NCERT Analog 11.14.17

EE23BTECH11007 - Aneesh K*

Question

A simple pendulum of length ℓ and having a bob of mass M is suspended in a car. The car is moving in a circular track of radius R with a uniform speed v. If the pendulum makes small oscillations in a radial direction about its equilibrium position, what will be its time period?

Table

Parameter	Description
v	Speed
R	Radius of circular track
M	Mass of bob
g	Acceleration due to gravity
a_c	Centrifugal acceleration
g_e	Effective gravitational acceleration $\sqrt{g^2 + a^2}$

Table: Parameters

Free Body Diagram

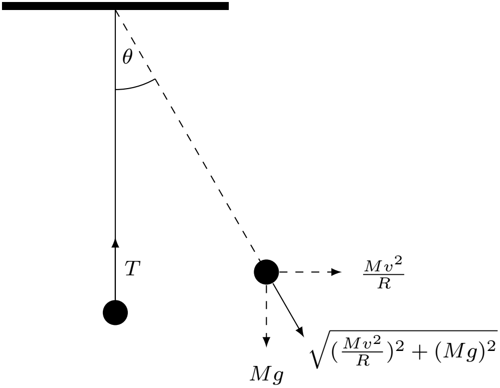

Figure: Free Body Diagram

Solution

From the free body diagram, restoring force:

$$F_r = -\left(\sqrt{\left(\frac{Mv^2}{R}\right)^2 + (Mg)^2}\right) \sin\theta\left(t\right) \tag{1}$$

Solution

For small oscillations, $\theta\left(t\right) << 1$.

$$\implies \sin\theta\left(t\right) \approx \theta\left(t\right) \tag{2}$$

$$\implies F_r \approx -\left(\sqrt{\left(\frac{Mv^2}{R}\right)^2 + \left(Mg\right)^2}\right)\theta\left(t\right) \tag{3}$$

$$\implies a = -\left(\sqrt{\left(\frac{v^2}{R}\right)^2 + g^2}\right)\theta\left(t\right) \tag{4}$$

$$I\frac{d^2\theta\left(t\right)}{dt^2} = -\left(\sqrt{\left(\frac{v^2}{R}\right)^2 + g^2}\right)\theta\left(t\right) \tag{5}$$

Solution

$$\text{Let } k = \frac{1}{\ell}\left(\sqrt{\left(\frac{v^2}{R}\right)^2 + g^2}\right) \tag{6}$$

$$\frac{d^2\theta\left(t\right)}{dt^2} + k\theta\left(t\right) = 0 \tag{7}$$

Solution

Taking Laplace transform:

$$s^2\Theta(s) - s\theta(0) - \theta'(0) + k\Theta(s) = 0 \tag{8}$$

Assuming $\theta(0) = 0$:

$$s^2\Theta(s) - \theta'(0) + k\Theta(s) = 0 \tag{9}$$

$$\implies \Theta(s) = \frac{\theta'(0)}{s^2 + k} \tag{10}$$

Solution

Taking inverse laplace transform using Bromwich integral:

$$\theta(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \Theta(s)e^{st}dt, c > 0 \tag{11}$$

$$= \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{\theta'(0)e^{st}}{s^2 + k}dt \tag{12}$$

Solution

Since the poles $s = j\sqrt{k}$ and $s = -j\sqrt{k}$ (both non repeated, i.e. $m = 1$) lie inside a semicircle for some $c > 0$. Using Jordans lemma and method of residues from **??**:

$$R_1 = \lim_{s \to j\sqrt{k}} \left(\left(s - j\sqrt{k} \right) \left(\frac{\theta'(0)}{s^2 + k} \right) e^{st} \right) \tag{13}$$

$$= \left(\frac{\theta'(0) e^{j\sqrt{k}t}}{2j\sqrt{k}} \right) \tag{14}$$

$$R_2 = \lim_{s \to -j\sqrt{k}} \left(\left(s + j\sqrt{k} \right) \left(\frac{\theta'(0)}{s^2 + k} \right) e^{st} \right) \tag{15}$$

$$= \left(\frac{-\theta'(0) e^{-j\sqrt{k}t}}{2j\sqrt{k}} \right) \tag{16}$$

Solution

$$\theta(t) = R_1 + R_2 \tag{17}$$

$$\theta(t) = \left(\frac{\theta'(0)\left(e^{j\sqrt{k}t} - e^{-j\sqrt{k}t}\right)}{2j\sqrt{k}}\right) \tag{18}$$

$$\implies \theta(t) = \frac{\theta'(0)}{\sqrt{k}} \sin\left(\sqrt{k}t\right) \tag{19}$$

$$\implies \theta(t) = \frac{\theta'(0)}{\sqrt{\frac{1}{\ell}\left(\sqrt{\left(\frac{v^2}{R}\right)^2 + g^2}\right)}} \sin\left(\sqrt{\frac{1}{\ell}\left(\sqrt{\left(\frac{v^2}{R}\right)^2 + g^2}\right)}t\right) \tag{20}$$

Plot

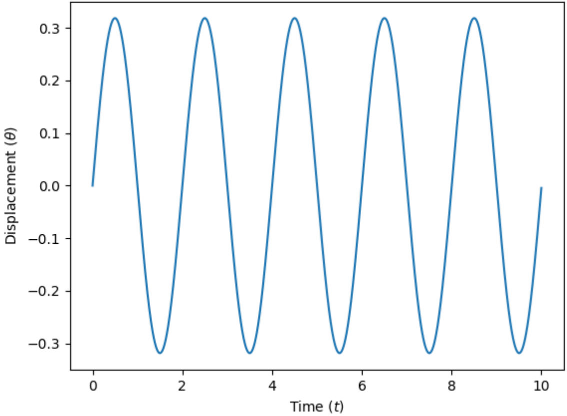

Figure: $\theta\left(t\right)$ vs t for $v = 1, R = 1, \ell = 1, g = 9.8, \theta'\left(0\right) = 1$

Code

```
#ifndef EE1205_EE1205_H
#define EE1205_EE1205_H

#include <stdlib.h>

// Bare basic version of Python's numpy.linspace(...)
double* linspace(double start, double stop, size_t count) {
    double* result = (double*) malloc(sizeof(double) * count);
    for (size_t i = 0; i < count; i++) {
        result[i] = start + (stop - start) * i / (count - 1);
    }
    return result;
}

#endif
```

Code

```
#include <stdio.h>
#include <math.h>
#include "ee1205.h"
// Length of pendulum
const double l = 1;
// Radius of track
const double R = 1;
// Velocity of car
const double v = 1;
// Acceleration due to gravity
const double g = 9.81;
double theta(double t) {
    double k = sqrt(1/l * sqrt(pow(pow(v, 2) / R, 2) + pow(g, 2)));
    return sin(k * t) / k;
}
int main () {
    // x-axis.
    double* t = linspace(0, 10, 10000);
    // Open output file.
    FILE* out;
    fopen_s(&out, "111417cout.txt", "w");
    // Write output to file.
    for (size_t i = 0; i < 10000; i++) fprintf(out, "%lf ", theta(t[i]));
    // Close file and free memory allocated for x-axis.
    fclose(out);
    free(t);
    return 0;
}
```

Code

```
import matplotlib.pyplot as plt
import numpy as np

t = np.linspace(0, 10, 10000)

x = np.loadtxt("111417cout.txt")

plt.plot(t, x)

plt.xlabel("Time ($t$)")
plt.ylabel("Displacement ($\\theta$)")

plt.savefig("plot.png")
```

Solution

$$\frac{2\pi}{T} = \sqrt{\frac{1}{\ell}\left(\sqrt{\left(\frac{v^2}{R}\right)^2 + g^2}\right)} \tag{21}$$

$$\implies T = 2\pi\sqrt{\frac{\ell}{\sqrt{\left(\frac{v^2}{R}\right)^2 + g^2}}} \tag{22}$$