Lecture 5

Dynamics theory review

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AEM 7130

Roadmap

1. Review Markov chains and dynamic models
2. Review theory for numerical methods

Building a dynamic economic model

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4. Transition equations: How do the state variables evolve over time?

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3. Payoff: What is the single-period payoff function? What's our reward?
4. Transition equations: How do the state variables evolve over time?
5. Planning horizon: When does our problem terminate? Never? 100 years?

Two types of solutions

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Open-loop: treat the model as a sequence of static optimization problems solved simultaneously

- Transitions act as constraints
- Ends up being just a potentially giant (but simple) non-linear optimization problem
- Drawback: solutions will be just a function of time so we can't introduce uncertainty, strategic behavior, etc

Two types of solutions

Feedback: treat the model as a single-period optimization problem with the immediate payoff and the *continuation value*

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Feedback: treat the model as a single-period optimization problem with the immediate payoff and the *continuation value*

- Yields a solution that is a function of states
- Permits uncertainty, game structures
- Drawback: need to solve for the continuation value function

Markov chains

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A stochastic process $\{x_t\}$ is said to have the **Markov property** if for all $k \geq 1$ and all t

$$Prob(x_{t+1}|x_t, x_{t-1}, \ldots, x_{t-k}) = Prob(x_{t+1}|x_t)$$

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The Markov property is necessary for the feedback representation

Markov chains

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A Markov chain is characterized by:

1. n-dimensional state space with vectors e_i, $i = 1, \ldots, n$ where e_i is an $n \times 1$ unit vector whose ith entry is 1 and all others are 0
2. An $n \times n$ *transition matrix* P which captures the probability of transitioning from one point of the state space to another point of the state space next period
3. $n \times 1$ vector π_0 whose ith value is the probability of being in state i at time 0: $\pi_{0i} = \text{Prob}(x_0 = e_i)$

Markov chains

P is given by

$$P_{ij} = \mathrm{Prob}(x_{t+1} = e_j | x_t = e_i)$$

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We need one assumption:

- For $i = 1, .., n$, $\sum_{j=1}^{n} P_{ij} = 1$ and π_0 satisfies: $\sum_{i=1}^{n} \pi_{0i} = 1$

Markov chains

Nice property of Markov chains:

We can use P to determine the probability of moving to another state in *two* periods by P^2 since

$$
\begin{aligned}
&\text{Prob}(x_{t+2} = e_j | x_t = e_i) \\
&= \sum_{h=1}^{n} \text{Prob}(x_{t+2} = e_j | x_{t+1} = e_h) \text{Prob}(x_{t+1} = e_h | x_t = e_i) \\
&= \sum_{h=1}^{n} P_{ih} P_{hj} = P_{ij}^2
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iterate on this to show that

$$\text{Prob}(x_{t+k} = e_j | x_t = e_i) = P_{ij}^k$$

Dynamic programming

Start with a general sequential problem to set up the basic recursive/feedback dynamic optimization problem

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Let $\beta \in (0,1)$, the economic agent selects a sequence of controls, $\{u_t\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t r(x_t, u_t)$$

subject to $x_{t+1} = g(x_t, u_t)$ and with x_0 given

Dynamic programming

Assume r is concave, continuously differentiable, and the state space is convex and compact

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We want to recover a *policy function* h which maps the current state x_t into the current control u_t, such that the sequence $\{u_s\}_{s=0}^{\infty}$ generated by iterating

$$
\begin{aligned}
u_t &= h(x_t) \\
x_{t+1} &= g(x_t, u_t),
\end{aligned}
$$

starting from x_0, solves our original optimization problem

Value functions

Consider a function $V(x)$, the **continuation value function** where

$$V(x_0) = \max_{\{u_s\}_{s=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t r(x_t, u_t)$$

subject to the transition equation: $x_{t+1} = g(x_t, u_t)$

The value function defines the maximum value of our original problem as a function of the state

Value functions

Suppose we know $V(x)$, then we can solve for the policy function h by solving for each $x \in X$

$$\max_u r(x, u) + \beta V(x')$$

where $x' = g(x, u)$ and primes on state variables indicate next period

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This is often easier

Bellman equations

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$h(x)$ maximizes the right hand side of the Bellman

Bellman equations

The policy function satisfies

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One of the workhorse solution methods exploits this recursion and contraction mapping properties of the Bellman operator to solve for $V(x)$

Solution properties

Under standard assuptions we have that

1. The solution to the Bellman equation, $V(x)$, is strictly concave
2. The solution is approached in the limit as $j \to \infty$ by iterations on: $V_{j+1}(x) = \max_u r(x, u) + \beta V_j(x')$, given any bounded and continuous V_0 and our transition equation
3. There exists a unique and time-invariant optimal policy function $u_t = h(x_t)$ where h maximizes the right hand side of the Bellman
4. The value function $V(x)$ is differentiable

Euler equations

Euler equations are dynamic efficiency conditions: they equalize the marginal effects of an optimal policy over time

E.g: set the current marginal benefit, energy from burning fossil fuels, with the future marginal cost, global warming

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1. We have a stock of capital K_t that depreciates at rate $\delta \in (0, 1)$
2. We can invest to increase our future capital I_t at cost $c(I_t)$ and effectiveness $\gamma \in (0, 1]$
3. Per-period payoff $U(Y_t)$ from consuming output $Y_t = f(K_t) = K_t$
4. Discount factor is $\beta \in (0, 1)$

Euler equations

The Bellman equation is

$$V(K_t) = \max_{I_t} \{u(K_t) - c(I_t) + \beta V(K_{t+1})\}$$
$$\text{subject to:} \quad K_{t+1} = (1-\delta)K_t + \gamma I$$

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The FOC with respect to investment is

$$c_I(I_t) = \beta \gamma V_K(K_{t+1})$$

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Advance both by one period since they must hold for all t

$$c_I(I_{t+1}) = \beta \gamma V_K(K_{t+2})$$
$$V_K(K_{t+1}) = u_K(K_{t+1}) + \beta \delta V_K(K_{t+2})$$

Euler equations

Substitute the time t and time $t+1$ FOCs into our time $t+1$ envelope condition

$$\frac{c'(I_t)}{\beta\,\gamma} = u'(K_{t+1}) + \beta\,\delta\frac{c'(I_{t+1})}{\beta\,\gamma}$$

$$\Rightarrow c'(I_t) = \beta\left[\gamma\,u'(K_{t+1}) + \delta\,c'(I_{t+1})\right]$$

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LHS is marginal cost of investment, RHS is marginal benefit of investment **along an optimal path**

Euler equations

$$\Rightarrow c'(I_t) = \beta \left[\gamma\, u'(K_{t+1}) + \delta\, c'(I_{t+1}) \right]$$

LHS: marginal cost of investment

RHS: marginal benefit of higher utility from more future output, and lower future investment cost because of higher capital stock

Euler equations: no-arbitrage

Euler equations are **no-arbitrage conditions**

Suppose we're on the optimal capital path and want to deviate by cutting back investment

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Euler equations are **no-arbitrage conditions**

Suppose we're on the optimal capital path and want to deviate by cutting back investment

Yields a marginal benefit today of saving us some investment cost

There are two costs associated with it:

1. Lower utility tomorrow because we will have a smaller capital stock
2. Greater investment cost tomorrow to return to the optimal capital trajectory

Euler equations: no-arbitrage

If this deviation (or deviating by investing more today) were profitable, we would do it

$\rightarrow$ the optimal policy must have zero additional profit opportunities: this is what the Euler equation defines

Basic theory

Here we finish up the basic theory pieces we need

We will focus on deterministic problems but this easily ports to stochastic problems

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Consider an infinite horizon problem for an economic agent

1. Payoff $r(s_t, u_t)$ in some period t is a function of the state vector s_t and control vector u_t
2. Transition equations are $s_{t+1} = g(s_t, u_t)$
3. Assume that $u \in U$ and $s \in S$
4. Payoff is bounded: $u(s_t, u_t)$.

Basic theory

Here the current state vector *completely* summarizes all the information of the past and is all the information the agent needs to make a forward-looking decision

$\rightarrow$ our problem has the Markov property

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Final two pieces

1. Stationarity: does not depend explicitly on time
2. Discounting: $\beta \in (0, 1)$, the future matters but not as much as today

Discounting and bounded payoffs ensures total value is bounded

Basic theory

Represent this payoff as

$$\sum_{t=0}^{\infty} \beta^t \boldsymbol{r}(\boldsymbol{s}_t, \boldsymbol{u}_t)$$

Basic theory

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$$\sum_{t=0}^{\infty} \beta^t r(s_t, u_t)$$

The value of the maximized discounted stream of payoffs is

$$V(s_0) = \max_{u_0 \in U(s_0)} r(s_t, u_t) + \beta \left[\max_{\{u_t\}_{t=1}^{\infty}} \sum_{t=t}^{\infty} \beta^t r(s_t, u_t) \right]$$
$$\text{subject to: } s_{t+1} = g(s_t, u_t)$$

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the terms inside the square brackets is the maximized discounted stream of payoffs beginning at state s_1

Basic theory

This means the problem can be written recursively as

$$V(s_0) = \max_{u_0 \in U(s_0)} r(s_t, u_t) + \beta V(s_1)$$
$$\text{subject to: } s_{t+1} = g(s_t, u_t)$$

which is our Bellman (we just exploited Bellman's principle of optimality)

Value function existence and uniqueness

Reformulate the problem as,

$$V(s) = \max_{s' \in \Gamma(s)} r(s, s') + \beta V(s'), \;\; \forall s \in S$$

where $\Gamma(s)$ is our set of feasible states next period

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If the following are true

1. $r(s_t, u_t)$ is real-valued, continuous and bounded
2. $\beta \in (0, 1)$
3. the feasible set of states for next period is non-empty, compact, and continuous

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then there exists a unique value function $V(s)$ that solves the Bellman equation

Intuitive sketch of the proof

Define an operator T as

$$T(W)(s) = \max_{s' \in \Gamma(s)} r(s, s') + \beta W(s'), \ \ \forall s \in S$$

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This operator takes some value function $W(s)$, maximizes it, and returns another $T(W)(s)$

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Therefore we simply search for the **fixed point** of $T(W)$ to solve our dynamic problem, but how do we find the fixed point?

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First we must show that a way exists by showing that $T(W)$ is a **contraction**:
as we iterate using the T operator, we will get closer and closer to the fixed point

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Blackwell's sufficient conditions for a contraction are

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1. Monotonicity: if $W(\boldsymbol{s}) \geq Q(\boldsymbol{s}) \ \ \forall \boldsymbol{s} \in S$, then $T(W)(\boldsymbol{s}) \geq T(Q)(\boldsymbol{s}) \ \ \forall \boldsymbol{s} \in S$

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2. Discounting: there exists a $\beta \in (0,1)$ such that $T(W+\boldsymbol{k})(\boldsymbol{s}) \leq T(W)(\boldsymbol{s}) + \beta\, \boldsymbol{k}$

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Why do we care this is a contraction?

Intuitive sketch of the proof

So we can take advantage of the contraction mapping theorem which states:

Intuitive sketch of the proof

So we can take advantage of the contraction mapping theorem which states:

1. T has a unique fixed point
2. $T(V^*) = V^*$
3. We can start from any arbitrary initial function W, iterate using T and reach the fixed point

Next up

Next: numerical methods for discrete time dynamics