2MAE404 MIMO control
Homework report 4

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1 Exercise 1 – Frequency domain models

1.1 System transfer matrix

In homework no. 3, the linearised dynamcis of the the matrices of the Flying Chardonnay drone around the equilibrium point corresponding to static hovering were considered:

$$\dot{x}(t) = A\Delta x + B\Delta u + E\Delta w = Ax + B(u - u_0) + Ew \tag{1}$$

The matrices were computed using the complex step method:

$$A = \begin{bmatrix} -0.01 & 0 & -10 & 0 & 10 & 0 \\ 0 & -0.005 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -0.01 & 0 & 0 & 0 & 20 & 0 \end{bmatrix} \tag{2}$$

$$B = \begin{bmatrix} 0 & 0 \\ -0.5 & -0.5 \\ 0 & 0 \\ -1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} \tag{3}$$

$$E = \begin{bmatrix} 0.01 & 0 \\ 0 & -0.005 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.01 & 0 \end{bmatrix} \tag{4}$$

Since the computation of the transfer matrix by hand would be a very time-consuming and error-prone task, the `tf` MATLAB function was used instead:

```
C = eye(6);
D = zeros(6,2);
sys = ss(A,B,C,D);
tf(sys)
```

The `tf` function is prone to numerical errors resulting in imperfectly cancelled out pole-zero pairs (also, some values in the A and B matrix were not exactly zero, also due to numerical error). However, the erroneous terms in the resulting transfer functions were quite easy to identify due to values many orders of magnitude smaller than the rest, e.g.:

$$H_{6,1}(s) = \frac{-s^4 - 0.01s^3 + 3.803 \cdot 10^{-15}s^2 + 0.1s + 3.938 \cdot 10^{-32}}{s^5 + 0.01s^4 - 20s^3 - 0.1s^2 - 6.776 \cdot 10^{-14}s - 3.388 \cdot 10^{-16}} \tag{5}$$

The simplification consists of ommiting the very small terms and reducing redundant pole-zero pairs:

$$H_{6,1}(s) \approx \frac{-s^4 - 0.01s^3 + 0.1s}{s^5 + 0.01s^4 - 20s^3 - 0.1s^2} = \frac{-s^3 - 0.01s^2 + 0.1}{s^4 + 0.01s^3 - 20s^2 - 0.1s} \tag{6}$$

The full transfer matrix, computed as the example above, is the following:

$$H(s) = \begin{bmatrix} \frac{20s^2-200}{s^5+0.01s^4-20s^3-0.1s^2} & \frac{-20s^2+200}{s^5+0.01s^4-20s^3-0.1s^2} \\ \frac{-0.5}{s+0.005} & \frac{-0.5}{s+0.005} \\ \frac{-1}{s^2} & \frac{1}{s^2} \\ \frac{-1}{s} & \frac{1}{s} \\ \frac{s^3+0.01s^2-0.1}{s^5+0.01s^4-20s^3-0.1s^2} & \frac{-s^3-0.01s^2+0.1}{s^5+0.01s^4-20s^3-0.1s^2} \\ \frac{-s^3-0.01s^2+0.1}{s^4+0.01s^3-20s^2-0.1s} & \frac{-s^3-0.01s^2+0.1}{s^4+0.01s^3-20s^2-0.1s} \end{bmatrix} \tag{7}$$

It is worth noting that the columns differ only by the sign of some of the terms. This is expected, since both inputs (i.e. thrusts) act on the system in the same way, differing only in induced pitching motion direction.

1.2 Disturbance transfer matrix

The next point is finding the disturbance matrix, the disturbance to the system dynamics being the wind vector w. The contribution of the w vector to $\dot{x}$ is Ew, as highlighted in Equation 1 and Figure 1. However, the assignment asks for the transfer matrix between w and x rather than $\dot{x}$. Luckily, thanks to the linearity of the system dynamics, this can be easily overcome by appending an integrator to the E matrix and adding it to the already integrated x signal rather than $\dot{x}$, as shown in Figure 2. The resulting transfer matrix is simply:

$$G_d(s) = \frac{1}{s}E = \frac{1}{s}\begin{bmatrix} 0.01 & 0 \\ 0 & -0.005 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.01 & 0 \end{bmatrix} = \begin{bmatrix} \frac{0.01}{s} & 0 \\ 0 & -\frac{0.005}{s} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{0.01}{s} & 0 \end{bmatrix} \tag{8}$$

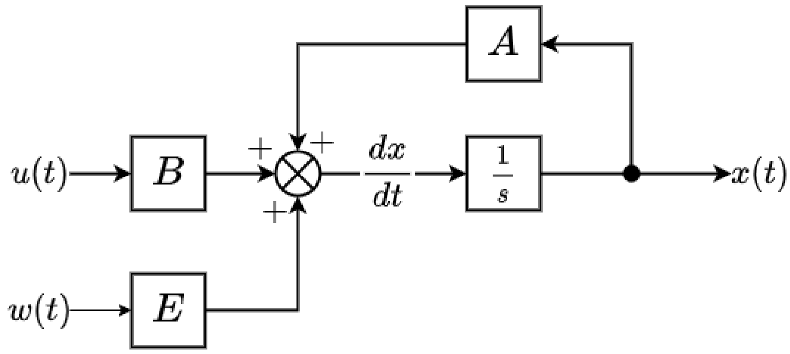

Figure 1: Diagram of the linearised dynamics of the Flying Chardonnay.

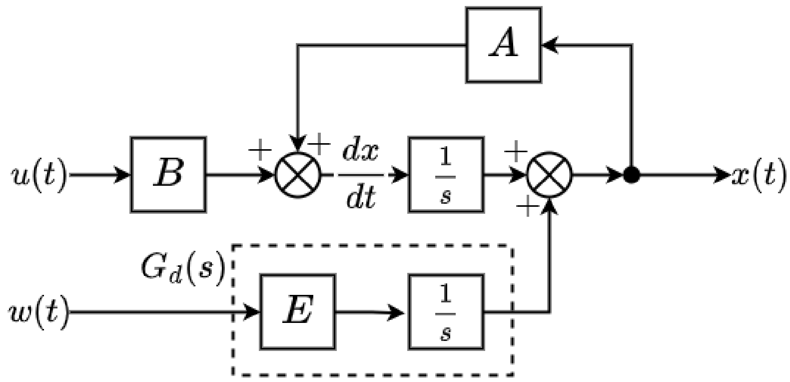

Figure 2: Equivalent diagram of the linearised dynamics of the Flying Chardonnay with the disturbance transfer function separated.

2 Excercise 2 – Linear model representations

2.1 Impulse matrix

The following linear state-space system matrices are given:

$$A = \begin{bmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{bmatrix} \tag{9}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{10}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{11}$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{12}$$

The equivalent Impulse Matrix $H(t)$ can be computed using the formula:

$$H(t) = Ce^{At}B + D\delta(t) = Ce^{At}B \tag{13}$$

The matrix exponential is defined using the Taylor series:

$$e^{At} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} \tag{14}$$

The first few powers of the A matrix, appearing in the terms of the Taylor expansion, are the following:

$$A^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{15}$$

$$A^1 = \begin{bmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{bmatrix} \tag{16}$$

$$A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{17}$$

It can be seen that the A matrix has a peculiar property – A^n is a zero matrix for $n \geq 2$. Therefore, the matrix exponential consists only of 2 (rather than infinitely many) non-zero terms:

$$e^{At} = \frac{t^0 A^0}{0!} + \frac{t^1 A^1}{1!} = I + At \tag{18}$$

The numerical values are therefore:

$$e^{At} = I + At = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 5t & -3t & 2t \\ 15t & -9t & 6t \\ 10t & 6t & 4t \end{bmatrix} = \begin{bmatrix} 1+5t & 3t & 2t \\ 15t & 1-9t & 6t \\ 10t & 6t & 1+4t \end{bmatrix} \tag{19}$$

The impulse matrix is therefore:

$$H(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1+5t & 3t & 2t \\ 15t & 1-9t & 6t \\ 10t & 6t & 1+4t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1+3t & -3t \\ 9t & 1-9t \end{bmatrix} \tag{20}$$

2.2 Markov parameters

The transfer matrix $H(s)$ (which is simply the Laplace transform of the impulse matrix $H(t)$) can be expressed using the following Laurent series:

$$H(s) = \mathcal{L}\left[H(t)\right] = \sum_{k=0}^{\infty} \frac{H_k}{s^k} = H_0 + \frac{H_1}{s} + \frac{H_2}{s^2} + ... \tag{21}$$

The H_k terms are called the **Markov parameters** of the system. They can be computed using the following equations:

$$H_0 = D \tag{22}$$

$$H_k = CA^{k-1}B \tag{23}$$

It is worth recalling that in the analysed system, A^n is a zero matrix for $n \geq 2$, and the D matrix is also zero, meaning that only 2 Markov parameters shall be non-zero:

$$H_1 = CB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{24}$$

$$H_2 = CAB = \begin{bmatrix} 3 & -3 \\ 9 & -9 \end{bmatrix} \tag{25}$$

2.3 Transfer matrix

The transfer matrix can be directly computed using the Markov parameters calculated before:

$$H(s) = \frac{H_1}{s} + \frac{H_2}{s^2} = \frac{1}{s}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{s^2}\begin{bmatrix} 3 & -3 \\ 9 & -9 \end{bmatrix} = \frac{1}{s^2}\begin{bmatrix} s+3 & -3 \\ 9 & s-9 \end{bmatrix} \tag{26}$$

2.4 Numerical transfer matrix

Since an arbitrary system can have infinitely many non-zero Markov parameters, the computation of the transfer matrix $H(s)$ based on these parameters is often not practical. However, the transfer matrix can be computed in MATLAB, using the `tf` function:

```
sys = ss(A,B,C,D);
tf(sys)
```

The result given by MATLAB is, as explained before in Excercise 1, "polluted" with small terms resulting from numerical errors:

$$H(s) \approx \frac{\begin{bmatrix} s^2 + 3s - 1.228 \cdot 10^{-14} & -3s + 1.469 \cdot 10^{-15} \\ 9s - 3.118 \cdot 10^{-14} & s^2 - 9s \end{bmatrix}}{s^3 + 4.125 \cdot 10^{-15}s^2 - 6.084 \cdot 10^{-14}s + 6.652 \cdot 10^{-29}} \tag{27}$$

The result can be simplified by ommiting the very small terms:

$$H(s) \approx \frac{1}{s^3}\begin{bmatrix} s^2 + 3s & -3s \\ 9s & s^2 - 9s \end{bmatrix} = \frac{1}{s^2}\begin{bmatrix} s+3 & -3 \\ 9 & s-9 \end{bmatrix} \tag{28}$$

The resulting matrix is identical to the one obtained in Equation 26, thus validating both methods of computation. It is important to note that results obtained using the `tf` function or similar methods can usually benefit from thresholding out the small non-physical terms, since they artificially increase the dimensionality of the system.

3 Excercise 3 – Noncommutative property of MIMO systems

Two transfer matrices, $G_1(s)$ and $G_2(s)$, are considered:

$$G_1(s) = \begin{bmatrix} 0 & \frac{-1}{s} \\ 0 & \frac{2}{s} \end{bmatrix} \tag{29}$$

$$G_2(s) = \begin{bmatrix} \frac{3}{s} & \frac{5}{s} \\ 0 & 0 \end{bmatrix} \tag{30}$$

The products $G_1(s)G_2(s)$ and $G_2(s)G_1(s)$ are:

$$G_1(s)G_2(s) = \begin{bmatrix} 0 & \frac{-1}{s} \\ 0 & \frac{2}{s} \end{bmatrix} \begin{bmatrix} \frac{3}{s} & \frac{5}{s} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{31}$$

$$G_2(s)G_1(s) = \begin{bmatrix} \frac{3}{s} & \frac{5}{s} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{-1}{s} \\ 0 & \frac{2}{s} \end{bmatrix} = \begin{bmatrix} 0 & \frac{7}{s^2} \\ 0 & 0 \end{bmatrix} \tag{32}$$

It can be seen that the products $G_1(s)G_2(s)$ and $G_2(s)G_1(s)$ are different. This illustrates that transfer matrices are not commutative (matrix multiplication in general is not commutative). This also implies that when the two systems are connected in series, the resulting dynamics depend on the order of this connection.

In a parallel connection, the outputs of both transfer matrices are summed. Since the transfer functions within them are linear, this is equivalent to the summation of the matrices themselves. Therefore, $G_1(s) + G_2(s)$ and $G_2(s) + G_1(s)$ are the same:

$$G_1(s) + G_2(s) = G_2(s) + G_1(s) = \begin{bmatrix} \frac{3}{s} & \frac{4}{s} \\ 0 & \frac{2}{s} \end{bmatrix} \tag{33}$$

This illustrates that in a parallel connection of MIMO systems, order does not matter.