Lecture 20, Thursday, November/16/2023

Outline

- Limits of products and quotients (Not in Thomas')
- Relative rates of growth (7.8)
- Big-Oh and little-Oh notation (7.8)

Limits of Products and Quotients

Consider computing the limit $\lim_{x\to a} f(x)h(x)$. Suppose $\lim_{x\to a} \frac{f(x)}{g(x)} = L \in \mathbb{R}$. Then

$$\lim_{x\to a} \big(L g(x)\big) h(x) = L \lim_{x\to a} g(x)h(x) = \lim_{x\to a} \frac{f(x)}{g(x)} \lim_{x\to a} g(x)h(x)$$
$$= \lim_{x\to a} \frac{f(x)}{g(x)} g(x)h(x) = \lim_{x\to a} f(x)h(x),$$

provided the limit on the left-hand side exists.

- Replacing $f(x)$ with $Lg(x)$ in a product (or quotient) will keep the limit unchanged, if it exists.
- It does not work with sums and differences, in general.
- $\lim_{x\to a}$ can be replaced with $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$.
- If $\lim_{x\to a} \frac{f(x)}{g(x)} = 1$, we may write $f(x) \sim g(x)$ as $x\to a$.

e.g.

- $\underline{\sin x \sim x \text{ as } x\to 0}$, since $\lim_{x\to 0} \frac{\sin x}{x} = 1$.
- $\underline{\tan x \sim x \text{ as } x\to 0}$, since $\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sec^2 x}{1} = 1$.
- $\underline{e^x - 1 \sim x \text{ as } x\to 0}$, since $\lim_{x\to 0} \frac{e^x-1}{x} = \lim_{x\to 0} \frac{e^x}{1} = e^0 = 1$.
- $\underline{\ln(1+x) \sim x \text{ as } x\to 0}$, since $\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1$.

• $\arctan x \sim x$ as $x \to 0$, since $\lim\limits_{x\to 0} \frac{\arctan x}{x} = \lim\limits_{x\to 0} \frac{\frac{1}{1+x^2}}{1} = 1.$

e.g. Since $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \lim\limits_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2}\lim\limits_{x\to 0} \frac{\sin x}{x} = \frac{1}{2}$,

we have $1-\cos x \sim \frac{1}{2}x^2$ as $x \to 0$.

e.g. Compute $\lim\limits_{x\to 0} \frac{\tan x - \sin x}{x^3}$

Sol 1: $\lim\limits_{x\to 0} \frac{\tan x - \sin x}{x^3} = \lim\limits_{x\to 0} \frac{\sin x - \sin x\cos x}{x^3 \cos x} = \lim\limits_{x\to 0} \frac{\overset{\sim x}{\sin x}\,(1-\cos x)}{x^3 \cos x}$

$= \lim\limits_{x\to 0} \frac{x\,\overset{\sim \frac{1}{2}x^2}{(1-\cos x)}}{x^3 \cos x} = \lim\limits_{x\to 0} \frac{x \cdot \frac{1}{2}x^2}{x^3 \cos x} = \frac{\frac{1}{2}}{1} = \frac{1}{2}.$

Sol 2: $\lim\limits_{x\to 0} \frac{\overset{\sim x}{\tan x} - \sin x}{x^3} = \lim\limits_{x\to 0} \frac{x - \overset{\sim x}{\sin x}}{x^3} = \lim\limits_{x\to 0} \frac{x-x}{x^3} = 0$

$$\boxed{\therefore\ \frac{1}{2} = 0.}$$

Relative Rates of Growth

Consider $P(t)$ being the population at time t. Intuitively, we may say that in long run $P(t) = e^t$ grows faster than $P(t) = t$, which grows faster than $P(t) = \ln t$. But we may say that $P(t) = t^2$ and $P(t) = kt^2$ $(k>0)$ belong to the same class (quadratic growth).

"eventually positive"

Def: Let f & g be functions which are positive for all sufficiently large inputs. We say that as $x \to \infty$, f grows:

(i) faster than g; (ii) slower than g;

(iii) at the same rate as g;

if $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)}$:

(i) $= \infty$; (ii) $= 0$; (iii) $= L \in \mathbb{R}_{>0}$.

Usually, f and g are nondecreasing, but they do not have to be in the definition.

e.g. (a) If f is eventually positive and $k \in \mathbb{R}_+$ is fixed, then

$$\lim_{x\to\infty} \frac{kf(x)}{f(x)} = k,$$

So kf and f grow at the same rate.

(b) x^x vs b^x: For any fixed $b > 1$,

$$\lim_{x\to\infty} \frac{x^x}{b^x} = \lim_{x\to\infty} \left(\frac{x}{b}\right)^x = \lim_{x\to\infty} e^{x \ln \frac{x}{b}} = \infty,$$

So x^x grows faster than b^x.

(c) b^x vs a^x : If $b>a>1$, then

$$\lim_{x\to\infty}\frac{b^x}{a^x}=\lim_{x\to\infty}\Big(\underbrace{\frac{b}{a}}_{>1}\Big)^x=\lim_{x\to\infty}e^{x\underbrace{\ln\frac{b}{a}}_{>0}}=\infty,$$

So b^x grows faster than a^x if $b>a>1$.

(d) a^x vs x^n $(a>1,\ n\in\mathbb{Z}_+:=\{1,2,3,\dots\})$:

$$\lim_{x\to\infty}\frac{a^x}{x^n}=\lim_{x\to\infty}\frac{a^x\ln a}{nx^{n-1}}=\lim_{x\to\infty}\frac{a^x(\ln a)^2}{n(n-1)x^{n-2}}=$$

L'Hôpital

$$\dots=\lim_{x\to\infty}\frac{a^x(\ln a)^n}{n!}=\infty$$

So a^x grows faster than x^n for any fixed $a\in(1,\infty)$ and $n\in\mathbb{Z}_+$.

(e) x^α vs x^β $(\alpha>\beta>0)$:

$$\lim_{x\to\infty}\frac{x^\alpha}{x^\beta}=\lim_{x\to\infty}x^{\alpha-\beta}=\lim_{x\to\infty}e^{(\alpha-\beta)\ln x}=\infty,$$

So x^α grows faster that x^β if $\alpha>\beta>0$.

Exercise Show that a^x grows faster than x^r for any fixed $a>1$ and $r>0$.

(f) x^r vs $\ln x$ ($r>0$):

$$\lim_{x\to\infty} \frac{x^r}{\ln x} \overset{\text{L'Hôpital}}{=} \lim_{x\to\infty} \frac{rx^{r-1}}{1/x} = \lim_{x\to\infty} rx^r = \lim_{x\to\infty} re^{r\ln x} = \infty,$$

So x^r grows faster than $\ln x$, $\forall$ fixed $r>0$.

(g) $\log_a$ vs $\log_b$ ($a>1$ and $b>1$):

Since $\displaystyle\lim_{x\to\infty} \frac{\log_a x}{\log_b x} = \lim_{x\to\infty} \frac{\ln x/\ln a}{\ln x/\ln b} = \frac{\ln b}{\ln a} > 0$,

logarithmic functions with base >1 ALL grow at the same rate.

(h) $\log_a x$ vs Constant function $g(x)=K$ ($a>1$, $K>0$):

Since $\displaystyle\lim_{x\to\infty} \frac{\log_a x}{K} = \lim_{x\to\infty} \frac{\ln x}{K\ln a} = \infty$, any log function with base >1 grows faster than a constant function.

As $x \to \infty$, the following list orders the functions from fast to slow growth rates:

1. x^x;
2. a^x, $a > 1$; (Exponential, bigger base means faster)
3. x^r, $r > 0$; (Power, bigger power means faster)
4. $\log_a x$, $a > 1$; (Logarithmic, same rate for all bases)
5. K. (Constant)

Growing at the Same Rate is a Transitive Relation

If f and g grow at the same rate, and g and h also grow at the same rate, then so do f and h, since

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L_1 > 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{g(x)}{h(x)} = L_2 > 0$$

(L_1, L_2 finite)

$$\Rightarrow \lim_{x\to\infty} \frac{f(x)}{h(x)} = \lim_{x\to\infty} \left(\frac{f(x)}{g(x)} \cdot \frac{g(x)}{h(x)} \right) = \underbrace{L_1 L_2}_{\text{finite}} > 0.$$

e.g. Show that $\sqrt{x^2+2020}$ and $(98\sqrt{x}-1)^2$ grow at the same rate.

Sol:

Similar ideas can be used to talk about the rates at which the functions approach 0:

e.g. • $\lim_{x\to 0} \sin x = \lim_{x\to 0} x = 0$, $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$\Rightarrow$ $f(x) = \sin x$ & $g(x) = x$ approach 0 at the same rate as $x \to 0$.

• $\lim_{x\to 2} \frac{(x-2)^2}{x-2} = \lim_{x\to 2} (x-2) = 0$

$\Rightarrow$ $f(x) = (x-2)^2$ approaches 0 faster than $g(x) = x-2$, as $x \to 2$ (or $g(x)$ approaches 0 slower than $f(x)$ as $x \to 2$).

Big-Oh and Little-Oh

Def: Let f and g be functions, both eventually positive.
Then we write:

- $f(x) = o(g(x))$ as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. ①
- $f(x) = O(g(x))$ as $x \to \infty$ if $\exists N, M \in \mathbb{R}$ such that

$$\frac{f(x)}{g(x)} \leq M, \quad \forall x \in [N, \infty). \quad ②$$

Remarks

- These are called little-oh and big-oh notation, respectively.
- ① states "f grows slower than g as $x \to \infty$", while
  ② means (roughly) "f grows at most as fast as g, as $x \to \infty$".
- If $f(x) = o(g(x))$ as $x \to \infty$, then $f(x) = O(g(x))$ as $x \to \infty$. (Why?).

- If f and g grow at the same rate as $x \to \infty$, then then $f(x) = O(g(x))$ as $x \to \infty$. (Why?)

- It is often useful to think of $o(g(x))$ and $O(g(x))$ as sets of functions, e.g.,
  - → $o(x^3)$ consists of all functions that grow slower than x^3;
  - → $O(e^x)$ contains all the functions that have equal or smaller growth rate as e^x.

e.g.

(a) $K = O(1)$ ← constant function.

(b) More generally, $Kf(x) = O(f(x))$, $(K>0)$ if f is eventually positive.

(c) $a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 = O(x^n)$ if $a_n > 0$, since

$$\lim_{x\to\infty} \frac{\sum_{i=0}^{n} a_i x^i}{x^n} = \lim_{x\to\infty} \sum_{i=0}^{n} a_i \underbrace{x^{i-n}}_{\to 0 \text{ unless } i=n} = a_n ,$$

showing that $\sum_{i=0}^{n} a_i x^i$ and x^n have the same growth rate.

(d) $x+\sin x = O(x)$, since

$$\lim_{x\to\infty} \frac{x+\sin x}{x} = \lim_{x\to\infty} \frac{1+\frac{\sin x}{x}}{1} = \frac{1+0}{1} = 1,$$

showing that $x+\sin x$ and x have the same growth rate.

(e) $\ln(\ln x) = o(\ln x)$, since

$$\lim_{x\to\infty} \frac{\ln(\ln x)}{\ln x} \overset{\text{L'Hô}}{=} \lim_{x\to\infty} \frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{1/x} = 0.$$

(f) From (e), we see that $\ln(\ln x) = O(\ln x)$ also holds (as $x\to\infty$).

Applications: Computational Complexity ?