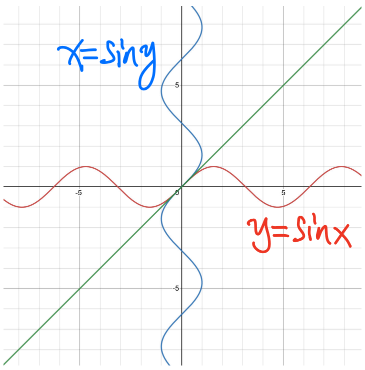
Lecture 17, Tuesday, November/07/2023

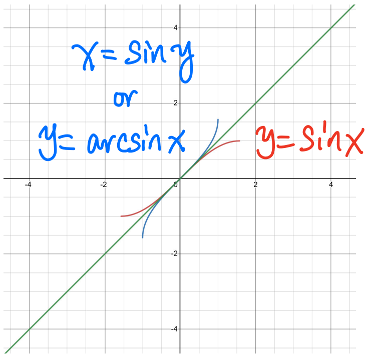
Outline

- Inverse differentiation (7.1)
- Inverse trigonometric functions (7.6)
- Natural logarithmic function (7.2)
  - ↳ Basic properties
  - ↳ Algebraic properties
  - ↳ Graph and range
  - ↳ $\ln(|f(x)|)$

Inverse Trigonometric Functions



The sine function is not injective on its natural domain $\mathbb{R}$, so it does not have an inverse in this case.



But $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ is increasing, so it is injective and has an inverse

$$\arcsin : [-1,1] \to [-\tfrac{\pi}{2}, \tfrac{\pi}{2}].$$

This is the inverse sine function, which is also denoted by $\sin^{-1}$.

Q: $\arcsin'(x) = ?$

Let $y_0 \in (-1,1)$ and let $x_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ be such that

$$\sin x_0 = y_0$$

Then, by inverse differentiation,

$$\arcsin'(y_0) = \frac{1}{\sin' x_0} = \frac{1}{\cos x_0} \overset{\cos x_0 > 0}{=} \frac{1}{\sqrt{1-\sin^2 x_0}} = \frac{1}{\sqrt{1-y_0^2}}$$

Hence:

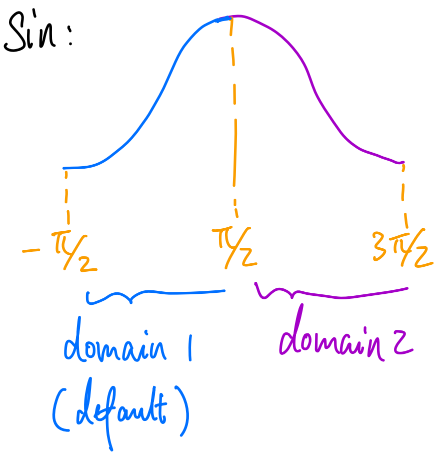
$$\arcsin' x = \frac{1}{\sqrt{1-x^2}}, \quad \forall x \in (-1,1). \qquad (*)$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C \quad \text{on } (-1,1)$$

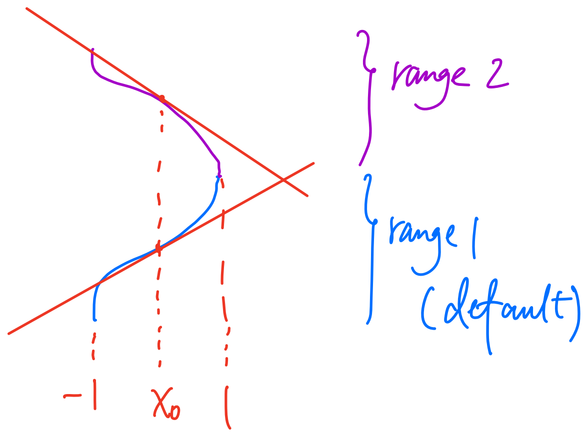
Remarks

- $\arcsin' x$ does not exist at $x = \pm 1$ (infinite slope).
- The formula (*) assumes that the domain of sine is $[-\frac{\pi}{2}, \frac{\pi}{2}]$. It may be different otherwise.

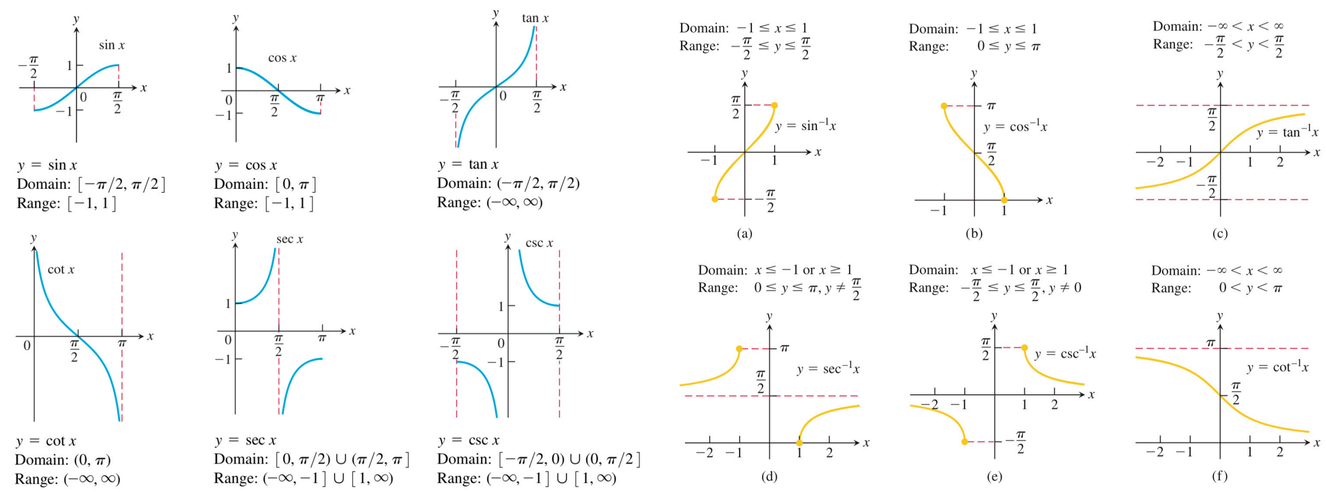
Sin:



arcsin:



The following figures summarize the six trigonometric functions and their inverses. Be aware of the given default domains.



We can use inverse differentiation to obtain the derivative formulae for other inverse trigonometric functions.

e.g. For $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$, the inverse function is $\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. For $y_0 \in \mathbb{R}$ and $\tan x_0 = y_0$,

$$\arctan'(y_0) = \frac{1}{\tan'(x_0)} = \frac{1}{\sec^2(x_0)} = \frac{1}{1+\tan^2(x_0)} = \frac{1}{1+y_0^2}.$$

Try to express in terms of y_0

Hence,

$$\arctan'(x) = \frac{1}{1+x^2}, \quad \forall x \in \mathbb{R},$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C \quad \text{on } \mathbb{R}$$

e.g. For $\sec : \underbrace{[0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi]}_{D} \to \underbrace{[1,\infty)\cup(-\infty,-1]}_{Y = \text{range}}$, we have inverse $\operatorname{arcsec}: Y \to D$. For any $y_0 \in Y \setminus \{-1,1\}$ with $\sec x_0 = y_0$,

$$\operatorname{arcsec}'(y_0) = \frac{1}{\sec'(x_0)} = \frac{1}{\underbrace{\sec(x_0)}_{y_0}\tan(x_0)}.$$

Since $\tan^2 x_0 + 1 = \sec^2 x_0$, we have

$$\tan x_0 = \begin{cases} \sqrt{\sec^2 x_0 - 1} = \sqrt{y_0^2-1}, & \text{if } x_0 \in (0,\frac{\pi}{2}) \\ -\sqrt{\sec^2 x_0 - 1} = -\sqrt{y_0^2-1}, & \text{if } x_0 \in (\frac{\pi}{2},\pi) \end{cases},$$

which means

$$\operatorname{arcsec}'(y_0) = \begin{cases} \frac{1}{y_0\sqrt{y_0^2-1}}, & \text{if } y_0 > 1 \\ \frac{-1}{y_0\sqrt{y_0^2-1}}, & \text{if } y_0 < -1 \end{cases} \qquad (**)$$

$$= \frac{1}{|y_0|\sqrt{y_0^2-1}}.$$

Hence,

$$\operatorname{arcsec}'(x) = \frac{1}{|x|\sqrt{x^2-1}}, \quad \forall x \text{ with } |x|>1.$$

Alternatively, from (**), one can also see that

$$\frac{d}{dx}\operatorname{arcsec}|x| = \frac{1}{x\sqrt{x^2-1}}, \quad \forall x \text{ with } |x|>1.$$

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \operatorname{arcsec}|x| + C \text{ on any interval } I \subseteq \mathbb{R}\setminus[-1,1].$$

Others

Since $\sin(y) = \cos(y - \frac{\pi}{2}) = \cos(\frac{\pi}{2} - y)$, if $x_0 = \sin y_0$ then

$$\arcsin x_0 = y_0 \quad \text{and} \quad \arccos x_0 = \frac{\pi}{2} - y_0$$

So

$$\boxed{\arcsin x + \arccos x = \frac{\pi}{2}, \quad \forall x \in [-1,1].}$$

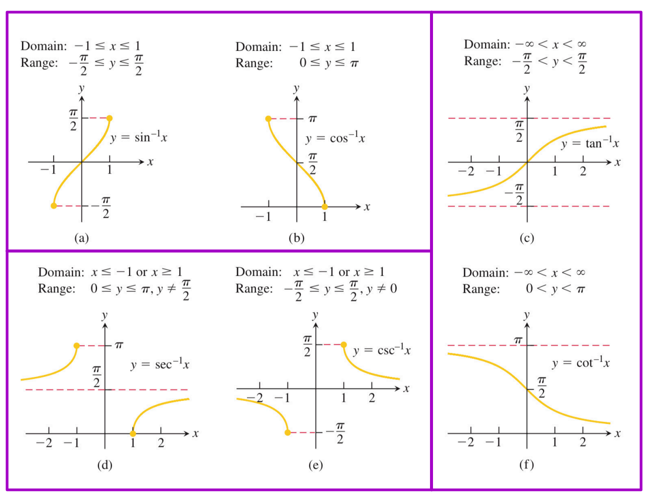
Similarly, one can show that

$$\boxed{\arctan x + \operatorname{arccot} x = \frac{\pi}{2}; \quad \operatorname{arcsec} x + \operatorname{arccsc} x = \frac{\pi}{2}.}$$

This means

$$0 = \frac{d}{dx}\left(\frac{\pi}{2}\right) = \arcsin' x + \arccos' x = \arctan' x + \operatorname{arccot}' x.$$
$$= \operatorname{arcsec}' x + \operatorname{arccsc}' x.$$

- $\sin : (-\pi/2, \pi/2) \to (-1,1)$, $\arcsin'(x) = \dfrac{1}{\sqrt{1-x^2}}$.
- $\cos : (0, \pi) \to (-1,1)$, $\arccos'(x) = \dfrac{-1}{\sqrt{1-x^2}}$.
- $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$, $\arctan'(x) = \dfrac{1}{1+x^2}$.
- $\cot : (0, \pi) \to \mathbb{R}$, $\operatorname{arccot}'(x) = \dfrac{-1}{1+x^2}$.
- $\sec : (0, \pi/2) \cup (\pi/2, \pi) \to (-\infty, -1) \cup (1, \infty)$,
  $$\operatorname{arcsec}'(x) = \frac{1}{|x|\sqrt{x^2-1}}.$$
- $\csc : (-\pi/2, 0) \cup (0, \pi/2) \to (-\infty, -1) \cup (1, \infty)$,
  $$\operatorname{arccsc}'(x) = \frac{-1}{|x|\sqrt{x^2-1}}.$$



Examples

1. $\displaystyle\int \frac{dx}{\sqrt{3-4x^2}} = \underline{\qquad\qquad\qquad} = \frac{1}{2}\arcsin\left(\frac{2}{\sqrt{3}}x\right) + C.$

2. $\displaystyle\int \frac{dx}{4x^2+4x+2} = \underline{\qquad\qquad\qquad} = \frac{1}{2}\arctan(2x+1) + C.$

3. $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \underline{\qquad\qquad\qquad} = \frac{1}{a}\operatorname{arcsec}\left|\frac{x}{a}\right| + C$

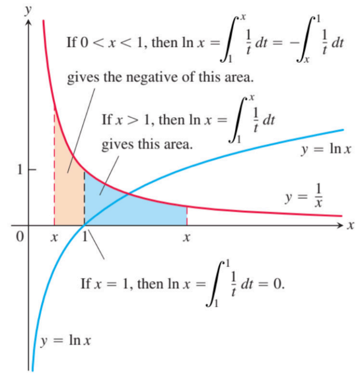
$(a>0)$ on (a,∞) or on $(-\infty,-a)$.

Natural Logarithmic Function

> Def: (Saint-Vincent, 1649)
>
> The (natural) logarithmic function is the function $\ln$ defined on $(0,\infty)$ by
>
> $$\ln(x) := \int_1^x \frac{1}{t}\,dt.$$

Hence, $\ln(x)$ is the signed area under the curve $y=\frac{1}{t}$ from $t=1$ to $t=x$, $x>0$.

1. Basic Properties



- $\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0$.
- $\ln' x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt \overset{\text{FTC I}}{=\!=} \frac{1}{x}$, $\forall x \in (0, \infty)$.
- By the point above, $\ln$ is differentiable on $(0,\infty)$, so it is also continuous on $(0,\infty)$.
- Since $\ln'(x) = \frac{1}{x} > 0$ $\forall x \in (0,\infty)$, $\ln$ is increasing on $(0,\infty)$.
- Since $\frac{1}{t}$ is decreasing,

$$\ln 4 = \int_1^4 \frac{1}{t}\,dt = \int_1^2 \frac{1}{t}\,dt + \int_2^3 \frac{1}{t}\,dt + \int_3^4 \frac{1}{t}\,dt$$

$$\geq \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{6+4+3}{12} > 1.$$

By IVT and monotonicity, there is a unique number x_0 such that $\ln(x_0) = 1$.

Def Define e to be the unique number in $(0,\infty)$ such that $\ln(e) = 1$. also called Euler's number.

Historical Remarks about e

- Bernoulli, 1683, in the context of compound interest.
- Euler, 1731, in the context of logarithm.

2. Algebraic Properties.

Theorem (Algebraic Properties of $\ln$) For any $b \in \mathbb{R}_{>0}$ and $x \in \mathbb{R}_{>0}$:

1. $\ln(bx) = \ln b + \ln x$.
2. $\ln \frac{b}{x} = \ln b - \ln x$.
3. $\ln \frac{1}{x} = -\ln x$.
4. $\ln(x^r) = r \ln x$, for any $r \in \mathbb{Q}$.

Rational number set

Proof: 1. Let $f(x) = \ln(bx)$, defined on $(0, \infty)$. Then

$$f'(x) = \frac{b}{bx} = \frac{1}{x} = \ln'(x),$$

So $\exists$ constant C such that $f(x) = \ln x + C$, $\forall x \in (0, \infty)$.

$$\Rightarrow \ln(bx) = \ln(x) + C, \quad \forall x \in (0, \infty)$$

Substituting $x=1$ yields

$$\ln b = \ln 1 + C = C,$$

So $C = \ln b$, i.e., $\ln(bx) = \ln x + \ln b$.

4. Let $f(x) = \ln(x^r)$, defined on $(0, \infty)$. Then

$$f'(x) = \frac{r x^{r-1}}{x^r} = r \frac{1}{x} = \frac{d}{dx}(r \ln x)$$

So $\exists$ constant C such that $f(x) = r \ln x + C$, $\forall x \in (0, \infty)$.

$\Rightarrow \ln(x^r) = r \ln x + C$.

Substituting $x = 1$ yields $\ln(1) = r \ln(1) + C$, i.e., $C = 0$.

So

$$\ln(x^r) = r \ln x.$$

2. $\ln(\frac{b}{x}) = \ln(bx^{-1}) = \ln b + \ln(x^{-1})$ (by 1)

$\quad = \ln b - \ln x$ (by 4).

3. $\ln(\frac{1}{x}) \overset{\text{by } 2}{=} \ln 1 - \ln x = -\ln x$. $\square$

3. Graph and Range

- Concavity

Since $\ln''(x) = \frac{d}{dx}(\frac{1}{x}) = -\frac{1}{x^2} < 0$ for all $x \in (0, \infty)$,

the curve $y = \ln x$ is <u>concave down</u> on $\mathbb{R}$.

- Range

Since $\ln 2 = \int_1^2 \frac{1}{t}\,dt > (2-1)\frac{1}{2} = \frac{1}{2}$, it follows that for any $n \in \mathbb{Z}_+$, $\ln(2^n) = n\ln 2 > \frac{n}{2}$. Since $\ln$ is increasing and $\frac{n}{2} \to \infty$ as $n \to \infty$, we see that $\ln(x)$ can exceed any fixed number M for all sufficently large x, which means $\lim_{x\to\infty} \ln x = \infty$.

Also,

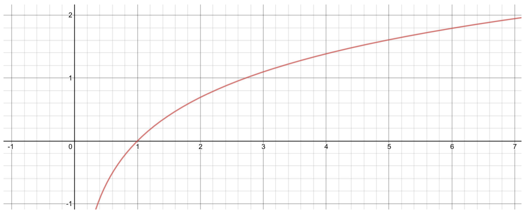
$$\lim_{x\to 0^+} \ln x = \lim_{y\to\infty} \ln \frac{1}{y} = \lim_{y\to\infty}(-\ln y) = -\infty,$$

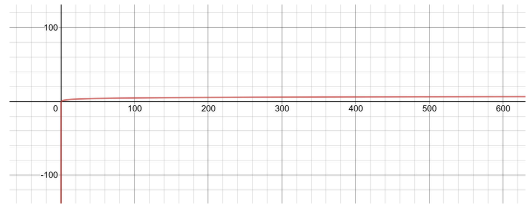
so $\lim_{x\to 0^+} \ln x = -\infty$, and $x=0$ is a vertical asymptote.

Since $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to 0^+} \ln x = -\infty$, using IVT, one can show that $\text{range}(\ln) = \mathbb{R}$.

- Limits of Derivatives

Since $\lim_{x\to\infty} \ln'(x) = \lim_{x\to\infty} \frac{1}{x} = 0$, the graph $y = \ln x$ tends to have a "flat" tangent line as x grows big, yet it is unbounded above.





4. Composite Functions: $\ln$ with $|f(x)|$.

If we take $g(x) = |x|$ with $D = \mathbb{R} \setminus \{0\}$, then $g'(x) = \frac{|x|}{x}$.

By the chain rule,

$$\frac{d}{dx} \ln |x| = \frac{1}{|x|} \cdot \frac{|x|}{x} = \frac{1}{x}.$$

This means that on the intervals $(-\infty, 0)$ and $(0, \infty)$, $\ln |x|$ is an antiderivative of $\frac{1}{x}$, i.e.,

$$\boxed{\int \frac{1}{x} dx = \ln |x| + C}$$

holds for any one of the two intervals above.

More generally, if $g(x) = |f(x)|$ where f is differentiable and never zero, then by the chain rule,

$$g'(x) = \underbrace{\frac{|f(x)|}{f(x)}} \cdot f'(x),$$

So

$$\frac{d}{dx} \ln |f(x)| = \frac{1}{|f(x)|} \cdot \left(\frac{|f(x)|}{f(x)} f'(x) \right) = \frac{f'(x)}{f(x)}.$$

That is,

$$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C.}$$

Note that it only works on an interval on which f is defined and never zero.

Extended Discussion: Relative Rates of Change

The quantity $\frac{f'(x_0)}{f(x_0)}$ is called the valative rate of change of f at $x = x_0$. We will see its meaning in class by a concrete example.

e.g. Population: ______________________.

If $f(x) > 0$ $\forall x$, then $\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$ gives the function of relative rate of change.