- Consider the example of the bouncing ball

- Recall the system

$x_1 = p$ (height) $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ (state)

$x_2 = v = \dot{p}$ (velocity)

$$\begin{cases} \dot{x} = f(x) = \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix}, & x \in \mathcal{C} = \{ x \in \mathbb{R}^2 ; x_1 \geq 0 \} \\ x^+ = g(x) = \begin{bmatrix} 0 \\ -\lambda x_2 \end{bmatrix}, & x \in \mathcal{D} = \{ x \in \mathbb{R}^2 ; x_1 = 0 \text{ and } x_2 \leq 0 \} \end{cases}$$

$\lambda \in [0,1]$ is a restitution factor

- Compute the solutions:

$$\frac{dx_1}{dt} = x_2 \quad \text{and} \quad \frac{dx_2}{dt} = -\gamma$$

Then one gets:

1) $x_2(t) = x_2(0) - \gamma t \quad \forall t \in [0, \infty)$

2) $x_1(t) - x_2(0) = x_2(0)\, t - \gamma \frac{t^2}{2}$

$$\Rightarrow \phi(t, 0) = \begin{bmatrix} x_1(0) + x_2(0)\, t - \gamma\, t^2/2 \\ x_2(0) - \gamma t \end{bmatrix}$$

- $\phi(t, 0) = \begin{bmatrix} x_1(0) + x_2(0)t - \gamma t^2/2 \\ x_2(0) - \gamma t \end{bmatrix}$

- $\exists\, t = t_1$ such that

$$x_1(0) + x_2(0)t - \gamma \frac{t^2}{2} = 0 \ ?$$

$\in \mathcal{E}$

- Consider $x_1(0) = 0$ and $x_2(0) > 0$:

t_1: $x_2(0)t - \gamma \frac{t^2}{2} = 0$

$t(x_2(0) - \gamma t/2) = 0$ $\nearrow t = 0$ or $\searrow t = \frac{2x_2(0)}{\gamma}$

- $$\phi(t,0) = \begin{bmatrix} x_2(0)t - \gamma t^2/2 \\ x_2(0) - \gamma t \end{bmatrix} \quad \forall\, t \in [0, t_1)$$

$$t_1 = \frac{2x_2(0)}{\gamma}$$

$$\Rightarrow \phi(t_1,0) = \begin{bmatrix} 0 \\ x_2(0) - \gamma \cdot 2 \cdot \frac{x_2(0)}{\gamma} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -x_2(0) \end{bmatrix} \in \mathcal{D}$$

$$\Rightarrow \phi(t_1,1) = g(\phi(t_1,0)) = \begin{bmatrix} 0 \\ \lambda x_2(0) \end{bmatrix} \begin{matrix} \notin \mathcal{D} \\ \in \mathcal{C} \end{matrix}$$

• Then one can compute $\phi(t,1)$:

$$\phi_2(t,1) = \phi_2(t_1,1) - \gamma(t-t_1)$$

$$= \lambda x_2(0) - \gamma t + \gamma t_1$$

$$= \lambda x_2(0) - \gamma t + 2x_2(0)$$

$$\phi_1(t,1) = \underbrace{\phi_1(t_1,1)}_{=0} + \int_{t_1}^{t} (\lambda x_2(0) - \gamma t + 2x_2(0))\,dt$$

$$= (\lambda x_2(0) + 2x_2(0))(t-t_1) - \gamma \frac{(t-t_1)^2}{2}$$

with $t_1 = 2\frac{x_2(0)}{\gamma}$

$\Rightarrow$ it follows:

$$\phi(t,1) = \begin{bmatrix} (\lambda x_2(0) + 2x_2(0))(t-t_1) - \frac{\gamma(t-t_1)^2}{2} \\ \lambda x_2(0) + 2x_2(0) - \gamma t \end{bmatrix}$$

- $\exists\ t = t_2$ such that

$$(t_2-t_1)\left[(\lambda x_2(0)+2x_2(0)) - \gamma\frac{(t_2-t_1)}{2}\right] = 0$$

$$t_2 = t_1 = \frac{2x_2(0)}{\gamma} \qquad t_2 = \frac{2(\lambda x_2(0)+2x_2(0))}{\gamma} + t_1$$

$$\rightarrow t_2 = \frac{2\lambda x_2(0)}{\gamma} + \frac{4x_2(0)}{\gamma} + \frac{2x_2(0)}{\gamma}$$

$$t_2 = \frac{2x_2(0)}{\gamma}\left[\lambda + 3\right]$$

$$\bullet \quad \phi(t,1) = \begin{bmatrix} (\lambda x_2(0) + 2x_2(0))(t-t_1) - \frac{\gamma(t-t_1)^2}{2} \\ \lambda x_2(0) + 2x_2(0) - \gamma t \end{bmatrix}$$

$$\forall t \in [t_1, t_2]$$

$$\bullet \quad \phi(t_2,1) = \begin{bmatrix} 0 \\ \lambda x_2(0) + 2x_2(0) - 2\lambda x_2(0) - 6x_2(0) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -\lambda x_2(0) - 4x_2(0) \end{bmatrix} \in \mathcal{D}$$

< 0 (since $x_2(0) > 0$)

$$\Rightarrow \phi(t_2,2) = g(\phi(t_2,1)) = \begin{bmatrix} 0 \\ (\lambda^2 + 4\lambda)\, x_2(0) \end{bmatrix}$$

> 0

$\notin \mathcal{D}, \in \mathcal{C}$

$\Rightarrow$ Then the solution is constituted by an alternation of flows and jumps until the time t (time of flow) gradually shrinks to zero and the solution jumps forever

$\Rightarrow$ One gets:

$$\mathrm{dom}\,\phi = [0, t_1] \times \{0\} \cup [t_1, t_2] \times \{1\} \cup \ldots$$

$\Rightarrow$ Zeno solution

- Consider the case $x_1(0) = 0$ and $x_2(0) = 0$. Then it follows:

$$\Rightarrow \phi(t,0) = \begin{bmatrix} -\gamma t^2/2 \\ -\gamma t \end{bmatrix} \longrightarrow 0$$

as $t \to \infty$

$$\forall t \in [0, \infty)$$

$$\phi(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathcal{C} \cap \mathcal{D}$$

$$\Rightarrow \phi(0,1) = g(\phi(0,0)) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \cdots = \phi(0,j)$$

$$\forall j \geq 1$$

$\Rightarrow \operatorname{dom} \phi = \{0\} \times \mathbb{N}$

$\Rightarrow$ The point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is both in $\mathcal{C}$ and $\mathcal{D}$

$\Rightarrow$ $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an equilibrium point

$\Rightarrow$ The solution does not flow but keeps jumping forever

$\Rightarrow$ Discrete solution

- Come back to the case

$$x_1(0) \geq 0 \quad x_2(0) \neq 0$$

- We can compute the time of the first jump (or equivalently first bounce).

- $$\phi(t,0) = \begin{bmatrix} x_1(0) + x_2(0)\,t - \gamma \frac{t^2}{2} \\ x_2(0) - \gamma t \end{bmatrix}$$

$\Rightarrow$ The time of the first bounce correspond to

$$x_1(0) + x_2(0)\,t - \gamma \frac{t^2}{2} = 0$$

$\Rightarrow$ Denote this time t_{fb}:

$$\Delta = x_2(0)^2 - 4\left(-\frac{\gamma}{2}\right) \cdot x_1(0)$$

$$= x_2(0)^2 + 2\gamma\, x_1(0) > 0$$

$$\Rightarrow \underbrace{t^1_{fb}}_{<0} = \frac{-x_2(0) + \sqrt{\Delta}}{-\gamma} \qquad \underbrace{t^2_{fb}}_{>0} = \frac{-x_2(0) - \sqrt{\Delta}}{-\gamma}$$

- One sets:

$$t_{fb} = \frac{x_2(0) + \sqrt{x_2(0)^2 + 2\gamma x_1(0)}}{\gamma}$$