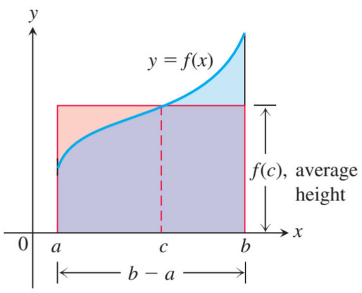
Lecture 13, Tuesday, October/24/2023

Outline

- MVT for definite integrals (5.4)
- Fundamental Theorems of Calculus (FTC, 5.4)
- Areas between curves

Mean Value Theorem for Definite Integrals

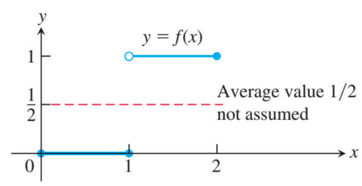


- If f is continuous on $[a,b]$, then some $c \in [a,b]$ must achieve the average height of f on $[a,b]$.

THEOREM ~~3~~ 5.4.3—The Mean Value Theorem for Definite Integrals If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

- Continuity condition cannot be skipped.



Proof:

1. It suffices to show that the average value $\frac{1}{b-a}\int_a^b f(x)dx$ is between $m := \min_{x\in[a,b]} f(x)$ and $M := \max_{x\in[a,b]} f(x)$, say $f(x_m) = m$ and $f(x_M) = M$. Then $\exists c$ between x_m and x_M such that

(Assume f is not constant, otherwise obvious.)

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

by IVT.

2. To show 1, apply the min-max inequality:

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

$$\Rightarrow \quad m \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq M. \qquad \square$$

Fundamental Theorems of Calculus

Let $v(t)$ be the velocity function of an object, say $v(t) \geq 0 \quad \forall t$.

Q1: What is $\int_1^5 v(t)\,dt$? A: ______________ .

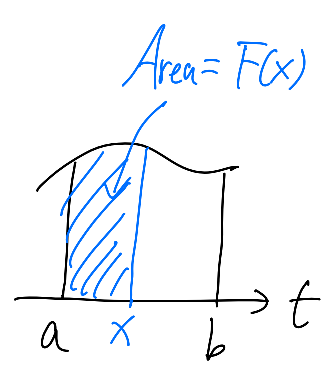
Q2: If $F(x) := \int_1^x v(t)\,dt$, what is $F(x)$?
A: ______________ .

Q3: What is $F'(x)$? A: ______________ .

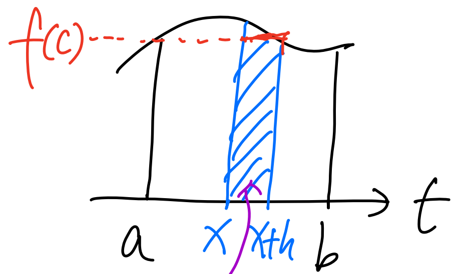
Let us look at a more general case.

- Suppose f is continuous on $[a,b]$.
- Let $F : [a,b] \to \mathbb{R}$, $F(x) := \int_a^x f(t)\,dt$.

Q: What is $F'(x)$ for $x \in (a,b)$?



- $F'(x) = \lim_{h\to 0} \dfrac{F(x+h) - F(x)}{h}$



$$= \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt}{h}$$

$$= \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h} \qquad ①$$

$\int_x^{x+h} f(t)\,dt$

This is average value of f between x and $x+h$, even if $h<0$.

- By MVT for integrals, $\exists\, c$ between x and $x+h$ such that

$$f(c) = \frac{1}{h}\int_x^{x+h} f(t)\,dt. \qquad ②$$

- Note that $c \to x$ as $h \to 0$.

- By ① & ②,

f is continuous

$$F'(x) = \lim_{h\to 0} f(c) = f\left(\lim_{h\to 0} c\right) = f(x).$$

- Similarly, $F_+'(a) = f(a)$ and $F_-'(b) = f(b)$.

This is summarized below.

(FTC1)

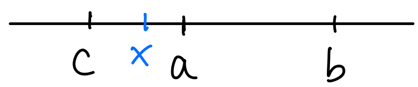
THEOREM 4—The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

Also works for $x=a$ and b with one-sided derivatives.

<u>Remark</u>: Suppose f is continuous on $[c,b]$, where $c<a$. Define $F(x) := \int_a^x f(t)\,dt$ for $x \in [c,b]$. Then $F'(x) = f(x)$ is still true for $x<a$.

(Why?)



<u>Remarks about FTC1</u>

- $\int_a^x f(t)\,dt = \int_a^x f(s)\,ds$: the inner variable is a dummy variable, so the choice of letter is not important.
  However, DO NOT write $\int_a^x f(x)\,dx$.

- FTC1 implies that any continuous function on I must have an antiderivative on I:
  $F(x) := \int_a^x f(t)\,dt$ is an antiderivative of f (with $F(a)=0$).

EXAMPLE ~~2~~ 5.4.2 Use the Fundamental Theorem to find dy/dx if

(a) $y = \int_a^x (t^3 + 1)\,dt$ **(b)** $y = \int_x^5 3t \sin t\,dt$

(c) $y = \int_1^{x^2} \cos t\,dt$ **(d)** $y = \int_{1+3x^2}^4 \frac{1}{2+t}\,dt$

Sol: (a) x^3+1.

(b) Let $F(x) = \int_5^x \underbrace{3t\sin t}_{f(t)}\,dt$. Then $y = -F(x)$,

and $\frac{dy}{dx} = (-F(x))' = -F'(x) \overset{FTC1}{=\!=} -f(x) = -3x\sin x$.

(c) Write $y = F(u) = \int_1^u \cos t\,dt$, $u = x^2$. Then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \cos u \cdot 2x = \cos x^2 \cdot 2x.$$

(d) Let $F(u) = \int_4^u \frac{1}{2+t}\,dt$, $u = 1+3x^2$.

Then $y = -F(u)$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = -F'(u)\frac{du}{dx} = \frac{-1}{2+u}\cdot 6x$$

$$= \frac{-6x}{3+3x^2} = \frac{-2x}{1+x^2}.$$

Remark: If $F(u) = \int_a^u f(t)\,dt$, then $\int_a^{g(x)} f(t)\,dt = (F\circ g)(x)$,

which means $\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = (F\circ g)'(x) = F'(g(x))g'(x)$

$\overset{FTC1}{=\!=} f(g(x))g'(x)$.

Let $v(t)$ be the velocity function (say $v(t) \geq 0, \forall t$).
Then $\int_a^b v(t)\,dt$ is the total distance travelled from $t=a$ to $t=b$. If $S(t)$ is the position at time t, then

$$\int_a^b v(t)\,dt = S(b) - S(a).$$

(Physical intuition)

More generally:

THEOREM 4 (Continued)—The Fundamental Theorem of Calculus, Part 2 (FTC2)
If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof: • Let $G(x) := \int_a^x f(t)\,dt$, and F be any antiderivative of f.

• Note that $\int_a^b f(t)\,dt = G(b) = G(b) - G(a)$.

$G_+'(a) = f(a)$ $\quad$ $G_-'(b) = f(b)$

• By FTC1, G is an antiderivative of f on $[a,b]$, so

$\exists$ constant C s.t. $G(x) = F(x) + C$ for all $x \in [a,b]$.

• Now $\int_a^b f(t)\,dt = G(b) - G(a) = F(b) + C - (F(a) + C) = F(b) - F(a)$.

$\square$

By FTC2, in order to calculate $\int_a^b f(x)\,dx$, it suffices to find an antiderivative F of f.

<u>Remark</u> FTC implies that differentiation and integration are "inverse operations":

$$\frac{d}{dx}\int_a^x f(t)\,dt \overset{\text{FTC1}}{=} f(x) \quad \text{and} \quad \int_a^x f'(t)\,dt \overset{\text{FTC2}}{=} \underbrace{f(x)-f(a)}_{\text{an antiderivative of } f'(x)}.$$

"Applying differentiation and integration one after another gives you the original function back (up to a constant shift)."

<u>Notation</u> We write $F(x)\Big|_{x=a}^{b}$ or $F(x)\Big|_a^b$ to mean

$$F(b)-F(a).$$

<u>e.g.</u> • $\int_0^{\pi} \cos x\,dx = \sin x\Big|_0^{\pi} = \sin\pi - \sin 0 = 0.$

• $\int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}.$

• $\int_1^9 \frac{x-1}{\sqrt{x}}\,dx = \int_1^9 \left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right) dx = \frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}}\Big|_1^9$

$$= \frac{2}{3}(27) - 6 - \frac{2}{3} + 2 = \frac{40}{3}.$$

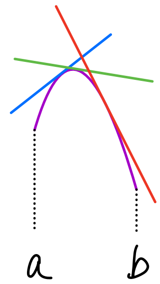
In-class Discussions

Q1: If $C(x)$ is the total cost for producing x units of product, what is the meaning of

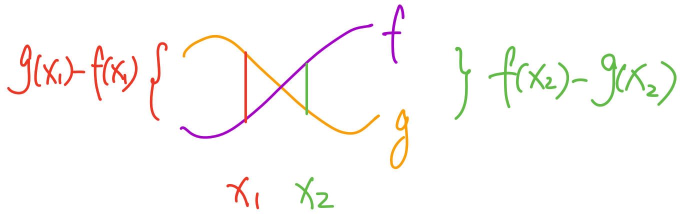
$$\int_a^b C'(x)\,dx \ ? \qquad (\text{Here } a<b.)$$

Q2: What is the average slope of all the tangent lines to the curve $y=f(x)$ over the interval $[a,b]$?

f differentiable



Areas Between Curves



Definition

Let f and g be functions that are integrable on $[a, b]$. Then the area A between the graph of $y = f(x)$ and the graph of $y = g(x)$, from $x = a$ to $x = b$, is defined by

$$A := \int_a^b |f(x) - g(x)|\, dx.$$

- For area between $y=f(x)$ and the x-axis, take $g(x) \equiv 0$, and the area becomes

$$A = \int_a^b |f(x)|\, dx.$$

- If f is further nonnegative in the case above, then

$$A = \int_a^b f(x)\,dx,$$

which is consistent with our previous discussion / intuition.

Remark Area A between curves $x=f(y)$ and $x=g(y)$, from $y=a$ to $y=b$, can be defined similarly:

$$A := \int_a^b |f(y)-g(y)|\, dy$$

5.6.4

EXAMPLE ~~4~~ Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Sol: $\underline{\qquad\qquad\qquad\qquad} = \frac{9}{2}$.

5.6.5

EXAMPLE ~~5~~ Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Sol: $\underline{\qquad\qquad\qquad\qquad} = \frac{10}{3}$.

Example

Find the area A between the graph of $y = f(x)$ and the graph of $y = g(x)$, from $x = a$ to $x = b$.

(c) $f(x) := (x-2)^2$ and $g(x) := 2x - 1$; $a = 0$ and $b = 8$;

Sol: $A = \int_0^8 |(x-2)^2 - 2x + 1|\,dx = \int_0^8 |(x-5)(x-1)|\,dx$

$= \underline{\qquad\qquad} = 40.$