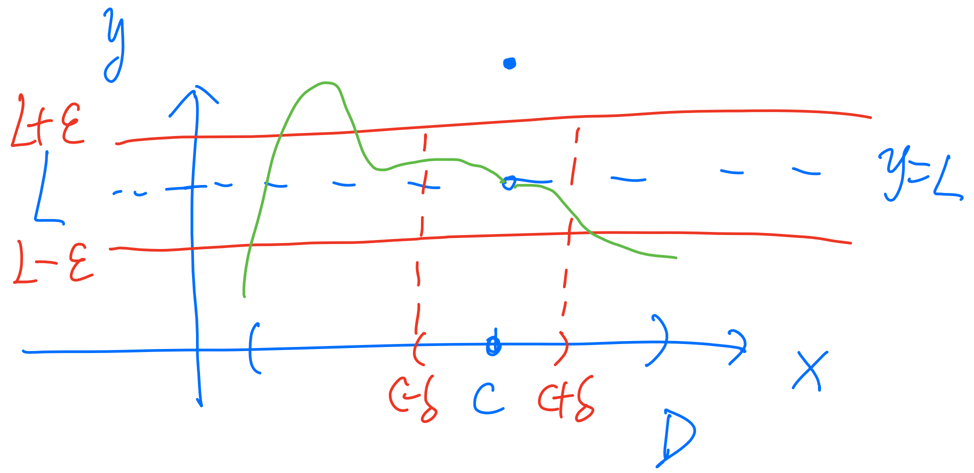
Lecture 2, Thursday, September/07/2023

Outline

- Formal definition of limits (2.3)
- Continuity (2.5)
- Limit properties / limit laws (2.2)
- Composition of continuous functions (2.5)
- Limit computation (basic)



Limits

$(c-a, c+a) \setminus \{c\}$ $\subseteq D$ for some $a>0$

Def: Let $f: D \to \mathbb{R}$ be a function defined on an open interval containing c, except possibly at c itself. Let $L \in \mathbb{R}$ (so $L \neq \pm\infty$). Then we write

$$\lim_{x \to c} f(x) = L$$

if, for all $\varepsilon > 0$ (epsilon), there exists a $\delta > 0$ (delta) such that, for all $x \in D$ with $0 < |x - c| < \delta$, we have

(for all x in domain close enough to c that are $\neq c$)

$$|f(x) - L| < \varepsilon.$$

($f(x)$ is close to L)

Alternative notation We could write $f(x) \to L$ as $x \to c$ to mean $\lim_{x \to c} f(x) = L$.

e.g. 1 Prove that $\lim_{x\to 5}(4x-6) = 14$. (Optional)

Proof: Let $\varepsilon > 0$ be fixed. We want $|4x-6-\underset{L}{14}| < \varepsilon$ whenever $0 < |x-\underset{c}{5}| < \delta$, and we want to choose such a δ. Note that

$$|4x-6-14| < \varepsilon \iff |4x-20| < \varepsilon \iff |x-5| < \frac{\varepsilon}{4}. \quad (*)$$

Set $\delta := \frac{\varepsilon}{4}$. By $(*)$, whenever x satisfies $0<|x-5|<\delta$, we have $|4x-6-14| < \varepsilon$. By definition of limits,

$$\lim_{x\to 5}(4x-6) = 14.$$

$\square$

e.g. 2 Prove that $\lim_{x\to c} x^2 = c^2$. (Optional)

Proof: Let $\varepsilon > 0$ (be arbitrary, but fixed).

Q: How should we choose δ?

- We want to achieve $|x^2 - c^2| < \varepsilon$, i.e.,

$$|x+c||x-c| < \varepsilon.$$

- Narrow the search first: Consider x with $|x-c| < 1$.

- Now, $|x+c| = |x-c+2c| \underset{\text{triangle inequality}}{\leq} |x-c| + 2|c| < 1 + 2|c|$

- If $|x-c| < \underset{\substack{= \\ \text{still trying to find}}}{\delta}$, then $|x+c||x-c| < |x+c|\delta < (1+2|c|)\delta$.
- Hence, if we set $\delta := \frac{\varepsilon}{1+2|c|}$, then $|x+c||x-c| < \varepsilon$.

Pick $\delta := \min\left(1, \frac{\varepsilon}{1+2|c|}\right)$. Then, if $|x-c| < \delta$, then $|x-c| < 1$ and $|x-c| < \frac{\varepsilon}{1+2|c|}$. By the argument above,

$$|x^2 - c^2| = |x+c||x-c| < (1+|c|)\frac{\varepsilon}{1+|c|} = \varepsilon.$$

By definition, $\lim_{x \to c} x^2 = c^2$. $\square$

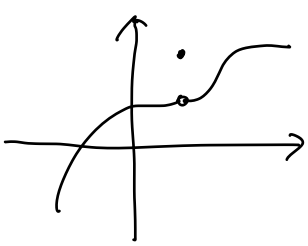
From e.g. 2 above, we see that even a very basic function may have a relatively complicated proof using δ-ε definition.

For this course, we will not test on δ-ε proofs.

Continuity

We have seen that in general, $\lim_{x \to c} f(x) \neq f(c)$.

e.g.



Def: Let $f: D \to \mathbb{R}$ be a function defined on an open interval containing c. We say that f is continuous at c if

This implies $c \in D$.

$$\lim_{x \to c} f(x) = f(c).$$

For this to be true, $\lim_{x\to c} f(x)$ must exist first.

A function defined on an open interval D is said to be continuous if it is continuous at every point in D.

The following functions are continuous (on domain D):

- Constant function: $f(x) = K$, where K constant; $D = \mathbb{R}$.
- Identity function: $f(x) = x$, $D = \mathbb{R}$.
- Absolute value function: $f(x) = |x|$, $D = \mathbb{R}$.
- Natural exponential function: $f(x) = e^x$, $D = \mathbb{R}$.
- Natural logarithmic function: $f(x) = \ln x$, $D = (0, \infty)$.

(Continued)

- Basic trigonometric functions: $f(x) = \sin x$, $g(x) = \cos x$, $D = \mathbb{R}$ for both.

All of these can be proven using δ-ε definition, which we omit.

By definition, limit of a continuous function can be computed by direct substitutions for any point in the domain: $\lim_{x \to 1} \sin x = \sin 1$.

Limit Properties

The following limit properties may make computations more convenient.

All of these can be proven using δ-ε definition, which we omit.

THEOREM ~~1~~ 2.2.1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, $M \neq 0$
6. *Power Rule:* $\lim_{x \to c} [f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x \to c} f(x) = L > 0$.)

For example, the first and fourth law states that

$$\lim_{x\to c}\left(f(x)+g(x)\right) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$$

and $$\left(\lim_{x\to c} f(x)g(x)\right) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right),$$

given that both $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist as real numbers.

By limit properties, limits of algebraic combinations $(+, -, \times, \div, (\cdot)^n, \sqrt[n]{\cdot})$ of continuous function can also be computed by substitutions (for points in the domains).

<u>e.g.</u> $$\lim_{x\to 2} \frac{\sqrt{\ln x}}{x^2-3} = \frac{\sqrt{\ln 2}}{4-3} = \sqrt{\ln 2}.$$

<u>Reason</u>: $$\lim_{x\to 2} \frac{\sqrt{\ln x}}{x^2-3} = \frac{\lim_{x\to 2}\sqrt{\ln x}}{\lim_{x\to 2}(x^2-3)} \quad \text{(limit law 5)}$$

$$= \frac{\sqrt{\lim_{x\to 2}\ln x}}{\lim_{x\to 2} x^2 - \lim_{x\to 2} 3} \quad \text{(limit law 7 and 2)}$$

$$= \frac{\sqrt{\lim_{x\to 2}\ln x}}{\left(\lim_{x\to 2} x\right)^2 - \lim_{x\to 2} 3} \quad \text{(limit law 6)}$$

$$= \frac{\sqrt{\ln 2}}{2^2 - 3} \qquad \text{(continuity)}$$

$$= \sqrt{\ln 2}.$$

Def. A polynomial is a function of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$$

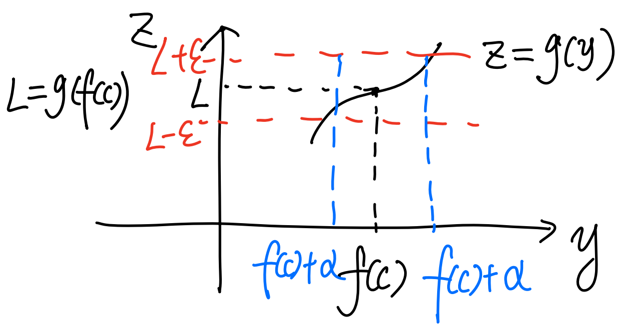
where $a_0, a_1, \dots, a_n$ are constants. A rational function is a function of the form $p(x)/q(x)$, where $p(x)$ and $q(x)$ are both polynomials (and $q(x)$ is not the zero function).

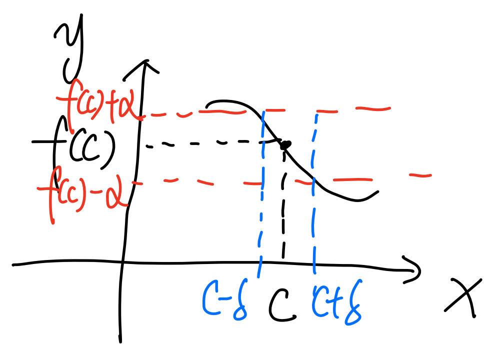
Remark: Since polynomials and rational functions are finite algebraic combinations of continuous functions $f(x) = K$ and $g(x) = x$, by limit laws, their limits (for points in the domain) can be found by direct substitutions. Hence they are continuous (in their domains).

Composition and continuity

> **Theorem** (2.5.9) If f is continuous at c and g is continuous at $f(c)$, then $g \circ f$ is continuous at c.

Proof: (Optional)





We show that $\lim_{x \to c} (g \circ f)(x) = (g \circ f)(c)$.

Let $z = g(y)$, $y = f(x)$, so $z = (g \circ f)(x)$.

Let $\varepsilon > 0$. Since g is continuous at $f(c)$, $\exists\, \alpha > 0$ "there exists"
such that $\forall y$ (in the domain of g), ($\forall$ means "for all")

$$|y - f(c)| < \alpha \quad \Rightarrow \quad |g(y) - g(f(c))| < \varepsilon. \qquad ①$$

Since f is continuous at c, $\exists\, \delta > 0$ such that $\forall x$ (in the domain of f),

$$|x - c| < \delta \quad \Rightarrow \quad |f(x) - f(c)| < \alpha. \qquad ②$$

Now ①, ② $\Rightarrow$ $\forall x$ (in the domain of f),

$$|x-c|<\delta \Rightarrow |g(f(x))-g(f(c))|<\varepsilon.$$

Hence, $\lim_{x\to c}(g\circ f)(x) = (g\circ f)(c)$.

$\square$

In fact, a more general result holds.

Theorem (2.5.10) If g is continuous at b, and $b=\lim_{x\to c} f(x)$, then

$$\lim_{x\to c} g(f(x)) = g(b) = g\left(\lim_{x\to c} f(x)\right).$$

Its proof is similar. A consequence of Thm 2.5.10 is that you may "move" the "$\lim_{x\to c}$" inside whenever the outside function is continuous (provided that the limit inside exists).

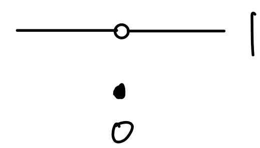
e.g. $g(x)=\sin x$, $f(x)=\begin{cases}1, & \text{if } x=0\\ x, & \text{if } x\neq 0\end{cases}$

By Theorem 2.5.10, since g is continuous,

$$\lim_{x\to 0} g(f(x)) = g\left(\lim_{x\to 0} f(x)\right) = g(0) = \sin 0 = 0.$$

Note that you cannot do the same in general if the "outside" function is not continuous.

e.g. Let $f(x)=x$ and $g(x)=\begin{cases} 0, & \text{if } x=0 \\ 1, & \text{if } x\neq 0 \end{cases}$



Then $g(f(x))=g(x)$, so $\lim\limits_{x\to 0} g(f(x)) = 1$.

But $g\left(\lim\limits_{x\to 0} f(x)\right) = g(0) = 0 \neq 1 = \lim\limits_{x\to 0} g(f(x))$.

Here the outside function g is not continuous.

Limit Computation

How to compute limit if we cannot make direct substitution?

- Eliminate zero denominator:

e.g.
$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2} = \lim_{x\to 0} \frac{(\sqrt{x^2+100}-10)(\sqrt{x^2+100}+10)}{x^2(\sqrt{x^2+100}+10)}$$

$$= \lim_{x\to 0} \frac{x^2}{x^2(\sqrt{x^2+100}+10)} = \lim_{x\to 0} \frac{1}{\sqrt{x^2+100}+10} = \frac{1}{20}.$$

- Squeeze theorem (Sandwich theorem)

2.2.4

THEOREM ~~4~~—The Sandwich Theorem Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

e.g. $\lim_{x \to 0} x^2 \sin \frac{1}{x} = ?$

As a final property of limit in this lecture, we have the following theorem.

2.2.5

THEOREM ~~5~~ If $f(x) \leq g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$