Lecture 9

Advanced Methods for Numerical Dynamic Models

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Roadmap

1. Regression
2. Endogenous grid method
3. Envelope condition method
4. Modified policy iteration

Chebyshev regression

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We can apply Chebyshev regression to even our regular tensor approaches, this has the advantage of dropping higher order terms which often oscillate due to error, giving us a smoother approximation

Chebyshev regression: practice

Go back to our original VFI example and convert it to a regression approach

```julia
using LinearAlgebra
using Optim
using Plots
params = (alpha = 0.75, beta = 0.95, eta = 2,
                steady_state = (0.75*0.95)^(1/(1 - 0.75)), k_0 = (0.75*0.95)^(1/(1 - 0.75))*.75,
                capital_upper = (0.75*0.95)^(1/(1 - 0.75))*1.5, capital_lower = (0.75*0.95)^(1/(1 - 0.75))/2,
                num_basis = 7, num_points = 9, tolerance = 0.0001, fin_diff = 1e-6, mpi_start = 5);

coefficients = zeros(params.num_basis);
coefficients[1:2] = [100 5];
```

Chebyshev regression: practice

```
cheb_nodes(n) = cos.(pi * (2*(1:n) .- 1)./(2n))
```

```
## cheb_nodes (generic function with 1 method)
```

```
grid = cheb_nodes(params.num_points) # [-1, 1] grid
```

```
## 9-element Array{Float64,1}:
##   0.984807753012208
##   0.8660254037844387
##   0.6427876096865394
##   0.3420201433256688
##   6.123233995736766e-17
##  -0.3420201433256687
##  -0.6427876096865394
##  -0.8660254037844387
##  -0.984807753012208
```

```
expand_grid(grid, params) = (1 .+ grid)*(params.capital_upper - params.capital_lower)/2 .+ params.capital_lower
```

```
## expand_grid (generic function with 1 method)
```

Chebyshev regression: practice

```
# Chebyshev polynomial function
function cheb_polys(x, n)
    if n == 0
        return 1                    # T_0(x) = 1
    elseif n == 1
        return x                    # T_1(x) = x
    else
        cheb_recursion(x, n) =
            2x.*cheb_polys.(x, n - 1) .- cheb_polys.(x, n - 2)
        return cheb_recursion(x, n) # T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)
    end
end;

basis_matrix = [cheb_polys.(grid, n) for n = 0:params.num_basis - 1];
basis_matrix = hcat(basis_matrix...);
basis_inverse = inv(basis_matrix'*basis_matrix)*(basis_matrix'); # pre-compute pseudoinverse for regressions
```

```
## 7×9 Array{Float64,2}:
##  0.111111   0.111111      0.111111   …   0.111111    0.111111      0.111111
##  0.218846   0.19245       0.142842      -0.142842   -0.19245      -0.218846
##  0.208821   0.111111     -0.0385885     -0.0385885   0.111111      0.208821
##  0.19245   -5.15976e-17  -0.19245        0.19245     5.02235e-18  -0.19245
##  0.170232  -0.111111     -0.208821      -0.208821   -0.111111      0.170232
```

Chebyshev regression: practice

```
shrink_grid(capital) = 2*(capital - params.capital_lower)/(params.capital_upper - params.capital_lower) - 1;
eval_value_function(coefficients, capital, params) =
    coefficients' * [cheb_polys.(shrink_grid(capital), n) for n = 0:params.num_basis - 1];
```

Chebyshev regression: practice

```
function loop_grid_regress(params, basis_inverse, basis_matrix, grid, capital_grid, coefficients)
    max_value = -.0*ones(params.num_points);
    consumption_store = -.0*ones(params.num_points);

    for (iteration, capital) in enumerate(capital_grid)
        function bellman(consumption)
            capital_next = capital^params.alpha - consumption
            cont_value = eval_value_function(coefficients, capital_next, params)
            value_out = (consumption)^(1-params.eta)/(1-params.eta) + params.beta*cont_value
            return -value_out
        end;

        results = optimize(bellman, 0.00*capital^params.alpha, 0.99*capital^params.alpha)
        max_value[iteration] = -Optim.minimum(results)
        consumption_store[iteration] = Optim.minimizer(results)
    end

    return max_value, consumption_store
end;
```

Chebyshev regression: practice

```julia
function solve_vfi_regress(params, basis_inverse, basis_matrix, grid, capital_grid, coefficients)

    max_value = -.0*ones(params.num_points);
    error = 1e10;
    value_prev = .1*ones(params.num_points);
    coefficients_store = Vector{Vector}(undef, 1)
    coefficients_store[1] = coefficients
    iteration = 1

    while error > params.tolerance
        max_value, consumption_store = loop_grid_regress(params, basis_inverse, basis_matrix, grid, capital_grid
        coefficients = basis_inverse*max_value
        error = maximum(abs.((max_value - value_prev)./(value_prev)))
        value_prev = deepcopy(max_value)
        if mod(iteration, 5) == 0
            println("Maximum Error of $(error) on iteration $(iteration).")
            append!(coefficients_store, [coefficients])
        end
        iteration += 1
    end

    return coefficients, max_value, coefficients_store
end;
```

Chebyshev regression: practice

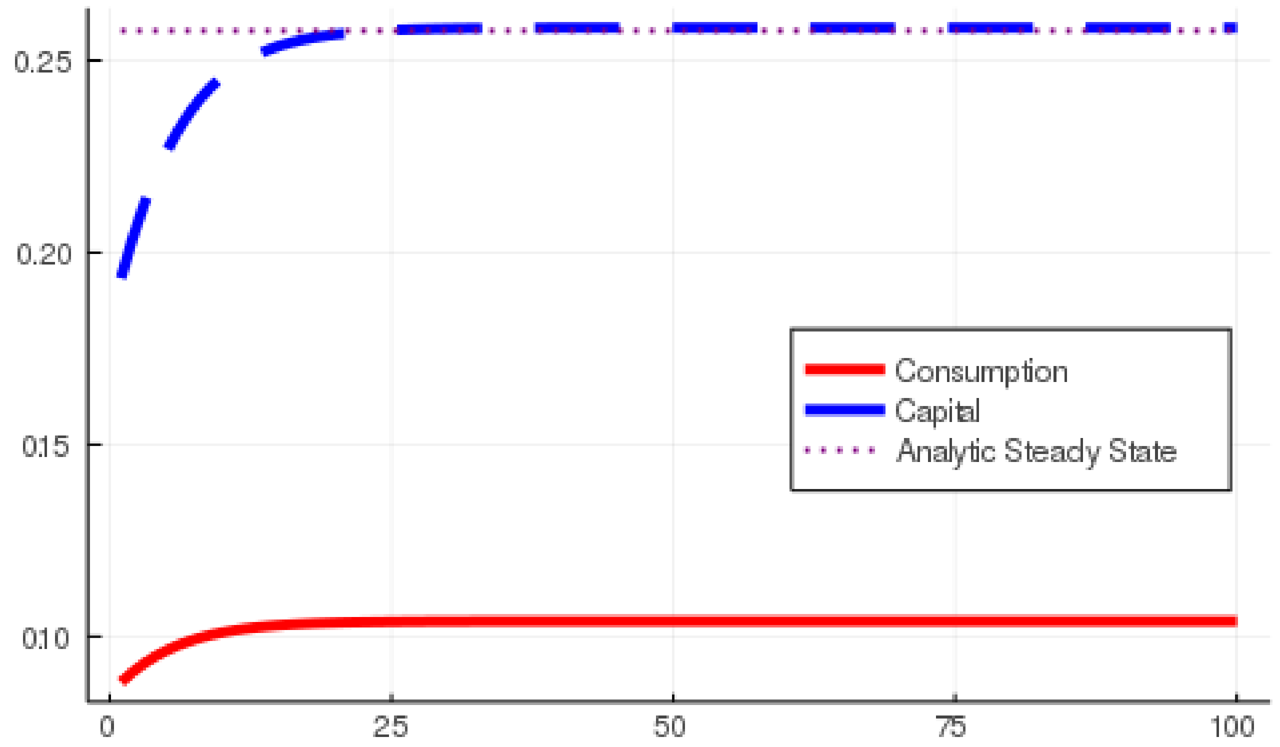
```
@time solution_coeffs, max_value, intermediate_coefficients =
    solve_vfi_regress(params, basis_inverse, basis_matrix, grid, capital_grid, coefficients)
```

```
## Maximum Error of 0.3365646232156377 on iteration 5.
## Maximum Error of 15.324437748784836 on iteration 10.
## Maximum Error of 0.19176452946373068 on iteration 15.
## Maximum Error of 0.07999511358219019 on iteration 20.
## Maximum Error of 0.04557549396818246 on iteration 25.
## Maximum Error of 0.029268260045591604 on iteration 30.
## Maximum Error of 0.02000715481671002 on iteration 35.
## Maximum Error of 0.014198326541472671 on iteration 40.
## Maximum Error of 0.01032384690730612 on iteration 45.
## Maximum Error of 0.007632084134370365 on iteration 50.
## Maximum Error of 0.005708492913566279 on iteration 55.
## Maximum Error of 0.004305925733500575 on iteration 60.
## Maximum Error of 0.003268177593053356 on iteration 65.
## Maximum Error of 0.0024920065993268197 on iteration 70.
## Maximum Error of 0.001906769094882636 on iteration 75.
## Maximum Error of 0.0014628021447215872 on iteration 80.
## Maximum Error of 0.0011244465442097609 on iteration 85.
## Maximum Error of 0.0008656712708535016 on iteration 90.
## Maximum Error of 0.0006672266517799315 on iteration 95.
## Maximum Error of 0.000514733393872197 on iteration 100.
```

Chebyshev regression: practice

```julia
function simulate_model(params, solution_coeffs, time_horizon = 100)
    capital_store = zeros(time_horizon + 1)
    consumption_store = zeros(time_horizon)
    capital_store[1] = params.k_0

    for t = 1:time_horizon
        capital = capital_store[t]

        function bellman(consumption)
            capital_next = capital^params.alpha - consumption
            cont_value = eval_value_function(solution_coeffs, capital_next, params)
            value_out = (consumption)^(1-params.eta)/(1-params.eta) + params.beta*cont_value
            return -value_out
        end;

        results = optimize(bellman, 0.0, capital^params.alpha)
        consumption_store[t] = Optim.minimizer(results)
        capital_store[t+1] = capital^params.alpha - consumption_store[t]
    end

    return consumption_store, capital_store
end;
```

Chebyshev regression: practice



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This yields closed form solutions to the model

$$\begin{aligned} k_{t+1} &= \beta\alpha\theta_t k_t^\alpha \\ c_t &= (1-\beta\alpha)\theta_t k_t^\alpha \end{aligned}$$

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$$\begin{aligned} k_{t+1} &= \beta\alpha\theta_t k_t^\alpha \\ c_t &= (1-\beta\alpha)\theta_t k_t^\alpha \end{aligned}$$

The endogenous grid method was introduced by Carroll (2006) for value function iteration

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Let's see how this works

Endogenous grid method

1. Choose a grid $\{k'_m, \theta_m\}_{m=1,\dots,M}$ on which the value function is approximated
2. Choose nodes ϵ_j and weights ω_j, $j = 1, \dots, J$ for approximating integrals.
3. Compute next period productivity, $\theta'_{m,j} = \theta_m^{\rho} exp(\epsilon_j)$.
4. Solve for b and $\{c_m, k_m\}$ such that
   - (inner loop) The quantities $\{c_m, k_m\}$ solve the following given $V(k'_m, \theta'_m)$:
     - $u'(c_m) = \beta E\left[V_k(k'_m, \theta'_{m,j})\right]$,
     - $c_m + k'_m = \theta_m f(k_m) + (1-\delta)k_m$
   - (outer loop) The value function $\hat{V}(k, \theta; b)$ solves the following given $\{c_m, k'_m\}$:
     - $\hat{V}(k_m, \theta_m; b) = u(c_m) + \beta \sum_{j=1}^{J} \omega_j \left[\hat{V}(k'_m, \theta'_{m,j}; b)\right]$

Endogenous grid method

Focus the inner loop of VFI:

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This means that we can pre-compute the expectations of the value function and value function derivatives and let $W(k', \theta) = E[V(k', \theta'; b)]$

We can then use the consumption FOC to solve for consumption, $c = [\beta W_k(k', \theta)]^{-1/\gamma}$ and then rewrite the resource constraint as,

$$(1-\delta)k + \theta k^{\alpha} = [\beta W_k(k', \theta)]^{-1/\gamma} + k'$$

Endogenous grid method

This is easier to solve than the necessary conditions we would get out of standard value function iteration

$$(k' - (1-\delta)k - \theta k^{\alpha})^{-\gamma} = \beta W_k(k', \theta')$$

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Can we make the algorithm better?

Endogenous grid method: turbo speed

Let's make a change of variables

$$Y \equiv (1-\delta)k + \theta k^{\alpha} = c + k'$$

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so we can rewrite the Bellman as

$$V(Y,\theta) = \max_{k'} \left\{ \frac{c^{1-\gamma}-1}{1-\gamma} + \beta E\left[V(Y',\theta')\right] \right\}$$
$$\text{s.t. } \; c = Y - k'$$
$$Y' = (1-\delta)k' + \theta'(k')^{\alpha}$$

Endogenous grid method: turbo speed

This yields the FOC

$$u'(c) = \beta E\left[V_Y(Y', \theta')(1 - \delta + \alpha\theta'(k')^{\alpha-1})\right]$$

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$$u'(c) = \beta E \left[V_Y(Y', \theta')(1 - \delta + \alpha\theta'(k')^{\alpha-1}) \right]$$

Y' is a simple function of k' (our grid points) so we can compute it, and the entire conditional expectation on the RHS, directly from the endogenous grid points

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Once we have converged on some $\hat{V}^*$ we then solve for k via $Y = (1 - \delta)k + \theta k^\alpha$ which does require a solver, but only once and after we have recovered our value function approximant

Endogenous grid method: practice

Let's solve our previous basic growth model using EGM

```
coefficients = zeros(params.num_basis);
coefficients[1:2] = [100 5];
```

Endogenous grid method: practice

```
function loop_grid_egm(params, capital_grid, coefficients)

    max_value = similar(capital_grid)
    capital_store = similar(capital_grid)

    for (iteration, capital_next) in enumerate(capital_grid)

        function bellman(consumption)
            cont_value = eval_value_function(coefficients, capital_next, params)
            value_out = (consumption)^(1-params.eta)/(1-params.eta) + params.beta*cont_value
            return value_out
        end;
        value_deriv = (eval_value_function(coefficients, capital_next + params.fin_diff, params) -
            eval_value_function(coefficients, capital_next - params.fin_diff, params))/(2params.fin_diff)
        consumption = (params.beta*value_deriv)^(-1/params.eta)
        max_value[iteration] = bellman(consumption)
        capital_store[iteration] = (capital_next + consumption)^(1/params.alpha)
    end

    grid = shrink_grid.(capital_store)
    basis_matrix = [cheb_polys.(grid, n) for n = 0:params.num_basis - 1];
    basis_matrix = hcat(basis_matrix...)
    return basis_matrix, capital_store, max_value

end
```

Endogenous grid method: practice

```julia
function solve_egm(params, capital_grid, coefficients)
    iteration = 1
    error = 1e10;
    max_value = -.0*ones(params.num_points);
    value_prev = .1*ones(params.num_points);
    coefficients_store = Vector{Vector}(undef, 1)
    coefficients_store[1] = coefficients
    while error > params.tolerance
        coefficients_prev = deepcopy(coefficients)
        current_poly, current_capital, max_value =
            loop_grid_egm(params, capital_grid, coefficients)
        coefficients = current_poly\max_value
        error = maximum(abs.((max_value - value_prev)./(value_prev)))
        value_prev = deepcopy(max_value)
        if mod(iteration, 5) == 0
            println("Maximum Error of $(error) on iteration $(iteration).")
            append!(coefficients_store, [coefficients])
        end
        iteration += 1
    end

    return coefficients, max_value, coefficients_store
end
```

Endogenous grid method: practice

```
@time solution_coeffs, max_value, intermediate_coefficients = solve_egm(params, capital_grid, coefficients)
```

```
## Maximum Error of 0.33984199435067963 on iteration 5.
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## Maximum Error of 0.00040153842119404194 on iteration 105.
```

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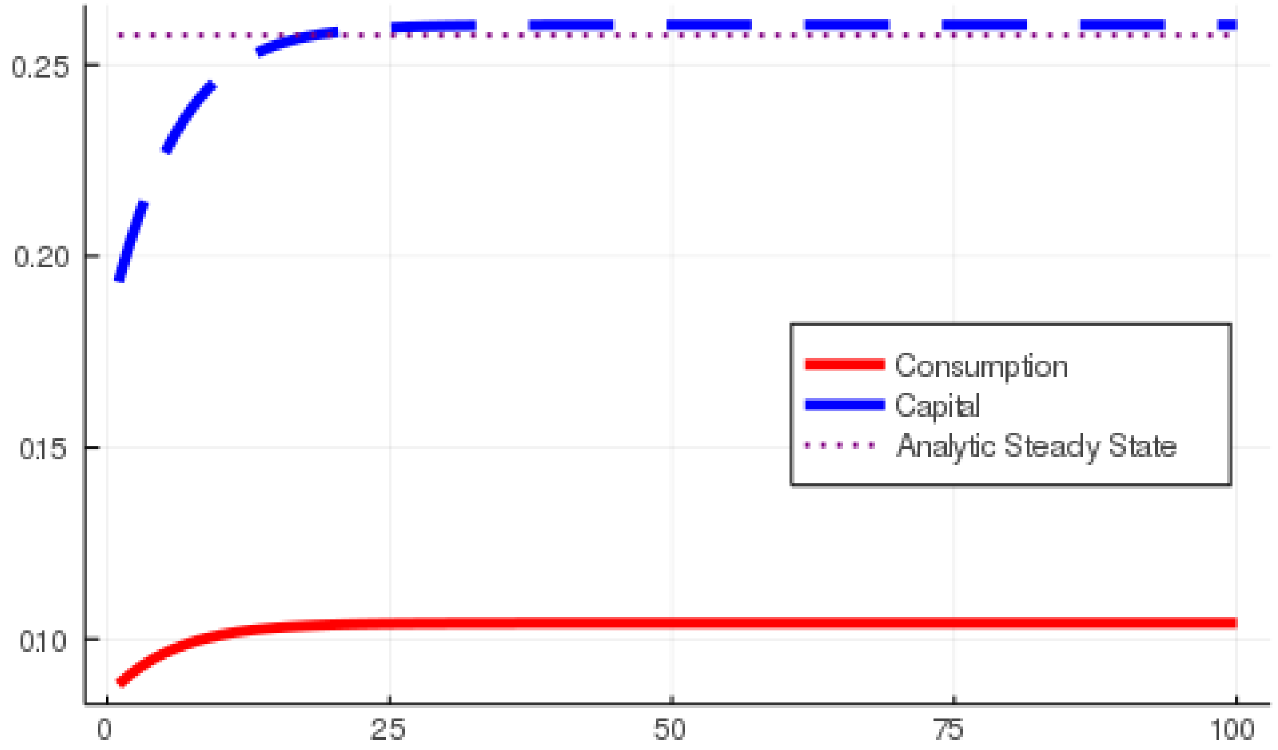

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These will end up being easier to solve and sometimes we can solve them in closed form

Envelope condition method

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We can then recover k' from the budget constraint given our current state

We never need to use a solver at any point in time!

Envelope condition method

The algorithm is

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   - (inner loop) The quantities $\{c_m, k'_m\}$ solve the following given $V(k_m)$:
   - $V_k(k_m) = u'(c_m)f'(k_m)$,
   - $c_m + k'_m = f(k_m)$
   - (outer loop) The value function $\hat{V}(k;b)$ solves the following given $\{c_m, k_m\}$:
   - $\hat{V}(k_m;b) = u(c_m) + \beta \sum_{j=1}^{J} \omega_j \left[\hat{V}(k'_m;b)\right]$

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However we will generally be able to solve a system of conditions via function iteration to recover the optimal controls as a function of current states and future states that are perfectly known at the current time

Envelope condition method

In more complex settings (e.g. elastic labor supply) we will not necessarily be able to solve for policies without a solver

However we will generally be able to solve a system of conditions via function iteration to recover the optimal controls as a function of current states and future states that are perfectly known at the current time

Thus at no point in time during the value function approximation algorithm do we need to interpolate off the grid or approximate expectations: this yields large speed and accuracy gains

Envelope condition method: practice

```julia
function loop_grid_ecm(params, basis_inverse, basis_matrix, grid, capital_grid, coefficients)

    max_value = similar(capital_grid);

    for (iteration, capital) in enumerate(capital_grid)

        function bellman(consumption)
            capital_next = capital^params.alpha - consumption
            cont_value = eval_value_function(coefficients, capital_next, params)
            value_out = (consumption)^(1-params.eta)/(1-params.eta) + params.beta*cont_value
            return value_out
        end;

        value_deriv = (eval_value_function(coefficients, capital + params.fin_diff, params) -
            eval_value_function(coefficients, capital - params.fin_diff, params))/(2params.fin_diff)
        consumption = (value_deriv/(params.alpha*capital^(params.alpha-1)))^(-1/params.eta)
        consumption = min(consumption, capital^params.alpha)
        max_value[iteration] = bellman(consumption)

    end

    return max_value
end
```

```
## loop_grid_ecm (generic function with 1 method)
```

Envelope condition method: practice

```
function solve_ecm(params, basis_inverse, basis_matrix, grid, capital_grid, coefficients)
    iteration = 1
    error = 1e10;
    max_value = similar(capital_grid);
    value_prev = .1*ones(params.num_points);
    coefficients_store = Vector{Vector}(undef, 1)
    coefficients_store[1] = coefficients
    while error > params.tolerance
        coefficients_prev = deepcopy(coefficients)
        max_value = loop_grid_ecm(params, basis_inverse, basis_matrix, grid, capital_grid, coefficients)
        coefficients = basis_inverse*max_value
        error = maximum(abs.((max_value - value_prev)./(value_prev)))
        value_prev = deepcopy(max_value)
        if mod(iteration, 5) == 0
            println("Maximum Error of $(error) on iteration $(iteration).")
            append!(coefficients_store, [coefficients])
        end
        iteration += 1
    end
    return coefficients, max_value, coefficients_store
end
```

```
## solve_ecm (generic function with 1 method)
```

Envelope condition method: practice

```
@time solution_coeffs, max_value, intermediate_coefficients =
    solve_ecm(params, basis_inverse, basis_matrix, grid, capital_grid, coefficients)
```

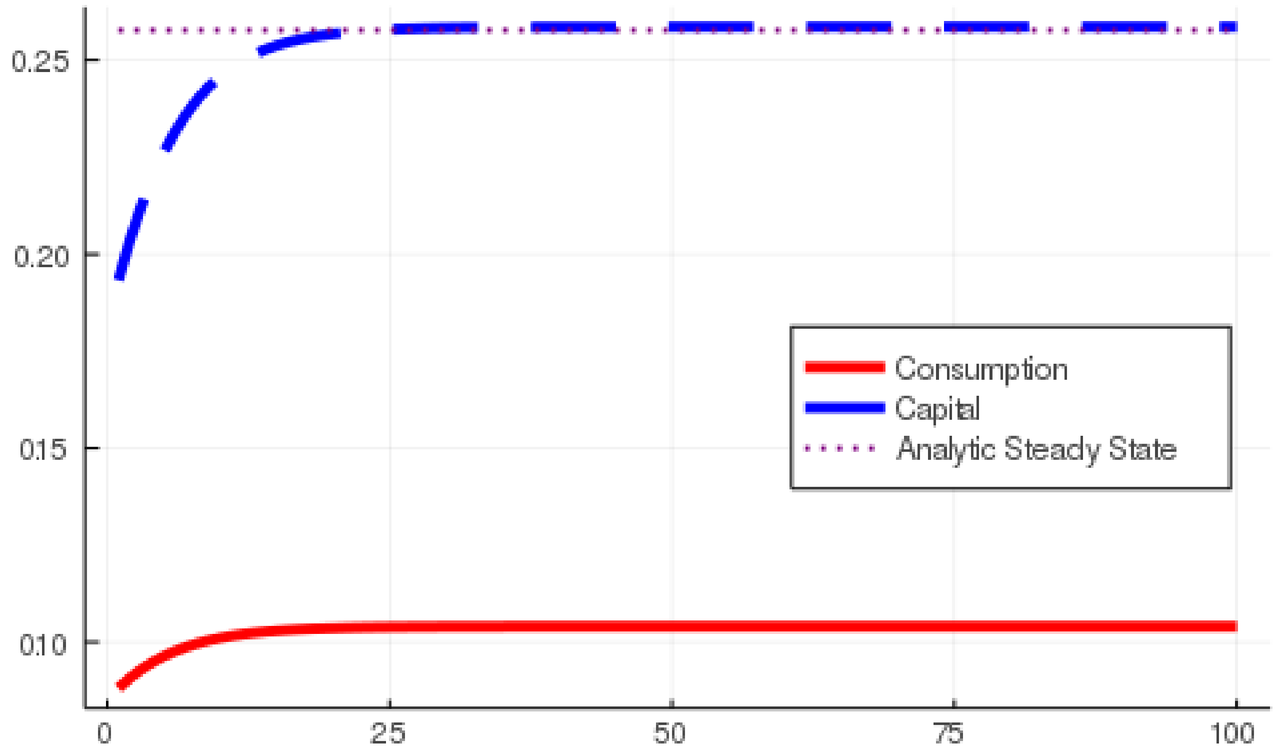
```
## Maximum Error of 0.35270640275290116 on iteration 5.
## Maximum Error of 8.805965931377644 on iteration 10.
## Maximum Error of 0.18965888767404476 on iteration 15.
## Maximum Error of 0.07943447581998832 on iteration 20.
## Maximum Error of 0.04532030555733413 on iteration 25.
## Maximum Error of 0.029127245675450643 on iteration 30.
## Maximum Error of 0.01992067971552436 on iteration 35.
## Maximum Error of 0.014141682933473048 on iteration 40.
## Maximum Error of 0.010285041765191065 on iteration 45.
## Maximum Error of 0.007604645927814628 on iteration 50.
## Maximum Error of 0.005688645480334269 on iteration 55.
## Maximum Error of 0.00429132835259861 on iteration 60.
## Maximum Error of 0.0032573086764123263 on iteration 65.
## Maximum Error of 0.0024838391761790304 on iteration 70.
## Maximum Error of 0.0019005891584954057 on iteration 75.
## Maximum Error of 0.0014581015347963073 on iteration 80.
## Maximum Error of 0.0011208568927165048 on iteration 85.
## Maximum Error of 0.0008629216689646948 on iteration 90.
## Maximum Error of 0.0006651156043508113 on iteration 95.
## Maximum Error of 0.0005131097078957133 on iteration 100.
```

Envelope condition method: practice

```julia
function simulate_model(params, solution_coeffs, time_horizon = 100)
    capital_store = zeros(time_horizon + 1)
    consumption_store = zeros(time_horizon)
    capital_store[1] = params.k_0

    for t = 1:time_horizon
        capital = capital_store[t]

        function bellman(consumption)
            capital_next = capital^params.alpha - consumption
            cont_value = eval_value_function(solution_coeffs, capital_next, params)
            value_out = (consumption)^(1-params.eta)/(1-params.eta) + params.beta*cont_value
            return -value_out
        end;

        results = optimize(bellman, 0.0, capital^params.alpha)
        consumption_store[t] = Optim.minimizer(results)
        capital_store[t+1] = capital^params.alpha - consumption_store[t]
    end

    return consumption_store, capital_store
end;
```

Envelope condition method: practice

```
time_horizon = 100;
consumption, capital = simulate_model(params, solution_coeffs, time_horizon);
plot(1:time_horizon, consumption, color = :red, linewidth = 4.0, label = "Consumption", legend = :right, size =
plot!(1:time_horizon, capital[1:end-1], color = :blue, linewidth = 4.0, linestyle = :dash, label = "Capital");
plot!(1:time_horizon, params.steady_state*ones(time_horizon), color = :purple, linewidth = 2.0, linestyle = :dot
```



Modified policy iteration

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This is called **modified policy iteration**

Modified policy iteration

It only change step 5 of VFI:

1. While convergence criterion $>$ tolerance
    - Start iteration p
    - Solve the right hand side of the Bellman equation
    - Recover the maximized values, conditional on $\Gamma(k_{t+1}; b^{(p)})$
    - Fit the polynomial to the values and recover a new vector of coefficients $\hat{b}^{(p+1)}$.
    - Compute $b^{(p+1)} = (1-\gamma)b^{(p)} + \gamma\hat{b}^{(p+1)}$ where $\gamma \in (0,1)$.
    - While MPI stop criterion $>$ tolerance
        - Use policies from last VFI iteration to re-fit the polynomial (no maximizing!)
        - Compute $b^{(p+1)}$ for iteration $p+1$ by $b^{(p+1)} = (1-\gamma)b^{(p)} + \gamma\hat{b}^{(p+1)}$ where $\gamma \in (0,1)$.

Modified policy iteration

Stop criteron can be a few things:

1. Fixed number of iterations
2. Stop when change in value function is sufficient small, QuantEcon suggests stopping MPI when

$$\max(V_p(x;c) - V_{p-1}(x;c)) - \min(V_p(x;c) - V_{p-1}(x;c)) < \epsilon(1-\beta)\beta$$

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   where the max and min are over the values on the grid

Also note: you should probably start MPI after a few VFI iterations unless you have a good initial guess

If your early policy functions are bad then starting MPI too early will blow up your problem

Modified policy iteration

```julia
function solve_vfi_regress_mpi(params, basis_inverse, basis_matrix, grid, capital_grid, coefficients)
    max_value = -.0*ones(params.num_points);
    error = 1e10;
    value_prev = .1*ones(params.num_points);
    value_prev_outer = .1*ones(params.num_points);
    coefficients_store = Vector{Vector}(undef, 1)
    coefficients_store[1] = coefficients
    iteration = 1
    while error > params.tolerance
        max_value, consumption_store =
            loop_grid_regress(params, basis_inverse, basis_matrix, grid, capital_grid, coefficients)
        coefficients = basis_inverse*max_value
        if iteration > params.mpi_start # modified policy iteration loop
            mpi_iteration = 1
            while maximum(abs.(max_value - value_prev)) -
                    minimum(abs.(max_value - value_prev)) >
                    (1 - params.beta)/params.beta*params.tolerance
                value_prev = deepcopy(max_value)
```

Modified policy iteration

```
     function bellman(consumption, capital)
        capital_next = capital^params.alpha - consumption
        cont_value = eval_value_function(coefficients, capital_next, params)
        value_out = (consumption)^(1-params.eta)/(1-params.eta) + params.beta*cont_value
        return value_out
      end
      max_value = bellman.(consumption_store, capital_grid) # greedy policy
      coefficients = basis_inverse*max_value
      if mod(mpi_iteration, 5) == 0
        println("MPI iteration $mpi_iteration on VFI iteration $iteration.")
      end
      mpi_iteration += 1
    end
  end
  error = maximum(abs.((max_value .- value_prev_outer)./(value_prev_outer)))
  value_prev_outer = deepcopy(max_value)

  if mod(iteration, 5) == 0
    println("Maximum Error of $(error) on iteration $(iteration).")
    append!(coefficients_store, [coefficients])
  end
  iteration += 1
end
return coefficients, max_value, coefficients_store
```

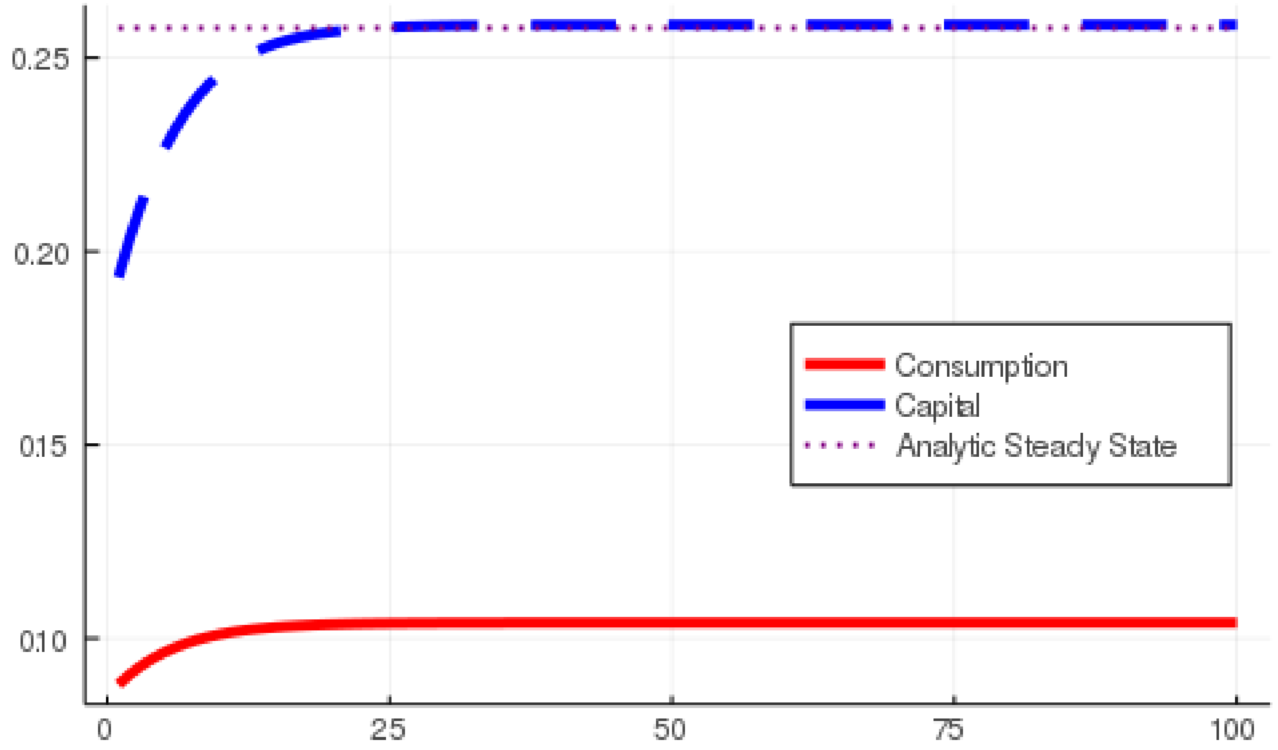
Modified policy iteration

Modified policy iteration

```
@time solution_coeffs, max_value, intermediate_coefficients =
    solve_vfi_regress_mpi(params, basis_inverse, basis_matrix, grid, capital_grid, coefficients)
```

```
## Maximum Error of 0.33656462321563774 on iteration 5.
## MPI iteration 5 on VFI iteration 6.
## MPI iteration 10 on VFI iteration 6.
## MPI iteration 15 on VFI iteration 6.
## MPI iteration 20 on VFI iteration 6.
## MPI iteration 25 on VFI iteration 6.
## MPI iteration 30 on VFI iteration 6.
## MPI iteration 35 on VFI iteration 6.
## MPI iteration 40 on VFI iteration 6.
## MPI iteration 45 on VFI iteration 6.
## MPI iteration 50 on VFI iteration 6.
## MPI iteration 55 on VFI iteration 6.
## MPI iteration 60 on VFI iteration 6.
## MPI iteration 65 on VFI iteration 6.
## MPI iteration 70 on VFI iteration 6.
## MPI iteration 5 on VFI iteration 7.
## MPI iteration 10 on VFI iteration 7.
## MPI iteration 15 on VFI iteration 7.
## MPI iteration 20 on VFI iteration 7.
## MPI iteration 25 on VFI iteration 7.
```

Modified policy iteration



Modified policy iteration

What was your speed up?

I got **6 times**: 0.6s $\rightarrow$ 0.1s