Optimal rendez-vous trajectory

Let us consider a space station S orbiting around the earth (orbit center: O) and a servicing vehicle M in the orbital plane. The objective is to dock the (active) chaser M on the (passive) target S.

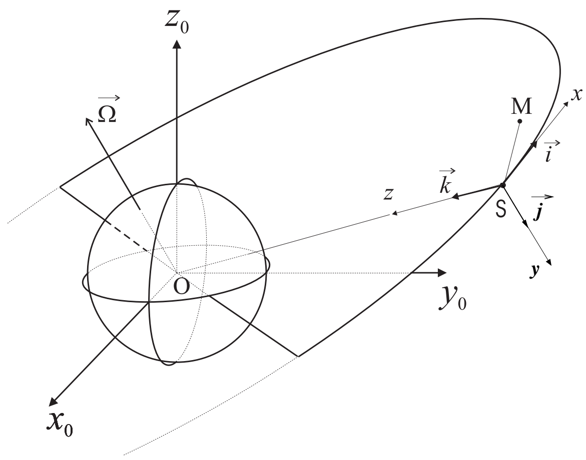

Figure 1: Frame definition

The differential equations governing the relative dynamics of M w.r.t. S in the orbital plane (simplified HILL-CLOHESSY-WHITSHIRE equations) projected in the local orbital frame $\left\{S, \vec{i}, \vec{j}, \vec{k}\right\}$ reads:

$$\ddot{x} - 2\omega\dot{z} = \varphi_x$$
$$\ddot{z} + 2\omega\dot{x} - 3\omega^2 z = \varphi_z$$

where x and z are the coordinates of M in the local orbital plane.

The 2 control inputs are the specific propulsion forces φ_x and φ_z (resp. tangential and radial) of M.

1) Give a state space representation of this system:

$$\dot{X} = \mathbf{A}X + \mathbf{B}u$$

where $X^T = (z, x, \dot{z}, \dot{x})$ and $u^T = (\varphi_z, \varphi_x)$.

2) Is the system controllable using:
 a) the radial thrust φ_z only ?
 b) the tangential thrust φ_x only ?
In the following, it is assumed that only one control input can be used. Which one (φ_x or φ_z) ?.

3) At the initial time $t = 0$, the space vehicle M has an initial state $X(0) = X_0$, assumed to be measured. The objective is to dock M on S at a given time $t = T$ while minimizing the fuel consumption of M. The problem is thus to find the state optimal trajectories $\widehat{X}(t)$ and the optimal control $\widehat{u}(t)$[1] such that $X(T) = \mathbf{0}$ while minimizing the performance index:

$$\mathscr{C} = \int_0^T \frac{1}{2}u^2 dt$$

3.1) In a first step, one wishes to develop a generic MATLAB function solving the two point boundary-value problem in the general case of a linear system, a quadratic index and a finite time horizon. Fill the code of the following `twopbvp.m` function:

[1] In this case u is a dimension 1 signal since only one control input is used.

```
function [K_t,P_t,phi_t]=twopbvp(T,t,a,b,q,r)
%TWOPBVP   Two point boundary-value Problem (LQ problem with finite horizon T
%          and null final state).
%
%        * K_t=twopbvp(T,t,A,B,Q,R) compute, at current time t (in [0, T[),
%          the optimal gain K_t for the LQ problem:
%
%          - System:
%              .
%              x = Ax + Bu  with negative state feedback: u(t)=-K_t(t)*x(t),
%
%          - Performance index
%                      /T
%              J = 0.5 | {x'Qx + u'Ru} dt
%                     /0
%          - Hard constraint:
%              x(T)=0
%
%        * [K_t,P_t]=twopbvp(T,t,A,B,Q,R) computes also P_t: the (semi-
%          definite positive) solution at current time t of the associated
%          Riccati equation:
%               .                            -1
%              P_t = -P_t A - A' P_t + P_t B R  B'P_t - Q
%
%        * [K_t,P_t,phi_t]=twopbvp(T,t,....) computes also the transistion
%          matrix phi_t at current time t on the optimal trajectory,
%          such that:
%                       x(t)=phi_t x(0)


error(nargchk(6,6,nargin));
error(abcdchk(a,b));

if t>T, disp('Input argument problem: t>T !!!');K_t=[];P_t=[];phi_t=[];return,end

[m,n] = size(a);[mb,nb] = size(b);[mq,nq] = size(q);[mr,nr] = size(r);
if (m ~= mq) | (n ~= nq)
       error('A and Q must be the same size');
end
if (mr ~= nr) | (nb ~= mr)
       error('B and R must be consistent');
end

H=[a -b*inv(r)*b';-q -a'];
eH_t=expm(H*(T-t));
...
```

3.2) Numerical application: $\omega = \frac{2\pi}{T_{orb}}\,(rd/s)$, $T = \frac{1}{4}T_{orb}$ and $T_{orb} = 5400\,(s)$.
For the 4 following initial states:

1. $z(0) = \dot{z}(0) = \dot{x}(0) = 0$ et $x(0) = 1000\,(m)$,
2. $z(0) = \dot{z}(0) = \dot{x}(0) = 0$ et $x(0) = -1000\,(m)$,
3. $x(0) = \dot{x}(0) = \dot{z}(0) = 0$ et $z(0) = 1000\,(m)$,
4. $x(0) = \dot{x}(0) = \dot{z}(0) = 0$ et $z(0) = -1000\,(m)$,

compute $\widehat{\mathscr{C}}$, $\widehat{u}(t)$, $\widehat{X}(t)$ and plot, using the given user-function `plotresults.m`:

- the 4 optimal state trajectories $\widehat{z}(t)\,,\widehat{x}(t)\,,100\widehat{\dot{z}}(t)\,,100\widehat{\dot{x}}(t)$ in the same graph,
- the optimal control response $\widehat{u}(t)$,
- the optimal trajectory in the local orbital plane $\widehat{x} = F(\widehat{z})$.

Comment these responses.
3.3) Knowing that:

- the chaser M has a total energy of 1 (with the same unit than the performance index $\mathscr{C}$),
- at $t = 0$, M is exactly on the same orbit than S ($z(0) = 0$) with a null relative velocity ($\dot{x}(0) = \dot{z}(0) = 0$),

what is the maximal distance $|x(0)|_{max}$ for the rendez-vous to be possible?
3.4) go back to questions 3.2) et 3.3) considering now that the 2 control inputs (φ_z and φ_x) can be used and the new performance index:

$$\mathscr{C} = \int_0^T \tfrac{1}{2}(\varphi_z^2 + \varphi_x^2)dt\ .$$