Representation and Analysis of Dynamical Systems

1h15

The exercises are independent.

Exercise 1: Model and analysis of a second-order system

Note: parameters are intentionally given without physical units. The numerical values are intentionally chosen in order to allow easy computation by hand.

We consider a non-linear mechanical system:

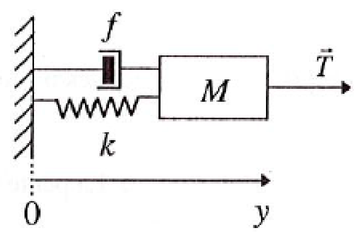


Figure 1: Mechanical system

The mass $M$is moving along an horizontal axis $Y(t)$ and submitted to an external force $T(t)$ anf two forces $F_1(t)$ and $F_2(t)$:

Inertia: $M\frac{d^2Y(t)}{dt^2} = T(t) + F_1(t) + F_2(t)$

Viscous damping: $F_1(t) = -a \times \left(\frac{dY(t)}{dt}\right) - b \times \left(\frac{dY(t)}{dt}\right)^2$ (a and b are positive parameters)

Nonlinear spring: $F_2(t) = -Y(t) - 2\big(Y(t)\big)^2$

Question 1:

Find the equilibrium point $Y(t) = Y_0 > 0$ when the external force is constant: $T(t) = T_0 = 1$

$$0 = 1 - 0 - Y_0 - 2 \times Y_0^2$$

$$2Y_0^2 + Y_0 - 1 = 0$$

positive solution: $Y_0 = 1/2$

Question 2:

We now consider the small variations $y(t)$ and $t(t)$ near the equilibrium point:

$$Y(t) = Y_0 + y(t), T(t) = T_0 + t(t)$$

Write the linearized differential equation between $y(t)$ and $t(t)$ near the equilibrium point

$$M\frac{d^2y(t)}{dt^2} = t(t) - a \times \left(\frac{dy(t)}{dt}\right) - y(t) - 2\left(\frac{1}{2} + y(t)\right)^2$$

$$M\frac{d^2y(t)}{dt^2} = t(t) - a \times \left(\frac{dy(t)}{dt}\right) - \frac{1}{2} - y(t) - 2\left(\frac{1}{2} + y(t)\right)^2$$

$$M\frac{d^2y(t)}{dt^2} + a \times \left(\frac{dy(t)}{dt}\right) + 3y(t) = t(t)$$

Question 3:

Write the transfer function of the system (input: force $t(t)$, output: elongation of the spring $y(t)$)

$$F(s) = \frac{Y(s)}{T(s)} = \frac{K}{\frac{s^2}{\omega^2{}_n} + \frac{2\sigma}{\omega_n}s + 1}$$

Give the expressions of K, ω_n and σ with respect to a and M.

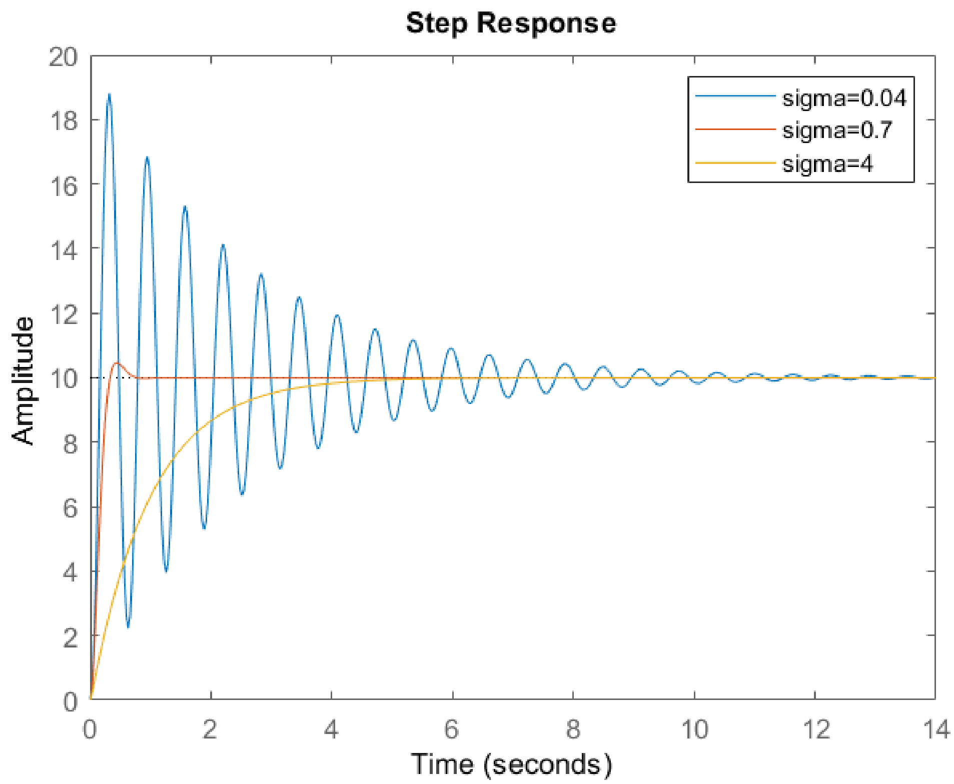
$$F(s) = \frac{Y(s)}{T(s)} = \frac{1}{Ms^2 + as + 3} = \frac{\frac{1}{3}}{\frac{M}{3}s^2 + \frac{a}{3}s + 1}$$

$$\begin{cases} K = 1/3 \\ \omega_n = \sqrt{\dfrac{3}{M}} \\ \sigma = \dfrac{a}{6}\sqrt{\dfrac{3}{M}} \end{cases}$$

Question 4:

We consider $K = 10$, $\omega_n = 10\ rad/s$. We suppose the input (torque T) is a step from $0N$ to 1N.

Plot the step response $y(t)$ for three different values of σ : $\sigma = 0.04\ , 0.7\ , 4$



correction criteria: static gain, large oscillations for $\sigma = 0.04$, small overshoot for $\sigma = 0.7$, slow response for $\sigma = 4$

Exercise 2: Bode diagram and closed loop bandwidth

Consider the open-loop transfer function:

$$F(s) = \frac{X(s)}{U(s)} = \frac{1}{s(s+10)(s+100)}$$

The Bode diagrams of this open-loop transfer function are given in figure 2:

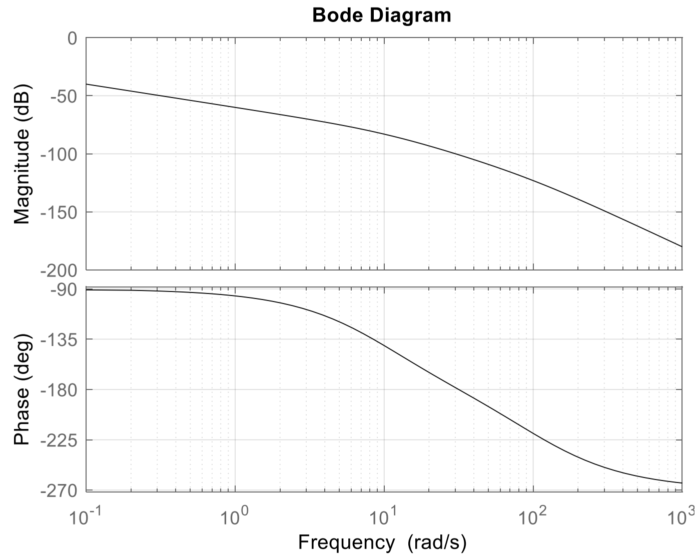


Figure 2: Bode diagram of F

The system is controlled according to the block diagram of figure 3

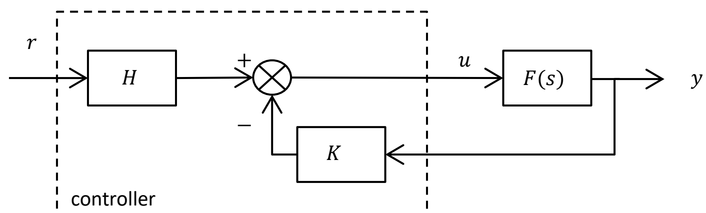
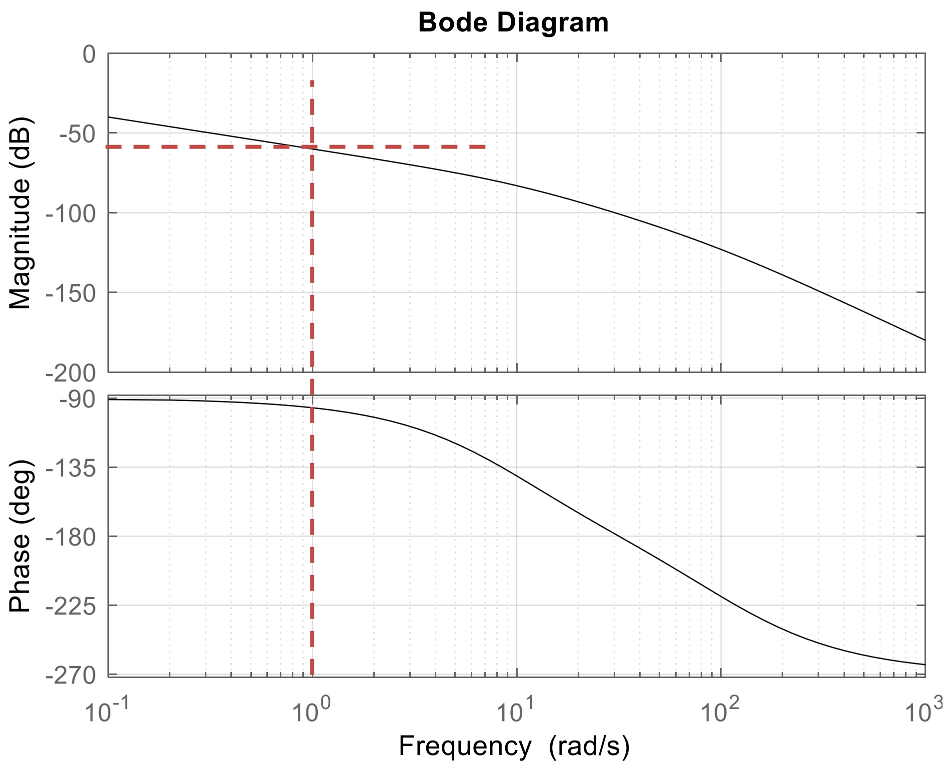


Figure 3: control

Question 1:

What is the value of the constant gain K that will ensure a closed loop bandwidth of $1\ rad.s^{-1}$? (An approximate response obtained from the Bode diagram is ok).



read on the Bode diagram: the gain where the cutoff frequency is 1 rad/s is approximately $-60dB$

$K = 1000$

Question 2:

What is the value of the constant gain H that will ensure a static gain of 1?

$$H = 1000$$

Exercise 3: Control

Consider the system given by its transfer function:

$$\mathrm{F(s)} = \frac{\mathrm{Y(s)}}{\mathrm{U(s)}} = \frac{1}{\mathrm{s}^2}$$

The final goal is to design a controller in order to follow a reference $\mathrm{Y_{ref}}$

Question 1:

We first consider a simple proportional feedback with gain K. Is it possible to find a value of K that gives an acceptable closed loop behavior?

No (1 pt)

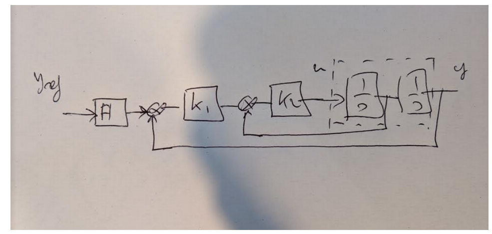
explanation (1 pt) (compute closed loop transfer function gives two poles with damping=0, bode diagram shows phase margin = 0)

Question 2:

We now consider the controller given by (H, K_1 and K_2 are pure gains):

$$u(t) = H\, y_{ref}(t) - K_1\, y(t) - K_2 \frac{dy(t)}{dt}$$

Draw the block diagram corresponding to this control law



See lecture notes!

Question 3:

Which values of K_1, K_2 and H will give a closed loop performance corresponding to a second order system with natural frequency of $1\ ra0d/s$ and damping of 0.5 and a static gain of 1?

$$CL(s) = \frac{Y(s)}{Y_{ref}(s)} = \frac{H}{s^2 + K_2 s + K_1}$$

$$K_1 = 1\,;K_2 = 1\,;H = 1$$