2MAE404 MIMO control
Homework report 7

(with later improvements)

Maciej Michałów   Marco Wijaya

December 2, 2022

1 Transmission Zeros, Zero-Pole Cancellation and Sigma Plots

1.1 Wind frequency resulting in highest amplitude

To compute the frequency resulting in the highest building vibration amplitude for a given amplitude of wind oscillations, a σ plot of the $G(s)$ transfer matrix has been created. The MATLAB code is as following:

```
G11 = tf(-0.25,[1 0.2 0.25]);
G12 = tf(0.25,[1 0.2 0.25]);
G21 = tf(-1, [1 1]);
G22 = tf(-1, [1 1]);
G = [G11 G12; G21 G22]; % full transfer matrix

[sv, omega] = sigma(G); % sigma(omega) computation
[~, i_max] = max(sv(1,:)); % index of max singular value
sv_max = sv(1,i_max); % max singular value
omega_max = omega(i_max); % omega corresponding to max singular value
```

The σ plot is shown in Figure 1. The highest singular value is $\sigma_{\max} = 11.15[\text{dB}] = 3.61[-]$ at a frequency of $\omega = 0.48[\text{rad/s}]$.

1.2 Wind direction resulting in highest amplitude

When analysing harmonic signals, it is useful to write them in complex form:

$$f(t) = A\sin(\omega t + \phi) = \operatorname{Im}\left(Ae^{j(\omega t+\phi)}\right) \tag{1}$$

where $A \in \mathbb{R}$ is the amplitude, $\omega \in \mathbb{R}$ is the frequency in [rad/s] and ψ is the phase shift (w.r.t. a pure sine wave) in [rad].

The whole complex signal $\overline{f}(t)$ is used, rather than just the imaginary part, to exploit the properties of complex representation to the fullest:

$$\overline{f}(t) = Ae^{j(\omega t+\phi)} \tag{2}$$

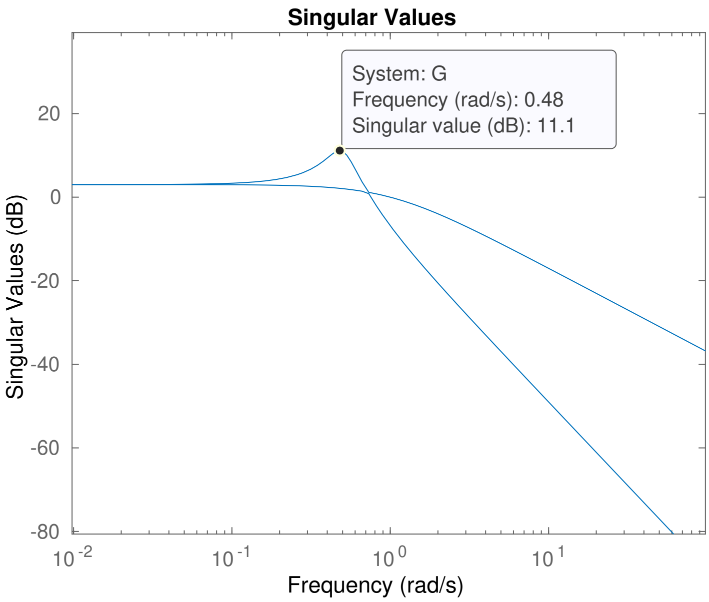

Figure 1: σ plot of $G(s)$ with maximal singular value marked.

A complex signal can be conveniently described with a complex amplitude $\hat{A}$, encoding both amplitude and phase information:

$$\overline{f}(t) = Ae^{j(\omega t+\phi)} = Ae^{j\phi}e^{j\omega t} = \hat{A}e^{j\omega t} \tag{3}$$

The response $y(t)$ of a SISO system to a sine input $u(t)$ with a given frequency ω is also a sine wave of frequency ω, but with the amplitude scaled and phase shifted according to the transfer function of the system at the given frequency, $F(j\omega)$. For complex sine signals, this is especially easy to write down:

$$\overline{y}(t) = F(j\omega)\overline{u}(t) \tag{4}$$

Rewriting both signals in terms of complex amplitude results in:

$$\hat{y}e^{j\omega t} = F(j\omega)\hat{u}e^{j\omega t} \tag{5}$$

The time-dependent parts of the equation cancel out, therefore describing the input-output relationship purely in terms of complex amplitudes and the transfer function:

$$\hat{y} = F(j\omega)\hat{u} \tag{6}$$

The same holds for MIMO systems, including the one studied in this excercise. The relationship between the input $u(t) = w(t)$ and output $y(t) = p(t)$ is therefore, in expanded form:

$$\hat{y} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = \begin{bmatrix} G_{11}(j\omega) & G_{12}(j\omega) \\ G_{21}(j\omega) & G_{22}(j\omega) \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix} = G(j\omega)\hat{u} \tag{7}$$

Note that $G(j\omega)$ is simply a 2×2 complex matrix and $\hat{u},\hat{y} \in \mathbb{C}^2$. The most convenient way of finding the maximum amplitude is by performing an SVD on the $G(j\omega)$ matrix:

$$G(j\omega) = U\Sigma V^* \tag{8}$$

where Σ is a diagonal matrix of singular values, while U and V^* are unitary complex matrices. The computation is performed in MATLAB:

```
Gjw = evalfr(G, omega_max*1j); % evaluate G(s) at s=j*omega
[U, S, V] = svd(Gjw); % perform SVD
```

The decomposition turned out to consist of matrices with the following structure:

$$G(j\omega) = \begin{bmatrix} e^{j\upsilon_1} & 0 \\ 0 & e^{j\upsilon_2} \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \tag{9}$$

The singular values are $\sigma_1 = 3.61$ (just like in the σ plot) and $\sigma_2 = 1.26$, while $\upsilon_1 = 1.78$ and $\upsilon_2 = 2.69$ are phase shifts. The input-output relationship of the system can be rewritten as:

$$\hat{y} = U\Sigma V^* \hat{u} \tag{10}$$

Since the highest gain of the system is σ_1, the input $\hat{u}$ should have the property of being multiplied only by σ_1. Since Σ is diagonal, this would suggest that $V^*\hat{u}$ has a non-zero first component and a 0 in the second one. Expanding analytically the SVD formulation of the input-output relation makes this even more obvious:

$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = \begin{bmatrix} \sigma_1 e^{j\upsilon_1} \left(\hat{u}_1/\sqrt{2} - \hat{u}_2/\sqrt{2}\right) \\ \sigma_2 e^{j\upsilon_2} \left(\hat{u}_1/\sqrt{2} + \hat{u}_2/\sqrt{2}\right) \end{bmatrix} \tag{11}$$

Since σ_1 only affects $\hat{y}_1$, $\hat{u}$ should be chosen in a way that results in $\hat{y}_2 = 0$. This is the case if the components of $\hat{u}$ solve the equation:

$$\hat{u}_1/\sqrt{2} + \hat{u}_2/\sqrt{2} = 0 \tag{12}$$

This is obviously equivalent to $\hat{u}_1 = -\hat{u}_2$, implying wind blowing in the northwest-southeast axis. The $\hat{u}$ vector has the form:

$$\hat{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} c \tag{13}$$

where $c \in \mathbb{C}$ is an arbitrary non-zero complex constant. For the following computations, $c = 1$ (meaning that $||\hat{u}|| = \sqrt{2}$) is assumed. The corresponding $\hat{y}$ is computed in MATLAB:

```
u_hat = [-1; 1]; % input direction
y_hat = Gjw*u_hat; % output direction
```

The result is approximately:

$$\hat{y} = \begin{bmatrix} 1.0417 - 4.9957j \\ 0 \end{bmatrix} \tag{14}$$

The corresponding gain is:

$$\frac{||\hat{y}||}{||\hat{u}||} = 3.61 \tag{15}$$

This matches the singular value σ_1, therefore confirming the results of the computation.

1.3 Building vibration direction

As already noticed in the previous section, the highest oscillations of the building at the given wind frequency correspond to a situation where $\hat{y}_2 = 0$, what translates to oscillation purely in the east-west axis. This can be additionally shown by plotting $y(t)$:

```
t = linspace(0,2*pi/omega_max,100000); % sampled time over 1 period
y = imag(y_hat*exp(1j*omega_max*t)); % y(t)

p_mag = sqrt(y(1,:).^2 + y(2,:).^2); % |p(t)|
[~, i_max] = max(p_mag); % index of max |p(t)|

% phase plot of oscillation over one period
figure;
plot(y(1,:), y(2,:), 'DisplayName', 'p(t)');
hold on;
plot([-1 1]*y(1,i_max), [-1 1]*y(2,i_max), 'x', 'DisplayName', 'p_{max}');
xlabel('p_e [n]');
ylabel('p_n [m]');
legend();
grid("on");
hold off;
```

The resulting plot is shown in Figure 2. The amplitude is consistent with the previous computation.

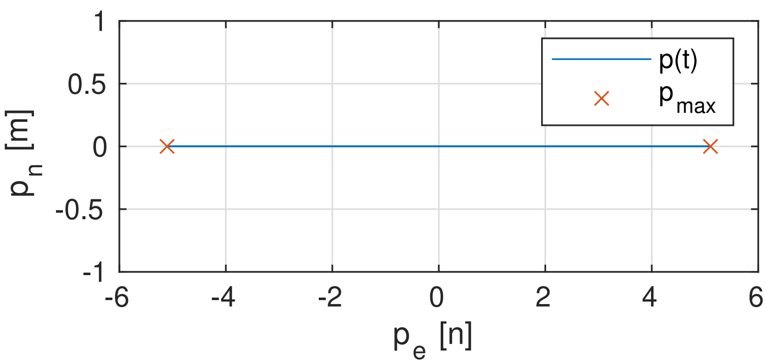

Figure 2: Parametric plot of the oscillation of $p(t)$ over 1 period. Note that there is no oscillation in the north-south axis, just as predicted.

1.4 Drone minimum altitude

Suppose that the building is being subject to a wind oscillation:

$$w(t) = \begin{bmatrix} w_n(t) \\ w_e(t) \end{bmatrix} = \begin{bmatrix} 2\cos(10\pi t) \\ 3\sin(10\pi t) \end{bmatrix} \tag{16}$$

The minimum height h of the drone (to observe the oscillation using a camera with a 60° field of view) depends only on the maximum displacement of the tip of the building, i.e. on the maximum value of $||p(t)||$. The complex amplitude of the input signal is

$\hat{w} = [2e^{j\pi/2}, 3]^T$, with the complex term in the first component encoding the phase of a cosine. Once again, the complex amplitudes of the oscillation can be computed using:

$$\hat{p} = G(j\omega)\hat{w} \tag{17}$$

This is performed using the following MATLAB code:

```
omega = 10*pi; % frequency [rad/s]
Gjw = evalfr(G, 1j*omega) % G(j*omega)
w_hat = [2*exp(1j*pi/2); 3]; % complex amplitude of w
p_hat = Gjw*w_hat; % complex amplitude of p
```

The maximum amplitude is (note that the computation must include the complex amplitudes, rather than their magnitudes, because the 2 axes are not in phase, and therefore achieve their peaks at different times):

$$\max ||p(t)|| = ||\hat{p}|| = 0.1147 \tag{18}$$

The obtained maximum displacement corresponds to an altitude:

$$h_{min} = \frac{||p||_{max}}{\tan(30°)} = \frac{0.1147}{0.577} = 0.1987 \tag{19}$$

Hence, the drone should be at least $h_{min} = 0.1987[m]$ above the top of the building.