Lecture 6

Projection methods

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AEM 7130

Roadmap

1. What is projection
2. How we approximate functions

Our basic dynamic model

An arbitrary infinite horizon problem can be represented using a Bellman equation

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We'll focus on value functions here, but we can approximate policy functions as well

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$$H(V) = V(S_t) - \max_{q_t} u(q_t) + \beta V(S_{t+1}(q_t))$$

We can find some function V that solves $H(V) = 0$

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We solve this by specifying some linear combination of **basis functions** $\Psi_i(\mathbf{S})$

$$V^j(\mathbf{S}|\theta) = \sum_{i=0}^{j} \theta_i \Psi_i(\mathbf{S})$$

with coefficients $\theta_0, \ldots, \theta_j$

Projection methods

We then define a residual

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We still have some choices to make:

What basis do we select?

How do we project (select the coefficients)?

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We can think of the problem as searching for some unknown conditional expectation $E[Y|X]$, given outcome variable Y and regressors X

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We don't know the true functional form, but we can approximate it using the first two monomials on X: 1 and X

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In OLS we use observed data, but in theory we use the operator $H(V)$

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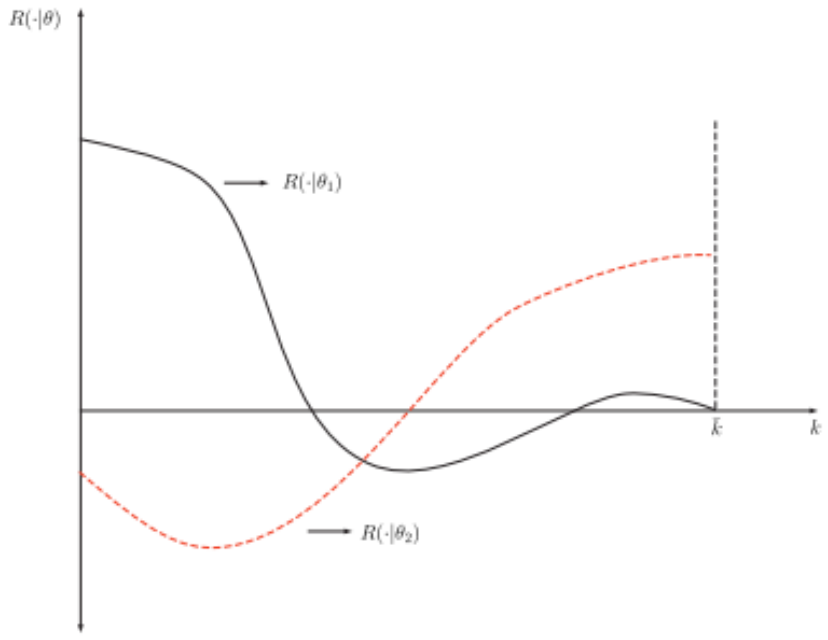
ρ tells us how close our residual function is to zero over the domain of our state space

Example residuals given different projections

Example: The figure shows two different residuals on some capital domain of $[0, \bar{k}]$

The residual based on the coefficient vector θ_1 is large for small values of capital but near-zero everywhere else

Figure 2: Residual Functions



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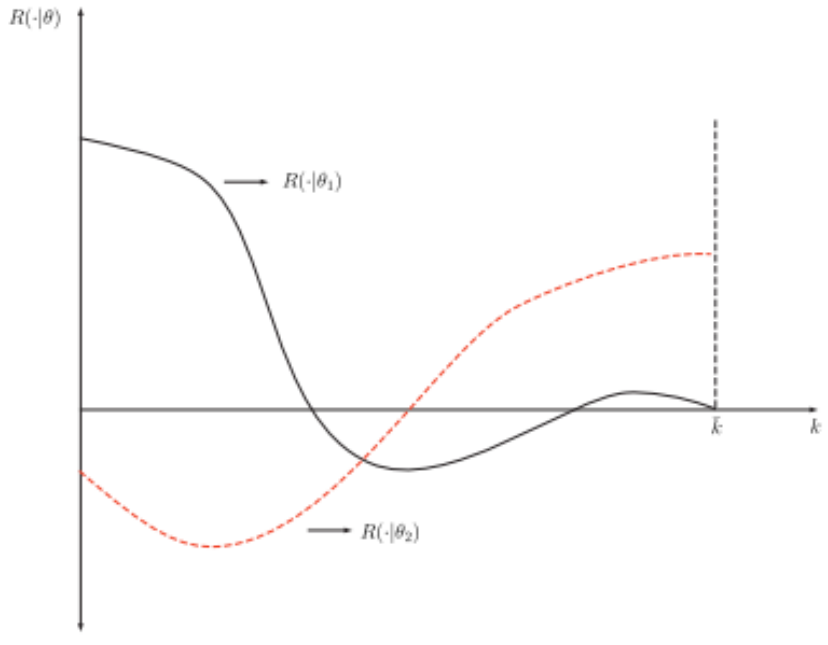

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Which is closer to zero over the interval? It will depend on our selection of ρ

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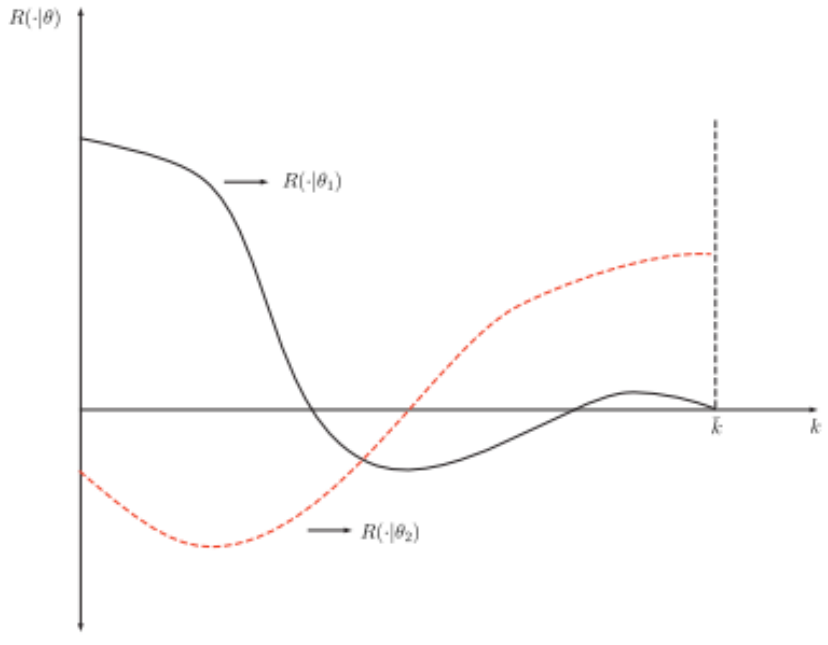

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The one-dimensional metric is defined as

$$\rho(R \cdot |\theta, 0) = \begin{cases} 0 & \text{if } \int_\Omega \phi_i(\mathbf{S}) R(\cdot|\theta) d\mathbf{S} = 0, i = 1, \ldots, j+1 \\ 1 & \text{otherwise} \end{cases}$$

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Where we want to solve for $\theta = \mathbf{argmin}\, \rho(R(\cdot|\theta), 0)$

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First lets begin with a simple example before moving into the most commonly used weight functions

Least squares projection

Suppose we selected the weight function to be

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Then we would be performing least squares! Why?

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The partial derivative weight function yields a metric function that solves the least squares problem!

Collocation

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What does this weight function mean?

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Since we have a finite set of points we do not need to solve difficult integrals but only a system of equations

$$R(\mathbf{S}_i|\theta) = 0, i = 1, \ldots, j+1$$

Rough idea of how we proceed

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But how do we implement collocation?

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We can solve this problem by *iterating* on the problem,
continually setting the residuals equal to zero, recovering new θs, and repeating

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4. Recover a set maximized continuation values at these points in our state space conditional on the previous value function approximant we used
5. Use these new maximized values to obtain updated coefficients solving the system of linear equations, and repeat the process until we have "converged"

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Therefore we can simulate anything we want and recover optimal policy functions given many different sets of initial conditions or realizations of random variables

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We often have continuous states in economics (capital, temperature, oil reserves, technology shocks, etc.), so we must have some way to reduce the infinity of points in our state space into something more manageable

We do so by selecting a specific finite number of points in our state space and use them to construct a *collocation grid* that spans the domain of our problem

Interpolation

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Note: The value function approximant is not very good outside the grid's domain since that would mean extrapolating beyond whatever information we have gained from analyzing our value function on the grid

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Each $\psi_j(x)$ is a basis function, and the coefficients c_j determine how they are combined at some point $\bar{x}$ to yield our approximation $\hat{V}(\bar{x})$ to $V(\bar{x})$

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This is what happens we select our number of grid points in the state space to be equal to the number of coefficients (which induces a Dirac delta weighting function)

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(Pseudo-)spectral methods

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The Stone-Weierstrass Theorem which states (for one dimension)

Suppose f *is a continuous real-valued function defined on the interval* $[a, b]$.
For every $\epsilon > 0$, $\exists$ *a polynomial* $p(x)$ *such that for all* $x \in [a, b]$ *we have* $||f(x) - p(x)||_{sup} \leq \epsilon$

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$||f(x) - p(x)||_{sup} \leq \epsilon$

What does the SW theorem say in words?

(Pseudo-)spectral methods

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Unfortunately we do not have infinite computational power so solving kinked problems with spectral methods is not advised

Note that the SW theorem *does not* say what kind of polynomial can approximate f arbitrarily well, just that some polynomial exists

Basis choice

What would be your first choice of basis?

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Logical choice: the monomial basis: $1, x, x^2, \ldots$

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Practice

code up a function `project_monomial(f, n, lb, ub)` that takes in some function `f`, degree of approximation `n`, lower bound `lb` and upper bound `ub`, and constructs a monomial approximation on an evenly spaced grid via collocation

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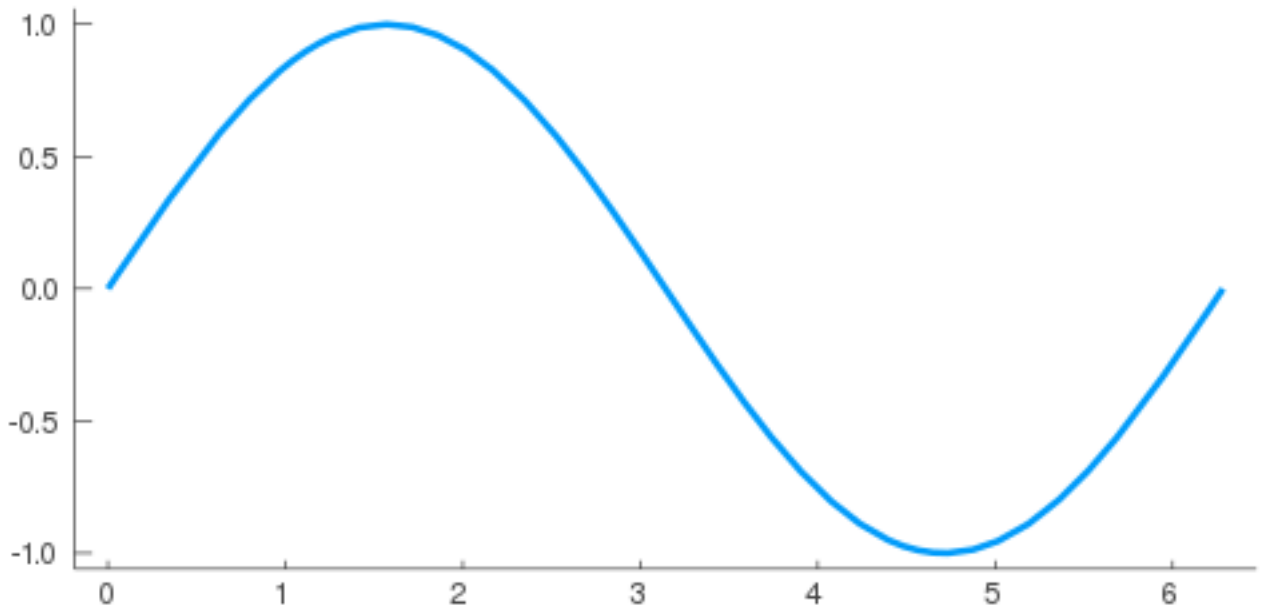
We will be plotting stuff, see http://docs.juliaplots.org/latest/generated/gr/ for example code using the `GR` backend

The monomial basis

Let's approximate `sin(x)`

```
using Plots
gr();
f(x) = sin.(x);

Plots.plot(f, 0, 2pi, line = 4, grid = false, legend = false, size = (500, 250))
```



Approximating sin(x)

```
function project_monomial(f, n, lb, ub)
    # solves Ψc = y → c = Ψ\y
    # Ψ = matrix of monomial basis functions evaluted on the grid

    coll_points = range(lb, ub, length = n)                    # collocation points
    y_values = f(coll_points)                                  # function values on the grid
    basis_functions = [coll_points.^degree for degree = 0:n-1] # vector of basis functions
    basis_matrix = hcat(basis_functions... )                   # basis matrix

    coefficients = basis_matrix\y_values                       # c = Ψ\y

    return coefficients

end;
coefficients_4 = project_monomial(f, 4, 0, 2pi);
coefficients_5 = project_monomial(f, 5, 0, 2pi);
coefficients_10 = project_monomial(f, 10, 0, 2pi)
```

```
## 10-element Array{Float64,1}:
##   0.0
##   0.9990725797458863
##   0.004015857153649684
##  -0.1738437387373486
##   0.007075663351639969
```

Approximating sin(x)

Now we need to construct a function `f_approx(coefficients, plot_points)` that takes in the `coefficients` vector, and an arbitrary vector of points to evaluate the approximant at (for plotting)

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```
function f_approx(coefficients, points)
    n = length(coefficients) - 1
    basis_functions = [coefficients[degree + 1] * points.^degree for degree = 0:n] # evaluate basis functions
    basis_matrix = hcat(basis_functions...)                                         # transform into matrix
    function_values = sum(basis_matrix, dims = 2)                                   # sum up into function value
    return function_values
end;
```

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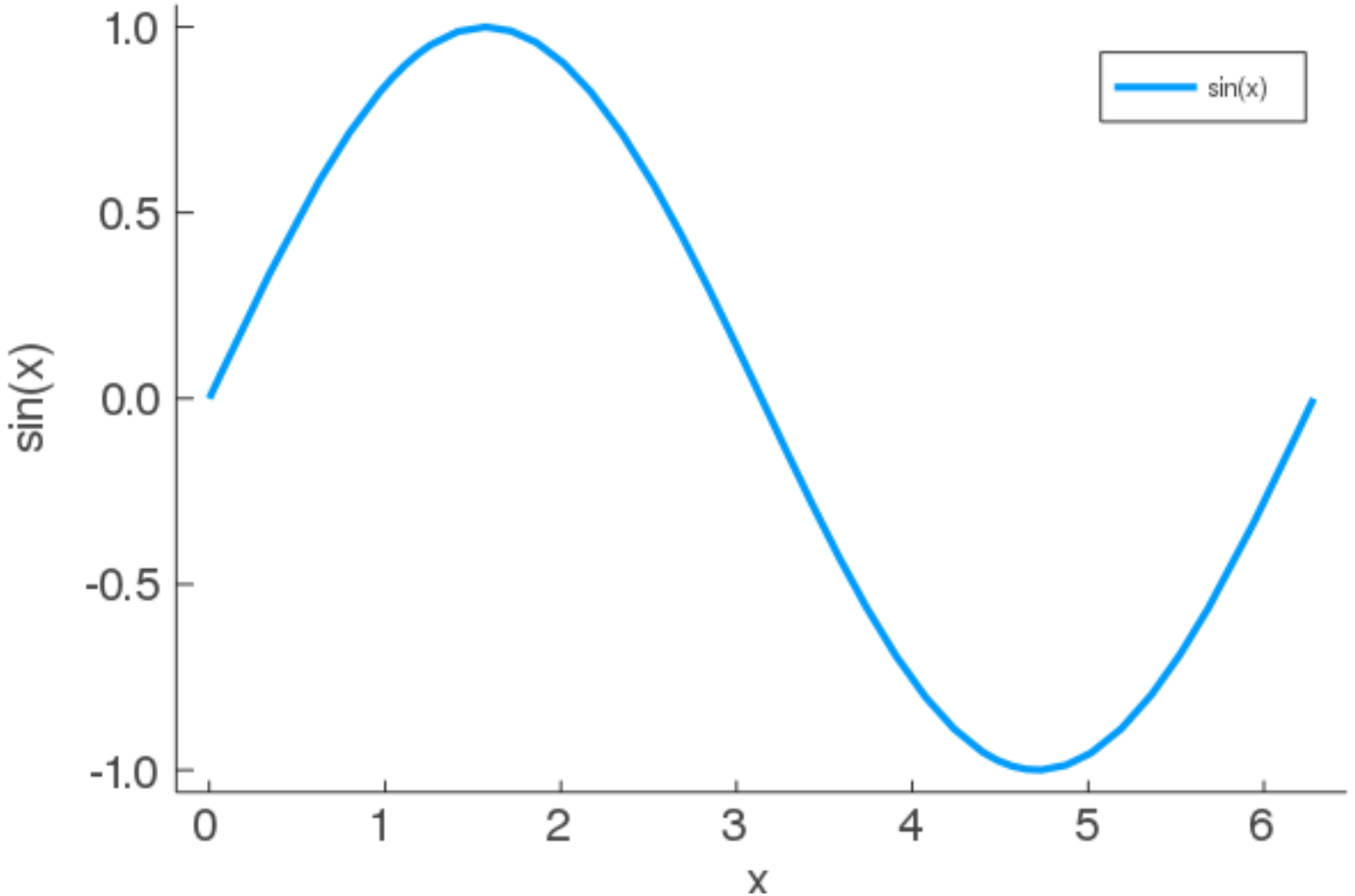
    function_values = sum(basis_matrix, dims = 2)                                   # sum up into function value
    return function_values
end;
```

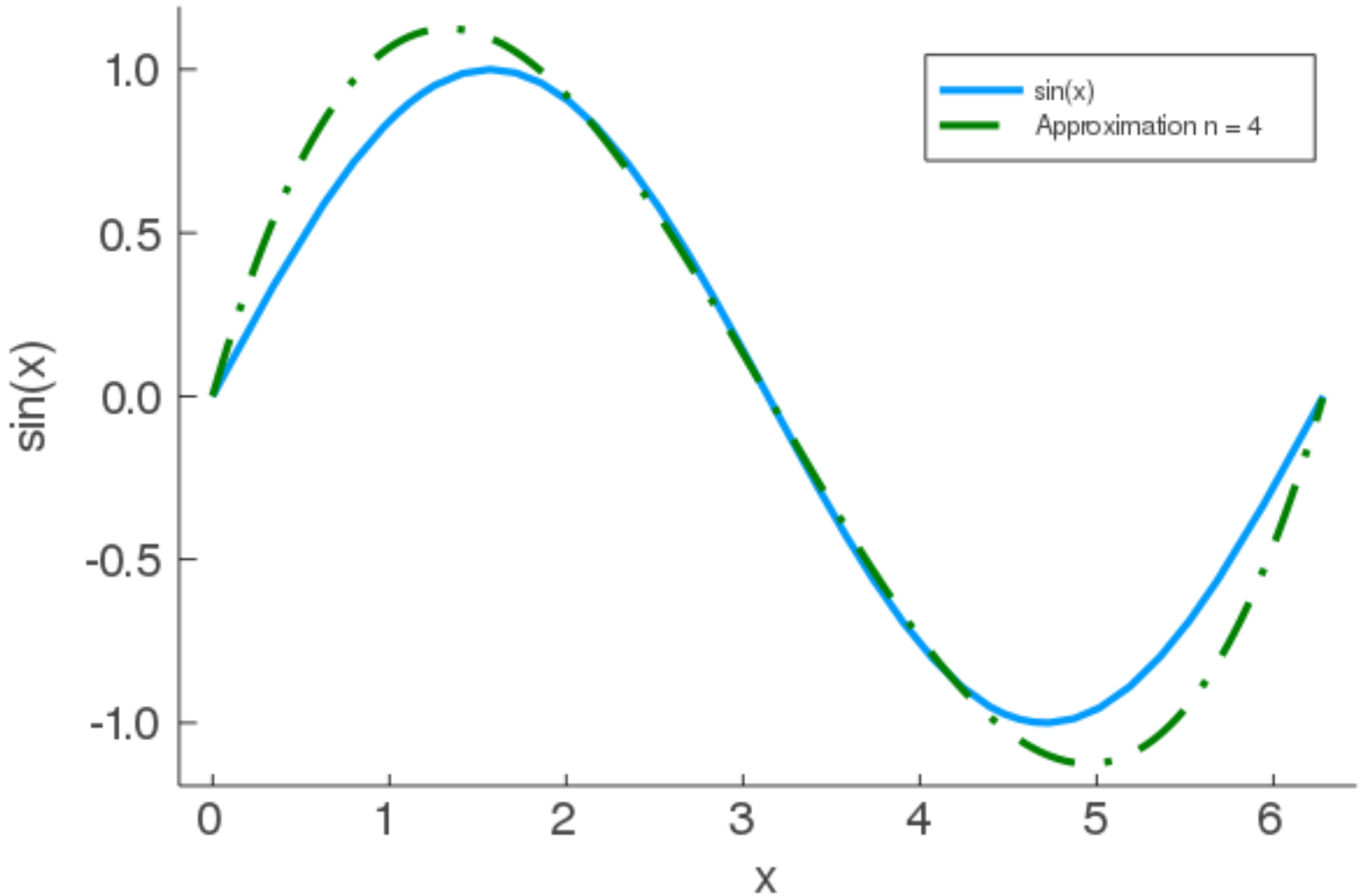
```
plot_points = 0:.01:2pi;
f_values_4 = f_approx(coefficients_4, plot_points);
f_values_5 = f_approx(coefficients_5, plot_points);
f_values_10 = f_approx(coefficients_10, plot_points)
```

```
## 629×1 Array{Float64,2}:
##   0.0
##   0.009999053610597868
```

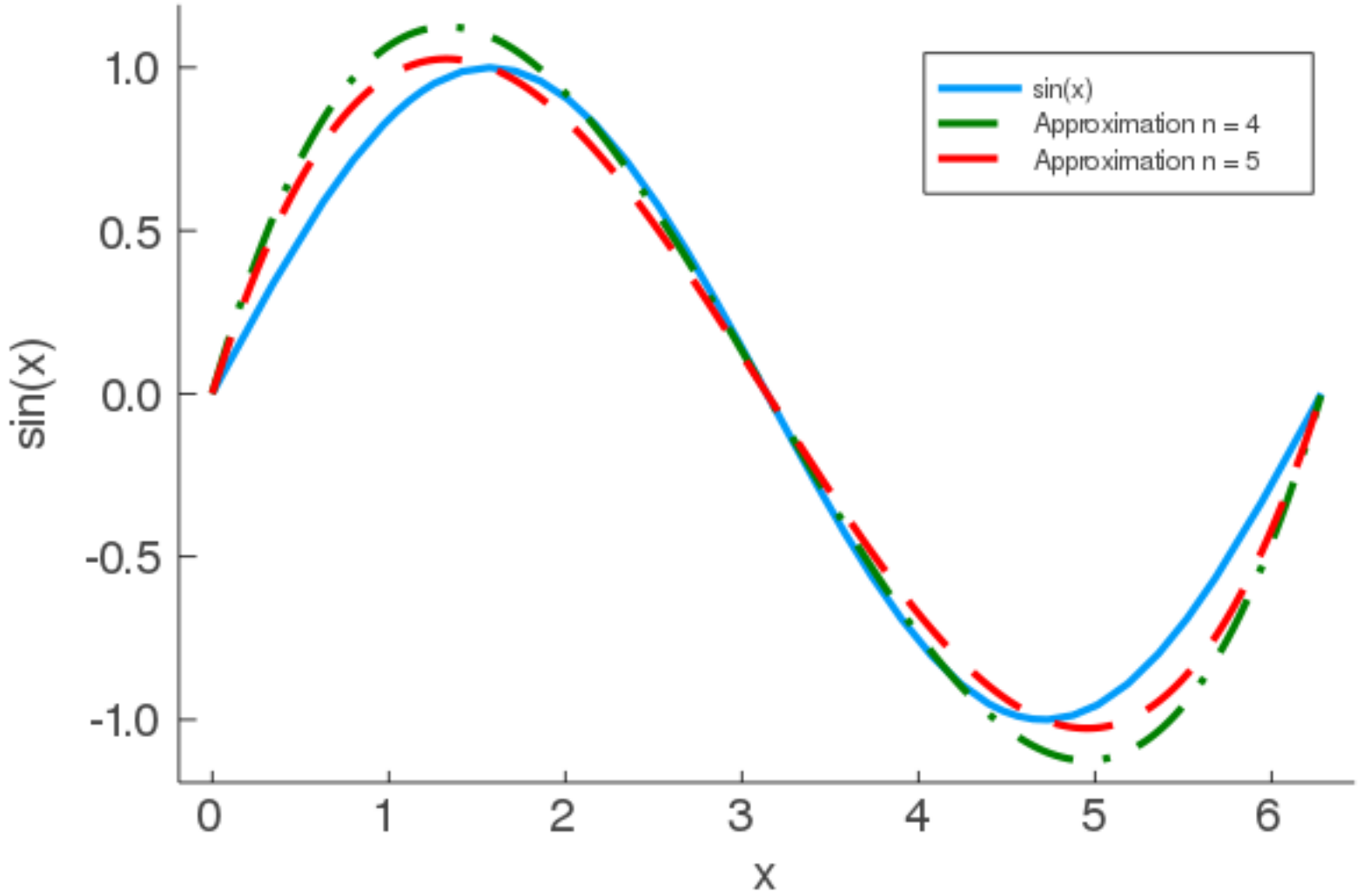
Plot



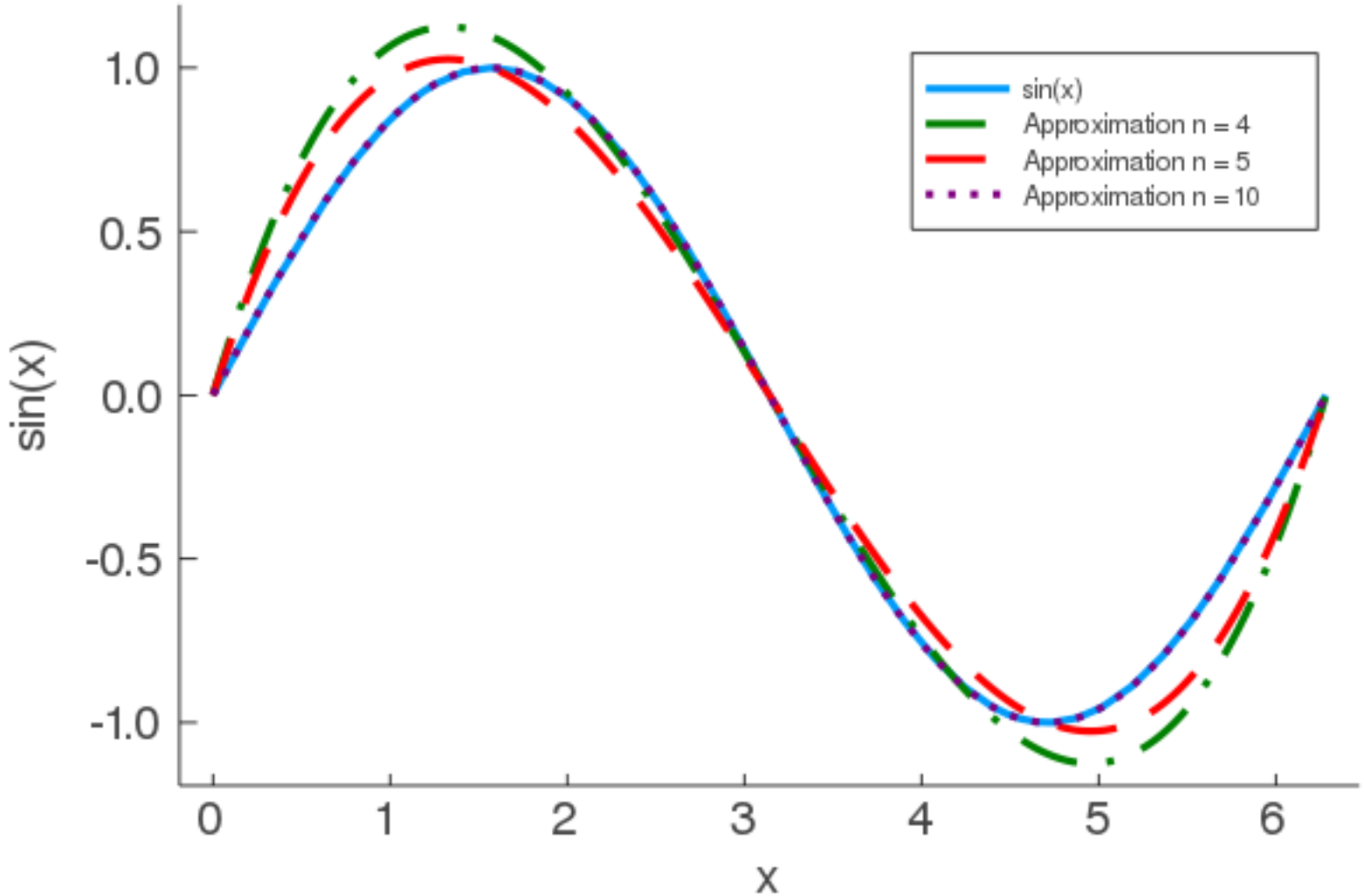
Plot



Plot


sin(x)
Approximation n = 4
Approximation n = 5
1.0
0.5
0.0
-0.5
-1.0
sin(x)
0
1
2
3
4
5
6
x

Plot


sin(x)
Approximation n = 4
Approximation n = 5
Approximation n = 10
1.0
0.5
0.0
-0.5
-1.0
sin(x)
0
1
2
3
4
5
6
x

Cool!

We just wrote some code that exploits Stone-Weierstrauss and allows us to (potentially) approximate any continuous function arbitrarily well as `n` goes to infinity

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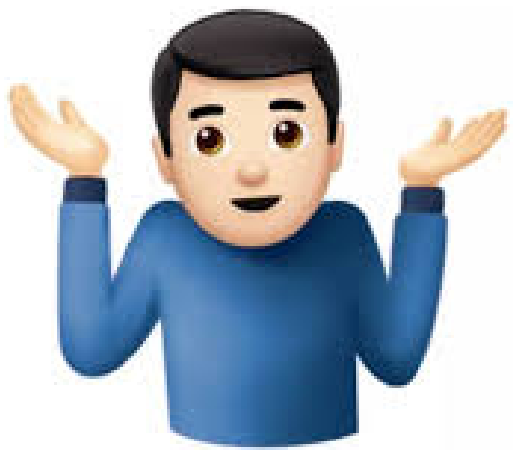

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Why?

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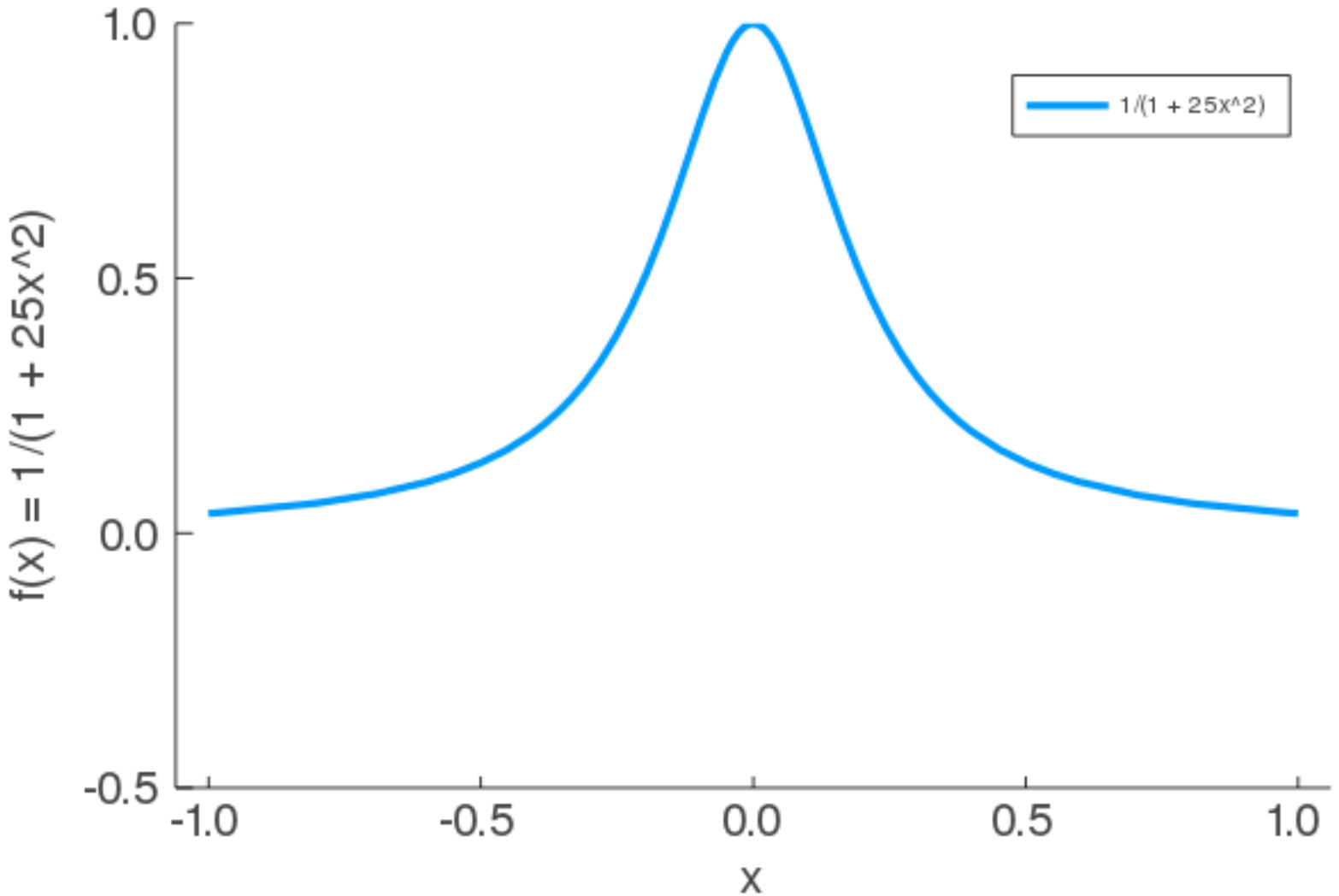
Why?

Try approximating **Runge's function**: `f(x) = 1/(1 + 25x^2)`

Runge's function

```
runge(x) = 1 ./ (1 .+ 25x.^2);
coefficients_5 = project_monomial(runge, 5, -1, 1);
coefficients_10 = project_monomial(runge, 10, -1, 1);
plot_points = -1:.01:1;
runge_values_5 = f_approx(coefficients_5, plot_points);
runge_values_10 = f_approx(coefficients_10, plot_points);
```

Runge's function



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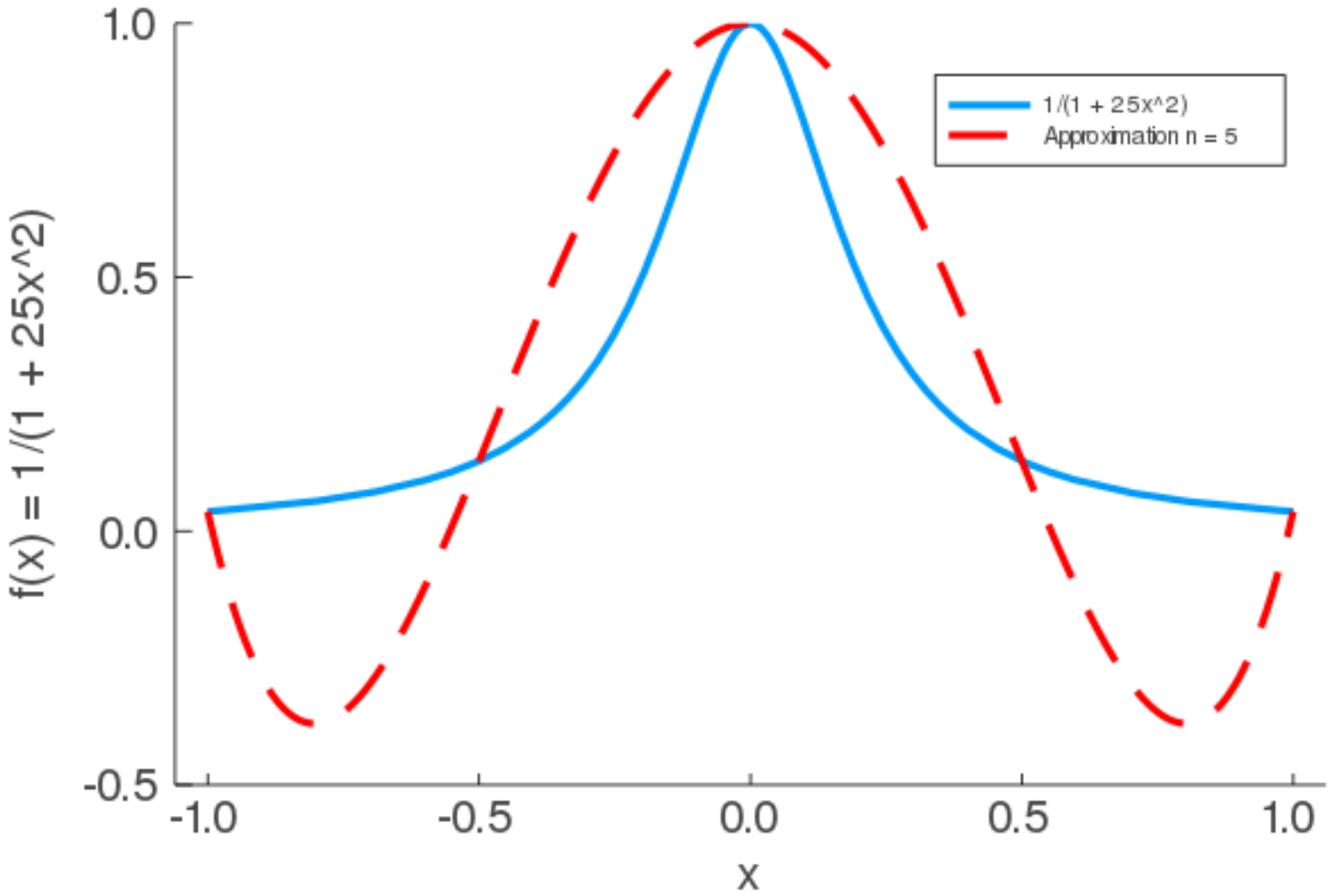
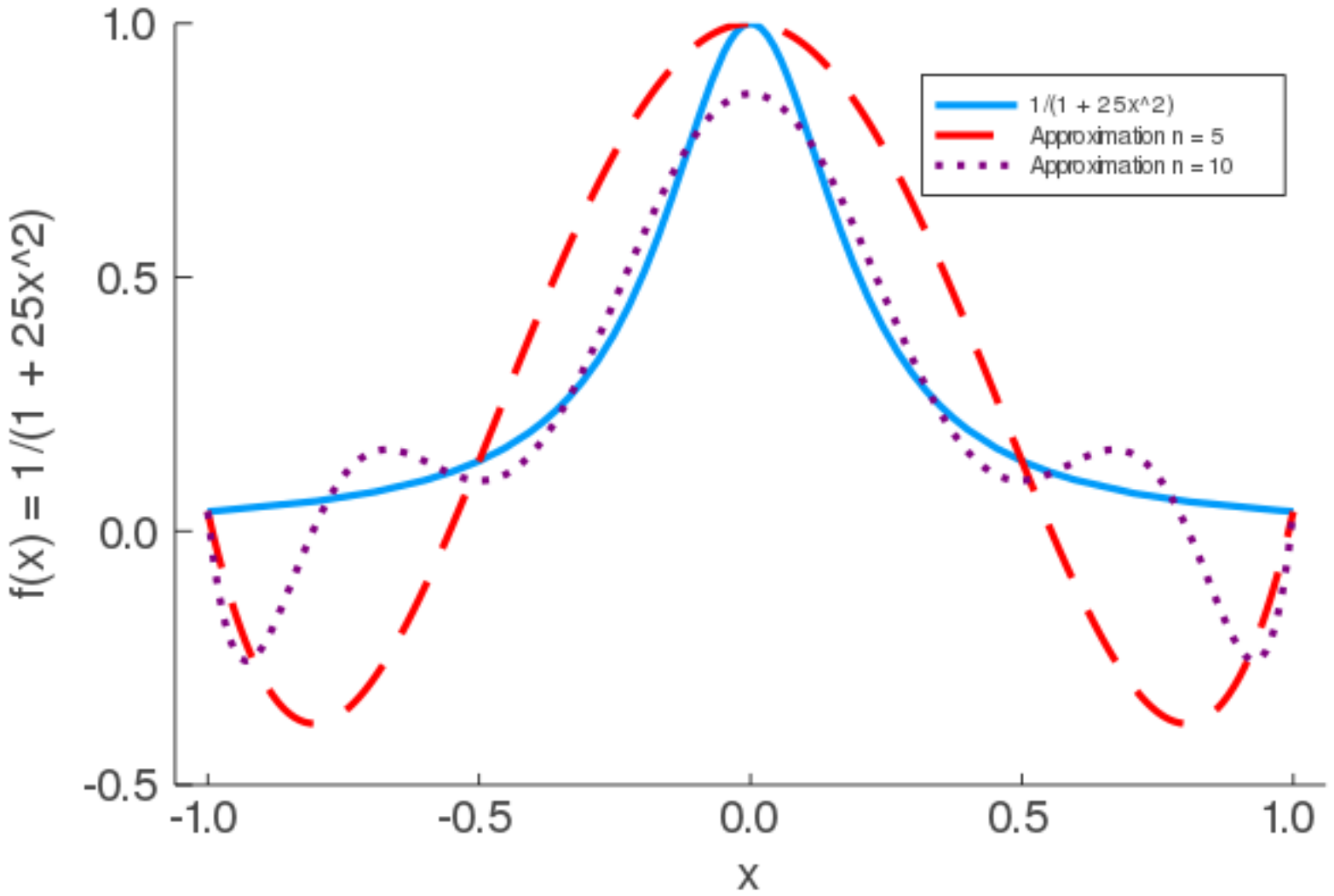
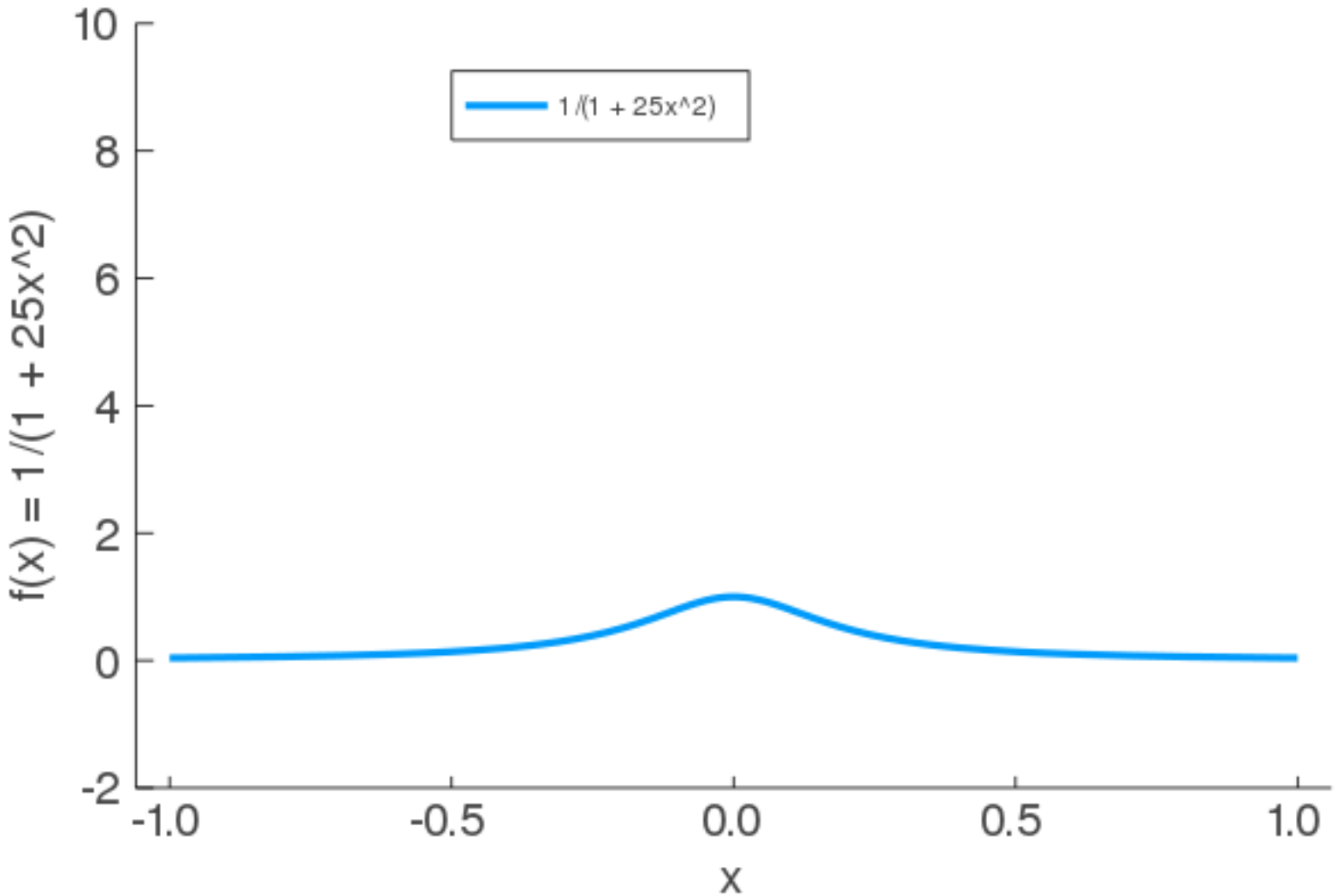


Runge's function



Maybe we can just crank up n?



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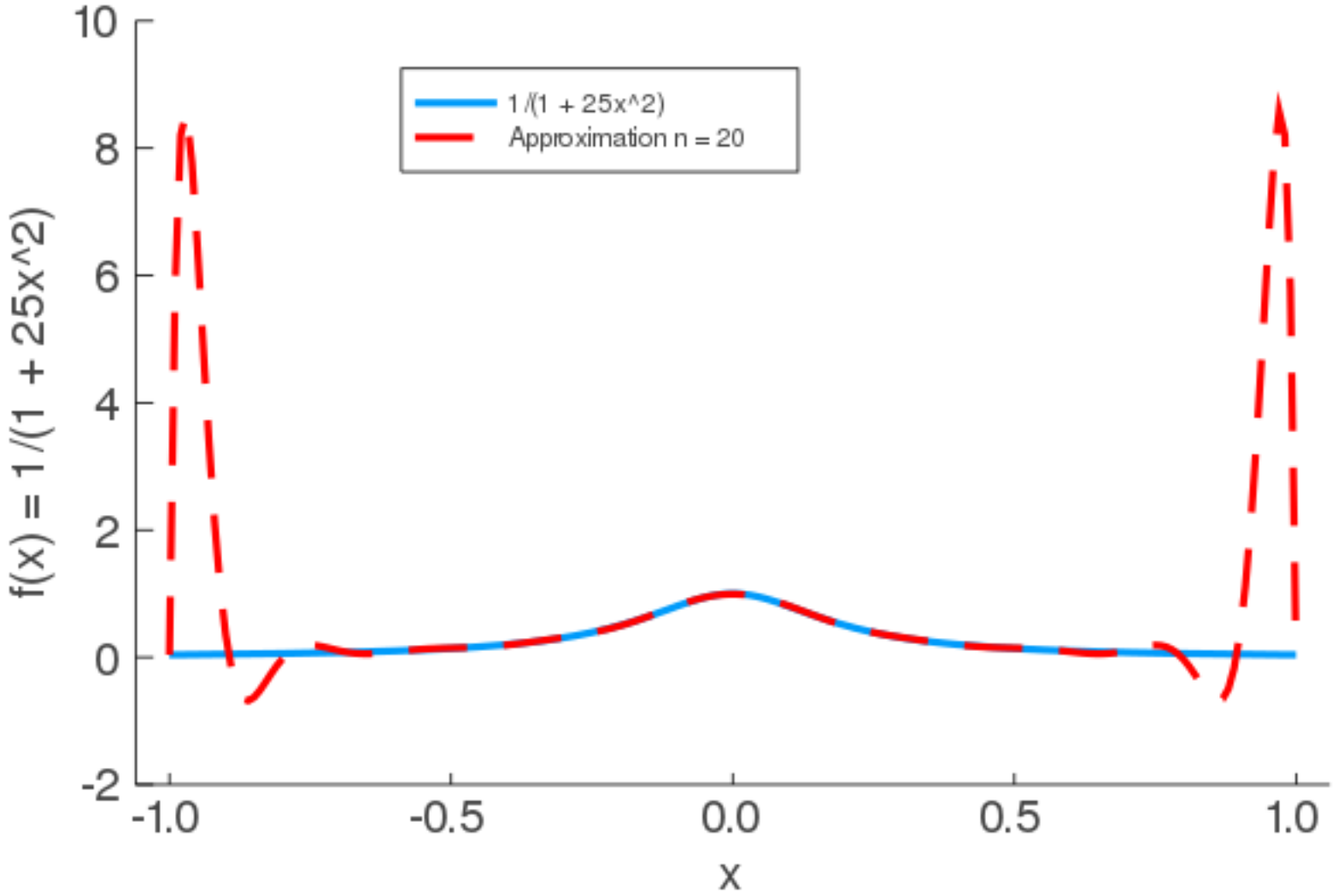

Monomials are not good

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The first 6 monomials can induce a condition number of 10^{10}, a substantial loss of precision

Second, they can vary dramatically in size, which leads to scaling/truncation errors

Monomials are not good

```
runge(x) = 1 ./ (1 .+ 25x.^2);
coefficients_10 = project_monomial(runge, 10, -1, 1);
points = rand(10);
n = length(coefficients_10) - 1;
basis_functions = [coefficients_10[degree + 1] * points.^degree for degree = 0:n];
basis_matrix = hcat(basis_functions...);
println("Condition number: $(cond(basis_matrix))")
```

```
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Ideally we want an **orthogonal basis:** when we add another element of the basis, it has sufficiently different behavior than the elements before it so it can capture features of the unknown function that the previous elements couldn't

The Chebyshev basis

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The Chebyshev polynomial closely approximates the *minimax polynomial*: the polynomial, given degree d, that minimizes any approximation error to the true function

The Chebyshev basis

Chebyshev polynomials are defined by a recurrence relation,

$$
\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_{n+1} &= 2xT_n(x) - T_{n-1}(x)
\end{aligned}
$$

and are defined on the domain $[-1, 1]$

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Chebyshev polynomials look similar to monomials but have better properties that are visually distinctive

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Write two functions: `cheb_polys(n, x)` and `monomials(n, x)` with a degree of approximation `n` and vector of points `x`, that return the values of the approximant at `x`

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Write a plotting function `plot_function(basis_function, x, n)` that takes an arbitrary basis function `basis_function` and plots all basis functions up to degree `n`

If you can't get the recurrence relation to work, you can use an alternative:

$$T_n(x) = cos(n\, arccos(x))$$

The two basis functions

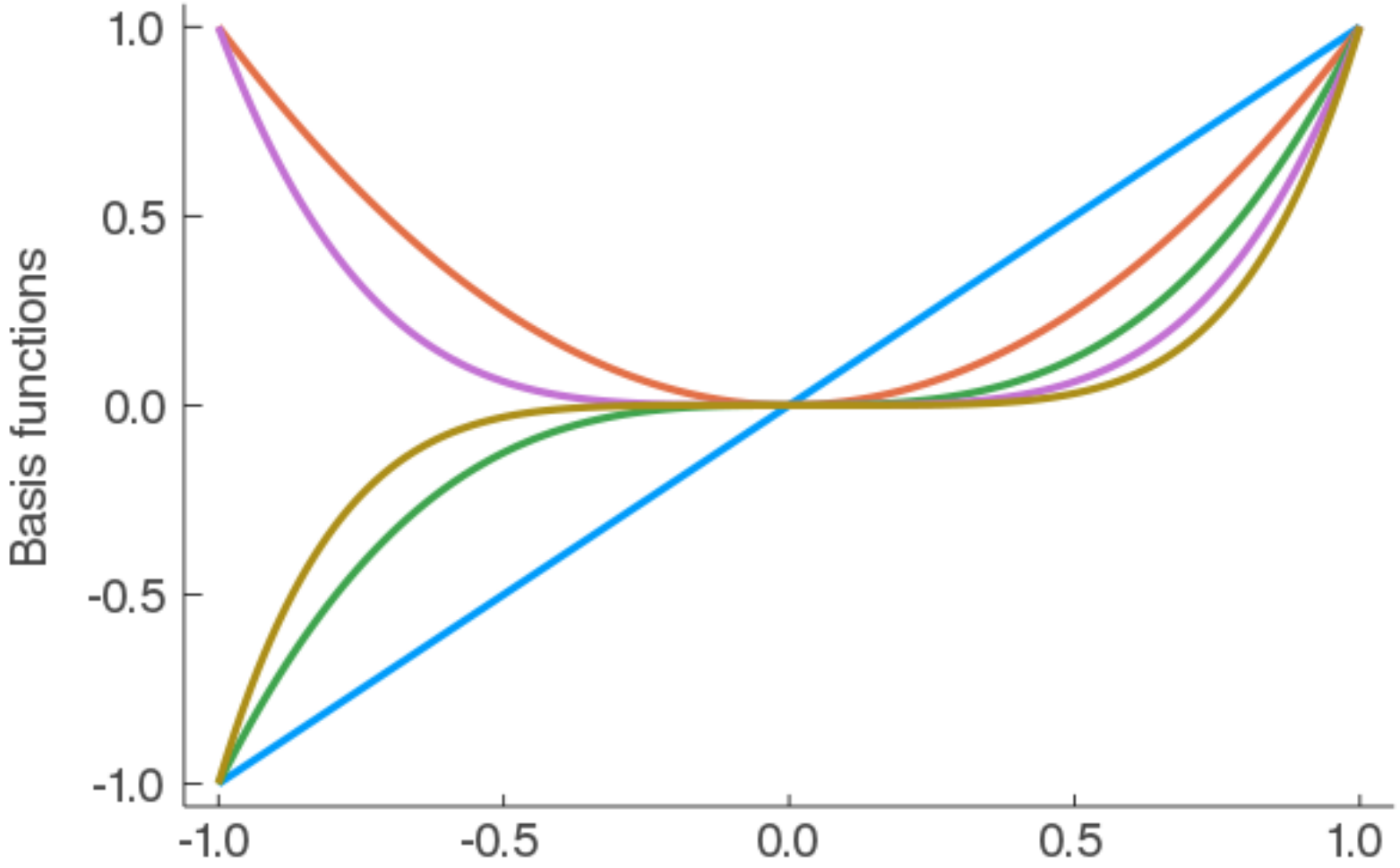
```
# Chebyshev polynomial function
function cheb_polys(x, n)
    if n == 0
        return x./x                 # T_0(x) = 1
    elseif n == 1
        return x                    # T_1(x) = x
    else
        cheb_recursion(x, n) =
            2x.*cheb_polys.(x, n - 1) .- cheb_polys.(x, n - 2)
        return cheb_recursion(x, n) # T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)
    end
end;

# Monomial function
monomials(x, n) = x.^n;
```

The plotting function

```
function plot_function(basis_function, x, n)

    for i = 1:n-1

        f_data = basis_function(x, i)

        if i == 1
            plot(x, f_data, linewidth = 4.0, xlabel = "x", ylabel = "Basis functions", label = "",
                 tickfontsize = 14, guidefontsize = 14, grid = false);
        else
            plot!(x, f_data, linewidth = 4.0, label = "");
        end

    end

    f_data = basis_function(x, n)
    plot!(x, f_data, linewidth = 4.0, label = "")

end;

x = -1:.01:1;
```

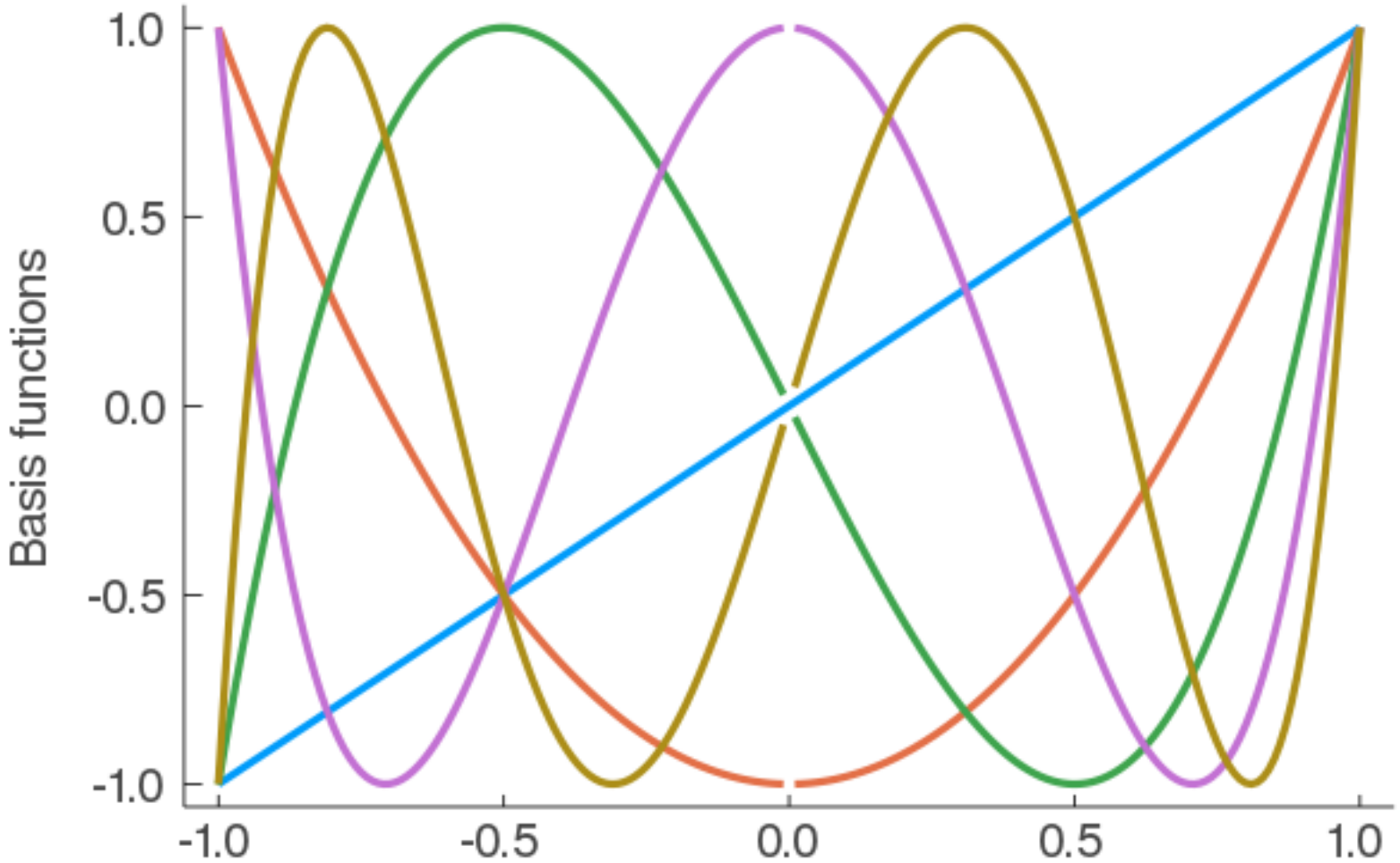
Monomials up to degree 5

```
plot_function(monomials, x, 5)
```



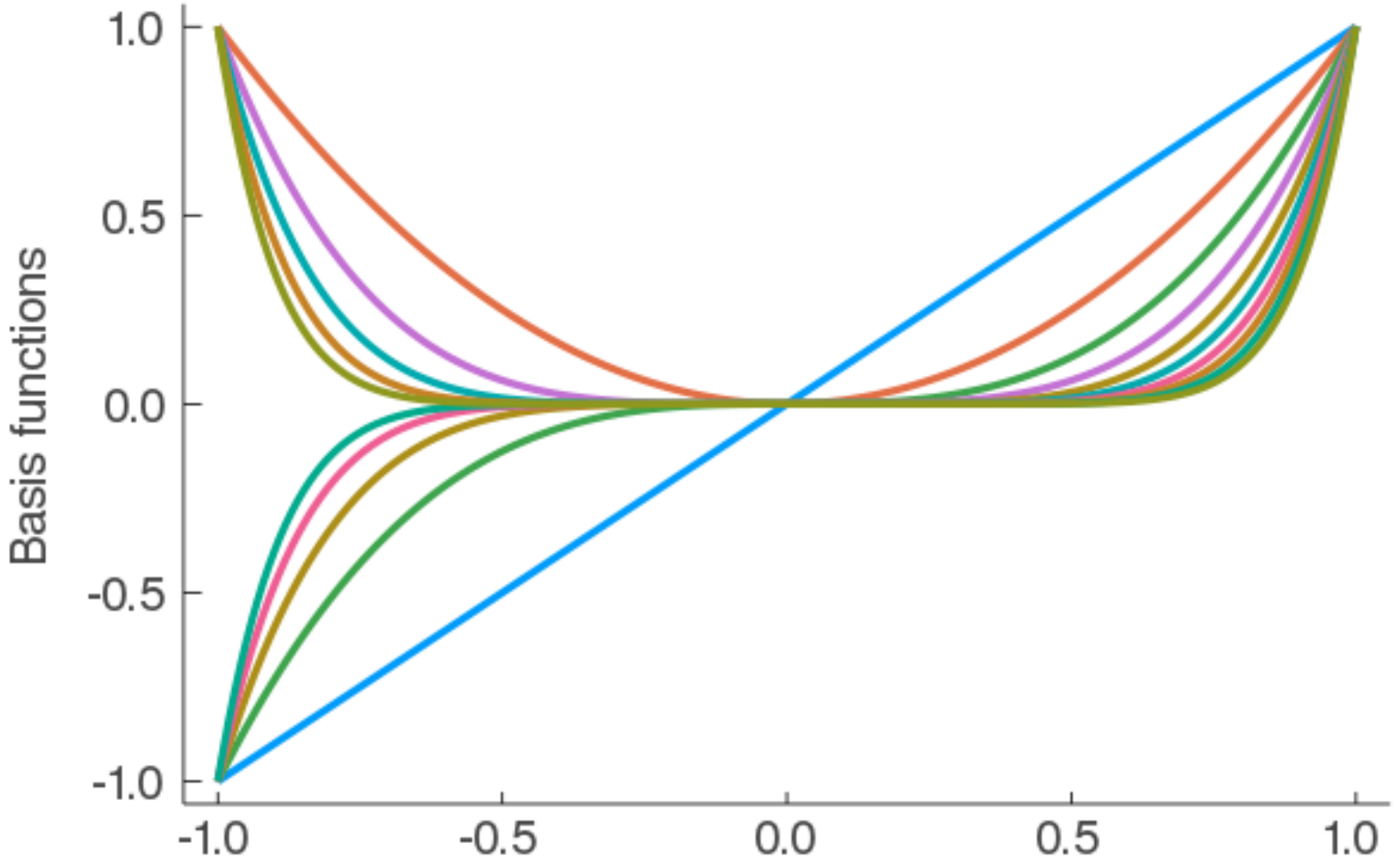
Chebyshev polynomials up to degree 5

```
plot_function(cheb_polys, x, 5)
```



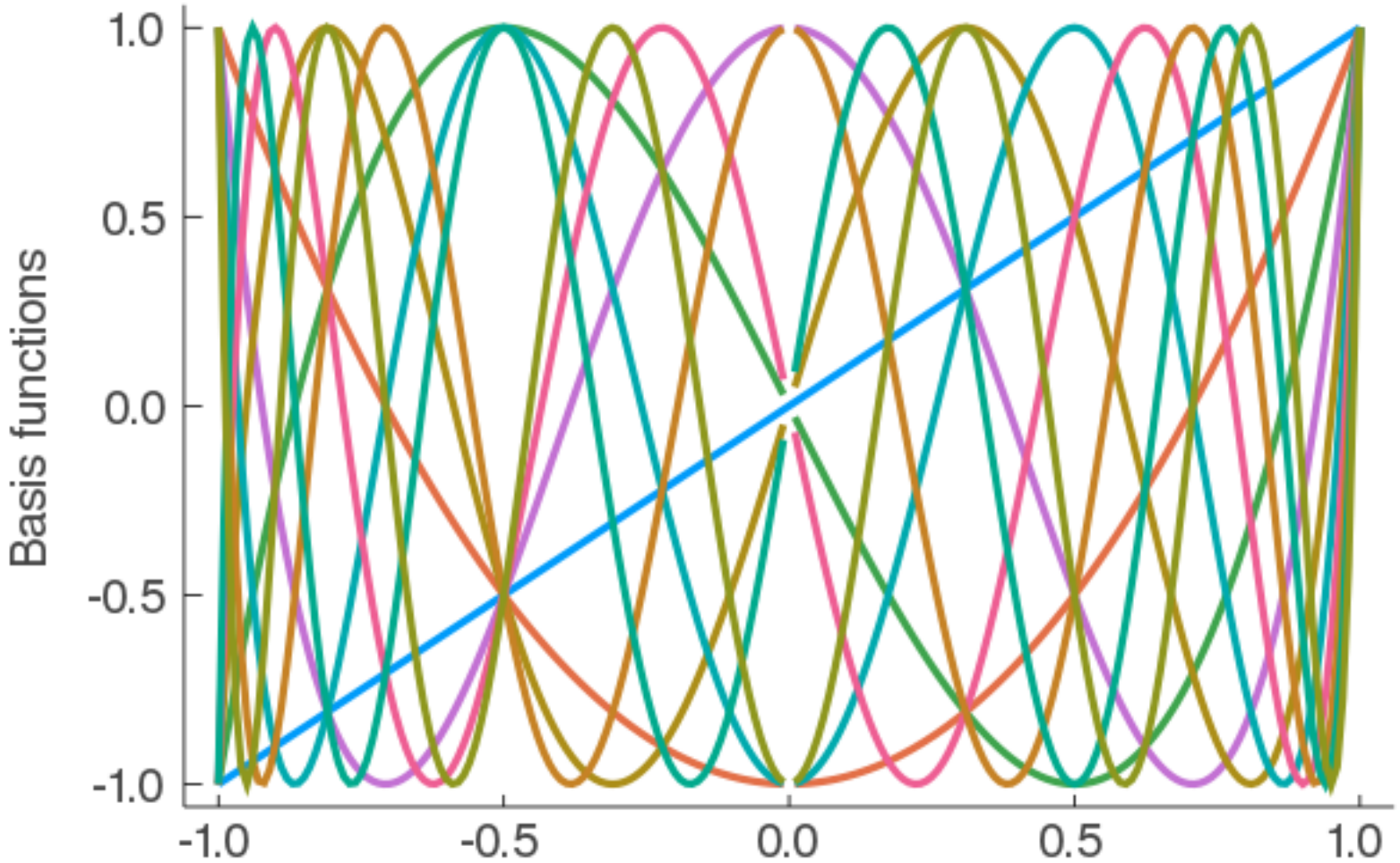
Monomials up to degree 10

```
plot_function(monomials, x, 10)
```



Chebyshev polynomials up to degree 10

```
plot_function(cheb_polys, x, 10)
```



Chebyshev polynomials rule

Chebyshev polynomials span the space

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This means that you can form any polynomial of degree equal to less than the Chebyshev polynomial you are using

It also guarantees that $\mathbf{\Phi}$ has full rank and is invertible

Chebyshev zeros and alternative rep

Also note that the Chebyshev polynomial of order n has n zeros given by

$$x_k = cos\left(\frac{2k-1}{2n}\pi\right), \quad k = 1, \ldots, n$$

which tend to cluster quadratically towards the edges of the domain

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which tend to cluster quadratically towards the edges of the domain

You can think about this as projecting sequentially finer uniform grids from a hemicircle onto the x-axis

Two important theorems

There are two important theorems to know about Chebyshev polynomials

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Chebyshev interpolation theorem: *If $f(x) \in \mathbb{C}[a,b]$, if $\{\psi_i(x), i = 0, \dots\}$ is a system of polynomials (where $\psi_i(x)$ is of exact degree i) orthogonal with respect to $\phi(x)$ on $[a,b]$ and if $p_j = \sum_{i=0}^{j} \theta_i \psi_i(x)$ interpolates $f(x)$ in the zeros of $\psi_{n+1}(x)$, then:*

$$\lim_{j\to\infty} \left(||f - p_j||_2\right)^2 = \lim_{j\to\infty} \int_a^b \phi(x)\big(f(x) - p_j\big)^2 dx = 0$$

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What does this say?

Two important theorems

If we have an approximation set of basis functions that are exact at the roots of the n^{th} order polynomials, then as n goes to infinity the approximation error becomes arbitrarily small and converges at a quadratic rate

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Unfortunately we cant store an infinite number of polynomials in our computer, we would like to know how big our error is after truncating our sequence of polynomials

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So the error in the approximation is as well!

Two important theorems

We often also have that Chebyshev approximations geometrically converge which give us the following convenient property:

$$|f(x) - f^j(x|\theta)| \sim O(\theta_j)$$

The truncation error by stopping at polynomial j is of the same order as the magnitude of the coefficient θ_j on the last polynomial

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Thus in many situations we can simply check the size of the last polynomial to gauge how accurate our approximation is

Boyd's moral principle

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John Boyd summarizes decades of experience with function approximation with his moral principle:

- When in doubt, use Chebyshev polynomials unless the solution is spatially periodic, in which case an ordinary fourier series is better
- Unless you are sure another set of basis functions is better, use Chebyshev polynomials
- Unless you are really, really sure another set of basis functions is better use Chebyshev polynomials

Grid point selection

We construct the basis function approximant by evaluating the function on a predefined grid in the domain of V

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If we have precisely n nodes, x_i, we then have

$$\sum_{j=1}^{n} c_j \phi_j(x_i) = V(x_i) \ \forall i = 1, 2, \ldots, n \quad \text{(interpolation conditions}$$

Grid point selection

We can write this problem more compactly as

$$\Phi c = y \qquad \text{(interpolation equation)}$$

where

- y is the column vector of $V(x_i)$
- c is the column vector of coefficients c_j
- Φ is an $n \times n$ matrix of the n basis functions evaluated at the n points

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How do we select our set of nodes x_i?

Chebyshev strikes again

A good selection of points are called *Chebyshev nodes*

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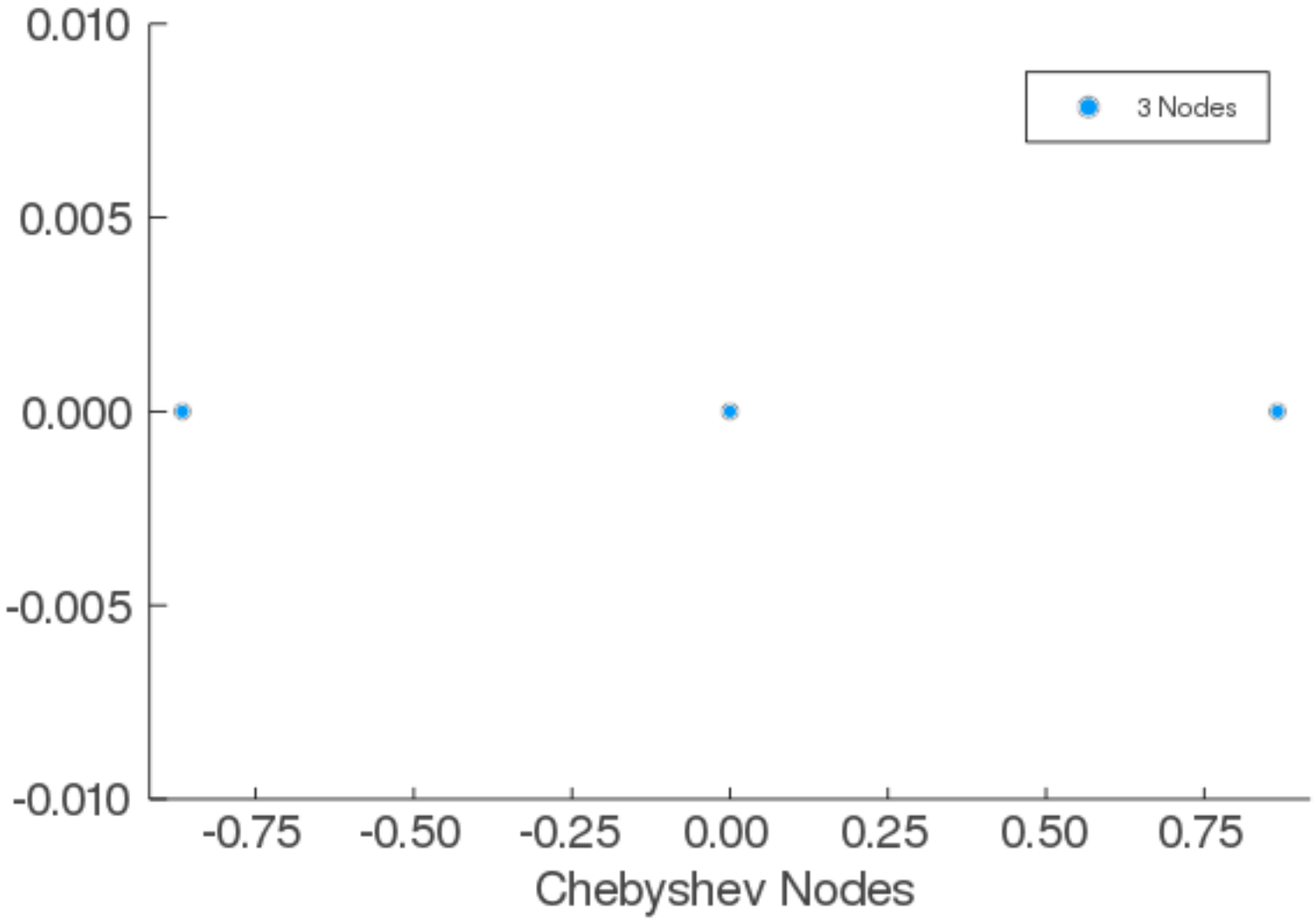
We can gain intuition by looking at a graph of where Chebyshev nodes are located, plot them yourself!

Chebyshev node function

```
cheb_nodes(n) = cos.(pi * (2*(1:n) .- 1)./(2n))
```

```
## cheb_nodes (generic function with 1 method)
```

Chebyshev node locations



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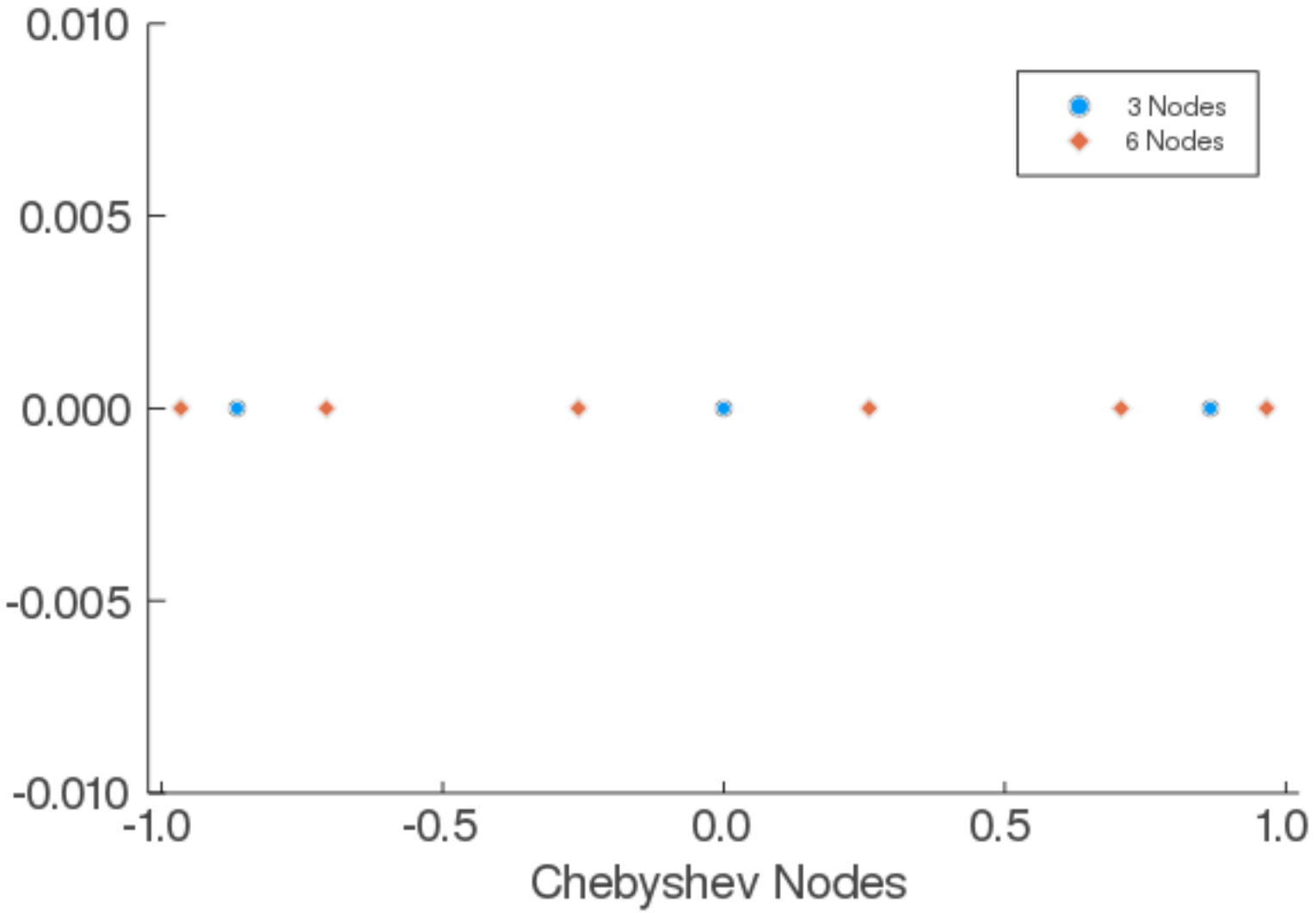

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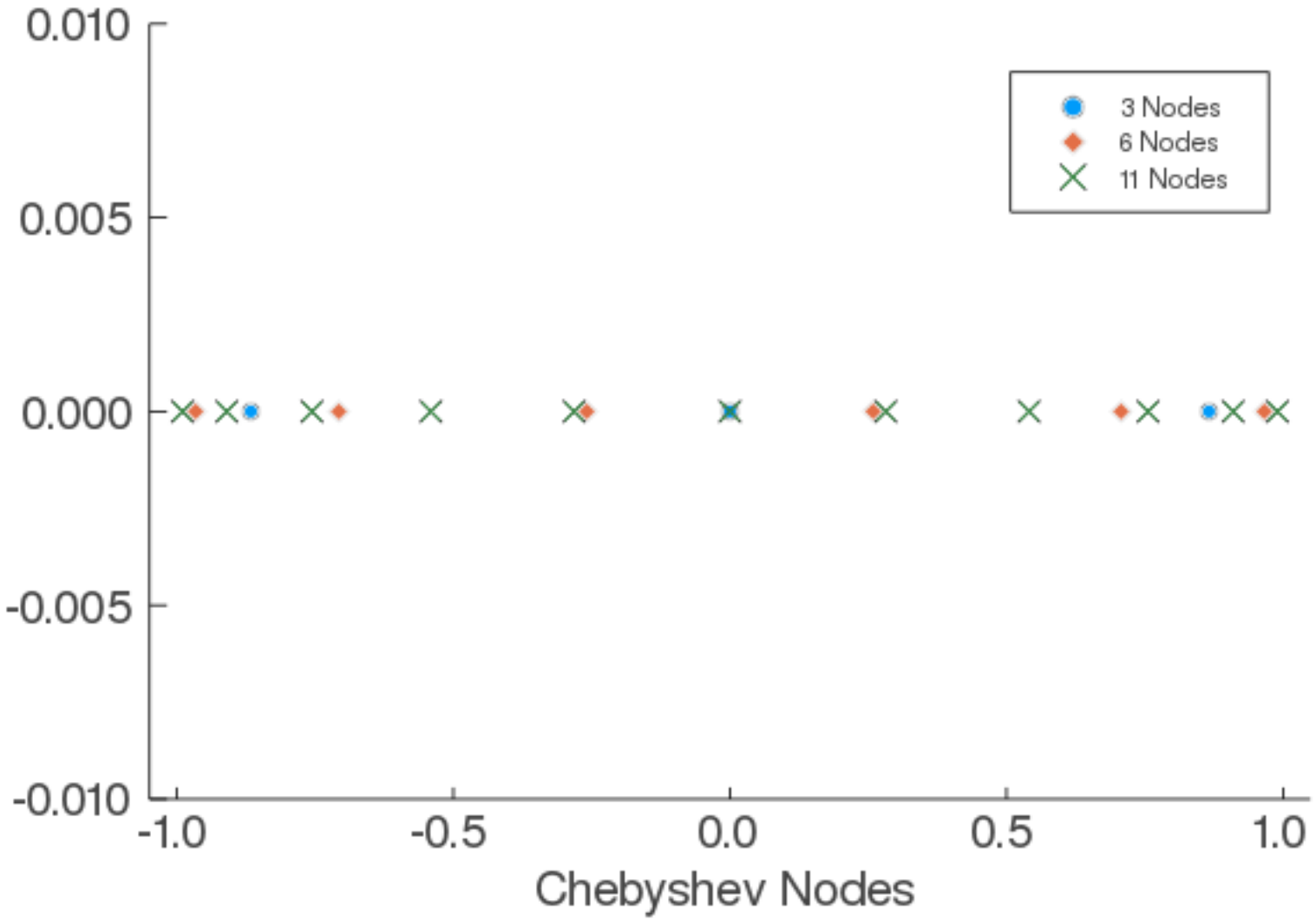

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Therefore, it's best to put more nodes in these areas to shore up this informational deficit and get good approximation quality near the edges of our domain

Discrete states

How do we handle a discrete state S_d when trying to approximate V?

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Just like you might expect, we effectively have a different function approximant over the continuous states for each value of S_d

Multi-dimensional approximation

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The tensor is just the product of every possibly polynomial pair which results in:
$[\phi_{1,1}\phi_{2,1}, \phi_{1,1}\phi_{2,2}, \phi_{1,1}\phi_{2,3}, \phi_{1,2}\phi_{2,1}, \phi_{1,2}\phi_{2,2}, \phi_{1,2}\phi_{2,3}, \phi_{1,3}\phi_{2,1}, \phi_{1,3}\phi_{2,2}, \phi_{1,3}\phi_{2,3}]$

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We can then solve for the 9 coefficients on these two dimensional polynomials

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$$\textbf{total } \# \textbf{ points} = (\textbf{points per state})^{\# \text{ states}}$$

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We will cover smart ways to deal with this later

Finite element methods

An alternative to pseudo-spectral methods is the class of **finite element methods**

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Finite element methods use basis functions that are non-zero over *subintervals* of the domain of our grid

For example, we can use **splines** (piecewise polynomials) over segments of our domains where they are spliced together at prespecified breakpoints, which are called knots

Finite element methods

The higher the order the polynomial we use, the higher the order of derivatives that we can preserve continuity at the knots

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If we have a quadratic spline, we can also preserve the first derivative's continuity at the knots, but the second derivative will be a discontinuous step function

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These are what ensure continuity of higher order derivatives at the knots as the degree of the spline grows

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In numerical models we typically don't use linear splines because we often care about the quality of approximation of higher order derivatives, cubic splines are more common

Cubic splines

Suppose we wish to approximate using a cubic spline on $N + 1$ knots

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We can obtain $3(N-1)$ equations by ensuring that the approximant is continuous at all interior knots, and its first and second derivatives are continuous at all interior knots $[3 \times (N+1-1-1)]$

This means that the value of the left cubic polynomial equals the value of the right cubic polynomial at each interior knot

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What is often used is that the approximant's first or second derivative matches that of the function at the end points

Splines can potentially handle lots of curvature

If the derivative is of interest for optimization, or to recover some variable of economic meaning, then we may need to have these derivatives preserved well at the knots

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Useful spline packages out there: `Dierckx`, `Interpolations`, `QuantEcon`

Code it up!

Let's code up our own linear spline approximation function `linear_spline_approx(f, knots)`, where `f` is the function we are approximating and `knots` are the knots

Have it return a function a function `spline_eval` that takes in `evaluation_points` as an argument where `evaluation_points` are the points we want to evaluate the spline approximant at

Hint: Linear splines are pretty easy, given two points (x_{i+1}, y_{i+1}) and (x_i, y_i), the spline in between these points is given by

$$y(x) = y_i + \frac{y_{i+1} - y_i}{x_{i+1} - x_i}(x - x_i)$$

Spline approximator

```
function linear_spline_approx(f, knots)
    function spline_eval(evaluation_points)
        prev_knot = knots[1] # initialize previous knot
        if !(typeof(evaluation_points) <: Number) # if using multiple points
            y_eval = similar(evaluation_points)
            y_index = 1
            for knot in knots[2:end]
                current_points = evaluation_points[prev_knot .≤ evaluation_points .< knot]
                y_eval[y_index:y_index + length(current_points) - 1] =
                    f(prev_knot) .+ (f(knot) - f(prev_knot))/(knot - prev_knot)*(current_points .- prev_knot)
                prev_knot = knot
                y_index += length(current_points)
            end
        else # if using just a single point
            for knot in knots[2:end]
                if prev_knot .≤ evaluation_points .< knot
                    y_eval = f(prev_knot) + (f(knot) - f(prev_knot))/(knot - prev_knot)*(evaluation_point - prev
                end
                prev_knot = knot
            end
        end
        return y_eval
    end
```

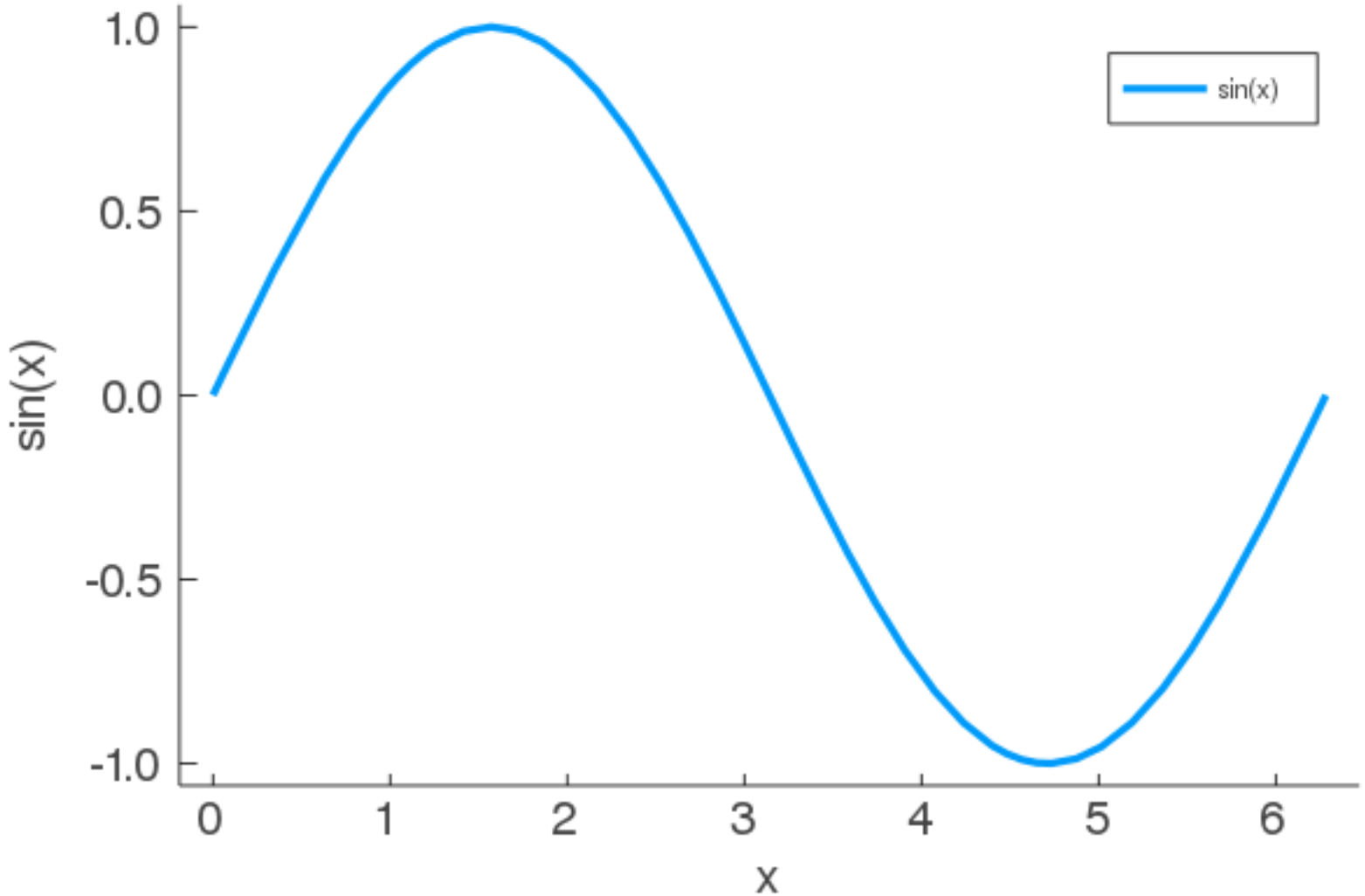
Plot

```
f(x) = sin(x)
```

```
## f (generic function with 1 method)
```

```
knots_coarse = 0:pi/2:2pi;
spline_func_coarse = linear_spline_approx(f, knots_coarse);
knots_fine = 0:pi/4:2pi;
spline_func_fine = linear_spline_approx(f, knots_fine);
knots_superfine = 0:pi/12:2pi;
spline_func_superfine = linear_spline_approx(f, knots_superfine);
x_vals =0:.05:2pi;
y_vals_coarse = spline_func_coarse(x_vals);
y_vals_fine = spline_func_fine(x_vals);
y_vals_superfine = spline_func_superfine(x_vals);
```

Plot



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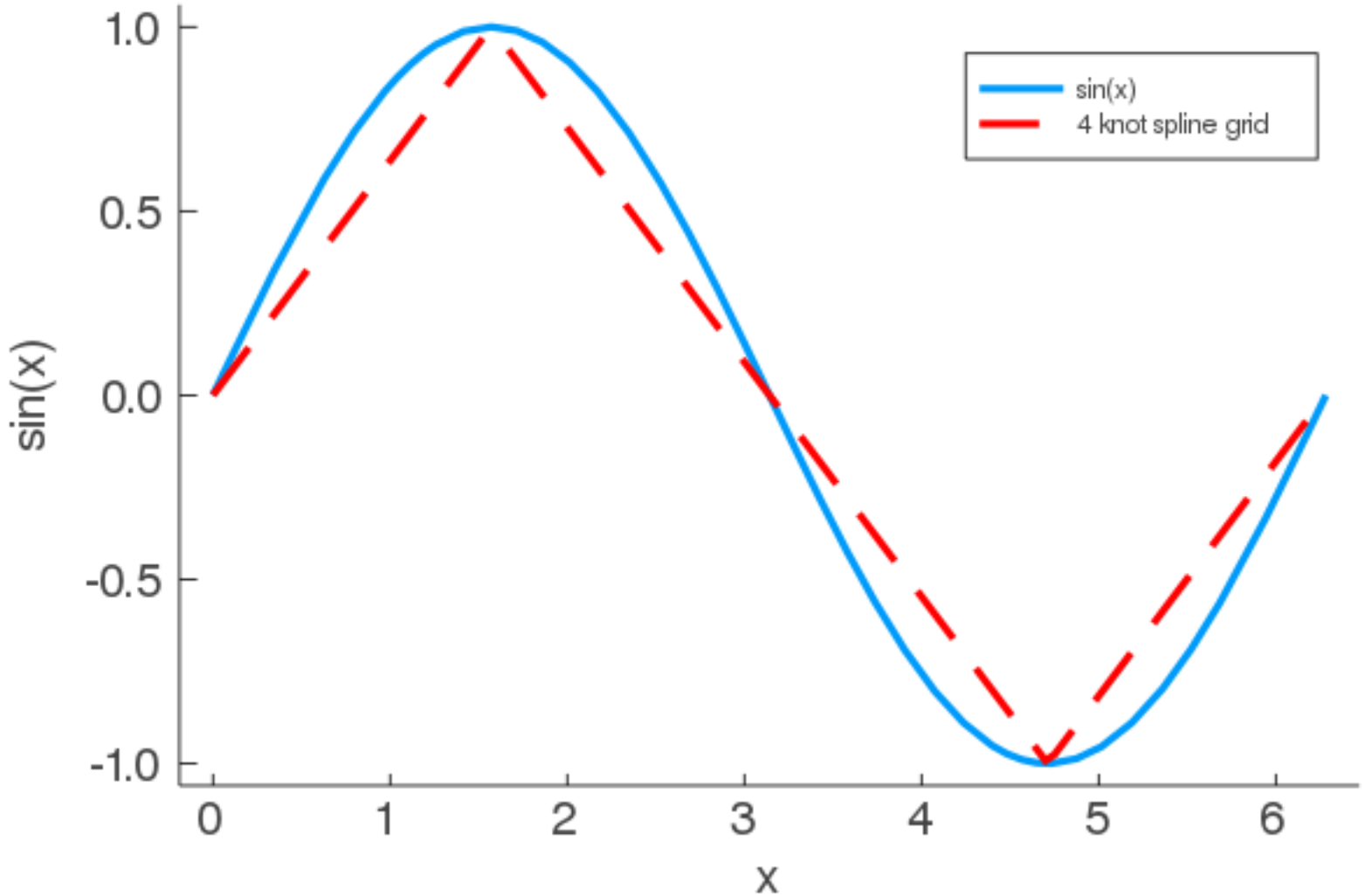

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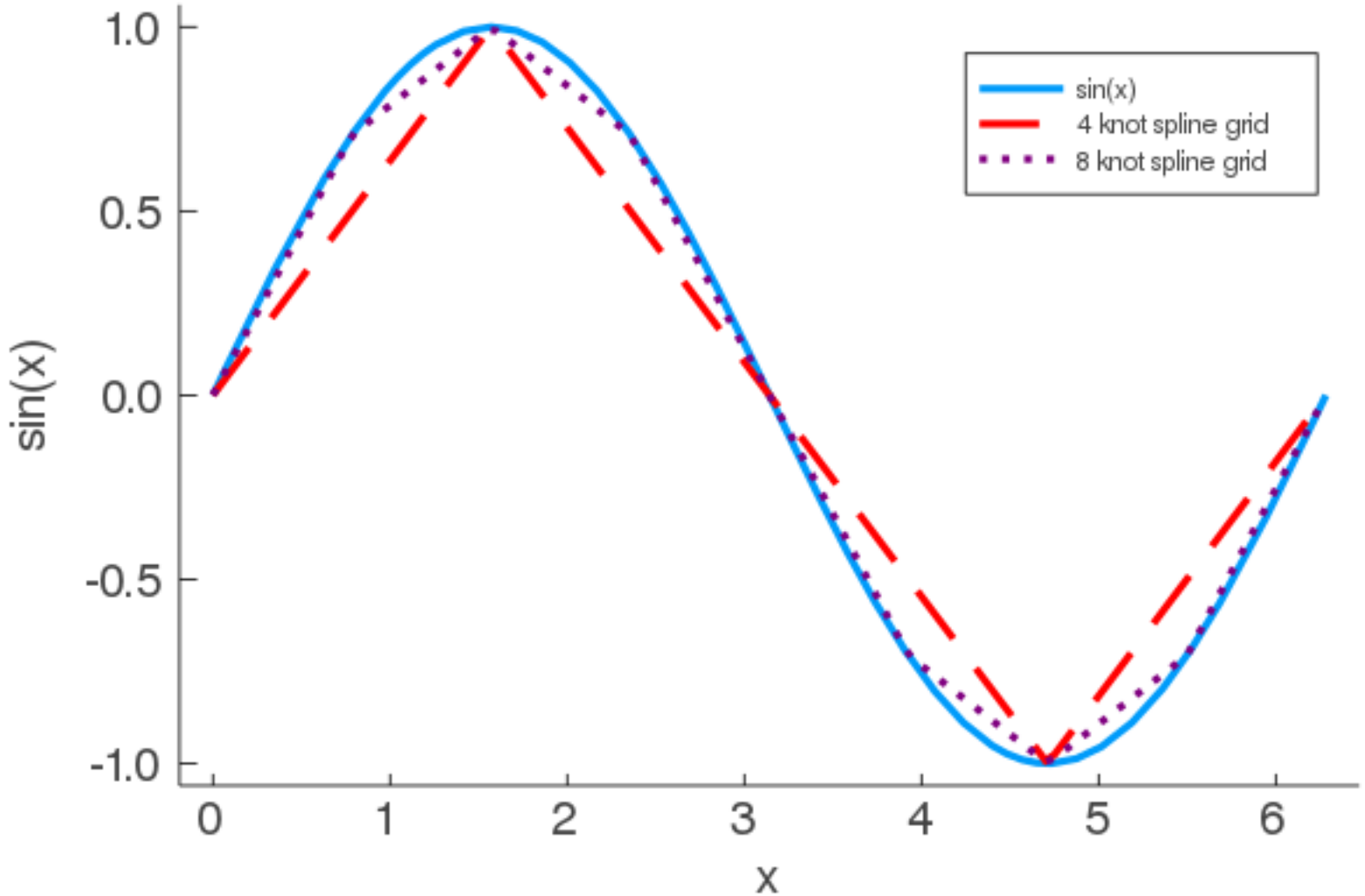
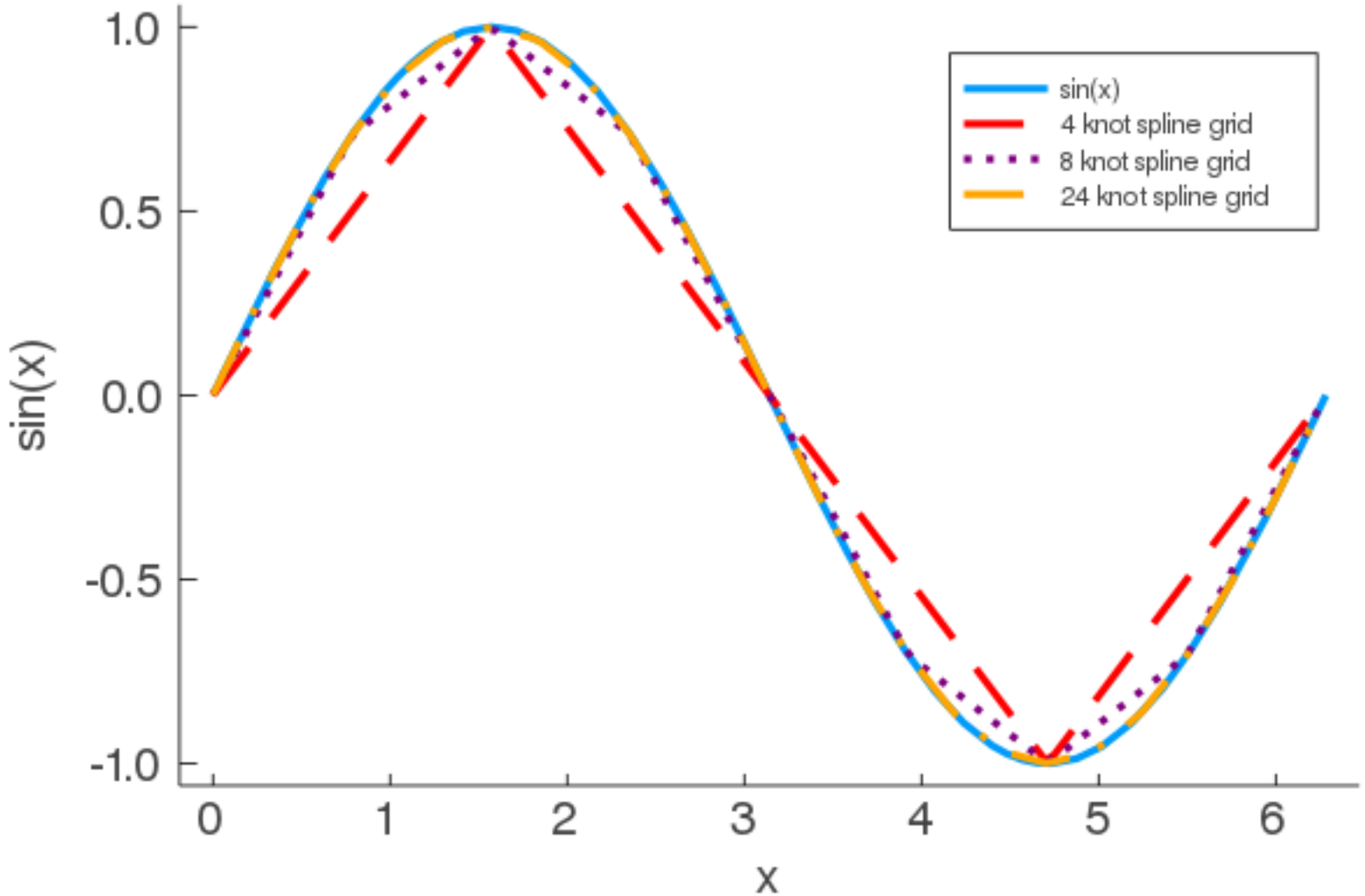

1.0
0.5
0.0
-0.5
-1.0
sin(x)
0
1
2
3
4
5
6
x
sin(x)
4 knot spline grid
8 knot spline grid

Plot


1.0
0.5
0.0
-0.5
-1.0
sin(x)
0
1
2
3
4
5
6
x
sin(x)
4 knot spline grid
8 knot spline grid
24 knot spline grid