Hybrid Dynamical Systems - Part III: Stability

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February 03, 2022 - ISAE - Toulouse

Outline

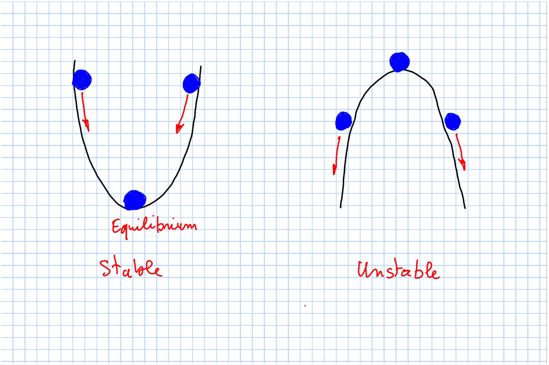
Stability analysis versus simulation

- Note that currently the computers are getting more and more powerful tools for simulating complex systems (as for example SIMULINK and MATLAB)
- Then, one could say that simulations combined with a good intuition could provide useful information on the behavior of the system under consideration (nonlinear or hybrid).
- However, it is impossible to directly conclude thanks simulation when trying to prove fundamental properties as stability (reachability, ...) of complex systems (as nonlinear or hybrid) since, for example some crucial cases may be missed.
  - ▷ The objective with the development of analysis tools is to provide mathematical proofs or certificates about the behavior of the considered system.
  - ▷ Then simulation can be used to verify, illustrate or complete the analysis part.

- The notion of stability and convergence play a key role in control theory and automation.
  - ▷ Asymptotic stability is a fundamental property of dynamical systems and its analysis is one of the main problems addressed by engineers since the early years of control theory.
- In the case of a linear system $\dot{x} = Ax$:
  - ▷ Indeed, stability of a linear systems implicitly refers to stability properties of the origin, which is always an equilibrium.
  - ▷ For linear systems we often talk about stability of "the system", rather than stability of a specific set or a point.
- Such a property, however, does not hold for nonlinear systems, and consequently for hybrid ones.

- For Hybrid systems differently from linear ones, the stability and convergence to certain sets plays a key role.
  - ▷ Asymptotic stability of a closed set, rather than of an equilibrium point, is relevant since solutions to a hybrid system do not typically converge to a single equilibrium point.
- In the lecture we will study
  - ▷ Stability of a very simple set, i.e. a point, and more specifically the origin $x_e = 0$
  - ▷ and next stability of a set
- We will see that the results are based on the concept of distance:
  - ▷ distance to a point (one retrieves the classical concept of Euclidean norm)
  - ▷ distance to a set (Definition later)

- Intuitively speaking the concept of stability of an equilibrium point x_e is related to the idea that solutions starting close enough evolve and remain arbitrarily close to x_e.
  - ▷ Hence, if we perturb a little the initial condition (where "little" means a ball of radius $\delta(\epsilon)$), then the whole evolution remains ϵ-close to x_e.
  - ▷ We will speak of Lyapunov stability



- Consider the hybrid dynamical system

$$\begin{cases} \dot{x} = f(x), & x \in \mathcal{C} \\ x^+ = g(x), & x \in \mathcal{D}. \end{cases} \tag{1}$$

- Note that the set of all maximal solutions to the hybrid system (1) is denoted as $\mathcal{S}_{\mathcal{H}}$
  - ▷ Maximal solution: related the idea to propagate further a solution as long as this is possible.
  - ▷ Definition 6 in the previous lecture.

Part II-Definition 6

A solution ϕ is *maximal* if there is no other solution ψ which is a continuation of ϕ, such that

$$\mathrm{dom}\phi \subset \mathrm{dom}\psi \quad \text{(strictly)}$$

and

$$\psi(t,j) = \phi(t,j)\ ,\ \forall (t,j) \in \mathrm{dom}\phi.$$

- We start with the notion of Lyapunov stability.
  - ▷ We are interested in the behavior of the solution ϕ when $\phi(0,0) \neq 0$ but close to $x_e = 0$.

Definition 1 - Lyapunov stability

The point $x_e = 0$ is Lyapunov stable for (1) if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that all maximal solutions ϕ to (1) ($\phi \in \mathcal{S}_{\mathcal{H}}$) satisfy

$$|\phi(0,0)| \leq \delta \implies |\phi(t,j)| \leq \varepsilon, \quad \forall (t,j) \in \mathrm{dom}\phi. \tag{2}$$

Note that δ depends on ϵ: $\delta(\epsilon)$.

- The origin is unstable if it is not Lyapunov stable.

- As for nonlinear or linear systems, this definition means that the solution remains bounded as soon as the initial condition is small as depicted in the following figure for a nonlinear system

$$\dot{x} = f(x), \quad y = x$$

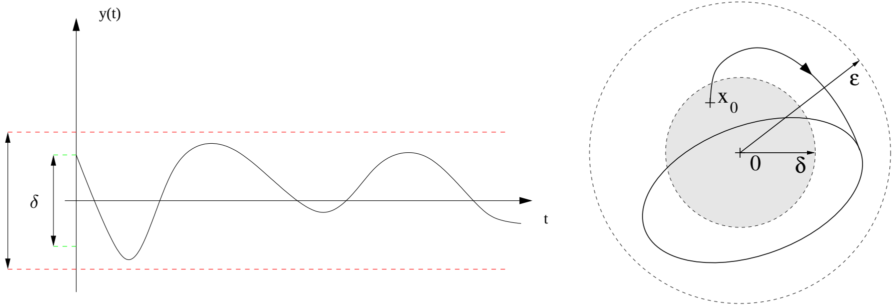


Figure: Stability.

- Consider the case of linear systems:
  - ▷ The unique solution starting from a generic initial condition $x_0 \in \mathbb{R}^n$ can be written as $\phi(t) = \Phi(t, t_0)x_0$,
  - ▷ Therefore the existence of a scalar $M > 0$ such that $|\Phi(t, t_0)| \leq M$ guarantees that we can choose a δ such that

$$|\phi(t)| \leq |\Phi(t, t_0)\phi(t_0)| \leq |\Phi(t, t_0)||\phi(t_0)| \leq M|\phi(t_0)| = M\delta = \varepsilon.$$

  - ▷ Then it is enough to pick $\delta = \dfrac{\varepsilon}{M}$.

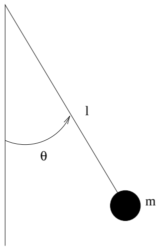
- Example. Consider the linear system

$$\dot{x} = -2x$$

with an initial condition $x(0)$.

  - ▷ One gets
$$x(t) = \Phi(t, t_0)\phi(t_0) = e^{-2t}x(0)$$
  - ▷ One gets
$$|e^{-2t}| \leq 1$$
  - ▷ It follows
$$|x(t)| \leq |x(0)|$$

- More difficult in the case of nonlinear systems which can present several equilibrium points.
- Example. Consider the inverse pendulum:



▷ One considers the state vector $x(t) = \begin{bmatrix}\theta(t) & \dot{\theta}(t)\end{bmatrix}^\top$:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\frac{g}{l}\sin(x_1(t)) - \frac{k}{m}x_2(t) \end{cases}$$

▷ $x_1 = 0$, $x_2 = 0$: the pendulum is in the bottom position. This equilibrium is stable.

▷ $x_1 = \pi$, $x_2 = 0$: the pendulum is in the up position. This equilibrium point is unstable.

- Definition 1 is not sufficient to ensure the convergence of the solution to the equilibrium point.
  - ▷ We need to add the notion of attractivity
- A second important concept that we introduce is associated to the asymptotic properties of solutions evolving forever, namely complete solutions
  - ▷ Definition 5 in the previous lecture

Part II - Definition 5

A solution ϕ is complete if $\mathrm{dom}\phi$ is unbounded.

Definition 2 - Global attractivity

The point $x_e = 0$ is globally attractive for system (1) if all maximal solutions ϕ are bounded and all complete ones satisfy

$$\lim_{\substack{t+j\to\infty \\ (t,j)\in \mathrm{dom}\,\phi}} |\phi(t,j)| = 0.$$

- Definition 2 means that there exists a neighborhood of x_e from which solutions in $\mathbb{R}^n$ that exist for arbitrarily long t and/or j converge to x_e as $t + j$ goes unbounded.
- Definition 2 only makes sense for maximal solutions which evolve forever, i.e. complete ones ($\mathrm{dom}\,\phi$ is unbounded).
- In fact, for non-complete solutions we cannot compute the limit as $t + j \to \infty$.

- The notion of completeness is very important.
- Example. Consider

$$\dot{x} = x^2$$

  - ▷ All its solutions read $\phi(t) = \frac{1}{\frac{1}{x(0)} - t}$
  - ▷ All its solutions starting from $x(0) \leq 0$ are complete and converge asymptotically to zero,
  - ▷ All solutions starting from $x(0) > 0$ diverge to infinity in finite time (they are not complete).
  - ▷ If we did not require all maximal solutions to be bounded in Definition 2, one could think that the equilibrium $x_e = 0$ is globally attractive, whereas there are solution escaping to infinity.

- The notion of global asymptotic stability (GAS) is defined as follows.

Definition 3 - Global asymptotic stability

The point $x_e = 0$ is globally asymptotically stable if it is
Lyapunov stable and globally attractive.

- Important remark. The combination of Lyapunov stability and global attractivity to form GAS is certainly nontrivial.

Example of a system globally attractor but unstable (1)

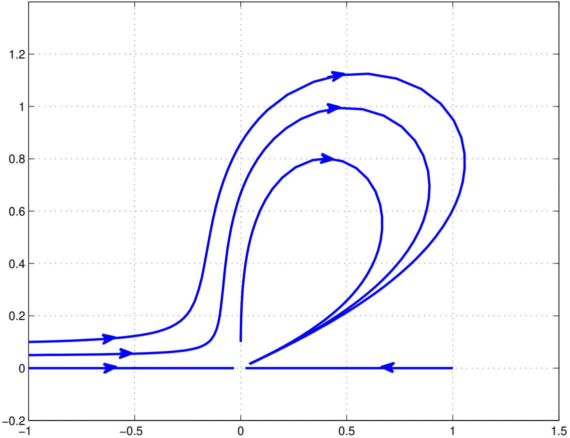
- Consider the following modified version of the butterfly system:

$$\begin{cases} \dot{x}_1(t) &= x_1^2(t)(x_2(t) - x_1(t)) + x_2^5(t) \\ \dot{x}_2(t) &= x_2^2(t)(x_2(t) - 2x_1(t)) \end{cases}$$

- The unique equilibrium point is $x_e = [0\ 0]^\top$.
- All the system trajectories converge toward x_e.
  - ▷ x_e is a globally attractive.

Example of a system globally attractor but unstable (2)

- The equilibrium point 0 is unstable as shown in the figure which follows.
- Actually, let us consider an initial condition close to 0, the trajectory $x(t)$ may be very large.



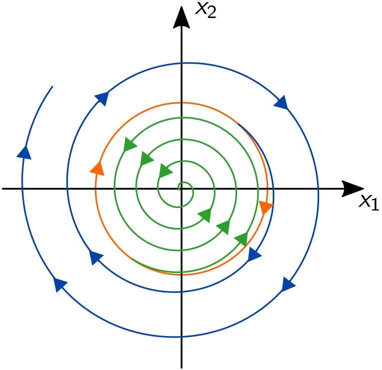
Lyapunov stable but not globally attractive (1)

- Consider the following continuous-time system

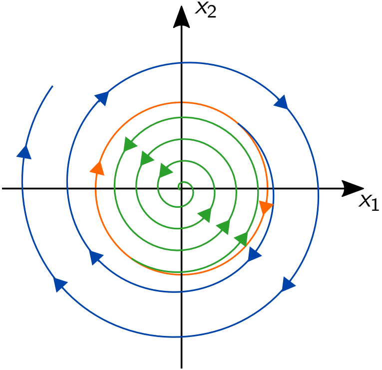
$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = f(x) = \begin{bmatrix} x_2 - \alpha(1 - |x|) \\ -x_1 - \alpha(1 - |x|) \end{bmatrix}$$

  with $\alpha > 0$.

- The solutions are depicted in figure below

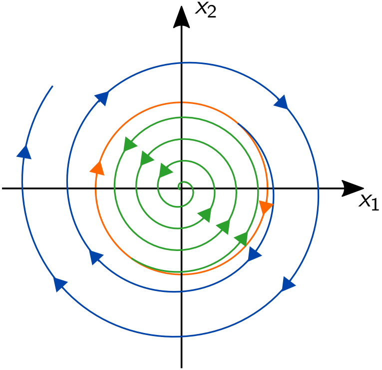


Lyapunov stable but not globally attractive (1)



- The solutions evolve on a limit cycle around the origin (red)
- The solutions converge to the origin inside the unit circle (green) .
- They will diverge when the initial condition is outside the unit circle (blue) .

Lyapunov stable but not globally attractive (2)



▷ The origin is Lyapunov stable but not globally attractive.

- Remark 1. Given any value of ε (even very large ones) we can always pick a δ such that solutions stay inside the ε-ball. More specifically, we can use any $\delta \leq 1$ to prove stability.
  - ▷ Lyapunov stability follows.
- Remark 2. Solutions starting sufficiently far from the origin diverge to infinity.
  - ▷ Lyapunov stability does not provide any guarantee one these solutions
  - ▷ Lyapunov stability is indeed a local property.
- The attractivity part requires boundedness of solution during transients, i.e. solutions could have very large overshoot and eventually converge to the equilibrium point.

- The notion of uniform global boundedness (UGB) or Lagrange stability is defined as follows.

Definition 4 - Uniform Global Boundedness or Lagrange stability

The point $x_e = 0$ is Lagrange stable (Uniformly Globally Bounded) for system (1) if for each $\delta > 0$, there exists $\varepsilon > 0$ such that

$$|\phi(0,0)| \leq \delta \implies |\phi(t,j)| \leq \varepsilon, \quad \forall (t,j) \in \mathrm{dom}\phi. \tag{3}$$

holds for all solutions.
Note that ϵ depends on δ: $\epsilon(\delta)$.

- Consider the previous example with "reversed-time" to illustrate Lagrange stability.
  - ▷ For doing this, we simply need to put a minus in front of the right-hand side so that $\dot{x}$ is reversed ($\dot{x} = -f(x)$)
- Solutions from any point in the state space converge to the unit circle (in red).

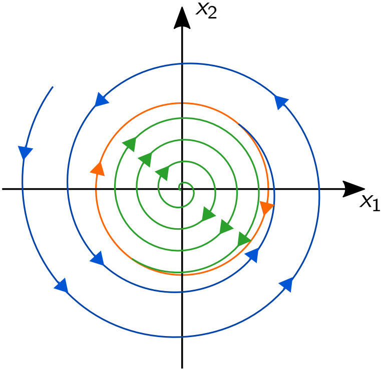
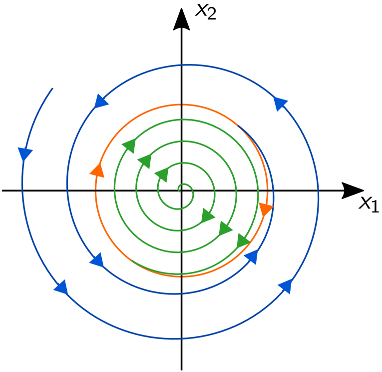


Figure: The origin is Lagrange stable but not globally attractive.

- The origin is Lagrange stable but not Lyapunov stable (due to green trajectories)
  - ▷ Solutions starting in the unit disc remain in there, and solutions starting outside the unit disc approach its boundary.
  - ▷ For any given δ, we can pick a large enough ε such that solutions remain inside the ε-ball.
  - ▷ In particular, any $\varepsilon \geq 1$ satisfies Lagrange stability.



Suggested example of a stable system but not attractor (1)

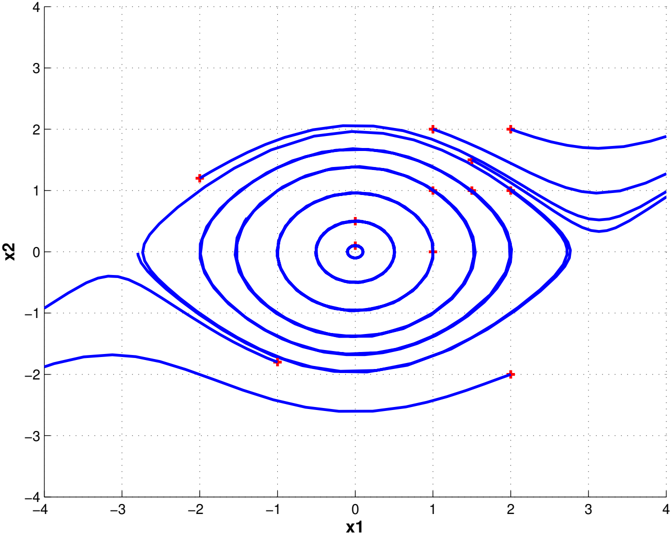
- Consider the following example

$$\begin{cases} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\sin(x_1(t)) \end{cases}$$

- Compute the equilibrium points.
- Show that an equilibrium point is stable but not attractive.

Suggested example of a stable system but not attractor (2)

- The origin is stable but not an attractor.



- The notion of Lyapunov stability is a local property
  - ▷ also related to the property of uniform local stability.
- The notion of global attactivity is a global convergence property
- Now we introduce the concept of Uniform Global Stability.

- In order to discuss about Uniform Global Stability and other concepts, it is useful to manipulate comparison functions.

Definition 5

A continuous function α of $[0, a]$ valued in $[0, +\infty]$ is said to be of class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to be of class $\mathcal{K}_\infty$ if $a = \infty$ et $\lim\limits_{r \to +\infty} \alpha(r) = +\infty$.

- Example 1. $\alpha(x) = tan^{-1}(x)$ is strictly increasing since one gets $\frac{\partial \alpha}{\partial x} = \frac{1}{1+x^2} > 0$. It belongs to class $\mathcal{K}$ but not to class $\mathcal{K}_\infty$ because $\lim_{x \to \infty} \alpha = \frac{\pi}{2} < \infty$.
- Example 2. $\alpha(x) = x^c$, $c > 1$, is strictly increasing since one gets $\frac{\partial \alpha}{\partial x} = cx^{c-1} > 0$. Furthermore, $\lim_{x \to \infty} \alpha = \infty$, thus it belongs to class $\mathcal{K}_\infty$.

Definition 6

The origin is Uniformly Globally Stable for system (1) if there exists a function α of class $\mathcal{K}_\infty$ such that all maximal solutions ϕ satisfy

$$|\phi(t,j)| \leq \alpha(|\phi(0,0)|), \quad \forall (t,j) \in \mathrm{dom}\phi$$

- UGS combines the small-signal properties of Lyapunov Stability with the large-signal properties of Lagrange stability.

Proposition 1

The origin is Uniformly Globally Stable if and only if it is Lyapunov stable and Lagrange Stable.

- In other words, one gets: UGS= LS+UGB
- The Uniformity within the UGS definition stands for the fact that the given bound is uniform over all solutions starting within a certain range from the origin.
  - ▷ A parallel uniformity requirement can be stated also in terms of the time it takes for a solution to reach a certain ball around the origin.
- One can formalize the notion of Uniform Global Attractivity (UGA).

Definition 7 - Uniformly Globally Attractive

The origin is Uniformly Globally Attractive for system (1) if for each $\varepsilon > 0$ and $M > 0$, there exists a time $T(M, \varepsilon)$ such that all maximal solutions ϕ to system (1) satisfy

$$|\phi(0,0)| \leq M \implies |\phi(t,j)| \leq \varepsilon \quad \forall (t,j) \in \mathrm{dom}\phi \text{ s.t. } t + j \geq T(M, \varepsilon).$$

- Note that uniform global attractivity (UGA) is more than simple global attractivity (GA):
  - ▷ It requires the existence of an upper bound T on the time for any solution starting in the M-ball to reach and stay forever in the ϵ-ball, centered at the origin.
  - ▷ Clearly, one expects this time T to become larger as M grows (solutions starting far away take longer to reach the origin) and as ϵ shrinks (solutions often converge asymptotically, therefore they slow down as they approach the origin).

- The notion of uniform global asymptotic stability (UGAS) is defined as follows.

Definition 8 - Uniform global asymptotic stabilty

The origin is UGAS for system (1) if it is Uniformly Globally Stable (UGS) and Uniformly Globally Attractive (UGA).

- In other words, one gets: UGAS=UGS+UGA= LS+UGB + UGA since UGS= LS+UGB from Proposition 1.
  - ▷ This is a stronger property than GAS (since GAS= LS+GA from Definition 3)
- Note that for classical continuous-time and discrete-time systems these two properties are actually equivalent.
  - ▷ For hybrid systems, this is not true in general, but it is true under certain regularity conditions on the data: choice of closed flow and jump sets $\mathcal{C}$ and $\mathcal{D}$

- Let us specify when GAS is equivalent to UGAS.

Theorem 1

If

- $\mathcal{C}$ and $\mathcal{D}$ are closed,
- f and g are continuous,

then

$$\boxed{\text{GAS} \iff \text{UGAS}}$$

Remark

The conditions in Theorem 1 guarantee existence of solutions.

- Example. Consider the previous example

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = f(x) = \begin{bmatrix} x_2 - \alpha(1 - |x|) \\ -x_1 - \alpha(1 - |x|) \end{bmatrix}$$

with $\alpha > 0$.

  - ▷ Note that one gets $\mathcal{C} = \mathbb{R}^2$ and $\mathcal{D} = \emptyset$.
  - ▷ As we have seen the origin is not GAS (since some trajectories are divergent).

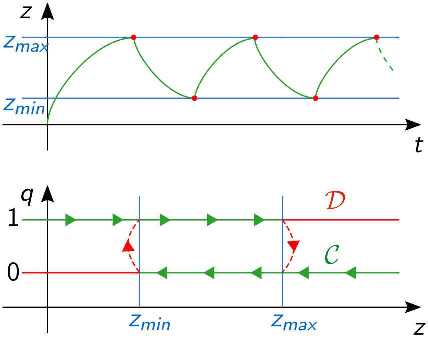
- Now consider a new selection for $\mathcal{C}$:

$$\mathcal{C} = \{x \in \mathbb{R}^2; |x| \leq \frac{1}{2}\}$$

  - ▷ One gets a smaller set of solutions (because $\mathcal{C}$ is smaller).
  - ▷ The origin is clearly GAS, thus from Theorem 1 and the regularity of the data, it is also UGAS.

- We have just studied the stability of a point (the origin).
  - ▷ Until now, we have proposed some definitions of stability and attractivity when the equilibrium point is the origin $(x_e = 0)$,
  - ▷ That can be easily generalized to any isolated point in the state space by extending the notion of distance.
  - ▷ Recall that $|\phi(0,0)|$ represent the norm of $\phi(0,0)$ or equivalently the distance of $\phi(0,0)$ to $x_e = 0$: Euclidean norm
- Hybrid dynamical systems have often timers; logical variables and these variables do not necessarily converge to zero.
  - ▷ Clearly, asymptotic stability of an equilibrium point becomes a special case.
  - ▷ Rather than establishing asymptotic stability of an equilibrium (the origin) we characterize the stability for a compact set (closed and bounded).

- Let us just provide some examples.
- Example 1. Consider the thermostat system (Example 3 in Introduction course).
  - ▷ The goal is to stabilize the temperature (variable z) in a given range by switching on and off a heater (variable q).
  - ▷ In this case one wants to characterize the stability of the set $\mathcal{A} = [z_{min}, z_{max}] \times \{0, 1\}$, in order to take into account the switching behavior of the heater.



- Example 2. Consider the sampled-data system (Example 4 in Introduction course).
  - ▷ The goal is to drive the plant state z to the origin, while the timer τ implements the sampling mechanism by repeatedly evolving in the interval $[0, T]$.
  - ▷ Given a positive scalar T, the state of the sampled-data system can be defined as

$$x = (z, u, \tau) \in \mathbb{R}^{n_p} \times \mathbb{R}^{n_u} \times [0, T]$$

    where n_p, n_u are the dimensions of the plant and controller states, respectively.
  - ▷ Given the timer dynamics, there is no possibility for the state x to converge to an isolated point.
  - ▷ The set we want to stabilize is then

$$\mathcal{A} = \{0\} \times \{\kappa(0)\} \times [0, T]$$

- Then we want to study the stability and attractivity of a set $\mathcal{A}$
- Before providing some definitions we need a preliminary ingredient:
  - ▷ The notion of distance of a point x from a set $\mathcal{A}$

Definition 9 - Distance

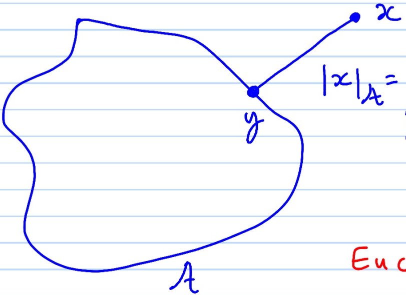
Given a set $\mathcal{A} \subset \mathbb{R}^n$, the distance of $x \in \mathbb{R}^n$ from the set $\mathcal{A}$ is defined as

$$|x|_{\mathcal{A}} = \inf_{y \in \mathcal{A}} |y - x|. \tag{4}$$

Remark

When $\mathcal{A} = \{0\}$ one retrieves $|x|_{\mathcal{A}} = |x|$.

Distance of a point x to a set A



$$|x|_A = \inf_{y \in A} |y - x|$$

Euclidean Norm

Remark: $A = \{0\} \Rightarrow |x|_A = |x|$

- Consider now the definition of Lyapunov stability
  - ▷ Extension of Definition 1 (for a point: $\mathcal{A} = \{0\}$)

Definition 10 - Lyapunov stability

A compact set $\mathcal{A}$ is Lyapunov stable for (1) if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that all maximal solutions ϕ to (1) ($\phi \in \mathcal{S}_{\mathcal{H}}$) satisfy

$$|\phi(0,0)|_{\mathcal{A}} \leq \delta \implies |\phi(t,j)|_{\mathcal{A}} \leq \varepsilon \quad \forall (t,j) \in \mathrm{dom}\phi. \tag{5}$$

- Indeed, all the previous definitions can be extended by replacing $|\,.\,|$ by $|\,.\,|_{\mathcal{A}}$.

- Similarly we the extension of all the previous definitions regarding global attractivity, global asymptotic stability, ...

Definition 11 (GA, GAS, Lagrange stability)

For system (1), a compact set $\mathcal{A}$ is

1. Globally attractive if all maximal solutions ϕ are bounded and all complete ones satisfy

$$|\phi(t,j)|_{\mathcal{A}} \to 0 \text{ as } t+j \to \infty, \quad \forall (t,j) \in \mathrm{dom}\phi. \tag{6}$$

2. Globally asymptotically stable if it is Lyapunov stable and globally attractive.

3. Lagrange stable if for each $\delta > 0$, there exists $\epsilon(\delta) > 0$ such that all maximal solutions ϕ satisfy

$$|\phi(0,0)|_{\mathcal{A}} \leq \delta \implies |\phi(t,j)|_{\mathcal{A}} \leq \varepsilon, \quad \forall (t,j) \in \mathrm{dom}\phi. \tag{7}$$

Definition 12 (UGS, UGA, UGAS)

For system (1), a compact set $\mathcal{A}$ is

1. Uniformly Globally Stable (UGS) if there exists a function α of class $\mathcal{K}_\infty$ such that all maximal solutions ϕ satisfy

$$|\phi(t,j)|_{\mathcal{A}} \leq \alpha(|\phi(0,0)|_{\mathcal{A}}), \quad \forall (t,j) \in \mathrm{dom}\phi$$

2. Uniformly Globally Attractive (UGA) if for each $\varepsilon > 0$ and $M > 0$, there exists a time $T(M,\varepsilon)$ such that all maximal solutions ϕ to system (1) satisfy

$$|\phi(0,0)|_{\mathcal{A}} \leq M \implies |\phi(t,j)|_{\mathcal{A}} \leq \varepsilon \quad \forall (t,j) \in \mathrm{dom}\phi \text{ s.t. } t+j \geq T(M,\varepsilon).$$

3. Uniformly Globally Asymptotically Stable (UGAS) if it is UGS and UGA.

- Let us state a basic assumption in order to simplify the study of stability.
  - ▷ More details and extensions to other cases are proposed in the book of Goebel, Sanfelice and Teel, 2012.

Assumption 1 - Basic assumptions

1. $\mathcal{C}$ and $\mathcal{D}$ are closed;
2. $\mathcal{A}$ is compact;
3. f and g are continuous.

  - ▷ Note that to prove the stability of a point ($\mathcal{A} = \{0\}$) items 1 and 2 were used.
  - ▷ The data of hybrid system is sufficiently regular.

Theorem 2

If Assumption 1 is satisfied, then

$$\boxed{\text{GAS} \iff \text{UGAS}}$$

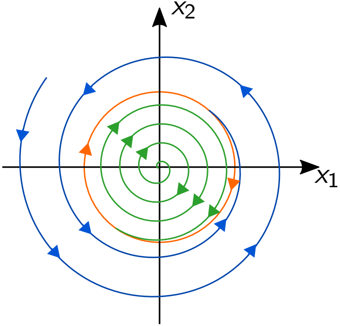
- Theorem 2 is an extension of Theorem 1 in the case of a compact set $\mathcal{A}$
- One retrieves items 1 and 2 used in the case $\mathcal{A} = \{0\}$ (see Theorem 1)
- Note that the GAS property of a compact set $\mathcal{A}$ is obtained if it is Lyapunov stable + Globally attractive (See Definitions 10 and 11)

- **Example.** Consider the previous example

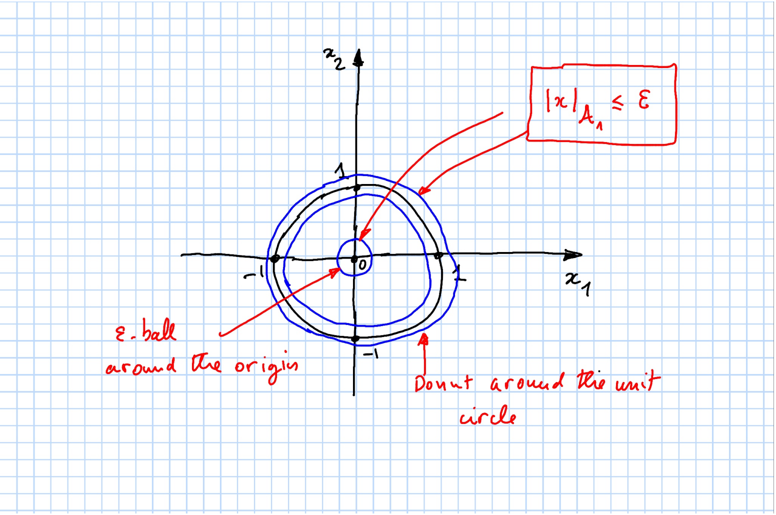
$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = f(x) = \begin{bmatrix} x_2 - \alpha(1 - |x|) \\ -x_1 - \alpha(1 - |x|) \end{bmatrix}$$

with $\alpha > 0$.

  - ▷ Remember that for the corresponding system with the reversed time dynamics $\dot{x} = -f(x)$ the unit disc is GAS (therefore UGAS).
  - ▷ In other words, the set $\mathcal{A}_0 = \{x \in \mathbb{R}^2; |x| \leq 1\}$ is UGAS



- Question. Since the origin is an equilibrium point and all the other solutions converge to the unit circle, is it possible to conclude that the set $\mathcal{A}_1 = \{x \in \mathbb{R}^2; |x| = 1 \text{ or } x = 0\}$ is also GAS?
- Answer. No
  - ▷ $\mathcal{A}_1$ is globally attractive, but it is not Lyapunov stable
  - ▷ Indeed, picking ϵ sufficiently small, so that the set where $|x|_{\mathcal{A}_1} \leq \epsilon$ is a disk around the origin union a thin donut around the unit circle,
  - ▷ There is no small enough δ that prevents solutions starting (arbitrarily) close to the origin to exit the set where $|x|_{\mathcal{A}_1} \leq \epsilon$, while spiraling out towards the unit circle.


x_2
x_1
1
-1
0
-1
1
$|x|_{A_1} \leq \varepsilon$
ε-ball
around the origin
Donut around the unit
circle

- To complete the results on the stability of an hybrid dynamical system, let us come back to the case of linear systems.
  - ▷ Asymptotic stability for linear systems coincides with exponential stability, due to the peculiar structure of solutions to ordinary differential equations (ODE).
- Consider a linear system $\dot{x} = Ax$
  - ▷ This system is exponentially stable if

$$\exists \lambda, k > 0$$

such that all solutions satisfy

$$|x(t)| \leq k e^{-\lambda t} |x(0)|, \quad \forall t \geq 0.$$

- We are then in measure to expand the global exponential stability (GES) for a set $\mathcal{A}$.

Definition 13 - Global exponential stability

A compact set $\mathcal{A}$ is globally exponentially stable for (1) if $\exists \lambda, k > 0$ such that all solutions $\phi \in \mathcal{S}_{\mathcal{H}}$ (maximal solutions ϕ to (1)) satisfy

$$|\phi(t,j)|_{\mathcal{A}} \leq k e^{-\lambda(t+j)} |\phi(0,0)|_{\mathcal{A}}, \quad \forall (t,j) \in \mathrm{dom}\phi.$$

- Define the notion of $\mathcal{KL}$-functions.

Definition 14 - Class-$\mathcal{KL}$ function

A two-arguments function $\beta(r,s)$ is of class $\mathcal{KL}$ if if it is of class $\mathcal{K}$ with respect to the first argument and of class $\mathcal{L}$ with respect to the second one. More precisely if

- for each $r \geq 0$, $\beta(r,\cdot)$ is continuous, non-decreasing and $\lim\limits_{s\to\infty} \beta(r,s) = 0$;
- for each $s \geq 0$, $\beta(\cdot,s)$ is continuous, non-increasing and $\lim\limits_{r\to 0^+} \beta(r,s) = 0$.

Examples of $\mathcal{KL}$-functions

- Consider $\beta(r,s) = \frac{r}{ksr+1}$ with $k > 0$
  - ▷ $\beta(r,s)$ is strictly increasing in r since $\frac{\partial \beta}{\partial r} = \frac{1}{(ksr+1)^2} > 0$
  - ▷ $\beta(r,s)$ is strictly decreasing in s since $\frac{\partial \beta}{\partial s} = \frac{-kr^2}{(ksr+1)^2} < 0$
  - ▷ $\lim\limits_{s\to\infty} \beta(r,s) = 0$ and $\lim\limits_{r\to 0^+} \beta(r,s) = 0$
  - ▷ $\beta(r,s)$ is of class $\mathcal{KL}$
- Consider $\beta(r,s) = r^c e^{-s}$, with $c > 0$.
  - ▷ $\beta(r,s)$ is of class $\mathcal{KL}$

- Now we can define the notion of $\mathcal{KL}$-asymptotic stability for system (1).

Definition 15 - $\mathcal{KL}$-asymptotic stability

A compact set $\mathcal{A}$ is $\mathcal{KL}$-asymptotically stable for (1) if there exists a function $\beta \in \mathcal{KL}$ such that all solutions $\phi \in \mathcal{S}_{\mathcal{H}}$ satisfy

$$|\phi(t,j)|_{\mathcal{A}} \leq \beta(|\phi(0,0)|_{\mathcal{A}}, t+j), \quad \forall (t,j) \in \mathrm{dom}\phi.$$

- The first argument of β is the distance from the attractor,
- The second argument of β is $t+j$

Meaning of Definition 15

- The condition

$$|\phi(t,j)|_{\mathcal{A}} \leq \beta(|\phi(0,0)|_{\mathcal{A}}, t+j), \quad \forall (t,j) \in \mathrm{dom}\phi.$$

based on a class-$\mathcal{KL}$ function allows a generalization of exponential bounds

$$|\phi(t,j)|_{\mathcal{A}} \leq ke^{-\lambda(t+j)}|\phi(0,0)|_{\mathcal{A}}, \quad \forall (t,j) \in \mathrm{dom}\phi.$$

  - ▷ β increases as the distance from the attractor increases,
  - ▷ β decreases as $t+j$ goes to infinity.

- Recall Assumption 1.

Assumption 1 - Basic assumptions

1. $\mathcal{C}$ and $\mathcal{D}$ are closed;
2. $\mathcal{A}$ is compact;
3. f and g are continuous.

- Then we have the following equivalence.

Theorem 3

If Assumption 1 is satisfied, then

$$\boxed{\text{GAS} \iff \text{UGAS} \iff \mathcal{KL}\text{-GAS}}$$

- Another interesting notion we can extend is the notion of
  - ▷ BIBO-stability (Bounded Input - Bounded Output stability) for linear systems
  - ▷ or ISS (Input-to-state stability) for nonlinear systems
- A natural question arises:

  What can be said about the behavior of system (1) when it is subject to a bounded (disturbance) input u?

The case of linear systems (1)

- First, let us see what can be the answer in the case of a linear system.
- The solution of the system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad A \, \text{Hurwitz},$$

reads:

$$x(t) = e^{At}x(0) + \int_{t=0}^{t} e^{(t-\tau)A}Bu(\tau)d\tau.$$

- Moreover since A is Hurwitz there exists $k, \lambda > 0$, such that $(\|.\| = |.|)$. $|e^{At}| \leq ke^{-\lambda t}$ Then, one obtains:

$$\begin{aligned} |x(t)| &\leq ke^{-\lambda t}|x(0)| + \int_{t=0}^{t} ke^{-\lambda(t-\tau)}|B||u(\tau)|d\tau. \\ &\leq ke^{-\lambda t}|x(0)| + \frac{k|B|}{\lambda} \sup_{0 \leq \tau \leq t} |u(\tau)| \end{aligned}$$

The case of linear systems (2)

- Then, one can deduce
  - ▷ a bounded input $\Rightarrow$ bounded states.
  - ▷ The bound on the states is proportional to the bound on the input.
- Is it the same for nonlinear or hybrid systems?
  - ▷ To address the ISS property of hybrid dynamical systems we exploit class-$\mathcal{KL}$ functions

- First formalize the definition of BIBO-stability

Definition 16 - BIBO stability

$\exists \lambda, k, \gamma > 0$ such that:

$$|x(t)| \leq ke^{-\lambda t}|x(0)| + \gamma ||u||, \quad \forall t \geq 0$$

- One gets then the definition of ISS (by using Definition 15 on $\mathcal{KL}$ functions

Definition 17 - ISS

$\exists \beta \in \mathcal{KL}$ and a function γ of class $\mathcal{K}_\infty$ such that:

$$|\phi(t,j)|_{\mathcal{A}} \leq \beta\left(|\phi(0,0)|_{\mathcal{A}}, t+j\right) + \gamma\left(||u||\right), \quad \forall (t,j) \in \mathrm{dom}\phi.$$

- Consider now 2 examples to illustrate what happens when analyzing stability properties of hybrid systems
- In particular, the main issues are related to the presence of maximal solutions.

Example 1 (1)

- The hybrid system (1) is defined by

$$\begin{array}{rcl} \dot{x} & = & f(x) := \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \forall x \in \mathcal{C} \\ x^+ & \in & g(x) := \emptyset, \quad \forall x \in \mathcal{D} := \emptyset. \end{array}$$

  with the following data:
  - ▷ The state $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$,
  - ▷ The flow set $\mathcal{C}$ is closed and bounded, and does not contain the origin,
  - ▷ The map set $\mathcal{D} = \emptyset$.
- Therefore no jump is allowed.
- Objective. We want to study the stability of the set $\mathcal{A} = \{0\}$.

Example 1 (2)

- Let us first study the solutions of the system:

$$\phi(t,0) = \begin{bmatrix} t \\ t \end{bmatrix} + \phi(0,0)$$

  - ▷ From any point in $\mathcal{C}$, all solutions flow in the same direction until they reach the boundary of the flow set and stop.
  - ▷ For any other point in the state space, solutions are not defined.
- It is important to note that all maximal solutions to this system are not complete since they do not evolve forever ($\mathtt{dom}\phi$ is not unbounded).

Example 1 (3)

- Question 1: is $\mathcal{A}$ Lyapunov stable?
- Recall Definition 1.

Definition 1 - Lyapunov stability

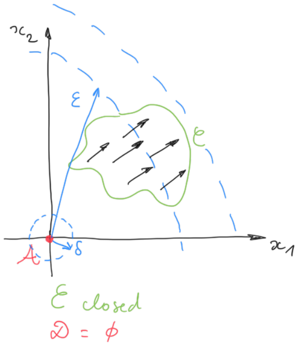
A point x_e is Lyapunov stable for (1) if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that all maximal solutions ϕ to (1) ($\phi \in \mathcal{S}_{\mathcal{H}}$) satisfy

$$|\phi(0,0)| \leq \delta \implies |\phi(t,j)| \leq \varepsilon, \quad \forall (t,j) \in \mathrm{dom}\phi. \tag{8}$$

Remark

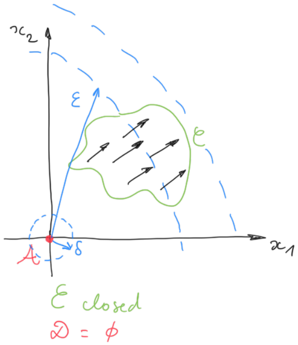
Even if solutions go away from the origin, they terminate as soon as they reach the boundary of $\mathcal{C}$, which is a bounded set.

Example 1 (4)



- This means that for any given value of $\varepsilon > 0$, we can always pick a small enough $\delta > 0$ such that solutions from the δ-ball remain in the ε-ball.
  - ▷ Pick ε such that the ε-ball around the origin contains $\mathcal{C}$
  - ▷ Then any solution terminates without escaping the ε-ball ($|\phi(t, j)| \le \varepsilon$).

Example 1 (5)



- Then from Definition 1, condition must hold for any smaller $\varepsilon > 0$, but it is enough to choose δ in such a way that the δ-ball around the origin does not intersect $\mathcal{C}$.
  - ▷ Then, there is no solution from the δ-ball,
  - ▷ Solutions remain in the ε-ball.
  - ▷ For any ε, we may always choose $\delta(\varepsilon) = \bar{\delta}$ and no solutions exist from the corresponding $\bar{\delta}$-ball, which proves (2) vacuously

Example 1 (6)

- Question 2: is $\mathcal{A}$ globally attractive?

Definition 2 - Global attractivity

$\mathcal{A} = \{0\}$ is globally attractive for system (1) if all maximal solutions ϕ are bounded and all complete ones satisfy

$$\lim_{\substack{t+j\to\infty \\ (t,j)\in\mathrm{dom}\,\phi}} |\phi(t,j)| = 0.$$

- If we follow carefully the definition of global attractivity, it is clear that it is true:
  - ▷ all solutions are bounded (indeed $\mathcal{C}$ is bounded)
  - ▷ and all complete solutions (there are none) converge to the origin (Indeed, nothing needs to be checked).

Example 1 (7)

- The previous case illustrates the importance of completeness of the solution and the fact that some strange behaviors can arrive for hybrid dynamical systems.
- To simplify the study, modify the jump dynamics as

$$x^+ = g(x) := \begin{bmatrix} 0 \\ 0 \end{bmatrix}, x \in \mathcal{D} := \mathbb{R}^2$$

- This modification adds some new solutions to the system but does not remove the previous ones.
- As a consequence, all maximal solutions are complete.

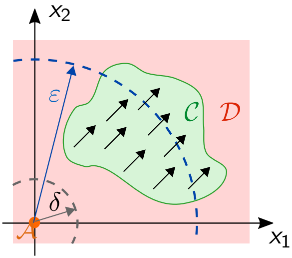
Example 1 (8)

- Let us study the solution to the new system:

$$\begin{aligned} \phi(t,0) &= \begin{bmatrix} t \\ t \end{bmatrix} + \phi(0,0) \\ \phi(0,j) &= g(\phi(0,j-1)) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathcal{D}, \forall j \geq 1 \end{aligned}$$

▷ All solutions flow for a finite amount of time (even 0), jump to the origin and stay there forever: They are complete and eventually discrete solutions.

Conclusion. $\mathcal{A} = \{0\}$ is attractive both for the original and the new system, since all solutions are bounded and all complete ones converge to the origin.



Example 2 (1)

- The hybrid system (1) is defined by

$$\begin{aligned} \dot{x} &= f(x) := \begin{bmatrix} z \\ 1 \end{bmatrix}, && \forall x \in \mathcal{C} := \{(z,\tau) : \tau \in [0, M]\} \\ x^+ &= g(x) := \begin{bmatrix} \lambda z \\ 0 \end{bmatrix}, && \forall x \in \mathcal{D} := \{(z,\tau) : \tau = M\}\,. \end{aligned}$$

  ▷ with the state $x = \begin{bmatrix} z \\ \tau \end{bmatrix} \in \mathbb{R}^2$,

  ▷ λ is a positive scalar such that $0 \leq \lambda < e^{-M}$

  ▷ Note that $\mathcal{C}$ and $\mathcal{D}$ read:

$$\mathcal{C} = \mathbb{R} \times [0, M] \text{ and } \mathcal{D} = \mathbb{R} \times \{M\}$$

- Objective. We want to study the stability of the set $\mathcal{A} = \{(z,\tau) : z = 0, \tau \in [0, M]\}$.

Example 2 (2)

- Let us study the solution to the system:

$$\left.\begin{array}{rcl}\phi_z(t,0) & = & e^t\phi_z(0,0)\\ \phi_\tau(t,0) & = & t+\phi_\tau(0,0)\end{array}\right\},\forall t\in[0,M-\phi_\tau(0,0)]$$

$$\phi(0,j)=g(\phi(0,j-1))=\left[\begin{array}{c}\lambda\phi_z(0,j-1)\\ 0\end{array}\right],j\geq 1-->\notin\mathcal{D}$$

- Take $t_1=M-\phi_\tau(0,0)$ such that $\phi_\tau(t_1,0)=M\in\mathcal{D}$, then one gets

$$\begin{array}{rcl}\phi(t_1,1) & = & g(\phi(t_1,0))=\left[\begin{array}{c}\lambda\phi_z(t_1,0)\\ 0\end{array}\right]=\left[\begin{array}{c}\lambda e^{t_1}\phi_z(0,0)\\ 0\end{array}\right]\\ \phi(t,1) & = & \left[\begin{array}{c}e^{(t-t_1)}\phi_z(t_1,1)\\ t-t_1+\phi_\tau(t_1,1)\end{array}\right]=\left[\begin{array}{c}e^{(t-t_1)}\lambda e^{t_1}\phi_z(0,0)\\ t-(M-\phi_\tau(0,0))+0\end{array}\right]\end{array}$$

- Take $t_2=2M-\phi_\tau(0,0)$ such that $\phi_\tau(t_2,1)=M\in\mathcal{D}$, then one gets

$$\begin{array}{rcl}\phi(t_2,2) & = & g(\phi(t_2,1))=\left[\begin{array}{c}\lambda\phi_z(t_2,1)\\ 0\end{array}\right]=\left[\begin{array}{c}\lambda^2 e^{t_2}\phi_z(0,0)\\ 0\end{array}\right]\\ \phi(t,2) & = & \left[\begin{array}{c}e^{(t-t_2)}\phi_z(t_2,1)\\ t-t_2+\phi_\tau(t_2,1)\end{array}\right]=\left[\begin{array}{c}e^{(t-t_2)}\lambda^2 e^{t_2}\phi_z(0,0)\\ t-(2M-\phi_\tau(0,0))+0\end{array}\right]\end{array}$$

Example 2 (3)

- Then one gets

$$
\begin{aligned}
\phi(t_j, j) &= g(\phi(t_j, 1-1)) = \begin{bmatrix} \lambda\phi_z(t_j, j-1) \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda^j e^{t_j}\phi_z(0,0) \\ 0 \end{bmatrix} \\
\phi(t, j) &= \begin{bmatrix} e^{(t-t_j)}\phi_z(t_j, j-1) \\ t - t_j + \phi_\tau(t_j, j-1) \end{bmatrix} = \begin{bmatrix} e^{(t-t_j)}\lambda^j e^{t_j}\phi_z(0,0) \\ t - (jM - \phi_\tau(0,0)) + 0 \end{bmatrix} \\
&= \begin{bmatrix} \lambda^j e^t \phi_z(0,0) \\ t - (jM - \phi_\tau(0,0)) \end{bmatrix}, \\
&\quad \forall t \in [jM - \phi_\tau(0,0), (j+1)M - \phi_\tau(0,0)]
\end{aligned}
$$

  - ▷ That means that τ revisits every point in the interval $[0, M]$.
  - ▷ That means that the hybrid systems does not possess an asymptotically stable equilibrium point.

- Compute $|\phi(t,j)|_{\mathcal{A}}$ (Recall $\mathcal{A} = \{(z,\tau) : z = 0, \tau \in [0, M]\}$):

$$|\phi(t,j)|_{\mathcal{A}} = |\phi_z(t,j)|_{\mathcal{A}} = |\phi_z(t,j)| = |\lambda^j e^t||\phi_z(0,0)|$$

Example 2 (4)

- Is $\mathcal{A}$ Lyapunov stable? YES
  - ▷ Note that one gets $\forall (t,j) \in \mathrm{dom}\phi$ one gets

$$t \leq (j+1)M --> |\lambda^j e^t| \leq \lambda^j e^{(j+1)M} < e^{-jM} e^{(j+1)M} = e^M$$
$$|\lambda^j e^t||\phi_z(0,0)| < e^M |\phi_z(0,0)|$$

  - ▷ Indeed, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$|\phi_z(0,0)| \leq \delta \implies |\lambda^j e^t||\phi_z(0,0)| \leq \varepsilon, \quad \forall (t,j) \in \mathrm{dom}\phi$$

  - ▷ Therefore for any ε such that

$$|\lambda^j e^t||\phi_z(0,0)| < e^M |\phi_z(0,0)| \leq \varepsilon,$$

there exists $\delta > 0$:

$$\delta \leq e^{-M} \varepsilon$$

Example 2 (5)

- Is $\mathcal{A}$ uniformly globally attractive (UGA)? YES
  - ▷ Recall the definition: if for each $\varepsilon > 0$ and $\Delta > 0$, there exists a time $T(\Delta, \varepsilon)$ such that all solutions ϕ to system (1) satisfy

$$|\phi_z(0,0)| \leq \Delta \implies |\phi_z(t,j)| \leq \varepsilon \quad \forall (t,j) \in \mathrm{dom}\phi \text{ s.t. } t+j \geq T(\Delta, \varepsilon).$$

  - ▷ Recall that we have $t \leq (j+1)M$, then one gets:

$$\begin{array}{ccc} t+j & \leq & (j+1)M + j \iff t+j-M \leq j(M+1) \\ & \iff & \\ j & \geq & (M+1)^{-1}(t+j-M) \end{array}$$

  - ▷ Recall that one gets: $|\lambda^j e^t||\phi_z(0,0)| < (\lambda e^M)^j e^M |\phi_z(0,0)|$ and for each $\varepsilon > 0$ and $\Delta > 0$ one gets:

$$|\lambda^j e^t||\phi_z(0,0)| < (\lambda e^M)^j e^M |\phi_z(0,0)| \leq (\lambda e^M)^j e^M \Delta \leq \varepsilon$$

    with $0 \leq \lambda e^M < 1$.

Example 2 (6)

- Let us choose $T(\Delta, \varepsilon)$ as

$$T = M + (M+1)T_0$$

- T_0 is a positive scalar satisfying

$$j \geq (M+1)^{-1}(t+j-M) \geq (M+1)^{-1}(T-M) = T_0$$

$$|\lambda^j e^t||\phi_z(0,0)| < (\lambda e^M)^{T_0} e^M |\phi_z(0,0)| \leq (\lambda e^M)^{T_0} e^M \Delta \leq \varepsilon$$

since $0 \leq \lambda e^M < 1$.

Example 2 (7)

- We have studied the case with λ is a positive scalar such that $0 \leq \lambda < e^{-M}$
  - ▷ We have seen that $\mathcal{A}$ is GAS (therefore UGAS) for any $\lambda < \lambda^* = e^{-M}$
- Indeed, $\lambda \in [0, 1]$ is a scalar modulating the attenuation of z across jumps, when the timer is reset to zero.
- It is clear that the UGAS of $\mathcal{A}$ depends on the value of λ.
  - ▷ In particular, since solutions flowing across the whole width of the flow set provide an amplification of $|z|$ corresponding to e^M, it follows that selecting $\lambda = \lambda^*$, provides a family of periodic solutions, not converging to zero and not diverging
  - ▷ $\mathcal{A}$ is unstable for any $\lambda > \lambda^*$.

Example 2 (7)

- We can depict this on the following pictures.

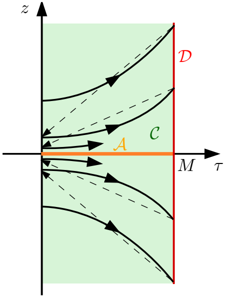


Figure: Left: case with $\lambda < \lambda^*$: asymptotic stability. Middle: case with $\lambda = \lambda^*$: emergence of an infinite number of hybrid periodic solutions. Right: case with $\lambda > \lambda^*$: diverging solutions (instability).

Suggestion

- Consider the previous example with the hybrid system (1) defined with state $x = (z, \tau)$ and with purely continuous dynamics

$$\dot{x} = f(x) := \begin{bmatrix} z \\ 1 \end{bmatrix}, \quad \forall x \in \mathcal{C} := \{(z, \tau) : \tau \in [0, T]\}, \tag{9}$$

  - ▷ The jump set $\mathcal{D} = \emptyset$ is empty, so that the jump dynamics plays no role.
  - ▷ Maximal solutions are not complete, since the state τ is only allowed to flow in the bounded set $[0, T]$ with dynamics $\dot{\tau} = 1$. In particular, the flow time is bounded by $t \leq T$.

- Objective. We want to study the stability of the set $\mathcal{A} = \{(z, \tau) : z = 0, \tau \in [0, T]\}$.

- Most of the definitions and results of the previous sections readily extend to the case of hybrid inclusions

$$\begin{array}{c} \dot{x} \in F(x) \ , \ x \in \mathcal{C} \\ x^{+} \in G(x) \ , \ x \in \mathcal{D} \end{array} \tag{10}$$

  - ▷ Recall that we discussed such an extension in the context of solution (see slide 26 of Part II: Notions of solution).
- Let us clarify here how the basic assumptions of continuity of f and g extend to the case of inclusions, where these two functions are replaced by set-valued maps F and G.
  - ▷ With set-valued maps, special care must be paid to avoiding defective selections such as the fact that for some x the set $F(x)$ possibly be unbounded or empty.

- We have then the following extension of Assumption 1.

Assumption 2 - Basic assumptions

The hybrid inclusion (10) satisfies the hybrid basic assumptions if sets $\mathcal{C}$ and $\mathcal{D}$ are closed and maps F and G satisfy the following properties:

1. maps F and G are locally bounded, namely for each compact set $\mathcal{K} \subset C \cup D$, both $F(\mathcal{K})$ and $G(\mathcal{K})$ are bounded;
2. the graph of F (namely the set of all pairs (y, x) such that $y \in F(x)$) and the graph of G are closed (this property corresponds to a generalization of continuity called *outer semicontinuity*);
3. $F(x)$ is nonempty for each $x \in C$ and $G(x)$ is nonempty for each $x \in D$;
4. $F(x)$ is convex for each $x \in C$.

- The four properties of F and G above, are a suitable generalization of the corresponding properties of continuity of f and g. For example, continuity implies that the graphs of f and g are closed sets.
- Then the extension of Theorem 3 on UGAS is as follows.

Theorem 3

If system (10) satisfies Assumption 2, then

$$\boxed{\text{UGAS of a compact set } \mathcal{A} \text{ is equivalent to GAS of } \mathcal{A}}$$

- We have studied the notion of stability (see Chapter 3 in the book of Goebel et al, for more details and examples).
- In particular, we considered in this lecture:
  - ▷ Lyapunov stability
  - ▷ global attractivity
  - ▷ uniform global attractivity
- We have studied the stability of a point (the origin).
- But due to the structure of Hybrid dynamical systems, we have seen that, stability and convergence to certain sets are fundamental.
- Rather than establishing asymptotic stability of an equilibrium (e.g., the origin) we characterize for a compact set $\mathcal{A}$
  - ▷ hybrid dynamical systems have often timers; logical variables and so on which do not necessarily converge to zero;
  - ▷ clearly, asymptotic stability of an equilibrium point becomes a special case.
- We have also seen that you need to have information about the solution of the system (complete, maximal).

- The elements provided in this lecture are the definition of stability and attractivity
- Trajectory-based stability analysis is possible because of the linear dynamics of the system.
- Main sources for this lecture: Chapter 3 of the book of Teel, notes of Luca Zaccarian, Christophe Prieur, Francesco Ferrante, Ricardo Sanfelice, ST
- Part IV: Lyapunov functions
  - ▷ We will see that the main tool needed to prove stability of the origin or a set $\mathcal{A}$ is the theory of Lyapunov.
  - ▷ In the following lecture, we will introduce Lyapunov tools which provide sufficient conditions for the UGAS of (nonlinear) hybrid systems.