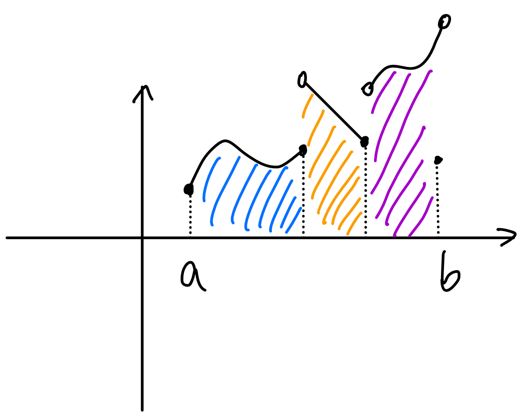
- Since all Riemann sums $S_P(f) := \sum_{k=1}^{n} f(c_k)\Delta x_k$ satisfy

$$L_P(f) \le S_P(f) \le U_P(f),$$

we know the limit of Riemann sums exists.

Theorem C If f is bounded on $[a,b]$ and is only discontinuous at finitely many points in $[a,b]$, then f is integrable on $[a,b]$.

Proof : Omitted.



Computation Using Riemann Sums

Suppose we know that f is integrable on $[a,b]$. Then we can compute the limit of Riemann sums by choosing any sequence of partitions P such that $\|P\| \to 0$.

In particular, we may choose $\Delta x_k = \Delta x = \frac{b-a}{n}$,
(P divides $[a,b]$ into n subintervals of equal length)
and take $n \to \infty$:

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x.}$$

Here, $c_k \in [x_{k-1}, x_k]$ can be chosen in any way (you may choose it to make the computation convenient).

e.g. 2 Evaluate $\int_0^2 (3x^2+x-5)\,dx$ by setting up a Riemann sum and taking the limit. (Note that the integral exists since $3x^2+x-5$ is continuous on $[0,2]$.)

Sol: ______________________ $= 0$.

Properties of Definite Integrals

Def: • For $a<b$, we define $\int_b^a f(x)dx := -\int_a^b f(x)dx$.

• Define $\int_a^a f(x) := 0$.

↳ The intuition behind the first definition is that all Δx_k's in a Riemann sum become negative of the original when x goes from b to a $(b>a)$.

TABLE 5.6 Rules satisfied by definite integrals

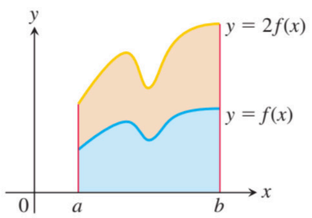
1. *Order of Integration:*	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A definition
2. *Zero Width Interval:*	$\int_a^a f(x)\,dx = 0$	A definition when $f(a)$ exists
3. *Constant Multiple:*	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any constant k
4. *Sum and Difference:*	$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	
5. *Additivity:*	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	
6. *Max-Min Inequality:*	If f has maximum value max f and minimum value min f on $[a, b]$, then $\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a)$.	
7. *Domination:*	$f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$	
	$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$	(Special case)

Linearity:

$$\int_a^b (\alpha f(x) + \beta g(x))\,dx = \alpha\int_a^b f(x)\,dx + \beta\int_a^b g(x)\,dx$$

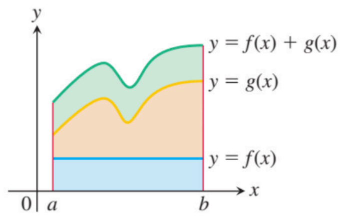
All properties above hold whenever the integrals exist: f does not have to be nonnegative or continuous.

Intuition behind the properties, in terms of area:



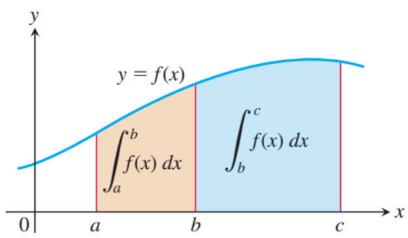
(b) *Constant Multiple:* $(k = 2)$

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$



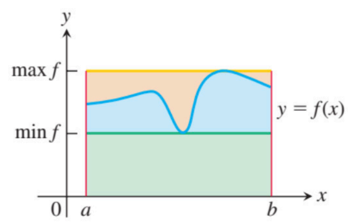
(c) *Sum: (areas add)*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



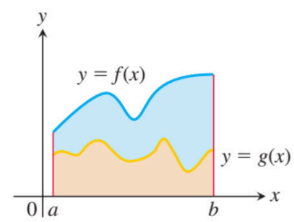
(d) *Additivity for Definite Integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



(e) *Max-Min Inequality:*

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a)$$



(f) *Domination:*

$f(x) \ge g(x)$ on $[a, b]$

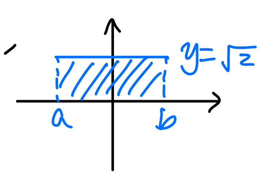
$$\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

- For additivity $\int_a^b + \int_b^c = \int_a^c$, b does not have to be in $[a,c]$:
  - ↳ e.g. $\int_3^6 f(x)dx + \int_6^5 f(x)dx = \int_3^5 f(x)dx$ is still true.

e.g.3 Prove that $0 \le \int_a^b \sqrt{1+\cos x}\,dx \le \sqrt{2}(b-a)$ if $b \ge a$.

Proof. Since $0 \le \sqrt{1+\cos x} \le \sqrt{2}$, $\forall x \in \mathbb{R}$, by Property 7,

$$\int_a^b 0\,dx \le \int_a^b \sqrt{1+\cos x}\,dx \le \int_a^b \sqrt{2}\,dx,$$



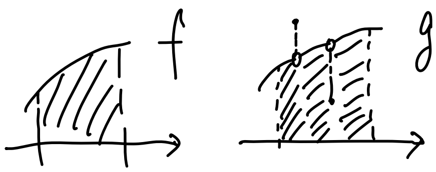
So $0 \le \int_a^b \sqrt{1+\cos x}\,dx \le \sqrt{2}(b-a)$. $\square$

The following is another property of definite integrals.

Theorem If f is integrable on $[a,b]$ and g is the same as f on $[a,b]$, except at finitely many points $x_1, \ldots, x_m$, then g is integrable on $[a,b]$, and

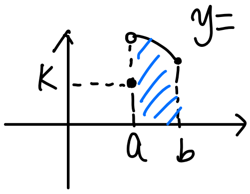
$$\int_a^b g(x)\,dx = \int_a^b f(x)\,dx.$$

Intuition

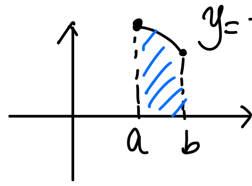


area does not change.

Consequence



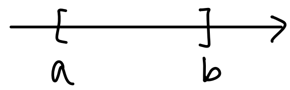
$y = \begin{cases} f(x), & \text{if } x \in (a,b] \\ K, & \text{if } x = a \end{cases}$ $=$



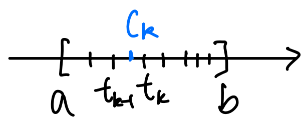
$y = f(x)$ $= \int_a^b f(x)dx$

Average Values / Means

Average velocity

- 

$v(t)$: velocity at time t (say $v(t) \geq 0$ $\forall t$)

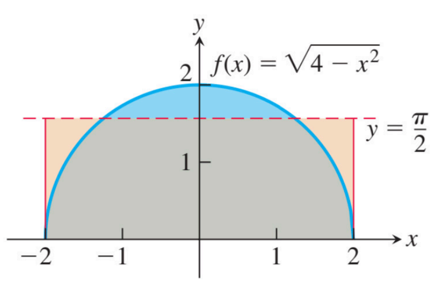
- 

$\sum_{k=1}^{n} v(c_k) \Delta t_k \approx$ total distance travelled from $t=a$ to $t=b$.

- $\int_a^b v(t)\,dt$ = exact distance travelled from $t=a$ to $t=b$.

- $\frac{1}{b-a} \int_a^b \underline{v(t)}\,dt$ = average $\underline{\text{velocity}}$ over (on) $[a,b]$.

Average height



- Area $= \int_{-2}^{2} \sqrt{4-x^2}\,dx$ $\left(= \frac{1}{2} 4\pi\right)$

- Average $\underline{\text{height}}$ $\quad$ height at x

$$= \frac{\text{Area}}{\text{base}} = \frac{1}{2-(-2)} \int_{-2}^{2} \underline{\sqrt{4-x^2}}\,dx$$

In general :

> **DEFINITION** If f is integrable on $[a, b]$, then its **average value on $[a, b]$**, also called its **mean**, is
>
> $$\text{av}(f) = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

Remark : f may not be nonnegative in the definition above.