An introduction to economic geography

EK Ricardian models and hat algebra

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Roadmap

1. The economics of space
2. A simple Eaton Kortum Model
3. Exact hat algebra to solve counterfactuals

The economics of space

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"Spatial Impossibility Theorems" tell us that there's no competitive equilibrium outcome with transport of goods and positive transport costs if space is homogeneous (utilities and production technologies are the same)

- Factor mobility perfectly substitutes for trade, never incur the trade cost

Spatial models

With spatial models we are now going to allow for:

- heterogeneity across space
- trade + frictions
- migration + frictions
- agglomeration economies
- dispersion forces (congestion)

Basically, more consideration of general equilibrium effects

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Often combines granular data (census blocks, households, cell-level) with a theoretical equilibrium model to simulate counterfactuals

Spatial models: macro-ish

These features allow us to better understand the impacts of things like:

- the costs of sea level rise (Desmet et al. 2021)
- climate change and the food problem (Nath 2021)
- climate and agricultural adaptation (Costinot, Donaldson, Smith 2016)
- the geography of environmental regulation (Hollingsworth, Jaworski, Kitchens, and Rudik, 2021)
- the climate adaptation value of migration (Cruz and Rossi-Hansberg 2021)

Spatial models: micro

These features better allow us to understand the impacts of things like:

- the social value of public transit (Severen 2021)
- amenity endogeneity (Almagro and Dominguez-Ilno, 2021)
- the value of density in Berlin (Ahlfeldt et al. 2015)
- the impact of Amazon HQ2 (Dingel and Tintelnot 2021)

Simple set up: a model of market access

We're going to start off with a simple **market access** / **gravity** model

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It is a version of the widely-used Eaton and Kortum (2002) Ricardian model

Simple set up: a model of market access

Here's what we are going to do:

1. Develop the static model
2. Show how to use exact hat algebra to simulate counterfactuals without data on time-invariant exogenous variables (Dekle et al. 2008)
    - Tomorrow's paper uses dynamic hat algebra (Caliendo et al. 2019)

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New computational piece here: we will be solving for things up to a normalization, we won't necessarily be able to get the levels of all variables

Households

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With the CES assumption, we have the price index in location i is:

$$P_i = \left(\int_0^N P(\omega)^{1-\sigma} d\omega \right)^{\frac{1}{1-\sigma}}$$

Households: equilibrium utility and wages

The consumer's indirect utility is given by their real wage:

$$V_i = \frac{w_i}{P_i}$$

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With perfectly mobile labor, indirect utility is equalized in equilibrium and wages w_i satisfy:

$$w_i = \bar{U} P_i$$

Firms

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This is the quantity of variety ω that can be produced using one unit of labor and capital in location i

Firms: EK assumption

We assume that $z_i(\omega)$ is a random variable distributed according to the Frechet distribution (Eaton and Kortum, 2002):

$$F_i\left(z\right) = \exp\left[-T_i(z)^{-\theta}\right]$$

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This distributional assumption is what will buy us tractability in our model later

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T_i measures fundamental productivity

Larger values of T_i shifts the distribution to the right and corresponds to larger absolute advantage

Firms: Production technology

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The input cost of producing variety ω in location i is:

$$c_i = (w_i)^{\gamma}(r)^{1-\gamma}$$

Firms: Trade and prices

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The price of a good from j in i is then:

$$P_{ij}(\omega) = \frac{1}{z_i(\omega)} c_i \tau_{ij} = \frac{1}{z_i(\omega)} (w_i)^{\gamma} (r)^{1-\gamma} \tau_{ij}$$

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Buyers purchase from the lowest-cost provider so the price in i is:

$$P_i(\omega) = \min_j \left\{ P_{ij}(\omega), j = 1, \ldots, N \right\}$$

Firms: Trade and prices

$P_{ij}(\omega)$ and the Frechet assumption on z give us a distribution of prices faced by i from j sellers $G_{ij}(p)$:

$$Pr[P_{ij} \leq p] \equiv G_{ij}(p) = 1 - F_j\left(\frac{1}{p}(w_i)^{\gamma}(r)^{1-\gamma}\tau_{ij}\right)$$

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The lowest price for a variety in i takes on the minimum across all countries j giving us the distribution for prices in i of varieties that i actually buys $G_n(p)$:

$$Pr[P_i \leq p] \equiv G_i(p) = 1 - \prod_{j=1}^{N}[1 - G_{ij}(p)]$$

Firms: Trade and prices

We then get our final expression for the distribution of prices in i:

$$G_i(p) = 1 - \exp(-\Phi_i p^\theta) \quad \text{where} \quad \Phi_i = \sum_{j=1}^{N} T_j (c_j \tau_{ij})^{-\theta}$$

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Φ_i summarizes how prices in i are governed by:

- Fundamental productivity in all j
- Input costs in all j
- Trade barriers with all j

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What does the distribution look like in autarky or with frictionless trade?

Firms: Prices

Two key properties of $G_i(p)$:

1. The probability that j is the lowest-cost seller in i is:

$$\pi_{ij} = \frac{T_j(c_j\tau_{ij})^{-\theta}}{\sum_{h=1}^{N} T_h(c_h\tau_{ih})^{-\theta}} \qquad \approx \text{ logit shares}$$

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2. The ideal price index in i is $(\sigma < 1 + \theta)$:

$$P_i = \gamma \Phi_i^{-1/\theta}$$

Firms: Prices

With a bit of math we can solve for the ideal price index as a function of input costs and parameters:

$$P_i = \kappa \left(\sum_{n=1}^{N} T_n \left[(w_n)^{\gamma} \tau_{in} \right]^{-\theta} \right)^{-1/\theta}$$

which is the cost of 1 unit of utility in location i, κ is a constant

Firms: Trade

$$P_i = \kappa\left(\sum_{n=1}^{N} T_n \left[(w_n)^{\gamma}\tau_{in}\right]^{-\theta}\right)^{-1/\theta}$$

Notice P_i is increasing in sellers' trade costs τ_{in} and input costs w_n, and decreasing in productivity T_n

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$(P_i)^{-\theta}$ is often called consumer market access (CMA_i) and captures location i's access to cheaper products

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Next, define a new variable, expenditure shares: the share of is total expenditures on products from location j: λ_{ij}

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Expenditure shares $\equiv$ probability of being the lowest-cost seller:

$$\lambda_{ij} = \frac{T_j\left((w_j)^\gamma \tau_{ij}\right)^{-\theta}}{\sum_{h=1}^{N} T_h\left((w_h)^\gamma \tau_{ih}\right)^{-\theta}}$$

Firms: Trade

Bilateral trade expenditures of location i on goods from location j are X_{ij} where:

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and we can show that:

$$X_{ij} = \kappa T_j Y_i \left(\frac{(w_j)^{\gamma} \tau_{ij}}{P_i} \right)^{-\theta}$$

i purchases more products from j if the costs of products from j are lower, i is richer, j is more productive

Exact hat algebra

How do we solve this model?

Suppose we want to know the effect of a change in productivity $\hat{T}_j = \frac{T_j^c}{T_j}$

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In this simple model trade costs are the time-invariant variable

Exact hat algebra

First, labor income $w_i L_i = \gamma Y_i$ can be written as a function of the sum of expenditures by j on products from i

$$w_i L_i = \gamma \sum_{j=1}^{N} \lambda_{ji} w_j L_j$$

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RHS is spending by all locations j on is products

Exact hat algebra

$$\lambda_{ij} = \frac{T_j\Big((w_j)^\gamma \tau_{ij}\Big)^{-\theta}}{\sum_{h=1}^N T_h\Big((w_h)^\gamma \tau_{ih}\Big)^{-\theta}}$$

$$w_i L_i = \gamma \sum_{j=1}^N \lambda_{ji} w_j L_j$$

Suppose we have a productivity shock $\hat{T}_j$, and that $\hat{\tau}_{ij} = \hat{L} = 1$

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Suppose we have a productivity shock $\hat{T}_j$, and that $\hat{\tau}_{ij} = \hat{L} = 1$

What we are going to do is manipulate these two expressions into hat form

Exact hat algebra

Let's start with the counterfactual labor market clearing condition:

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$$w_i^c L_i^c = \gamma \sum_{j=1}^{N} \lambda_{ji}^c w_j^c L_j^c$$

$\lambda_{ji}^c w_j^c L_j^c$ is just how much j spends on products from i so that:

$$w_i^c L_i^c = \gamma \sum_{j=1}^{N} X_{ji}^c$$

Exact hat algebra

Next let's put labor income into hat form:

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$$\hat{w}_i \hat{L}_i = \frac{w_i^c L_i^c}{w_i L_i} = \gamma \sum_{j=1}^{N} \frac{X_{ji}^c}{w_i L_i} = \gamma \sum_{j=1}^{N} \frac{X_{ji}^c}{w_i L_i} \frac{X_{ji}}{X_{ji}} = \gamma \sum_{j=1}^{N} \frac{X_{ji}}{w_i L_i} \hat{X}_{ji}$$

The change in income due to $\hat{T}_j$ is given by the change in bilateral expenditures, as well as the factual/observed income and bilateral expenditures which are just data

Exact hat algebra

Now let's look at the counterfactual expenditure shares:

$$\lambda_{ij}^c = \frac{T_j^c\left((w_j^c)^\gamma \tau_{ij}\right)^{-\theta}}{\sum_{h=1}^N T_h^c\left((w_h^c)^\gamma \tau_{ih}\right)^{-\theta}}$$

And remember that trade costs do not change in this example

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And remember that trade costs do not change in this example

Next, put it in hat form

Exact hat algebra

$$\hat{\lambda}_{ij} = \frac{\lambda_{ij}^c}{\lambda_{ij}} = \frac{\frac{T_j^c\left((w_j^c)^\gamma \tau_{ij}\right)^{-\theta}}{T_j\left((w_j)^\gamma \tau_{ij}\right)^{-\theta}}}{\frac{\sum_{h=1}^N T_h^c\left((w_h^c)^\gamma \tau_{ih}\right)^{-\theta}}{\sum_{h=1}^N T_h\left((w_h)^\gamma \tau_{ih}\right)^{-\theta}}} = \frac{\hat{T}_j\left((\hat{w}_j)^\gamma\right)^{-\theta}}{\frac{\sum_{h=1}^N T_h^c\left((w_h^c)^\gamma \tau_{ih}\right)^{-\theta}}{\sum_{h=1}^N T_h\left((w_h)^\gamma \tau_{ih}\right)^{-\theta}}} = \frac{\hat{T}_j\left((\hat{w}_j)^\gamma\right)^{-\theta}}{\hat{P}_i^{-\theta}}$$

The change in expenditure shares is given by the changes in productivity, wages, and prices

Exact hat algebra

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The change in expenditure shares is given by the changes in productivity, wages, and prices

We still need to solve for $\hat{P}_i^{-\theta}$

Exact hat algebra

We can do so with the price index expression:

$$P_i = \kappa \left(\sum_{j=1}^{N} T_j \left[(w_j)^\gamma \tau_{ij} \right]^{-\theta} \right)^{-1/\theta}$$

Which gives us:

$$\hat{P}_i^{-\theta} = \frac{\left(\sum_{j=1}^{N} T_j^c \left[(w_j^c)^\gamma \tau_{ij} \right]^{-\theta} \right)}{\left(\sum_{h=1}^{N} T_h \left[(w_h)^\gamma \tau_{ih} \right]^{-\theta} \right)}$$

Exact hat algebra

Next we make a few (potentially odd) substitutions

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Since j is not in the denominator, we can treat the denominator as a constant and move it inside the sum over j:

$$\hat{P}_i^{-\theta} = \sum_{j=1}^{N} \frac{\left(T_j^c \left[(w_j^c)^\gamma \tau_{ij}\right]^{-\theta}\right)}{\left(\sum_{h=1}^{N} T_h \left[(w_h)^\gamma \tau_{ih}\right]^{-\theta}\right)}$$

Exact hat algebra

Next., multiply and divide each term in the sum by the factual productivity and wage terms in order to start working toward hat expressions:

$$\hat{P}_i^{-\theta} = \sum_{j=1}^{N} \left[\frac{\left(T_j^c \left[(w_j^c)^\gamma \tau_{ij} \right]^{-\theta} \right)}{\left(\sum_{h=1}^{N} T_h \left[(w_h)^\gamma \tau_{ih} \right]^{-\theta} \right)} \frac{T_j \left[w_j^\gamma \right]^{-\theta}}{T_j \left[w_j^\gamma \right]^{-\theta}} \right]$$

Exact hat algebra

We can then switch out the factual and counterfactual terms up top:

$$\hat{P}_i^{-\theta} = \sum_{j=1}^{N} \left[\frac{\left(T_j \left[(w_j)^\gamma \tau_{ij}\right]^{-\theta}\right)}{\left(\sum_{h=1}^{N} T_h \left[(w_h)^\gamma \tau_{ih}\right]^{-\theta}\right)} \frac{T_j^c \left[(w_j^c)^\gamma\right]^{-\theta}}{T_j \left[w_j^\gamma\right]^{-\theta}} \right]$$

and this gives us:

$$\hat{P}_i^{-\theta} = \sum_{j=1}^{N} \left[\lambda_{ij} \hat{T}_j \left(\hat{w}_j^\gamma\right)^{-\theta} \right]$$

The change in prices is just data and other hat variables

Exact hat algebra

Here's the three sets of equations again, they are all in terms of changes or data, so we do not need to solve for model fundamentals:

$$\hat{w}_i \hat{L}_i = \gamma \sum_{j=1}^{N} \frac{X_{ji}}{w_i L_i} \hat{X}_{ji}$$

$$\hat{\lambda}_{ij} = \frac{\hat{T}_j \Big((\hat{w}_j)^{\gamma} \Big)^{-\theta}}{\hat{P}_i^{-\theta}}$$

$$\hat{P}_i^{-\theta} = \sum_{j=1}^{N} \left[\lambda_{ij} \hat{T}_j \Big(\hat{w}_j^{\gamma} \Big)^{-\theta} \right]$$

Exact hat algebra

Given $\hat{T}_i$ we need to identify $\hat{w}_i, \hat{L}_i, \hat{P}_i, \hat{\lambda}_{ij}$

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Need another N equations to identify the change in equilibrium

These N equations are:

$$\frac{\hat{w}_i}{\hat{P}_i} = \frac{\hat{w}_j}{\hat{P}_j} = \widehat{\bar{U}} \ \forall i, j$$

and

$$\hat{L} = 1$$

Exact hat algebra

$$\hat{P}_i^{-\theta} = \sum_{j=1}^{N} \left[\lambda_{ij} \hat{T}_j \left(\hat{w}_j^{\gamma} \right)^{-\theta} \right]$$

along with the change in equilibrium requirement that:

$$\frac{\hat{w}_i}{\hat{P}_i} = \widehat{\bar{U}} \ \ \forall i$$

allows us to iterate and solve for $\hat{w}_i$ up to a normalization

Exact hat algebra

$$\hat{\lambda}_{ij} = \frac{\hat{T}_j\big((\hat{w}_j)^\gamma\big)^{-\theta}}{\hat{P}_i^{-\theta}}$$

then lets us solve for $\hat{\lambda}_{ij}$ given $\hat{P}_i$ and $\hat{w}_j$

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Last we need expenditures and labor using the first condition:

$$\hat{w}_i \hat{L}_i = \gamma \sum_{j=1}^{N} \frac{X_{ji}}{w_i L_i} \hat{X}_{ji}$$

Exact hat algebra

$$\hat{w}_i \hat{L}_i = \gamma \sum_{j=1}^{N} \frac{X_{ji}}{w_i L_i} \hat{X}_{ji}$$

Expenditures by i X_i will be equal to labor income so we have that:

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The definition of expenditure shares gives us:

$$\hat{\lambda}_{ij} = \frac{\hat{X}_{ij}}{\hat{X}_i}$$

Exact hat algebra

Combine these two to get:

$$\frac{\hat{X}_{ih}}{\hat{\lambda}_{ih}} = \gamma \sum_{j=1}^{N} \frac{X_{ji}}{w_i L_i} \hat{X}_{ji}$$

which is a set of n equations and n unknown $\hat{X}_{ji}$ for any h we choose

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We have solved for all the unobserved hat terms given some change in productivity $\hat{T}_i$

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With these we can simulate any change in productivity, and the only two model parameters we needed to take a stand on were γ and θ