Representation and Analysis of Dynamic Systems
Lab 3-4-5

Figure 1 : the Quanser Aero experiment configured as a half quadrotor system

The Quanser Aero Experiment experiment can be configured as a half-quadrotor system, as shown in Figure 1. In this set up, both the front and back rotors are horizontal to the ground and only motions about the yaw axis are enabled (i.e. the pitch axis is locked). By changing the direction and speed of the rotors, users can change the pitch axis angle.

Unmanned quadrotor vehicles are used for wide-variety of applications. Using tethered half-quadrotor systems will allow us to focus on the modeling, control, and parameter estimation in yaw-axis motion of quadrotors, which can then be applied to full quadrotor system.

Topics covered:

• Find linear equations of motion describing the half-quadrotor yaw motions based on rotor voltage

• Derive the transfer function model

• Derive the linear state-space representation

1 SYSTEM

1.1 Electrical

The Quanser Aero has two thruster modules, each of which is driven by a DC motor. The motor armature circuit schematic for one of the thrusters is shown in Figure 2 and the electrical and mechanical parameters are given in. The DC motor shaft is connected to the propeller hub. The hub is a collet clamp used to mount the propeller to the motor and has a moment of inertia of J_h. A propeller is attached to the output shaft with a moment of inertia of J_p.

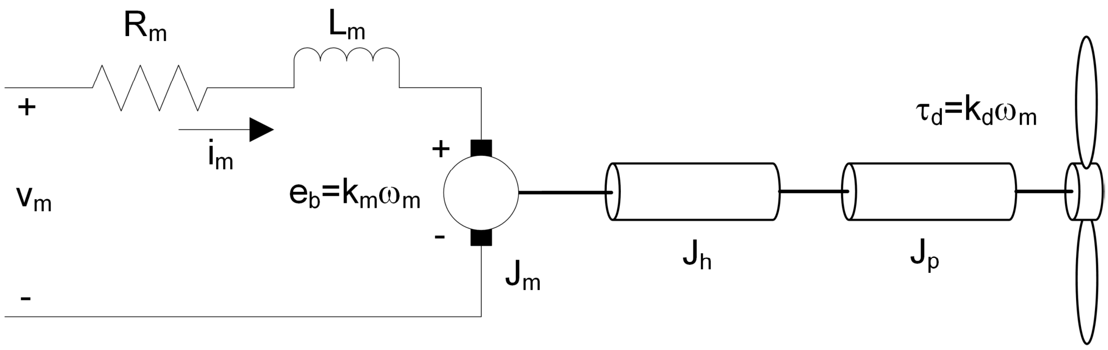

Figure 2 :

The back-emf (electromotive) voltage $E_b(t)$ depends on the speed of the motor shaft,Ω_m, and the back-emf constant of the motor, k_m. It opposes the current flow $I(t)$. The back emf voltage is given by:

$$E_b(t) = k_m \Omega(t) \tag{1}$$

The torque exerted by drag and air resistance has been simplified into an experimentally derived coefficient of the speed of the propeller. In this model the drag torque T_d which opposes the motor torque is given by (for more details see the next chapter):

$$\mathrm{T}(t) = k_1 \Omega_m(t) + k_2 \Omega_m(t)^2 \tag{2}$$

Using Kirchoff's Voltage Law, we can write the following equation:

$$V_m(t) - R_m I_m(t) - L_m \frac{dI_m(t)}{dt} - k_m \Omega_m(t) = 0 \tag{3}$$

The motor shaft equation is expressed as:

$$J_{eq} \frac{d\Omega_m(t)}{dt} = T_m(t) - T(t) \tag{4}$$

where J_{eq} is the total moment of inertia acting on the motor shaft and T_m is the applied torque from the DC motor. Based on the current applied, the torque is:

$$T_m(t) = k_m I_m(t) \tag{5}$$

Symbol	Description	Value / Unit
Variables		
V_m	Controlled input voltage	V
E_b	Back electromotive force (emf)	V
I_m	Motor current	A
Ω_m	Motor and propeller rotation speed	$rad.s^{-1}$
T_d	Resistant torque	$N.m$
T_m	Applied torque from the DC motor	$N.m$
DC motor constants		
R_m	Terminal resistance	$8.4\ \Omega$
k_t	Torque constant	$0.042\ N.m.A^{-1}$
k_m	Motor back-emf constant	$0.042\ N.m.A^{-1}$
J_m	Rotor inertia	$4.0 \times 10^{-6}\ kg.m^2$
L_m	Rotor inductance	$1.16\ mH$
Propeller constants		
k_1, k_2	Drag / air resistance coefficient	See Table 2
J_h	Propeller hub inertia	$3.04 \times 10^{-9} kg.m^2$
J_p	Propeller inertia	$7.2 \times 10^{-6} kg.m^2$

Table 1 : Electrical and propeller variables and constants

1.2 Propeller

The thrust is generated by two counter-rotating APC 5.0x4.6 propellers, models LP05046E/EP protected by a housing device. The full performance (thrust and torque) has been measured on a test bench (Figure 3) in a range of $200\ rad.s^{-1}$ to $600\ rad.s^{-1}$. Unsurprisingly the performance is not symmetrical according to the rotation direction (direct or reverse) and is highly reduced by the housing (Figure 4 and Figure 5).

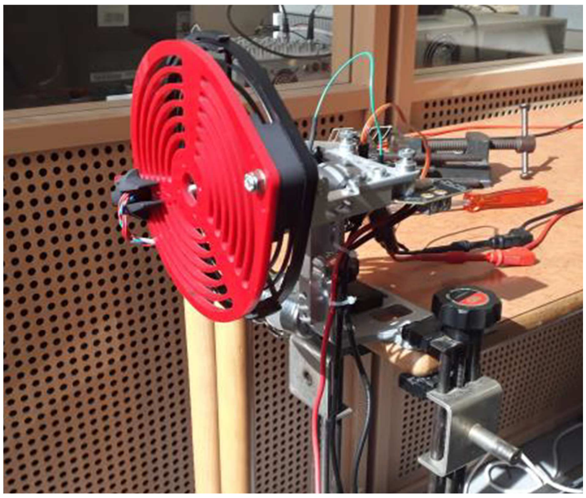

Figure 3 : performance test bench

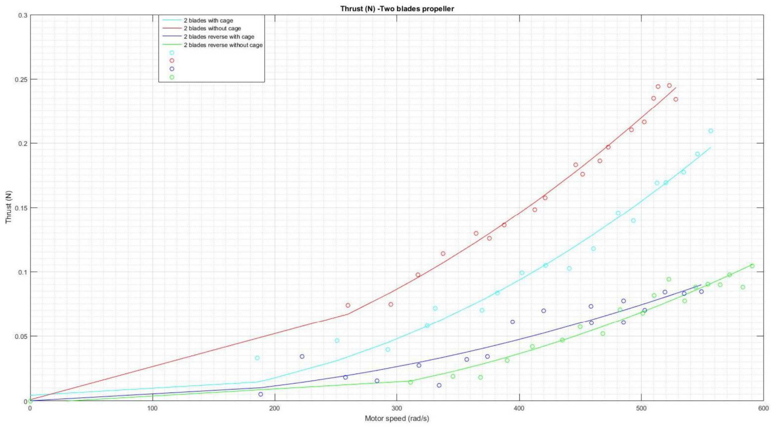

Figure 4 : propeller thrust

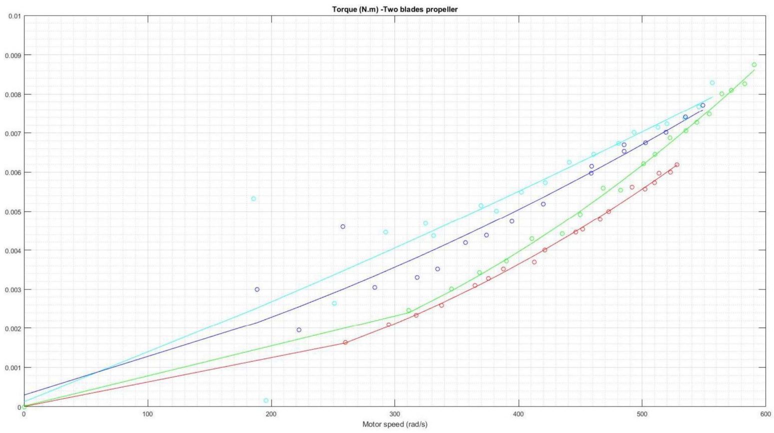

Figure 5 : propeller torque

A second order regression model has been computed from the experimental data. The result is given in

Configuration	$Thrust = f(Motor\ speed)$ $(N) \qquad (rad.s^{-1})$	$Torque = f(Motor\ speed)$ $(N.m) \qquad (rad.s^{-1})$
2 blades with cage	$F = 0.0\ \Omega_m + 6.23 \times 10^{-7}\Omega_m^2$	$\tau_d = 1.2 \times 10^{-5}\Omega_m + 3.54 \times 10^{-9}\Omega_m^2$

Table 2 : propeller efficiency model

According to this experimental result the relationship between the propeller rotation speed and the thrust is:

$$F(t) = k_3\Omega_m(t) + k_4\Omega_m(t)^2 \tag{6}$$

1.3 mechanical

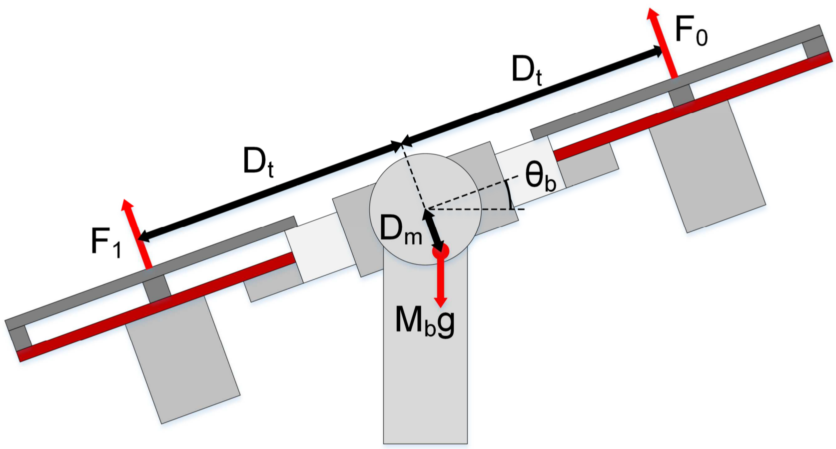

Figure 6 : free body diagram of 1-dof Quanser Aero

The free-body diagram of a Quanser Aero in the 1-DOF configuration that pivots about the pitch axis is shown in Figure 6. The center of gravity of the rotating part is at a small distance D_m of the center of rotation, ensuring a natural horizontal stable position.

Symbol	Description	Value / Unit
Variables		
T_t	Total torque acting on the rigid body	rad
F_0, F_1	Thrust forces	N
θ_b	Rigid body orientation	rad
Mechanical dimensions		
M_b	Aero body mass	$1.15\ kg$
D_t	Thrust displacement	$0.158\ m$
D_m	Center of mass displacement	$0.0071\ m$
J_p	Inertia along the pitch axis	$2.15 \times 10^{-2}\ kg.m^{-2}$
D_p	Pitch axis viscosity coefficient	$0.0071\ N.m.s.rad^{-1}$

Table 3 : Quanser Aero system parameters

The torques acting on the rigid body system can be described by the equation:

$$T_t - M_b g\ (D_m \sin(\theta_b) - T_f = J_p \frac{d^2\theta_b(t)}{dt^2} \tag{7}$$

Where D_m is the distance below the plane of Aero body of the center of mass as depicted in Figure 6. The thrust force is given by:

$$T_t = F_0(t)D_t - F_1(t)D_t \tag{8}$$

The viscous resisting torque T_f depends only on rotation speed:

$$T_f = D_p \frac{d\theta_b}{dt} \tag{9}$$

In this configuration a constant thrust F is provided through a constant voltage. This is supposed to mimic the constant thrust necessary to compensate gravity for the generic free flight of a quadrotor. The orientation is generated by a differential thrust $f(t)$:

$$\begin{cases} F_0(t) = F + f(t) \\ F_1(t) = F - f(t) \end{cases} \tag{10}$$

1.4 Sensors

A set of sensors provides the following measurements:

- Two optical encoders for pitch and roll angles of rotation
- Those encoders provide also rotation speed
- Two tachometers for blades rotation speed
- One Inertial Measurement Unit placed at the center of rotation provides acceleration and rotation speed

Lab 3

2 MODELLING AND ANALYSIS OF A SINGLE MOTOR

Question 1: model of the propulsive device

Put together the electrical and mechanical differential equations of the system. Inputs is $V(t)$, output is $F(t)$.

Most of the equations are linear differential equations that can be represented by a combination of gain, integrator, sum (or difference). The nonlinear equations (2) and (6) will be represented by a double square. Complete the following graph with correct constants and signal names.

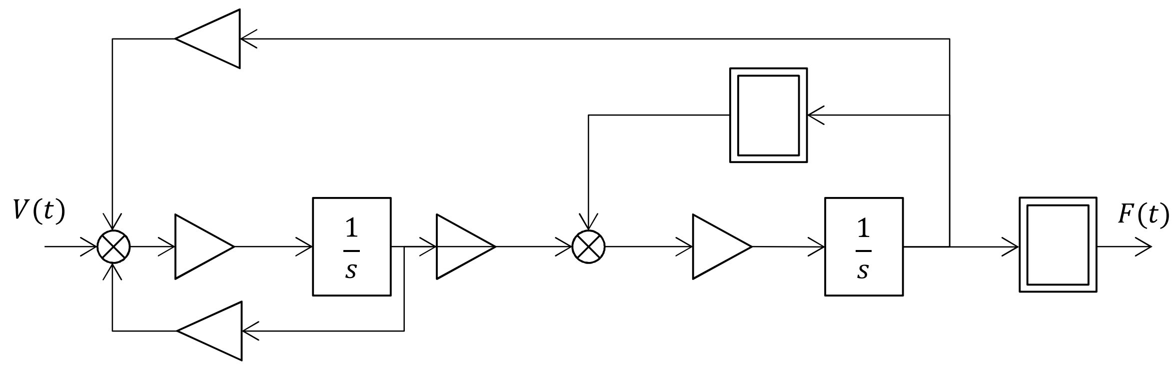

Figure 8 : propulsive block diagram model

Question 2: find the equilibrium point

The system will be controlled with small variation around a constant voltage $V = 10V$. Let us first find the conditions of the equilibrium point: for a constant input voltage V what is the constant current $i(t) = I$, the constant rotation speed $\omega(t) = \Omega$ and the constant force $F(t) = F$? You will set the derivative at zero in equations (3) and (4).

Question 3: Linearization

We can now compute the linearized model near the equilibrium point. We will now consider small variations:

$$\begin{cases} V(t) & = & V + v(t) \\ I(t) & = & I + i(t) \\ \Omega(t) & = & \Omega + \omega(t) \\ \mathrm{F}(t) & = & \mathrm{F} + f(t) \end{cases} \tag{11}$$

Write the linearized equations and differential equations involving $v(t)$, $i(t)$, $\omega(t)$ and $f(t)$.

From those equations find:

- the transfer function $G(s) = \frac{f(s)}{v(s)}$

- the state space model. For the state choose $x = [\omega(t) \quad i(t)]^T$

Question 3 bis: create and convert Matlab Linear Systems Models

Create two **Matlab** models: the transfer function **Gtf** and the state space model **Gss**.

Doc/Matlab/Control System Toolbox/Dynamic System Models/Linear System Representation/ss, tf

You can try to convert the transfer function to a state space model and vice versa. Compare the transfer functions and state space models you obtain.

Doc/Matlab/Control System Toolbox/Dynamic System Models/Model Transformation/ss,tf

Question 4: create the Simulink nonlinear model

Create a **Simulink** model (name: **Electrical.slx**). Your model will have one input (the voltage) and three outputs (force, rotation speed, current). Use the **Simulink/Source/In** and **Simulink/Sinks/Out** blocks.

Notice that this system has two integrators, which can be initialized. By default **Matlab/Simulink** will consider the output of each integrator as internal states, initially set at zero.

Question 5: find the equilibrium point

We will use the **trim** function to find the equilibrium point. In general a nonlinear system may have more than one equilibrium point. With the **trim** function you can find the equilibrium point $[xtrim, utrim, ytrim]$ in the neighborhood of $[x0, u0, y0]$. You can select which component of $x0$, $u0$ or $y0$ you want to be kept constant with the set of index $[ix, iu, iy]$.

Doc/Simulink/Modeling/Transform Models/Trimming and Linearization/trim

Compare the rotation speed with the one you have obtained at question 2.

Question 6: find the linearized model

At the equilibrium point you can now find the linearized model with the **linmod** function. Create a state space model Gss with the **ss** function and convert into a transfer function model Gtf function.

Doc/Simulink/Modeling/Transform Models/Trimming and Linearization/**linmod**

Compare the linearized model with the one obtained at question 3.

Question 7: Analyze the system's properties

You have obtained a model (transfer function, state space model, and corresponding Matlab models). We can now analyze the system in time and frequency domain and check for stability.

Doc/Matlab/Control System Toolbox/Linear Analysis/Time-Domain Analysis/**step**, **impulse**

Doc/Matlab/Control System Toolbox/Linear Analysis/Frequency-Domain Analysis/**bode**,**nyquist**, **nichols**

Doc/Matlab/Control System Toolbox/Linear Analysis/Stability Analysis/**pole**, **zero**, **damp**, **pzmap**

3 MODELLING AND ANALYSIS OF THE “QUADROTOR CONFIGURATION”

We now consider the full system with two propellers and one degree of freedom: the pitch angle.

Question 8: the Simulink simulation model

Create a new Simulink for the model of the full system. You will just have to duplicate the Electrical model (each one will be placed in a Subsystem) and program equations (7) to (10).

To extract the states of the Simulink model you can use the following function:

x = Simulink.BlockDiagram.getInitialState(‘Mechanical’)

Into the variable s you now have the full description of the internal states of the model. (In our case, each integrator output is a state).

Question 9: find the equilibrium point

Find the equilibrium point of the full system when the input is set at 0V. (Same as question 5)

Question 10: find the linearized model

Find the linearized model (same as question 6). What is the dimension of the linearized system?

Question 11: observability and governability

Compute the governability and observability matrixof the linearized mode. (ctrb and obsv Matlab functions).

Doc/Matlab/Control System Toolbox/Matrix Computation/**ctrb**, **obsv**

How many states are controlable/observable? To answer this question you must compute the rank of the controllability/observability matrix. Be aware that this rank is computed by checking the relative magnitude of the singular values. For a robust result you must relax the tolerance criteria:

Co = ctrb(A,B)

tol = 1e-8

co = **rank**(Co,tol)

Convert the state space model into a transfer function model. Look at the poles and the zeros: conclusion, interpretation?

Doc/Matlab/Control System Toolbox/Dynamic System Models/ Linear System Representation/**ss**, **tf**, **zpk**

Question 12: model reduction

Find the minimum realization of the system (pole zero cancelation)

Doc/Matlab/Control System Toolbox/Dynamic System Models/ Model Reduction/**minreal**

What is the order of the reduced model?

4 SPEED CONTROL