Lecture 8, Thursday, Sept/28/2023

Outline

- The Mean value theorem (4.2)
- Monotonicity of functions (4.3)
  - ↳ Intervals of monotonicity
  - ↳ First derivative test

Mean Value Theorem

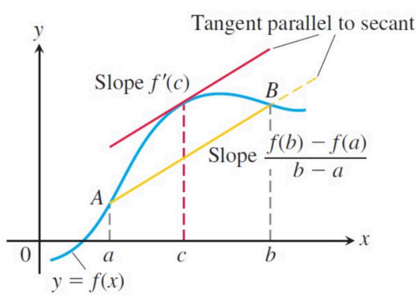
The mean value theorem is one of the most fundamental and important theorems in calculus.

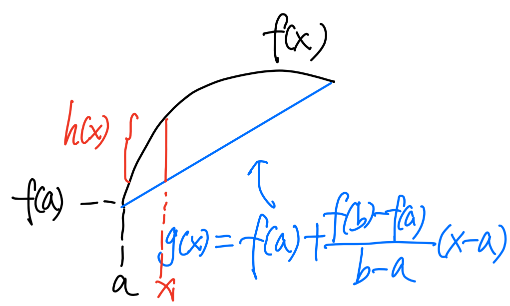
Theorem (Mean Value Theorem) (MVT)

Suppose that a function f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Alternative form:
- $f(b) - f(a) = f'(c)(b-a)$
- $f(b) = f(a) + f'(c)(b-a)$





Proof : ________________________ .

A Physical Consequence of the MVT

Suppose that $f(t)$ represents the distance travelled until time t. Then the mean value theorem implies that if we pick two moments $t = a$ and $t = b$, there has to be some moment $t = c$ in between at which the instantaneous speed is equal to the average speed between $t = a$ and $t = b$.

https://mrhonner.com/archives/10268

Mr Honner ABOUT

Calculus Gave Me a Speeding Ticket

Published by **MrHonner** on April 9, 2013

Years ago, one sunny Sunday afternoon, I was driving home from visiting friends at college and received a speeding ticket. I didn't realize it at the time, but calculus played an important role in my citation.

You see, this was no ordinary speeding ticket, the kind where a police officer paces the offender or uses radar to measure a vehicle's speed. My speed was calculated from an airplane high above the road. And the Mean Value Theorem clinched the case.

Aerial speed enforcement works like this: large marks painted on the road divide the highway into quarter-mile intervals. A pilot flying overhead uses a stopwatch to time a suspected speeder from one mark to the next. Say the pilot records a time of 12 seconds; a simple calculation converts one quarter mile per 12 seconds into 75 miles per hour; this information, the *average speed on this interval*, is radioed to the police on the ground who then stop and ticket the driver.

Mathematical Consequences of the MVT

COROLLARY 1 If $f'(x) = 0$ at each point x of an open interval $\overset{I}{\cancel{(a, b)}}$, then $f(x) = C$ for all $x \in \underset{I}{\cancel{(a, b)}}$, where C is a constant.

*: A corollary is a consequence of a theorem.

We know that a constant function has zero derivative everywhere.

Corollary 1 states that constant functions are the only such functions.

Proof : ______________________ .

COROLLARY 2 If $f'(x) = g'(x)$ at each point x in an open interval $\cancel{(a, b)}$ I, then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in \cancel{(a, b)}$ I. That is, $f - g$ is a constant function on $\cancel{(a, b)}$ I.

Proof : ____________________________ .

e.g.1 If $f'(x) = \sin x$, what is $f(x)$?

Sol: Since $(-\cos x)' = \sin x$, $f(x) = -\cos x + C$ for some $C \in \mathbb{R}$.

e.g.2 If $f'(x) = 5x^2 - \frac{1}{\sqrt{x}}$ and $f(1) = 0$, what is $f(x)$?

Sol: ____________________________ .

$$f(x) = \frac{5}{3}x^3 - 2\sqrt{x} + \frac{1}{3}.$$

The MVT can be used to prove inequalities.

e.g.3 Prove that:

(a) $|\sin y - \sin x| \leqslant |y-x|$ for all $x, y \in \mathbb{R}$.

(b) $\dfrac{1}{2\sqrt{x+1}} < \sqrt{x+1} - \sqrt{x} < \dfrac{1}{2\sqrt{x}}$ for all $x>0$.

Proof: (a) May assume $y>x$, since it is clear for $y=x$, and you may swap x and y first in the case where $y<x$.

Since sin is differentiable (and hence continuous) everywhere, by the MVT, $\exists c \in (x,y)$ such that

$$\sin y - \sin x = \sin'(c)(y-x) = \cos(c)(y-x).$$

Now

$$|\sin y - \sin x| = |\cos(c)(y-x)| = |\cos(c)||y-x| \leqslant |y-x|$$

(b) ____________________________.

(This can be shown easily without the use of MVT; here we give a proof using the MVT.)

Monotonicity

DEFINITIONS Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.

2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

$\geqslant$ nondecreasing / weakly increasing

$\leqslant$ nonincreasing / weakly decreasing

Def A function that is increasing or decreasing on I is said to be monotonic (or monotone) on I.

COROLLARY ~~3~~ 4.3.3 Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.

If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

Proof: ______________________________ .

The same proof technique can be used to show that the statements above still hold if (a, b) and $[a, b]$ are replaced by (a, ∞) and $[a, \infty)$, respectively.

or $(-\infty, b)$ $[-\infty, b]$

<u>e.g. 4</u> Prove that $f(x)=\sqrt{x}$ is increasing on $[0,\infty)$.

<u>Proof</u>: · f is continuous on $[0,\infty)$, and

$$f'(x)=\frac{1}{2\sqrt{x}}>0 \quad , \quad \forall x\in(0,\infty).$$

· By Corollary 4.3.3 (extension), the statement follows. $\square$

Intervals of Monotonicity

Suppose that $x_1, x_2, \dots, x_n$ are all critical points of a function f, with $x_1 < x_2 < \dots < x_n$. Let x_i and x_{i+1} be two consecutive critical points. If f' is continuous on $[x_i, x_{i+1}]$, then f' is either entirely positive or entire negative on (x_i, x_{i+1}). (Why?) By corollary 4.3.3:

- f is increasing on $[x_i, x_{i+1}]$, if $f'(c)>0$ for some $c\in(x_i, x_{i+1})$.
- f is decreasing on $[x_i, x_{i+1}]$, if $f'(c)<0$ for some $c\in(x_i, x_{i+1})$.

Similar statements can be stated for the intervals $(-\infty, x_1]$ and $[x_n, \infty)$.

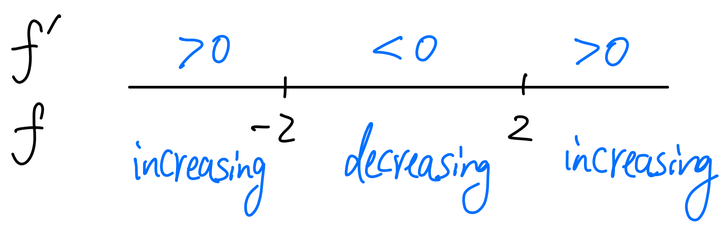
e.g. For the function $f(x) = x^3 - 12x - 5$, we have

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x+2)(x-2).$$

Critical points: $x_1 = -2$, $x_2 = 2$.

By the remarks on the previous page, we only need to pick a sample point on each interval to test the monotonicity of f:

Since $f'(-3) > 0$, $f'(0) < 0$ and $f'(3) > 0$, we have the following picture:



First Derivative Test

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

Remarks For the conditions above, more formally:

1. There exists $a>0$ such that $f'(x)<0$ for all $x\in(c-a,c)$ and $f'(x)>0$ for all $x\in(c,c+a)$;

2. There exists $a>0$ such that $f'(x)>0$ for all $x\in(c-a,c)$ and $f'(x)<0$ for all $x\in(c,c+a)$;

3. There exists $a>0$ such that $f'(x)$ is always positive or alway negative for all $x\in(c-a,c+a)\setminus\{c\}$.

Proof: 1. By Corollary 4.3.3, f is decreasing on $[c-a,c]$ and increasing on $[c,c+a]$, which means that $f(c)<f(x)$ for all $x\in[c-a,c)$ and $f(c)<f(x)$ for all $x\in(c,c+a]$, so $f(c)\le f(x)$

for all $x \in (c-a, c+a)$, i.e., f has a local minimum at c.

2, 3 : Similar. $\square$

Q: Can you think of an example that shows the test won't work if the continuity assumption is removed?

e.g. 5 Find all absolute and local extrema for

$$f(x) = x^{4/3} - 4x^{1/3}.$$

Sol: $f'(x) = \frac{4}{3}x^{\frac{1}{3}} - \frac{4}{3}x^{-\frac{2}{3}} = \frac{4}{3}x^{-\frac{2}{3}}(x-1)$

for all $x \neq 0$, so $f'(x) = 0 \iff x = 1$.

For $x = 0$, Since

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^{4/3} - 4h^{1/3}}{h} = \lim_{h \to 0} h^{1/3} - 4h^{-2/3}$$

does not exist, $f'(0)$ does not exist; hence:

Critical points: $x = 0$ and $x = 1$.

Since $f'(-1) < 0$, $f'(\frac{1}{2}) < 0$, $f'(2) > 0$, we have

f'	<0		<0		>0
f	decreasing	0	decreasing	1	increasing

- By the first derivative test, the only local extremum occurs at $x=1$, which is a local minimum.
- Since f is decreasing on $(-\infty, 1)$ and increasing on $(1, \infty)$, $f(1)$ is also an absolute minimum.
- f has no absolute maximum, since otherwise there would have been a local maximum.
- $f(1) = 1-4 = -3$. $\leftarrow$ Abs. min, occured at $x=1$.

Q: True or False? The function f defined by

$$f(x) = \begin{cases} x^2\left(1+\sin\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{otherwise} \end{cases},$$

has a local minimum at $x=0$.