

The Origami Package

Computing Veechgroups of Origamis

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Chapter 1

The Veechgroup of Origamis

1.1 Introduction

This package provides calculations with origamis. An origami (also known as square-tiled surface) is a finite covering of a torus which is ramified at most over one point. It can be described in the following way from two permutations $\sigma_x, \sigma_y \in S_d$. We take d squares Q_1, \dots, Q_d and glue the lower side of Q_i to the upper side of $Q_{\sigma_y(i)}$ and the right side of Q_i to the left side of $Q_{\sigma_x(i)}$. We require origamis to be connected and thus the group generated by σ_x and σ_y acts transitively on $\{1, \dots, d\}$. In this package we identify an origami with a pair of permutations, which acts transitively on $\{1, \dots, d\}$ up to simultaneous conjugation. We introduce a new type of origamis, namely origami objects, which are created by this two permutations and its degree. The degree of an origami is the number of squares. Origamis are stored as such objects. We are mainly interested in the Veechgroup of an origami. It can be shown that the Veechgroup of an origami is a subgroup of $SL_2(\mathbb{Z})$ of finite index. So we fix two generators of $SL_2(\mathbb{Z})$

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

1.2 The Free Group

In this package we fix the free group F generated by \tilde{S} and \tilde{T} . We consider the canonical epimorphism $\pi : F \rightarrow SL_2(\mathbb{Z})$ with $\pi(\tilde{S}) = S$ and $\pi(\tilde{T}) = T$.

1.3 The Origami Object

In this section we describe the main functions of this package.

1.3.1 Origami

▷ `Origami(permX, permy, d)`

(function)

This function generates a new origami object with $\sigma_x = \text{permX}$, $\sigma_y = \text{permY}$ and degree d .

Example

```
gap> Origami((1,2), (2,3), 3);
Origami((1,2), (2,3), 3)
```

▷ `OrigamiWithoutTest(permX, permy, d)` (function)

This function does the same as `Origami`, but in contrast it does not test, whether the origami describes a connected surface. ▷ `ActionOfSpecialLinearGroup(matrix, origami)` (operation)

The group $SL_2(\mathbb{Z})$ acts on the set of origamis of fixed degree. The following methods for this operation are installed. ▷ `ActionOfSpecialLinearGroup(word, origami)` (method)

Given a word $word$ in the free group $Group(\tilde{S}, \tilde{T})$ this function computes $\pi(word) \in SL_2(\mathbb{Z})$ and returns $\pi(word).origami$. The word is given as a string, as shown in the following example.

Example

```
gap> ActionOfSpecialLinearGroup("ST", Origami((1,3,5), (1,3)(2,4,5), 5));
Origami((1,3)(2,5,4), (2,4,5,3), 5)
```

▷ `ActionOfSpecialLinearGroup(matrix, origami)` (method)

Given matrix in $SL_2(\mathbb{Z})$ this function returns $matrix.origami$. The word is given as a string, as shown in the following example.

Example

```
gap> ActionOfSpecialLinearGroup([ [ 0, -1 ], [ 1, 1 ] ], Origami((1,3,5),
>(1,3)(2,4,5), 5));
Origami((1,3)(2,5,4), (2,4,5,3), 5)
```

▷ `ActionOfF2ViaCanonical(origami, word)` (function)

Given a word $word$ in the free group $Group(\tilde{S}, \tilde{T})$ this function computes $\pi(word) \in SL_2(\mathbb{Z})$ and returns $\pi(word).origami$. But in contrast to `ActionOfSpecialLinearGroup` the result is stored in the canonical representation. ATTENTION: the order of arguments is switched compared to the order of the arguments in the function `ActionOfSpecialLinearGroup`.

Example

```
gap> ActionOfF2ViaCanonical(Origami((1,2), (1,3), 3), "S");
Origami((1,2), (2,3), 3)
```

▷ `RightActionOfF2ViaCanonical(origami, word)` (function)

This function computes the right action of the projection of a word $word$ in the free group $Group(\tilde{S}, \tilde{T})$ on an origami $origami$. It returns $origami.\pi(word) = \pi(word)^{-1}.origami$, where the left action is the common action of $SL_2(\mathbb{Z})$ on origamis of a given degree. This action has the same orbits as the left action. For the Veechgroup computation both actions can be used and give the same result. In contrast to `ActionOfSpecialLinearGroup` the result is stored in the canonical representation. ATTENTION: the order of arguments is switched compared to the order of the arguments in the function `ActionOfSpecialLinearGroup`.

Example

```
gap> RightActionOff2ViaCanonical(Origami((2,3), (1,3,2), 3), "T");
Origami((1,2), (2,3), 3)
```

▷ CanonicalOrigamiViaDelecroixAndStart(*origami*, *start*)

(function)

This function calculates a canonical representation of an origami depending on a given number *start* (Between 1 and the degree of the origami). To determine a canonical numbering the algorithm starts at the square with number *start*. This square is labeled as 1 in the new numbering. The algorithm walks along the origami in the following way and numbers the squares in the order, they are visited. First it walks in horizontal direction until it reaches the square with number *start* again. Then it walks one step up (in vertical direction) and then again a loop in horizontal direction. This will be repeated until the vertical loop is complete or all squares have been visited. If there are unvisited squares, we continue with the smallest number (in the new numbering), that has not been in a vertical loop. An origami is connected, so that number exists. This function is used to determine a canonical origami independent of the starting point. The idea for the algorithm derives from the algorithm to_standard_form implemented in the SageMath package `surface_dynamics`. (REFERENZ HINZUFÜGEN!)

Example

```
gap> CanonicalOrigamiAndStart(Origami((1,10,7,6,8,9,2)(4,5),
> (1,8,5,4)(2,10,6)(3,9,7), 10), 1);
Origami((1,2,3,4,5,6,7)(8,9), (1,5,8,9)(2,4,7)(3,10,6), 10)
```

▷ CanonicalOrigamiViaDelecroix(*origami*)

(function)

This function computes a canonical representation of an origami. It calculates the representation from CanonicalOrigamiViaDelecroixAndStart with all squares as start squares, independent of the given representation. Then it takes the minimum with respect to the order which GAP automatically uses to compare pairs of permutations. Two origamis are equal (up to renumbering of the squares) if they are described by the same permutations in their canonical representation.

Example

```
gap> CanonicalOrigami(Origami((1,10,7,6,8,9,2)(4,5), (1,8,5,4)(2,10,6)(3,9,7), 10));
Origami((2,3,4,5,6,7,8)(9,10), (1,2,6)(3,5,7)(4,8,9,10), 10)
```

▷ OrigamiFamily

(family)

Since origamis do not fit in any existing family in GAP, we introduce a new family for origami objects called OrigamiFamily. ▷ HorizontalPerm(*origami*)

(attribute)

This function returns the horizontal permutation σ_y of an origami *origami*.

Example

```
gap> HorizontalPerm(Origami((1,3,5), (1,3)(2,4,5), 5));
(1,3,5)
```

▷ VerticalPerm(*origami*)

(attribute)

This function returns the vertical permutation σ_x of an origami *origami*.

Example

```
gap> VerticalPerm( Origami((1,3,5), (1,3)(2,4,5), 5));
(1,3)(2,4,5)
```

▷ DegreeOrigami(origami)

(attribute)

This function returns the degree of an *origami*.

Example

```
gap> DegreeOrigami(Origami((1,3,5), (1,3)(2,4,5), 5));
5
```

▷ Stratum(origami)

(attribute)

This function calculates the stratum of an origami *origami*. The stratum of an origami is a list of the nonzero degrees of the singularities. For a singularity with cone angle $2\pi k$ the degree of the singularity is $k - 1$.

Example

```
gap> Stratum(Origami((1,6,4,7,5,3)(2,8), (1,4,5,3,8,2,6), 8));
[ 1, 5 ]
```

▷ Genus(origami)

(attribute)

This function calculates the genus of the origami surface.

Example

```
gap> Genus( Origami((1,2,3,4),(1,2)(3,4), 4) );
2
```

▷ VeechGroup(origami)

(attribute)

This function calculates the Veechgroup of an origami *origami*. The Veechgroup G of *origami* is a subgroup $SL_2(\mathbb{Z})$ of finite index. The group is stored as a ModularSubgroup from the **ModularSubgroup** package. It is represented by two permutations σ_S and σ_T describing how the generators S and T of $SL_2(\mathbb{Z})$ act on the cosets of G in $SL_2(\mathbb{Z})$. E.g, if $SH_i = H_j$ and H_i, H_j are the cosets associated to the integers i, j , respectively, then $\sigma_S(i) = j$. The algorithm was introduced by Gabriela Weitze-Schmithüsen in [WS04]. You get the coset permutations using the ModularSubgroup package as in the following example.

Example

```
gap> SAction(VeechGroup(Origami((1,2,5)(3,4,6), (1,2)(5,6), 6)));
(1,3)(2,5)(4,7)(6,8)(9,10)
gap> TAction(VeechGroup(Origami((1,2,5)(3,4,6), (1,2)(5,6), 6)));
(1,2,4)(3,6)(5,8,7,9,10)
```

▷ Cosets(origami)

(attribute)

This function calculates the right cosets of the Veechgroup of an origami *origami* as a list of words in S and T .

Example

```
gap> Cosets(Origami((1,2,5)(3,4,6), (1,2)(5,6), 6));
[ <identity ...>, S, T, T^-1, S*T, S*T^-1, T*S, T^-1*S, S*T*S, S*T^-1*S ]
```

▷ `EquivalentOrigami(origami1, origami2)` (function)

This function tests whether *origami1* is equal to *origami2* up to renumbering of the squares.

Example

```
gap> EquivalentOrigami(Origami((1,4)(2,6,3), (1,5)(2,3,6,4), 6), Origami((1,4,3)
>(2,5), (1,5,3,4)(2,6), 6));
true

gap> EquivalentOrigami(Origami((1,4)(2,6,3), (1,5)(2,3,6,4), 6), Origami((1,2,5)
>(3,4,6), (1,2)(5,6), 6));
false
```

1.4 List of Origamis

The following functions generate lists of Origamis.

1.4.1 OrigamiList

▷ `OrigamiList(d)` (function)

This Function calculates a list of all origamis, with a given degree *d*.

Example

```
gap> OrigamiList(2);
[ Origami(), (1,2), 2), Origami((1,2), (), 2), Origami((1,2), (1,2), 2) ]
```

▷ `OrigamiListWithStratum(d, stratum)` (function)

This function calculates a list of all origamis with a given degree *d* and stratum *stratum*.

Example

```
gap> OrigamiListWithStratum(5, [1,1]);
[ Origami((1,2), (1,3)(2,4,5), 5), Origami((1,2), (1,3,2,4,5), 5),
  Origami((1,2,3), (1,2)(3,4,5), 5), Origami((1,2,3), (2,3,4,5), 5),
  Origami((1,2,3), (2,4,5,3), 5), Origami((1,2)(3,4,5), (2,3), 5),
  Origami((1,2)(3,4,5), (1,2,3), 5), Origami((1,2)(3,4,5), (2,3,4,5), 5),
  Origami((1,2)(3,4,5), (2,3,5,4), 5), Origami((1,2)(3,4,5), (1,2,3,4,5), 5),
  Origami((1,2)(3,4,5), (1,2,3,5,4), 5), Origami((1,2,3,4), (3,4,5), 5),
  Origami((1,2,3,4), (3,5,4), 5), Origami((1,2,3,4), (1,2,3)(4,5), 5),
  Origami((1,2,3,4), (1,3,2)(4,5), 5), Origami((1,2,3,4), (2,3,4,5), 5),
  Origami((1,2,3,4), (2,5,4,3), 5), Origami((1,2,3,4), (1,2,4,5,3), 5),
  Origami((1,2,3,4), (1,3,5,4,2), 5), Origami((1,2,3,4,5), (3,5), 5),
  Origami((1,2,3,4,5), (1,2)(3,4,5), 5), Origami((1,2,3,4,5), (1,2)(3,5,4), 5),
  Origami((1,2,3,4,5), (2,4,3,5), 5), Origami((1,2,3,4,5), (2,5,3,4), 5) ]
```

1.5 Using SageMath functions

The SageMath Package `surface_dynamics` from Vincent Delecroix provides methods for origamis. To use the functions of this chapter, Sage must be installed on your operation system.

1.5.1 VeechgroupBySage

▷ `VeechgroupBySage(origami)` (function)

This function executes the SageMath method `Veechgroup` to *origami* and returns its result as GAP object.

Example

▷ `NormalFormBySage(origami)` (function)

This function executes the SageMath method `NormalForm` to *origami* and returns its result as GAP object.

Example

▷ `IsHyperellipticBySage(origami)` (function)

This function executes the SageMath method `IsHyperelliptic` to *origami* and returns its result as GAP object.

Example

▷ `IsPrimitiveBySage(origami)` (function)

This function executes the SageMath method `IsPrimitive` to *origami* and returns its result as GAP object.

Example

▷ `ReduceBySage(origami)` (function)

This function executes the SageMath method `Reduce` to *origami* and returns its result as GAP object.

Example

▷ `AbsolutePeriodGeneratorsBySage(origami)` (function)

This function executes the SageMath method `AbsolutePeriodGenerators` to *origami* and returns its result as GAP object.

Example

▷ `LatticeOfAbsolutePeriodsBySage(origami)` (function)

This function executes the SageMath method `LatticeOfAbsolutePeriods` to *origami* and returns its result as GAP object.

Example

▷ `LatticeOfQuotientsBySage(origami)` (function)

This function executes the SageMath method `LatticeOfQuotients` to *origami* and returns its result as **GAP** object.

Example

▷ `OptimalDegreeBySage(origami)` (function)

This function executes the SageMath method `OptimalDegree` to *origami* and returns its result as **GAP** object.

Example

▷ `PeriodGeneratorsBySage(origami)` (function)

This function executes the SageMath method `PeriodGenerators` to *origami* and returns its result as **GAP** object.

Example

▷ `WidthsAndHeightsBySage(origami)` (function)

This function executes the SageMath method `WidthsAndHeights` to *origami* and returns its result as **GAP** object.

Example

▷ `SumOfLyapunovExponentsBySage(origami)` (function)

This function executes the SageMath method `SumOfLyapunovExponents` to *origami* and returns its result as **GAP** object.

Example

▷ `LyapunovExponentsApproxBySage(origami)` (function)

This function executes the SageMath method `LyapunovExponentsApprox` to *origami* and returns its result as **GAP** object.

Example

▷ `IntermediateCoversBySage(origami)` (function)

This function executes the SageMath method `NormalFormBySage` to *origami* and returns its result as **GAP** object.

Example

References

- [WS04] Gabriela Weitze-Schmithuesen. An algorithm for finding the veech group of an origami. *Experimental Mathematics*, 13(4):459–472, 2004. [7](#)