Left-hand rule: error bound

MVT, for some $t_{i-1,x} \in (x_{i-1}, x)$

$$\int_{x_{i-1}}^{x_i} f(x)\,dx = \int_{x_{i-1}}^{x_i} \left[f(x_{i-1}) + f'(t_{i-1,x})(x - x_{i-1}) \right] dx$$

$$= f(x_{i-1})\Delta x + \int_{x_{i-1}}^{x_i} f'(t_{i-1,x})(x - x_{i-1})\,dx \qquad (*)$$

$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f(x_{i-1})\Delta x \right| = \left| \sum_{i=1}^{n} \left(\int_{x_{i-1}}^{x_i} f(x)\,dx - f(x_{i-1})\Delta x \right) \right|$$

$$\leq \sum_{i=1}^{n} \left| \int_{x_{i-1}}^{x_i} f(x)\,dx - f(x_{i-1})\Delta x \right| \qquad \text{(Triangle inequality)}$$

$$\leq \sum_{i=1}^{n} \left| \int_{x_{i-1}}^{x_i} f'(t_{i-1,x})(x - x_{i-1})\,dx \right| \qquad \text{(by } (*)\text{)}$$

$$\leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \left| f'(t_{i-1,x})(x - x_{i-1}) \right| dx \qquad \text{(Property of integral)}$$

$$\leq \sum_{i=1}^{n} \max|f'| \int_{x_{i-1}}^{x_i} (x - x_{i-1})\,dx$$

$$\Delta x = \frac{b-a}{n}$$

$$= K \sum_{i=1}^{n} \frac{1}{2}(x - x_{i-1})^2 \Big|_{x = x_{i-1}}^{x_i} = \frac{K}{2} \sum_{i=1}^{n} (\Delta x)^2$$

$$= \frac{K}{2} \cdot n \cdot \frac{(b-a)^2}{n^2} = \frac{K}{2n}(b-a)^2.$$

- Right-hand rule's error bound can be shown similarly.
- Midpoint rule's error bound can be shown similarly, but requires Taylor's thm (will be taught in MAT1002).

* Taylor's theorem is a generalized MVT.
* This proof is for reference only; no need to remember.