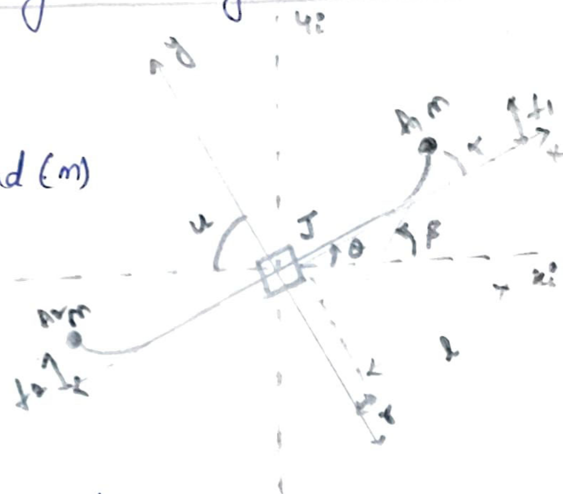
CONTROL OF FLEXIBLE STRUCTURES.

Chapter 1: MATLAB labworks.

1.1 Bamoss: Flexible structure modelling, analysis and contro



Given:

$u \rightarrow$ driving torque (Nm)

$\theta \longrightarrow$ angular position of hub (rd)

f_1 & $f_2 \longrightarrow$ lateral directon at free end (m)

$\alpha \rightarrow$ angular deviation (rd)

$\beta = \theta + \alpha$

the initial condition given.

$\theta(t=0) = 0.18$ deg.

$f_1(t=0) = 4mm$

$f_2(t=0) = 0.5mm$

$\dot{\theta}(t=0) = 20$ deg/s.

$\dot{f}_1(t=0) = -7cm/s$.

$\dot{f}_2(t=0) = 18m/s$

Assumptions

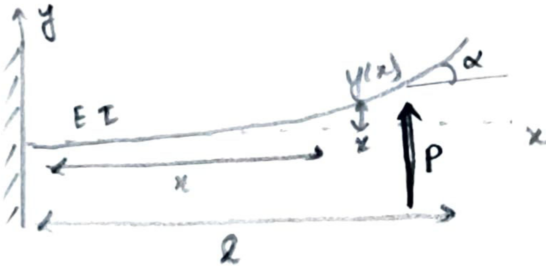
* only motion in horizontal plane
* masses & inertia of beam are neglected
* only small motion considered

Given Numerical data:

$J = 0.015 kgm^2$ $l = 0.286m$ $E = 200\times10^9 N/m^2$ $m = 0.3kg$ $r = 0.05m$

$h = 0.64mm$ $b = 4cm$

1.11 Dynamic modelling



Considering a single cantilever beam with tip load P at its free end.

$\Rightarrow$ to compute the lateral deflection f at the free end we consider the Bending moment at any point x on Beam due to deflection y.

$$\boxed{M(x) = EI\frac{d^2y}{dx^2} = P(l-x)}$$

$EI \longrightarrow$ is the material property of the beam to find the deflection $y(x)$ at any point we integrate twice the moment equation with initial conditions of $y(0) = 0$ & $y'(0) = 0$ (2)

$$\int_0^{y(x)}\!\!\int EI \frac{d^2y}{dx^2} = \int_0^{x}\!\!\int P(l-x)$$

$$\int_0^{y(x)} \frac{dy}{dx} - \cancel{y'(0)} = \int_0^{x} \frac{Plx}{EI} - \frac{Px^2}{2EI}$$

$$y - \cancel{y(0)} = \frac{Plx^2}{2EI} - \frac{Px^3}{6EI}.$$

$$\boxed{y(x) = \frac{Px^2}{2EI}\left(l - \frac{x}{3}\right)}$$

to find the deflection f at $x = l$.

$$f = y(l) = \frac{Pl^2}{2EI}\left(l - \frac{l}{3}\right) \qquad l =$$

$$\boxed{f = \frac{Pl^3}{3EI}}$$

In order to find the stiffness k of the beam. we force equation

$$P = k f.$$

$$\therefore \boxed{K = \frac{P}{f} = \frac{3EI}{l^3}}$$

given value $h = 0.64$ cm $\qquad b = 4$cm
$h = 64 \times 10^{-5}$ m. $\qquad = 0.04$ m.

$$I = \frac{bh^3}{12} = \frac{0.04 \times 64 \times 10^{-5}}{12}. = 8.73 \times 10^{-13}\, m^4.$$

$E = 200 \times 10^9$ N/m² $\qquad l = 0.283$ m

$$K = \frac{3 \times 200 \times 10^9 \times 8.73 \times 10^{-13}}{(0.283)^3}.$$

$$\boxed{k = 23.11 \text{ N/m}}$$

to get the notion of curvature. we calculate the α where

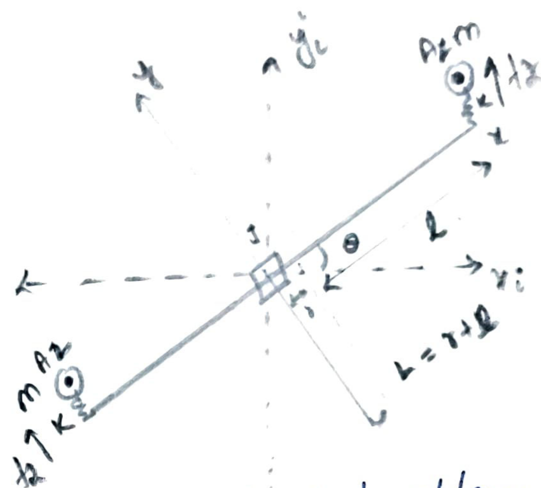
$$\tan\alpha = \frac{dy}{dx}. = \frac{Plx}{EI} - \frac{Px^2}{2EI}.$$

at $x = l$ ∴ $\tan\alpha = \frac{Pl^2}{EI} - \frac{PL^2}{2EI}$ $\boxed{\alpha = \frac{Pl^2}{2EI}}$

and WKT $f = \frac{Pl^3}{3EI}$ & therefore α can be written as a function of f as ~~given~~

$$\boxed{\alpha = \frac{3f}{2l}}$$

⟹ Now consider beam as spring acting b/w the mass m and the tip of beam.



Now considering the Lagrange ~~equation~~ Approach.

Consider the kinetic energy eq.

$$T = \frac{1}{2}\dot{x}^T M \dot{x}$$

$$\boxed{T = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}mV_{A_1}^2 + \frac{1}{2}mV_{A_2}^2}$$

the sum of KE from all the components.

$$\boxed{V_{A_i} = \frac{d\vec{OA_i}}{dt}\Big|_{R_i} = \frac{d\vec{OA_i}}{dt}\Big|_{R_b} + \Omega \times \vec{OA_i}}$$

as V_{A_2} is in to be obtained in inertial frame. we (change of inertia fram (i) to body (b) frame)

As it is in the fram $\vec{x}$ & $\vec{y}$. $\vec{OA}$ can be written $\boxed{\vec{OA} = L\vec{x} + f_1\vec{y}}$

∴ $$V_{A_1} = \begin{bmatrix} 0 \\ \dot{f}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} \times \begin{bmatrix} L \\ f_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cancel{-f_1\dot{\theta}} \\ \dot{f}_1 + L\dot{\theta} \\ 0 \end{bmatrix}$$ → negligible

$$\boxed{V_{A_1}^2 = (\dot{f}_1 + L\dot{\theta})^2}$$

Illy. $$V_{A_2} = \begin{bmatrix} 0 \\ \dot{f}_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\dot{\theta} \end{bmatrix} \times \begin{bmatrix} L \\ f_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \cancel{\dot{\theta} f_2} \\ \dot{f}_2 - L\dot{\theta} \\ 0 \end{bmatrix}$$ negligible

$$\boxed{V_{A_2}^2 = (\dot{f}_2 - L\dot{\theta})^2}$$

Substituting the value ⟹ $T = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}m(\dot{f}_1 + L\dot{\theta})^2 + \frac{1}{2}m(\dot{f}_2 - L\dot{\theta})^2$.

$$= \frac{1}{2}\left[J\dot{\theta}^2 + m\dot{f}_1^2 + 2m\dot{f}_1 L\dot{\theta} + m(\dot{\theta}L)^2 + m\dot{f}_2^2 - 2m\dot{f}_2 L\dot{\theta} + m(\dot{\theta}L)^2\right]$$

$$= \frac{1}{2}\left[(J + 2mL^2)\dot{\theta}^2 + m\dot{f}_1^2 + m\dot{f}_2^2 + 2mL\dot{f}_1\dot{\theta} - 2m\dot{f}_2 L\dot{\theta}\right]$$

Considering $J + 2mL$ as whole body multiline. (4)

$$J_T = J + 2mL^2.$$

F even I dont know

$\therefore$
$$T = \frac{1}{2}\left[J_T\dot\theta^2 + m\dot f_1^2 + m\dot f_2^2 + 2mL\dot f_1\dot\theta \mp 2mL\dot f_2\dot\theta \right].$$

this can be written as a matrix multiplication form

$$\boxed{T = \frac{1}{2}\begin{bmatrix}\dot\theta & \dot f_1 & \dot f_2\end{bmatrix}\begin{bmatrix} J_T & mL & -mL \\ mL & m & 0 \\ -mL & 0 & m\end{bmatrix}\begin{bmatrix}\dot\theta \\ \dot f_1 \\ \dot f_2\end{bmatrix}}$$

corresponds to mass matrix (M)

this is the expression for kinetic energy.

Considering the potential energy given by the spring. therefore,

$$\boxed{V = \frac{1}{2}k f_1^2 + \frac{1}{2}k f_2^2}$$

this can be written in matrix form.

$$\boxed{V = \frac{1}{2}\begin{bmatrix}\theta & f_1 & f_2\end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k\end{bmatrix}\begin{bmatrix}\theta \\ f_1 \\ f_2\end{bmatrix}}$$

corresponds to stiffness matrix (K)

Now, to Considering the input impact on each state we consider the notion of work.
As the torque works only along θ. ~~which is~~ and first component of F correspond to θ,
~~along~~ ~~x~~ ~~axis~~ we can write F as.

$$F = \begin{bmatrix}1 \\ 0 \\ 0\end{bmatrix}$$

therefore the dynamic model of the system can be written as.

$$M\ddot q + kq = Fu.$$

$$\begin{bmatrix} J_T & mL & -mL \\ mL & m & 0 \\ -mL & 0 & m\end{bmatrix}\begin{bmatrix}\ddot\theta \\ \ddot f_1 \\ \ddot f_2\end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k\end{bmatrix}\begin{bmatrix}\theta \\ f_1 \\ f_2\end{bmatrix} = \begin{bmatrix}1 \\ 0 \\ 0\end{bmatrix}u.$$

Given the values of $m = 0.3\,kg$ $\quad l = 0.286\,m$ $\quad k = 23.11\,N/m$ (5)

$J_T = J + 2mL^2$ $\quad J = 0.015\,kgm^2$

$L = l + r = 0.286 + 0.05$
$= \underline{0.336\,m}$

$J_T = 0.015 + 2 \times 0.3 \times (0.336)^2$ $\quad mL = 0.3 \times 0.336$
$= \underline{0.0827}\ kgm^2$ $\quad = \underline{0.1008}$

$$\boxed{\begin{bmatrix} 0.0827 & 0.1008 & -0.1008 \\ 0.1008 & 0.3 & 0 \\ -0.1008 & 0 & 0.3 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{f_1} \\ \ddot{f_r} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 23.11 & 0 \\ 0 & 0 & 23.11 \end{bmatrix} \begin{bmatrix} \theta \\ f_1 \\ f_r \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u}$$ ✗

this is the dynamic model of the system.

Now, Moving on to Open-loop modal analysis.

Considering the lagrange expression of the system. we can find the eigen values and eigen vectors that dictate the system properties by performing open loop modal analysis. we obtain the poles. of the system through this process and observe the modal behavior for each value.

To perform the modal analysis we consider.

$$\det|M\omega^2 - k| = 0.$$

which is obtained by considering a generic solution $q(t) = \sum_{i=1}^{n} \phi_i \lambda_i e^{j(\omega_i t + \varphi_i)}$ for all t. on substituting in the dynamic equation we obtain

$$[-M\omega_i^2 + k]\phi_i = 0$$

which can be written as eigen value problem.

∴ to find the solution we consider

$$\det|M\omega^2 - k| = 0.$$

on substituting the matrices we obtained.

$$\det\left|M\omega^2 - k\right| = \left|\begin{bmatrix} J_T\omega^2 & mL\omega^2 & -mL\omega^2 \\ mL\omega^2 & m\omega^2-k & 0 \\ -mL\omega^2 & 0 & m\omega^2-k \end{bmatrix}\right| = 0.$$

$$\Rightarrow J_T\omega^2\left((m\omega^2-k)^2\right) - mL\omega^2\left(mL\omega^2 \times (m\omega^2-k)\right) - mL\omega^2\left(+mL\omega^2(m\omega^2-k)\right) = 0$$

$$\Rightarrow (m\omega^2-k)\left[J_T\omega^2(m\omega^2-k) - (mL\omega^2)^2 - (mL\omega^2)^2\right] = 0.$$

$$\Rightarrow (m\omega^2-k)\left[J_T\omega^2(m\omega^2-k) - 2(mL\omega^2)^2\right] = 0$$

$$\Rightarrow (m\omega^2-k)(\omega^2)\left[J_T m\omega^2 - J_T k - 2m^2L^2\omega^2\right] = 0$$

$$\Rightarrow (m\omega^2-k)(\omega^2)\left((J_T m - 2m^2L^2)\omega^2 - J_T k\right] = 0$$

$$\Rightarrow (m\omega^2-k)(\omega^2)\left(mJ\omega^2 - J_T k\right) = 0.$$

Now considering each factor we obtain the following eigen values.

$$\begin{cases} \omega_1 = 0 \text{ rad/s.} \\ \omega_2 = \pm\sqrt{\dfrac{k}{m}} \\ \omega_3 = \pm\sqrt{\dfrac{kJ_T}{mJ}}. \end{cases}$$

$\longrightarrow$ substituting values.

$k = 23.11$ N/m $\quad J = 0.015$ kg m^2

$m = 0.3$ kg $\quad J_T = 0.0827$ kg

$$\omega_2 = \pm\sqrt{\frac{23.11}{0.3}} = \pm 8.77 \text{ rad/s.}$$

$$\omega_3 = \pm\sqrt{\frac{23.11 \times 0.082}{0.3 \times 0.015}} \pm 20.52 \text{ rad/s}$$

$s = j\omega. \qquad s^2 = -\omega^2$

therefore we obtain the poles to be at 0 with 2 poles $\pm 8.77j$ rad/s. and $\pm 20.52j$ rad/s.

to find the eigen vectors we use the form

$[M\omega^2 - k]\phi = 0.$ at each ω.

considering $\omega^2 = 0$, $\rightarrow |M\omega^2 - k|\phi = 0$

$$\begin{bmatrix} J_T\omega^2 & mL\omega^2 & -mL\omega^2 \\ mL\omega^2 & m\omega^2-k & 0 \\ -mL\omega^2 & 0 & m\omega^2-k \end{bmatrix}\begin{bmatrix}\phi_{11} \\ \phi_{12} \\ \phi_{13}\end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -k & 0 \\ 0 & 0 & -k \end{bmatrix}\begin{bmatrix}\phi_{11} \\ \phi_{12} \\ \phi_{13}\end{bmatrix} = 0$$

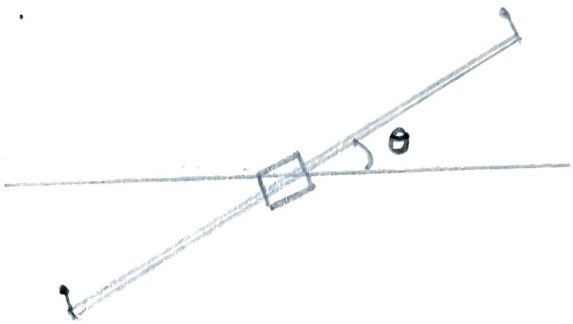
we can see that ϕ_1 can take any value and satisfy the equation.

$$0 \times \phi_{11} = 0 \longrightarrow \phi_{11} \rightarrow \text{any value}$$
$$-k\phi_{12} = 0 \qquad \phi_{12} = 0$$
$$-k\phi_{13} = 0 \qquad \phi_{13} = 0.$$

$\therefore$ we get the first eigen vector.

$$\boxed{\phi_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

Here we can observe that at $\omega^2 = 0$, we see no vibrations due to f_1 & f_2 which shows that this is a rigid mode.



$\omega^2 = 0$
rigid mode.

purely rotational motion with no vibration.

Now consider $\omega^2 = k/m$;

$$[M\omega^2 - k]\phi = 0.$$

$$\begin{bmatrix} J_T\frac{k}{m} & Lk & -Lk \\ Lk & \cancel{k-k} & 0 \\ -Lk & 0 & \cancel{k-k} \end{bmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \\ \phi_{23} \end{bmatrix} = 0$$

the 3 equations.

$$J_T\frac{k}{m}\phi_{21} + Lk\phi_{22} - Lk\phi_{23} = 0$$
$$Lk\phi_{21} + \cancel{(k-k)}\phi_{22} = 0$$
$$-Lk\phi_{21} + \cancel{(k-k)}\phi_{23} = 0.$$

we can see that $\phi_{21} = 0$

substituting $\phi_{21} = 0$ in first equation

we get $\phi_{22} = \phi_{23}$.

$\therefore$ the eigen vector is

$$\boxed{\phi_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}$$

Here we can observe that at $\omega^2 = k/m$, the vibration due to f_1 & f_2 are symmetrical. therefore this represents the symmetrical flexible mode, we can also observe that the rigid mode is absent hence no rotation about the hub (8)

flexible symmetric mode

$\omega^2 = \frac{k}{m}$.

vibrations observed to the symmetrically without rotation about centre.

Now considering $\omega^2 = \frac{k J_T}{m J}$.

$$(M\omega^2 - K)\phi_3 = 0.$$

$$\omega^2 \begin{bmatrix} J_T & mL & -mL \\ mL & \frac{m\omega^2 - k}{\omega^2} & 0 \\ -mL & 0 & \frac{m\omega^2-k}{\omega^2} \end{bmatrix} \begin{bmatrix} \phi_{31} \\ \phi_{32} \\ \phi_{33} \end{bmatrix} = 0.$$

$$J_T\phi_{31} + mL\phi_{32} - mL\phi_{33} = 0$$

$$mL\phi_{31} + \left(m - m\frac{J}{J_T}\right)\phi_{32} = 0$$

$$-mL\phi_{31} + \left(m - m\frac{J}{J_T}\right)\phi_{33} = 0$$

from the two equation, we observe that $\phi_{32} = -\phi_{33}$.

Also.

$$\phi_{32} = \frac{-LJ_T}{J_T - J}\phi_{31}$$

$$\phi_{33} = \frac{LJ_T}{J_T - J}\phi_{31}.$$

Now considering $\phi_{31} = 1$.

we get $\phi_{32} = \frac{-LJ_T}{J_T - J}$ $\qquad \phi_{33} = \frac{LJ_T}{J_T - J}$.

$$= \frac{-0.336 \times 0.0827}{0.0827 - 0.015} \qquad \phi_{33} = +0.4104.$$

$$\phi_{32} = -0.4104$$

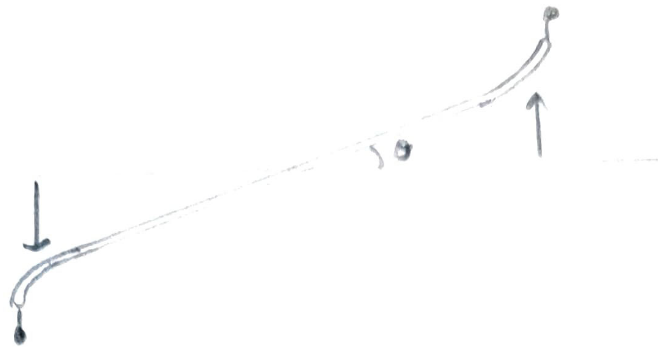
$$\boxed{\phi_3 = \begin{bmatrix} 1 \\ -0.4104 \\ +0.4104 \end{bmatrix}}$$

Here we observe that at $\omega^2 = \frac{kJ_T}{mJ}$. we can see the vibrations due to f1 & f2 are antisymmetrical and the.

presence of rigid mode signifies some rotation about hub. (9)
therefore we can say this mode is anti symmetrical flexible mode.

antisymmetrical flexible mode

$$\omega^2 = \frac{k J_T}{mJ}.$$



anti symmetrical deflection with rotation about hub.

To study the controllability of the system

we consider the three mode shapes

$$\phi_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \phi_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \phi_3 = \begin{bmatrix} 1 \\ -0.4104 \\ 0.4104 \end{bmatrix}$$

Now the overall mode shape matrix

$$\phi = [\phi_1\ \phi_2\ \phi_3] \Rightarrow \phi = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -0.41 \\ 0 & 1 & 0.41 \end{bmatrix}$$

to find the controllability of the system, we transformed the states on to the modes of the system.

$$q = \phi \eta .$$

$$\therefore \quad \phi^T M \phi \ddot{\eta} + \phi^T K \phi \eta = \phi^T F u$$

In the modal basis state space, we can analyse the controllability of the system by observing the matrix $\phi^T F$.

$$\phi^T F = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -0.41 \\ 0 & 1 & 0.41 \end{bmatrix}^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & -0.41 & 0.41 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\phi^T F = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} \longrightarrow \text{Notion of controllability}$$

We can observe that the 1st and 3rd mode corresponding to the rigid mode and anti symmetric flexible mode are controllable!
While the 2nd mode corresponding to the symmetrical flexible modes are not controllable! ??

From Open loop modal analysis we observed the poles of system to be $0, 0, \pm 8.77j, \pm 20.52j$.

Now to find zero of system $\longrightarrow$ Closed loop modal Analysis.

The Idea behind closed loop analysis is that when a gain of Closed loop approaches infinity the poles of the closed loop will tend to the Open loop zero! (As observed in rootlocus plots).

Therefore we feedback each state for each transfer function of each output to observe the respective zeros.

Considering $\frac{\theta}{u}(s)$ function:
↳ we use control law. $u = -k_\theta \theta$. where $k_\theta \rightarrow \infty$

$$M\ddot{q} + kq = F(-k_\theta \theta)$$

$$\therefore \begin{bmatrix} J_T & mL & -mL \\ mL & m & 0 \\ -mL & 0 & m \end{bmatrix} \ddot{q} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} q = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -k_\theta & 0 & 0 \end{bmatrix} q.$$

$$= \begin{bmatrix} -k_\theta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} q.$$

$$\underbrace{\begin{bmatrix} J_T & mL & -mL \\ mL & m & 0 \\ -mL & 0 & m \end{bmatrix}}_{M_{new}} \ddot{q} + \underbrace{\begin{bmatrix} +k_\theta & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}}_{k_{new}} q = 0$$

now performing the Modal analysis. $|M_{new}\omega^2 - k_{new}| = 0$

$$\left\| \begin{matrix} J_T\omega^2 + k_\theta & mL\omega^2 & -mL\omega^2 \\ mL\omega^2 & m\omega^2 - k & 0 \\ -mL\omega^2 & 0 & m\omega^2 - k \end{matrix} \right\| = 0$$

$$\cancel{(J_T w^2 + k\theta)}\left[(mw^2-k)^2\right] - mLw^2(mLw^2(mw^2-k)) - mLw^2(mlw^2(mw^2-k)) = 0$$

$$k\theta(mw^2-k)^2 - (mlw^2)^2(mw^2-k) - (mlw)^2(mw^2-k) = 0.$$

$$(mw^2-k)\left[k\theta(mw^2-k) - \cancel{2(mlw^2)^2}\right] = 0$$

$$k\theta(mw^2-k)^2 = 0.$$

From this we observe that the poles of the closed loop system are

$$w_1^2 = w_2^2 = \frac{k}{m}$$

substituting

$$w_1 = w_2 = \pm 8.77.$$

these are also the zeros of the open loop system.

$\therefore$ the transfer function b/w θ and u can be written

$$w_1^2 = 0 \quad w_2^2 = \frac{k}{m} \quad w_3^2 = \frac{kJ_T}{mJ} \longrightarrow \text{~~poles~~. frequency.}$$

$$w_4^2 = w_5^2 = \frac{k}{m} \longrightarrow \text{~~zeros~~. freqn.}$$

$s = wj$

$$\frac{\theta(s)}{u} = \frac{\cancel{(s^2+w_4^2)}(s^2+w_5^2)}{(s^2+w_1^2)\cancel{(s^2+w_2^2)}(s^2+w_3^2)} \quad (\text{as } w_4 = w_2).$$

$$\frac{\theta(s)}{u} = \frac{(s^2+w_5^2)}{(s^2+w_1^2)(s^2+w_3^2)} = \frac{(s^2+k/m)}{s^2\left(s^2+\frac{kJ_T}{mJ}\right)} \quad \boxed{\frac{\theta}{s} = \frac{J(ms^2+k)}{s^2(mJs^2+kJ_T)}}$$

substituting the values.

$$\frac{\theta(s)}{u} = \frac{0.015(0.3s^2+23.11)}{s^2(0.3\times0.015\,s^2+23.11\times0.0827)}$$

$$\boxed{\frac{\theta(s)}{u} = \frac{4.5\times10^{-3}s^2+0.34}{s^2(4.5\times10^{-3}s^2+1.911)}} \approx \boxed{\frac{\theta(s)}{u} = \frac{s^2+75.5}{s^2(s^2+424.66)}}$$

Now considering the transfer function $\frac{\beta(s)}{u}$.

↳ we consider the control law $u = -k_\beta \beta$ where $K_\beta \to \infty$

$$M\ddot{q} + kq = F(-k_\beta \beta)$$

~~$F(-k_\beta\beta)$ [0 0 −kβ] β [0 0 −kβ] β~~

$$\text{WKT} \to \beta = \theta + \alpha = \theta + \frac{3f_1}{2l} \Longrightarrow k_\beta\beta = \left(\theta + \frac{3f_1}{2l}\right)k_\beta \quad \boxed{\therefore k_\beta.\beta = k_\beta\theta + \frac{3k_\beta f_1}{2l}}$$

$$F(-k\beta\beta) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -k\beta & -\frac{3k\beta}{2l} & 0 \end{bmatrix} \begin{bmatrix} \theta \\ x_1 \\ x_2 \end{bmatrix}$$

$$F(-k_\beta \beta) = \begin{bmatrix} -k\beta & -\frac{3k\beta}{2l} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} q \longrightarrow$$ substituting in the equation.

Now
$$\underbrace{\begin{bmatrix} J_T & mL & -mL \\ mL & m & 0 \\ -mL & 0 & m \end{bmatrix}}_{M_{new}} \ddot{q} + \underbrace{\begin{bmatrix} k\beta & +\frac{3k\beta}{2l} & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}}_{k_{new}} q = 0.$$

now performing the modal analysis $|M_{new}\omega^2 - k_{new}| = 0$

$$\left|\begin{bmatrix} J_T\omega^2 + k\beta & \omega^2 mL + \frac{3k\beta}{2l} & -mL\omega^2 \\ mL\omega^2 & m\omega^2 - k & 0 \\ -mL\omega^2 & 0 & m\omega^2 - k \end{bmatrix}\right| = 0.$$

$$(\cancel{J_T\omega^2} + k\beta)\left[(m\omega^2 - k)^2\right] - \left(\cancel{mL\omega^2} + \frac{3k\beta}{2l}\right)\left[mL\omega^2(m\omega^2 - k)\right] - mL\omega^2\left(mL\omega^2(m\omega^2 - k)\right) = 0$$

$$(m\omega^2 - k)\left[k\beta(m\omega^2 - k) - \frac{3k\beta}{2l}(mL\omega^2) - (mL\omega^2)^2\right] = 0$$

$$(m\omega^2 - k)\left[k\beta\left[m\omega^2 - k - \frac{3mL\omega^2}{2l}\right] - \cancel{(mL\omega^2)^2}\right] = 0$$

$$(m\omega^2 - k)\,k\beta\left(m\omega^2\left(1 - \frac{3L}{2l}\right) - k\right) = 0$$

$\therefore$ the poles of the closed loop systems are.

$$\omega_1^2 = \frac{k}{m} \quad \text{and} \quad \omega^2 = \frac{k}{m\left(1 - \frac{3L}{2l}\right)} = \frac{2kl}{m(2l - 3L)} = \frac{2kl}{m(2l - 3l - 3r)}$$

$\omega^2 = \omega_1$ $\qquad \omega^2 = \frac{-2kl}{(3r + l)m}$ $\qquad \omega^2 = -\omega_T$

substituting,

$$\omega_1 = \pm\sqrt{\frac{k}{m}} = \cancel{f}8.77 \qquad \omega_2 = \sqrt{\frac{2kl}{(3r+l)m}}\, i = \cancel{f}9.99i.$$

$\omega^2 = -\omega_6^2$

these are the closed loop poles that represent the open loop zeros.

therefore the transfer function of the system.

$\omega_1^2 = 0 \qquad \omega_2^2 = \frac{k}{m} \qquad \omega_3^2 = \frac{kJ_T}{mJ} \longrightarrow$ poles.

$\omega_4^2 = \frac{k}{m} \qquad \omega_5^2 = \frac{-2kl}{m(3r+l)} \longrightarrow$ zeros.

$\omega^2 = -\omega_5^2$

$s = j\omega$

$s^2 = -\omega^2 = \omega_7^2$

$(s^2 - \omega_7^2)$

∴ the transfer function of the system.

$$\frac{\beta(s)}{u} = \frac{\cancel{(s^2+\omega_4^2)}(s^2+\omega_5^2)}{(s^2+\omega_1^2)\cancel{(s^2+\omega_2^2)}(s^2+\omega_3^2)} \qquad (\omega_2^2 = \omega_4^2)$$

$$\boxed{\frac{\beta(s)}{u} = \frac{s^2+\omega_5^2}{(s^2+\omega_1^2)(s^2+\omega_3^2)}}$$

stupid

substituting the values.

$$\frac{\beta(s)}{u} = \frac{s^2 + \frac{2kl}{m(3r+l)}}{s^2\left(s^2+\frac{kJ_T}{mJ}\right)} = \frac{\frac{m(3r+l)s^2 - 2kl}{m(3r+l)}}{s^2\frac{(mJs^2+kJ_T)}{mJ}}$$

$$\boxed{\frac{\beta(s)}{u} = \frac{mJ(3r+l)s^2 - 2klJ}{(3r+l)s^2(mJs^2+kJ_T)}}$$

substituting the value of each parameters.

$$\frac{\beta(s)}{u} = \frac{0.3\times0.015(3\times0.05+0.286)s^2 - 2\times23.11\times0.286\times0.015}{(3\times0.05+0.286)s^2(0.3\times0.015s^2 + 23.11\times0.0827)}$$

$$\frac{\beta(s)}{u} = \frac{1.962\times10^{-3}s^2 - 0.198}{0.436\ s^2(4.5\times10^{-3}s^2+1.911)}$$

$$\frac{\beta(s)}{u} = \frac{4.5\times10^{-3}s^2 - 0.454}{s^2(4.5\times10^{-3}s^2+1.911)}$$

$$\boxed{\frac{\beta(s)}{u} = \frac{s^2 - 100.8}{s^2(s^2+424.66)}}$$

this is the transfer function of $\frac{\beta}{u}$.

Modelling and Control of Flexible Systems

BAMOSS Marked Seminar

- Akash Sharma

- Deeksha Kota


```
clear
clc
close all
format short e
```

Required Parameters

```
% Lames:
l=0.286;     % m
h=0.04;      % m
e=0.00064;    % m
E=200e9;      % N/m2

J=0.015;    % Kgm
L=0.335;     % m
m=0.30;      % Kg

%Calcul raideur quivalente:
I=h*e^3/12;
k=3*E*I/l^3;
we=sqrt(k/m);  % pulsation encastre de la lame:

% Inertie totale:
Jt=J+2*m*L^2;

% Matrice de masse:
M=[Jt m*L -m*L;m*L m 0; -m*L 0 m];
% Matrice de raideur:
K=[0 0 0;0 k 0;0 0 k];
% Matrice d'amortissement:
D=[0 0 0;0 0.001 0;0 0 0.001];
% Matrice d'entree
F=[1;0;0];
```

State-Space Representation

```
% State-Space matrices are defined based on the Lagrange equation

A = [  zeros(3)  eye(3)
     -inv(M)*K -inv(M)*D];
```

```
B = [zeros(3,1)
     inv(M)*F];

C = [1 0 0 0 0 0
     0 0 0 1 0 0
     0 0 0 1 3/(2*l) 0];

D = zeros(3,1);

sys = ss(A,B,C,D)
```

```
sys =

  A =
            x1        x2        x3        x4        x5        x6
   x1        0         0         0         1         0         0
   x2        0         0         0         0         1         0
   x3        0         0         0         0         0         1
   x4        0     500.5    -500.5         0   0.02233  -0.02233
   x5        0    -242.4     167.7         0  -0.01082  0.007482
   x6        0     167.7    -242.4         0  0.007482  -0.01082

  B =
          u1
   x1      0
   x2      0
   x3      0
   x4  66.67
   x5 -22.33
   x6  22.33

  C =
         x1     x2     x3     x4     x5     x6
   y1     1      0      0      0      0      0
   y2     0      0      0      1      0      0
   y3     0      0      0      1  5.245      0

  D =
       u1
   y1   0
   y2   0
   y3   0

Continuous-time state-space model.
```

sys.a (same thing) — Not whole system.

Question A.5

```
damp(A)
```

```
         Pole              Damping       Frequency      Time Constant
                                       (rad/TimeUnit)     (TimeUnit)

  0.00e+00                -1.00e+00      0.00e+00             Inf
  0.00e+00                -1.00e+00      0.00e+00             Inf
 -9.15e-03 + 2.02e+01i     4.52e-04      2.02e+01         1.09e+02
 -9.15e-03 - 2.02e+01i     4.52e-04      2.02e+01         1.09e+02
 -1.67e-03 + 8.64e+00i     1.93e-04      8.64e+00         6.00e+02
 -1.67e-03 - 8.64e+00i     1.93e-04      8.64e+00         6.00e+02
```

These eigen values are frequencies of different modes of the free undamped system. They match the ones obtained through the calculations by hand. Complex poles come in symmetric pairs according to the equation $s = j\omega$. Therefore, for 3 values of ω^2, we get 6 complex conjugate poles. These poles are shown in the first column. After drawing the modal shapes, we find that the frequencies of each mode.

$\omega_1 = 0$ **--> Rigid modes**

$\omega_2 = \sqrt{\left(\frac{k}{m}\right)} = 8.64$ **--> Symmetric Flexible Modes**

$\omega_3 = \sqrt{\left(\frac{k}{m}\frac{J_t}{J}\right)} = 20.2$ **--> Anti-Symmetric Flexible Modes**

Controllability and Observability Analysis

The 2 MATLAB functions **ctrbf** and **obsvf** are used to determine controllability and observability of each of the modes.

The **ctrbf** function converts the matrices into a staircase form where the **number** of controllable and uncontrollable modes can be identified through the $\overline{A}$ and $\overline{B}$ matrices. Once this is known, by indexing the $\overline{A}$ matrix, we can find exactly which modes are controllable and uncontrollable. The sub-matrices A_c and A_{uc} that carry information about the controllability of the modes, are sliced from the $\overline{A}$ matrix according to this configuration.

$\overline{A} = \begin{bmatrix} A_{\text{uc}} & 0 \\ A_{21} & A_c \end{bmatrix} \quad \overline{B} = \begin{bmatrix} 0 \\ B_c \end{bmatrix}$

Controllability Analysis

```
states = 6;
[Abar_con,Bbar_con,Cbar_con,~,k_con] = ctrbf(A,B,C);
Abar_con
```

```
Abar_con = 6×6
   2.1714e+01  -8.0777e+00  -3.1992e-14   1.0668e-15   8.1561e-15   4.0876e-17
   6.7627e+01  -2.1717e+01  -5.7605e-14   1.2025e-15   1.4548e-16   1.6334e-16
  -3.4398e-13   1.1296e-13   2.6777e-03   1.0000e+00   9.3093e-15   4.1774e-18
  -1.1632e-13  -2.7590e-13  -6.1011e+01  -4.2679e-02   2.8905e+01   9.4713e-17
   6.2571e-15  -2.4396e-15  -7.8688e-02  -5.5102e-05   3.7279e-02   1.0000e+00
   1.7551e-12   3.2355e-12   7.3675e+02   5.1592e-01  -3.4905e+02  -1.5574e-02
```

```
sum(k_con) % Number of controllable modes
```

```
ans =
     4
```

```
uncon_states = states - sum(k_con);
```

```
Auc = Abar_con(1:uncon_states,1:uncon_states);
Ac = Abar_con(uncon_states+1:states,uncon_states+1:states);

damp(Ac)
```

```
        Pole              Damping       Frequency      Time Constant
                                      (rad/TimeUnit)    (TimeUnit)

 -9.15e-03 + 2.02e+01i    4.52e-04      2.02e+01         1.09e+02
 -9.15e-03 - 2.02e+01i    4.52e-04      2.02e+01         1.09e+02
  8.54e-08               -1.00e+00      8.54e-08        -1.17e+07
 -8.54e-08                1.00e+00      8.54e-08         1.17e+07
```

Controllable modes - rigid modes & anti-symmetric modes

```
damp(Auc)
```

```
        Pole              Damping       Frequency      Time Constant
                                      (rad/TimeUnit)    (TimeUnit)

 -1.67e-03 + 8.64e+00i    1.93e-04      8.64e+00         6.00e+02
 -1.67e-03 - 8.64e+00i    1.93e-04      8.64e+00         6.00e+02
```

Uncontrollable modes - symmetric modes

Obervability Analysis

The strategy used to obtain insight on observability using th **obsvf** function is the same as that of controllability, the only difference being that observability has to be checked with respect to **one specific output at a time**. Hence, the system is indexed for each output one by one. The configuration followed in this case is as follows.

$$\bar{A} = \begin{bmatrix} A_{\text{no}} & A_{12} \\ 0 & A_o \end{bmatrix} \quad \bar{B} = \begin{bmatrix} B_{\text{no}} \\ B_o \end{bmatrix}$$

Observability with respect with θ

```
sys1 = sys(1,:);

[Abar_obs1,Bbar_obs1,Cbar_obs1,~,k_obs1] = obsvf(sys1.A,sys1.B,sys1.C);
Abar_obs1
```

```
Abar_obs1 = 6×6
   5.6425e-04  -1.0000e+00   6.8390e-17   6.3021e-16            0            0
   7.4705e+01  -3.8976e-03   1.1469e-16   8.8744e-16            0            0
  -4.0119e-14  -1.3076e-18  -4.4620e-05  -4.1006e+02            0            0
  -3.2716e-18  -1.0091e-16   1.0000e+00  -1.8252e-02            0            0
   5.6843e-14  -1.7347e-18  -9.5067e-18   7.0785e+02            0            0
            0            0            0            0   1.0000e+00            0
```

```
sum(k_obs1) % Number of observable modes
```

```
ans =
```

```
     4
```

```
unob_states = states - sum(k_obs1);

Ano = Abar_obs1(1:unob_states,1:unob_states);
Ao = Abar_obs1(unob_states+1:states,unob_states+1:states);


damp(Ao)
```

```
         Pole              Damping       Frequency       Time Constant
                                       (rad/TimeUnit)     (TimeUnit)

  0.00e+00                -1.00e+00       0.00e+00              Inf
  0.00e+00                -1.00e+00       0.00e+00              Inf
 -9.15e-03 + 2.02e+01i     4.52e-04       2.02e+01           1.09e+02
 -9.15e-03 - 2.02e+01i     4.52e-04       2.02e+01           1.09e+02
```

Observable modes - rigid modes & anti-symmetric modes

```
damp(Ano)
```

```
         Pole              Damping       Frequency       Time Constant
                                       (rad/TimeUnit)     (TimeUnit)

 -1.67e-03 + 8.64e+00i     1.93e-04       8.64e+00           6.00e+02
 -1.67e-03 - 8.64e+00i     1.93e-04       8.64e+00           6.00e+02
```

Unobservable modes - symmetric modes

Observability with respect with $\dot{\theta}$

```
sys2 = sys(2,:);

[Abar_obs2,Bbar_obs2,Cbar_obs2,~,k_obs2] = obsvf(sys2.A,sys2.B,sys2.C);
Abar_obs2
```

```
Abar_obs2 = 6×6
  -2.3165e+01            0  -6.6515e+01   7.3589e-14  -1.9908e-13            0
            0            0            0            0            0  -1.0000e+00
   9.1898e+00            0   2.3162e+01  -2.5596e-14  -4.5444e-13            0
  -9.3428e-14            0  -3.1937e-14  -4.4620e-05  -4.1006e+02            0
   1.2850e-16            0   1.2953e-16   1.0000e+00  -1.8252e-02            0
   1.4211e-13            0            0  -1.6137e-13   7.0785e+02            0
```

```
sum(k_obs2) % Number of observable modes
```

```
ans =
     3
```

```
unob_states = states - sum(k_obs2);

Ano = Abar_obs2(1:unob_states,1:unob_states);
Ao = Abar_obs2(unob_states+1:states,unob_states+1:states);
```

```
damp(Ao)
```

```
        Pole              Damping       Frequency      Time Constant
                                      (rad/TimeUnit)     (TimeUnit)

  0.00e+00               -1.00e+00       0.00e+00            Inf
 -9.15e-03 + 2.02e+01i    4.52e-04       2.02e+01          1.09e+02
 -9.15e-03 - 2.02e+01i    4.52e-04       2.02e+01          1.09e+02
```

Observable modes - 1 rigid mode & anti-symmetric modes

```
damp(Ano)
```

```
        Pole              Damping       Frequency      Time Constant
                                      (rad/TimeUnit)     (TimeUnit)

 -1.67e-03 + 8.64e+00i    1.93e-04       8.64e+00          6.00e+02
 -1.67e-03 - 8.64e+00i    1.93e-04       8.64e+00          6.00e+02
  0.00e+00               -1.00e+00       0.00e+00            Inf
```

Unobservable modes - 1 rigid mode & symmetric modes

Observability with respect with $\dot{\beta}$

```
sys3 = sys(3,:);

[Abar_obs3,Bbar_obs3,Cbar_obs3,~,k_obs3] = obsvf(sys3.A,sys3.B,sys3.C);
Abar_obs3
```

```
Abar_obs3 = 6×6
  -6.9389e-16  -9.1795e-01   3.3518e-03  -3.4969e-01  -7.3670e-06  -1.8729e-01
   2.5395e-14  -2.9548e-03   1.6613e+02   1.6018e+00  -5.7440e+02   1.9396e-02
   1.7395e-16   3.9672e-01  -2.0852e+00  -8.2931e-01   4.7631e+00  -4.3351e-01
   3.3107e-14   1.6564e-15   2.1688e+02   2.0832e+00  -4.9427e+02   1.1072e-02
  -2.1901e-18  -3.9007e-17   1.7257e-17   4.7214e-01  -1.0403e-02  -8.8153e-01
  -3.1597e-17  -6.8216e-15   1.3159e-14   1.4714e-14   1.6085e+02  -6.3268e-03
```

```
sum(k_obs3) % Number of observable modes
```

```
ans =
     5
```

```
unob_states = states - sum(k_obs3);

Ano = Abar_obs3(1:unob_states,1:unob_states);
Ao = Abar_obs3(unob_states+1:states,unob_states+1:states);


damp(Ao)
```

```
        Pole              Damping       Frequency      Time Constant
                                      (rad/TimeUnit)     (TimeUnit)
```

```
-9.15e-03 + 2.02e+01i     4.52e-04       2.02e+01         1.09e+02
-9.15e-03 - 2.02e+01i     4.52e-04       2.02e+01         1.09e+02
 1.65e-15                -1.00e+00       1.65e-15        -6.05e+14
-1.67e-03 + 8.64e+00i     1.93e-04       8.64e+00         6.00e+02
-1.67e-03 - 8.64e+00i     1.93e-04       8.64e+00         6.00e+02
```

Observable modes - 1 rigid mode, symmetric modes & anti-symmetric modes

```
damp(Ano)
```

```
    Pole        Damping       Frequency      Time Constant
                            (rad/TimeUnit)     (TimeUnit)

  -6.94e-16     1.00e+00       6.94e-16          1.44e+15
```

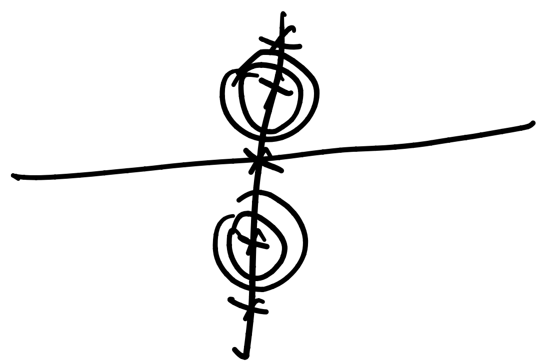
Unobservable modes - 1 rigid mode

Poles & Zeros of Transfer Functions

Transfer between u & θ

```
poles = pole(sys1)
```

```
poles = 6×1 complex
   0.0000e+00 + 0.0000e+00i
   0.0000e+00 + 0.0000e+00i
  -9.1483e-03 + 2.0250e+01i
  -9.1483e-03 - 2.0250e+01i
  -1.6667e-03 + 8.6432e+00i
  -1.6667e-03 - 8.6432e+00i
```



- **2 poles at** ω_1
- **2 poles at** ω_2
- **2 poles at** ω_3

```
zeros = tzero(sys1)
```

```
zeros = 4×1 complex
  -1.6667e-03 + 8.6432e+00i
  -1.6667e-03 - 8.6432e+00i
  -1.6667e-03 + 8.6432e+00i
  -1.6667e-03 - 8.6432e+00i
```

- **4 zeros at** ω_2

```
zpk(sys1) % Transfer function between u and theta
```

```
ans =

          66.667 (s^2 + 0.003333s + 74.71)^2
  -----------------------------------------------------
  s^2 (s^2 + 0.003333s + 74.71) (s^2 + 0.0183s + 410.1)
```

```
Continuous-time zero/pole/gain model.
```

```
sys1_min = minreal(zpk(zeros,poles,66.67)) % Minimum realization of this system
```

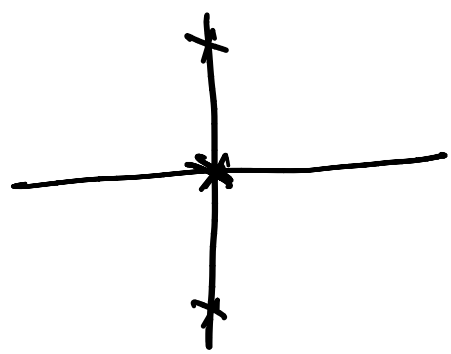
```
sys1_min =

  66.67 (s^2 + 0.003333s + 74.71)
  -------------------------------
    s^2 (s^2 + 0.0183s + 410.1)

Continuous-time zero/pole/gain model.
```

The poles at symmetric modes are cancelled by the zeros.

Transfer between u and $\dot{\theta}$

```
poles = pole(sys2)
```

```
poles = 6×1 complex
   0.0000e+00 + 0.0000e+00i
  -9.1483e-03 + 2.0250e+01i
  -9.1483e-03 - 2.0250e+01i
  -1.6667e-03 + 8.6432e+00i
  -1.6667e-03 - 8.6432e+00i
   0.0000e+00 + 0.0000e+00i
```



- **2 poles at** ω_1
- **2 poles at** ω_2
- **2 poles at** ω_3

```
zeros = tzero(sys2)
```

```
zeros = 5×1 complex
  -1.6667e-03 + 8.6432e+00i
  -1.6667e-03 - 8.6432e+00i
  -1.6667e-03 + 8.6432e+00i
  -1.6667e-03 - 8.6432e+00i
   0.0000e+00 + 0.0000e+00i
```

- **4 zeros at** ω_2
- **1 zero at** ω_1

```
zpk(sys2) % Transfer function between u and theta_dot
```

```
ans =

          66.667 (s^2 + 0.003333s + 74.71)^2
  ---------------------------------------------------
  s (s^2 + 0.003333s + 74.71) (s^2 + 0.0183s + 410.1)

Continuous-time zero/pole/gain model.
```

```
sys2_min = minreal(zpk(zeros,poles,1)) % Minimum realization of this system
```

```
sys2_min =

  (s^2 + 0.003333s + 74.71)
  -------------------------
  s (s^2 + 0.0183s + 410.1)

Continuous-time zero/pole/gain model.
```

1 rigid mode and the symmetric modes are cancelled by the zeros.

Transfer between u and $\dot{\beta}$

```
poles = pole(sys3)
```

```
poles = 6×1 complex
   0.0000e+00 + 0.0000e+00i
  -9.1483e-03 + 2.0250e+01i
  -9.1483e-03 - 2.0250e+01i
  -1.6667e-03 + 8.6432e+00i
  -1.6667e-03 - 8.6432e+00i
   0.0000e+00 + 0.0000e+00i
```

- **2 poles at** ω_1
- **2 poles at** ω_2
- **2 poles at** ω_3

```
zeros = tzero(sys3)
```

```
zeros = 5×1 complex
  -9.9319e+00 + 0.0000e+00i
   9.9363e+00 + 0.0000e+00i
  -1.6667e-03 + 8.6432e+00i
  -1.6667e-03 - 8.6432e+00i
   0.0000e+00 + 0.0000e+00i
```

- **2 zeros at** ω_2
- **1 zero at** ω_1
- **1 zero at** $\omega_4 = -9.932$
- **1 non-minimum zero at** $\omega_5 = 9.936$

```
zpk(sys3) % Transfer function between u and beta_dot
```

```
ans =

  -50.466 (s+9.932) (s-9.936) (s^2 + 0.003333s + 74.71)
  -----------------------------------------------------
   s (s^2 + 0.003333s + 74.71) (s^2 + 0.0183s + 410.1)

Continuous-time zero/pole/gain model.
```

```
sys3_min = minreal(zpk(zeros,poles,1)) % Minimum realization of this system
```

```
sys3_min =
```

```
     (s+9.932) (s-9.936)
  -------------------------
  s (s^2 + 0.0183s + 410.1)

Continuous-time zero/pole/gain model.
```

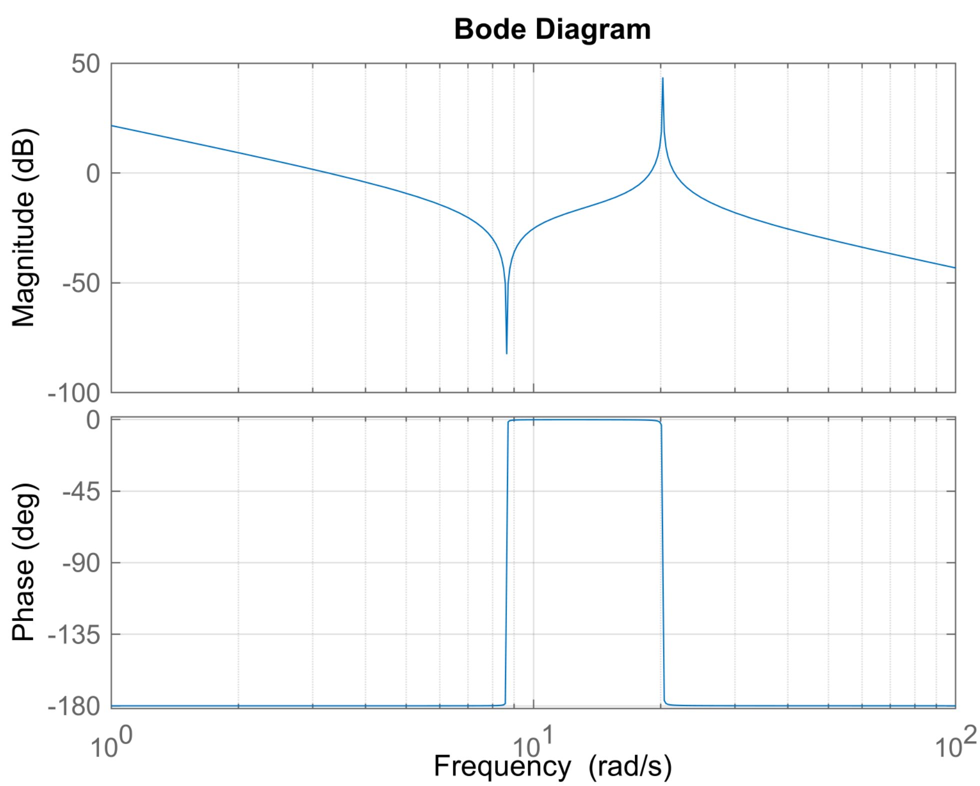
1 rigid mode and the symmetric modes are cancelled by the zeros.

Question A.6

In order to interpret the bode plots of the 3 transfers, the behaviour is analyzed at all the relevant frequencies of each system in increasing order. The effect of amplification or attenuation along with resonance is highlighted along with changes in the gain magnitude slope and the phase.

Transfer between u and θ

```
bode(sys1)
grid on
```



$\omega = 0$ (2 poles)

These poles attenuate the input signal

Magnitude plot slope = -40db/decade

Phase difference = -180°

$\omega = 8.64$ (2 zeros)

These zeros cause anti-resonance at this frequency

Magnitude plot slope = 0db/decade

Phase difference = 0°

$\omega = 20.2$ (2 poles)

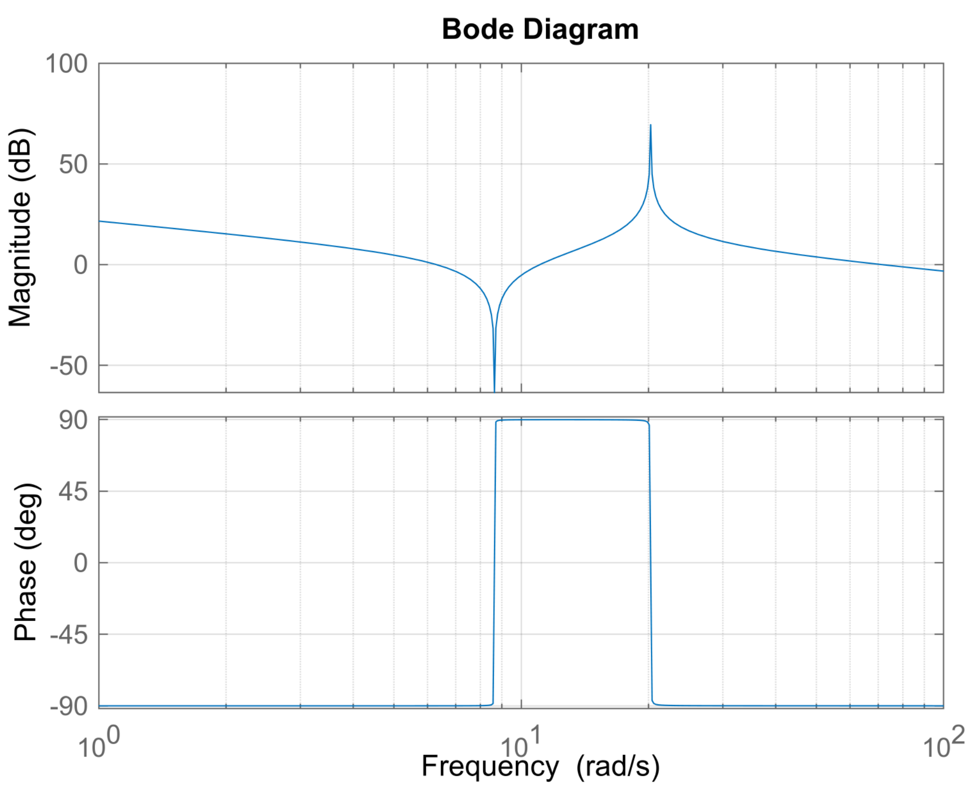
These zeros cause resonance at this frequency

Magnitude plot slope = -40db/decade

Phase difference = -180°

Transfer between u and $\dot{\theta}$

```
bode(sys2)
grid on
```



$\omega = 0$ (1 pole)

This pole attenuates the input signal

Magnitude plot slope = -20db/decade

Phase difference = -90°

$\omega = 8.64$ (2 zeros)

These zeros cause anti-resonance at this frequency

Magnitude plot slope = 20db/decade

Phase difference = 90°

$\omega = 20.2$ (2 poles)

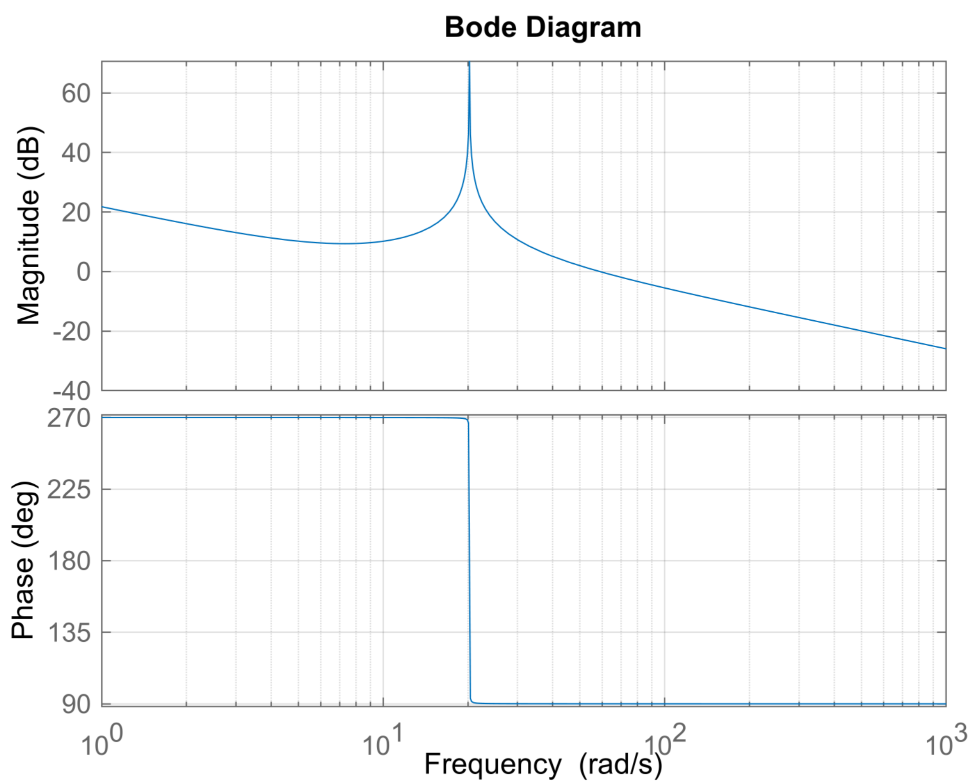
These zeros cause resonance at this frequency

Magnitude plot slope = -20db/decade

Phase difference = -90°

Transfer between u and $\dot{\beta}$

```
bode(sys3)
grid on
```



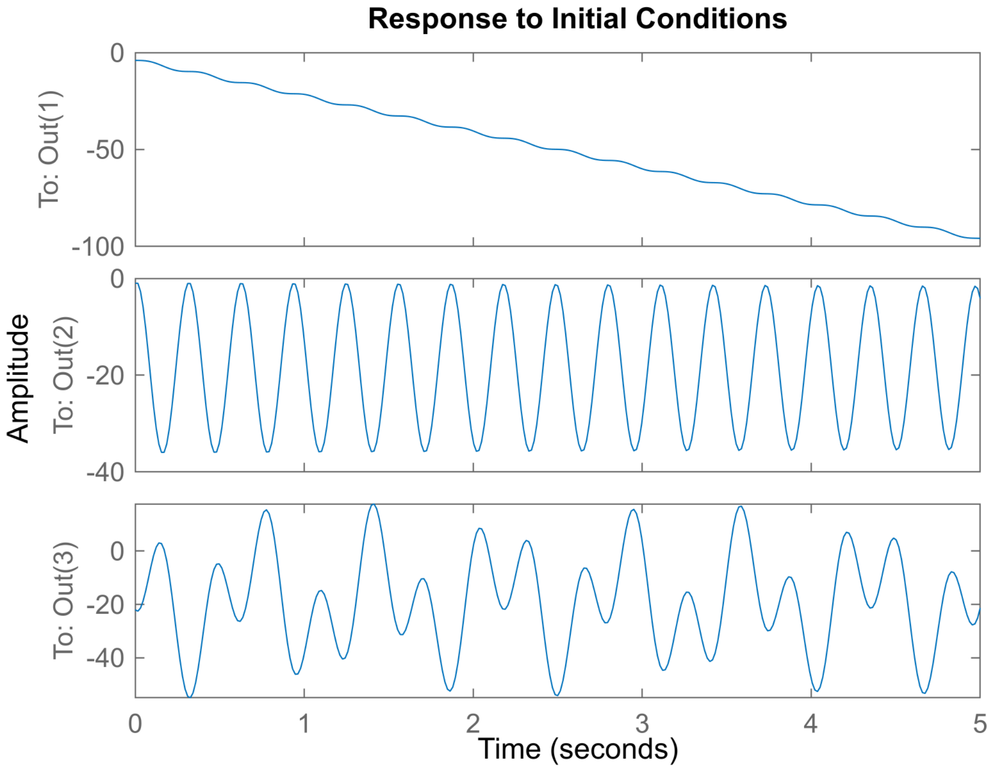
- We know that $\beta \approx \theta$. Therefore, it's bode plot is similar to that of θ. The bode plot of $\dot{\beta}$ can be explained by looking at β. $\dot{\beta}$ is just β with an added zero at s=0. Therefore, the phase increases from -180° to -90° which is the same as 270°.

- This bode plot shows that non-minimum zeros do not produce any anti-resonance.
- We only observe resonance due to the anti-symmetric modes.

One important thing to note is that all the resonances and anti-resonances are quite high which is due to poor damping of the system.

Question A.7

```
x0 = [deg2rad(-4),2e-3,0.0e-3,deg2rad(-1),-7e-2,18e-2];  % Initial Conditions
Tfinal = 5;  % Final time to plot
initial((180/pi)*eye(3)*sys,x0,Tfinal)
```



Output θ

The rigid and anti-symmetric modes are observable in the θ graph. The rigid modes correspond to just rotation of the structure about its axis which is seen as a continuously increasing θ. The slight vibration of θ is because of the anti-symmetric vibrations. We can also approximate the slope of this line as follows.

$\dot{\theta} \approx -\frac{100}{\text{Tfinal}} = -\frac{100}{5} = -20$ We see that theta reaches about 100 rad/s in 5 seconds. Therefore, $\theta = \dot{-20}$

Output $\dot{\theta}$

Only 1 rigid mode and the anti-symmetric modes are observable in the $\dot{\theta}$ graph. The frequency of the oscillations seen in this graph is equal to ω_3 and $\dot{\theta}$ oscillates about the value of -20 which is due to the contribution of the observed rigid mode.

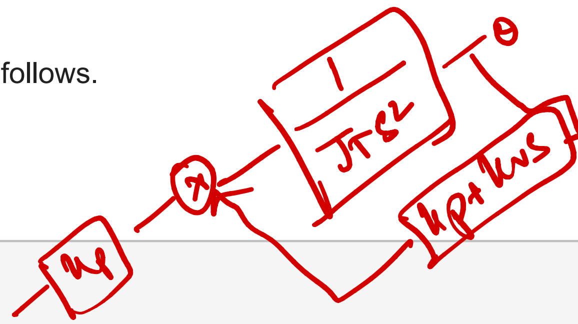
Output $\dot{\beta}$

In the $\dot{\beta}$ graph, all modes except 1 rigid mode are observable. The 1 observable rigid mode causes the output to oscillate about -20 again while the symmetric and anti-symmetric modes merge by the superimposition of 2 sine waves of frequencies ω_2 and ω_3 respectively. For the first few seconds, the non-minimum phase is also observed.

Question B.1

The objective is to tune the gains by assuming a rigid body model and verifying whether the resulting control law is able to control the flexible system as well.

The gain expressions were calculated by hand and are as follows.



$K_p = J_t\omega^2$

$K_d = 2J_t\xi\omega$

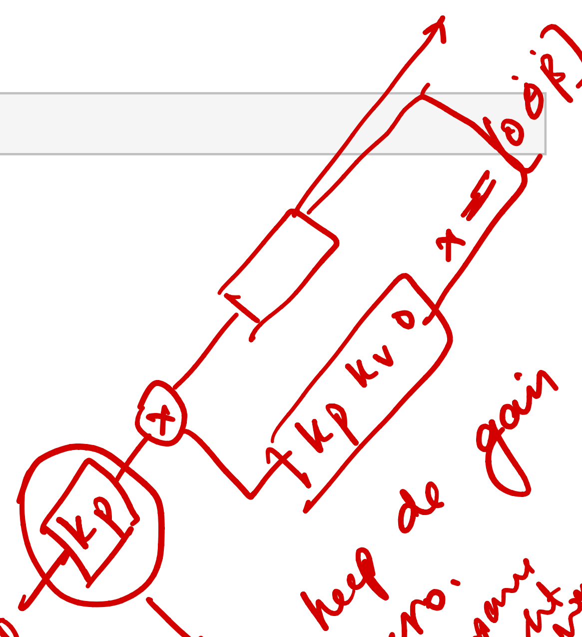
```
omega = 2;
epsilon = 0.707;

Kp = omega^2*Jt;
Kd = 2*epsilon*omega*Jt;
```

These gains are feedback gains on θ and $\dot{\theta}$ respectively while $\dot{\beta}$ has not feedback. Therefore, the gain vector becomes -

$K = \begin{bmatrix} K_p & K_d & 0 \end{bmatrix}$

```
CL_sys = feedback(sys,[Kp Kd 0])
```

* Kp



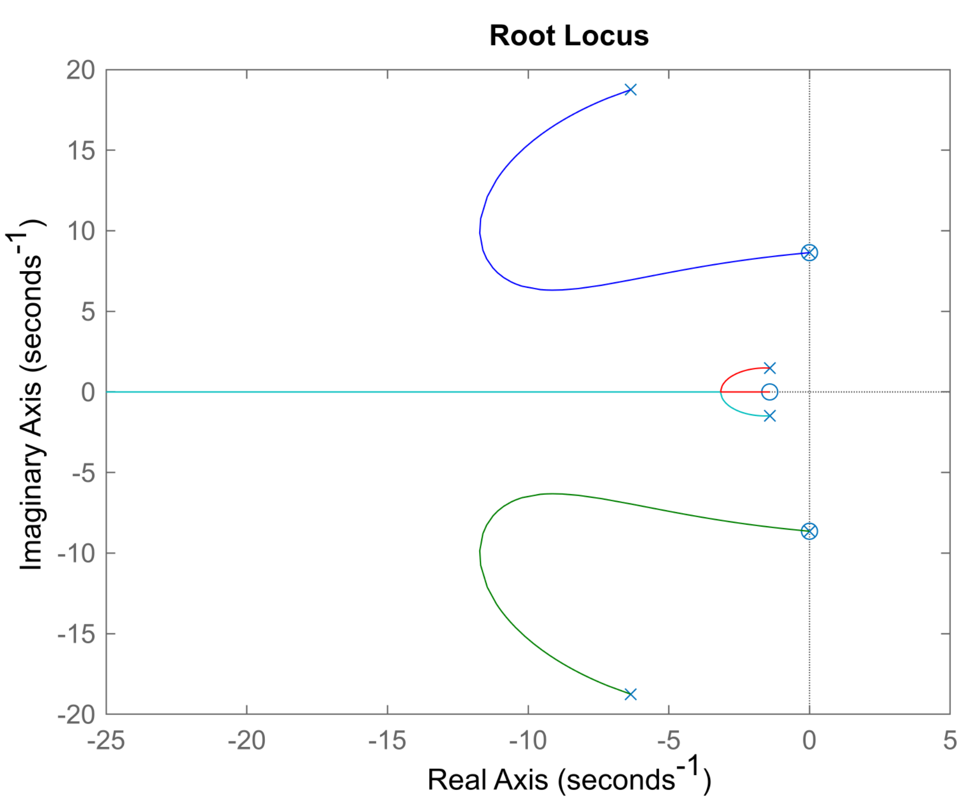
to keep dc gain zero. compare right way.

```
CL_sys =

  A =
             x1         x2         x3         x4         x5         x6
   x1         0          0          0          1          0          0
   x2         0          0          0          0          1          0
   x3         0          0          0          0          0          1
   x4    -21.96      500.5     -500.5     -15.52    0.02233   -0.02233
   x5     7.355     -242.4      167.7        5.2   -0.01082   0.007482
   x6    -7.355      167.7     -242.4       -5.2   0.007482   -0.01082

  B =
             u1
```

```
    x1       0
    x2       0
    x3       0
    x4   66.67
    x5  -22.33
    x6   22.33

  C =
           x1      x2      x3      x4      x5      x6
    y1      1       0       0       0       0       0
    y2      0       0       0       1       0       0
    y3      0       0       0       1   5.245       0

  D =
        u1
    y1   0
    y2   0
    y3   0

Continuous-time state-space model.
```

```
OL_sys = [Kp Kd 0]*CL_sys; % Open-loop system with PD controller
rlocus(OL_sys)
```

rlocus(OLsys,'rx')



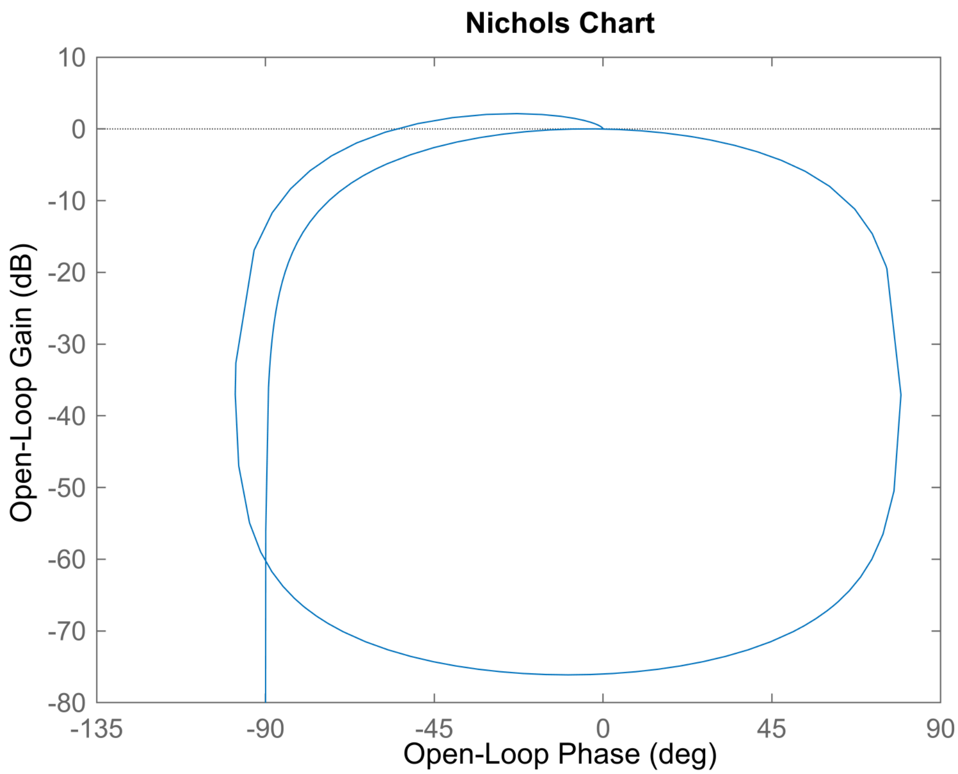
We have the following observations from this graph -

- The poles of the anti-symmetric modes are placed further in the stable half of the plane.
- The rigid mode poles are controlled in a similar way.
- The poles of the symmetric modes do not change location. This confirms the previous results that **symmetric modes are uncontrollable.**

- Since the final system is stable, this proves that the tuned gains for a rigid model can be used for a flexible model as well.

```
nichols(OL_sys)
```

ngrid;

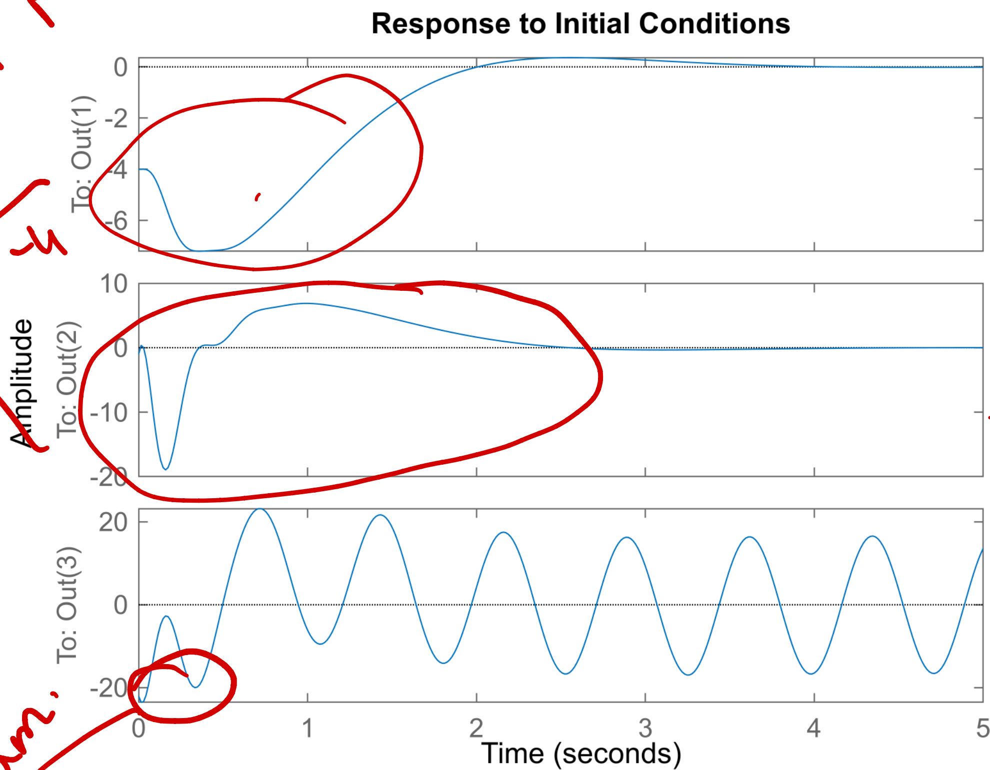


different?

Question B.2

```
initial((180/pi)*eye(3)*CL_sys,x0,Tfinal)
```

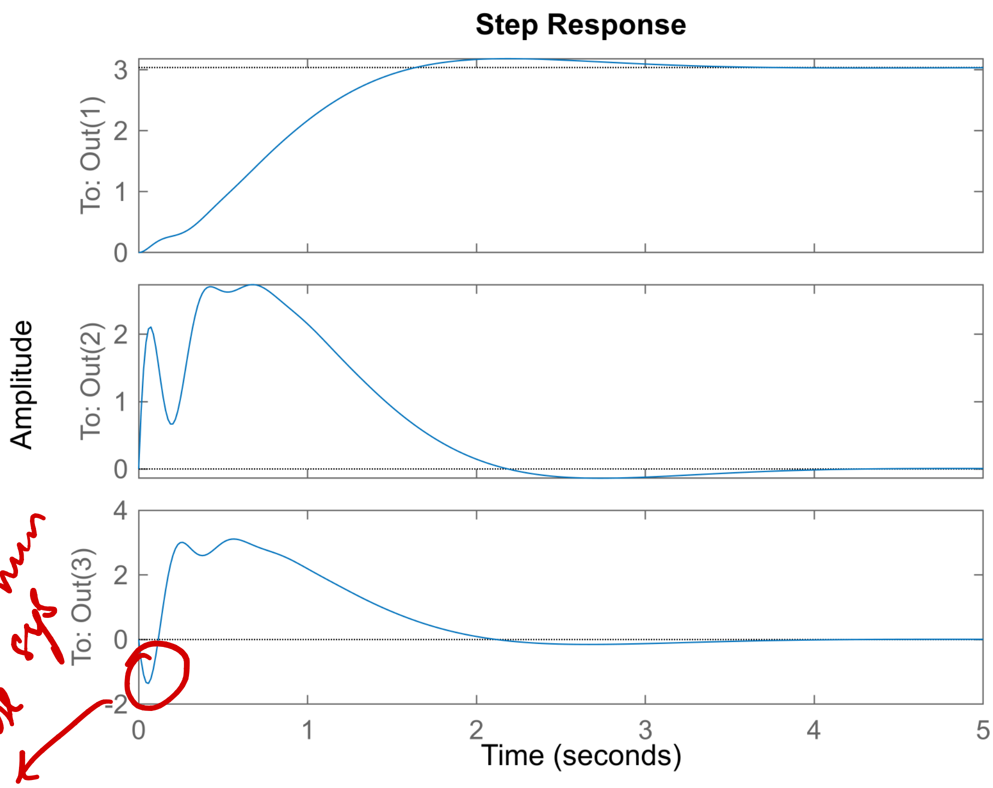
dc gain mods.



non minimum.

- The first 2 graphs show a controlled response since all the observed modes in these graphs are controllable. (rigid and anti-symmetric modes)
- The 3rd graph, however, is able to observe the symmetric mode which is uncontrollable. Therefore, we see continuous oscillations about the null-point.

```
step(CL_sys)
```



non minimum phase sys

no vibration?
step dont scribe
sufficiently the
symmetric flexible
mode

Question B.3

```
Ts = 0.058; % Sampling Time
sys_d = c2d(sys,Ts); % Discrete system
rlocus([Kp Kd 0]*sys_d)
```

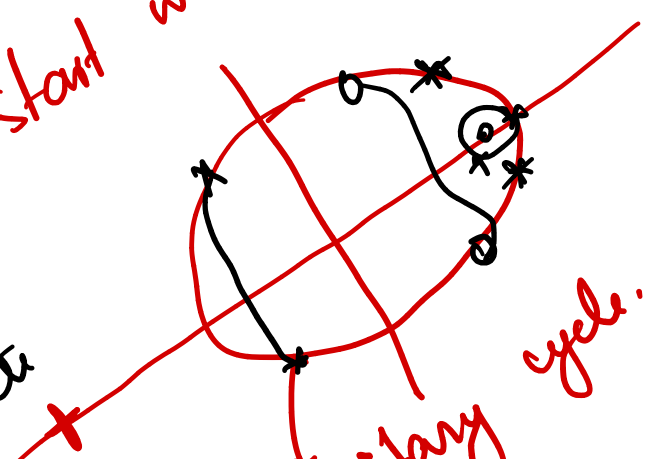
zgrid.

rlocus(—, 1, 'r+')

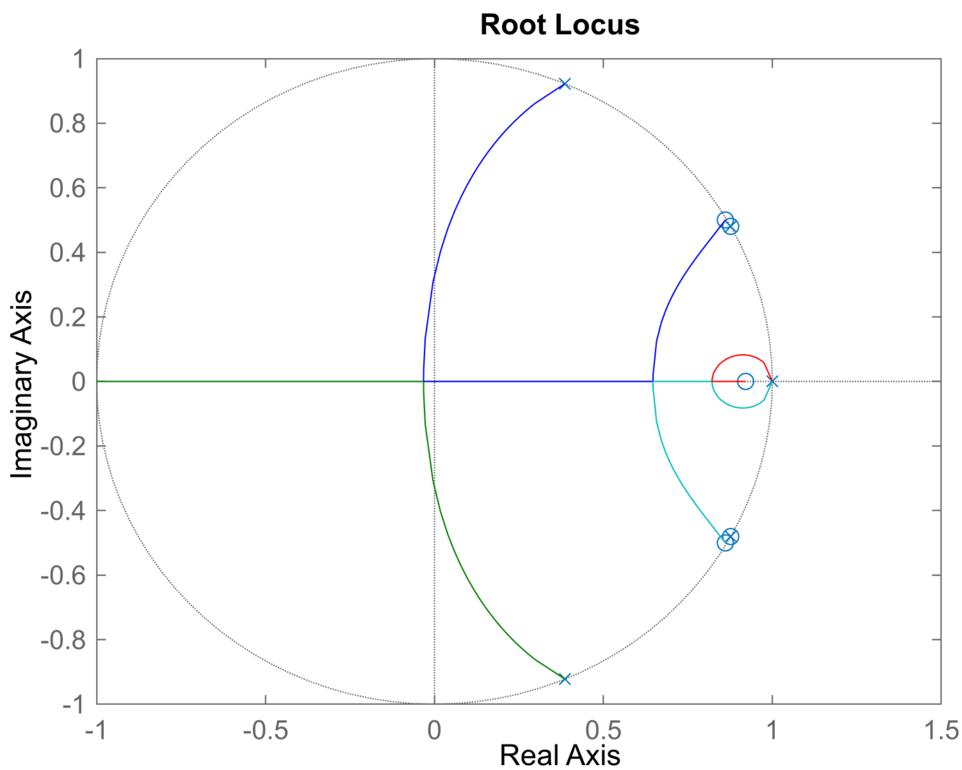
closed loop poles.

start with T=0.1

time too high.
↓
more delay
↓
unstability in descrete sys.

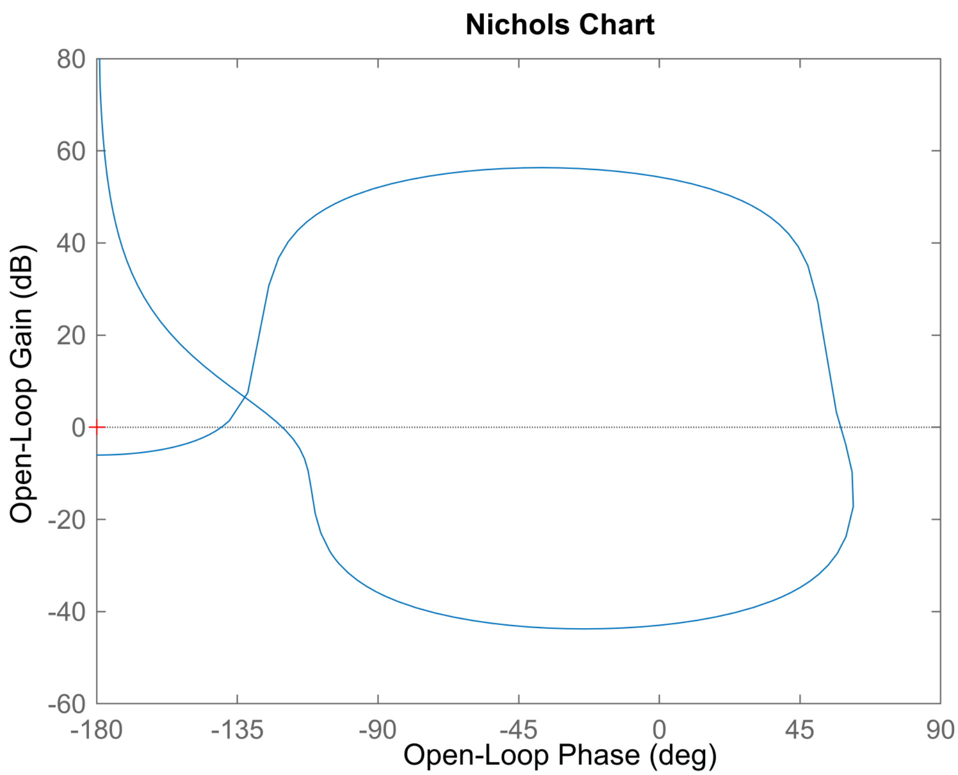


unitary cycle.



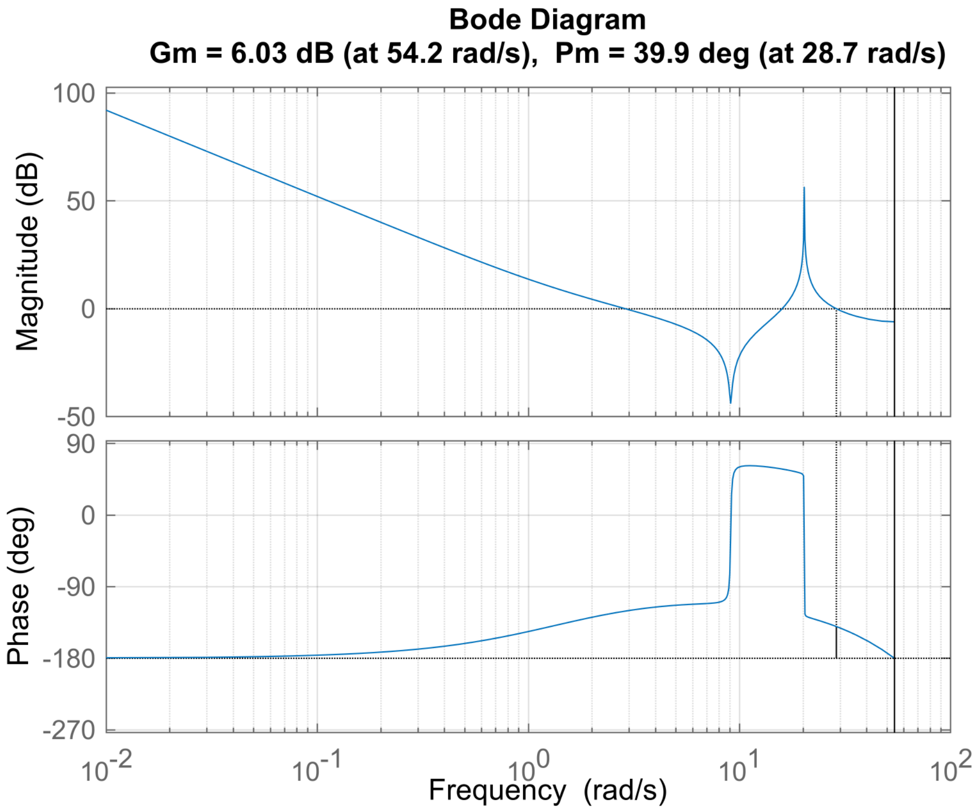
In a discrete system, the region of a unit circle is the region of stability which corresponds to the left-half plane in continuous systems. The P-D control law is still able to stabilize the discrete system since all poles are within this region.

```
nichols([Kp Kd 0]*sys_d)
```



It was found that a sampling period $T_s = 0.058$ s corresponds to a gain margin.

```
margin([Kp Kd 0]*sys_d)
grid on
```



Question B.4

Since the feedback is now on θ and $\dot{\beta}$, the K vector now becomes -

$K = \begin{bmatrix} K_p & 0 & K_d \end{bmatrix}$

$\dot{\beta} = \dot{\theta} + \dot{\alpha}$

Since the system is assumed to be rigid, we can write,

$\dot{\beta} \approx \dot{\theta}$

Therefore, the same Kp and Kd tuned values can be used here as well.

```
CL_sys = feedback(sys,[Kp 0 Kd])
```

$* \; K_p$

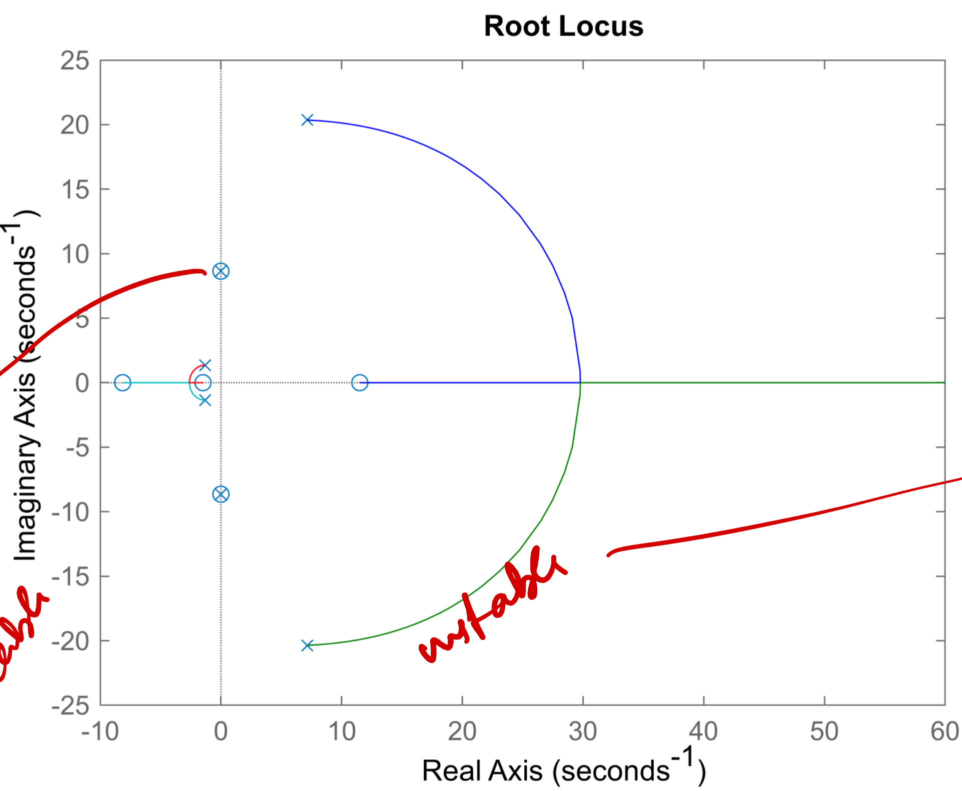
```
CL_sys =

  A =
             x1         x2         x3         x4         x5         x6
   x1         0          0          0          1          0          0
   x2         0          0          0          0          1          0
   x3         0          0          0          0          0          1
   x4    -21.96      500.5     -500.5     -15.52     -81.39   -0.02233
   x5     7.355     -242.4      167.7        5.2      27.26   0.007482
   x6    -7.355      167.7     -242.4       -5.2     -27.27   -0.01082
```

```
B =
          u1
   x1      0
   x2      0
   x3      0
   x4  66.67
   x5 -22.33
   x6  22.33

C =
        x1     x2     x3     x4     x5     x6
   y1    1      0      0      0      0      0
   y2    0      0      0      1      0      0
   y3    0      0      0      1  5.245      0

D =
        u1
   y1    0
   y2    0
   y3    0

Continuous-time state-space model.
```

```
OL_sys = [Kp 0 Kd]*CL_sys; % Open-loop system with PD controller
rlocus(OL_sys)
```

sgrid

rlocus(OL, 1, 'r+')



zero

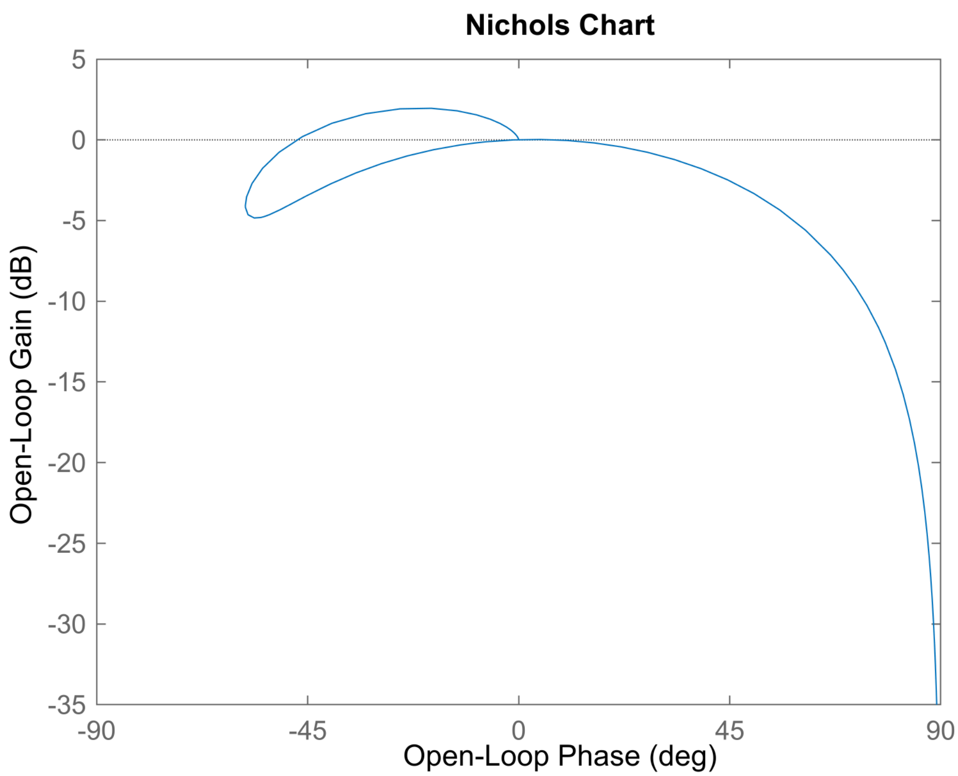
don't move + uncontrollable

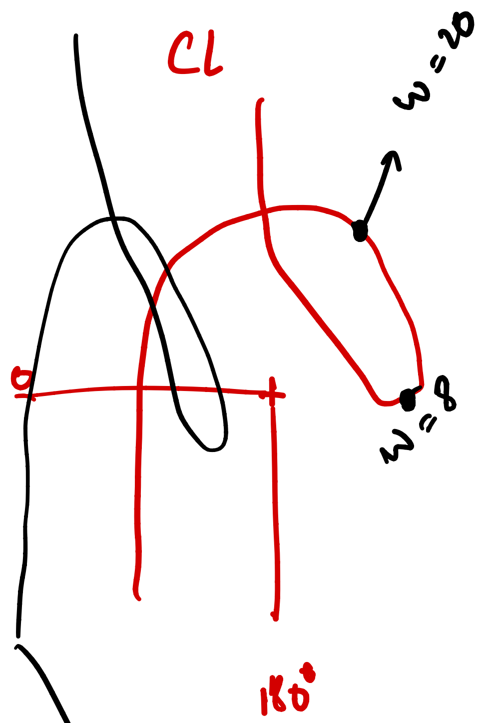
unstable

use filter!

By measuring $\dot{\beta}$, the location of our actuator and sensor is no longer the same. **This means the system is not colocated**. Such systems are difficult to control as can be seen in this root locus plot where the poles become unstable.

```
nichols(OL_sys)
```





shift the resonance to stabilize.

Filter Design

To design the filter, the rltool was used to stabilize the open loop system.

```
s = tf('s');
filter = -1.7692*(s-1.957)/(s+7.711)
```

at $\omega = 2$ rad/s
$\zeta_q = 0.7$

```
filter =

  -1.769 s + 3.462
  ----------------
     s + 7.711

Continuous-time transfer function.
```

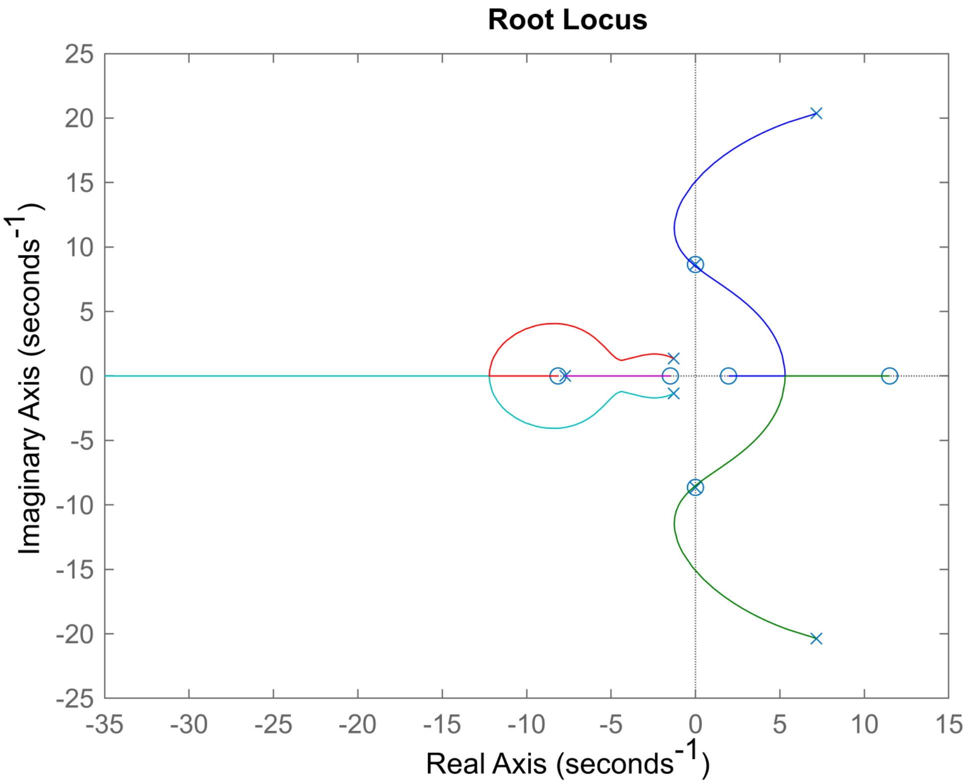
```
CL_sys = feedback(sys,[Kp 0 Kd])xKp.
OL_sys = filter*OL_sys;
rlocus(OL_sys)
```

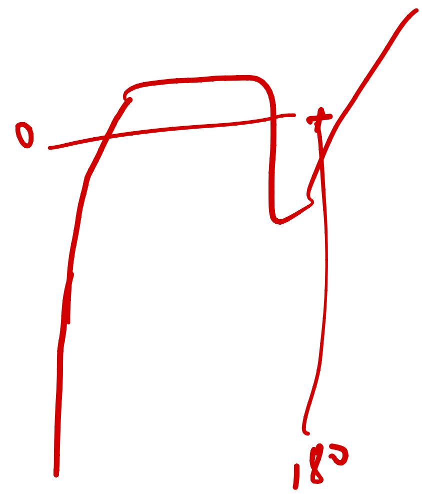
have to act after 2 rad before 20 rad.

filter = tf(1, [1/7^2 1/7 1])

choose $\omega_n = 7$ rad/s.

$$\frac{1}{\frac{s^2}{\omega_n^2} + \frac{2\zeta s}{\omega_n} + 1}$$





check root locus.

As we can see, this filter has moved part of the root locus to the stable half plane which means for a few select gains, the system can be stable.

do not change the behavior of the rigid mode.

Conclusion

Modal Analysis is a useful and efficient way to find poles(open-loop modal analysis) and zeros(closed-loop modal analysis). The resulting poles are all the flexible modes of the system that is defined.

While designing flexible systems, the notion of colocation is vital to design a controller for it. The actuators and sensors have to be positioned together to obtain better controllability, observability and stability.

Not well damped & can use H_∞ for better performan

symmetric dont move

but antisym go to left half plane.

$M\ddot{q} + Kq = Fu$

$\hookrightarrow$ modal analysis $\det(M\omega^2 - K) = 0$

$\hookrightarrow$ $\omega^2 = \omega_0^2$ $\quad \cancel{\omega = \pm \omega_0}$

$\downarrow$

$s = j\omega \longrightarrow s^2 = -\omega^2$

$(s^2 + \omega_0^2)$

modes are controllable / observable

$\downarrow$

Not mode shapes.

Closed loop modal analysis.

$u = -k_\theta \theta$

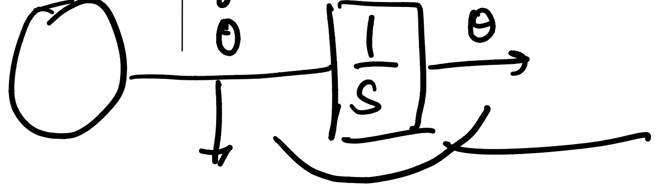
$M\ddot{q} + k_{new} q = 0$

$\det(M\omega^2 - k_{new}) = 0$

$\frac{\theta}{u} = \frac{\alpha(s^2 + \omega_i^2)}{*}$

$u = -k_p \beta$

can not observe in the velocity.
the rigid mode.



not observable $\longrightarrow$ $\therefore$ one rigid mode not obs.

__/___/__