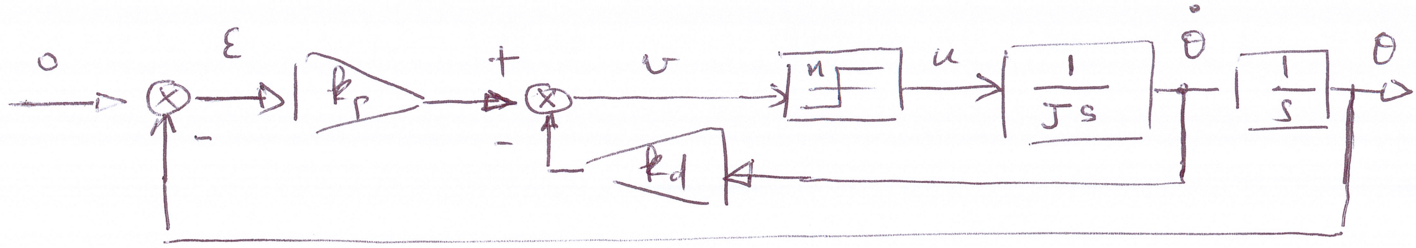
1

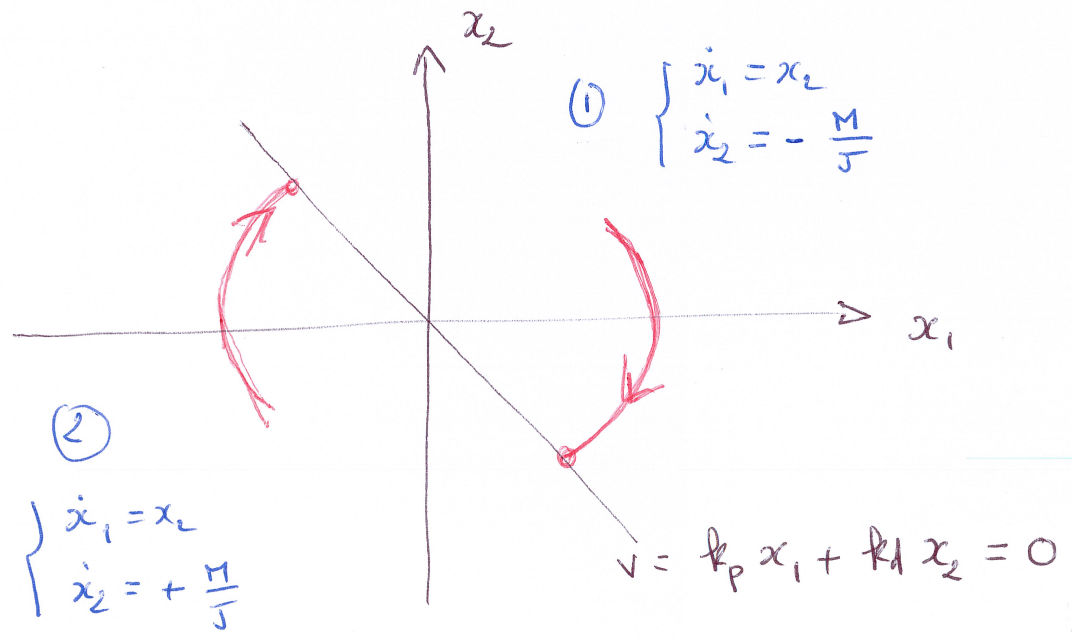


$$\begin{cases} x_1 = \varepsilon = -\theta \\ x_2 = \dot\varepsilon = -\dot\theta \end{cases} \qquad J\ddot\theta = u \quad \Rightarrow \quad \dot x_2 = -\frac{1}{J}u$$

$$u = M\,\mathrm{sign}(v)$$

$$v = k_p\varepsilon - k_d\dot\theta = k_p\varepsilon + k_d\dot\varepsilon = k_p x_1 + k_d x_2$$

$$\Rightarrow \boxed{\begin{cases} \dot x_1 = x_2 \\ \dot x_2 = -\frac{M}{J}\,\mathrm{sign}(k_p x_1 + k_d x_2) \end{cases}}$$



Let us integrate in R1 :

$$\dot x_2 = -\frac{M}{J} \Rightarrow x_2 = x_{20} - \frac{M}{J}t \Rightarrow t = J\,\frac{x_{20} - x_2}{M}$$

$$\dot x_1 = x_{20} - \frac{M}{J}t \Rightarrow x_1 = x_{10} + x_{20}t - \frac{M}{2J}\left(\frac{J}{M}(x_{20} - x_2)\right)^2$$

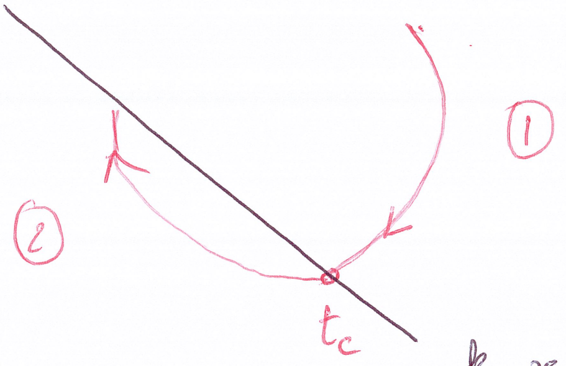
$$\Rightarrow x_1 = x_{10} + x_{20}\frac{J}{M}(x_{20} - x_2) - \frac{J}{2M}(x_{20} - x_2)^2$$

$$\boxed{x_1 = x_{10} + \frac{J}{2M}\left(x_{20}^2 - x_2^2\right)}$$

(2)

$\Rightarrow \begin{cases} \text{Parabolic trajectories} \\ \text{Axis : } x_2 = 0 \end{cases}$ Maximum for $\underline{x_2 = 0}$

Connecting trajectories



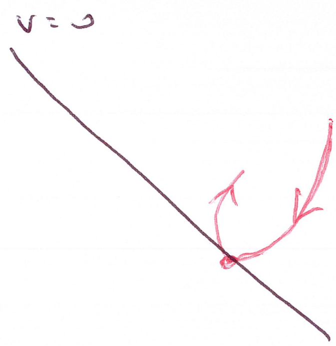
$k_p x_1 + k_d x_2 = 0 \quad = V$

At time t_c the trajectory from R1 reaches the line $k_p x_1 + k_d x_2 = 0$

$\begin{cases} \text{Before commutation : } \dot{x}_2(t_c^-) = -\frac{M}{J} \\ \text{After commutation : } \dot{x}_2(t_c^+) = +\frac{M}{J} \end{cases}$

$\Rightarrow$ discontinuity on $\dot{x}_2$

This discontinuity could make commutation impossible if the trajectory in R2 immediately goes back in R1



Note that before commutation we have

$(t_c^-) = k_p x_1(t_c^-) + k_d x_2(t_c^-) > 0$

The commutation is possible iff after commutation we have:

$$V(t_c^+) < 0$$

$\Rightarrow$ V must be a decreasing function at t_c

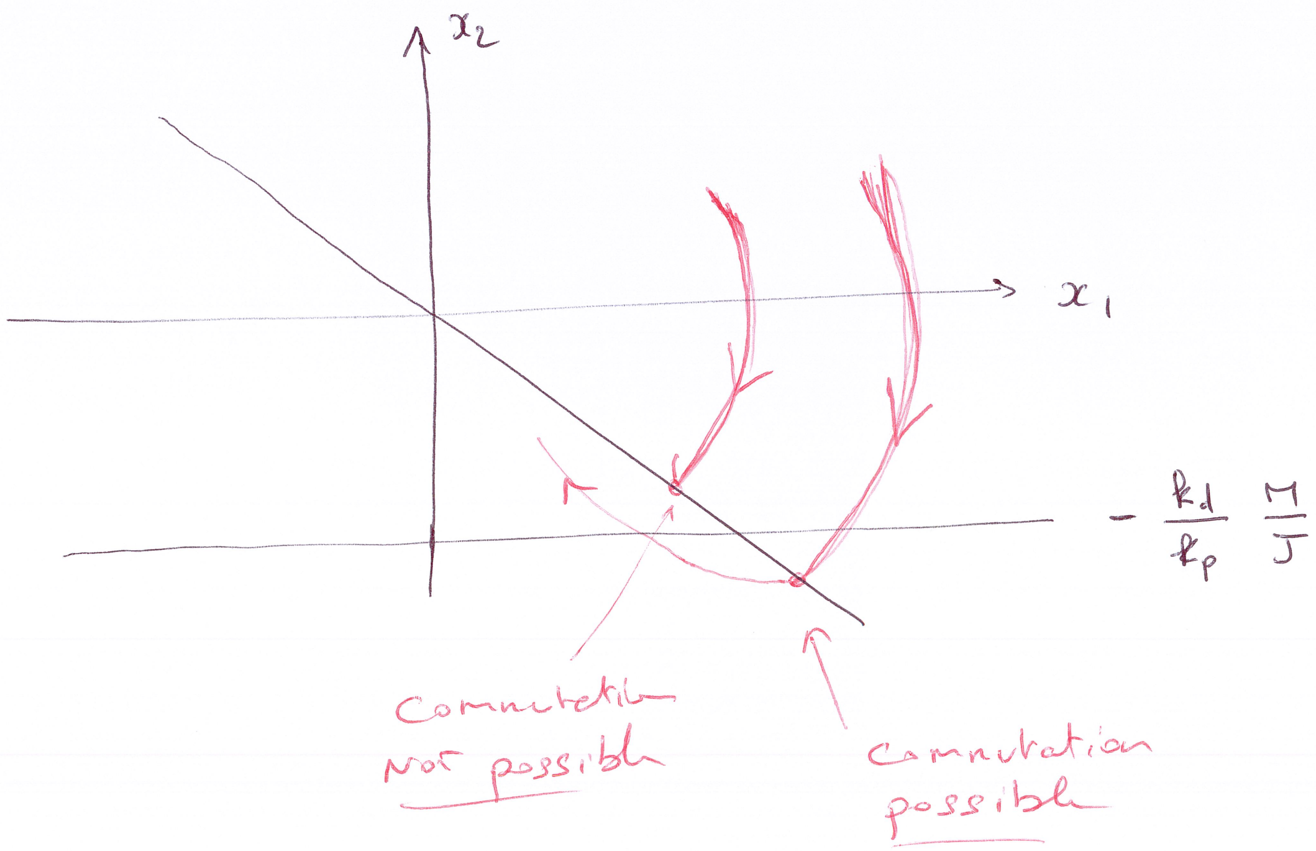
$\Rightarrow$ $\dot{V}(t_c^+) < 0$

We have $\dot{V}(t_c^+) = k_p \underbrace{\dot{x}_1(t_c^+)}_{x_2(t_c^+)} + k_d \underbrace{\dot{x}_2(t_c^+)}_{\frac{M}{J}}$

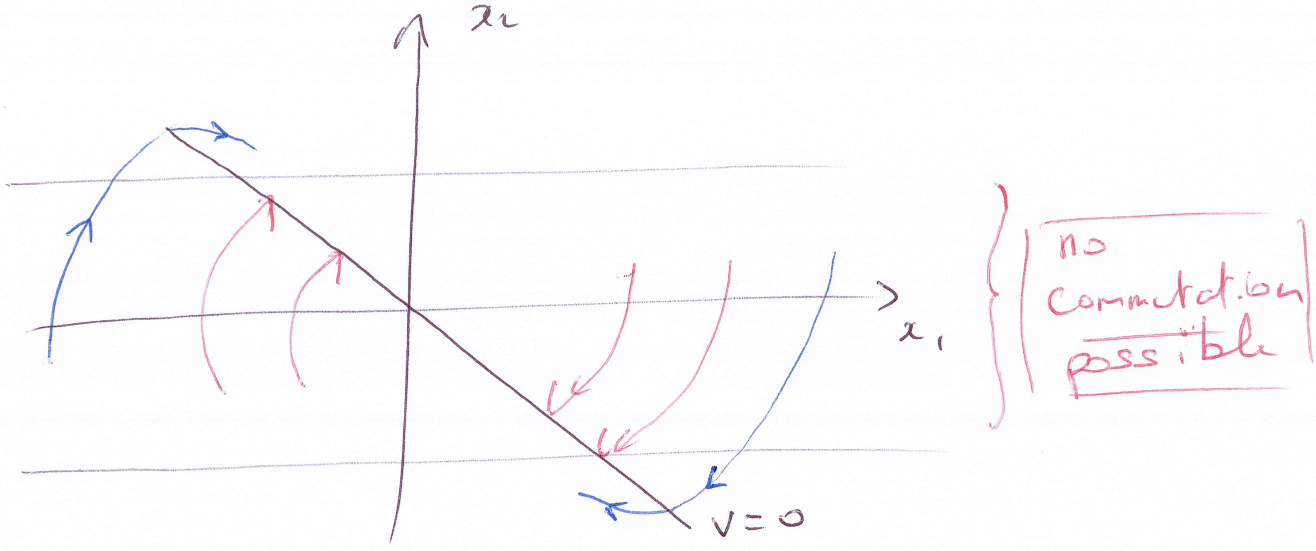
Rem: $x_2(t_c^+) = x_2(t_c) = x_{2c}$

$\Rightarrow k_p \, x_{2c} + k_d \, \frac{M}{J} < 0$

$$\boxed{x_{2c} < -\frac{k_d}{k_p}\frac{M}{J}}$$

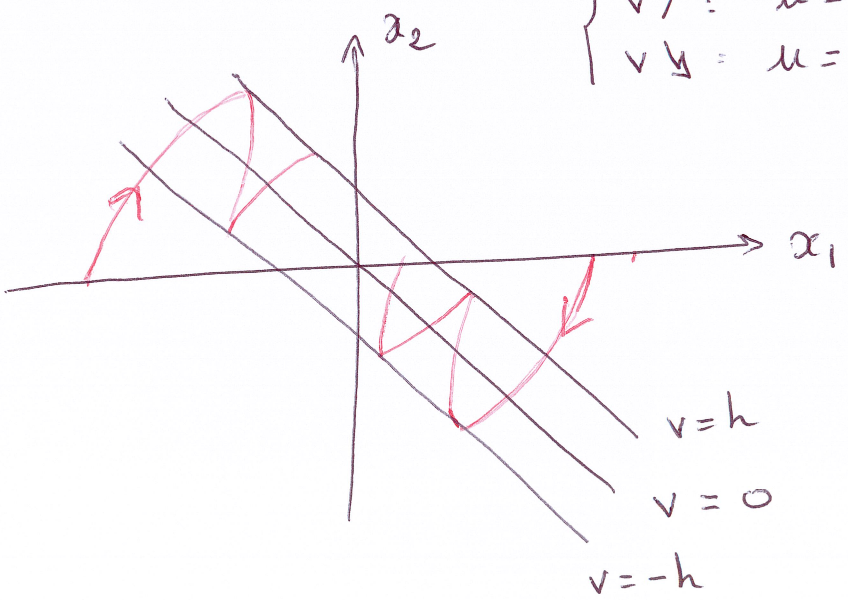


By symetry we have :



However, in practice the relay is not perfect and there is some <u>hysteresis</u>.

$$\begin{cases} v \nearrow : & u = 1 \text{ if } v > h \\ v \searrow : & u = -1 \text{ if } v < -h \end{cases}$$



commutation is then <u>possible</u>

h very small. $\Rightarrow$ we observe <u>sliding</u> on $v = k_p x_1 + k_d x_2 \simeq 0$

more precisely : $|v| \leq h$

during the sliding mode we have:

$$k_p x_1 + k_d x_2 = k_p \varepsilon + k_d \dot{\varepsilon} \simeq 0$$

$$\Rightarrow \boxed{\dot{\varepsilon} = -\frac{k_p}{k_d}\varepsilon} \Leftrightarrow \boxed{\dot{\varepsilon} = -\frac{1}{\tau}\varepsilon, \quad \tau = \frac{k_d}{k_p}}$$

$\varepsilon \longrightarrow 0$ independently of J !

$\Rightarrow$ robust design.

$\Rightarrow$ It is then interesting to enforce sliding condition through the maximization of: $\frac{k_d}{k_p}\frac{M}{J}$

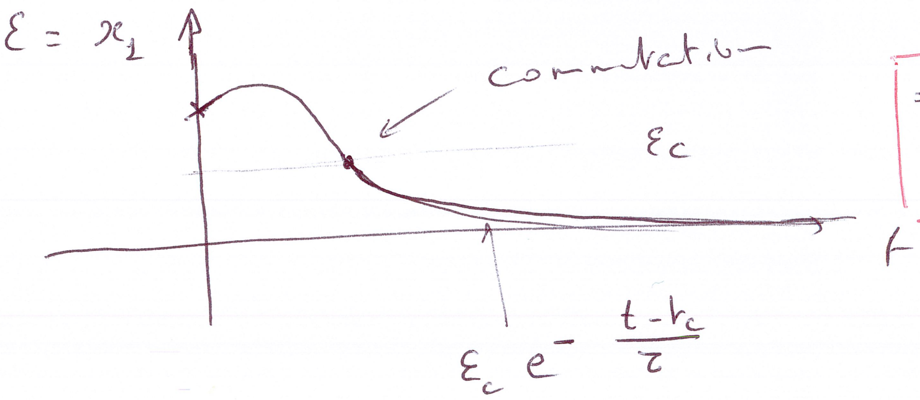
$\Rightarrow$ 2 options:
- augment M
- augment $\frac{k_d}{k_p}$

* option 1 requires enhanced actuator: might not be possible

- option 2: augment $\frac{k_d}{k_p} = \tau$

  $\hookrightarrow$ this will slow down the sliding dynamics since $\tau \nearrow$

$\Rightarrow \boxed{\text{compromise}}$

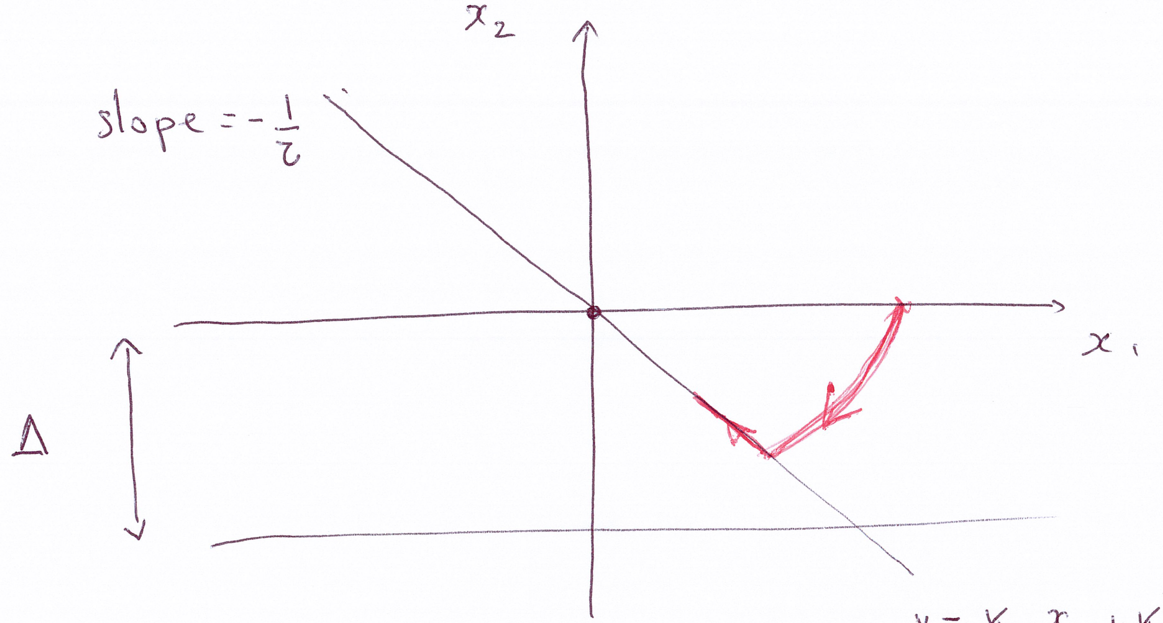


$\Rightarrow$ variable k_p, k_d

Variable K_p, K_d

- We initial choose K_p & K_d to enforce sliding conditions on a large domain.

$\Rightarrow$ high value for $\tau = \frac{K_d}{K_p}$



$v = K_p x_1 + K_d x_2 = 0$

$\Leftrightarrow x_2 = -\frac{K_p}{K_d} x_1 = -\frac{1}{\tau} x_1$

$$\Delta = \frac{K_d}{K_p} \frac{M}{J} = \tau \frac{M}{J}$$

- We update $\tau = \frac{K_d}{K_p}$ once sliding has started : τ is decreased progressively from τ_0 to τ_F