Lecture 16, Thursday, November/02/2023

Outline

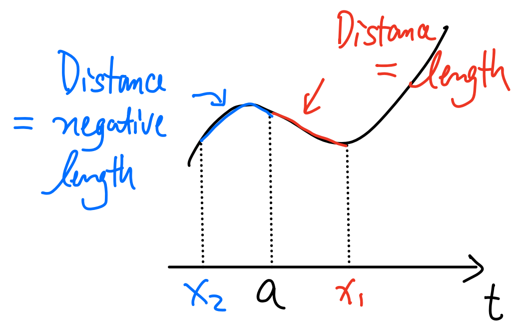
- Arc length differential (6.3)
- Work (6.5)
- Fluid forces (6.5)
- Inverse differentiation (7.1)

Arc Length Differentials

Given $y=f(t)$ defined on some interval I, fix $a\in I$. Define

$$s(x) := \int_a^x \sqrt{1+f'(t)^2}\,dt$$

for $x\in I$. Then $s(x)$ is the signed distance from $(a,f(a))$ to $(x,f(x))$ along the curve $y=f(t)$. (When $x<a$, this signed distance is negative.)



- By FTC I, $s'(x) = \sqrt{1+f'(x)^2}$,
- $ds = \sqrt{1+f'(x)^2}\,dx$ is called the arc length differential
- Symbolically,

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{(dx)^2+(dy)^2}.$$

Work

- If a constant force F moves an object along a straight line for a distance d, then the work done by F to the object is $W = Fd$.
- If F is in newton and d is in meter, then W is in joule.
- If F moves an object along the x-axis, and $F = F(x)$ depends on the x-position, and F is continuous on $[a,b]$:

  ↳ Partition $[a,b]$ with $x_0, x_1, \dots, x_n$.

  ↳ The work done from x_{k-1} to x_k is approximately $F(x_k^*)\Delta x_k$, $x_k^* \in [x_{k-1}, x_k]$.

  ↳ Total work done from a to b is approximately
  $$\sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

DEFINITION The **work** done by a variable force $F(x)$ in moving an object along the x-axis from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)\,dx. \tag{2}$$

Hooke's Law states that the force required to stratch or compress a spring is directly proportional to its distance x away from the natural position of the spring:

$$F(x) = kx,$$

where k is the spring constant, with unit being force unit per length unit.

EXAMPLE ~~3~~ 6.5.3 A spring has a natural length of 1 m. A force of 24 N holds the spring stretched to a total length of 1.8 m.

(a) Find the ~~force~~ spring constant k.

(b) How much work will it take to stretch the spring 2 m beyond its natural length?

(c) How far will a 45-N force stretch the spring?

Sol: (a) By Hooke's law,

______________________________ ,

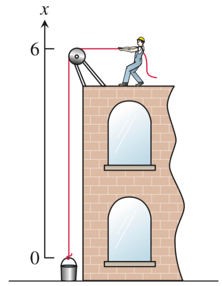
$\therefore\ k = 30$ (N/m).

(b) $W = \dfrac{\qquad}{\qquad} = 60$ J.

(c) Solving $30x = 45$ gives $x = 1.5$ (m).

(1.5 meters beyond the natural position.)

EXAMPLE ~~4~~ 6.5.4 A 2-kg bucket is lifted from the ground into the air by pulling in 6 m of rope at a constant speed (Figure 6.38). The rope weighs 0.1 kg/m. How much work was spent lifting the bucket and rope? (Take $g = 9.8\ m/s^2$.)



Sol: • Force required to lift up an object equals it weight.

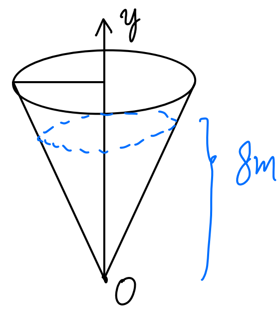
• At height x, the weight of the bucket & rope to be lifted is ____________________.

• $W =$ ______________________________

$= 13.8g = 135.24\ (J)$.

e.g. (Pumping liquid from container).

A conic tank has height 10m and base radius 5m, with vertex pointing down. The y-axis is the axis of the cone, with vertex at $y=0$. The tank is filled with olive oil with weight-density $8820\ N/m^3$; the surface of the oil is 8m from the bottom of the tank. How much work is required to pump the oil to the rim of the tank?



Sol: • Let y represent the oil level.

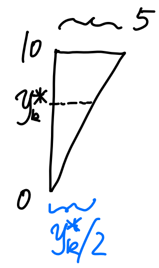
• "Cut" the oil into horizontal thin slices S_k with thickness Δy_k (in meter).

• If y_k^* is the approximative oil level of S_k ($y_k^* \in [y_{k-1}, y_k]$), then work required to lift S_k is approximately

$$W_k = \underbrace{F_k}_{\text{Weight}(S_k)} \cdot \underbrace{(10 - y_k^*)}_{\text{distance to rim}} = \text{weight-density}(S_k) \cdot \text{vol}(S_k) \cdot (10 - y_k^*)$$

$\exists\, y_k^* \in (y_{k-1}, y_k)$ s.t. k^{th} conical frustum has volume $\pi\left(\frac{y_k^*}{2}\right)^2 \Delta y_k$

$$\approx 8820 \cdot \left(\pi \left(\frac{y_k^*}{2}\right)^2 \Delta y_k\right) \cdot (10 - y_k^*).$$



• So total work required is approximately $\sum_{k=1}^{n} W_k$.

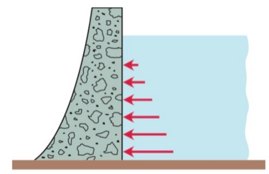
• Exact work required is

$$\int_0^8 8820\pi \frac{1}{4} y^2 (10 - y)\, dy = 2205\pi \int_0^8 (10y^2 - y^3)\, dy$$

$$= 2205\pi \left(\frac{10}{3} 8^3 - \frac{1}{4} 8^4\right) \quad \text{(joules)}.$$

Fluid Forces

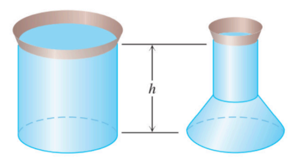
- The same object will generally experience bigger force if it is put in a deeper position in the same fluid.



- For standing-still fluid, the pressure p at depth h is $p = wh$, where w is the weight-density of the fluid.

  (newton/m^3, slug)

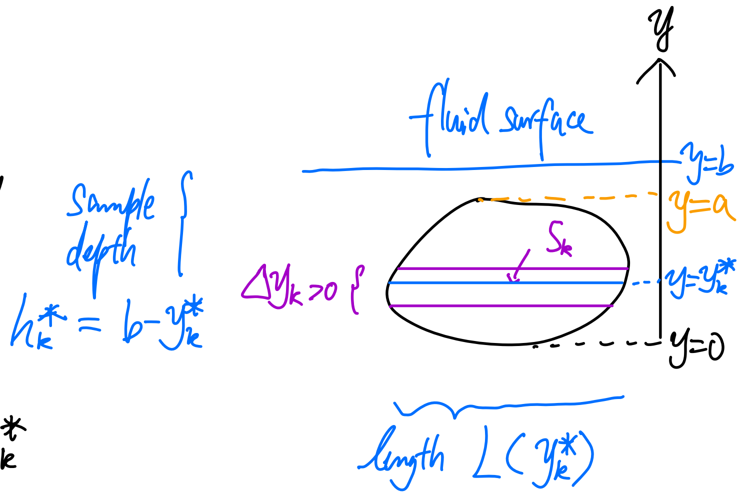
- For a container with a horizontal base, the total force applied by the fluid to the base is $F = pA = whA$, where A is the base area.



Same force to base

- If a flat plate is submerged vertically, the the pressure against it will depend on the depth of the part of the plate. To calculate the fluid force against it, we may use integrals.

- "Cut" the plate into horizontal thin slices S_k :
  - ↳ Width $= \Delta y_k$
  - ↳ Sample depth $h_k^* = b - y_k^*$
  - ↳ Length of S_k at sample depth : $L(y_k^*)$
  - ↳ Area of $S_k \approx L(y_k^*) \Delta y_k$
  - ↳ Force exerted against $S_k \approx w h_k^* L(y_k^*) \Delta y_k$



- Force exerted against plate is approximately

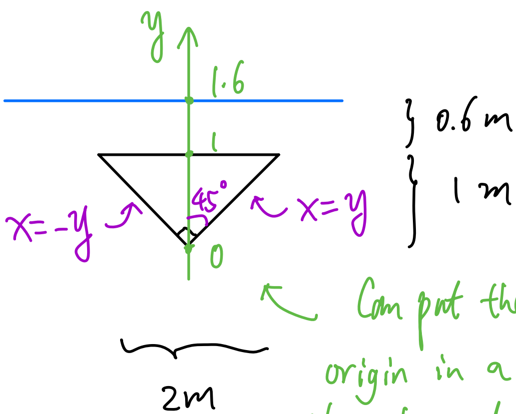
$$\sum_{k=1}^{n} w \underbrace{h_k^*}_{b-y_k^*} L(y_k^*) \Delta y_k$$

or to be exact,

$$\int_0^a w(b-y) L(y)\, dy .$$

6.5.6

EXAMPLE ~~6~~ A flat isosceles right-triangular plate with base 2 m and height 1 m is submerged vertically, base up, 0.6 m below the surface of a swimming pool. Find the force exerted by the water against one side of the plate.



Can put the origin in a place to make computation convenient

S_k : 

$L(y_k^*) = 2y_k^*$

$W = 1000\,kg/m^3 \cdot 9.8\,m/s^2 = 9800\,N/m^3.$

$h_k^* = 1.6 - y_k^*$

$\therefore F = \underline{\hspace{10cm}} \approx 9147\,N.$

Inverse Functions and Differentiation

Def: A function f with domain D is one-to-one (or injective) if $f(x_1) \neq f(x_2)$ for all $x_1, x_2 \in D$ with $x_1 \neq x_2$.

e.g.

- $f(x) := x^2$ is one-to-one on $[0,1]$ but not one-to-one on $[-1,1]$.

Definition Let $f: D \to \text{range}(f)$ be one-to-one (injective). The inverse function of f is the function $f^{-1}: \text{range}(f) \to D$ defined by

$$f^{-1}(y_0) := x_0, \quad \text{where } f(x_0) = y_0.$$

Remarks

- Inverse of f is only defined when f is injective.
- The range of f^{-1} is $D = \text{domain}(f)$.
- By definition, $\forall x_0 \in D$, $f^{-1}(f(x_0)) = x_0$, and $\forall y_0 \in \text{range}(f)$, $f(f^{-1}(y_0)) = y_0$, So $f^{-1} \circ f$ and $f \circ f^{-1}$ are both identity

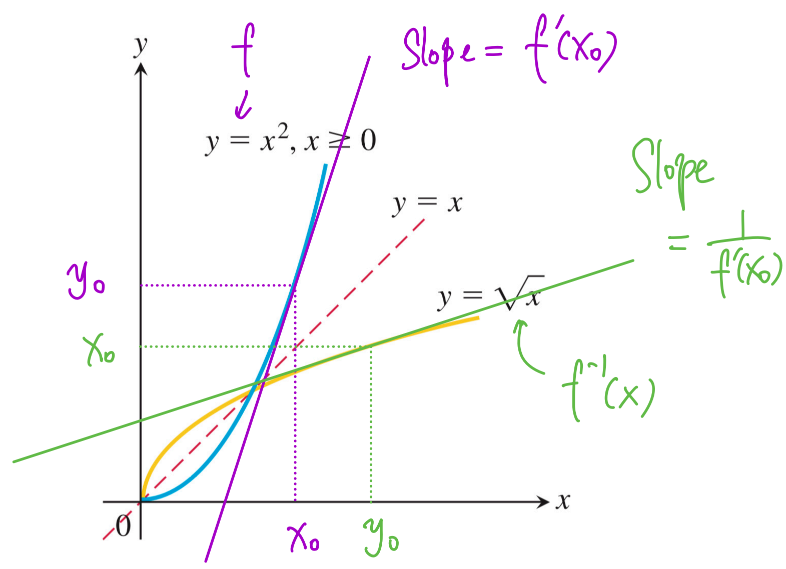
functions (a function that maps an element back to itself.)

- $f:[0,2]\to[0,4]$, $f(x)=x^2$ has inverse $f^{-1}:[0,4]\to[0,2]$, $f^{-1}(y)=\sqrt{y}$.
- Note that x & y are dummy variables, in the sense that $f^{-1}(x)=\sqrt{x}$ and $f^{-1}(y)=\sqrt{y}$ represent the same function rule.

- If f is monotonic on D, then f is injective on D, so inverse $f^{-1}:\text{range}(f)\to D$ must exist.

<u>Fact</u> (without proof).

- If $f: I\to \text{range}(f)$ is continuous and f^{-1} exists, then f^{-1} is also continuous.



<u>Theorem</u> (Inverse differentiation rule)

Let $f: I \to \text{range}(f)$ be one-to-one, where I is an interval.

If f is differentiable and f' is never zero on I, then f^{-1} is differentiable, and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} \quad \forall y_0 \in \text{range}(f),$$

where $f(x_0) = y_0$.

i.e., $(f^{-1})'(y_0) = \dfrac{1}{f'(f^{-1}(y_0))}$

<u>Proof</u>: (Optional)

- Let $Y := \text{range}(f)$. Fix any interior point y_0 in Y.
- For any y_0+k, $\exists$ a unique x_0+h in D such that

$$f(x_0+h) = y_0+k.$$

- $$\lim_{k\to 0} \frac{f^{-1}(y_0+k) - f^{-1}(y_0)}{k} = \lim_{k\to 0} \frac{f^{-1}(f(x_0+h)) - f^{-1}(f(x_0))}{k}$$

$$= \lim_{k\to 0} \frac{x_0+h-x_0}{k} = \lim_{k\to 0} \frac{h}{y_0+k-y_0}$$

$$= \lim_{k\to 0} \frac{h}{f(x_0+h)-f(x_0)} \qquad 1$$

- View h as a function of k. Since f is continuous, so is f^{-1} (by the fact above). Hence $h \to 0$ as $k \to 0$.

(To see this,

$$h = x_0 + h - x_0 = f^{-1}(y_0+k) - f^{-1}(y_0) \to 0 \text{ as } k \to 0,$$

since $\lim_{k\to 0} f^{-1}(y_0+k) = f^{-1}(\lim_{k\to 0}(y_0+k)) = f^{-1}(y_0)$.)

- By ①,

$$\lim_{k\to 0} \frac{f^{-1}(y_0+k) - f^{-1}(y_0)}{k} = \lim_{h\to 0} \frac{h}{f(x_0+h) - f(x_0)} = \frac{1}{\underbrace{\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}}_{f'(x_0),\ \neq 0.}}.$$

- Hence, $(f^{-1})'(y_0)$ exists and $= \frac{1}{f'(x_0)}$.

- Proof is similar if y_0 is an endpoint of Y.

$\square$

e.g. Consider $f(x) = x^3 - 2$ defined on $(0, \infty)$.

(a) Find f^{-1} and its domain.

(b) Find $(f^{-1})'$.

Sol: (a) The domain of f^{-1} is $\text{range}(f) = (-2, \infty)$.

$y = x^3 - 2 \iff (y+2)^{1/3} = x$.

Swapping x and y gives $y = (x+2)^{1/3}$, so $f^{-1}(x) = (x+2)^{1/3}$.

(b) Method 1 : direct $\quad (f^{-1})'(x) = \frac{1}{3}(x+2)^{-\frac{2}{3}}$.

Method 2 : inverse differentiation