Lecture 6

Function approximation

Ivan Rudik
AEM 7130

Roadmap

1. Projection theory
2. Spectral and finite element methods

Projection theory

Projection methods

We often need to approximate functions in economics, a common way to do this is via projection

Main idea: build some function $\hat{V}$ indexed by coefficients that approximates the function we are interested in

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What do I mean by approximately?

The coefficients of $\hat{V}$ are selected to minimize some residual function that tells us how far away our approximation is to the true function on some set of points

Projection methods

How do we do this?

Projection methods

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First we specify our approximating function: some linear combination of **basis functions** $\Psi_i(\mathbf{S})$

$$V^j(\mathbf{S}|c) = \sum_{i=0}^{j} c_i \Psi_i(\mathbf{S})$$

with coefficients $c_0, \ldots, c_j$

Projection methods

We then choose some residual function R which is a function of V^j:

$$R(\mathbf{S}|c) = H(V^j(\mathbf{S}|c))$$

and select the coefficient values to minimize the residual, given some measure of distance

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What basis do we select?

How do we project (select the coefficients / residual function)?

Projection methods

Let's work a simple example to get intuition

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Ordinary least squares linear regression

Projection methods

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Ordinary least squares linear regression

We can think of the problem as searching for some unknown conditional expectation $E[Y|X]$, given outcome variable Y and regressors X

Projection methods

We don't know the true functional form of $E[Y|X]$, but we can approximate it using the first two monomials on X: 1 and X

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OLS is within the class of projection methods

Projection classes are defined by metrics

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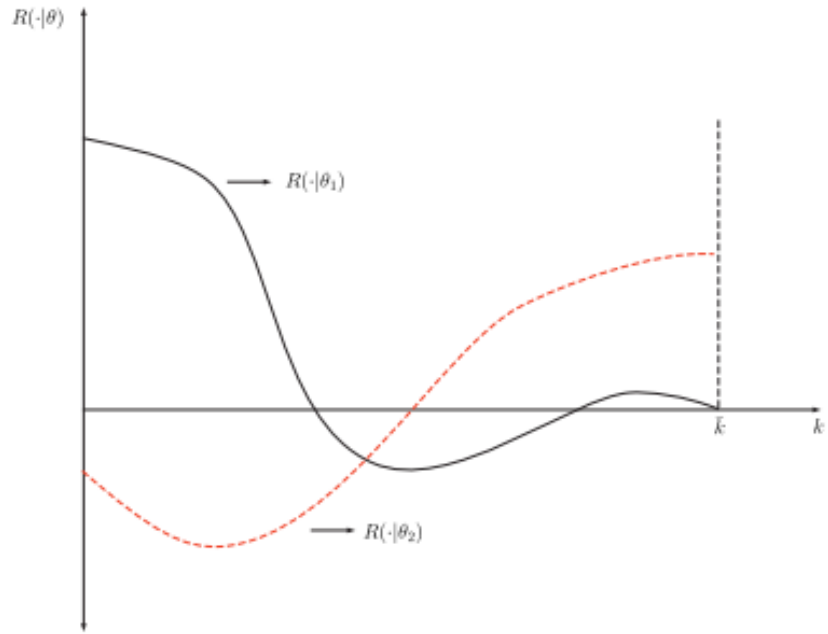
ρ tells us how close our residual function is to zero over the domain of our state space

Example residuals given different projections

Example: The figure shows two different residuals on some capital domain of $[0, \bar{k}]$

The residual based on the coefficient vector c_1 is large for small values of capital but near-zero everywhere else

Figure 2: Residual Functions



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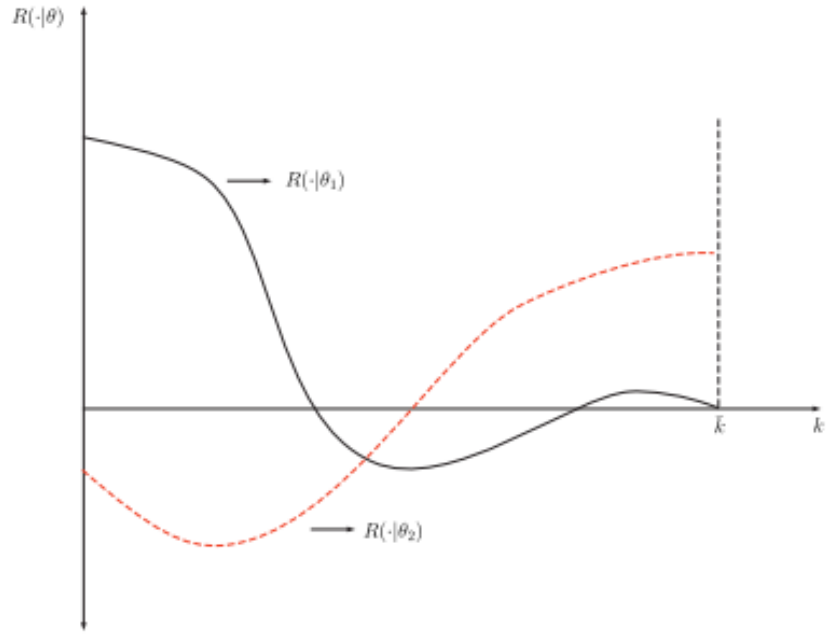

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Which is closer to zero over the interval? It will depend on our selection of ρ

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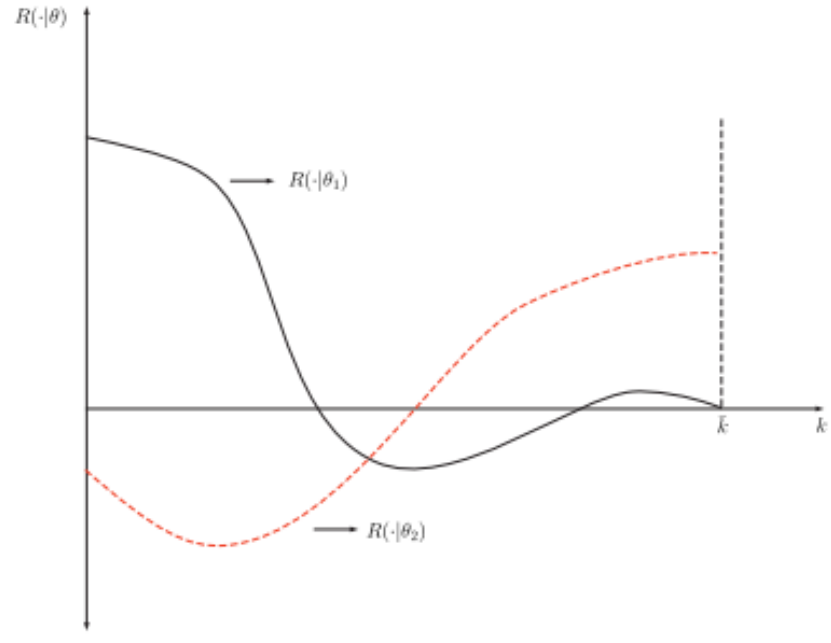

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The one-dimensional metric is defined as

$$\rho(R \cdot |c, 0) = \begin{cases} 0 & \text{if } \int_\Omega \phi_i(\mathbf{S}) R(\cdot|c) d\mathbf{S} = 0, i = 1, \ldots, j+1 \\ 1 & \text{otherwise} \end{cases}$$

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Where we want to solve for $c = \operatorname{argmin} \rho(R(\cdot|c), 0)$

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We can then change our problem to simply solving a system of integrals ensuring the metric is zero

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First lets begin with a simple example before moving into the most commonly used weight functions

Least squares projection

Suppose we selected the weight function to be

$$\phi_i(\mathbf{S}) = \frac{\partial R(\mathbf{S}|c)}{\partial c_{i-1}}$$

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Then we would be performing least squares! Why?

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$$\min_{\mathbf{c}} \int R^2(\cdot|c) d\mathbf{S}$$

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OLS minimizes residuals weighted by how they change in the coefficients

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Here our weight function is

$$\phi_i(\mathbf{S}) = \delta(\mathbf{S} - \mathbf{S}_i)$$

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What does this weight function mean?

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In class we will mostly be using collocation for function approximation

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We often have continuous states in economics (capital, technology, etc.), so how does collocation manage to work?

Using our knowledge of how the value function behaves at the limited set of points on our grid, we can interpolate our approximating function at all points off the grid points, but *within* the domain of our grid

Interpolation

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Each $\psi_j(x)$ is a basis function, and the coefficients c_j determine how they are combined at some point $\bar{x}$ to yield our approximation $\hat{V}(\bar{x})$ to $V(\bar{x})$

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This is what happens we select our number of grid points in the state space to be equal to the number of coefficients (which induces a Dirac delta weighting function)

Interpolation

Solve a system of equations, *linear in* c_j that equates the approximating function at the grid points to the recovered values

$$\mathbf{\Psi c} = \mathbf{v}$$

where Ψ is the matrix of basis functions, c is a vector of coefficients, and v is a vector of the recovered values

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We can recover c by left dividing by Ψ which yields

$$\mathbf{c} = \mathbf{\Psi}^{-1}\mathbf{v}$$

Interpolation

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We will learn how to interpolate using two different approaches:

1. Spectral methods
2. Finite element methods

Spectral methods

(Pseudo-)spectral methods

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When using spectral methods we virtually always use polynomials

Why?

(Pseudo-)spectral methods

The Stone-Weierstrass Theorem which states (for one dimension)

Suppose f is a continuous real-valued function defined on the interval $[a, b]$. For every $\epsilon > 0$, $\exists$ a polynomial $p(x)$ such that for all $x \in [a, b]$ we have $||f(x) - p(x)||_{sup} \leq \epsilon$

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What does the SW theorem say in words?

(Pseudo-)spectral methods

For any continuous function $f(x)$, we can approximate it arbitrarily well with some polynomial $p(x)$, as long as $f(x)$ is continuous

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Note that the SW theorem *does not* say what kind of polynomial can approximate f arbitrarily well, just that some polynomial exists

Basis choice

What would be your first choice of basis?

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Logical choice: the monomial basis: $1, x, x^2, \ldots$

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Logical choice: the monomial basis: $1, x, x^2, \ldots$

It is simple, and SW tells us that we can uniformly approximate any continuous function on a closed interval using them

Basis choice

Practice

code up a function `project_monomial(f, n, lb, ub)` that takes in some function `f`, degree of approximation `n`, lower bound `lb` and upper bound `ub`, and constructs a monomial approximation on an evenly spaced grid via collocation

Basis choice

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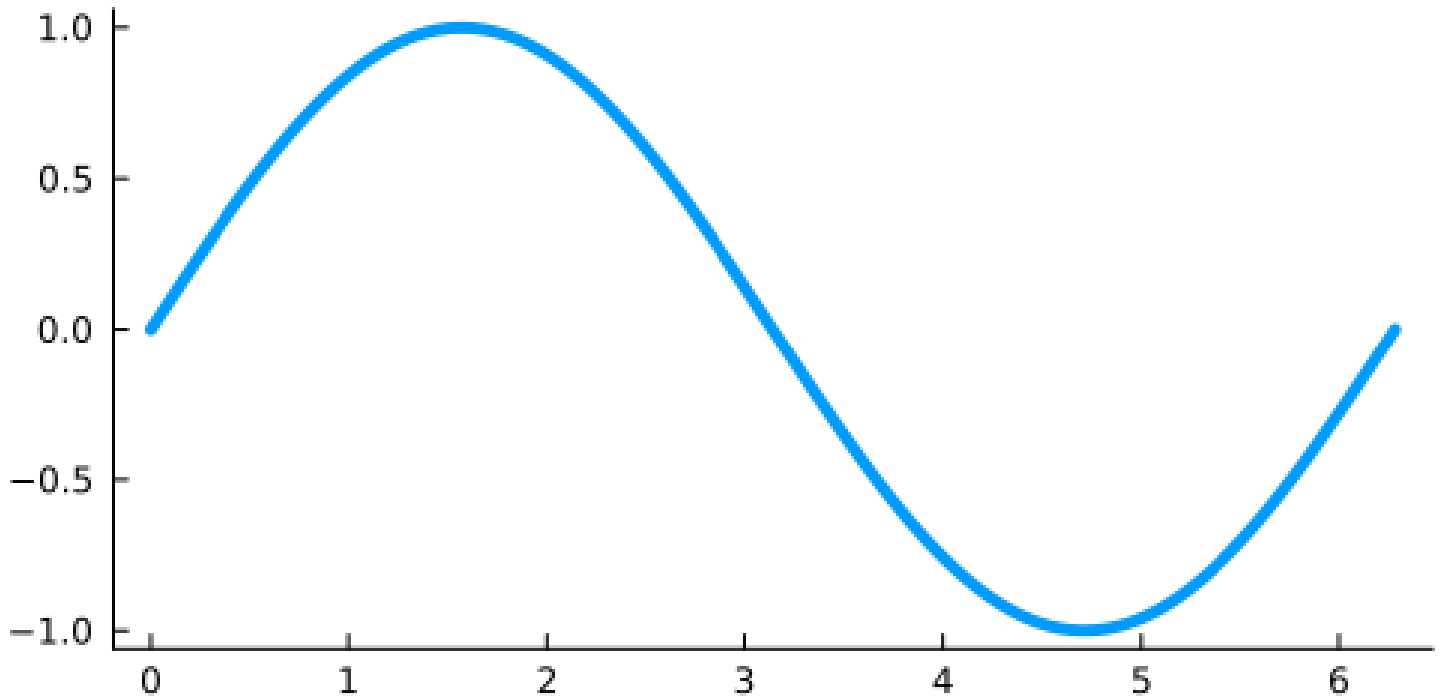
We will be plotting stuff, see http://docs.juliaplots.org/latest/generated/gr/ for example code using the `GR` backend

Approximate `sin(x)` with monomials

```
using Plots
gr();
f(x) = sin.(x);

Plots.plot(f, 0, 2pi, line = 4, grid = false, legend = false, size = (500, 250))
```



Approximating sin(x)

```
function project_monomial(f, n, lb, ub)
    # solves Ψc = y → c = Ψ\y
    # Ψ = matrix of monomial basis functions evaluted on the grid

    coll_points = range(lb, ub, length = n)                    # collocation points
    y_values = f(coll_points)                                  # function values on the grid
    basis_functions = [coll_points.^degree for degree = 0:n-1] # vector of basis functions
    basis_matrix = hcat(basis_functions...)                    # basis matrix

    coefficients = basis_matrix\y_values                       # c = Ψ\y

    return coefficients

end
```

```
## project_monomial (generic function with 1 method)
```

Approximating sin(x)

```
coefficients_4 = project_monomial(f, 4, 0, 2pi);
coefficients_5 = project_monomial(f, 5, 0, 2pi);
coefficients_10 = project_monomial(f, 10, 0, 2pi)
```

```
## 10-element Vector{Float64}:
##   0.0
##   0.9990725797453611
##   0.0040158571557465 94
##  -0.17384373874058445
##   0.007075663354237428
##   0.004040763229673627
##   0.0016747985986872982
##  -0.0006194667844639566
##   6.485272688675296e-5
##  -2.293696012668126e-6
```

Approximating sin(x)

Now we need to construct a function `f_approx(coefficients, plot_points)` that takes in the `coefficients` vector, and an arbitrary vector of points to evaluate the approximating function at (for plotting)

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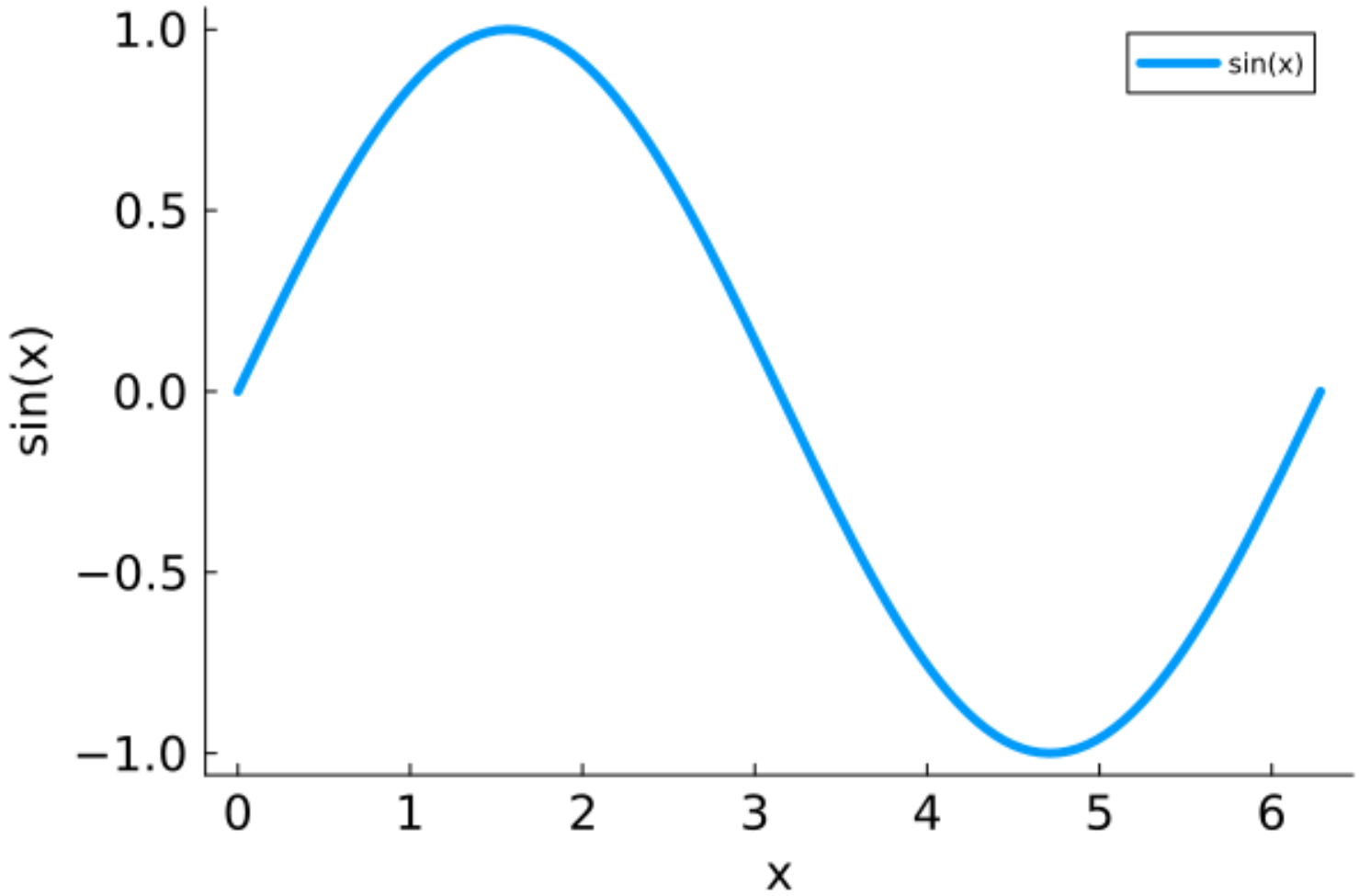
```
function f_approx(coefficients, points)
    n = length(coefficients) - 1
    basis_functions = [coefficients[degree + 1] * points.^degree for degree = 0:n] # evaluate ba
    basis_matrix = hcat(basis_functions...)                                         # transform t
    function_values = sum(basis_matrix, dims = 2)                                   # sum up into
    return function_values
end;
```

Approximating sin(x)

```
plot_points = 0:.01:2pi;
f_values_4 = f_approx(coefficients_4, plot_points);
f_values_5 = f_approx(coefficients_5, plot_points);
f_values_10 = f_approx(coefficients_10, plot_points)
```

```
## 629×1 Matrix{Float64}:
##   0.0
##   0.00999095361059282
##   0.019981668333002568
##   0.029971103713540295
##   0.03995822109621549
##   0.04994198367413662
##   0.059921356542025114
##   0.06989530674981242
##   0.07986280335729017
##   0.08982281748978363
##   ⋮
##  -0.08303623493737788
##  -0.07307101015972606
##  -0.06309900382674627
```

Plot



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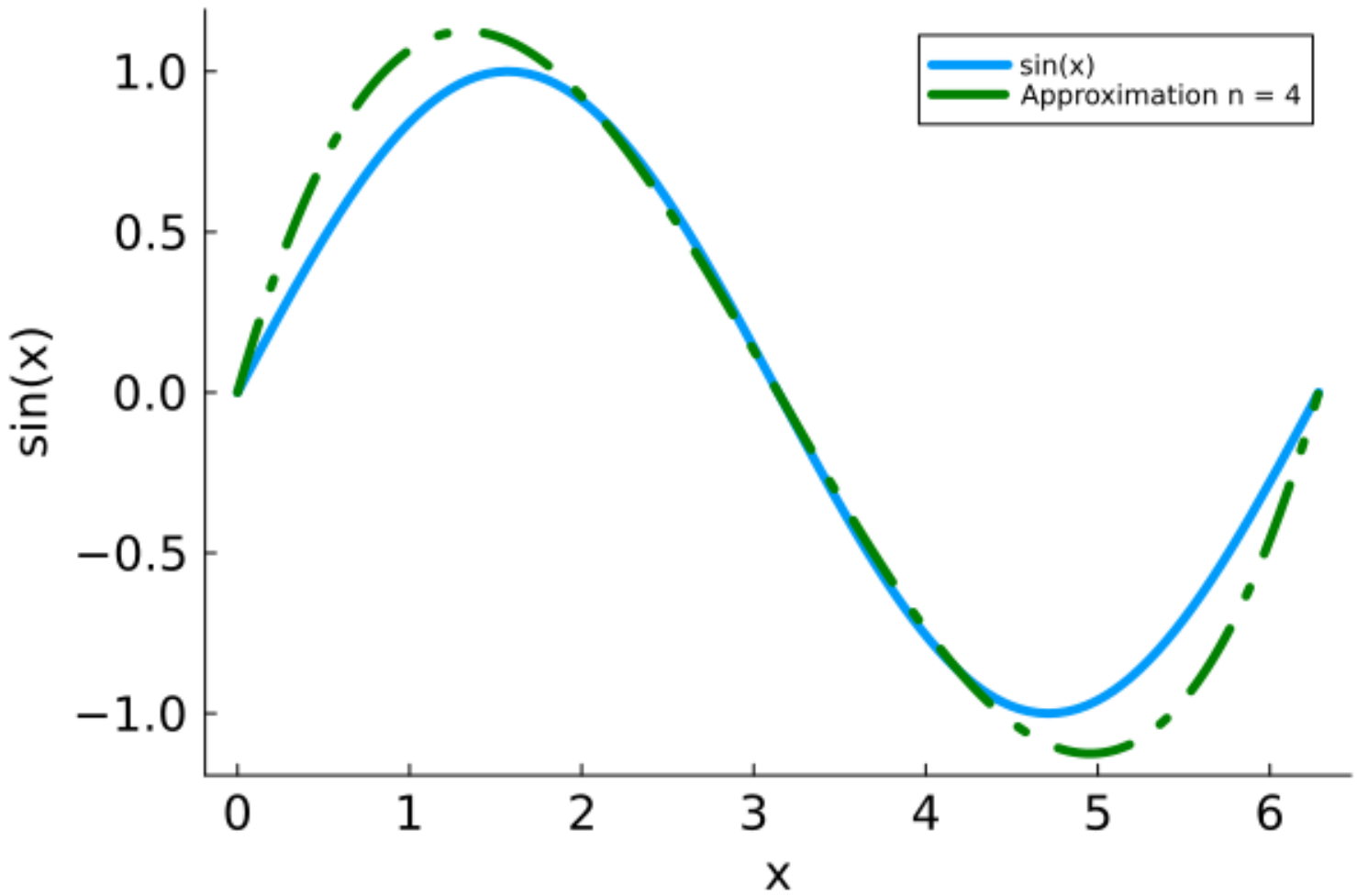

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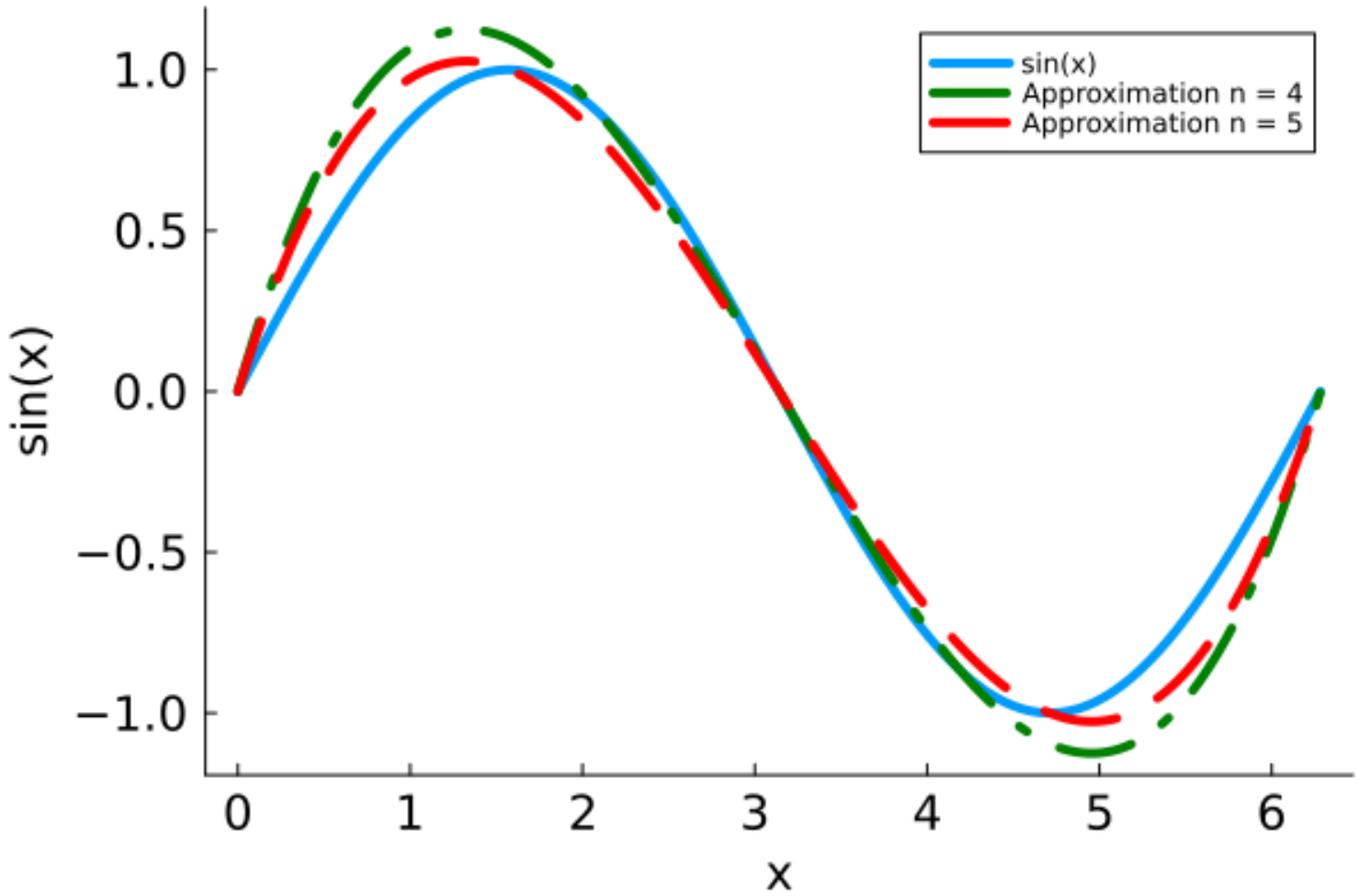
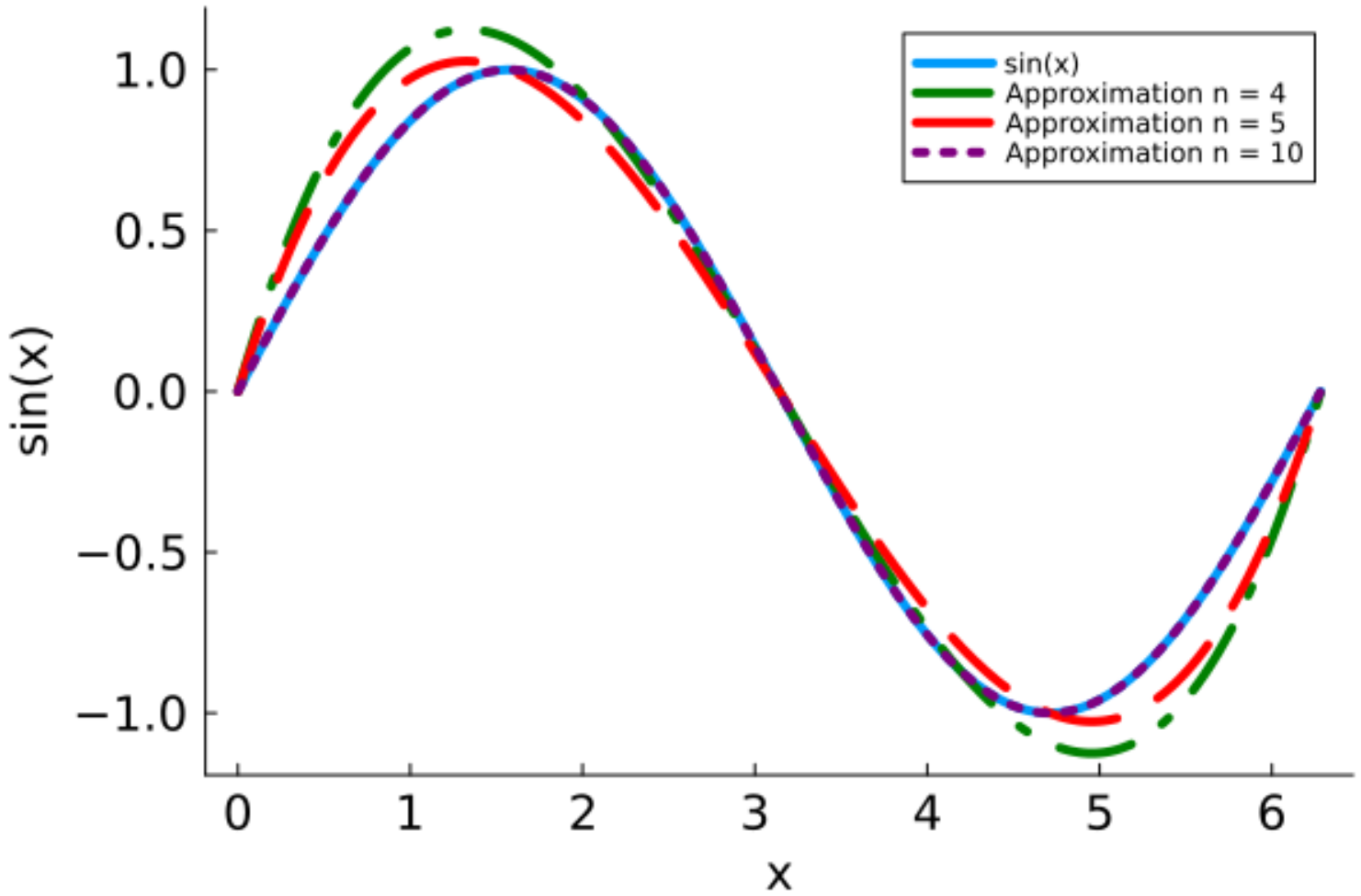

sin(x)
Approximation n = 4
Approximation n = 5
1.0
0.5
0.0
−0.5
−1.0
sin(x)
0
1
2
3
4
5
6
x

Plot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Cool!

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To approximate *any* function we'd need to feed in some basis function `g(x, n)` as opposed to hard-coding it like I did in the previous slides

Cool!

We just wrote some code that exploits Stone-Weierstrauss and allows us to (potentially) approximate any continuous function arbitrarily well as `n` goes to infinity

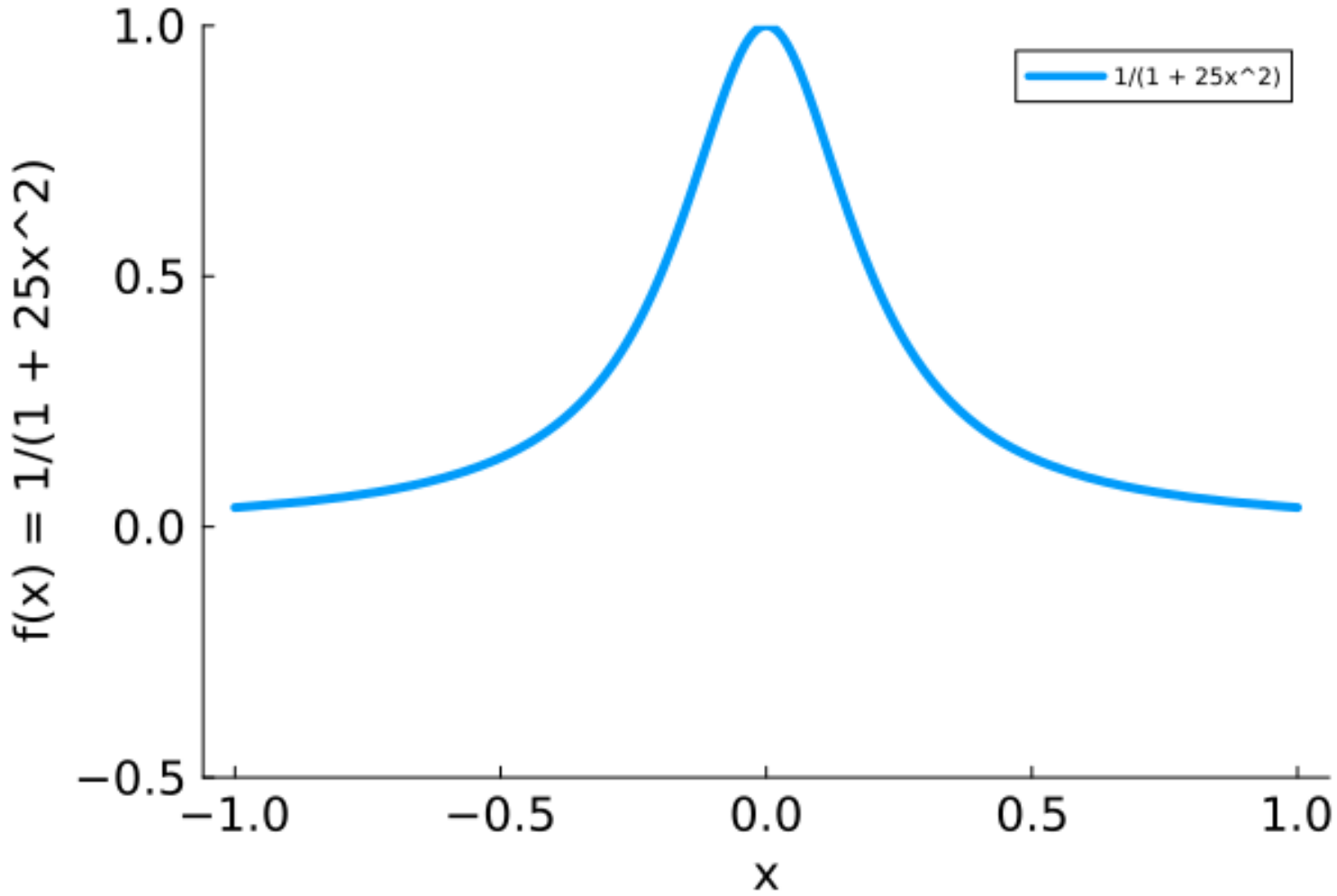
To approximate *any* function we'd need to feed in some basis function `g(x, n)` as opposed to hard-coding it like I did in the previous slides

Now try approximating **Runge's function**: `f(x) = 1/(1 + 25x^2)`

Runge's function

```
runge(x) = 1 ./ (1 .+ 25x.^2);
coefficients_5 = project_monomial(runge, 5, -1, 1);
coefficients_10 = project_monomial(runge, 10, -1, 1);
plot_points = -1:.01:1;
runge_values_5 = f_approx(coefficients_5, plot_points);
runge_values_10 = f_approx(coefficients_10, plot_points);
```

Runge's function



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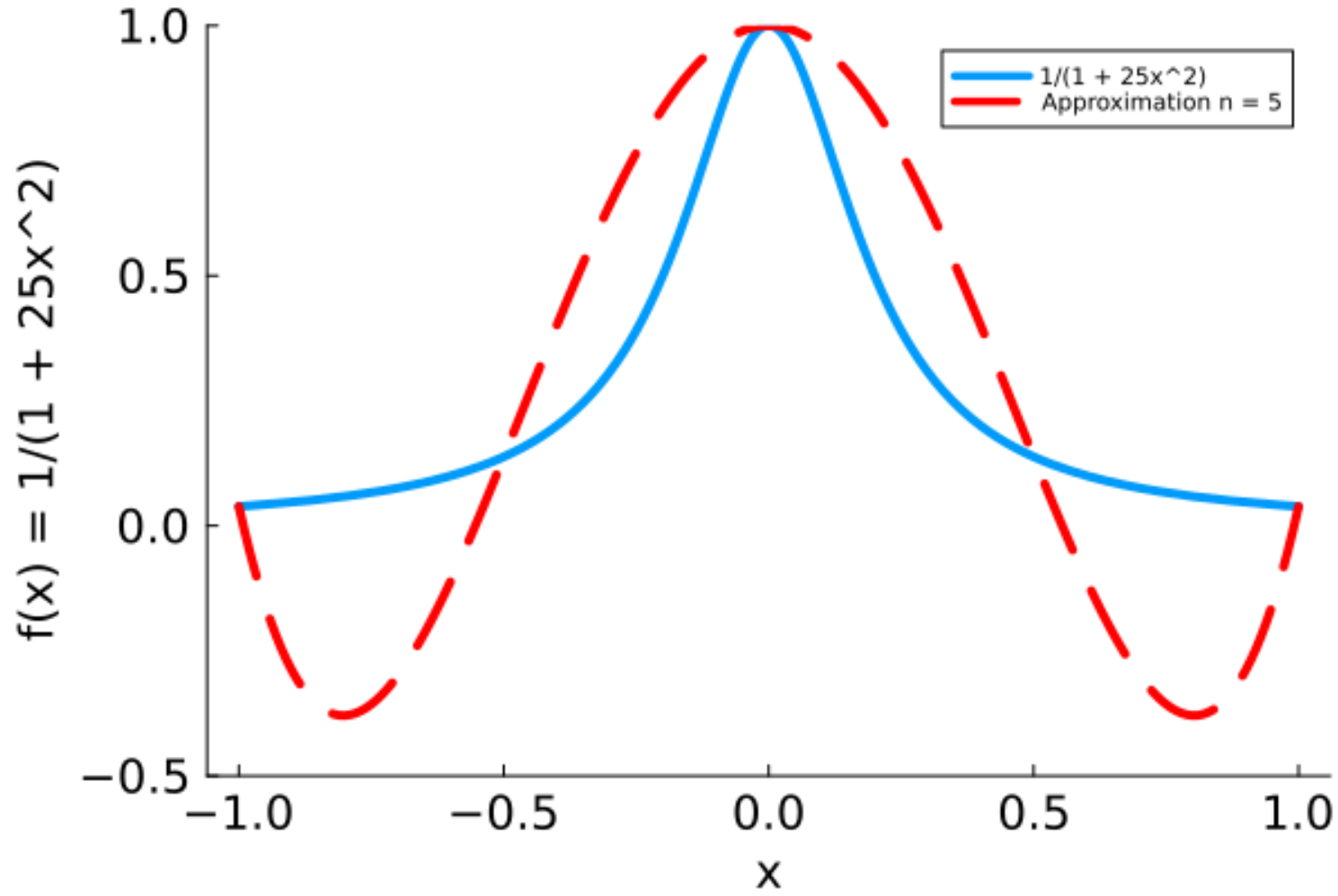

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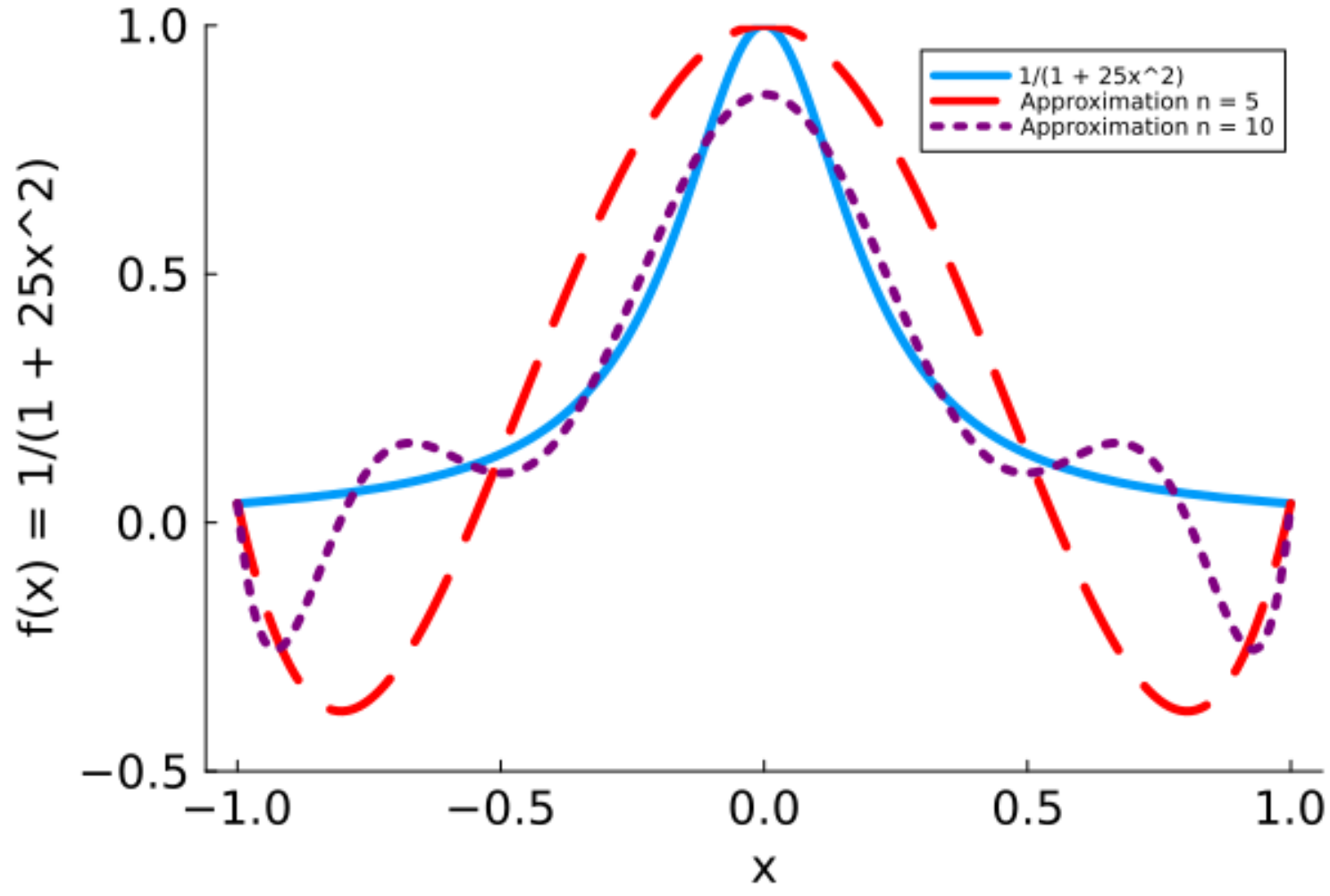

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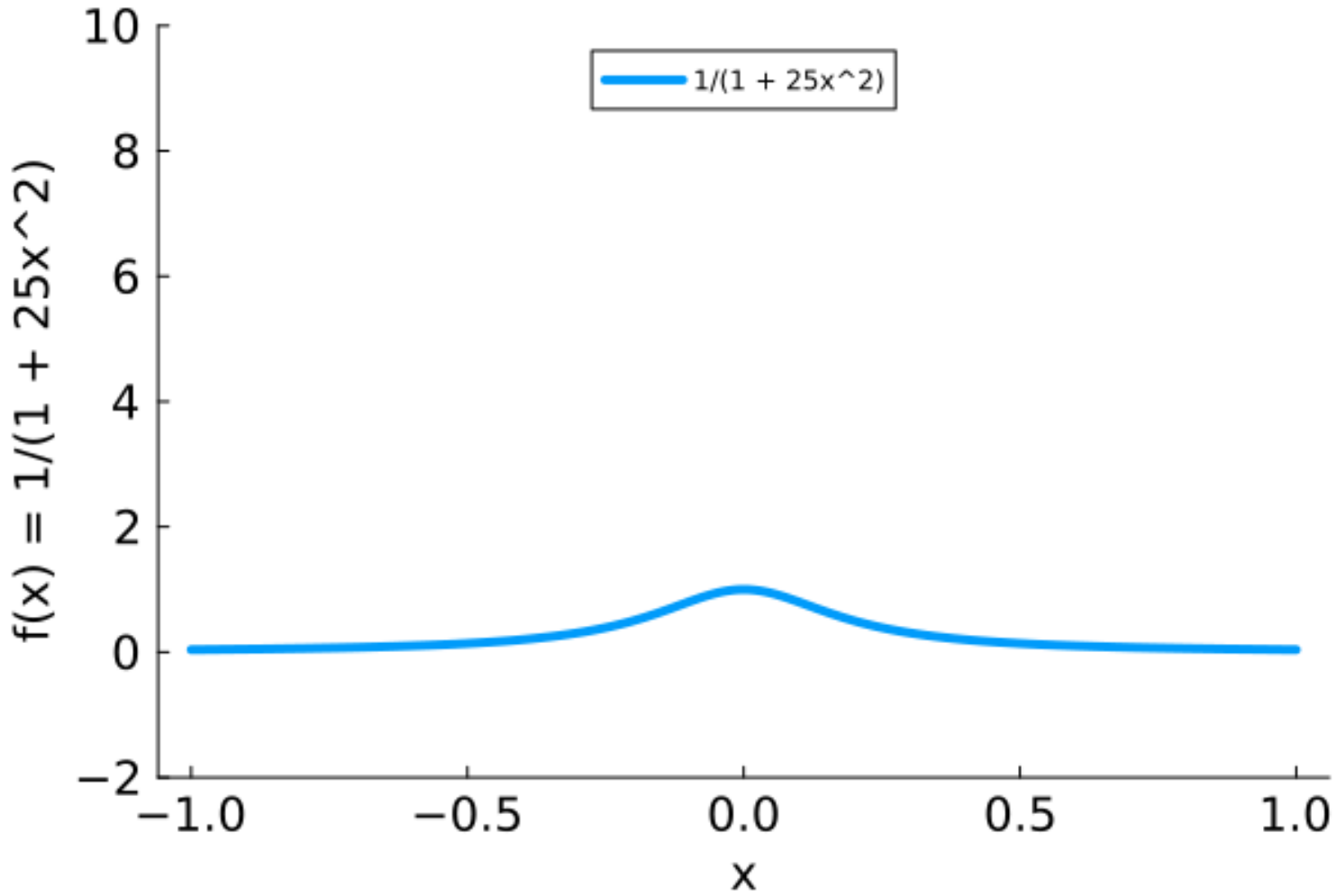

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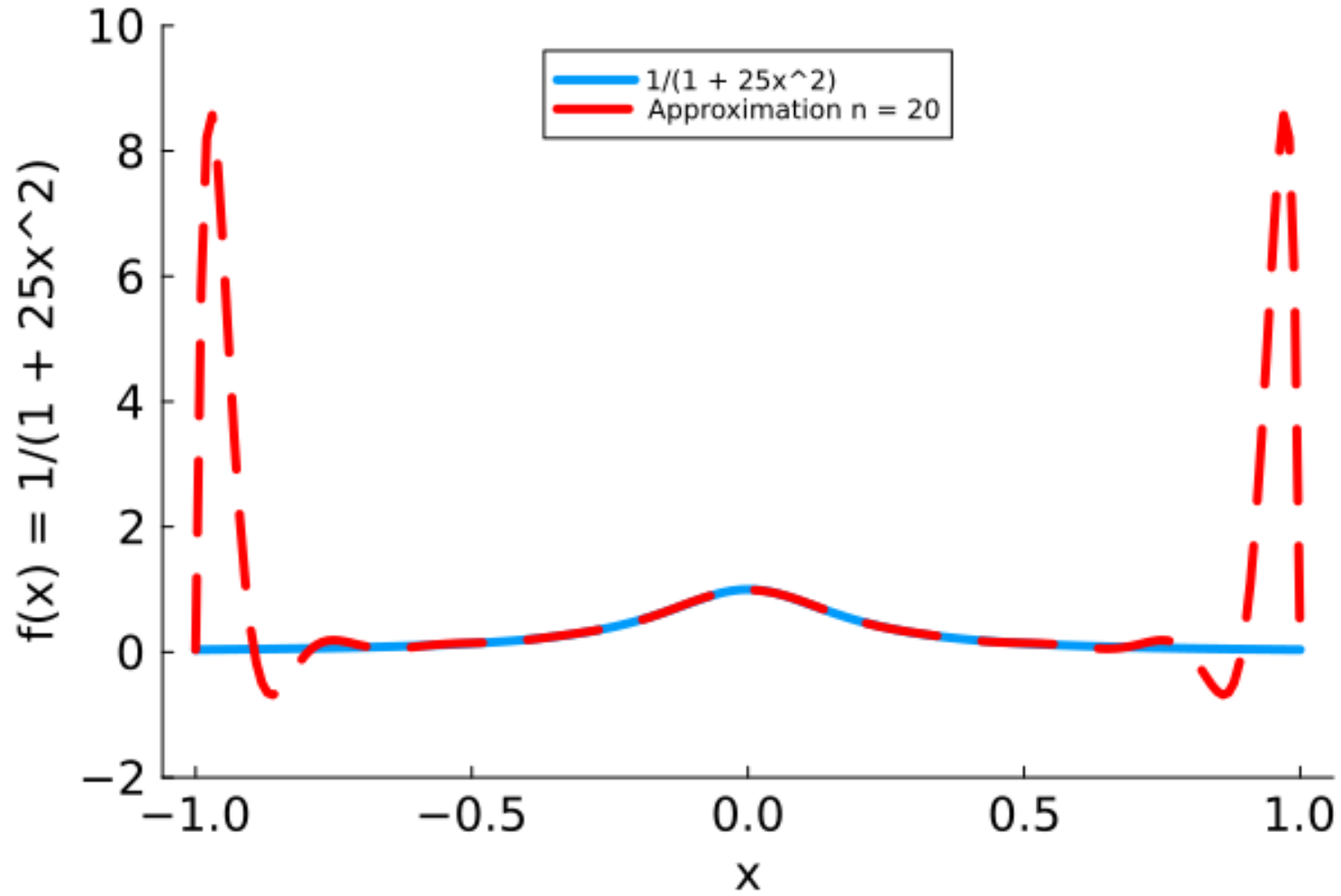

Monomials are not good

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The matrix of monomials, Φ, is often ill-conditioned, especially as the degree of the monomials increases

The first 6 monomials can induce a condition number of 10^{10}, a substantial loss of precision

Second, they can vary dramatically in size, which leads to scaling/truncation errors

Monomials are not good

```
runge(x) = 1 ./ (1 .+ 25x.^2);
coefficients_10 = project_monomial(runge, 10, -1, 1);
points = rand(10);
n = length(coefficients_10) - 1;
basis_functions = [coefficients_10[degree + 1] * points.^degree for degree = 0:n];
basis_matrix = hcat(basis_functions...);
println("Condition number: $(cond(basis_matrix))")
```

```
## Condition number: 6.388871948025998e20
```

Monomials are not good

Example: x^{11} goes from .0001 to about 90 when moving x from 0.5 to 1.5

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Ideally we want an **orthogonal basis:** when we add another element of the basis, it has sufficiently different behavior than the elements before it so it can capture features of the unknown function that the previous elements couldn't

The Chebyshev basis

Most frequently used is the Chebyshev basis

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1. They are easy to compute

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It has nice approximation properties:

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3. They are bounded between $[-1, 1]$

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Chebyshev polynomials are often selected because they minimize the oscillations that occur when approximating functions like Runge's function

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Chebyshev polynomials are often selected because they minimize the oscillations that occur when approximating functions like Runge's function

The Chebyshev polynomial closely approximates the **minimax polynomial**: the polynomial, given degree d, that minimizes any approximation error to the true function

The Chebyshev basis

Chebyshev polynomials are defined by a recurrence relation,

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_{n+1} = 2xT_n(x) - T_{n-1}(x)$$

and are defined on the domain $[-1, 1]$

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Chebyshev polynomials look similar to monomials but have better properties that are visually distinctive

The Chebyshev basis

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$$T_0(x) = 1$$
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Write two functions: `cheb_polys(n, x)` and `monomials(n, x)` with a degree of approximation `n` and vector of points `x`, that return the values of the approximating function at `x`

If you can't get the recurrence relation to work, you can use an alternative:

$$T_n(x) = cos(n\, arccos(x))$$

The Chebyshev basis

Next, a plotting function `plot_function(basis_function, x, n)` that takes an arbitrary basis function `basis_function` and plots all basis functions up to degree `n`

The two basis functions

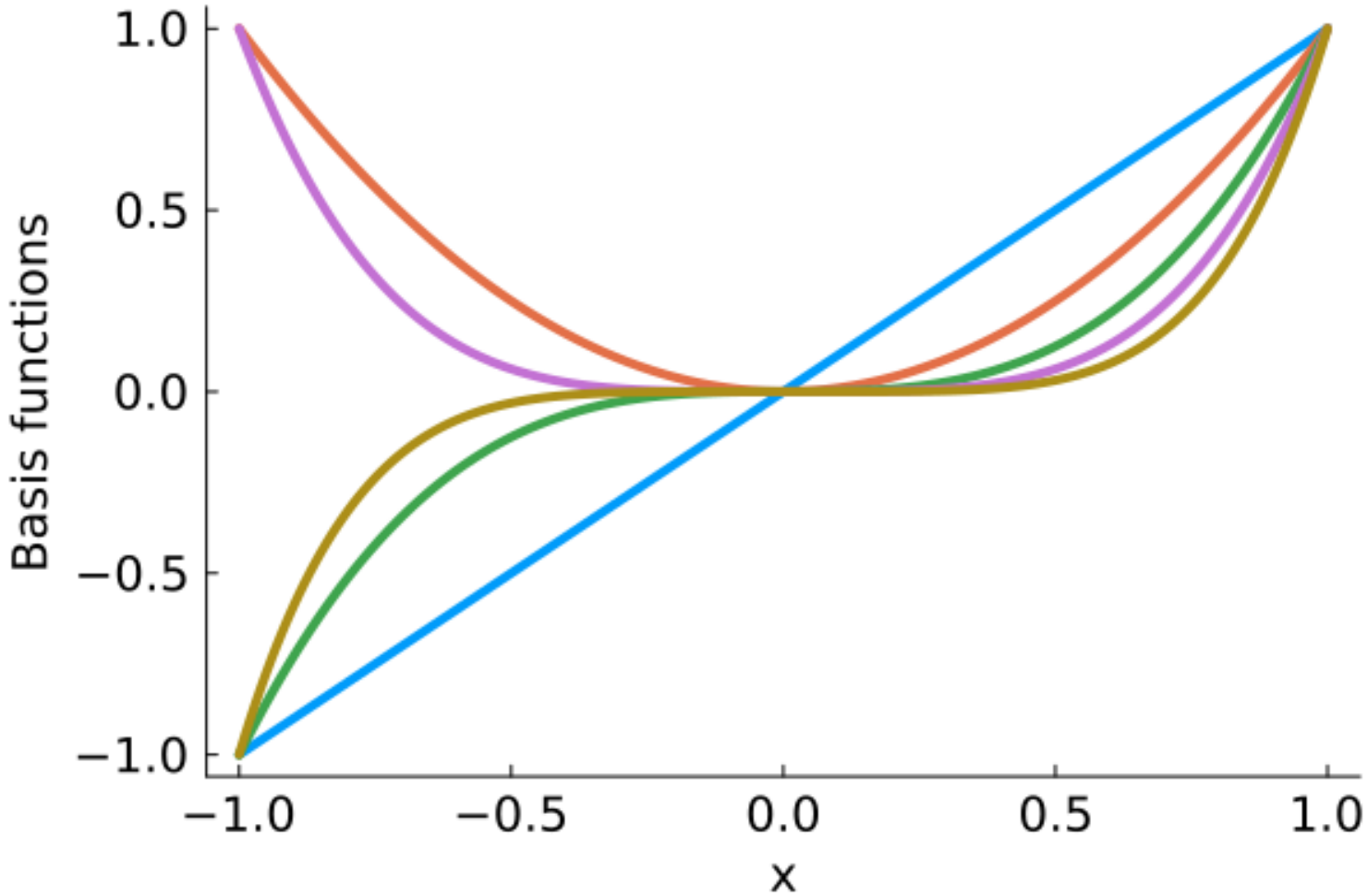
```
# Chebyshev polynomial function
function cheb_polys(x, n)
    if n == 0
        return x ./ x                # T_0(x) = 1
    elseif n == 1
        return x                     # T_1(x) = x
    else
        cheb_recursion(x, n) =
            2x .* cheb_polys.(x, n-1) .- cheb_polys.(x, n-2)
        return cheb_recursion(x, n) # T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)
    end
end;

# Monomial function
monomials(x, n) = x.^n;
```

The plotting function

```
function plot_function(basis_function, x, n)

    for i = 1:n-1

        f_data = basis_function(x, i)

        if i == 1
            plot(x, f_data, linewidth = 4.0, xlabel = "x", ylabel = "Basis functions", label = "
                 tickfontsize = 14, guidefontsize = 14, grid = false);
        else
            plot!(x, f_data, linewidth = 4.0, label = "");
        end

    end

    f_data = basis_function(x, n)
    plot!(x, f_data, linewidth = 4.0, label = "")

end;

x = -1:.01:1;
```

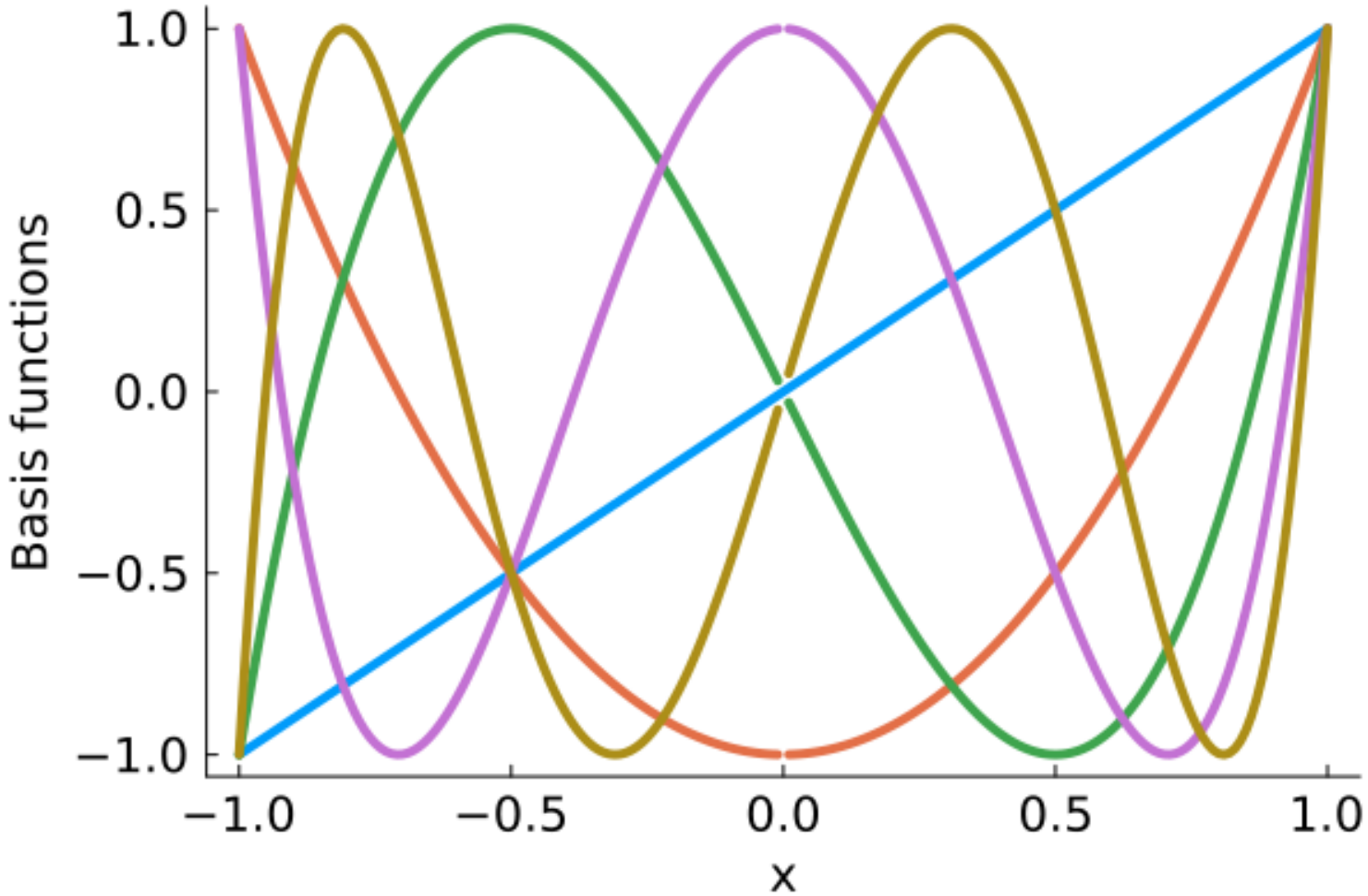
Monomials up to degree 5

```
plot_function(monomials, x, 5)
```



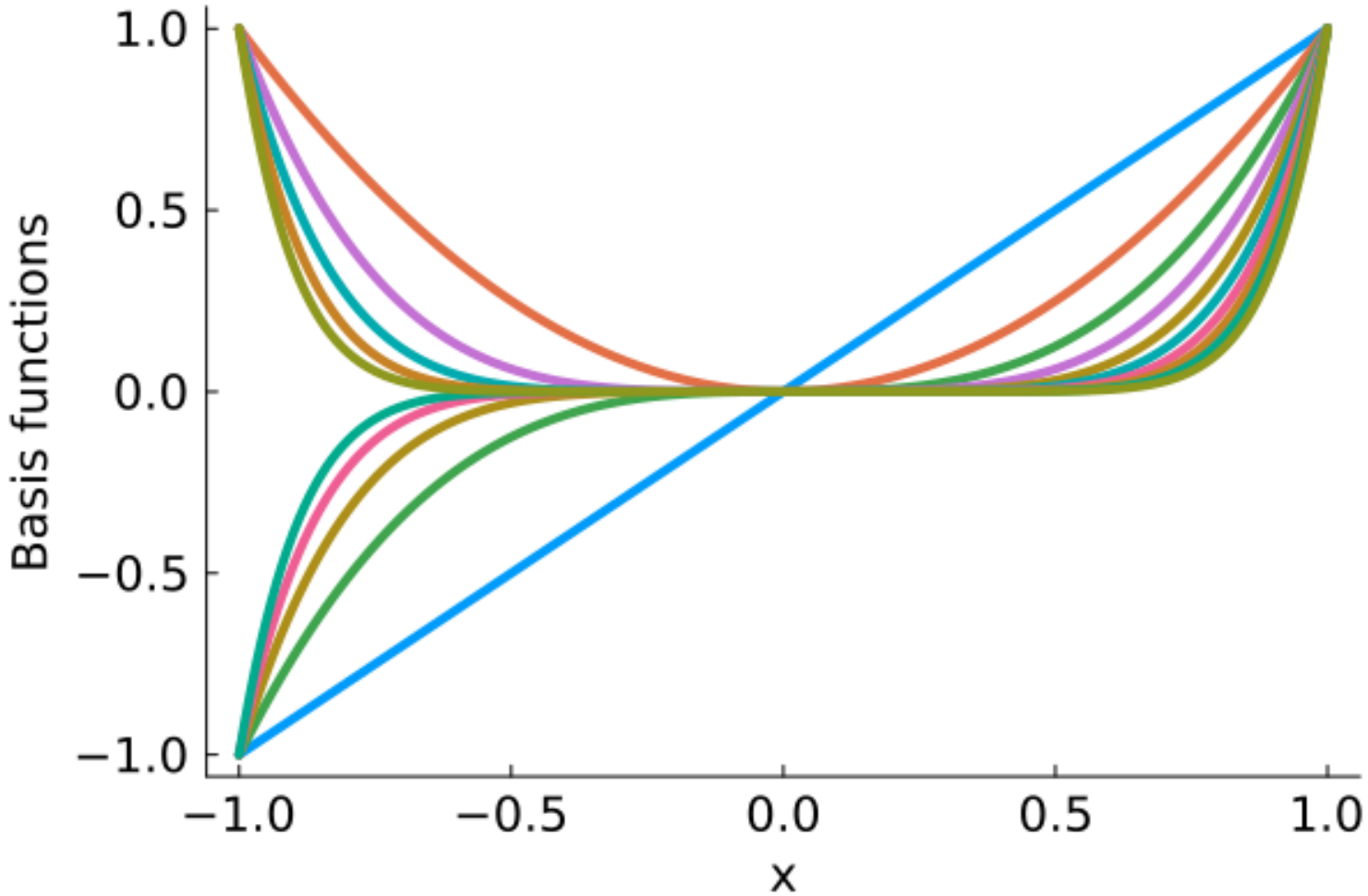
Chebyshev polynomials up to degree 5

```
plot_function(cheb_polys, x, 5)
```



Monomials up to degree 10

```
plot_function(monomials, x, 10)
```



Chebyshev polynomials up to degree 10

```
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```

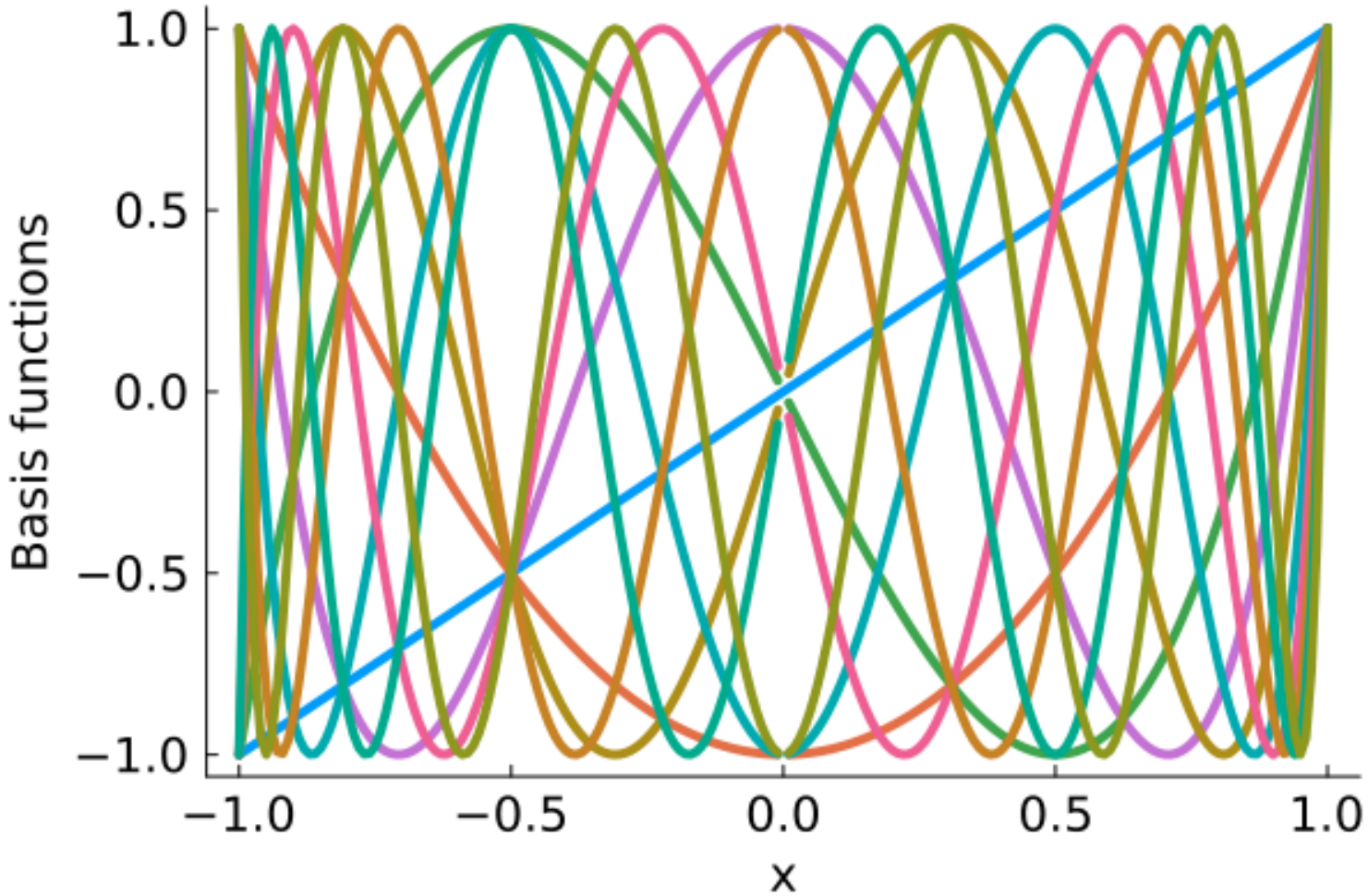

Chebyshev polynomials

Chebyshev polynomials span the space

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It also guarantees that Φ has full rank and is invertible

Two important theorems

There are two important theorems to know about Chebyshev polynomials

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Chebyshev interpolation theorem: *If $f(x) \in \mathbb{C}[a,b]$, if $\{\psi_i(x), i = 0, \dots\}$ is a system of polynomials (where $\psi_i(x)$ is of exact degree i) orthogonal with respect to $\phi(x)$ on $[a,b]$ and if $p_j = \sum_{i=0}^{j} c_i \psi_i(x)$ interpolates $f(x)$ in the zeros of $\psi_{n+1}(x)$, then:*

$$\lim_{j \to \infty} \left(||f - p_j||_2 \right)^2 = \lim_{j \to \infty} \int_a^b \phi(x) \big(f(x) - p_j \big)^2 dx = 0$$

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$$\lim_{j\to\infty} \left(||f - p_j||_2 \right)^2 = \lim_{j\to\infty} \int_a^b \phi(x) \big(f(x) - p_j\big)^2 dx = 0$$

What does this say?

Two important theorems

If we have an approximation set of basis functions that are exact at the roots of the n^{th} order polynomials, then as n goes to infinity the approximation error becomes arbitrarily small and converges at a quadratic rate

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Unfortunately we cant store an infinite number of polynomials in our computer, we would like to know how big our error is after truncating our sequence of polynomials

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So the error in the approximation is as well!

Two important theorems

We often also have that Chebyshev approximations geometrically converge which give us the following convenient property:

$$|f(x) - f^j(x|c)| \sim O(c_j)$$

The truncation error by stopping at polynomial j is of the same order as the magnitude of the coefficient c_j on the last polynomial

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Thus in many situations we can simply check the size of the last polynomial to gauge how accurate our approximation is

Boyd's moral principle

Chebyshev polynomials are the most widely used basis

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This is not purely theoretical but also from practical experience

Boyd's moral principle

John Boyd summarizes decades of experience with function approximation with his moral principle:

- When in doubt, use Chebyshev polynomials unless the solution is spatially periodic, in which case an ordinary fourier series is better
- Unless you are sure another set of basis functions is better, use Chebyshev polynomials
- Unless you are really, really sure another set of basis functions is better use Chebyshev polynomials

Grid point selection

We construct the approximating function by evaluating the basis functions on a predefined grid in the domain of V

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If we have precisely n nodes, x_i, we then have

$$\sum_{j=1}^{n} c_j \phi_j(x_i) = V(x_i) \ \forall i = 1, 2, \ldots, n$$

Grid point selection

We can write this problem more compactly as

$$\Phi c = y \qquad \text{(interpolation equation)}$$

where

- y is the column vector of $V(x_i)$
- c is the column vector of coefficients c_j
- Φ is an $n \times n$ matrix of the n basis functions evaluated at the n points

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How do we select our set of nodes x_i?

Chebyshev strikes again

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They are given by

$$x_k = cos\left(\frac{2k-1}{2n}\pi\right), \ k = 1, \ldots, n$$

for some Chebyshev polynomial of degree n

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Mathematically, these also help reduce error in our approximation

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Mathematically, these also help reduce error in our approximation

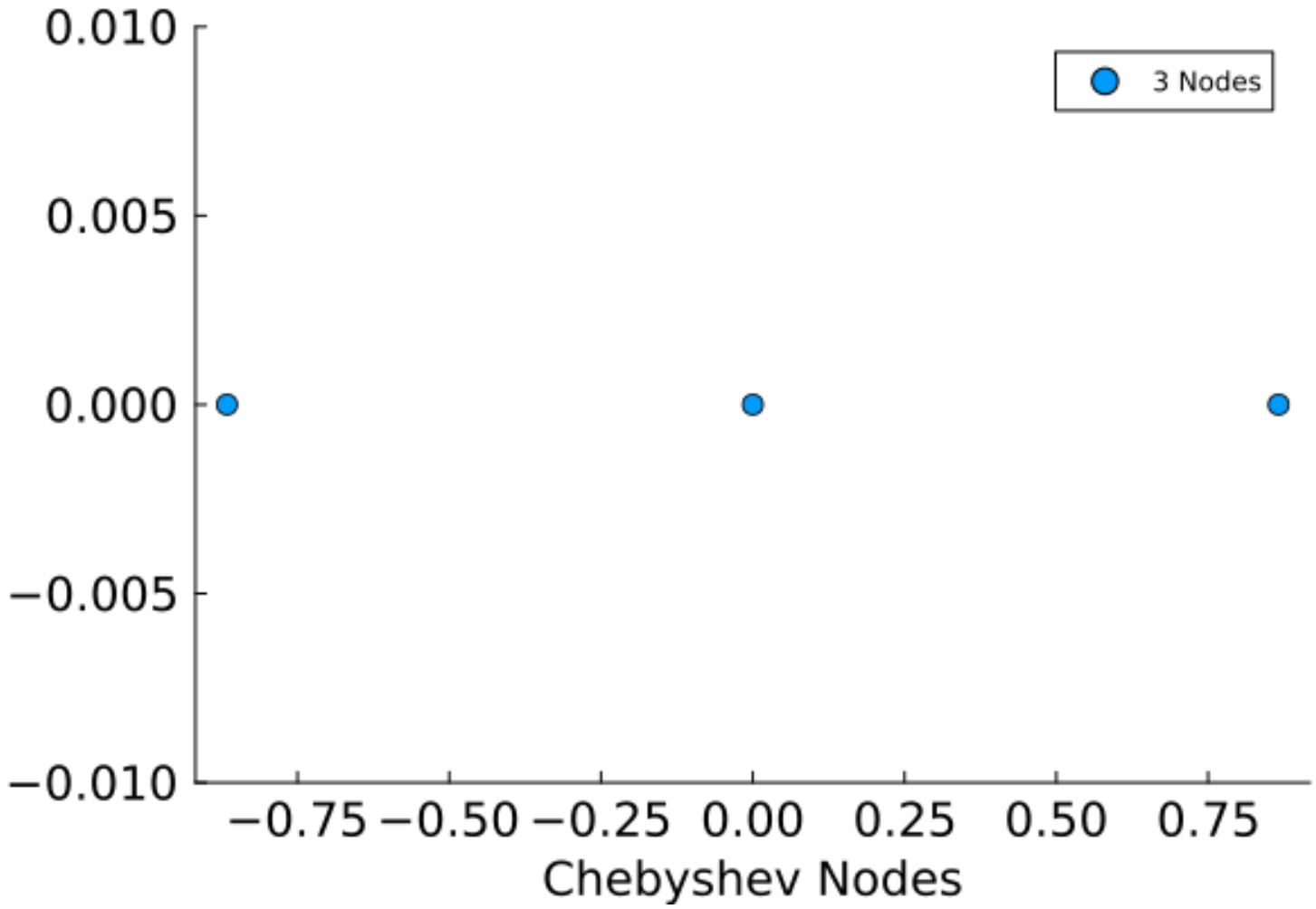
We can gain intuition by looking at a graph of where Chebyshev nodes are located, plot them yourself!

Chebyshev node function

```
cheb_nodes(n) = cos.(pi * (2*(1:n) .- 1)./(2n))
```

```
## cheb_nodes (generic function with 1 method)
```

Chebyshev node locations



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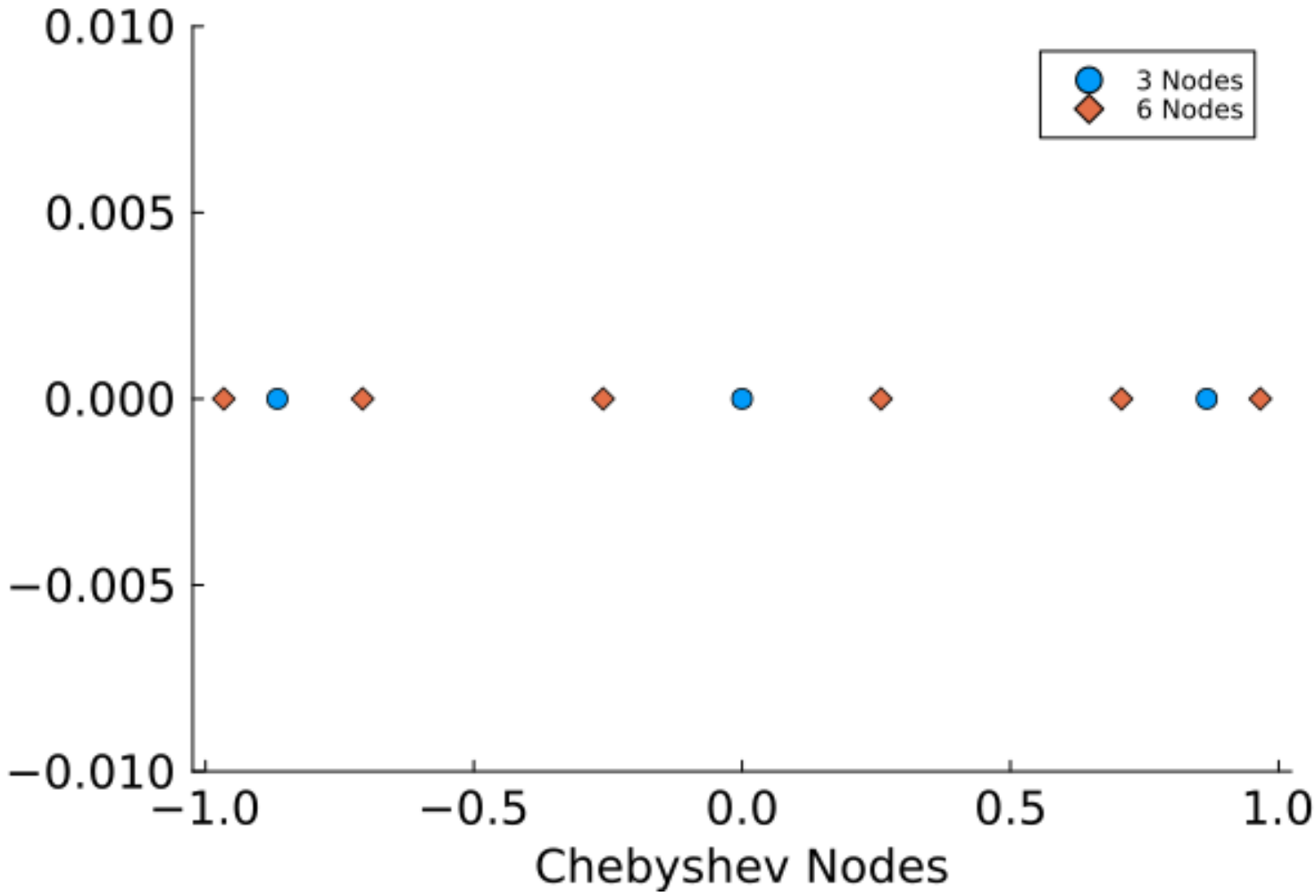

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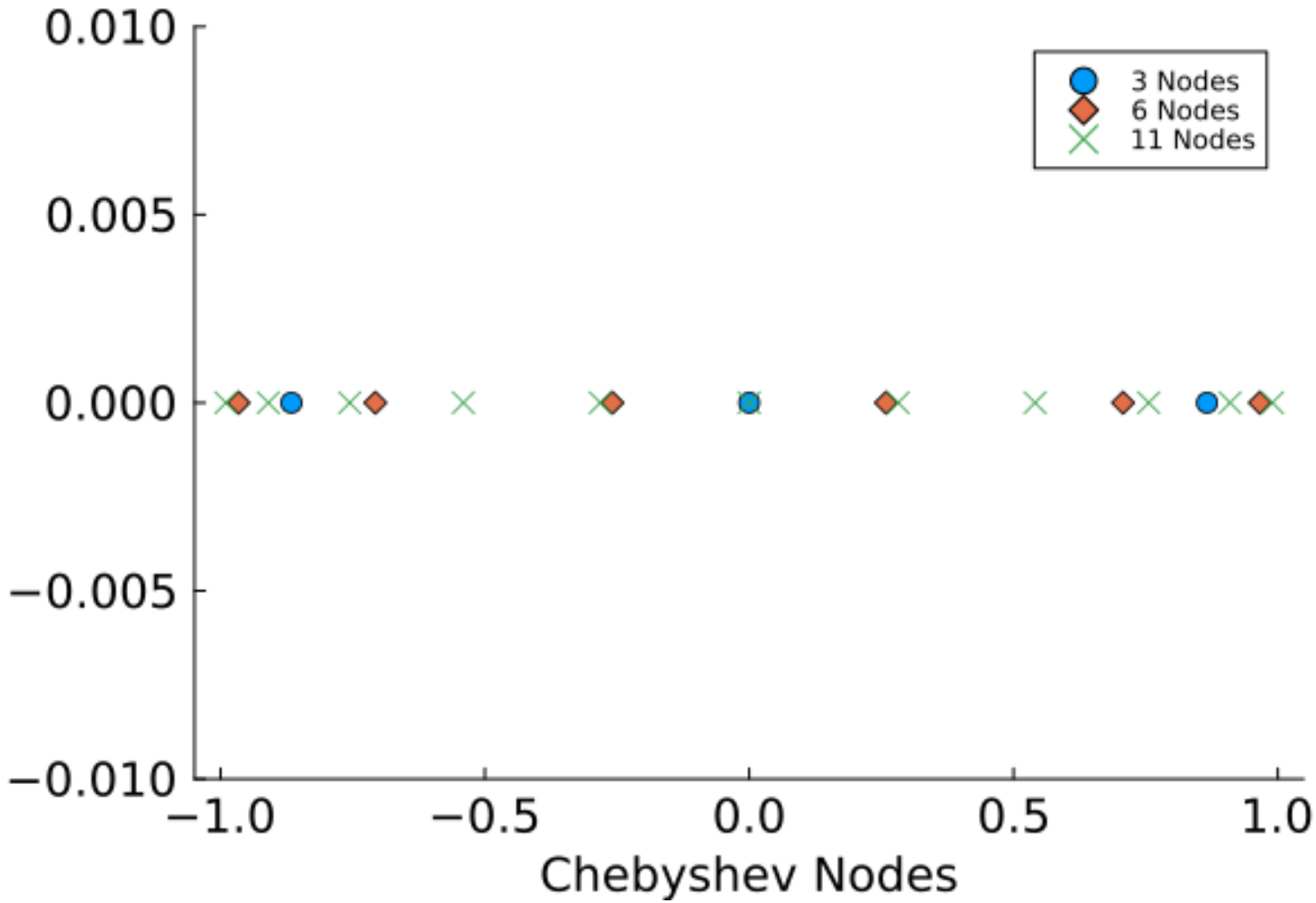

Chebyshev zeros and alternative rep

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Imagine areas of our approximating function near the center of our domain but not at a node

These areas benefit from having multiple nodes on both the left and right

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This provides more information for these off-node areas and allows them to be better approximated because we know whats happening nearby in several different directions

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If we moved to an area closer to the edge of the domain, there may only one node to the left or right of it providing information on what the value of our approximating function should be

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If we moved to an area closer to the edge of the domain, there may only one node to the left or right of it providing information on what the value of our approximating function should be

Therefore, it's best to put more nodes in these areas to shore up this informational deficit and get good approximation quality near the edges of our domain

Discrete states

How do we handle a discrete state S_d when trying to approximate V?

Discrete states

How do we handle a discrete state S_d when trying to approximate V?

Just like you might expect, we effectively have a different approximating function over the continuous states for each value of S_d

Multi-dimensional approximation

Thus far we have displayed the Chebyshev basis in only one dimension

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Construct a vector of polynomials $[\phi_{2,1},\ \phi_{2,2},\ \phi_{2,3}]$ for dimension 2

Multi-dimensional approximation

The tensor is just the product of every possibly polynomial pair which results in:

$$\begin{bmatrix}\phi_{1,1}\phi_{2,1}, & \phi_{1,1}\phi_{2,2}, & \phi_{1,1}\phi_{2,3}, \\ \phi_{1,2}\phi_{2,1}, & \phi_{1,2}\phi_{2,2}, & \phi_{1,2}\phi_{2,3}, \\ \phi_{1,3}\phi_{2,1}, & \phi_{1,3}\phi_{2,2}, & \phi_{1,3}\phi_{2,3}\end{bmatrix}$$

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$$\phi_{1,3}\phi_{2,1},\ \phi_{1,3}\phi_{2,2},\ \phi_{1,3}\phi_{2,3}]$$

We can then solve for the 9 coefficients on these two dimensional polynomials

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$$\text{total } \# \text{ points} = (\text{points per state})^{\# \text{ states}}$$

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Exponential complexity is costly, often called the **Curse of dimensionality**

Finite element methods

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For example, we can use **splines** (piecewise polynomials) over segments of our domains where they are spliced together at prespecified breakpoints, which are called knots

Finite element methods

The higher the order the polynomial we use, the higher the order of derivatives that we can preserve continuity at the knots

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For example, a linear spline yields an approximating function that is continuous, but its first derivatives are discontinuous step functions unless the underlying value function happened to be precisely linear

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For example, a linear spline yields an approximating function that is continuous, but its first derivatives are discontinuous step functions unless the underlying value function happened to be precisely linear

If we have a quadratic spline, we can also preserve the first derivative's continuity at the knots, but the second derivative will be a discontinuous step function

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These are what ensure continuity of higher order derivatives at the knots as the degree of the spline grows

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Each end of a linear segment must equal the function value at the knots

Finite element methods

We have two conditions and two unknowns for each segment: this is a simple set of linear equations that we can solve

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In numerical models we typically don't use linear splines because we often care about the quality of approximation of higher order derivatives, cubic splines are more common

Cubic splines

Suppose we wish to approximate using a cubic spline on $N + 1$ knots

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We can obtain $3(N - 1)$ equations by ensuring that the approximating function is continuous at all interior knots, and its first and second derivatives are continuous at all interior knots $[3 \times (N + 1 - 1 - 1)]$

This means that the value of the left cubic polynomial equals the value of the right cubic polynomial at each interior knot

Cubic splines

Ensuring the approximating function equals the function's value at all of the nodes adds another $N + 1$ equations

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We therefore have a total of $4N - 2$ equations for $4N$ coefficients

Cubic splines

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What is often used is that the approximating function's first or second derivative matches that of the function at the end points

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Useful spline packages out there: `Dierckx`, `Interpolations`, `QuantEcon`

Code it up!

Let's code up our own linear spline approximation function `linear_spline_approx(f, knots)`, where `f` is the function we are approximating and `knots` are the knots

Have it return a function a function `spline_eval` that takes in `evaluation_points` as an argument where `evaluation_points` are the points we want to evaluate the spline approximating function at

Hint: Linear splines are pretty easy, given two points (x_{i+1}, y_{i+1}) and (x_i, y_i), the spline in between these points is given by

$$y(x) = y_i + \frac{y_{i+1} - y_i}{x_{i+1} - x_i}(x - x_i)$$

Spline approximator

```
function linear_spline_approx(f, knots)
    function spline_eval(evaluation_points)
        prev_knot = knots[1] # initialize previous knot
        if !(typeof(evaluation_points) <: Number) # if using multiple points
            y_eval = similar(evaluation_points)
            y_index = 1
            for knot in knots[2:end]
                current_points = evaluation_points[prev_knot .<= evaluation_points .< knot]
                y_eval[y_index:y_index + length(current_points) - 1] =
                    f(prev_knot) .+ (f(knot) - f(prev_knot))/(knot - prev_knot)*(current_points
                prev_knot = knot
                y_index += length(current_points)
            end
        else # if using just a single point
            for knot in knots[2:end]
                if prev_knot .<= evaluation_points .< knot
                    y_eval = f(prev_knot) + (f(knot) - f(prev_knot))/(knot - prev_knot)*(evaluat
                end
                prev_knot = knot
            end
        end
```

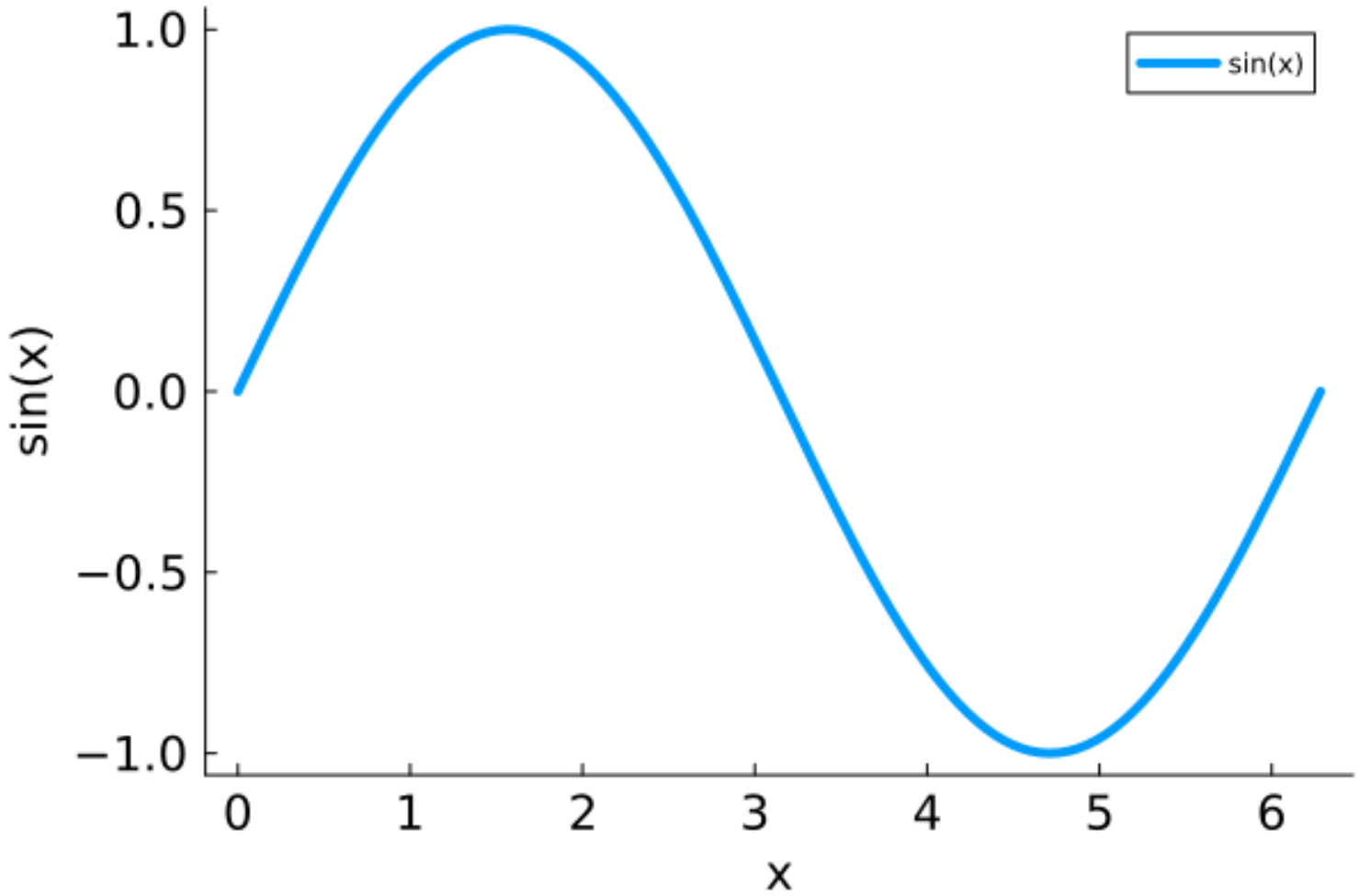
Plot

```
f(x) = sin(x)
```

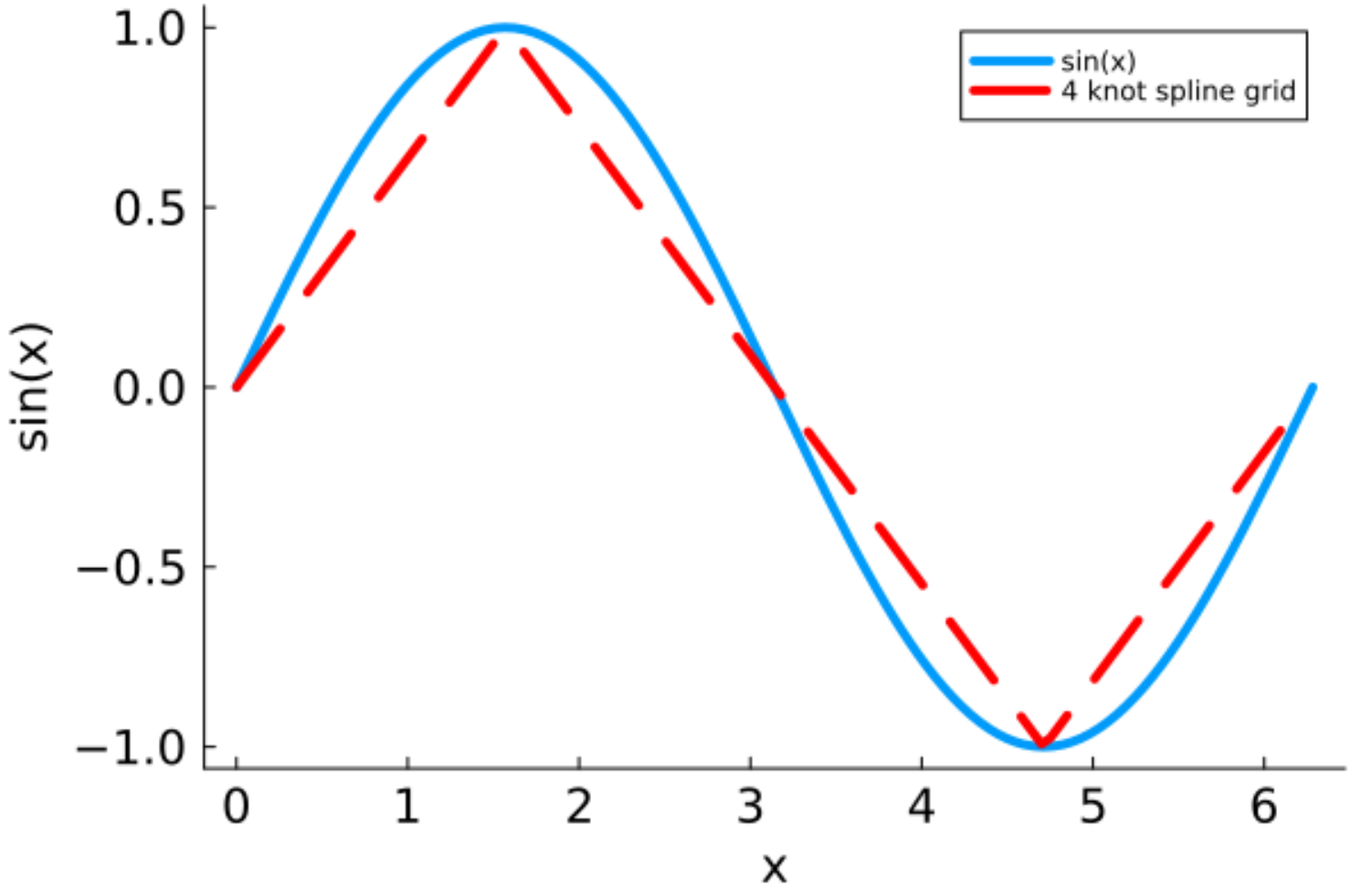
```
## f (generic function with 1 method)
```

```
knots_coarse = 0:pi/2:2pi;
spline_func_coarse = linear_spline_approx(f, knots_coarse);
knots_fine = 0:pi/4:2pi;
spline_func_fine = linear_spline_approx(f, knots_fine);
knots_superfine = 0:pi/12:2pi;
spline_func_superfine = linear_spline_approx(f, knots_superfine);
x_vals =0:.05:2pi;
y_vals_coarse = spline_func_coarse(x_vals);
y_vals_fine = spline_func_fine(x_vals);
y_vals_superfine = spline_func_superfine(x_vals);
```

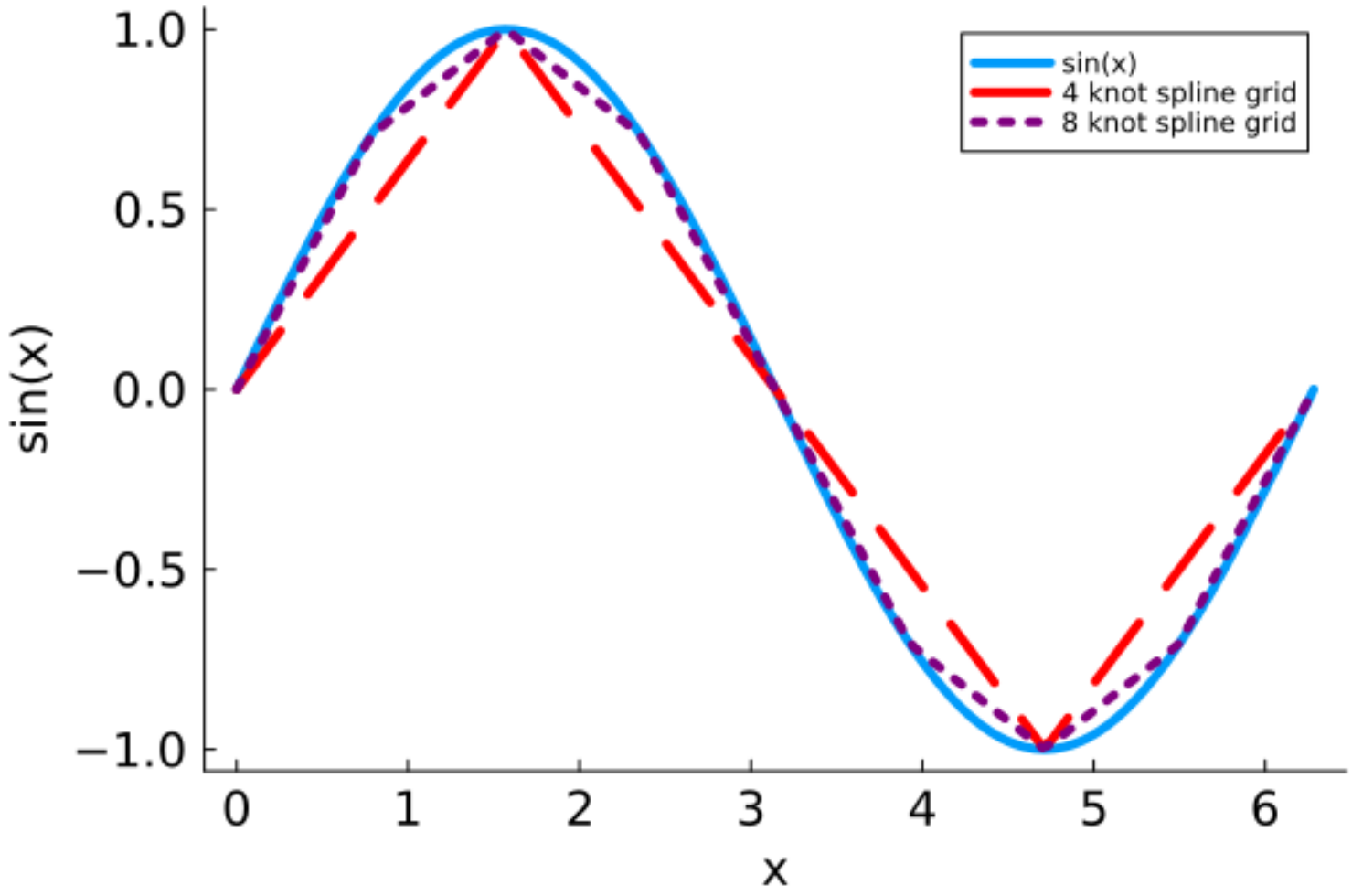
Plot



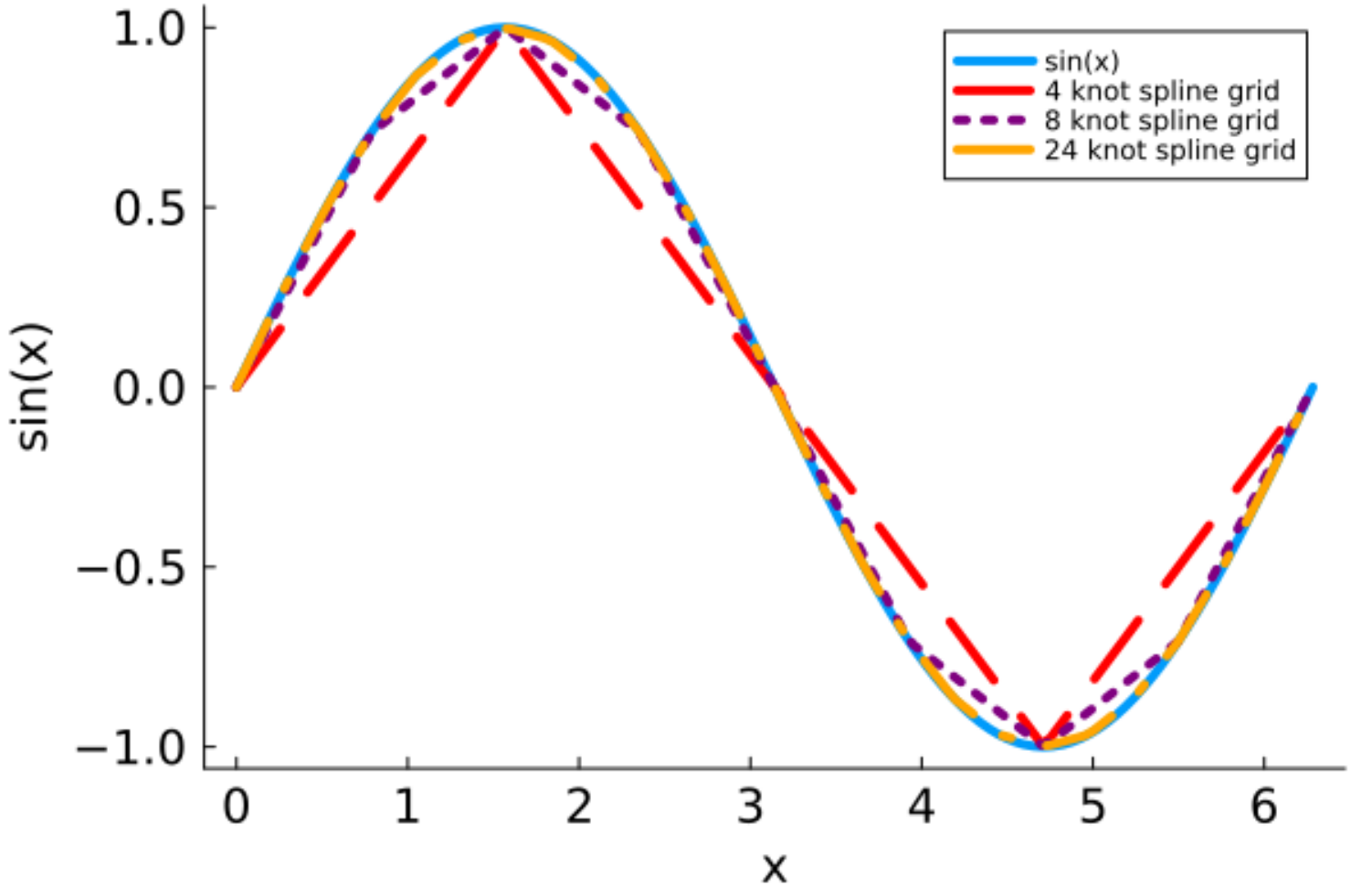
Plot



Plot


sin(x)
4 knot spline grid
8 knot spline grid
1.0
0.5
0.0
−0.5
−1.0
sin(x)
0
1
2
3
4
5
6
x

Plot


sin(x)
4 knot spline grid
8 knot spline grid
24 knot spline grid
1.0
0.5
0.0
−0.5
−1.0
sin(x)
0
1
2
3
4
5
6
x