Lecture 6

Projection methods

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AEM 7130

Roadmap

1. What is projection
2. How we approximate functions

Our basic dynamic model

An arbitrary infinite horizon problem can be represented using a Bellman equation

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We'll focus on value functions here, but we can approximate policy functions as well

Projection methods

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Rearrange the Bellman equation and define a new functional H that maps the problem into a more general framework

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$$H(V) = V(S_t) - \max_{q_t} u(q_t) + \beta V(S_{t+1}(q_t))$$

We can find some function V that solves $H(V) = \mathbf{0}$

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We solve this by specifying some linear combination of **basis functions** $\Psi_i(\mathbf{S})$

$$V^j(\mathbf{S}|\theta) = \sum_{i=0}^{j} \theta_i \Psi_i(\mathbf{S})$$

with coefficients $\theta_0, \ldots, \theta_j$

Projection methods

We then define a residual

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We still have some choices to make:

What basis do we select?

How do we project (select the coefficients)?

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We can think of the problem as searching for some unknown conditional expectation $E[Y|X]$, given outcome variable Y and regressors X

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In OLS we use observed data, but in theory we use the operator $H(V)$

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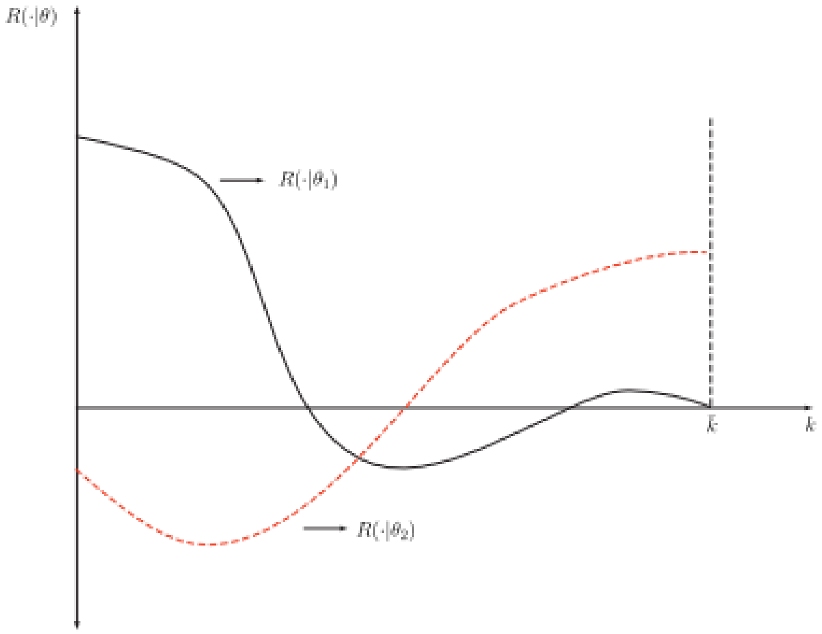
ρ tells us how close our residual function is to zero over the domain of our state space

Example residuals given different projections

Example: The figure shows two different residuals on some capital domain of $[0, \bar{k}]$

The residual based on the coefficient vector θ_1 is large for small values of capital but near-zero everywhere else

Figure 2: Residual Functions



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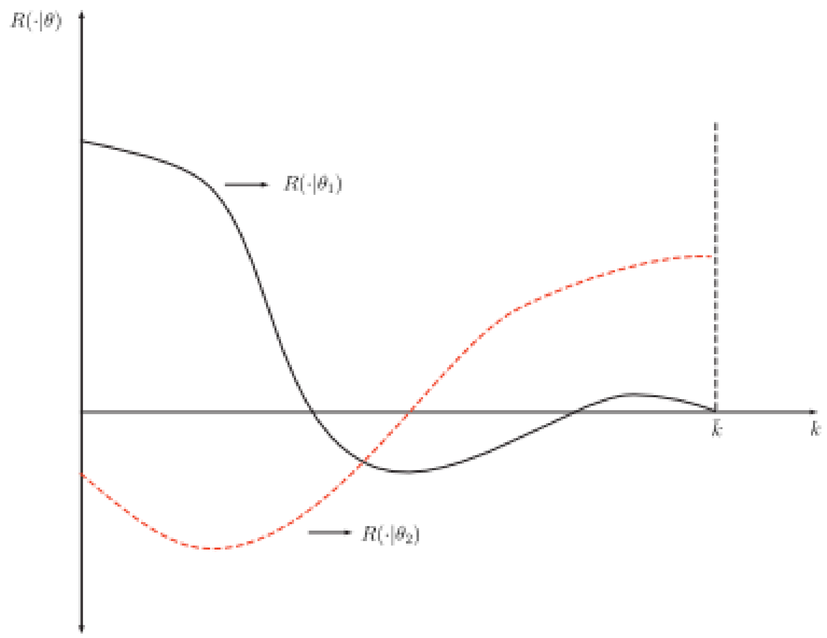

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Which is closer to zero over the interval? It will depend on our selection of ρ

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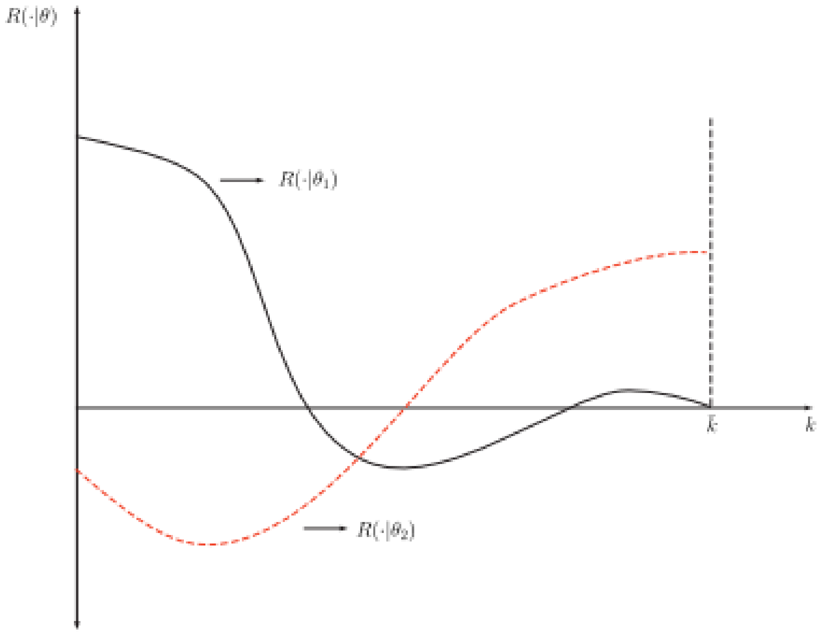

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The one-dimensional metric is defined as

$$\rho(R \cdot |\theta, 0) = \begin{cases} 0 & \text{if } \int_\Omega \phi_i(\mathbf{S}) R(\cdot|\theta) d\mathbf{S} = 0, i = 1, \ldots, j+1 \\ 1 & \text{otherwise} \end{cases}$$

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Where we want to solve for $\theta = \mathbf{argmin}\, \rho(R(\cdot|\theta), 0)$

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First lets begin with a simple example before moving into the most commonly used weight functions

Least squares projection

Suppose we selected the weight function to be

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Then we would be performing least squares! Why?

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The partial derivative weight function yields a metric function that solves the least squares problem!

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What does this weight function mean?

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Since we have a finite set of points we do not need to solve difficult integrals but only a system of equations

$$R(\mathbf{S}_i|\theta) = 0, i = 1, \ldots, j + 1$$

Rough idea of how we proceed

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But how do we implement collocation?

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We can solve this problem by *iterating* on the problem,
continually setting the residuals equal to zero, recovering new θs, and repeating

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4. Recover a set maximized continuation values at these points in our state space conditional on the previous value function approximant we used
5. Use these new maximized values to obtain updated coefficients solving the system of linear equations, and repeat the process until we have "converged"

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Therefore we can simulate anything we want and recover optimal policy functions given many different sets of initial conditions or realizations of random variables

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We do so by selecting a specific finite number of points in our state space and use them to construct a *collocation grid* that spans the domain of our problem

Interpolation

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Note: The value function approximant is not very good outside the grid's domain since that would mean extrapolating beyond whatever information we have gained from analyzing our value function on the grid

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Each $\psi_j(x)$ is a basis function, and the coefficients c_j determine how they are combined at some point $\bar{x}$ to yield our approximation $\hat{V}(\bar{x})$ to $V(\bar{x})$

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This is what happens we select our number of grid points in the state space to be equal to the number of coefficients (which induces a Dirac delta weighting function)

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$$\mathbf{c} = (\Psi'\Psi)^{-1}\Psi'\mathbf{y}$$

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Why?

The Stone-Weierstrass Theorem which states (for one dimension)

Suppose f is a continuous real-valued function defined on the interval $[a, b]$.
For every $\epsilon > 0, \ \exists$ a polynomial $p(x)$ such that for all $x \in [a, b]$ we have
$||f(x) - p(x)||_{sup} \leq \epsilon$

(Pseudo-)spectral methods

Spectral methods apply all of our basis functions to the entire domain of our grid: they are global

When using spectral methods we virtually always use polynomials

Why?

The Stone-Weierstrass Theorem which states (for one dimension)

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What does the SW theorem say in words?

(Pseudo-)spectral methods

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Note that the SW theorem *does not* say what kind of polynomial can approximate f arbitrarily well, just that some polynomial exists

Basis choice

What would be your first choice of basis?

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Practice

code up a function `project_monomial(f, n, lb, ub)` that takes in some function `f`, degree of approximation `n`, lower bound `lb` and upper bound `ub`, and constructs a monomial approximation on an evenly spaced grid via collocation

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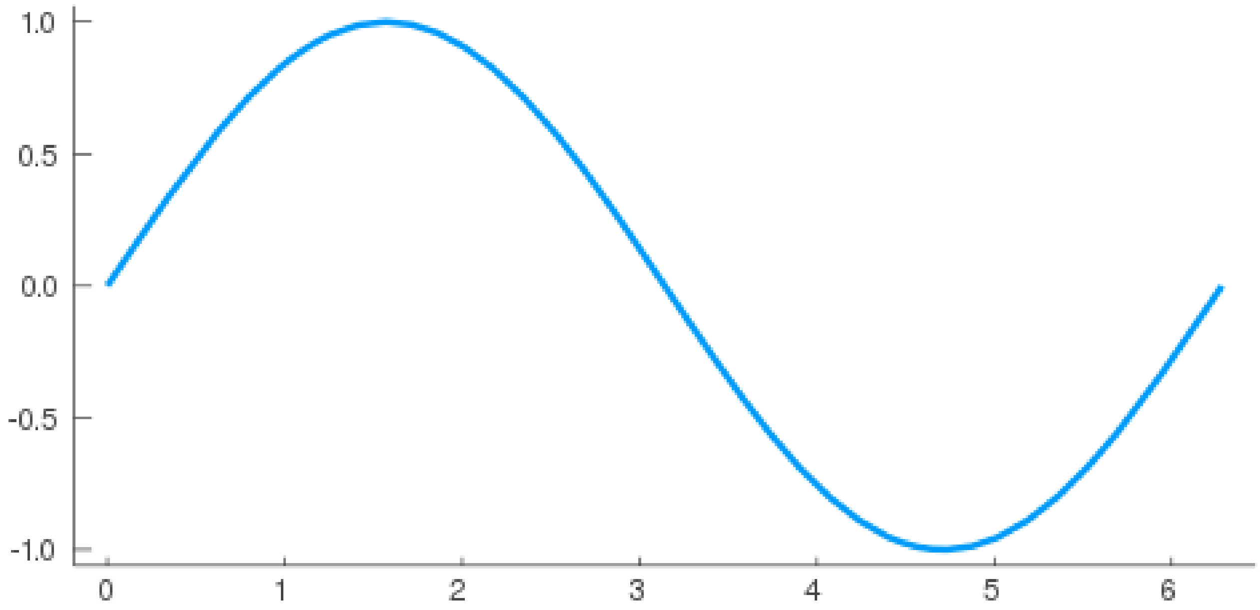
We will be plotting stuff, see http://docs.juliaplots.org/latest/generated/gr/ for example code using the `GR` backend

The monomial basis

Let's approximate `sin(x)`

```
using Plots
gr();
f(x) = sin.(x);

Plots.plot(f, 0, 2pi, line = 4, grid = false, legend = false, size = (500, 250))
```



Approximating sin(x)

```
function project_monomial(f, n, lb, ub)
    # solves Ψc = y → c = Ψ\y
    # Ψ = matrix of monomial basis functions evaluted on the grid

    coll_points = range(lb, ub, length = n)                          # collocation points
    y_values = f(coll_points)                                        # function values on the grid
    basis_functions = [coll_points.^degree for degree = 0:n-1]      # vector of basis functions
    basis_matrix = hcat(basis_functions...)                          # basis matrix

    coefficients = basis_matrix\y_values                             # c = Ψ\y

    return coefficients

end;
coefficients_4 = project_monomial(f, 4, 0, 2pi);
coefficients_5 = project_monomial(f, 5, 0, 2pi);
coefficients_10 = project_monomial(f, 10, 0, 2pi)
```

```
## 10-element Array{Float64,1}:
##   0.0
##   0.9990725797458863
##   0.004015857153649684
##  -0.1738437387373486
```

Approximating sin(x)

Now we need to construct a function `f_approx(coefficients, plot_points)` that takes in the `coefficients` vector, and an arbitrary vector of points to evaluate the approximant at (for plotting)

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function f_approx(coefficients, points)
    n = length(coefficients) - 1
    basis_functions = [coefficients[degree + 1] * points.^degree for degree = 0:n] # evaluate basis functions
    basis_matrix = hcat(basis_functions...)                                        # transform into matrix
    function_values = sum(basis_matrix, dims = 2)                                  # sum up into function value
    return function_values
end;
```

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    function_values = sum(basis_matrix, dims = 2)                                  # sum up into function value
    return function_values
end;
```
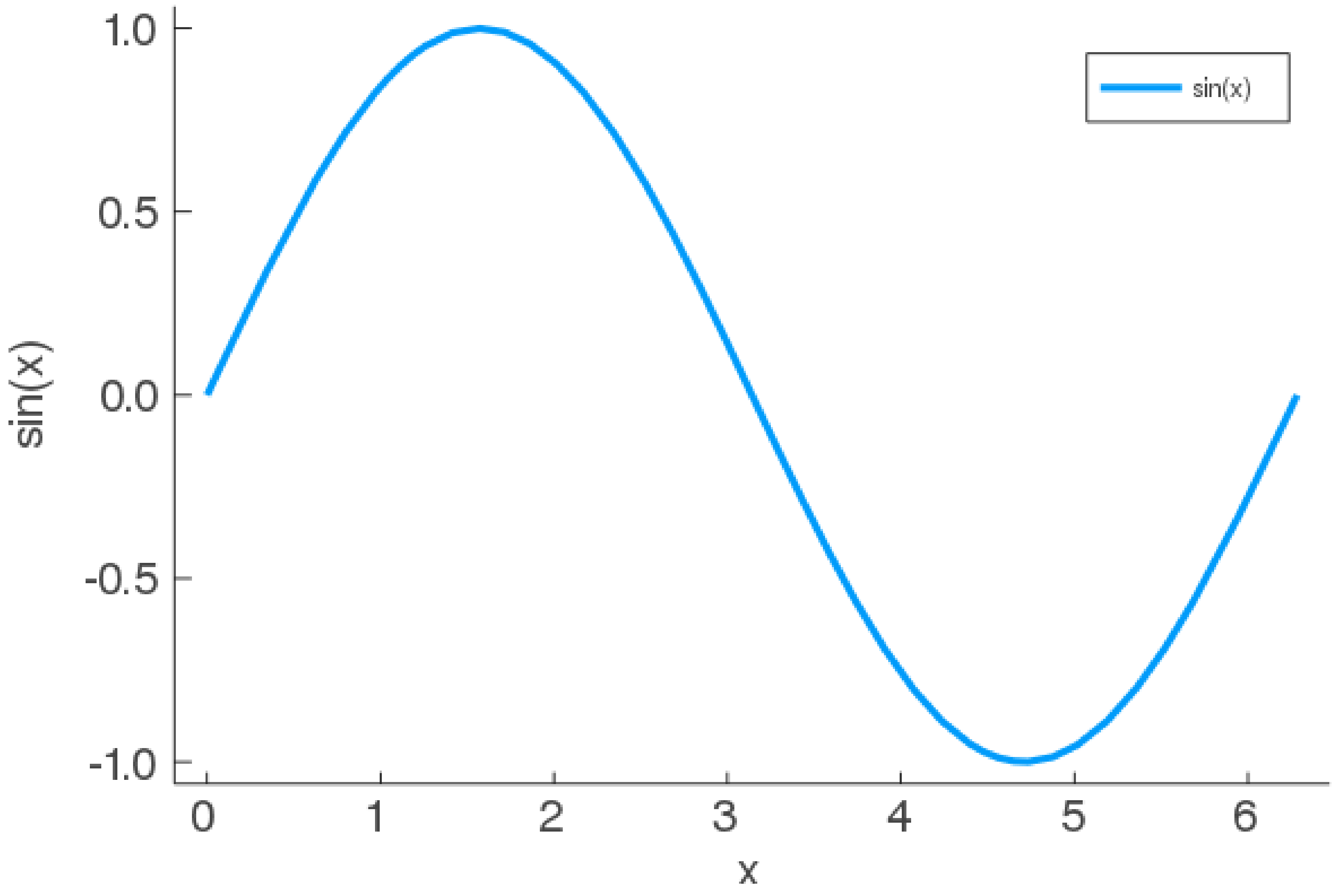
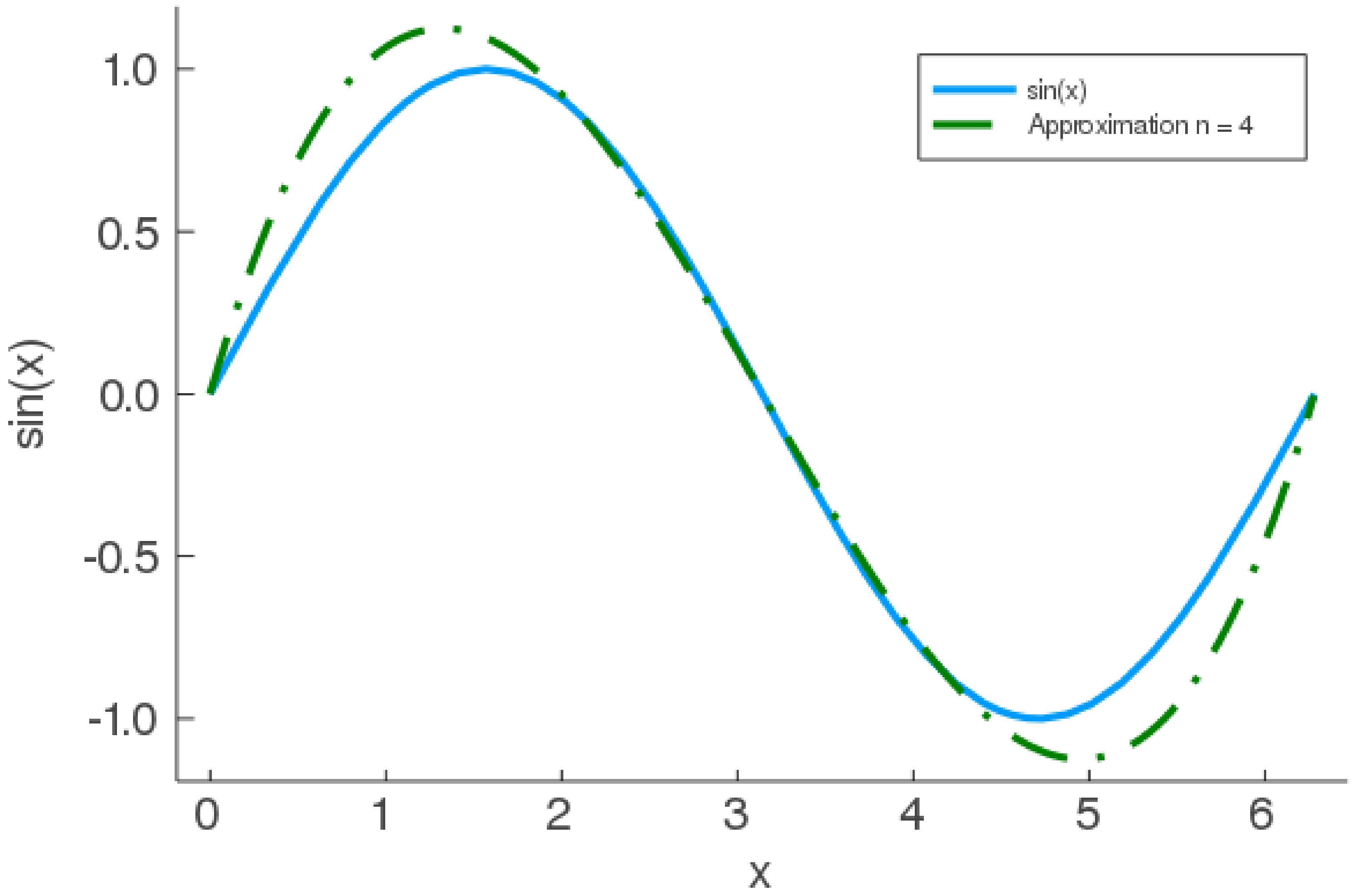

```
plot_points = 0:.01:2pi;
f_values_4 = f_approx(coefficients_4, plot_points);
f_values_5 = f_approx(coefficients_5, plot_points);
f_values_10 = f_approx(coefficients_10, plot_points)
```

```
## 629×1 Array{Float64,2}:
##   0.0
##   0.009999095361059786
```

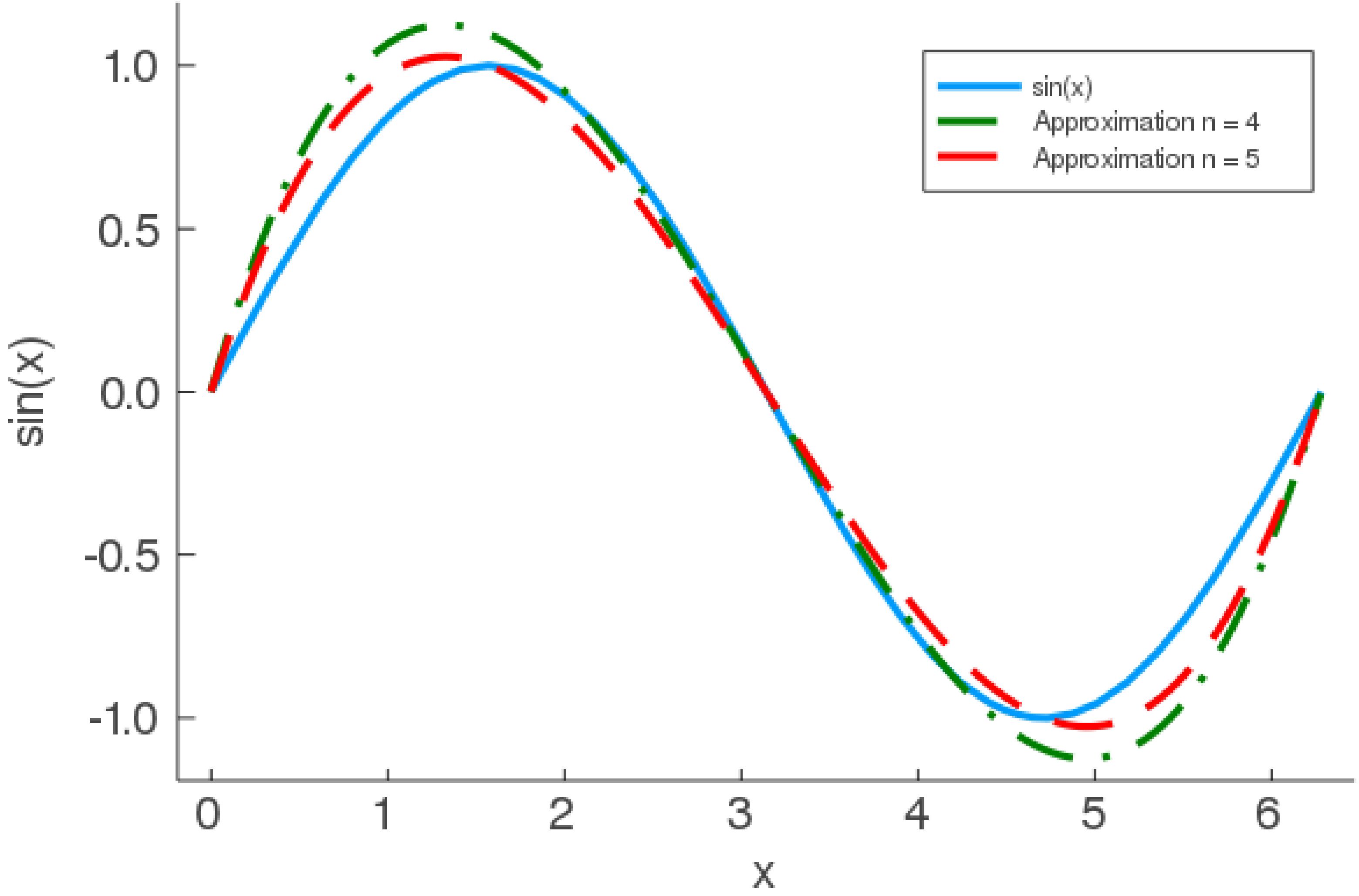
Plot



Plot

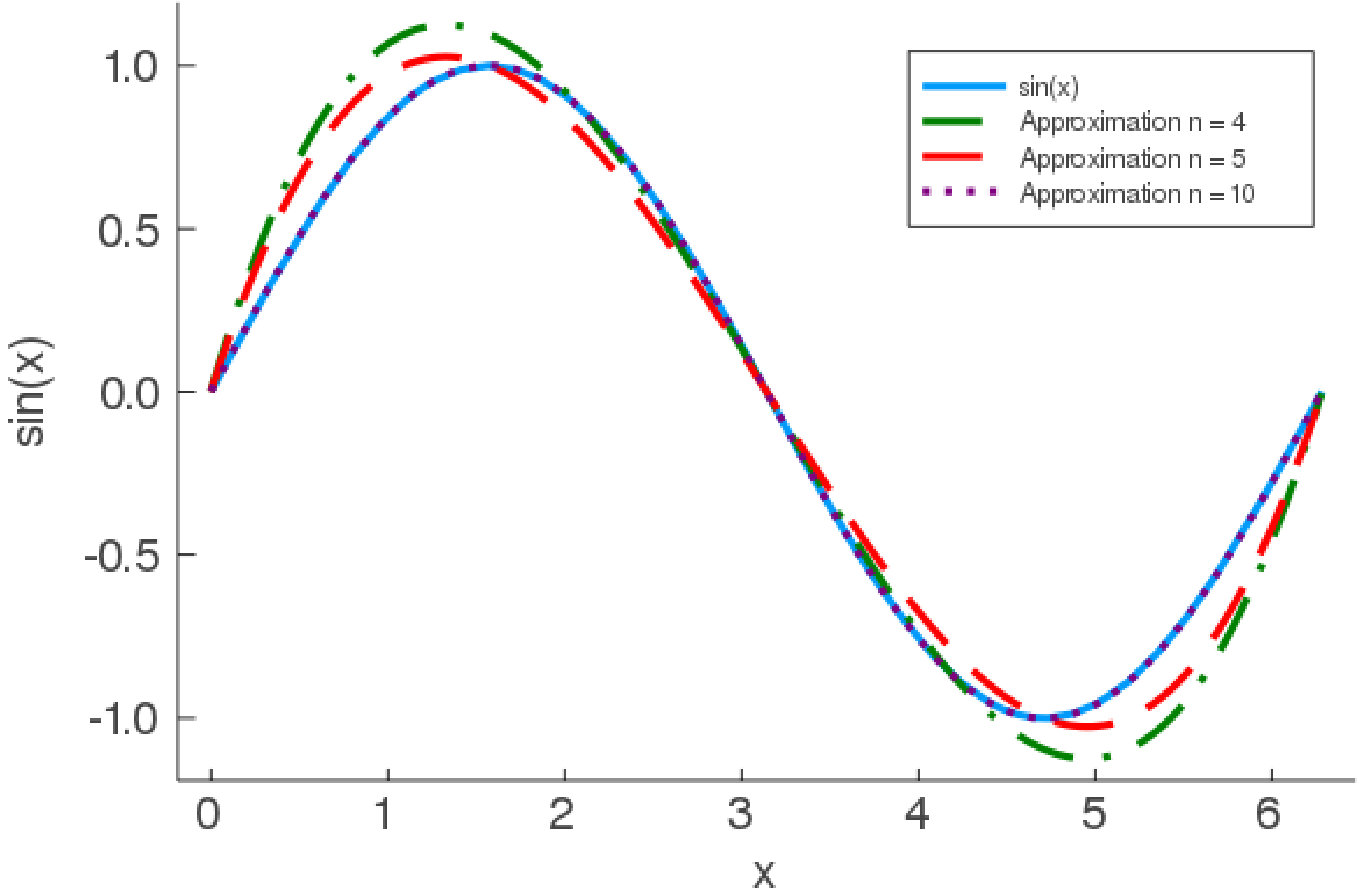


Plot



sin(x)
Approximation n = 4
Approximation n = 5
1.0
0.5
0.0
-0.5
-1.0
sin(x)
0
1
2
3
4
5
6
x

Plot



1.0
0.5
0.0
-0.5
-1.0
sin(x)
0
1
2
3
4
5
6
x
sin(x)
Approximation n = 4
Approximation n = 5
Approximation n = 10

Cool!

We just wrote some code that exploits Stone-Weierstrauss and allows us to (potentially) approximate any continuous function arbitrarily well as `n` goes to infinity

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