

# Lecture 18, Thursday, November 09/2023

## Outline

- Natural logarithmic function (7.2)
  - ↳  $\int \frac{f'(x)}{f(x)} dx$
  - ↳ Logarithmic differentiation
- Natural exponential function (7.3)
  - ↳ Definition and basic properties
  - ↳ Monotonicity and graph
  - ↳ Algebraic properties
- General powers and general exponential functions (7.3)
  - ↳ Definition of general powers
  - ↳ Algebraic properties
  - ↳  $e$  as a limit and its value
    - ↳ Compound interest
  - ↳ Differentiation rules

Relative Rates of Change :  $\frac{f'(x)}{f(x)}$

Last time, we saw that

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C,$$

which works on any one interval where  $f(x)$  is never zero.

e.g.1 Find  $\int \tan x dx$  and  $\int \sec x dx$ .

Sol :  $\int \tan x dx = \underline{\hspace{10em}}$   
 $\underline{\hspace{10em}} = -\ln|\cos x| + C$   
 $(= \ln|\cos x|^{-1} + C = \ln|\sec x| + C)$

Valid on  
intervals  
with  $\cos$  never  
zero.

$\int \sec x dx = \underline{\hspace{10em}} = \ln|\underbrace{\sec x + \tan x}_{\text{works on intervals on which this is never zero.}}| + C$

Formulae:

$$\int \tan x dx = -\ln|\cos x| + C = \ln|\sec x| + C,$$

$$\int \sec x dx = \ln|\sec x + \tan x| + C.$$

e.g. 2  $\int_0^{\pi/6} \tan(2x) dx = \dots = \frac{1}{2} \ln 2.$

(T.2, e.g. 4)

## Logarithmic Differentiation

e.g. 3 Find  $y'$  for  $x > 0$ , where  $y = \frac{x^{3/4} \sqrt{x^2+1}}{(3x+2)^5}$ .

Sol: • Would be tedious to differentiate directly.

• For  $x > 0$ ,  $y > 0$ .

•  $\ln y = \frac{3}{4} \ln x + \frac{1}{2} \ln(x^2+1) - 5 \ln(3x+2)$ .

• Applying  $\frac{d}{dx}$  both sides:

$$\frac{y'}{y} = \frac{3}{4} \frac{1}{x} + \frac{1}{2} \frac{2x}{x^2+1} - 5 \frac{3}{3x+2}.$$

$$y' = y \left( \frac{3}{4} \frac{1}{x} + \frac{1}{2} \frac{2x}{x^2+1} - 5 \frac{3}{3x+2} \right) = \frac{x^{3/4} \sqrt{x^2+1}}{(3x+2)^5} \left( \frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2} \right).$$

This approach may be used to differentiate  $F(x)$  of the form

$$\begin{aligned} \text{"P" = "Product"} &\rightarrow \prod_{k=1}^n (f_k(x))^{m_k} = f_1(x)^{m_1} f_2(x)^{m_2} \dots f_n(x)^{m_n}, \\ \text{"P" = "Product"} & \end{aligned}$$

where  $f_k(x) > 0$  and  $m_k$  can be negative:

*Side note:*  
 $\Sigma$  = "Sigma"  
 $S$  = "Sum"

$$\cdot y = F(x) ; \quad \ln y = \ln \left( \prod_{k=1}^n f_k(x)^{m_k} \right) = \sum_{k=1}^n m_k \ln(f_k(x))$$

$$\cdot \frac{y'}{y} = \sum_{k=1}^n m_k \frac{f'_k(x)}{f_k(x)} \quad \Rightarrow \quad y' = y \sum_{k=1}^n m_k \frac{f'_k(x)}{f_k(x)} .$$

If  $f_k(x) < 0$  on the interval of integration for some  $k$  above,  
we may replace  $F(x)$  with  $|F(x)|$  when working with  $\ln$ :

$$\frac{d}{dx} \ln |F(x)| = \frac{F'(x)}{F(x)} \quad \Rightarrow \quad F'(x) = F(x) \frac{d}{dx} \ln |F(x)| .$$

E.g. 4 Let  $y = \frac{(2x+1)^3}{(3x-1)^5}$ . Find  $y'$  on  $(-\infty, \frac{1}{3})$  using logarithmic differentiation.

Sol : •  $\ln|y| = \dots$

•  $\frac{d}{dx} \ln|y| = \frac{y'}{y}$ .

•  $y' = y \left( \frac{d}{dx} \ln|y| \right) = \dots = \frac{(2x+1)^3}{(3x-1)^5} \left( \frac{6}{2x+1} - \frac{15}{3x-1} \right)$ .

# Natural Exponential Function

## 1. Definition and Basic Properties

- Since  $\ln$  is injective on  $(0, \infty)$  and has range  $\mathbb{R}$ , it has an inverse function.

Def. The (natural) exponential function is the inverse function of  $\ln$ , denoted by  $\exp: \mathbb{R} \rightarrow (0, \infty)$ .

- Since  $\underbrace{l = \ln e}_{\text{def of } e}$ , we have  $\exp(l) = \exp(\ln e) = e$ .
- Since  $e > 0$ ,  $e^r$  is defined for all rational power  $r$ .  
 $(1 < e < 4)$
- Since  $r = r \ln e = \ln e^r$ ,  $\exp(r) = \exp(\ln e^r) = e^r$ :

$$\exp(r) = e^r, \quad \forall r \in \mathbb{Q}.$$

This is a property from definitions of  $\exp$  and  $e$ .

- For irrational power  $x$ , we have not defined  $e^x$  formally yet.

We can define it as follows:

Def:  $e^x := \exp(x)$ ,  $\forall x \in \mathbb{R} \setminus \mathbb{Q}$ .

This is  
a definition.

- Hence, we may also write  $e^x$  to mean  $\exp(x)$ . Consequently,

$$e^{\ln x} = \exp(\ln x) = x, \quad \forall x \in (0, \infty);$$

$$\ln(e^x) = \ln(\exp(x)) = x, \quad \forall x \in \mathbb{R}.$$

- Let  $y = \exp(x)$ . For any fixed  $x_0 \in \mathbb{R}$ , let  $y_0 = \exp(x_0)$ . Then

$$\exp'(x_0) = \frac{1}{\ln(y_0)} = \frac{1}{1/y_0} = y_0 = \exp(x_0).$$

inverse differentiation rule

$$\exp'(x) = \exp(x), \quad \text{or} \quad \frac{d}{dx} e^x = e^x, \quad \forall x \in \mathbb{R}.$$

## 2. Monotonicity and Graph

- Since  $\exp'(x) = \exp''(x) = \exp(x) > 0 \quad \forall x \in \mathbb{R}$ , the  $\exp$  function is increasing and concave up on  $\mathbb{R}$ .

- Using limit definitions, one can prove that  $\lim_{x \rightarrow -\infty} e^x = 0$  and  $\lim_{x \rightarrow \infty} e^x = \infty$ . (Proof: Skipped.)

### 3. Algebraic Properties

Theorem (Algebraic Properties of exp) For any real numbers  $x_1, x_2$  and  $x$ :

$$1. e^{x_1} e^{x_2} = e^{x_1+x_2}.$$

$$2. e^{-x} = \frac{1}{e^x}.$$

$$3. e^{x_1}/e^{x_2} = e^{x_1-x_2}.$$

$$4. (e^x)^r = e^{rx}, \forall r \in \mathbb{Q}.$$

Proof:  $\underline{1}$ . Let  $y_i = e^{x_i}$  for  $i \in \{1, 2\}$ . Then

$$e^{x_1} e^{x_2} = y_1 y_2 = e^{\ln(y_1 y_2)} = e^{\ln y_1 + \ln y_2} = e^{x_1+x_2}.$$

*ln property 1*

$\underline{4}$ . Let  $y = e^x$ . Then

$$(e^x)^r = y^r = e^{\ln(y^r)} = e^{r \ln y} = e^{rx}.$$

*ln property 4*

$$\underline{3}. e^{x_1-x_2} = e^{x_1} e^{-x_2} = e^{x_1} (e^{x_2})^{-1} = e^{x_1}/e^{x_2}.$$

*exp property 1      exp property 4*

$$\underline{2}. e^{-x} = e^{0-x} = \frac{e^0}{e^x} = \frac{1}{e^x}.$$

*exp property 3*

□

## General Powers

With  $e^x$  defined for all  $x \in \mathbb{R}$ , we can now define  $a^x$  for any  $a \in (0, \infty)$  and  $x \in \mathbb{R}$ .

Def.: For any  $a \in (0, \infty)$  and  $x \in \mathbb{R}$ , define

$$a^x := e^{x \ln a}.$$

Algebraic property 4 for  $\ln$  and  $\exp$  now also holds for irrational values of  $r$ . That is,

$$(i) \ln(x^r) = r \ln x, \quad \forall x \in (0, \infty), \forall r \in \mathbb{R}.$$

$$(ii) (e^x)^r = e^{rx}, \quad \forall x \in \mathbb{R}, \forall r \in \mathbb{R}.$$

Both can be verified using algebraic properties/definitions:

(i) holds because  $\ln(x^r) \stackrel{\text{def}}{=} \ln(e^{r \ln x}) = r \ln x$ .

(ii) can be shown using (i). (Details: exercise.)

### 1. Algebraic Properties

The algebraic properties for  $e^x$  also hold for  $a^x$ .

For any  $a \in (0, \infty)$  and any real numbers  $x, x_1, x_2$  and  $r$ :

$$1. a^{x_1} a^{x_2} = a^{x_1+x_2}.$$

$$2. a^{-x} = 1/a^x$$

$$3. a^{x_1}/a^{x_2} = a^{x_1-x_2}$$

$$4. (a^x)^r = a^{rx}$$

The proofs follow from definitions and properties for  $\ln$  and  $\exp$ .

Proof: 1.  $a^{x_1} a^{x_2} \stackrel{\text{def}}{=} e^{x_1 \ln a} e^{x_2 \ln a} \stackrel{\text{exp property}}{=} e^{(x_1+x_2) \ln a} \stackrel{\text{def}}{=} a^{x_1+x_2}.$

2, 3, 4: exercise. □

## 2. The Value of $e$

So far, we see that  $1 < e < 4$  (recall:  $e$  is defined to be the unique number such that  $\ln e = 1$ ). To see a more accurate approximation, we note the following limit.

Theorem  $e = \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}}.$

By looking at values of  $(1+x)^{\frac{1}{x}}$  with  $x$  closed to 0, we get

$$e \approx 2.71828.$$

How do we prove the limit?

Proof: •  $(1+x)^{1/x} = e^{\frac{1}{x} \ln(1+x)}$  by definition.

•  $\lim_{x \rightarrow 0} \frac{1}{x} \ln(1+x) = \underline{\hspace{2cm}} = 1$ .

• Since  $\exp$  is continuous,

$$\lim_{x \rightarrow 0} (1+x)^{1/x} = \lim_{x \rightarrow 0} e^{\frac{1}{x} \ln(1+x)} = e^{\lim_{x \rightarrow 0} \frac{1}{x} \ln(1+x)} = e^1 = e.$$

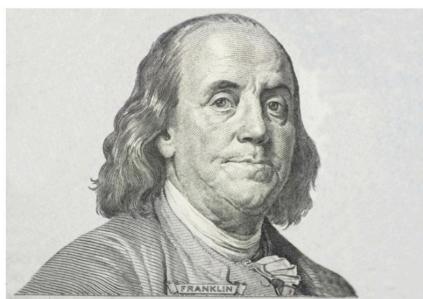
□

By letting  $y = \frac{1}{x}$ , we have  $(1+x)^{\frac{1}{x}} = (1+\frac{1}{y})^y$ . As  $x \rightarrow 0^+$ ,  $y \rightarrow \infty$ , so  $\lim_{y \rightarrow \infty} (1+\frac{1}{y})^y = \lim_{x \rightarrow 0^+} (1+x)^{\frac{1}{x}} = e$ . This gives an alternative limit form of  $e$ :

$$e = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x.$$

## Compound Interests and Euler's Number

"Money makes money. And the money that money makes, makes money." — Benjamin Franklin.



Suppose you have \$1, and you put it in a bank with annual interest  $100\%$ . Suppose the interest is compounded  $n$  times a year. Then at the end of the year, you will get:

- $1 + 1 = 2$  dollars if  $n = 1$ .
- $(1 + \frac{1}{2})^2 = 2.25$  dollars if  $n = 2$ .
- $(1 + \frac{1}{4})^4 = 2.4414$  dollars if  $n = 4$ .
- $(1 + \frac{1}{n})^n$  dollars in general.

When  $n$  is big, you should expect  $e$  dollars in your account at the end of the year.

### 3. Differentiation Rules

With general powers, we can define two types of function.

1. (General exponential function, with base  $a$ )

Fix  $a \in (0, \infty)$ . Define  $f(x) = a^x$ ,  $D = \mathbb{R}$ .

2. (General power function)

Fix  $a \in \mathbb{R}$ . Define  $f(x) = x^a$ ,  $D = (0, \infty)$ .

For power functions, we have been using the rule

$$\frac{d}{dx} x^a = ax^{a-1} \quad \text{for } x \in (0, \infty).$$

We have only proved it for  $a \in \mathbb{Q}$ . Now we can prove it generally.

Proof: . . .  $\frac{d}{dx} x^a = \frac{d}{dx} e^{alnx} = \underbrace{e^{alnx}}_{x^a} \frac{d}{dx}(alnx)$

$$= x^a \cdot a \frac{1}{x} = ax^{a-1}.$$

□

Similar techniques can be used to find  $\frac{d}{dx} x^x$  (for  $x > 0$ ):

$$\frac{d}{dx} x^x = \underline{\hspace{2cm}} = x^x(1 + \ln x).$$

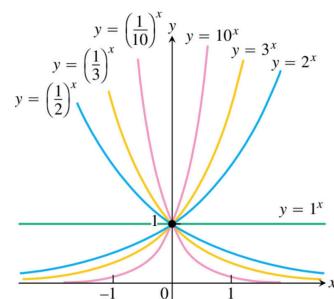
For a fixed  $a \in (0, \infty)$ , let  $f(x) = a^x$ . Then

$$f(x) = e^{x \ln a}, \text{ so } f'(x) = e^{x \ln a} \cdot \ln a = a^x \cdot \ln a.$$

$$\boxed{\frac{d}{dx} a^x = a^x \cdot \ln a.}$$

In other words, provided that  $a \neq 1$ ,

$$\boxed{\int a^x dx = \frac{a^x}{\ln a} + C, \text{ on } \mathbb{R}.}$$



The graph of  $f_a(x) = a^x$  ( $a \neq 1$ ):

- $f'_a(x) = a^x \cdot \ln a$   $\begin{cases} > 0, & \text{if } a > 1 \\ < 0, & \text{if } 0 < a < 1 \end{cases}$ , so  $f_a$  is  $\begin{cases} \uparrow \text{on } \mathbb{R}, & \text{if } a > 1 \\ \downarrow \text{on } \mathbb{R}, & \text{if } 0 < a < 1. \end{cases}$

- $f''_a(x) = a^x (\ln a)^2 > 0 \quad \forall x$ , so  $f_a$  is  $\overset{\text{Haf}}{\leftarrow}$  concave up on  $\mathbb{R}$ .

### Exercise

Suppose that  $P_0$  dollars are deposited into an account with annual interest rate  $r$ , compounded  $n$  times a year, where  $n$  is very big. Approximate the balance in the account  $t$  years after the deposit using a limit. amount of \$  
( $n \rightarrow \infty$ )