# Representation and Analysis of Dynamical Systems

## 1h20 – without documentation

**The exercises are independent.**

### Exercise 1: Model and analysis of a pendulum

Consider a pendulum of mass $m > 0$, friction $c > 0$ and length $l > 0$ and (Figure 1) and its nonlinear mechanical equation:

$$ml\ddot{\theta} = -mg\sin\theta - cl\dot{\theta}$$

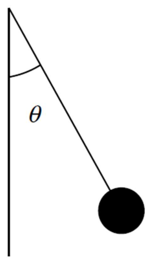



***Figure 1: Pendulum***

***Question 1.1:***

Choose appropriate state variables and write the state-space representation.

$$\begin{cases} x_1 = \theta \\ x_2 = \dot{\theta} \end{cases}$$

$$\begin{cases} 0 = x_2 \\ 0 = -\dfrac{g}{l}\sin(x_1) - \dfrac{k}{m}x_2 \end{cases}$$

***Question 1.2:***

Find all equilibrium points of the system.

$$(x_1 x_2) = (n\,\pi, 0) \text{ with } n = 0, \pm 1, \pm 2, \ldots$$

***Question 1.3:***

Linearize the system around the equilibrium points, and determine if the system equilibria are locally asymptotically stable.

$$\frac{d}{dt}\Delta x = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}(-1)^n & -\frac{k}{m} \end{bmatrix}\Delta x$$

### Exercise 2: Stability issue (the 3 questions are independent)

Consider the open-loop transfer function of the system:

$$F(s) = \frac{X(s)}{U(s)} = \frac{10}{(s+1)\left(\frac{s^2}{100} + \frac{s}{10} + 1\right)}$$

***<u>Question 2.1:</u>***

Draw the approximate Bode diagrams of the system.

phase must reach -270°

frequencies $\omega_1 = 1$ and $\omega_2 = 10$ must be seen

damping >1 means no resonnant mode

***<u>Question 2.2:</u>***

The phase F(s) is equal to -135° for ω = 7 rad/s.

Compute the value of the gain Kc to obtain a controlled system with a phase margin of 45°.

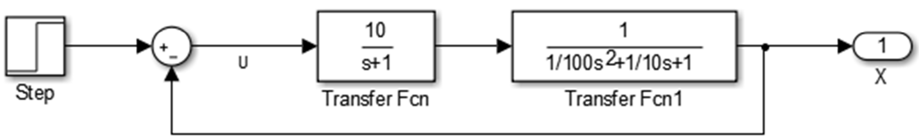
Gain for ω = 7 rad/s = 1.63 => Kc = 1/1.63 = 0.612

***<u>Question 2.3:</u>***

Consider Kc = 1 (this is not the answer to the previous question).

Compute the static error of the closed-loop system (see Figure 2) in response to a unitary step.

error = 1/(1+Kc*Kstat)=1/11



### Exercise 3: Control

Consider the system known by its transfer function:

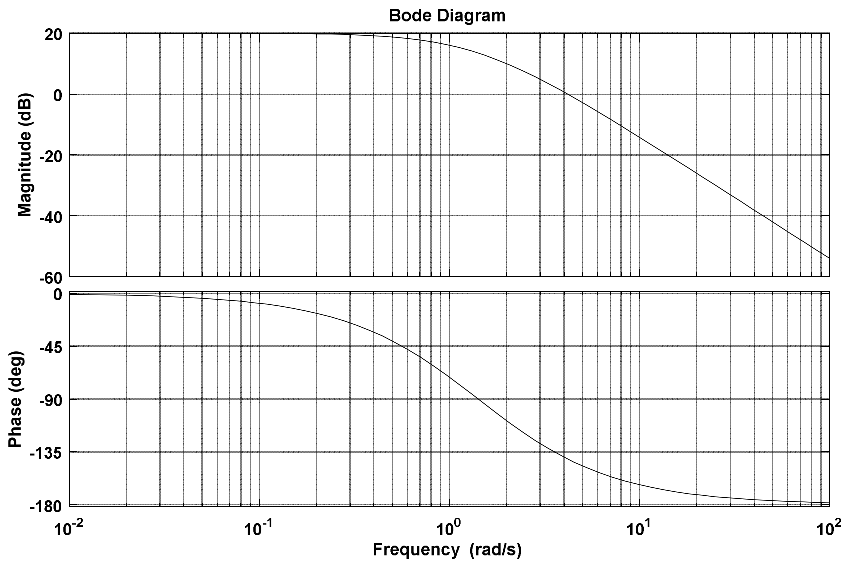
$$F(s) = \frac{Y(s)}{U(s)} = \frac{200}{s^3 + 13\,s^2 + 32\,s + 20}$$

The final goal is to design a controller in order to achieve a closed loop bandwidth of approximately $4\,rad/s$

***Question 3.1:***

The first goal is to design a proportional controller $C_1(s) = k_p$

We first approximate $F(s)$ by $F_a(s) = \frac{20}{s^2+3\,s+2}$ . The Bode diagram of $F_a(s)$ is given below:




3.1.1 Explain why $F(s)$ can be approximated by $F_a(s)$ for the design of the closed loop proportional controller.

neglect the fast dynamic
fast dynamic faster than expected closed loop performance

3.1.2 Which value of $k_p$ fulfills the requirement of a closed loop bandwidth of approximately $4\,rad/s$

$$k_p = 1$$

3.1.3 Demonstrate that the closed loop system ($C_1(s)$ and $F_a(s)$) can be approximated by a second order with natural frequency $\omega_0 \approx 4{,}6\,rad/s$ and damping $\sigma \approx 0.3$

$$CL = \frac{k_p F_a(s)}{1 + k_p F_a(s)} = \frac{20}{s^2 + 3\,s + 22} \approx \frac{10}{s^2 + 2 \times 0.3 \times 3.3 \times s + 4.6^2}$$

$\omega_0 \approx 4.6\,;\ \sigma \approx 0.3$

3.1.4 Explain why this controller with the real system ($C_1(s)$ and $F(s)$) will result to a much less damped controlled system

because phase is reduced (see Bode diagram)

***Question 3.2***

The proportional controller $C_1(s) = k_p$ is replaced by $C_2(s) = k_p \frac{1+s/4}{1+s/8}$

Explain why this controller will improve the damping of the closed loop system and won't affect that much the bandwidth.

Draw approximate Bode diagram. Show that phase ↗ and magnitude not much affected