Lecture 7

Continuous time dynamic models

Ivan Rudik
AEM 7130

Roadmap

1. The theory behind continuous time models
2. Numerical methods for solving continuous time model

Model setup

Consider a problem where each period an agent obtains flow utility $J(x(t), u(t))$, where x is our state and u is our control

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The agent's objective is to maximize the total payoff, subject to the transitions of the states

$$\max_{u, x_T} \int_0^T J(x(t), u(t))\, dt$$
$$\textbf{subject to: } \dot{x}(t) = g(x(t), u(t)),\ x(0) = x_0,\ x(T) = x_T$$

This is an open-loop solution so we optimize our entire policy trajectory from time $t = 0$

Hamiltonians

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This function is called the *Hamiltonian*

$$H(x(t), u(t), \lambda(t)) \equiv J(x(t), u(t)) + \lambda(t) g(x(t), u(t))$$

Hamiltonians

Pontryagin's Maximum Principle states that the following conditions are necessary for an optimal solution,

$$\begin{aligned}
\frac{\partial H(x(t),u(t),\lambda(t))}{\partial u} &= 0 \ \forall t \in [0,T] && \text{(Maximality)}\\
\frac{\partial H(x(t),u(t),\lambda(t))}{\partial x} &= -\dot{\lambda}(t) && \text{(Co-state)}\\
\frac{\partial H(x(t),u(t),\lambda(t))}{\partial \lambda} &= \dot{x}(t) && \text{(State transitions)}\\
x(0) &= x_0 && \text{(Initial condition)}\\
\lambda(T) &= 0 && \text{(Transversality)}
\end{aligned}$$

What do these conditions mean?

Necessary conditions

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The decisionmaker can use her control to increase the contemporaneous flow utility and reap immediate rewards, or to alter the state variable to increase future rewards

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It defines how the shadow value of our state transition, called the *co-state variable*, evolves over time

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Why?

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If it didn't, we could increase value by accumulating more of the stock variable
$\rightarrow$ what we were doing cannot be optimal

We can re-write the co-state equation as

$$\frac{\partial J}{\partial x} + \lambda(t)\frac{\partial g}{\partial x} + \dot{\lambda}(t) = 0$$

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The immediate value is made up of the actual utility payoff (first term),
and the future utility payoff payoff from how increasing the stock today affects the stock in the future (second term)

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We need additional optimality conditions to use as constraints to impose the optimal path

Pinning down the optimal path

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We do this using **transversality conditions**

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Usually terminal conditions free and initial conditions are not

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It must be zero, why?

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If H were positive, the value is increasing and the agent can do better by delaying ending the problem

If H were negative, the value is decreasing and the agent can do better by ending the problem earlier

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If it were positive the policymaker could profitably deviate by altering the level of the stock. Finally, these are all necessary conditions of the problem.

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Present value refers to the value with respect to a specific period that we call the present

Current value terms

Our previous necessary conditions apply to present value Hamiltonians,
but let us analyze a current value Hamiltonian to avoid including time terms,

$$H^{cv}(x(t), u(t), \mu(t)) \equiv e^{rt} H(x(t), u(t), \lambda(t), t) = e^{rt} J(x(t), u(t), t) + e^{rt} \lambda(t) g(x(t), u(t))$$

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We can then re-write our necessary conditions in current value by substituting in for the shadow value (which implies that $\dot{\lambda}(t) = -re^{-rt}\mu(t) + e^{-rt}\dot{\mu}(t)$), and for $\partial H/\partial x = e^{-rt}\,\partial H^{cv}/\partial x$ into our co-state condition

$$e^{-rt}\frac{\partial H^{cv}(x(t),u(t),\mu(t))}{\partial x} = e^{-rt}\left[r\mu(t) - \dot{\mu}(t)\right]$$

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If discounting is high (large r), then the current shadow value must change quicker in order to compensate the policymaker for leaving stock for the future

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n determines the number of differential equations that we have

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IVPs are defined by the function being pinned down at one end or the other

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If $n > 1$ then we can have a *boundary value problem* where we impose n conditions on y

$$g_i(y(t_0)) = 0, \quad i = 1, \ldots, .n',$$
$$g_i(y(T)) = 0, \quad i = n' + 1, \ldots, n$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$

Numerical methods for continuous time models

In general we have that

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for some set of points t_i, $t_0 \leq t_i \leq T, 1 \leq i \leq n$

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In general we have that

$$g_i(y(t_i)) = 0$$

for some set of points t_i, $t_0 \leq t_i \leq T$, $1 \leq i \leq n$

Often we set $T = \infty$ so we will need some condition in the limit: $\lim_{t\to\infty} y(t)$

Numerical methods for continuous time models

Note two more things:

1. We are implicitly assuming that these n conditions are independent, otherwise we will not have a unique solution
2. IVPs and BVP are fundamentally different: IVPs are problems where the auxiliary conditions that pin down the solution are all at one point, in BVPs they can be at different points, this has significant implications for how we can solve the problems

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In general you can always transform a nth-order ODE into n first-order ODEs

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Our goal is to find for each t_i, a value Y_i that closely approximates $y(t_i)$

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This approximates the solution only at the grid points, but then we can interpolate using standard procedures to get the approximate solution off the grid points

Euler's method

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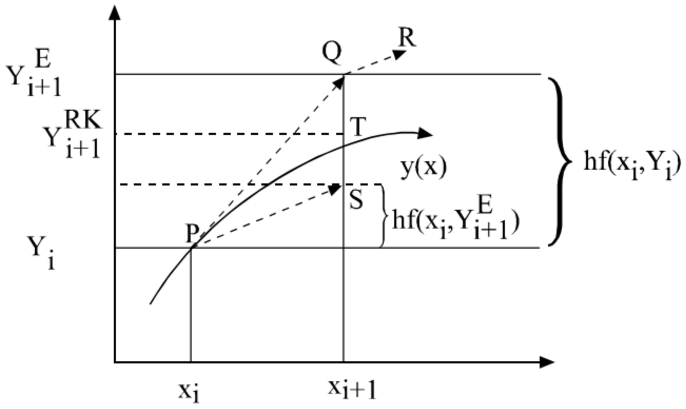
Euler's method is the difference equation

$$Y_{i+1} = Y_i + hf(t_i, Y_i)$$

where Y_0 is given by the initial condition

Euler's method



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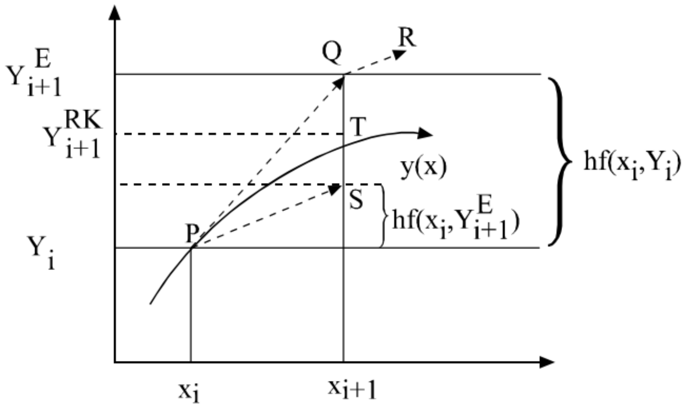

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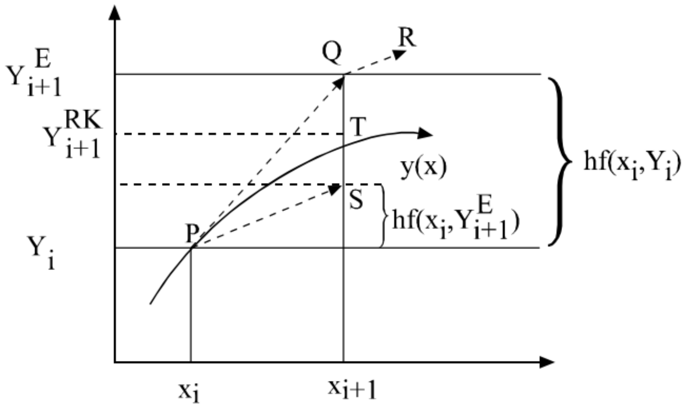

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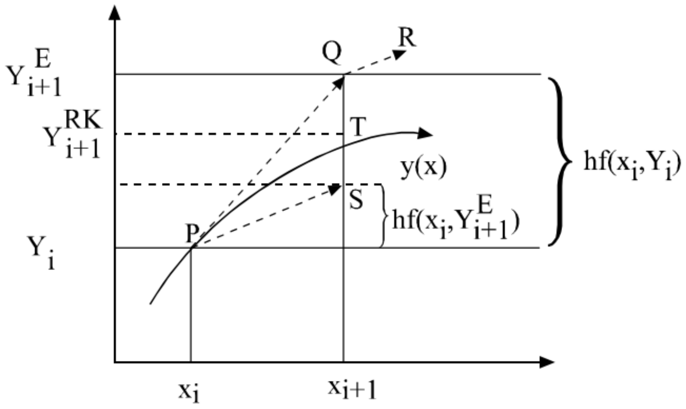

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The Euler estimate of $y(t_{i+1})$ is then Y_{i+1}^{E}

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If $y(t)$ is the true solution, the second order Taylor expansion around t_i is

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(\xi)$$

for some $\xi \in [t_i, t_{i+1}]$

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For small h, $y(x)$ should be a close approximation to the solution of the truncated Taylor expansion,so Y_i should be a good approximation to $y(t_i)$

Euler's method

This approach approximated $y(t)$ with a linear function with slope $f(t_i, Y_i)$ on the interval $[t_i, t_{i+1}]$

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We can motivate Euler's method with an integration argument instead of a Taylor expansion argument

The fundamental theorem of calculus tells us that

$$y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} f(s, y(s))ds$$

Euler's method

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As $h \to 0$, we are back in the ODE world

Euler's method errors

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This difference equation has solution $Y_i = (1 + h)^i$ and implies the approximation is $Y(t) = (1 + h)^{t/h}$

Euler's method errors

Thus the relative error between the two is

$$log(|Y(t)/y(t)|) = \frac{t}{h} log(1+h) - t = \frac{t}{h}(h - h^2 + \dots) - t = -th + \dots$$

where excluded terms have order higher than h

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In general we can show that Euler's method has linear convergence

Suppose the solution to $y'(t) = f(t, y(t)), y(t_0) = y_0$ *is* C^3 *on* $[t_0, T]$*, that* f *is* C^2 *, and that* f_y *and* f_{yy} *are bounded for all* y *and* $t_0 \leq t \leq T$*. Then the error of the Euler scheme with step size* h *is* $O(h)$

Implicit Euler method

We expanded y around t_i, but we could always expand around t_{i+1} so that we have

$$y(t_i) = y(t_{i+1}) - hy'(t_{i+1}) = y(t_{i+1}) - hf(t_{i+1}, y(t_{i+1}))$$

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Notice that now Y_{i+1} is only implicitly defined in terms of t_i and Y_i so we will need to solve a non-linear equation in Y_{i+1}

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Because of this we can typically use larger h's with the implicit Euler method

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We could instead use the slope at (t_{i+1}, Y_{i+1}^{E}) but this will give the same problem but in the opposite direction, we will undershoot

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A first-order Runge-Kutta method will take the average of these two slopes to arrive at the formula

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There are higher order Runge-Kutta rules that have even more desirable properties outlined in Judd (1998)

Boundary Value Problems

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BVPs have auxiliary conditions that are imposed at different points in time so we lose the local nature of the problem and our solutions must now be global in nature

Boundary Value Problems

Consider the following BVP

$$\begin{aligned} \dot{x} &= f(x, y, t) \\ \dot{y} &= g(x, y, t) \\ x(t_0) = x_0, &\quad y(T) = y_T \end{aligned}$$

where $x \in \mathbb{R}^n, y \in \mathbb{R}^m$

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We cannot use standard IVP approaches because at $\boldsymbol{t_0}$ or $\boldsymbol{T}$ we only know the value of either $\boldsymbol{x}$ or $\boldsymbol{y}$ but not both

Thus we cannot find the next value of both of them using only local information: we need alternative approaches

Boundary Value Problems: Shooting

The core method for solving BVPs is called **shooting**

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There are two components to a shooting method

Shooting

First we guess some $y(0) = y_0$ and then solve the IVP problem with methods we've already used

$$\begin{aligned} \dot{x} &= f(x, y, t) \\ \dot{y} &= g(x, y, t) \\ x(t_0) = x_0, \quad y(0) &= y_0 \end{aligned}$$

to find some $y(T)$ which we call $Y(T, y_0)$ since it depends on our initial guess y_0

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We want to find a y_0 such that $y_T = Y(T, y_0)$

This is a nonlinear equation in y_0 so we need to solve nonlinear equations

Shooting

We can write the algorithm as

1. Initialize: Guess y_0^i. Choose a stopping criterion $\epsilon > 0$
2. Solve the IVP for $x(T), y(T)$ given the initial condition $y_0 = y_0^i$
3. If $||y(T) - y_T|| < \epsilon$, STOP. Else choose y_0^{i+1} based on the previous values of y and go back to step 1

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In the outer layer (step 2) we solve the nonlinear equation $Y(T, y_0) = y_T$

We can use any nonlinear solver here, typically we do this by defining a subroutine that computes $Y(T, y_0) - y_T$ as a function of y_0 and then sends that subroutine to a rootfinding program

Example: Lifecycle model

A simple lifecycle model is given by

$$\begin{aligned} \max_{c(t)} \int_0^T e^{-rt} u(c(t)) dt \\ s.t. \quad \dot{A}(t) = f(A(t)) + w(t) - c(t) \\ A(0) = A(T) = 0. \end{aligned}$$

$u(c(t))$ is utility from consumption, $w(t)$ is the wage rate, $A(t)$ are assets and $f(A(t))$ is the return on invested assets

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We assume that assets are initially and terminally zero where the latter would come about naturally from a transversality condition

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$$\dot{\lambda}(t) = r\lambda(t) - \lambda(t)f'(A(t))$$

The maximum principle implies that $u'(c(t)) = \lambda(t)$

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The issue here is that we never know A and λ at either $t = 0$ or $t = T$

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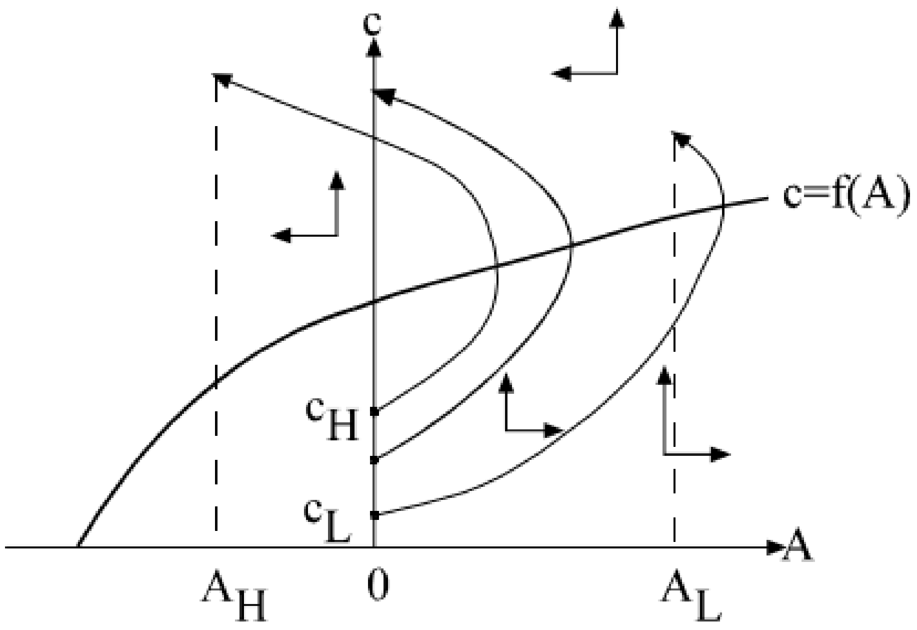
The issue here is that we never know A and λ at either $t = 0$ or $t = T$

We can use the maximum principle to convert the costate condition into a condition on consumption

$$\dot{c}(t) = -\frac{u'(c(t))}{u''(c(t))}\left[f'(A(t)) - r\right]$$

Example: Lifecycle model

The Figure shows the phase diagram assuming that $f'(A) > r$ for all A



If $A(T) < 0$ when we guess $c(0) = c_H$, but $A(T) > 0$ when we guess $c(0) = c_L$, we know the correct guess lies in between and we can solve for it using the bisection method

Reverse shooting for ∞ horizon problems

The standard infinite horizon optimal control problem is

$$\begin{aligned}
\max_{u(t)} \int_0^\infty & e^{-rt}\pi(x(t),u(t))dt \\
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We replace it with a transversality condition that $\lim_{t\to\infty} e^{-rt}|\lambda(t)^T x(t)| \leq \infty$

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But this implies something very convenient: that the initial state corresponding to any terminal state is not very sensitive to the value of the terminal state

Thus what we will do is not guess the value of the initial condition and integrate forward, we will guess the terminal condition and integrate **backward**

Example: Reverse shooting for ∞ horizon problems

Consider the simplest growth model

$$\begin{aligned} \max_{c(t)} \int_0^\infty & e^{-rt} u(c(t)) dt \\ \dot{k}(t) &= f(k(t)) - c(t) \\ s.t. \quad k(0) &= k_0, \end{aligned}$$

where c is consumption, k is the capital stock, and f is production

Example: Reverse shooting for ∞ horizon problems

We can use Pontryagin's necessary conditions to get that consumption and capital are governed by the following differential equations

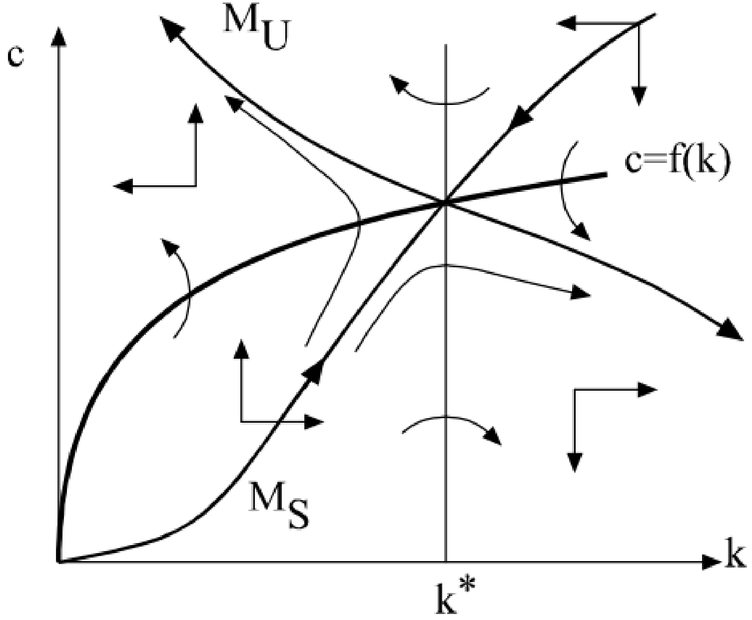
$$\dot{c}(t) = -\frac{u'(c(t))}{u''(c(t))}(f'(k) - r)$$
$$\dot{k}(t) = f(k(t)) - c(t),$$

with boundary conditions,

$$k(0) = k_0, \quad 0 < \lim_{t\to\infty} |k(t)| \leq \infty$$

Example: Reverse shooting for ∞ horizon problems

Assume u and f are concave, the Figure shows the phase diagram for the problem



We have a steady state at $k = k^*$, this occurs when $f'(k(t)) = r$ and $c^* = f(k^*)$

Example: Reverse shooting for ∞ horizon problems

For this problem there exists a stable manifold M_S and an unstable manifold M_U so that the steady state is **saddle point stable**

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For this problem there exists a stable manifold M_S and an unstable manifold M_U so that the steady state is **saddle point stable**

Both are invariant manifolds because any system that starts on either of these manifolds will continue to move along the manifold

However M_S is stable because it will converge to the steady state while M_U diverges away from the steady state

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Suppose we start with $k_0 < k^*$, if we guess $c(0)$ too big we will cross the k isoquant and have a falling capital stock, but if we guess $c(0)$ too small we will get a path that crosses the c isoquant and results in a falling consumption level

Example: Reverse shooting for ∞ horizon problems

This gives us our algorithm

1. Initialize: set $c_H = f(k_0)$ and set $c_L = 0$, choose a stopping criterion $\epsilon > 0$
2. Set $c_0 = \frac{1}{2}(c_L + c_H)$
3. Solve the IVP with initial conditions $c(0) = c_0, k(0) = k_0$. Stop the IVP at the first t when $\dot{c}(t) < 0$ or $\dot{k}(t) < 0$, denote this T
4. If $|c(T) - c^*| < \epsilon$, STOP. If $\dot{c}(t) < 0$, set $c_L = c_0$, else set $c_H = c_0$. Go to step 2.

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This algorithm makes sense but the phase diagram shows why it will have trouble finding the stable manifold

Any small deviation from M_S is magnified and results in a path that increasingly gets far away from M_S

Unless we happen to pick a point precisely on the stable manifold we will move away from it, so it is hard to search for the solution since changes in our guesses will lead to wild changes in terminal values

Example: Reverse shooting for ∞ horizon problems

Now suppose we wanted to find a path on M_U, notice that the flow pushes points *toward* M_U so the deviations are smushed together

Example: Reverse shooting for ∞ horizon problems

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We don't actually want to solve for a path on M_U but this gives us some insight

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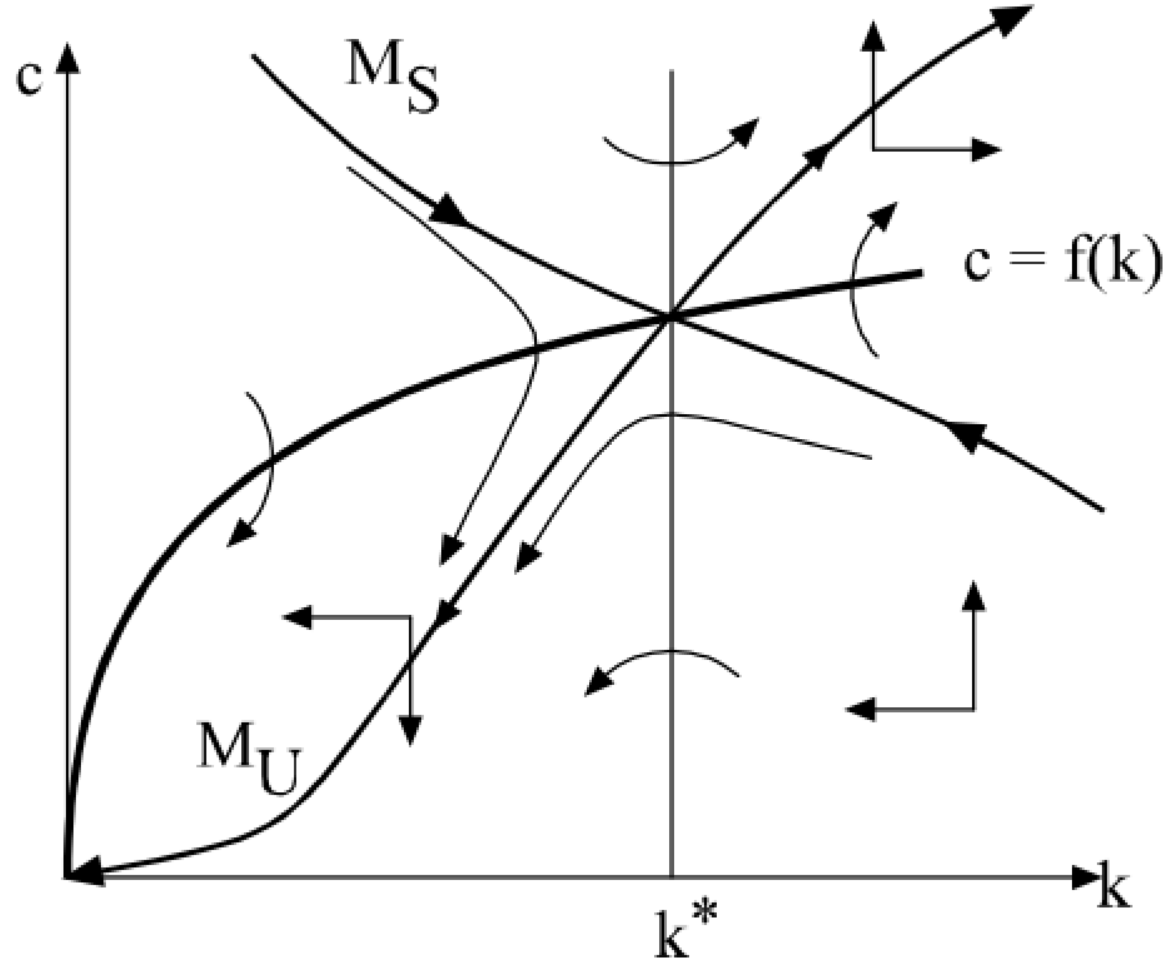
This allows us to exploit how paths tend to converge toward the stable manifold while going away from the steady state

Reverse shooting for ∞ horizon problems



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We can express this as a policy function $C(k)$ such that $c(t) = C(k(t))$

Optimality requires that the capital stock converge to the steady state, and the only optimal $(c(t), k(t))$ pairs are on the stable manifold so the stable manifold **is the policy function** $C(k)$