NCERT Analog Assignment

EE23BTECH11007 - Aneesh Kadiyala*

Question 11.14.17: A simple pendulum of length ℓ and having a bob of mass M is suspended in a car. The car is moving in a circular track of radius R with a uniform speed v. If the pendulum makes small oscillations in a radial direction about its equilibrium position, what will be its time period?
Solution:

TABLE 0
PARAMETERS

Parameter	Description
v	Speed
R	Radius of circular track
M	Mass of bob
g	Acceleration due to gravity
a_c	Centrifugal acceleration
g_e	Effective gravitational acceleration $\sqrt{g^2+a^2}$

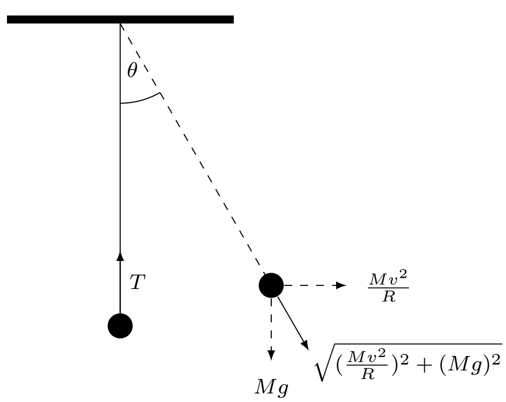

Fig. 0. Free Body Diagram

From the figure, restoring force:

$$F_r = -\left(\sqrt{\left(\frac{Mv^2}{R}\right)^2 + (Mg)^2}\right)\sin\theta(t) \tag{1}$$

For small oscillations, $\theta(t) << 1$.

$$\implies \sin\theta(t) \approx \theta(t) \tag{2}$$

$$\implies F_r \approx -\left(\sqrt{\left(\frac{Mv^2}{R}\right)^2 + (Mg)^2}\right)\theta(t) \tag{3}$$

$$\implies a = -\left(\sqrt{\left(\frac{v^2}{R}\right)^2 + g^2}\right)\theta(t) \tag{4}$$

$$l\frac{d^2\theta(t)}{dt^2} = -\left(\sqrt{\left(\frac{v^2}{R}\right)^2 + g^2}\right)\theta(t) \tag{5}$$

$$\text{Let } k = \frac{1}{\ell}\left(\sqrt{\left(\frac{v^2}{R}\right)^2 + g^2}\right) \tag{6}$$

$$\frac{d^2\theta(t)}{dt^2} + k\theta(t) = 0 \tag{7}$$

$$\tag{8}$$

Taking Laplace transform:

$$s^2\Theta(s) - s\theta(0) - \theta'(0) + k\Theta(s) = 0 \tag{9}$$

Assuming $\theta(0) = 0$:

$$s^2\Theta(s) - \theta'(0) + k\Theta(s) = 0 \tag{10}$$

$$\implies \Theta(s) = \frac{\theta'(0)}{s^2+k} \tag{11}$$

Taking inverse laplace transform using Bromwich integral:

$$\theta(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \Theta(s)e^{st}dt, c > 0 \tag{12}$$

$$= \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{\theta'(0)e^{st}}{s^2+k}dt \tag{13}$$

Since the poles $s = j\sqrt{k}$ and $s = -j\sqrt{k}$ (both non repeated, i.e. $m = 1$) lie inside a semicircle for some

$c > 0$. Using Jordans lemma and method of residues from **??**:

$$R_1 = \lim_{s \to j\sqrt{k}} \left(\left(s - j\sqrt{k}\right) \left(\frac{\theta'(0)}{s^2 + k} \right) e^{st} \right) \tag{14}$$

$$= \left(\frac{\theta'(0) e^{j\sqrt{k}t}}{2j\sqrt{k}} \right) \tag{15}$$

$$R_2 = \lim_{s \to -j\sqrt{k}} \left(\left(s + j\sqrt{k}\right) \left(\frac{\theta'(0)}{s^2 + k} \right) e^{st} \right) \tag{16}$$

$$= \left(\frac{-\theta'(0) e^{-j\sqrt{k}t}}{2j\sqrt{k}} \right) \tag{17}$$

$$\theta(t) = R_1 + R_2 \tag{18}$$

$$\theta(t) = \left(\frac{\theta'(0) \left(e^{j\sqrt{k}t} - e^{-j\sqrt{k}t} \right)}{2j\sqrt{k}} \right) \tag{19}$$

$$\implies \theta(t) = \frac{\theta'(0)}{\sqrt{k}} \sin\left(\sqrt{k}t\right) \tag{20}$$

$$\implies \theta(t) = \frac{\theta'(0)}{\sqrt{\frac{1}{\ell}\left(\sqrt{\left(\frac{v^2}{R}\right)^2 + g^2}\right)}} \sin\left(\sqrt{\frac{1}{\ell}\left(\sqrt{\left(\frac{v^2}{R}\right)^2 + g^2}\right)} t \right) \tag{21}$$

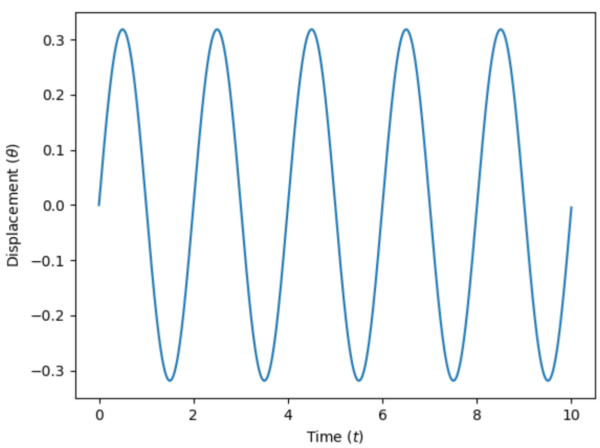

Fig. 0. $\theta(t)$ vs t for $v = 1, R = 1, \ell = 1, g = 9.8, \theta'(0) = 1$

$$\frac{2\pi}{T} = \sqrt{\frac{1}{\ell}\left(\sqrt{\left(\frac{v^2}{R}\right)^2 + g^2}\right)} \tag{22}$$

$$\implies T = 2\pi \sqrt{\frac{\ell}{\sqrt{\left(\frac{v^2}{R}\right)^2 + g^2}}} \tag{23}$$