Lecture 8

Continuous time dynamic models

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AEM 7130

Roadmap

1. The theory behind continuous time models
2. Numerical methods for solving continuous time model

Model setup

Consider a problem where each period an agent obtains flow utility $J(x(t), u(t))$, where x is our **state** and u is our **control**

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Suppose there is a finite horizon with a **terminal time** T

Model setup

The agent's objective is to maximize the total payoff, subject to the transitions of the states

$$\max_{u, x_T} \int_0^T J(x(t), u(t))\, dt$$
$$\text{subject to: } \dot{x}(t) = g(x(t), u(t)),\ x(0) = x_0,\ x(T) = x_T$$

This is an open-loop solution so we optimize our entire policy trajectory from time $t = 0$

We will not be solving for functions of states, but functions of time: $u(t), x(t)$

Hamiltonians

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This function is called the *Hamiltonian*:

$$H(x(t), u(t), \lambda(t)) \equiv J(x(t), u(t)) + \lambda(t) g(x(t), u(t))$$

It is a function that treats the transitions as quasi-constraints so it appears similar to the Lagrangian you know

Hamiltonians

Pontryagin's Maximum Principle states that the following conditions are necessary for an optimal solution:

$$
\begin{aligned}
\frac{\partial H(x(t), u(t), \lambda(t))}{\partial u} &= 0 \ \forall t \in [0, T] && \text{(Maximality)} \\
\frac{\partial H(x(t), u(t), \lambda(t))}{\partial x} &= -\dot{\lambda}(t) && \text{(Co-state)} \\
\frac{\partial H(x(t), u(t), \lambda(t))}{\partial \lambda} &= \dot{x}(t) && \text{(State transitions)} \\
x(0) &= x_0 && \text{(Initial condition)} \\
\lambda(T) &= 0 && \text{(Transversality)}
\end{aligned}
$$

What do these conditions mean?

Necessary conditions

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The decisionmaker can use her control to increase the contemporaneous flow utility and reap immediate rewards, or to alter the state variable to increase future rewards

$$H(x(t),u(t),\lambda(t)) \equiv \underbrace{J(x(t),u(t))}_{\text{current flow}} + \underbrace{\lambda(t)g(x(t),u(t))}_{\text{change in future value}}$$

Necessary conditions

$$\frac{\partial H(x(t), u(t), \lambda(t))}{\partial u} = 0 \ \forall t \in [0, T]$$

The **maximality condition**: in every instant, we select the control so that we can no longer increase our total payoff

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The **maximality condition**: in every instant, we select the control so that we can no longer increase our total payoff

It effectively sets the net marginal benefits of the control to zero

Necessary conditions

$$\frac{\partial H(x(t), u(t), \lambda(t))}{\partial \lambda} = \dot{x}(t)$$

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Taking the derivative of the Hamiltonian with respect to the shadow value, just like a Lagrangian, yields this constraint back

Necessary conditions

$$\frac{\partial H(x(t), u(t), \lambda(t))}{\partial x} = -\dot{\lambda}(t)$$

The **co-state condition** defines how the shadow value of our state transition, called the **co-state variable**, evolves over time

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Then the shadow value of that stock (λ) must decrease along an optimal trajectory

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Suppose we increase today's stock of x by one unit and this increases the instantaneous change in our value (H)

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Why?

Necessary conditions

If it didn't, we could increase value by accumulating more of the stock variable
$\rightarrow$ there is a profitable deviation and what we were doing cannot be optimal

We can re-write the co-state equation as

$$\frac{\partial J}{\partial x} + \lambda(t)\frac{\partial g}{\partial x} + \dot{\lambda}(t) = 0$$

Necessary conditions

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We must have that the a unit of the stock's value must change (third term), so that is exactly offsets the change in value from increasing the stock in the immediate instant of time

The immediate value is made up of the actual utility payoff (first term), and the future utility payoff payoff from how increasing the stock today affects the stock in the future (second term)

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Many different paths are consistent with these differential equations, depends on the constant of integration

We need additional optimality conditions to use as constraints to impose the optimal path

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We do this using **transverality conditions**

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Usually terminal conditions are free and initial conditions are not

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If it were positive the policymaker could profitably deviate by altering the level of the stock. Finally, these are all necessary conditions of the problem

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Present value refers to the value with respect to a specific period that we call the present

Current value terms

Our previous necessary conditions apply to present value Hamiltonians,
but let us analyze a current value Hamiltonian to avoid including time terms,

$$\begin{aligned} H^{cv}(x(t), u(t), \mu(t)) &\equiv e^{rt} H(x(t), u(t), \lambda(t), t) \\ &= e^{rt} J(x(t), u(t), t) + e^{rt}\lambda(t) g(x(t), u(t)) \end{aligned}$$

$\mu(t)$ is the shadow value λ brought into current value terms: $\mu(t) = e^{rt}\lambda(t)$

Current value terms

We can then re-write our necessary conditions in current value by substituting in for:

- the shadow value (which implies that $\dot{\lambda}(t) = -re^{-rt}\mu(t) + e^{-rt}\dot{\mu}(t)$)
- $\partial H/\partial x = e^{-rt}\,\partial H^{cv}/\partial x$ into our co-state condition:

$$e^{-rt}\,\frac{\partial H^{cv}(x(t), u(t), \mu(t))}{\partial x} = e^{-rt}\left[r\mu(t) - \dot{\mu}(t)\right]$$

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Before, the present value form of the co-state condition required the change in the present shadow value precisely equal the effect of the state variable on instantaneous value

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If discounting is high (large r), then the current shadow value must change quicker in order to compensate the policymaker for leaving stock for the future

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n determines the number of differential equations that we have

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IVPs are defined by the function being pinned down at one end or the other

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If $n > 1$ then we can have a *boundary value problem* where we impose n conditions on y

$$g_i(y(t_0)) = 0, \quad i = 1, \ldots, .n',$$
$$g_i(y(T)) = 0, \quad i = n' + 1, \ldots, n$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$

Numerical methods for continuous time models

In general we have that

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Often we set $T = \infty$ so we will need some condition in the limit: $\lim_{t\to\infty} y(t)$

Numerical methods for continuous time models

Note two more things:

1. We are implicitly assuming that these n conditions are independent, otherwise we will not have a unique solution
2. IVPs and BVP are fundamentally different: IVPs are problems where the auxiliary conditions that pin down the solution are all at one point, in BVPs they can be at different points, this has significant implications for how we can solve the problems

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for $x, y \in \mathbb{R}$

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of two first-order ODEs

In general you can always transform a nth-order ODE into n first-order ODEs

Finite difference methods for IVPs

We solve IVPs using finite-difference methods

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Assume the grid is uniformly spaced: $t_i = t_0 + ih,\ i = 0, 1, \ldots, N$ where h is the mesh size

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Assume the grid is uniformly spaced: $t_i = t_0 + ih,\ i = 0, 1, \ldots, N$ where h is the mesh size

Our goal is to find for each t_i, a value Y_i that closely approximates $y(t_i)$

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To do this, we replace our differential equation with a difference system on the grid

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This approximates the solution only at the grid points, but then we can interpolate using standard procedures to get the approximate solution off the grid points

Euler's method

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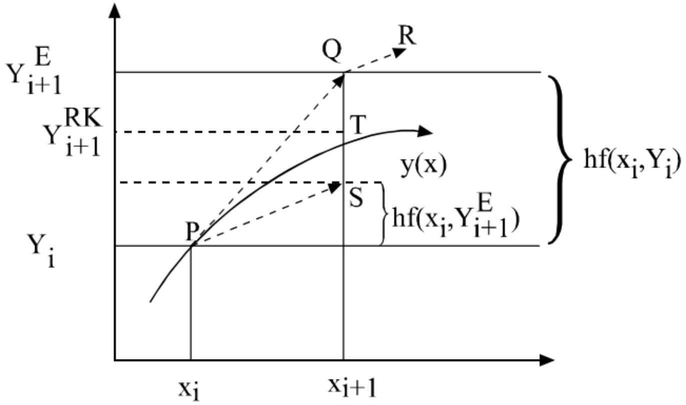
Euler's method is the difference equation

$$Y_{i+1} = Y_i + hf(t_i, Y_i)$$

where Y_0 is given by the initial condition

Euler's method



Suppose the current iterate is $P = (t_i, Y_i)$ and $y(t)$ is the true solution

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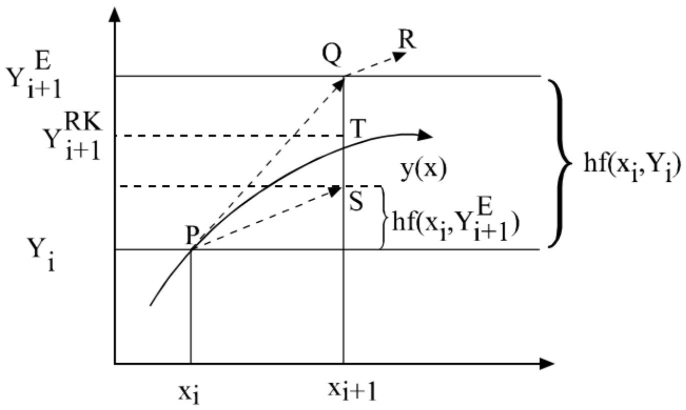

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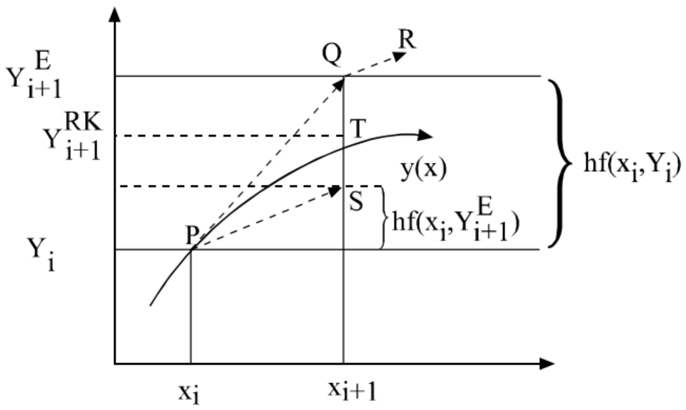

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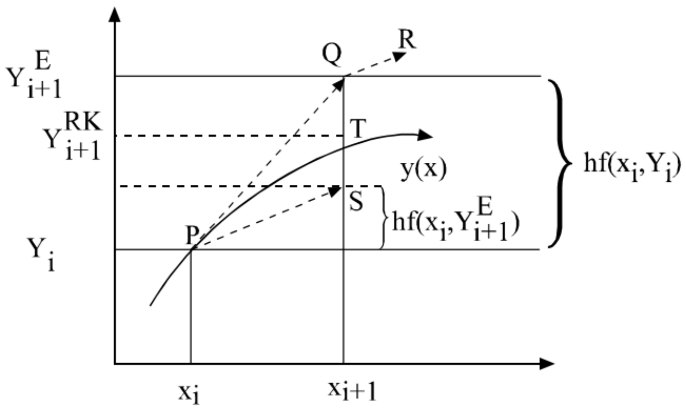

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The Euler estimate of $y(t_{i+1})$ is then Y_{i+1}^E

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If $y(t)$ is the true solution, the second order Taylor expansion around t_i is

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(\xi)$$

for some $\xi \in [t_i, t_{i+1}]$

Euler's method

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(\xi)$$

If we drop the second order term and assume $f(t_i, Y_i) = y'(t_i)$ and $Y_i = y(t_i)$ we have exactly Euler's formula

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For small $h, y(x)$ should be a close approximation to the solution of the truncated Taylor expansion,so Y_i should be a good approximation to $y(t_i)$

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We can motivate Euler's method with an integration argument instead of a Taylor expansion argument

The fundamental theorem of calculus tells us that

$$y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} f(s, y(s))ds$$

Euler's method

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If we approximate the integral with $hf(t_i, y(t_i))$, a box of width h and height $f(t_i, y(t_i))$, then $y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i))$ which implies the Euler method difference equation above if $Y_i = y(t_i)$

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As $h \to 0$, we are back in the ODE world

Euler's method errors

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This difference equation has solution $Y_i = (1+h)^i$ and implies the approximation is $Y(t) = (1+h)^{t/h}$

Euler's method errors

Thus the relative error between the two is

$$\log(|Y(t)/y(t)|) = \frac{t}{h}\log(1+h) - t = \frac{t}{h}(h - h^2 + \dots) - t = -th + \dots$$

where excluded terms have order higher than h

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Thus the relative error in the Euler approximation has order h and as h goes to zero so does the approximation error

Euler's method errors

In general we can show that Euler's method has linear convergence

Suppose the solution to $y'(t) = f(t, y(t)), y(t_0) = y_0$ *is* C^3 *on* $[t_0, T]$, *that* f *is* C^2 , *and that* f_y *and* f_{yy} *are bounded for all* y *and* $t_0 \leq t \leq T$. *Then the error of the Euler scheme with step size* h *is* $O(h)$

Euler's method code

Euler's method code

```
function euler_ode(df, t0, y0, h, n)

    t = zeros(n+1)
    y = zeros(n+1)

    # set the initial values
    t[1] = t0
    y[1] = y0

    # use Euler's method to approximate the solution at each step
    for i in 1:n
        t[i+1] = t[i] + h
        y[i+1] = y[i] + h * df(t[i], y[i])
    end

    return (t, y)

end
```

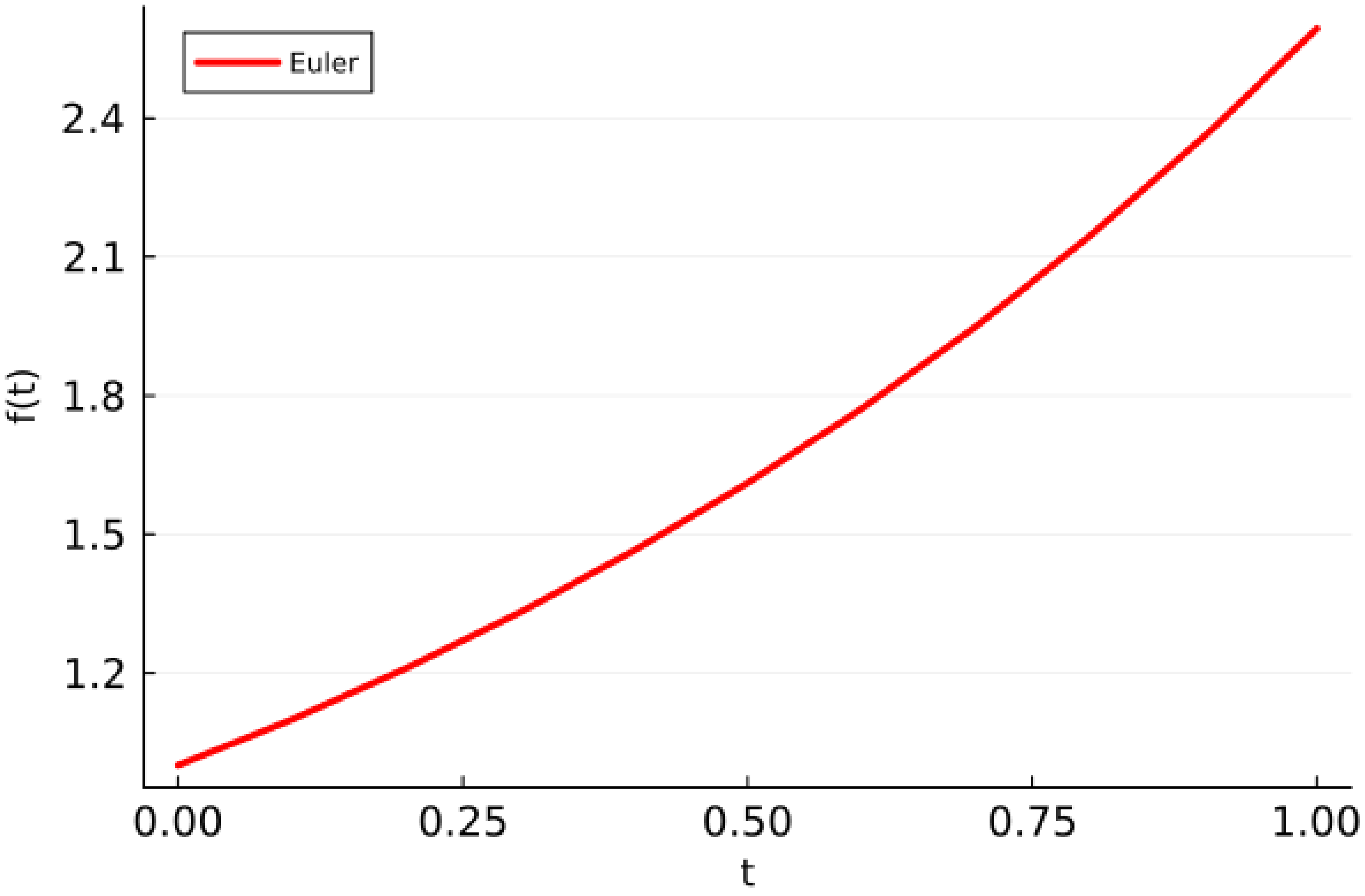
```
## euler_ode (generic function with 1 method)
```

Euler's method code

```
# dy/dt = y -> y = C_0*exp(t)
df(t, y) = y
t1, y1 = euler_ode(df, 0., 1., .1, 10)
```

Define df/dt and send it to the `euler_ode` function

Euler's method code



Implicit Euler method

We expanded y around t_i, but we could always expand around t_{i+1} so that we have

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This yields the implicit Euler method

$$Y_{i+1} = Y_i + hf(t_{i+1}, Y_{i+1})$$

Notice that now Y_{i+1} is only implicitly defined in terms of t_i and Y_i so we will need to solve a non-linear equation in Y_{i+1}

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Thus the implicit Euler method will get us better approximation properties, often times much better

Because of this we can typically use larger h's with the implicit Euler method

Implicit Euler's method code

```
# rootfinding portion of implicit Euler
function find_euler_root(df, y, t, h, y0, tol)

    y_new = y0
    y_old = y0
    error = Inf

    while error > tol
        y_new = y + h * df(t, y_new)
        error = abs((y_new - y_old)/y_old)
        y_old = deepcopy(y_new)
    end

    return y_new

end
```

```
## find_euler_root (generic function with 1 method)
```

Implicit Euler's method code

```
function euler_implicit_ode(df, t0, y0, h, n, tol = 1e-6)

    t = zeros(n+1)
    y = zeros(n+1)
    t[1] = t0
    y[1] = y0

    for i in 1:n
        t[i+1] = t[i] + h
        y[i+1] = find_euler_root(df, y[i], t[i+1], h, y[i], tol)
    end

    return (t, y)

end
```

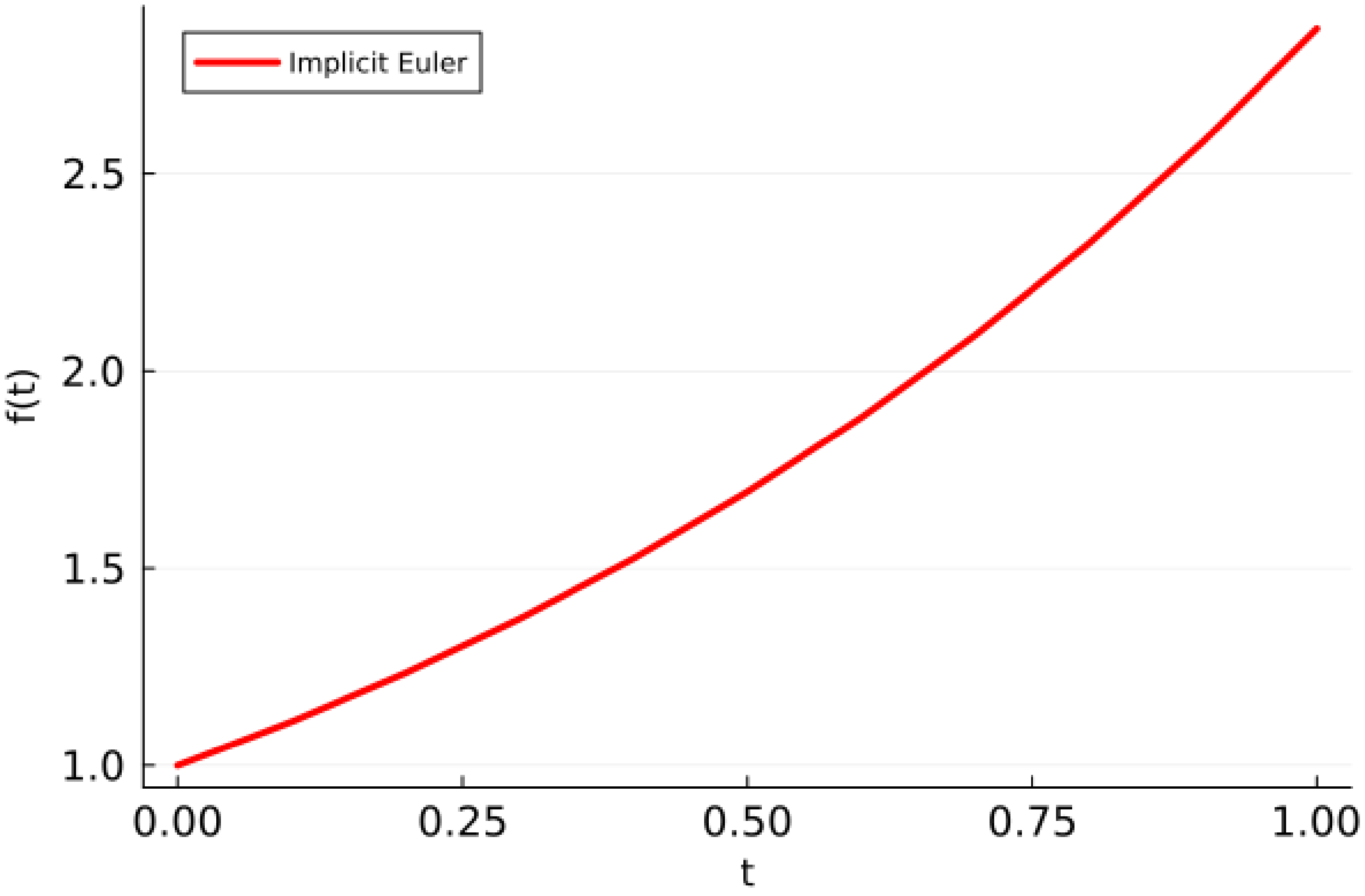
```
## euler_implicit_ode (generic function with 2 methods)
```

Implicit Euler's method code

```
df(t, y) = y
t2, y2 = euler_implicit_ode(df, 0., 1., .1, 10, 1e-6)
```

Define df/dt and send it to the `euler_implicit_ode` function

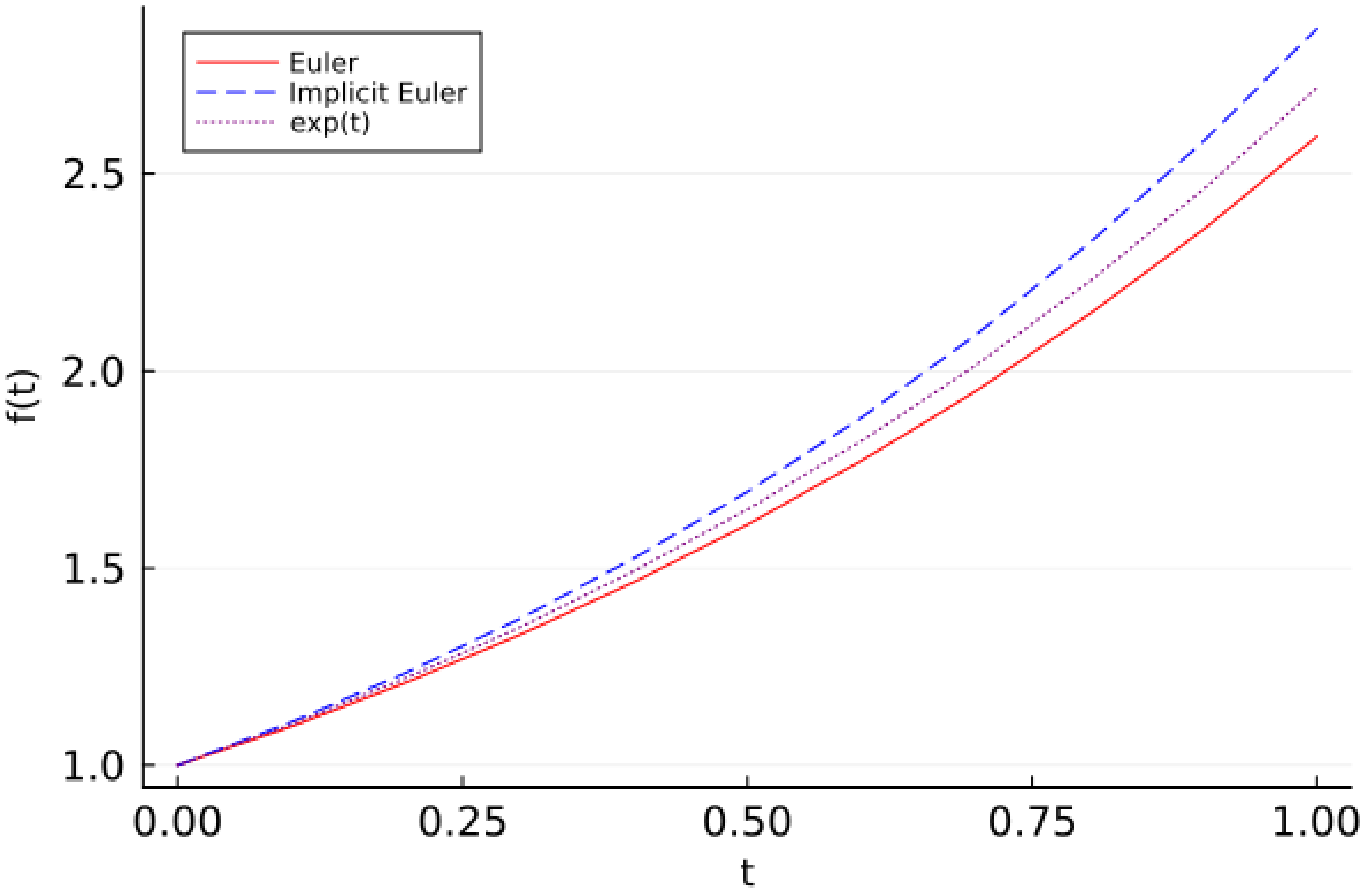
Implicit Euler's method code



Comparison

```
df(t, y) = y
t1, y1 = euler_ode(df, 0., 1., .1, 10)
t2, y2 = euler_implicit_ode(df, 0., 1., .1, 10, 1e-6)
y_real = exp.(t1)
```

Comparison



Runge-Kutta methods

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For example if y is concave we will overshoot the true value

We could instead use the slope at (t_{i+1}, Y_{i+1}^E) but this will give the same problem but in the opposite direction, we will undershoot

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A first-order Runge-Kutta method will take the average of these two slopes to arrive at the formula

$$Y_{i+1} = Y_i + \frac{h}{2}[f(t_i, Y_i) + f(t_{i+1}, Y_{i+1})]$$

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This converges quadratically to the true solution in h, but now uses two evaluations per step

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This has the same flavor as forward vs central finite differences

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This converges quadratically to the true solution in h, but now uses two evaluations per step

This has the same flavor as forward vs central finite differences

There are higher order Runge-Kutta rules that have even more desirable

Runge-Kutta code

```
function find_euler_root_rk(df, y, t, tp, h, y0, tol)

    y_new = y0
    y_old = y0
    error = Inf

    while error > tol
        y_new = y + h/2 * (df(t, y) + df(tp, y_new))
        error = abs((y_new - y_old)/y_old)
        y_old = deepcopy(y_new)
    end

    return y_new

end
```

```
## find_euler_root_rk (generic function with 1 method)
```

Runge-Kutta code

```
function euler_rk_ode(df, t0, y0, h, n, tol = 1e-6)

    t = zeros(n+1)
    y = zeros(n+1)
    t[1] = t0
    y[1] = y0

    for i in 1:n
        t[i+1] = t[i] + h
        y[i+1] = find_euler_root_rk(df, y[i], t[i], t[i+1], h, y0, tol)
    end

    return (t, y)

end
```

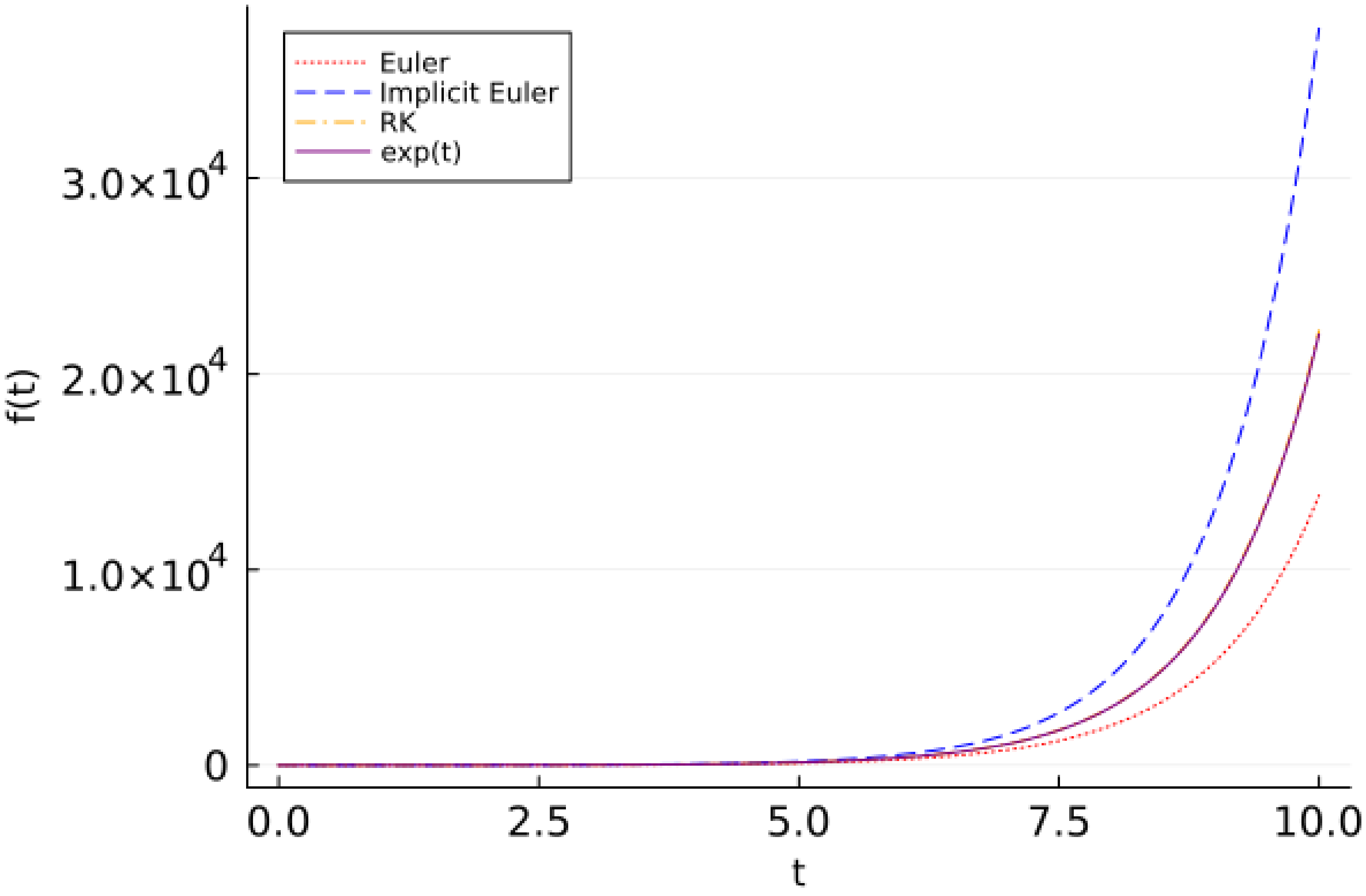
```
## euler_rk_ode (generic function with 2 methods)
```

Comparison

```
df(t, y) = y
t1, y1 = euler_ode(df, 0., 1., .1, 100)
t2, y2 = euler_implicit_ode(df, 0., 1., .1, 100, 1e-7)
t2, y3 = euler_rk_ode(df, 0., 1., .1, 100, 1e-7)
y_real = exp.(t1)
```

Check the time/memory with `@btime` in `BenchmarkTools`

Comparison: RK has minimal error



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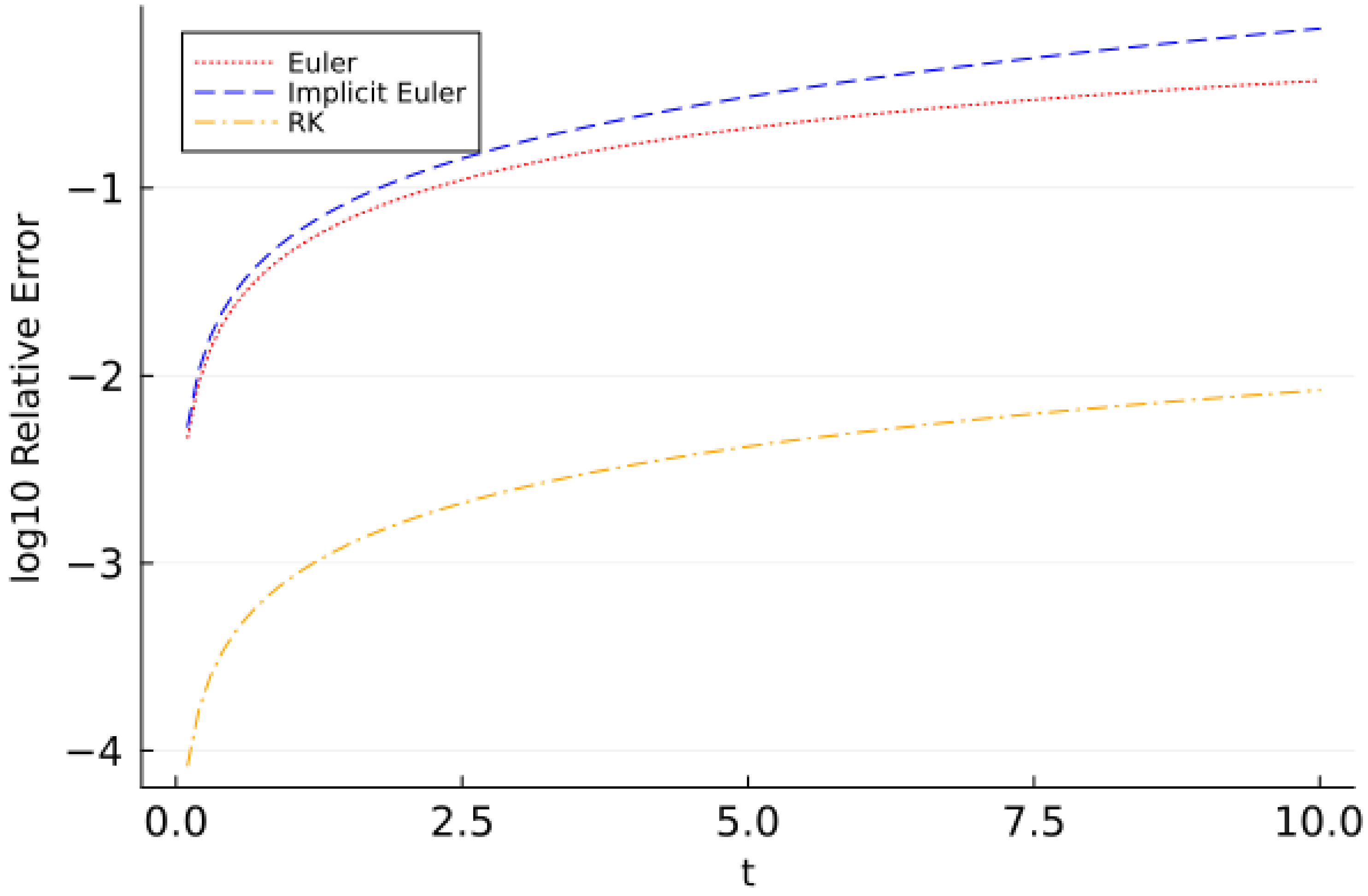

Boundary Value Problems

IVPs are easy to solve because the solution depends only on local conditions so we can use local solution algorithms which are convenient

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BVPs have auxiliary conditions that are imposed at different points in time so we lose the local nature of the problem and our solutions must now be global in nature

Boundary Value Problems

Consider the following BVP

$$\begin{aligned} \dot{x} &= f(x, y, t) \\ \dot{y} &= g(x, y, t) \\ x(t_0) = x_0, \quad y(T) &= y_T \end{aligned}$$

where $x \in \mathbb{R}^n, y \in \mathbb{R}^m$

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where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$

We cannot use standard IVP approaches because at t_0 or T we only know the value of either x or y but not both

Thus we cannot find the next value of both of them using only local information: we need alternative approaches

Boundary Value Problems: Shooting

The core method for solving BVPs is called **shooting**

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But we do get some information from where we end up at $y(T)$ and can use that information to update our guesses for $y(t_0)$ until we are sufficiently close to y_T

There are two components to a shooting method

Shooting

First we guess some $y(0) = y_0$ and then solve the IVP problem with methods we've already used

$$\begin{aligned} \dot{x} &= f(x, y, t) \\ \dot{y} &= g(x, y, t) \\ x(t_0) = x_0, \quad y(0) &= y_0 \end{aligned}$$

to find some $y(T)$ which we call $Y(T, y_0)$ since it depends on our initial guess y_0

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We can write the algorithm as

1. Initialize: Guess y_0^i. Choose a stopping criterion $\epsilon > 0$
2. Solve the IVP for $x(T), y(T)$ given the initial condition $y_0 = y_0^i$
3. If $||y(T) - y_T|| < \epsilon$, STOP. Else choose y_0^{i+1} based on the previous values of y and go back to step 1

Shooting

This is an example of a two layer algorithm

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The inner layer (step 1) uses an IVP method that solves $Y(T, y_0)$ for any y_0

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In the outer layer (step 2) we solve the nonlinear equation $Y(T, y_0) = y_T$

We can use any nonlinear solver here, typically we do this by defining a subroutine that computes $Y(T, y_0) - y_T$ as a function of y_0 and then sends that subroutine to a rootfinding program

Example: Lifecycle model

A simple lifecycle model is given by

$$\begin{aligned} \max_{c(t)} \int_0^T & e^{-rt} u(c(t)) dt \\ s.t. \quad \dot{A}(t) &= f(A(t)) + w(t) - c(t) \\ A(0) &= A(T) = 0. \end{aligned}$$

$u(c(t))$ is utility from consumption, $w(t)$ is the wage rate, $A(t)$ are assets and $f(A(t))$ is the return on invested assets

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$u(c(t))$ is utility from consumption, $w(t)$ is the wage rate, $A(t)$ are assets and $f(A(t))$ is the return on invested assets

We assume that assets are initially and terminally zero where the latter would come about naturally from a transversality condition

Example: Lifecycle model

The Hamiltonian is

$$H = u(c(t)) + \lambda(t)\left[f(A(t)) + w(t) - c(t)\right]$$

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$$\dot{\lambda}(t) = r\lambda(t) - \lambda(t)f'(A(t))$$

The maximum principle implies that $u'(c(t)) = \lambda(t)$

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The issue here is that we never know A and λ at either $t = 0$ or $t = T$

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This gives us a two equation system of differential equations (1 for the A transition, 1 for the costate condition) and the boundary conditions on A are what pin down the problem

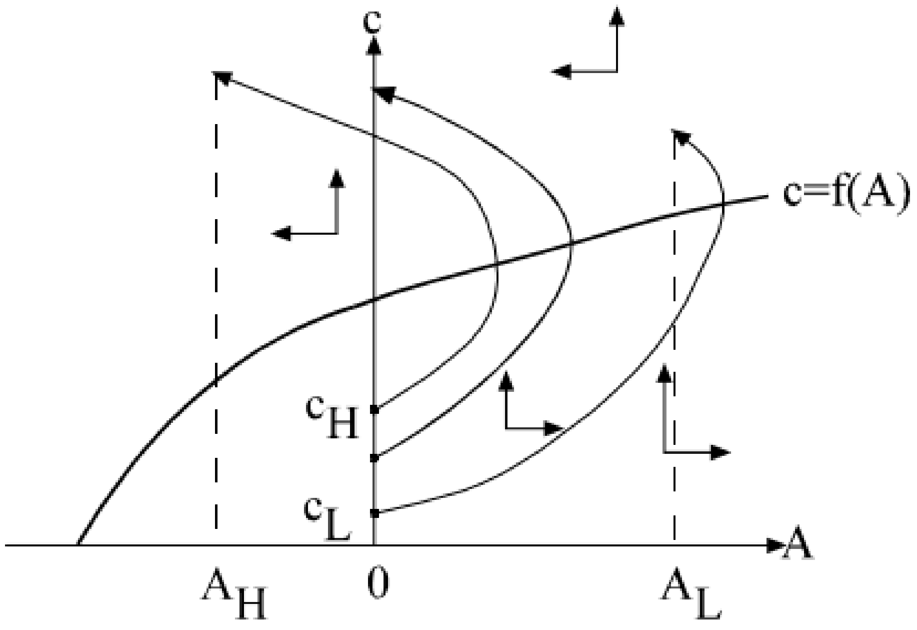
The issue here is that we never know A and λ at either $t = 0$ or $t = T$

We can use the maximum principle to convert the costate condition into a condition on consumption

$$\dot{c}(t) = -\frac{u'(c(t))}{u''(c(t))}\left[f'(A(t)) - r\right]$$

Example: Lifecycle model

The Figure shows the phase diagram assuming that $f'(A) > r$ for all A



If $A(T) < 0$ when we guess $c(0) = c_H$, but $A(T) > 0$ when we guess $c(0) = c_L$, we know the correct guess lies in between and we can solve for it using the

Example: Lifecycle model

Let's code it up

$$\dot{A}(t) = f(A(t)) + w(t) - c(t) \qquad \dot{c}(t) = -\frac{u'(c(t))}{u''(c(t))}\left[f'(A(t)) - r\right]$$

- $f(A(t)) = 1.05A(t)$
- $u(c(t)) = \log(c(t))$
- $w(t) = 5$
- $r = .02$

Example: Lifecycle model

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$$\dot{A}(t) = f(A(t)) + w(t) - c(t) \qquad \dot{c}(t) = -\frac{u'(c(t))}{u''(c(t))}\left[f'(A(t)) - r\right]$$

- $f(A(t)) = 1.05A(t)$
- $u(c(t)) = \log(c(t))$
- $w(t) = 5$
- $r = .02$

```
df(t,a,c) = (1.05*a + 5 - c, -1 * (1/c) / (-1/c^2) * (1.05 - .02))
```

```
## df (generic function with 2 methods)
```

Example: Lifecycle model

We need a 2 variable ODE solver next

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```
function euler_ode(df, t0, a0, c0, h, n)

    t = zeros(n+1)
    a = zeros(n+1)
    c = zeros(n+1)

    t[1] = t0
    a[1] = a0
    c[1] = c0

    for i in 1:n
        t[i+1] = t[i] + h
        a[i+1] = a[i] + h * df(t[i], a[i], c[i])[1]
        c[i+1] = c[i] + h * df(t[i], a[i], c[i])[2]
    end

    return t, a, c

end
```

Example: Lifecycle model

Last, wrap it in bisection method

```
function solve_bvp(df, t0, a0, aend, c0low, c0high, h, n, tol = 1e-6)

    t = zeros(n+1)
    a = zeros(n+1)
    c = zeros(n+1)

    while abs.(c0low - c0high) > tol

        c0guess = (c0low + c0high)/2
        t, a, c = euler_ode(df, t0, a0, c0guess, h, n)
        anew = a[end]

        if sign(anew) > 0
            c0low = c0guess
        else
            c0high = c0guess
        end

    end
```

Example: Lifecycle model

Now we have to find the initial bounds, one where $A(T) > 0$, one where $A(T) < 0$

```
aend = 0.
a0 = 0.
t0 = 0.
h = .01
n = 100
c0low = 1     # low c0 guess
c0high = 10   # high c0 guess
```

Example: Lifecycle model

```
a0high = euler_ode(df, t0, a0, c0high, h, n)[2][end]
```

```
## -19.07974813179398
```

```
a0low = euler_ode(df, t0, a0, c0low, h, n)[2][end]
```

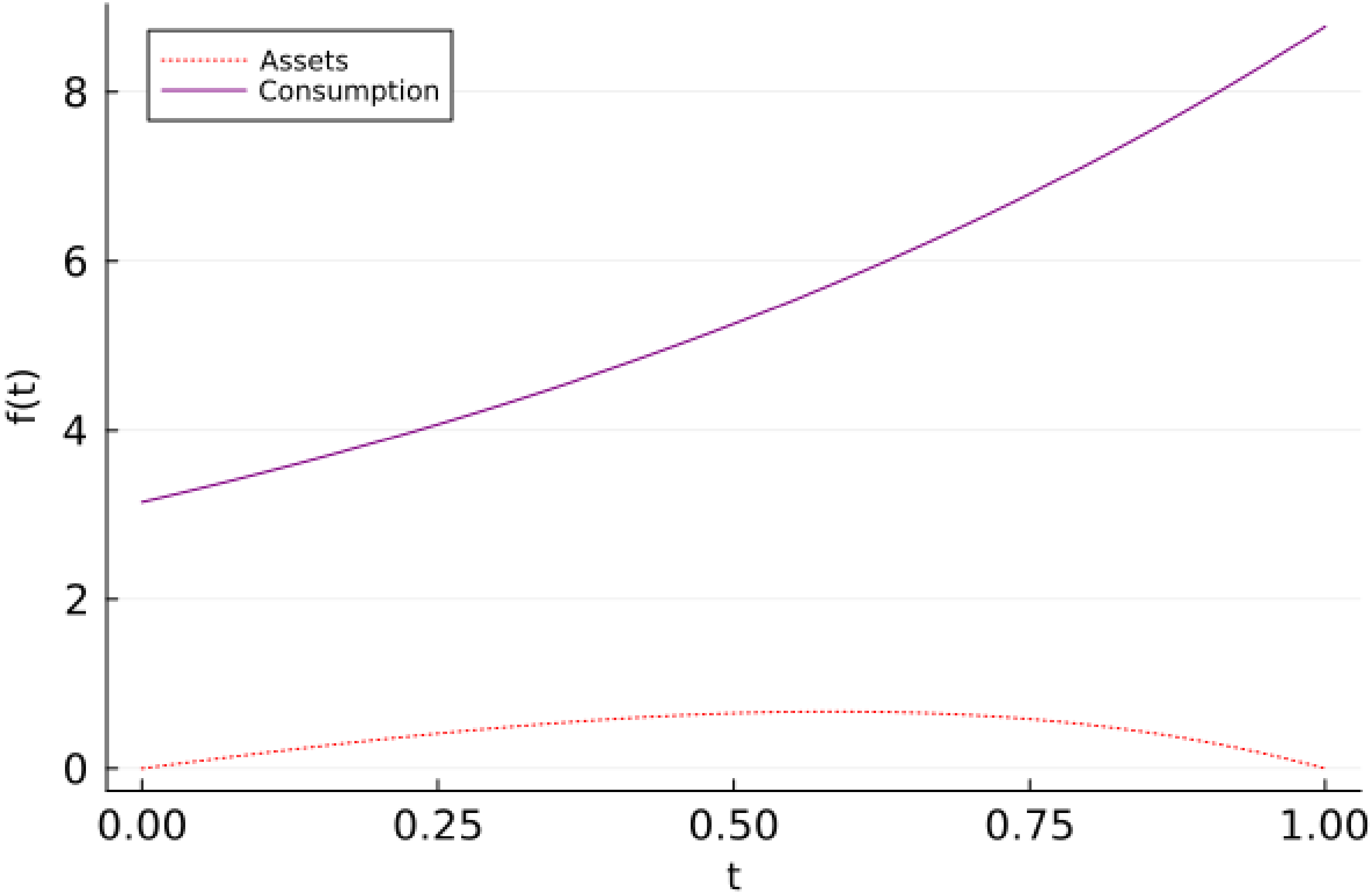
```
## 5.986527176940143
```

As expected, too high consumption $C(0)$ yields negative assets $A(T)$, too low consumption $C(0)$ yields positive assets $A(T)$

The $C(0)$ that solves the problem will fall somewhere in between

Example: Lifecycle model

```
t, a, c = solve_bvp(df, t0, a0, aend, c0low, c0high, h, n)
```



Reverse shooting for ∞ horizon problems

The standard infinite horizon optimal control problem is

$$\begin{aligned} \max_{u(t)} \int_0^\infty & e^{-rt}\pi(x(t), u(t))dt \\ s.t. \quad \dot{x}(t) &= f(x(t), u(t)) \\ x(0) &= x_0. \end{aligned}$$

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We still have $x(0) = x_0$ as before, but we no longer have the terminal condition

Reverse shooting for ∞ horizon problems

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$$
\begin{aligned}
\max_{u(t)} \int_0^\infty e^{-rt} \pi(x(t), u(t)) dt \\
s.t. \quad \dot{x}(t) = f(x(t), u(t)) \\
x(0) = x_0.
\end{aligned}
$$

We still have $x(0) = x_0$ as before, but we no longer have the terminal condition

We replace it with a transversality condition that $\lim_{t\to\infty} e^{-rt}|\lambda(t)^T x(t)| \leq \infty$

Reverse shooting for ∞ horizon problems

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But this implies something very convenient: that the initial state corresponding to any terminal state is not very sensitive to the value of the terminal state

Thus what we will do is not guess the value of the initial condition and integrate forward, we will guess the terminal condition and integrate

Example: Reverse shooting for ∞ horizon problems

Consider the simplest growth model

$$\begin{aligned} \max_{c(t)} \int_0^{\infty} e^{-rt} u(c(t)) dt \\ \dot{k}(t) = f(k(t)) - c(t) \\ s.t. \quad k(0) = k_0, \end{aligned}$$

where c is consumption, k is the capital stock, and f is production

Example: Reverse shooting for ∞ horizon problems

We can use Pontryagin's necessary conditions to get that consumption and capital are governed by the following differential equations

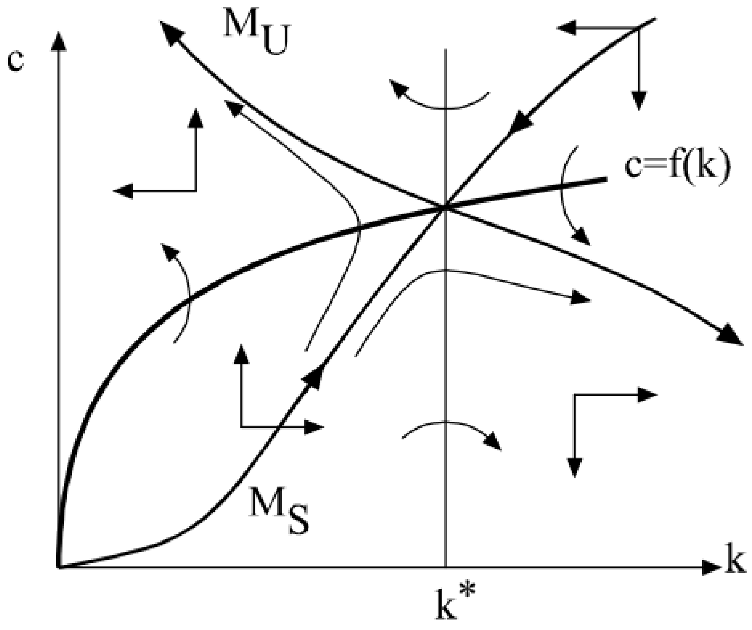
$$\dot{c}(t) = -\frac{u'(c(t))}{u''(c(t))}(f'(k) - r)$$
$$\dot{k}(t) = f(k(t)) - c(t),$$

with boundary conditions,

$$k(0) = k_0, \quad 0 < \lim_{t\to\infty} |k(t)| \leq \infty$$

Example: Reverse shooting for ∞ horizon problems

Assume u and f are concave, the Figure shows the phase diagram for the problem



We have a steady state at $k = k^*$, this occurs when $f'(k(t)) = r$ and

Example: Reverse shooting for ∞ horizon problems

For this problem there exists a stable manifold M_S and an unstable manifold M_U

so that the steady state is **saddle point stable**

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For this problem there exists a stable manifold M_S and an unstable manifold M_U
so that the steady state is **saddle point stable**

Both are invariant manifolds because any system that starts on either of these manifolds will continue to move along the manifold

However M_S is stable because it will converge to the steady state while M_U diverges away from the steady state

Example: Reverse shooting for ∞ horizon problems

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Suppose we start with $k_0 < k^*$, if we guess $c(0)$ too big we will cross the k isoquant and have a falling capital stock, but if we guess $c(0)$ too small we will get a path that crosses the c isoquant and results in a falling consumption level

Example: Reverse shooting for ∞ horizon problems

This gives us our algorithm

1. Initialize: set $c_H = f(k_0)$ and set $c_L = 0$, choose a stopping criterion $\epsilon > 0$
2. Set $c_0 = \frac{1}{2}(c_L + c_H)$
3. Solve the IVP with initial conditions $c(0) = c_0, k(0) = k_0$. Stop the IVP at the first t when $\dot{c}(t) < 0$ or $\dot{k}(t) < 0$, denote this T
4. If $|c(T) - c^*| < \epsilon$, STOP. If $\dot{c}(t) < 0$, set $c_L = c_0$, else set $c_H = c_0$. Go to step 2.

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Any small deviation from M_S is magnified and results in a path that increasingly gets far away from M_S

Unless we happen to pick a point precisely on the stable manifold we will move away from it, so it is hard to search for the solution since changes in our guesses will lead to wild changes in terminal values

Example: Reverse shooting for ∞ horizon problems

Now suppose we wanted to find a path on M_U, notice that the flow pushes points *toward* M_U so the deviations are smushed together

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Example: Reverse shooting for ∞ horizon problems

Now suppose we wanted to find a path on M_U, notice that the flow pushes points *toward* M_U so the deviations are smushed together

If we wanted to compute a path that lies near the unstable manifold, we could simply pick a point near the steady state as the initial condition and integrate the system

We don't actually want to solve for a path on M_U but this gives us some insight

Example: Reverse shooting for ∞ horizon problems

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We can do this easily by reversing time

$$\begin{aligned}\dot{c}(t) &= \frac{u'(c(t))}{u''(c(t))}(f'(k) - r) \\ \dot{k}(t) &= -(f(k(t)) - c(t))\end{aligned}$$

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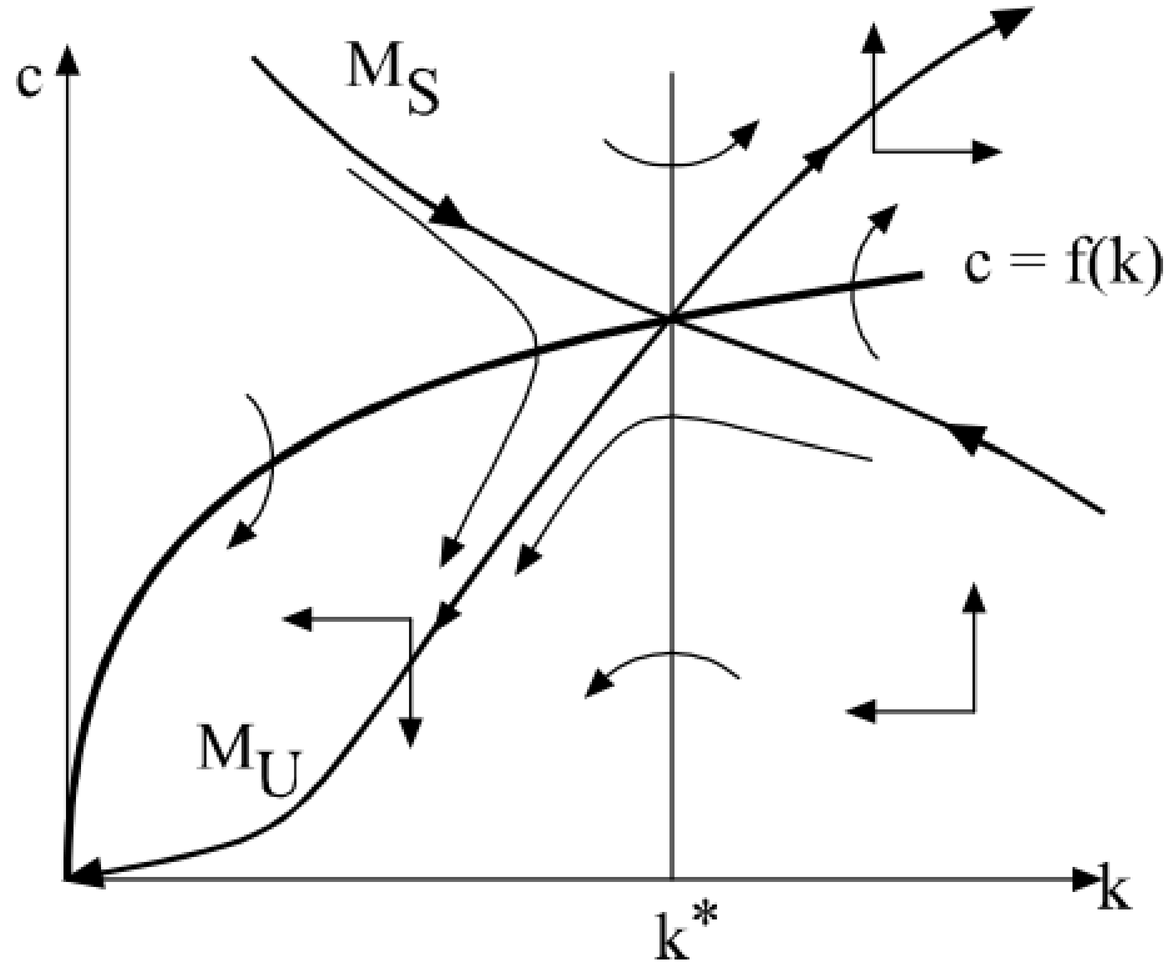
This allows us to exploit how paths tend to converge toward the stable

Reverse shooting for ∞ horizon problems



Reverse shooting example

Let's code it up

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This is $k = 625, c = 25$

Pretend we only knew $f(k) = c$ (from $\dot{K} = 0$) and solve by searching over terminal capital

Example: Lifecycle model

```
df(t,k,c) = (
  -(sqrt(k) - c),
  -(-1 * (1/c) / (-1/c^2) * (0.5*k^(-0.5) - .02))
  )
```

```
## df (generic function with 2 methods)
```

Reverse shooting example

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```
function euler_ode(df, t0, k0, c0, h, n)

    t = zeros(n+1)
    k = zeros(n+1)
    c = zeros(n+1)

    t[1] = t0
    k[1] = k0
    c[1] = c0

    for i in 1:n
        t[i+1] = t[i] + h
        k[i+1] = max(1e-6, k[i] + h * df(t[i], k[i], c[i])[1])
        c[i+1] = max(1e-6, c[i] + h * df(t[i], k[i], c[i])[2])
    end

    return t, k, c

end
```

Reverse shooting example

Last, wrap it in bisection method

```
function solve_bvp_rev(df, t0, k0, klow, khigh, h, n, tol = 1e-6)

    t = zeros(n+1)
    k = zeros(n+1)
    c = zeros(n+1)

    while abs.(klow - khigh) > tol
        kguess = (klow + khigh)/2
        t, k, c = euler_ode(df, t0, kguess, sqrt(kguess), h, n)
        anew = k[end]
        if anew < k0
            klow = kguess
        else
            khigh = kguess
        end
    end

    return t, k, c
```

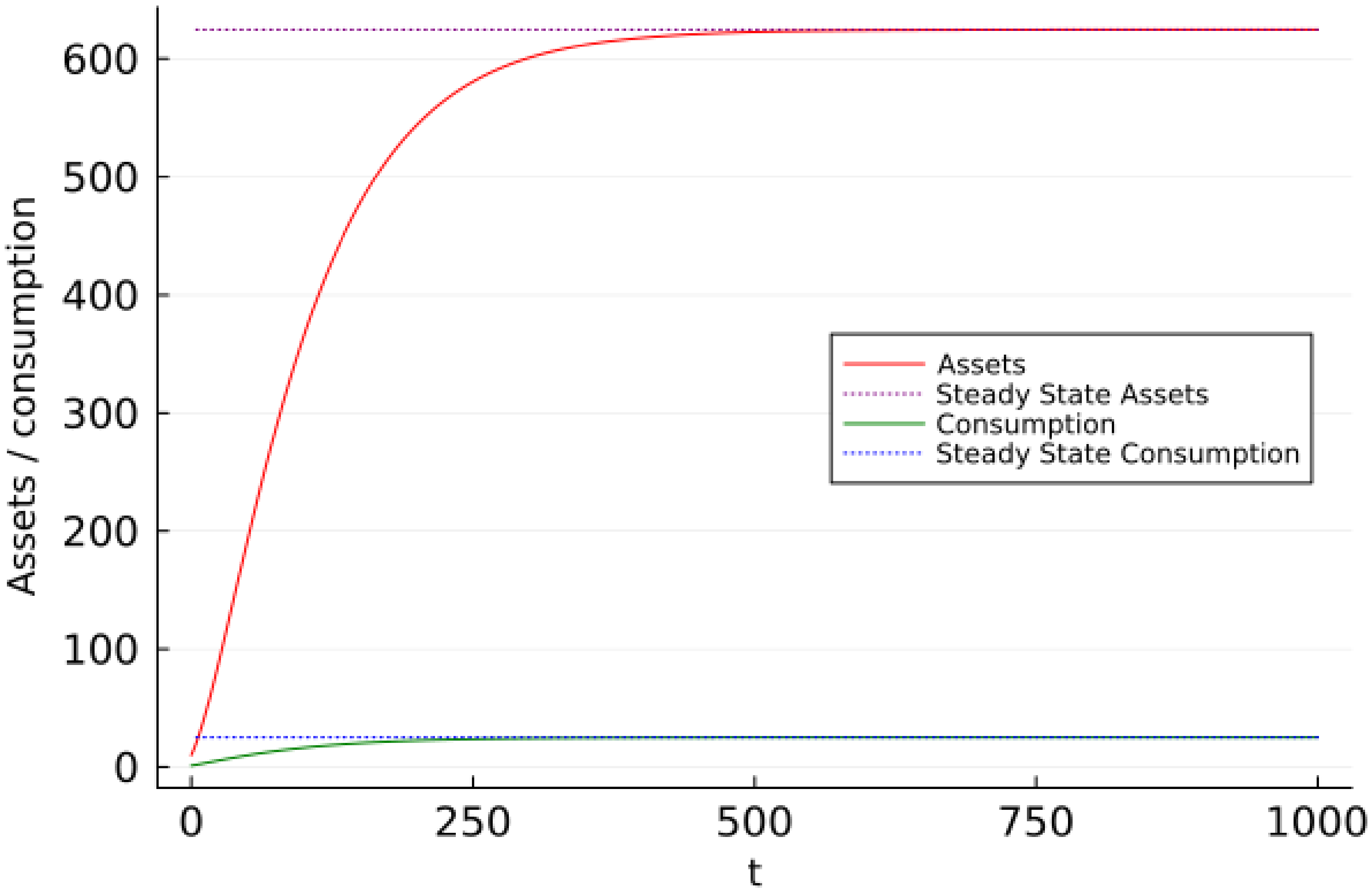
Reverse shooting example

Now we have to find the initial bounds, one where $A(T) > 0$, one where $A(T) < 0$

```
k0 = 10 # initial condition to hit
t0 = 0. # time starts at 0

klow = 100    # below closed-form solution of k = 625
khigh = 1000  # above closed-form solution of k = 625

aend = (.5 / .02)^2 # closed-form solution
cend = sqrt(aend) # closed-form solution

h = .1
n = 10000 # make the horizon long to approx infinite-horizon
```

Reverse shooting example

```
t, k, c = solve_bvp_rev(df, t0, k0, klow, khigh, h, n)
```



Reverse shooting example

```
k0 = 800
t, k, c = solve_bvp_rev(df, t0, k0, klow, khigh, h, n)
```