Lectures on Dynamic Systems and Control

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Chapter 27

Poles and Zeros of MIMO Systems

27.1 Introduction

You are familiar with the definitions of poles, zeros, and their multiplicities for the scalar transfer functions associated with SISO LTI systems. Also recall the interpretation (stated here for the CT case, but the analogous statement holds in the DT case) of a pole frequency p_0 as being a "generated frequency" of the system, in the sense that an exponential of the form $ce^{p_0 t}$ for $t \geq 0$ (and for some nonzero constant c) is present in the *output* even when the input for $t \geq 0$ comprises a sum of exponentials that does *not* contain a term with this frequency. Similarly, the frequency ζ_0 of a zero may be interpreted as an "absorbed frequency", in the sense that even when the *input* comprises a sum of exponentials that contains a term of the form $ce^{\zeta_0 t}$ for $t \geq 0$, the output does *not* contain a component at this frequency.

For the case of the $p \times m$ transfer function *matrix* $H(s)$ that describes the zero-state input/output behavior of an m-input, p-output LTI (CT) system, the definitions of poles and zeros are more subtle. We would still like them to respectively have the interpretations of generated and absorbed frequencies, in some sense, but that still leaves us with many choices. We begin by discussing the case of diagonal transfer function matrices. (We continue to use the notation of CT systems in what follows, but the DT story is identical.)

27.2 Poles and Zeros for Diagonal $H(s)$

It is clear what we would want our eventual definitions of poles and zeros for multi-input, multi-output (MIMO) systems to specialize to in the case where $H(s)$ is nonzero only in its *diagonal* positions, because this corresponds to completely decoupled scalar transfer functions. For this diagonal case, we would evidently like to say that the poles of $H(s)$ are the poles of the individual diagonal entries of $H(s)$, and similarly for the zeros.

Example 27.1 Given the 3×3 transfer matrix

$$H(s) = \text{diagonal}\left(\frac{s+2}{(s+3)^2} \, , \; \frac{s}{(s+2)(s+3)} \, , \; 0 \right)$$

we would say that $H(s)$ has poles at -3 of multiplicity 2 and 1 respectively, and a pole at -2 of multiplicity 1; and that it has zeros at -2, at 0, and at ∞, all with multiplicity 1.

Note from the above example that in the MIMO case we can have poles and zeros at the same frequency (e.g. those at -2 in the example), without any cancellation! Also note that a pole or zero is not necessarily characterized by a single multiplicity; we may instead have a *set* of multiplicity indices (e.g. as needed to describe the pole at -3 in the above example). The diagonal case makes clear that we do *not* want to define a pole or zero location of $H(s)$ in the general case to be a frequency where *all* entries of $H(s)$ respectively have poles or zeros. The particular $H(s)$ that we have shown in the example has a *normal rank* (i.e. for most values of s) of 2, and this rank drops at precisely the locations of the zeros of the individual entries.

27.3 MIMO Poles

We might consider defining a pole location as follows:

- **(Pole Location)** $H(s)$ has a pole at a frequency p_0 if *some* entry of $H(s)$ has a pole at $s = p_0$.

This choice would still have the significance of a generated frequency, for an appropriately chosen input and output. The above definition is indeed the one that is picked. The full definition also shows us how to determine the set of multiplicities associated with each pole frequency. For completeness — but not because we expect you to understand the motivation for it, or to remember and use it — we state the prescription here:

- **(Pole Multiplicities)** Determine the largest multiplicity $\nu_1(p_0)$ with which the pole p_0 occurs among the 1×1 minors of $H(s)$, then the largest multiplicity $\nu_2(p_0)$ of the pole p_0 among the 2×2 minors of $H(s)$, and so on. Stop at minors of size $k \times k$ if k is the first size for which $\nu_k(p_0) \geq \nu_{k+1}(p_0)$ (this k will actually depend on p_0, so we should really write $k(p_0)$, but we omit the argument in the interest of keeping the notation streamlined), or if all minors of larger size vanish. The set of multiplicities associated with the pole at p_0 is now given by the set of numbers $\nu_1(p_0)\,,\, \nu_2(p_0) - \nu_1(p_0)\,,\, \cdots\,,\, \nu_k(p_0) - \nu_{k-1}(p_0)$.

(Caution: For all the computations with minors described above and later in these notes, any common factors between the expressions obtained for the numerator and denominator of a minor must first be cancelled out, of course.) You should verify that you get the expected values for pole multiplicities when you apply this definition to the preceding example of a diagonal $H(s)$.

Determining Poles from a State-Space Realization

Given this definition of poles (and their multiplicities) for MIMO transfer functions, what can be said about the relation of the poles of $H(s)$ to properties of a realization (A, B, C, D) of this transfer function? What is clear is that the poles of

$$H(s) = C(sI - A)^{-1}B + D$$

must be contained among the eigenvalues of A, because the denominator terms in the entries of $H(s)$ are all $a(s) = \det(sI - A)$, apart from possible cancellations between $a(s)$ and the entries of

$C(\mathrm{adj}[sI - A])B$. In fact, the poles of $H(s)$ must be contained among the *reachable and observable* eigenvalues of A, as only the reachable and observable part of the realization contributes to the transfer function. What can be shown, although this is more than we are equipped to do in this course, is that the poles of $H(s)$ are precisely *equal* — in location and multiplicity — to the reachable and observable eigenvalues of A. In fact, the multiplicity indices associated with a pole of $H(s)$ are precisely the sizes of the Jordan blocks associated with the corresponding eigenvalue of A.
You can verify from the preceding facts that:

- the **characteristic polynomial of a minimal realization** of $H(s)$ — which we may refer to as the **pole polynomial** — equals the least common multiple of the denominators of all possible minors (of all sizes) in $H(s)$.

Example 27.2 Consider the 2×2 transfer function

$$H_1(s) = \begin{bmatrix} \frac{1}{s+3} & 1 \\ 0 & \frac{1}{s+3} \end{bmatrix} .$$

Its only polar frequency is at -3. The largest multiplicity of this pole in the 1×1 minors is 1, and its largest multiplicity in the 2×2 minor (there is only one minor of this size) is 2. Hence the multiplicities of the pole at -3 are 1 and $2 - 1 = 1$. The characteristic polynomial of a minimal realization of $H_1(s)$ is $(s + 3)^2$.

Now consider the transfer function

$$H_2(s) = \begin{bmatrix} \frac{1}{s+3} & \frac{1}{s+3} \\ \frac{1}{s+3} & \frac{1}{s+3} \end{bmatrix} .$$

Its only polar frequency is again at -3. The largest multiplicity of this pole in the 1×1 minors is 1, and its 2×2 minor vanishes. Hence the pole at -3 has a multiplicity of just 1, and the characteristic polynomial of a minimal realization of $H_2(s)$ is simply $(s + 3)$.

You should verify that the above results are consistent with the minimal realizations produced by Gilbert's method. Suppose

$$H_3(s) = \begin{bmatrix} \frac{1}{(s-1)(s+3)^2} & \frac{1}{(s-1)^2(s+3)} \end{bmatrix} .$$

Verify that this transfer matrix has a pole at 1 of multiplicity 2, and a pole at -3 of multiplicity 2. The characteristic polynomial of a minimal realization of $H_3(s)$ is thus $(s - 1)^2(s + 3)^2$.

27.4 MIMO Zeros

We have already established, with guidance from the diagonal case, that a zero should *not* be defined as a frequency where *all* entries of $H(s)$ have zeros. It is also not satisfying in the general MIMO case (although it is correct in the diagonal case) to define a zero location as a frequency where *some* entry of $H(s)$ has a zero. Among the objections to this definition are the following:

(i) although such a frequency can be hidden from a particular output even when it is present in a particular input (since it is "absorbed" by the corresponding entry of $H(s)$), this frequency will in general *not* be hidden from *all* outputs, and is therefore not really "absorbed" in a MIMO sense;

(ii) we will not in general have the desirable feature that the zeros of an invertible $H(s)$ will be poles of $H^{-1}(s)$.

A much more satisfactory definition of a zero is the following:

- **(Zero Location)** $H(s)$ has a zero at a frequency ζ_0 if it *drops rank* at $s = \zeta_0$.

This particular definition corresponds to what is termed a *transmission zero*, and is the only definition of interest to us in this course. Consider, for example, the case of an $H(s)$ of full column rank (as a rational matrix — i.e. there is no rational vector $u(s) \neq 0$ such that $H(s)u(s) = 0$), and assume it is finite at $s = \zeta_0$, i.e. has no poles at ζ_0. Then $H(s)$ drops rank at $s = \zeta_0$ iff $H(\zeta_0)u_0 = 0$ for some $u_0 \neq 0$.

As we have seen, however, a MIMO transfer function can have poles and zeros at the same frequency, so a more general characterization of rank loss is needed to enable us to detect a drop in rank even at frequencies where some entries of $H(s)$ have poles. This is provided by the following test, which is restricted to the case of full-column-rank $H(s)$, but an obvious transposition will handle the case where $H(s)$ has full row rank, and somewhat less obvious extensions will handle the general case:

- **(Zero Location — refined)** A rational matrix $H(s)$ of full column rank has a zero at $s = \zeta_0$ if there is a rational vector $u(s)$ such that $u(\zeta_0)$ is finite and nonzero, and $\lim_{s\to\zeta_0}[H(s)u(s)] = 0$.

Example 27.3 Consider

$$H(s) = \begin{pmatrix} 1 & \frac{1}{s-3} \\ 0 & 1 \end{pmatrix},$$

It is clear that $H(s)$ has a pole at $s = 3$, but it may not be immediately obvious that it also has a zero at $s = 3$. Observe that for s approaching 3, the second column of $H(s)$ approaches alignment with the first column, so the rank of $H(s)$ approaches 1, i.e. there is a rank drop at $s = 3$. To confirm this, pick

$$u(s) = \begin{pmatrix} -1 \\ s-3 \end{pmatrix}$$

and verify that $\lim_{s\to 3} H(s)u(s) = 0$ even though $u(3)$ is (finite and) nonzero.

As suggested earlier, one of the nice features of our definition of zeros is that, for an invertible $H(s)$, they become poles of the inverse. In this example,

$$H^{-1}(s) = \begin{pmatrix} 1 & -\frac{1}{s-3} \\ 0 & 1 \end{pmatrix}$$

which evidently has a pole $s = 3$.

There is also a prescription for establishing the multiplicities of the zeros, and again we state it for completeness, but *not* with the expectation that you learn to work with it:

- **(Zero Multiplicities)** Determine the *largest* multiplicity with which ζ_0 occurs as a *pole* among the 1×1 minors or, if it doesn't appear as a pole, then determine the *smallest* multiplicity with which it occurs as a *zero* of *every* 1×1 minor; denote this by $\nu_1(\zeta_0)$. Continue similarly with the 2×2 minors, and so on, stopping with minors of size r equal to the rank of $H(s)$ (beyond which size all minors vanish). Let ℓ denote the first size for which $\nu_\ell(\zeta_0) < \nu_{\ell-1}(\zeta_0)$ (this ℓ will actually depend on ζ_0, so we should denote it by $\ell(\zeta_0)$, but we omit the argument to keep the notation simple). Then the set of multiplicities associated with the zero at ζ_0 is given by $\nu_{\ell-1}(\zeta_0) - \nu_\ell(\zeta_0)\,,\, \nu_\ell(\zeta_0) - \nu_{\ell+1}(\zeta_0)\,,\, \cdots\,,\, \nu_{r-1}(\zeta_0) - \nu_r(\zeta_0)$.

Given these definitions of the poles and zeros (and their multiplicities) for MIMO transfer functions, it can be shown that for an *invertible* $H(s)$ the total number of poles (summed over all frequencies, including infinity, and with multiplicities accounted for) equals the total number of zeros (again summed over all frequencies, including infinity, and with multiplicities accounted for). However, for non-invertible square $H(s)$ and for non-square $H(s)$, there *will be more poles than zeros* — an interesting difference from the scalar case. In fact, if the coefficients of the rationals in $H(s)$ are picked "randomly", then a square $H(s)$ will typically (or "generically") be invertible and will have zeros, while a non-square $H(s)$ will typically *not* have zeros. (Of course, the coefficient values in our idealized models of systems are not picked randomly, so the non-generic cases are of interest too.)

Determining Zeros from a Minimal Realization

What can be said about the relation of the zeros of $H(s)$ to properties of a minimal realization (A, B, C, D) of this transfer function? (The non-minimal parts of a realization do not contribute to the transfer matrix, and therefore play no role in determining poles and transmission zeros.) The answer is provided by the following nice result (which we shall demonstrate immediately below, but only for those zero locations that are not also pole locations, because the general proof requires tools beyond those developed here):

- **(Finite Zeros from a Minimal State-Space Model)** Given a *minimal* state-space realization (A, B, C, D) of $H(s)$, the finite zeros of $H(s)$, in both location and multiplicity, are the same as the finite zeros of the *system matrix*

$$\begin{pmatrix} sI - A & -B \\ C & D \end{pmatrix} .$$

(The finite zeros of the system matrix are defined as before, namely as the finite values of s for which the matrix drops rank.)

Thus, the locations of the finite zeros of $H(s)$ are the values of s for which the system matrix of a minimal realization drops rank. Note that the system matrix has no finite poles to confound our determination of which values of s correspond to rank loss. (If the realization is not minimal, then the system matrix has additional zeros, corresponding to the unobservable and/or unreachable eigenvalues of the realization. These zeros, along with the transmission zeros, comprise what are referred to as the *invariant zeros* of the system.)

To demonstrate the above result for the special case where pole and zero locations do not coincide, we begin with the identity

$$\begin{pmatrix} I & 0 \\ -C(sI - A)^{-1} & I \end{pmatrix} \begin{pmatrix} sI - A & -B \\ C & D \end{pmatrix} = \begin{pmatrix} sI - A & -B \\ 0 & H(s) \end{pmatrix} . \tag{27.1}$$

Several facts can be deduced from this identity, including the following:

- If ζ_0 is not an eigenvalue of A and thus not a pole of $H(s)$, the first matrix in the above identity is well-defined and invertible at $s = \zeta_0$, so the other two matrices in the identity must have the same rank at $s = \zeta_0$. Therefore, since $\zeta_0 I - A$ is invertible, it follows in this case that $H(s)$ drops rank at $s = \zeta_0$ iff the system matrix drops rank at $s = \zeta_0$. This is the result we were aiming to demonstrate.
- The above identity also shows that the rank of $H(s)$ as a rational matrix (where this rank may be defined as the size of the largest non-vanishing minor of $H(s)$, and is also the rank that $H(s)$ has for most values of s) is n less than the rank of the system matrix, where n is the order of the realization (A, B, C, D). It follows that $H(s)$ has full column (respectively, row) rank as a rational matrix iff the system matrix has full column (row) rank as a rational (or polynomial) matrix.
- For *square* $H(s)$, we can take determinants on both sides of the above identity, and thereby conclude that
$$\det(\text{system matrix}) = \det(sI - A)\ \det H(s)$$
Thus, if $\det H(s)$ is a non-zero rational, then the zeros of $H(s)$ are precisely the roots of the polynomial $\det(sI - A)\ \det H(s)$. For this reason, the product of the pole polynomial of $H(s)$ and of $\det H(s)$ — in the case where $\det H(s) \neq 0$ — may be referred to as the **zero polynomial** of $H(s)$.

The problem of finding the values of s where a matrix of the form $s\mathcal{E} - \mathcal{A}$ drops rank, with $\mathcal{E}$ possibly singular or even non-square, is referred to as a *generalized eigenvalue problem*, and the corresponding values of s are referred to as generalized eigenvalues. The problem of computing the transmission zeros of a system using the system matrix of an associated minimal realization is evidently of this type. Good numerical routines (e.g. the "qz" algorithm in Matlab) exist for solving the generalized eigenvalue problem.

Exercise Suppose
$$H(s) = \begin{pmatrix} 1 & \frac{1}{s-3} \\ 0 & 1 \end{pmatrix}$$
Find a minimal realization of this transfer function, and use the associated system matrix to establish that $H(s)$ has a single pole and a single zero at $s = 3$.

Zero Directions

Now let us consider in more detail the particular but important case where $H(s)$, and therefore the system matrix of a minimal realization of it, have full column rank as rational matrices. For this case, rank loss in the system matrix at $s = \zeta_0$ corresponds to having
$$\begin{pmatrix} \zeta_0 I - A & -B \\ C & D \end{pmatrix} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{27.2}$$
The observability of the realization ensures (by the modal observability test) that $u_0 \neq 0$ in the above equation, and the assumption that the system matrix — or equivalently $H(s)$ — has full column-rank as a rational matrix ensures that $x_0 \neq 0$. The vector x_0 in this equation is referred to as the *state zero direction* associated with ζ_0, and u_0 is the *input zero direction*. The dynamical significance of the state and input zero directions is given by the following result:

- **(Dynamical Interpretation of Zero Location and Zero Directions)** Suppose ζ_0 is a zero location of $H(s)$ and x_0, u_0 are associated state and input zero directions computed from the system matrix of a minimal realization of $H(s)$. Then, with initial condition $x(0) = x_0 \neq 0$ and input $u(t) = u_0 e^{\zeta_0 t} \neq 0$ for $t \geq 0$, the state response of the system (A, B, C, D) is $x(t) = x_0 e^{\zeta_0 t} \neq 0$ and the output response is identically 0, i.e. $y(t) \equiv 0$, for $t \geq 0$.

The proof of the above statement is by simple verification. Thus note that $x(t) = x_0 e^{\zeta_0 t}$ and $u(t) = u_0 e^{\zeta_0 t}$ satisfy the state equation $\dot{x}(t) = Ax(t) + Bu(t)$, in view of the top row of (27.2). Therefore the (unique) state trajectory obtained by choosing x_0 as the initial condition $x(0)$ and choosing the input as $u(t) = u_0 e^{\zeta_0 t}$ is precisely $x(t) = x_0 e^{\zeta_0 t}$. The corresponding output is $y(t) = Cx(t) + Du(t)$, and the bottom row of (27.2) shows that this expression evaluates to 0.

The above result shows that a MIMO zero still has an interpretation as an absorbed frequency. The components of the input zero direction vector u_0 specify the proportions in which the exponential $e^{\zeta_0 t}$ should be present at the corresponding inputs of the system to ensure — when the initial condition is picked to be the state zero direction vector x_0 — that this exponential appears in *none* of the outputs. For the case where ζ_0 is not a pole of $H(s)$, we can use (27.1) to deduce that $H(\zeta_0)u_0 = 0$.

One can similarly develop "duals" of the preceding results to focus on the loss of *row* rank rather than column rank, invoking left zero directions rather than the right zero directions that we have introduced above, but we omit the details and summarize the results in Table 27.1. Also, there are natural (but notationally cumbersome) generalizations of the above construction to expose the dynamical significance of having a zero with multiplicities larger than 1.

Example 27.4 A transfer function matrix is given by

$$H(s) = \begin{bmatrix} \frac{s-1}{s-2} & \frac{2}{s+1} \\ \frac{s}{s+1} & 0 \end{bmatrix}.$$

The reader should be able to verify that there is a pole at -1 with multiplicity 2, and a pole at 2 with multiplicity 1. The normal rank of $H(s)$ is 2. At ∞, $H(\infty)$ has rank 1 which implies that the system has a zero at ∞. This transfer function matrix also loses rank at $s = 0$. The third zero (note that since the transfer function matrix is square there is an equal number of poles and zeros) must be at the location of the pole $s = 2$. To see this, we define

$$u(s) = \begin{bmatrix} -2(s-2) \\ (s+1)(s-1) \end{bmatrix}.$$

It is clear that $u(2)$ is finite and

$$\lim_{s\to 2} H(s)u(s) = \lim_{s\to 2} \begin{bmatrix} 0 \\ \frac{-2s(s-2)}{s+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which confirms that $H(s)$ has a zero at 2.

Another way of determining the finite zeros is to obtain a realization and analyze the system's matrix. Using Gilbert's realization, we get

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} u \end{aligned}$$

The system matrix is

$$\begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = \begin{bmatrix} s-2 & 0 & 0 & -1 & 0 \\ 0 & s+1 & 0 & -1 & 0 \\ 0 & 0 & s+1 & 0 & -1 \\ 1 & 0 & 2 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix},$$

and its determinant is

$$\det \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = 2s(s-2),$$

from which it is clear that the finite zeros are 0 and 2.

$H(s)$ is $p \times m$ full column rank transfer matrix $H(s) = \left[\begin{array}{cc} A & B \\ C & D \end{array}\right]$ minimal	$H(s)$ is $p \times m$ full row rank transfer matrix $H(s) = \left[\begin{array}{cc} A & B \\ C & D \end{array}\right]$ minimal	Comments
ζ_0 is a zero of $H(s)$ if rank $H(\zeta_0) < m$ **Equivalently** there exists u_0 such that $H(\zeta_0)u_0 = 0$	ζ_0 is a zero of $H(s)$ if rank $H(\zeta_0) < p$ **Equivalently** there exists u_0 such that $u_0^T H(\zeta_0) = 0$	characterizes zeros that are different from poles
ζ_0 is a zero of $H(s)$ if there exists $u(s)$ such that $u(\zeta_0)$ is finite and $\lim_{s \to \zeta_0} H(s)u(s) = 0$	ζ_0 is a zero of $H(s)$ if there exists $u(s)$ such that $u(\zeta_0)$ is finite and $\lim_{s \to \zeta_0} u(s)^T H(s) = 0$	characterizes all zeros
ζ_0 is a zero of $H(s)$ if rank $\begin{pmatrix} \zeta_0 I - A & -B \\ C & D \end{pmatrix} < n + m$	ζ_0 is a zero of $H(s)$ if rank $\begin{pmatrix} \zeta_0 I - A & -B \\ C & D \end{pmatrix} < n + p$	characterizes finite zeros
ζ_0 is a zero of $H(s)$ if there exists $\begin{pmatrix} x_0 \\ u_0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ such that $\begin{pmatrix} \zeta_0 I - A & -B \\ C & D \end{pmatrix} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$	ζ_0 is a zero of $H(s)$ if there exists $\begin{pmatrix} x_0 \\ u_0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ such that $\begin{pmatrix} x_0^T & u_0^T \end{pmatrix} \begin{pmatrix} \zeta_0 I - A & -B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$	characterizes finite zeros
ζ_0 is a zero of $H(s)$ if there exists $x_0 \neq 0$, $u_0 \neq 0$ such that if $x(0) = x_0$ and $u(t) = e^{\zeta_0 t} u_0$ the solution of $\dot{x} = Ax + Bu$ $y = Cx + Du$ satisfies $y(t) \equiv 0$	ζ_0 is a zero of $H(s)$ if there exists $x_0 \neq 0$, $u_0 \neq 0$ such that if $x(0) = x_0$ and $u(t) = e^{\zeta_0 t} u_0$ the solution of $\dot{x} = A^T x + C^T u$ $y = B^T x + D^T u$ satisfies $y(t) \equiv 0$	characterizes finite zeros

Table 27.1: Duality between right and left zeros