Modeling and control of flexible structures

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This document contains:

- a copy of the slides used during this course,
- the statement of the labworks,
- an example of a written examination.

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Modeling and Control of Flexible Mechanical Systems.

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Outline I

Outline II

6. MATLAB/SIMULINK labworks

Outline

1. Dynamic modeling of flexible mechanical systems
   - LAGRANGE equations
   - Effective masses approach
   - Application to the 6 d.o.fs connexion of a flexible appendage on a rigid body
2. Positivity and actuator/sensor collocation
3. Control of flexible systems
4. Model (or controller) reduction
5. Applications : dynamic isolation of a space telescope
6. MATLAB/SIMULINK labworks

Outline

Lagrange equations

Considering a mechanical system (multi-body system with rigid and/or flexible links), LAGRANGE approach consists in derived :

- select the vector $\mathbf{q} \in \mathbb{R}^n$ of the n kinematic variables. $\mathbf{q}$ is the list of the variables to be known to describe the geometric "shape" (or configuration) of the system at each time t, for instance : joint displacements for multi-body systems and/or mesh node displacements for a flexible body (using a mesh tool),
- express the kinetic energy : $T = T(\mathbf{q}, \dot{\mathbf{q}})$,
- express the potential energy : $V = V(\mathbf{q})$,
- express the Lagrangian $L = T - V$ and compute the LAGRANGE derivation : for each component $q_i = \mathbf{q}(i)$, $i = 1, \cdots, n$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L((\mathbf{q},\dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L(\mathbf{q},\dot{\mathbf{q}})}{\partial \mathbf{q}} = \mathbf{F}_g \ .} \qquad (1)$$

$\mathbf{F}_g(i)$ is the generalized force (or torque) working along q_i ; $\mathbf{F}_g \in \mathbb{R}^n$
$\mathbf{F}(i) = \frac{\partial W_{ext}}{\partial \mathbf{q}(i)}$ where W_{ext} is the work of the **external** forces (and torques). In the case of vibration analysis around a nominal configuration $\mathbf{q}_0$ (i.e. $\mathbf{q} = \mathbf{q}_0 + \delta\mathbf{q}$), displacements (position variation $\delta\mathbf{q}$ and rate $\delta\dot{\mathbf{q}}$) are assumed to be small. Quadratic (and higher order terms) in $\delta\dot{\mathbf{q}}(i)^2$, $\delta\dot{\mathbf{q}}(i)\delta\dot{\mathbf{q}}(j)$, $\delta\mathbf{q}(i)\delta\dot{\mathbf{q}}(j)$, $\cdots$ are neglected in the expression of displacements and velocities (involved to compute T and V). Then, T and V can be expressed as quadratic forms of $\delta\dot{\mathbf{q}}$ and $\delta\mathbf{q}$, respectively :

- $T = \frac{1}{2}\delta\dot{\mathbf{q}}^T\mathbf{M}(\mathbf{q}_0)\delta\dot{\mathbf{q}}$ where $\mathbf{M}(\mathbf{q}_0)_{n\times n}$ is the **mass matrix** (DP),
- $V = \frac{1}{2}\delta\mathbf{q}^T\mathbf{K}(\mathbf{q}_0)\delta\mathbf{q}$ where $\mathbf{K}(\mathbf{q}_0)_{n\times n}$ is the **stiffness matrix** (SDP).

Simplified notations : $\mathbf{q}_0 = \mathbf{0}$, δ is omitted, $\mathbf{M}(\mathbf{q}_0) \to \mathbf{M}$ and $\mathbf{K}(\mathbf{q}_0) \to \mathbf{K}$.
Then the LAGRANGE derivation (1) is obvious and reads :

$$\boxed{\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F}} \,. \tag{2}$$

Modal analysis

The **modal analysis** consists in finding the general solution of the differential equation (2) without the right-hand term ($F = 0$ that is to say : the free response without external sollicitations) :

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0} \,. \tag{3}$$

It can be shown that this solution, at each time t, is a combination of n harmonic responses :

$$\mathbf{q}(t) = \sum_{i=1}^{n} \boldsymbol{\phi}_i \, \lambda_i \mathrm{e}^{j(\omega_i t + \varphi_i)} \ , \quad \forall t$$

- $\omega_i \in \mathbb{R}$ is the **frequency** of mode $\#\ i$,
- $\boldsymbol{\phi}_i \in \mathbb{R}^n$ is the **modal shape** of mode $\#\ i$,
- $\lambda_i \in \mathbb{R}$ and $\varphi_i \in \mathbb{R}$ are the gain and the phase lag of mode $\#\ i$ (not important : depend on initial conditions $\mathbf{q}(t=0)$ and $\dot{\mathbf{q}}(t=0)$).

Computation of characteristic parameters ω_i and ϕ_i :
ω_i and ϕ_i are solutions of the eigenvalue/eigenvector problem :

$$\boxed{(\mathbf{M}\omega_i^2 - \mathbf{K})\phi_i = \mathbf{0}\ , \quad \forall\, i}\ . \tag{4}$$

a) solve in ω^2 the characteristic equation :

$$\det(\mathbf{M}\omega^2 - \mathbf{K}) = 0 \ \Rightarrow\ n \text{ solutions } \omega_i^2\ , \quad i = 1, \cdots, n\ ,$$

b) for each ω_i, find the eigen-vector ϕ_i such that (4) holds.

Modal coordinates representation : Let us perform the mapping $\mathrm{q} = \mathbf{\Phi}\boldsymbol{\eta}$ in (2) where :

- $\mathbf{\Phi} = [\phi_1, \phi_2, \cdots, \phi_n]$ is the modal shapes matrix,
- $\boldsymbol{\eta} = [\eta_1, \eta_2, \cdots, \eta_n]^T$ is the modal coordinates vector,

then $\mathbf{M}\mathbf{\Phi}\,\ddot{\boldsymbol{\eta}} + \mathbf{K}\mathbf{\Phi}\,\boldsymbol{\eta} = \mathbf{F}$ and pre-multiplying by $\mathbf{\Phi}^T$:

$$\mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi}\,\ddot{\boldsymbol{\eta}} + \mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi}\,\boldsymbol{\eta} = \mathbf{\Phi}^T\mathbf{F}\ . \tag{5}$$

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The mapping $\mathbf{\Phi}$ simultaneously diagonalizes matrices $\mathbf{M}$ and $\mathbf{K}$:

- $\mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi} = \mathrm{diag}([m_1, \cdots, m_i, \cdots, m_n])$: m_i is the modal mass of mode $\#\ i$,
- $\mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi} = \mathrm{diag}([k_1, \cdots, k_i, \cdots, k_n])$: $k_i = m_i\omega_i^2$ is the modal stiffness of mode $\#\ i$.

$\Rightarrow$ (5) reads also : $\forall\, i, \quad m_i(\ddot{\eta}_i + \omega_i^2\eta_i) = \phi_i^T\mathbf{F}$.
Damping modelling : Damping in flexible structures is still an open problem. A practical assumption (BASILE assumption commonly adopted) considers an arbitrary damping coefficient ξ (5% and 0.5% in aeronautical and space engineering, respectively) for all the modes in the modal representation :

$$\forall\, i, \quad m_i(\ddot{\eta}_i + 2\xi\omega_i\dot{\eta}_i + \omega_i^2\eta_i) = \phi_i^T\mathbf{F}\ . \tag{6}$$

$\Rightarrow$ $\mathbf{\Phi}$ diaganalyses also the damping matrix D :

$$\mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi}\,\ddot{\boldsymbol{\eta}} + \mathbf{\Phi}^T\mathbf{D}\mathbf{\Phi}\,\dot{\boldsymbol{\eta}} + \mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi}\,\boldsymbol{\eta} = \mathbf{\Phi}^T\mathbf{F}\ .$$

Then (2) becomes :

$$\boxed{\mathbf{M}\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F}}. \tag{7}$$

State-space representation

The model (7) is a $2n$ order differential equation.
Assuming that generalized forces are provided by m actuators (input vector $\mathbf{u} \in \mathbb{R}^m$) distibuted at m points (nodes) in the structure, then $\mathbf{F}$ can be expressed : $\boxed{\mathbf{F} = \mathcal{B}\mathbf{u}}$ where $\boldsymbol{\mathcal{B}} = [\boldsymbol{\mathcal{B}}_1, \cdots, \boldsymbol{\mathcal{B}}_m]$ ($\boldsymbol{\mathcal{B}} \in \mathbb{R}^{n\times m}$) is the participation matrix of the input $\mathbf{u}$ on the vector $\mathbf{q}$ ($W_{ext} = \mathbf{q}^T \boldsymbol{\mathcal{B}}\mathbf{u}$).
In the same way one can consider some position outputs $\mathbf{y}^P \in \mathbb{R}^{p_P}$ or velocity output $\mathbf{y}^V \in \mathbb{R}^{p_V}$ through the projection matrices $\boldsymbol{\mathcal{C}}^P \in \mathbb{R}^{p_P\times n}$ and $\boldsymbol{\mathcal{C}}^V \in \mathbb{R}^{p_V\times n}$ of $\mathbf{q}$ and $\dot{\mathbf{q}}$, respectively ($\mathbf{y}^P = \boldsymbol{\mathcal{C}}^P\mathbf{q}$ and $\mathbf{y}^V = \boldsymbol{\mathcal{C}}^V\dot{\mathbf{q}}$).
Selecting $\mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix}$ for the state vector, a state space representation of transfert between $\mathbf{u}$ and $\mathbf{y} = [\mathbf{y}^{P^T}, \mathbf{y}^{V^T}]^T$ is :

$$\left[\begin{array}{c} \dot{\mathbf{x}} \\ \hline \mathbf{y}^P \\ \mathbf{y}^V \end{array}\right] = \left[\begin{array}{cc|c} \mathbf{0}_{n\times n} & \mathbf{1}_n & \mathbf{0}_{n\times m} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} & \mathbf{M}^{-1}\boldsymbol{\mathcal{B}} \\ \hline \boldsymbol{\mathcal{C}}^P & \mathbf{0}_{p_P\times n} & \mathbf{0}_{p_P\times m} \\ \mathbf{0}_{p_V\times n} & \boldsymbol{\mathcal{C}}^V & \mathbf{0}_{p_V\times m} \end{array}\right] \left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{u} \end{array}\right] . \qquad (8)$$

State-space representation

From the modal coordinates representation (6), another (block-diagonal) state space representation is :

$$\left[\begin{array}{c} \dot{\widetilde{\mathbf{x}}} \\ \hline \mathbf{y}^P \\ \mathbf{y}^V \end{array}\right] = \left[\begin{array}{cccc|c} \mathbf{A}_1 & \mathbf{0}_{2\times 2} & \cdots & \mathbf{0}_{2\times 2} & \mathbf{B}_1 \\ \mathbf{0}_{2\times 2} & \mathbf{A}_2 & \ddots & \vdots & \mathbf{B}_2 \\ \vdots & \ddots & \ddots & \mathbf{0}_{2\times 2} & \vdots \\ \mathbf{0}_{2\times 2} & \cdots & \mathbf{0}_{2\times 2} & \mathbf{A}_n & \mathbf{B}_n \\ \hline \mathbf{C}_1^P & \mathbf{C}_2^P & \cdots & \mathbf{C}_n^P & \mathbf{0}_{p_P\times m} \\ \mathbf{C}_1^V & \mathbf{C}_2^V & \cdots & \mathbf{C}_n^V & \mathbf{0}_{p_V\times m} \end{array}\right] \left[\begin{array}{c} \widetilde{\mathbf{x}} \\ \hline \mathbf{u} \end{array}\right] . \qquad (9)$$

- $\widetilde{\mathbf{x}} = [\eta_1, \dot{\eta}_1, \eta_2, \dot{\eta}_2, \cdots, \eta_n, \dot{\eta}_n]^T$,
- $\mathbf{A}_i = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\xi\omega_i \end{bmatrix}$, $\mathbf{B}_i = \begin{bmatrix} \mathbf{0}_{1\times m} \\ 1/m_i \boldsymbol{\phi}_i^T \boldsymbol{\mathcal{B}} \end{bmatrix}$,
- $\mathbf{C}_i^P = [\boldsymbol{\mathcal{C}}^P \phi_i \quad \mathbf{0}_{p_P\times 1}]$, $\mathbf{C}_i^V = [\mathbf{0}_{p_V\times 1} \quad \boldsymbol{\mathcal{C}}^V \phi_i]$.

Rk : if $\boldsymbol{\phi}_i^T \boldsymbol{\mathcal{B}} = \mathbf{0}_{1\times m}$ then the mode $\#\, i$ is uncontrollable from $\mathbf{u}$ (this test is valid only if all frequencies ω_i are simple).

Transfer function of SISO flexible systems

Let us consider a Single Input Single Output conservative ($\xi = 0$) flexible system between the force input u and the position at a node N_o of the structure (output $y = \boldsymbol{\mathcal{C}}\mathbf{q}$), then :

$$F_P(\mathrm{s}) = \frac{Y(\mathrm{s})}{U(\mathrm{s})} = G\frac{N(\mathrm{s})}{\Pi_{i=1}^{n}(\mathrm{s}^2+\omega_i^2)} \ .$$

The poles $\pm j\omega_i$ (or the eigenvalues of A) can be identified from the modal analysis of $\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$. Such an analysis is a **real** modal analysis that can be done because the system is conservative ($\xi = 0$). If $\xi > 0$, then a **complex** modal analysis is required (out of the scope...).

The numerator $N(\mathrm{s})$ can be identified from a **asymptotic closed-loop (real) modal analysis** feedbacking the position y or the acceleration $\ddot{y}$ (but not $\dot{y}$) to u though an infinite loop gain.k.
The transfer from u to $\dot{y}$ (velocity of node N_o) is : $F_v(\mathrm{s}) = \mathrm{s}F_p(\mathrm{s})$.
The transfer from u to $\ddot{y}$ (acceleration of node N_o) is : $F_a(\mathrm{s}) = \mathrm{s}^2F_p(\mathrm{s})$.

Asymptotic closed-loop modal analysis

Indeed : $\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \boldsymbol{\mathcal{B}}u$.

- **Position feedback :** $u = -ky = -k\boldsymbol{\mathcal{C}}\mathbf{q}$ with $k \to \infty$ leads to the 2-nd order closed-loop system :

  $$\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{K} + k\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{C}})\mathbf{q} = \mathbf{0} \ .$$

  Real modal analysis :
  $\lim_{k\to\infty}\det(\mathbf{M}\omega^2 - \mathbf{K} - k\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{C}}) = \det(\mathbf{M})\Pi_{i=1}^{n}(\omega^2 - \bar{\omega}_i^2)$ with
  - $\bar{\omega}_i^2 \to \infty(a + jb)$ (asymptotic branches),
  - $\bar{\omega}_i^2 \to m(< n)$ finite values z_i^2 corresponding to the roots of $N(\mathrm{s})$ $\Rightarrow \boxed{N(\mathrm{s}) = \Pi_{i=1}^{m}(\mathrm{s}^2 + z_i^2)}$. $N(\mathrm{s})$ is also a function of s^2.

  This is the property of the root locus of the transfer $F_p(\mathrm{s})$ where the closed-loop poles are driven to ∞ or to the zeros of $N(\mathrm{s})$ when $k \to \infty$.

  The position feedback introduces active stiffness in the system.

- **Acceleration feedback :** $u = -k\ddot{y} = -k\mathcal{C}\ddot{\mathbf{q}}$ with $k \to \infty$ leads to the 2-nd order closed-loop system :

$$(\mathbf{M} + k\mathcal{B}\mathcal{C})\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}\ .$$

  Real modal analysis :
  $\lim_{k\to\infty} \det(\omega^2 - (\mathbf{M} + k\mathcal{B}\mathcal{C})^{-1}\mathbf{K}) = \Pi_{i=1}^{n}(\omega^2 - \bar{\omega}_i^2)$ with :
  - $\bar{\omega}_i^2 \to 0$ (with a pole/zero cancellation in case of a rigide mode),
  - $\bar{\omega}_i^2 \to \infty(a + jb)$ (asymptotic branches),
  - $\bar{\omega}_i^2 \to z_i^2$.

  The acceleration feedback introduces active mass in the system.

- **Velocity feedback :** $u = -k\dot{y} = -k\mathcal{C}\dot{\mathbf{q}}$ with $k \to \infty$ leads to the 2-nd order closed-loop system :

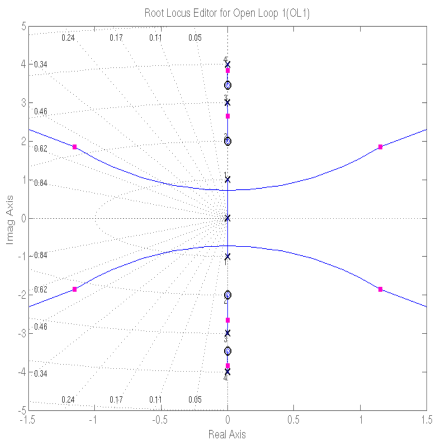
$$\mathbf{M}\ddot{\mathbf{q}} + k\mathcal{B}\mathcal{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}\ .$$

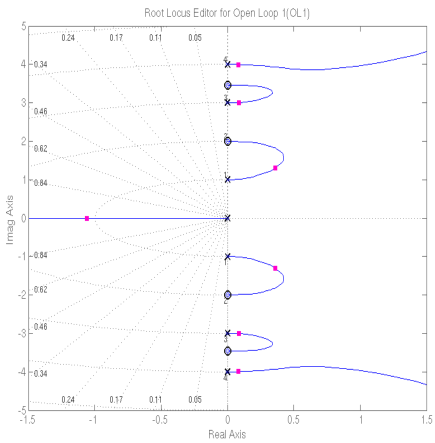
  A real modal analysis is not possible.

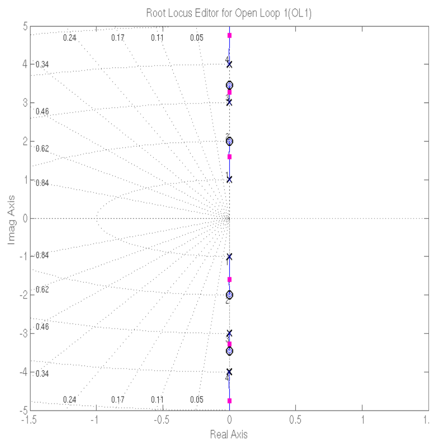
  The velocity feedback introduces active damping in the system.

Illustration : closed-loop dynamics with position, velocity and acceleration feebacks : Example :

$$F_p(\mathrm{s}) = \frac{(\mathrm{s}^2+4)(\mathrm{s}^2+12)}{\mathrm{s}^2(\mathrm{s}^2+1)(\mathrm{s}^2+9)(\mathrm{s}^2+16)}$$





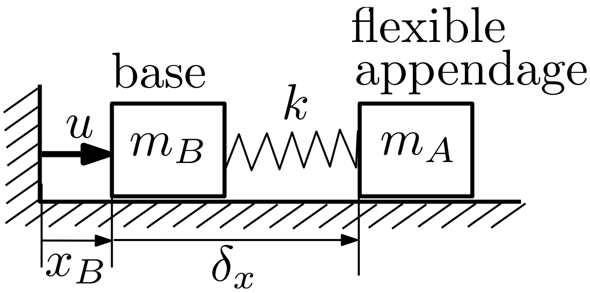


Position | Velocity | Acceleration

In this example, the velocity feedback destabilizes all the modes except the rigid mode.

Exercise :

Considering the spring mass system depicted below and choosing $\mathbf{q} = [x_B,\ \delta x]^T$:
• build the model using the LAGRANGE approach,
• perform the modal analysis of this system,
• using MATLAB and a state space representation, analyse the transfer between u and x_B (functions `damp`, `bode`, `tf`,...).



Dynamic modeling of flexible mechanical systems | Positivity and actuator/sensor collocation | Control of flexible systems

Outline

Effective masses approach

Principle : generalization of the relationship $f = m\ddot{x}$ to flexible bodies characterized by a dynamic mass $m \rightarrow M(\mathrm{s})$ (s is the LAPLACE symbolic variable).
$\Rightarrow$ to describe the relationship between the acceleration $\ddot{x}_B$ of a particular point of the flexible body (the anchorage point or the base) and the force u applied to the flexible body at this point. This relationship does not consider the internal deformations δx of the body.

Frequency (LAPLACE) domain relation : $\boxed{M(\mathrm{s})\ddot{X}_B(\mathrm{s}) = U(\mathrm{s})}$.

Example : considering the previous spring-mass system, it can be easily shown that :

$$\boxed{M(\mathrm{s}) = m_B + m_A \frac{\omega_c^2}{\mathrm{s}^2+\omega_c^2}} \quad \text{with } \omega_c = \sqrt{\frac{k}{m_A}}$$

ω_c is called the **cantilevered** frequency : i.e. the frequency of the flexible appendage when it is cantilevered on the base.

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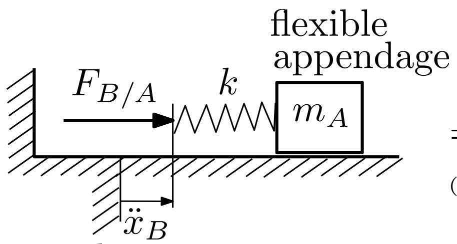
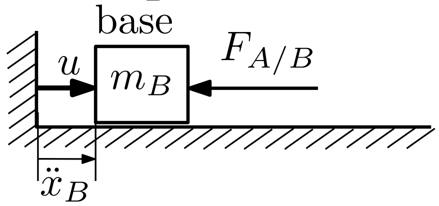
- This representation has a direct physical meaning : holding the system by the base : one can "feel" m_B kilogramms rigidly attached and m_A kilogramms vibrating at the frequency ω_c.
- One can also write : $M(\mathrm{s}) = m_B + M^A(\mathrm{s})$ with $M^A(\mathrm{s}) = m_A \frac{\omega_c^2}{\mathrm{s}^2+\omega_c^2}$.
  $M^A(\mathrm{s})$ is the **effective** mass of the appendage considered without the base. That is the relation between the force applied by the base to the appendage $F_{B/A}$ and the acceleration $\ddot{x}_B$ of the base (or the appendage anchorage point). Indeed :

- 

$$\Rightarrow F_{B/A} = M^A(\mathrm{s})\ddot{X}_B$$
(from: $m_A(\ddot{X}_B + \delta\ddot{X}) = F_{B/A} = -k\frac{\delta\ddot{X}}{\mathrm{s}^2}$)

- 

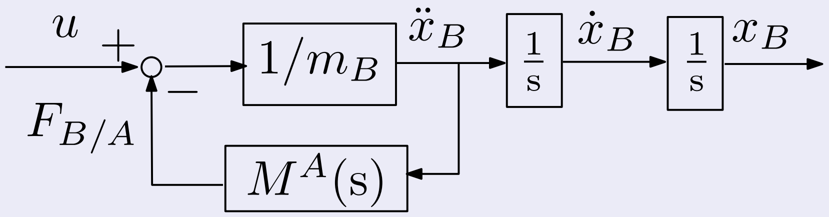
$$\Rightarrow U + F_{A/B} = m_B\ddot{X}_B$$

Then the connection of the appendage A on the base B simply reads :

$$U(\mathrm{s}) = (m_B + M^A(\mathrm{s}))\ddot{X}_B = M(\mathrm{s})\ddot{X}_B \; .$$

Inverse dynamic model : the effective masses approach allows to represent the inverse dynamics model (input$=u$, output$=\ddot{x}_B$) of the system $(A+B)$ by a block-diagram where the appendage direct model $M^A(\mathrm{s})$ is in feedback on the inverse dynamic model of the base $1/m_B$. Indeed : $\frac{1}{m_B+M^A(\mathrm{s})} = \frac{1/m_B}{1+M^A(\mathrm{s})/m_B}$.

Inverse dynamic model of the system $A+B$: $\frac{1}{M(\mathrm{s})\mathrm{s}^2}$.



Cantilevered and free frequencies

$$M(\mathrm{s}) = m_B + m_A\frac{\omega_c^2}{\mathrm{s}^2+\omega_c^2} \quad \text{with } \omega_c = \sqrt{\frac{k}{m_A}}$$

$$M(\mathrm{s}) = m_B\frac{\mathrm{s}^2+\omega_f^2}{\mathrm{s}^2+\omega_c^2} \quad \text{with } \omega_f = \sqrt{\frac{k(m_B+m_A)}{m_Bm_A}}$$

- ω_c is the **cantilevered frequency** of the appndage A,
- ω_f is the **free frequency** of the composite $A+B$.

$\boxed{\omega_f > \omega_c}$ for all base ($\forall\, m_B$).

- the denominateur of $M(\mathrm{s})$ (or the numerator of $M^{-1}(\mathrm{s})$) exhibits the cantilevered frequencies,
- the numerator of $M(\mathrm{s})$ (or the denominator of $M^{-1}(\mathrm{s})$) exhibits the free frequencies.

State-space representation

State-space realization of $M^A(\mathrm{s}) = m_A \frac{\omega_c^2}{\mathrm{s}^2+\omega_c^2}$: for instance a companion relization :

$$\left[\begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \\ \hline F_{B/A} \end{array}\right] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -\omega_c^2 & 0 & 1 \\ \hline m_A\omega_c^2 & 0 & 0 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ \hline \ddot{x}_B \end{array}\right]$$

State-space realization of the transfer between u and x_B (see block diagram slide 21) :

$$\left[\begin{array}{c} \dot{x}_B \\ \ddot{x}_B \\ \dot{x}_1 \\ \dot{x}_2 \\ \hline x_B \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{m_A}{m_B}\omega_c^2 & 0 & \frac{1}{m_B} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{m_A+m_B}{m_B}\omega_c^2 & 0 & \frac{1}{m_B} \\ \hline 1 & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} x_B \\ \dot{x}_B \\ x_1 \\ x_2 \\ \hline u \end{array}\right]$$

Rk : the following representation is also used :

$$\begin{cases} m_B\ddot{x}_B + m_A\omega_c^2 x_1 = u \\ \ddot{x}_1 + \omega_c^2 x_1 = \ddot{x}_B \end{cases}$$

In the next section, the effective masses approach is used to build the 6 d.o.fs inverse dynamic model of spacecraft composed of a main rigid body (hub) and a flexible appendage. Due to cross coupling terms between axes, the geometry of the spacecraft must be taken into account (that is the vector between the hub center of mass G and the anchorage point P of the flexible appendage on the hub) but the principle of the block-diagram description presented in slide 21 is still valid and can be augmented to connect several flexible appendages A_i cantilevered to the hub at anchorage points P_i.

Outline

1. **Dynamic modeling of flexible mechanical systems**
   - LAGRANGE equations
   - Effective masses approach
   - Application to the 6 d.o.fs connexion of a flexible appendage on a rigid body
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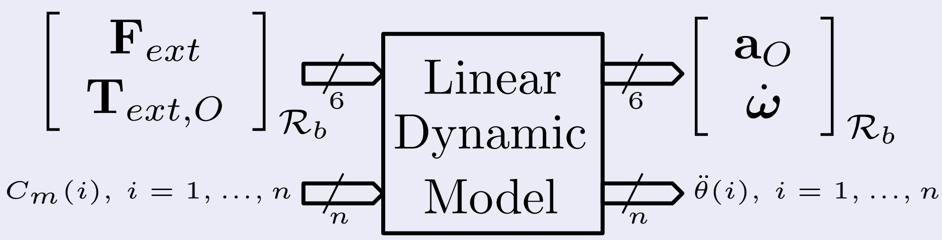
Objectives

To built efficiently the linear dynamic model of any open-chain mechanical system. This model, called inverse dynamic model, gives the relationship between :

- input : external wrench (forces $\mathbf{F}_{ext}$ and torques $\mathbf{T}_{ext,O}$) applied on the main body (hub) at a reference point O and torques $C_m(i),\ i = 1, ...n$ applied inside revolute joints,
- output : inertial accelerations of the main body $[\mathbf{a}_O^T\ \dot{\boldsymbol{\omega}}^T]^T$ and relative angular acceleration $\ddot{\theta}_i, i = 1, .., n$ around the n revolute joints.

General inverse dynamic model at point O in $\mathcal{R}_b$: $[\mathbf{P}_O^{Sat}(\mathrm{s})]^{-1}$.



Context

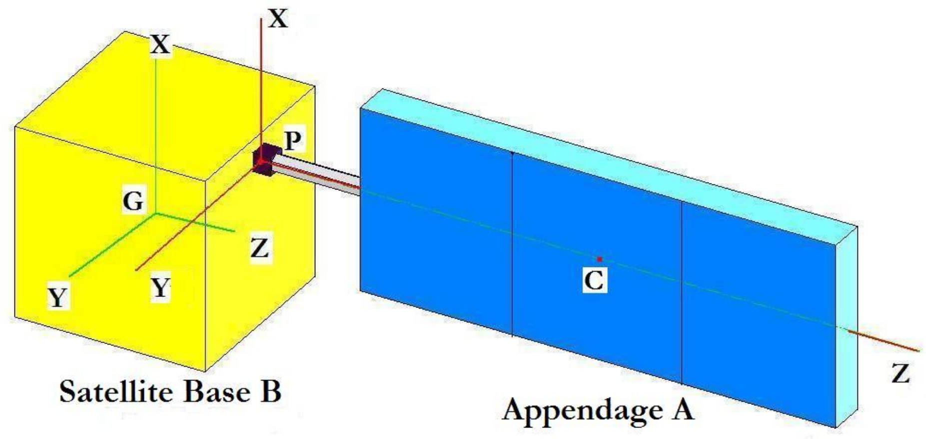


FIGURE: A simple spacecraft model with a flexible appendage $\mathcal{A}$ connected to the rigid hub $\mathcal{B}$ at point P.

Dynamic model of the base $\mathcal{B}$ at point G

From NEWTON's and EULER's equations, the dynamic model of the base $\mathcal{B}$ at its center of mass G reads :

$$\begin{bmatrix} \mathbf{F}_{ext} \\ \mathbf{T}_{ext,G} \end{bmatrix} = \mathbf{D}_G^{\mathcal{B}} \begin{bmatrix} \mathbf{a}_G \\ \dot{\boldsymbol{\omega}} \end{bmatrix} = \begin{bmatrix} m^{\mathcal{B}}\mathbf{1}_3 & \mathbf{0}_{3\times 3} \\ 0 & \mathbf{J}_G^{\mathcal{B}} \end{bmatrix} \begin{bmatrix} \mathbf{a}_G \\ \dot{\boldsymbol{\omega}} \end{bmatrix} \tag{10}$$

where

- $m^{\mathcal{B}}$ is the mass of the body B,
- $\mathbf{J}_G^{\mathcal{B}}$ is the inertia matrix (in $kg.m^2$) at point G of the body $\mathcal{B}$ in the frame $\mathcal{R}_b$,
- $\boldsymbol{\omega}$ is the absolute angular velocity vector of the body .
- and $\dot{\boldsymbol{\omega}} = \frac{d\boldsymbol{\omega}}{dt} |_{\mathcal{R}_b} = \frac{d\boldsymbol{\omega}}{dt} |_{\mathcal{R}_i}$.

Rk : linearization of :

$$\mathbf{T}_{ext,G} = \mathbf{J}_G^{\mathcal{B}}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{J}_G^{\mathcal{B}}\boldsymbol{\omega}$$

Kinematic model between point G and P

$$\mathbf{v}_P = \mathbf{v}_G + \overrightarrow{PG} \times \boldsymbol{\omega} = \mathbf{v}_G + ({}^*\overrightarrow{PG})\boldsymbol{\omega} \qquad (11)$$

where $({}^*\overrightarrow{PG})$ is the antisymmetric matrix associated with the vector $\overrightarrow{PG}$. That is, if $[x,\ y,\ z]^T_{\mathcal{R}_c}$ is the coordinate vector of $\overrightarrow{GP}$ projected in any frame $\mathcal{R}_c$ then $({}^*\overrightarrow{GP})$ reads :

$$({}^*\overrightarrow{GP}) = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}_{\mathcal{R}_c} \quad , \quad ({}^*\overrightarrow{PG}) = \begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix}_{\mathcal{R}_c} .$$

Then, the six d.o.f. twists $\boldsymbol{\nu}_G$ and $\boldsymbol{\nu}_P$ of the body $\mathcal{B}$ respectively at points G and P are given by :

$$\underbrace{\begin{bmatrix} \mathbf{v}_G \\ \boldsymbol{\omega} \end{bmatrix}}_{\boldsymbol{\nu}_G} = \underbrace{\begin{bmatrix} \mathbf{1}_3 & ({}^*\overrightarrow{GP}) \\ \mathbf{0}_{3\times 3} & \mathbf{1}_3 \end{bmatrix}}_{\boldsymbol{\tau}_{GP}} \underbrace{\begin{bmatrix} \mathbf{v}_P \\ \boldsymbol{\omega} \end{bmatrix}}_{\boldsymbol{\nu}_P} \qquad (12)$$

$\boldsymbol{\tau}_{GP}$ is called the (6×6) kinematic model between the points G and P.

Kinematic model between point G and P

Relation between inertial accelerations at points P and G :

$$\mathbf{a}_P = \frac{d\mathbf{v}_P}{dt}\,|_{\mathcal{R}_i} \quad \text{and} \quad \mathbf{a}_G = \frac{d\mathbf{v}_G}{dt}\,|_{\mathcal{R}_i}$$

It is well-known that :

$$\mathbf{a}_P = \mathbf{a}_G + \dot{\boldsymbol{\omega}} \times \overrightarrow{GP} + \boldsymbol{\omega} \times \left(\left(\frac{d\overrightarrow{GP}}{dt} \right) |_{\mathcal{R}_G} + \boldsymbol{\omega} \times \overrightarrow{GP} \right)$$

Assumptions :

- rigid body : $\left(\frac{d\overrightarrow{GP}}{dt}\right) |_{R_G} = 0$
- small motions : nonlinear terms can be neglected.

Then : $\mathbf{a}_P = \mathbf{a}_G + ({}^*\overrightarrow{PG})\dot{\boldsymbol{\omega}}$ and

$$\begin{bmatrix} \mathbf{a}_G \\ \dot{\boldsymbol{\omega}} \end{bmatrix} = \boldsymbol{\tau}_{GP} \begin{bmatrix} \mathbf{a}_P \\ \dot{\boldsymbol{\omega}} \end{bmatrix} = \begin{bmatrix} \mathbf{1}_3 & ({}^*\overrightarrow{GP}) \\ \mathbf{0}_{3\times 3} & \mathbf{1}_3 \end{bmatrix} \begin{bmatrix} \mathbf{a}_P \\ \dot{\boldsymbol{\omega}} \end{bmatrix}$$

Kinematic model between point G and P.

Relationship between the 6 d.o.f external force vectors at point G and at point P :
External force power along any virtual velocity fields :

$$P_{ext} = \begin{bmatrix} \mathbf{v}_G \\ \boldsymbol{\omega} \end{bmatrix}^T \begin{bmatrix} \mathbf{F}_{ext} \\ \mathbf{T}_{ext,G} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_P \\ \boldsymbol{\omega} \end{bmatrix}^T \begin{bmatrix} \mathbf{F}_{ext} \\ \mathbf{T}_{ext,P} \end{bmatrix} \tag{13}$$

Then :

$$\begin{bmatrix} \mathbf{F}_{ext} \\ \mathbf{T}_{ext,P} \end{bmatrix} = \boldsymbol{\tau}_{GP}^T \begin{bmatrix} \mathbf{F}_{ext} \\ \mathbf{T}_{ext,G} \end{bmatrix} = \begin{bmatrix} \mathbf{1}_3 & \mathbf{0}_{3\times 3} \\ -({}^*\overrightarrow{GP}) & \mathbf{1}_3 \end{bmatrix} \begin{bmatrix} \mathbf{F}_{ext} \\ \mathbf{T}_{ext,G} \end{bmatrix} \tag{14}$$

Transport of the dynamic model of B from point G to point P

The direct dynamic model $\mathbf{D}_P^{\mathcal{B}}$ of the base $\mathcal{B}$ at point P becomes :

$$\begin{bmatrix} \mathbf{F}_{ext} \\ \mathbf{T}_{ext,P} \end{bmatrix} = \boldsymbol{\tau}_{GP}^T \begin{bmatrix} m^{\mathcal{B}}\mathbf{1}_3 & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \mathbf{J}_G^{\mathcal{B}} \end{bmatrix} \boldsymbol{\tau}_{GP} \begin{bmatrix} \mathbf{a}_P \\ \dot{\boldsymbol{\omega}} \end{bmatrix} = \mathbf{D}_P^{\mathcal{B}} \begin{bmatrix} \mathbf{a}_P \\ \dot{\boldsymbol{\omega}} \end{bmatrix}$$

Thus the transport of the direct dynamic model of a body $\mathcal{B}$ from a point G to a point P reads :

$$\mathbf{D}_P^{\mathcal{B}} = \boldsymbol{\tau}_{GP}^T \mathbf{D}_G^{\mathcal{B}} \boldsymbol{\tau}_{GP} = \begin{bmatrix} m^{\mathcal{B}}\mathbf{1}_3 & m^{\mathcal{B}}({}^*\overrightarrow{GP}) \\ -m^{\mathcal{B}}({}^*\overrightarrow{GP}) & \mathbf{J}_G^{\mathcal{B}} - m^{\mathcal{B}}({}^*\overrightarrow{GP})^2 \end{bmatrix}. \tag{15}$$

Connection with a rigid appendage

Let us consider a rigid appendage $\mathcal{A}$ cantilevered to the base $\mathcal{B}$ at point P : reaction force $\mathbf{F}_{B/A}$ and torque $\mathbf{T}_{\mathcal{B}/\mathcal{A},P}$ at point P between the base and the appendage must be taken into account :

$$\begin{bmatrix} \mathbf{F}_{ext} - \mathbf{F}_{\mathcal{B}/\mathcal{A}} \\ \mathbf{T}_{ext,P} - \mathbf{T}_{\mathcal{B}/\mathcal{A},P} \end{bmatrix} = \mathbf{D}_P^{\mathcal{B}} \begin{bmatrix} \mathbf{a}_P \\ \dot{\boldsymbol{\omega}} \end{bmatrix} .$$

Let $\mathbf{D}_P^{\mathcal{A}}$ the dynamic model of $\mathcal{A}$ at point P :

$$\begin{bmatrix} \mathbf{F}_{\mathcal{B}/\mathcal{A}} \\ \mathbf{T}_{\mathcal{B}/\mathcal{A},P} \end{bmatrix} = \mathbf{D}_P^{\mathcal{A}} \begin{bmatrix} \mathbf{a}_P \\ \dot{\boldsymbol{\omega}} \end{bmatrix} \quad \left(\mathbf{D}_P^{\mathcal{A}} = \boldsymbol{\tau}_{CP}^T \begin{bmatrix} m^{\mathcal{A}}\mathbf{1}_3 & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{J}_C^{\mathcal{A}} \end{bmatrix} \boldsymbol{\tau}_{CP} \right)$$

Then the dynamic model of the composite $A + B$ at point P reads :

$$\begin{bmatrix} \mathbf{F}_{ext} \\ \mathbf{T}_{ext,P} \end{bmatrix} = (\mathbf{D}_P^{\mathcal{A}} + \mathbf{D}_P^{\mathcal{B}}) \begin{bmatrix} \mathbf{a}_P \\ \dot{\boldsymbol{\omega}} \end{bmatrix} = (\mathbf{D}_P^{\mathcal{A}} + \boldsymbol{\tau}_{GP}^T \mathbf{D}_G^{\mathcal{B}} \boldsymbol{\tau}_{GP}) \begin{bmatrix} \mathbf{a}_P \\ \dot{\boldsymbol{\omega}} \end{bmatrix}$$

Connection with a rigid appendage

It could be more interesting to express the whole dynamic model at the center of mass G of the base B, Then :

$$\begin{bmatrix} \mathbf{F}_{ext} \\ \mathbf{T}_{ext,G} \end{bmatrix} = (\boldsymbol{\tau}_{PG}^T \mathbf{D}_P^{\mathcal{A}} \boldsymbol{\tau}_{PG} + \mathbf{D}_G^{\mathcal{B}}) \begin{bmatrix} \mathbf{a}_G \\ \dot{\boldsymbol{\omega}} \end{bmatrix} = (\mathbf{D}_G^{\mathcal{A}} + \mathbf{D}_G^{\mathcal{B}}) \begin{bmatrix} \mathbf{a}_G \\ \dot{\boldsymbol{\omega}} \end{bmatrix}$$

This equation introduces the dynamic model of the appendage at the point G : $(\mathbf{D}_G^{\mathcal{A}})$.
The inverse dynamic model (used for AOCS design) :

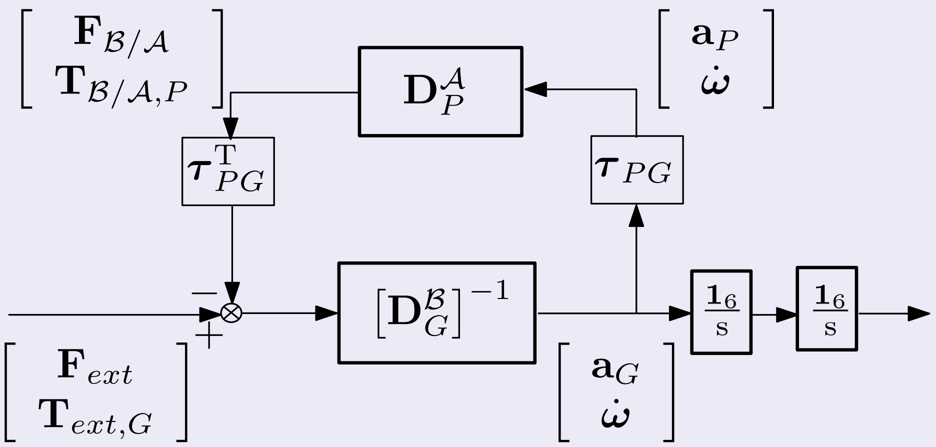
$$\begin{bmatrix} \mathbf{a}_G \\ \dot{\boldsymbol{\omega}} \end{bmatrix} = (\mathbf{D}_G^{\mathcal{B}} + \mathbf{D}_G^{\mathcal{A}})^{-1} \begin{bmatrix} \mathbf{F}_{ext} \\ \mathbf{T}_{ext,G} \end{bmatrix} = (\mathbf{D}_G^{Sat})^{-1} \begin{bmatrix} \mathbf{F}_{ext} \\ \mathbf{T}_{ext,G} \end{bmatrix}$$

It can be shown that :

$$(\mathbf{D}_G^{\mathcal{B}} + \mathbf{D}_G^{\mathcal{A}})^{-1} = (\mathbf{D}_G^{\mathcal{B}})^{-1} \left(\mathbf{1}_6 + \boldsymbol{\tau}_{PG}^T \mathbf{D}_P^{\mathcal{A}} \boldsymbol{\tau}_{PG} (\mathbf{D}_G^{\mathcal{B}})^{-1} \right)^{-1} \qquad (16)$$

Minimal block-diagram representation

Block Diagram of the inverse Dynamic Model : $\frac{\mathbf{1}_6}{s^2}\,[\mathbf{D}_G^{\mathcal{B}+\mathcal{A}}]^{-1}$.



In this block diagram, all the blocks and link are expressed **in the same reference frame** : the main body $\mathcal{B}$ frame : $\mathcal{R}_b$.

Connection with a rotation transformation

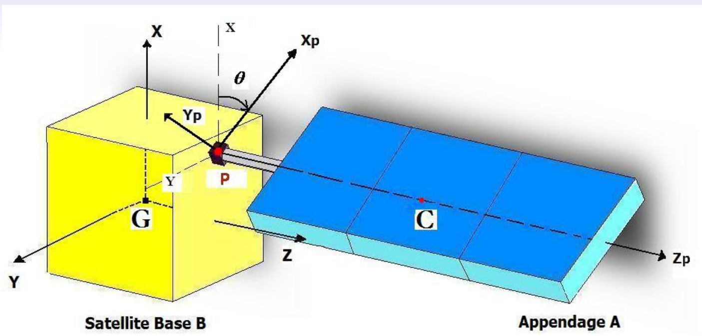


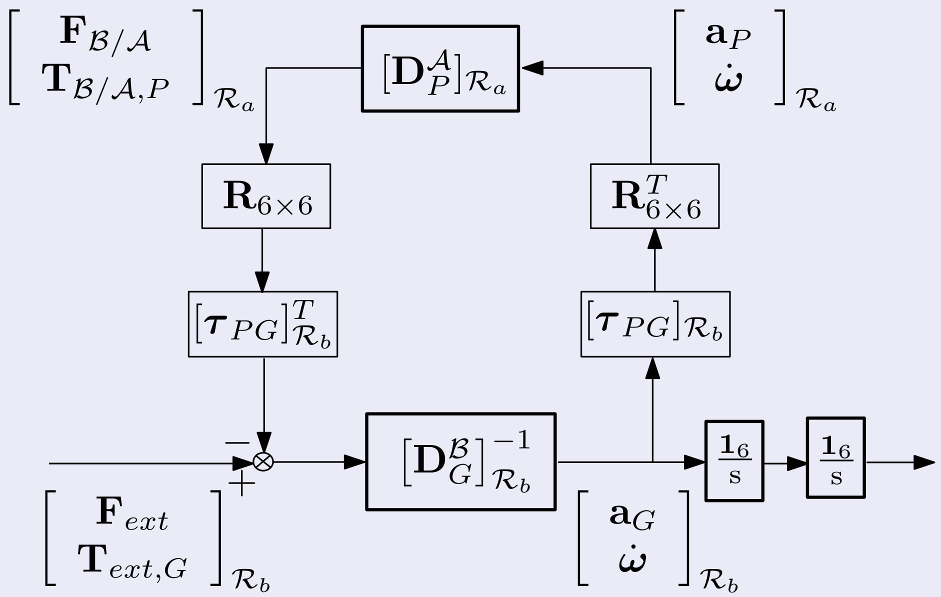
FIGURE: Rigid connection between the hub and a rotated appendage.

$$[\mathbf{D}_G^{\mathcal{A}}]_{\mathcal{R}_b} = [\boldsymbol{\tau}_{PG}]^T_{\mathcal{R}_b} \underbrace{\begin{bmatrix}\mathbf{P}_{a/b} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{P}_{a/b}\end{bmatrix}}_{\mathbf{R}_{6\times6}} [\mathbf{D}_P^{\mathcal{A}}]_{\mathcal{R}_a} \begin{bmatrix}\mathbf{P}_{a/b} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{P}_{a/b}\end{bmatrix}^T [\boldsymbol{\tau}_{PG}]_{\mathcal{R}_b};$$

$$\mathbf{P}_{a/b} = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Connection with a rotation transformation

Inverse Dynamic Model with a rotation between the Main Body frame and the appendage frame $\frac{\mathbf{1}_6}{s^2}\,[\mathbf{D}_G^{\mathcal{B}+\mathcal{A}}]^{-1}_{\mathcal{R}_b}$.



Cantilever hybrid model of the flexible appendage

Cantilever Hybrid Model :

$$\begin{bmatrix} \mathbf{F}_{B/A} \\ \mathbf{T}_{B/A,P} \end{bmatrix} = \mathbf{D}_P^{\mathcal{A}} \begin{bmatrix} \mathbf{a}_P \\ \dot{\boldsymbol{\omega}} \end{bmatrix} + \mathbf{L}_P^T \ddot{\boldsymbol{\eta}} \tag{17}$$

$$\ddot{\boldsymbol{\eta}} + \mathrm{diag}(2\xi_i\omega_i)\dot{\boldsymbol{\eta}} + \mathrm{diag}(\omega_i^2)\boldsymbol{\eta} = -\mathbf{L}_P \begin{bmatrix} \mathbf{a}_P \\ \dot{\boldsymbol{\omega}} \end{bmatrix} \tag{18}$$

where : $\mathbf{L}_P = [\mathbf{l}_P^{1\,T} \quad \mathbf{l}_P^{2\,T} \quad \cdots \quad \mathbf{l}_P^{k\,T}]^T$.
$\mathbf{l}^i_{P(1\times6)}$, ω_i, ξ_i are the modal contribution[1] at point P, the frequency, and the damping ratio of the flexible mode i respectively, for $i = 1, ..., k$ (k is the number of flexible modes taken into account).
$\boldsymbol{\eta}$ is the vector of flexible modal coordinates.

1. if modal contribution matrix is given at point C (the appendage center of mass) and denoted $\mathbf{L}_C$, then one can write : $\mathbf{L}_P = \mathbf{L}_C\boldsymbol{\tau}_{CP}$.

State space realization (preferred)

$$\begin{bmatrix} \dot{\boldsymbol{\eta}} \\ \ddot{\boldsymbol{\eta}} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{k\times k} & \mathbf{1}_k \\ -\mathbf{K}_{k\times k} & -\mathbf{D}_{k\times k} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta} \\ \dot{\boldsymbol{\eta}} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{k\times 6} \\ -\mathbf{L}_{P_{k\times 6}} \end{bmatrix} \begin{bmatrix} \mathbf{a}_P \\ \dot{\boldsymbol{\omega}} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{F}_{B/A} \\ \mathbf{T}_{B/A,P} \end{bmatrix} = \begin{bmatrix} -\mathbf{L}_P^T\mathbf{K} & -\mathbf{L}_P^T\mathbf{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta} \\ \dot{\boldsymbol{\eta}} \end{bmatrix} + \underbrace{(\mathbf{D}_P^{\mathcal{A}} - \mathbf{L}_P^T\mathbf{L}_P)}_{\mathbf{D}_{P_0}^{A}} \begin{bmatrix} \mathbf{a}_P \\ \dot{\boldsymbol{\omega}} \end{bmatrix}$$

where $\mathbf{D} = \text{diag}(2\xi_i\omega_i)$ and $\mathbf{K} = \text{diag}(\omega_i^2)$ and $\mathbf{D}_{P_0}^{\mathcal{A}}$ is the residual mass of the appendage rigidly connected to the base.

The characteristic parameters ω_i, ξ_i, $\mathbf{l}_p^i$ and $\mathbf{D}_P^{\mathcal{A}}$ of the appendage are the output data of Finite Element Software used to model such an appendage considered alone and cantilevered at point P.
Indeed, equations (17) and (18) can be rewritten :

$$\begin{bmatrix} \mathbf{D}_P^{\mathcal{A}} & \mathbf{L}_P^T \\ \mathbf{L}_P & \mathbf{1}_k \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_r \\ \ddot{\boldsymbol{\eta}} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{6\times 6} & \mathbf{0}_{6\times k} \\ \mathbf{0}_{k\times 6} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \star \\ \dot{\boldsymbol{\eta}} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{6\times 6} & \mathbf{0}_{6\times k} \\ \mathbf{0}_{k\times 6} & \mathbf{K} \end{bmatrix} \begin{bmatrix} \star \\ \boldsymbol{\eta} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{ext} \\ \mathbf{0}_{k\times 1} \end{bmatrix}$$

with $\ddot{\mathbf{q}}_r = \begin{bmatrix} \mathbf{a}_P \\ \dot{\boldsymbol{\omega}} \end{bmatrix}$ and $\mathbf{F}_{ext} = \begin{bmatrix} \mathbf{F}_{\mathcal{B}/\mathbf{A}} \\ \mathbf{T}_{\mathcal{B}/\mathcal{A},P} \end{bmatrix}$.
This equation is in the form of (7) provided by the Lagrange method.

This state-space representation allows the direct transfer matrix $\mathbf{M}_P^{\mathcal{A}}(\mathrm{s})$ between force and acceleration of the appendage at point P (also called **dynamic mass matrix** or **effective mass model**) to be computed :

$$\begin{bmatrix} \mathbf{F}_{\mathcal{B}/\mathbf{A}} \\ \mathbf{T}_{\mathcal{B}/\mathcal{A},P} \end{bmatrix} = \mathbf{M}_P^{\mathcal{A}}(\mathrm{s}) \begin{bmatrix} \mathbf{a}_P \\ \dot{\boldsymbol{\omega}} \end{bmatrix} \quad (19)$$

with : $\mathbf{M}_P^{\mathcal{A}}(\mathrm{s}) = \mathbf{D}_P^{\mathcal{A}} - \mathbf{L}_P^T\mathbf{L}_P +$
$\begin{bmatrix} -\mathbf{L}_P^T\mathbf{K} & -\mathbf{L}_P^T\mathbf{D} \end{bmatrix}_{6\times k} \begin{bmatrix} \mathrm{s}\mathbf{1}_k & -\mathbf{1}_k \\ \mathbf{K} & (\mathrm{s}\mathbf{1}_k + \mathbf{D}) \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0}_{k\times 6} \\ -\mathbf{L}_{P_{k\times 6}} \end{bmatrix}.$

In the case where flexible mode damping ratios are neglected ($\mathbf{D} = 0$) :

Transfer matrix

$$\mathbf{M}_P^{\mathcal{A}}(\mathrm{s}) = \mathbf{D}_{P_0}^{\mathcal{A}} + \sum_{i=1}^{k} \mathbf{D}_{P_i}^{\mathcal{A}} \frac{\omega_i^2}{\mathrm{s}^2 + \omega_i^2}$$

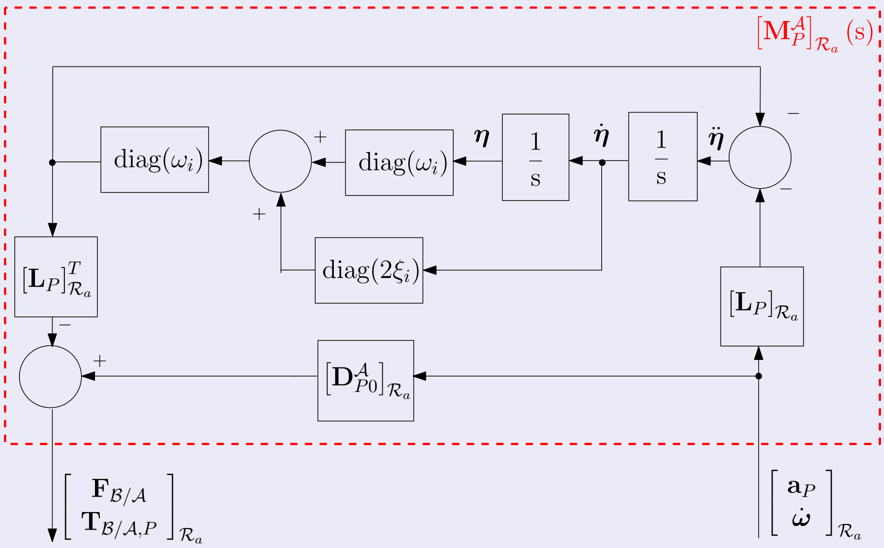
where :

- $\mathbf{D}_{P_i}^{\mathcal{A}} = {\mathbf{l}_P^i}^T \mathbf{l}_P^i$ is rank-1 effective-mass matrix of the ith mode,
- $\mathbf{D}_P^{\mathcal{A}}$ is the static gain (DC gain) of $\mathbf{M}_P^{\mathcal{A}}(\mathrm{s})$.

Dynamic model of the whole system

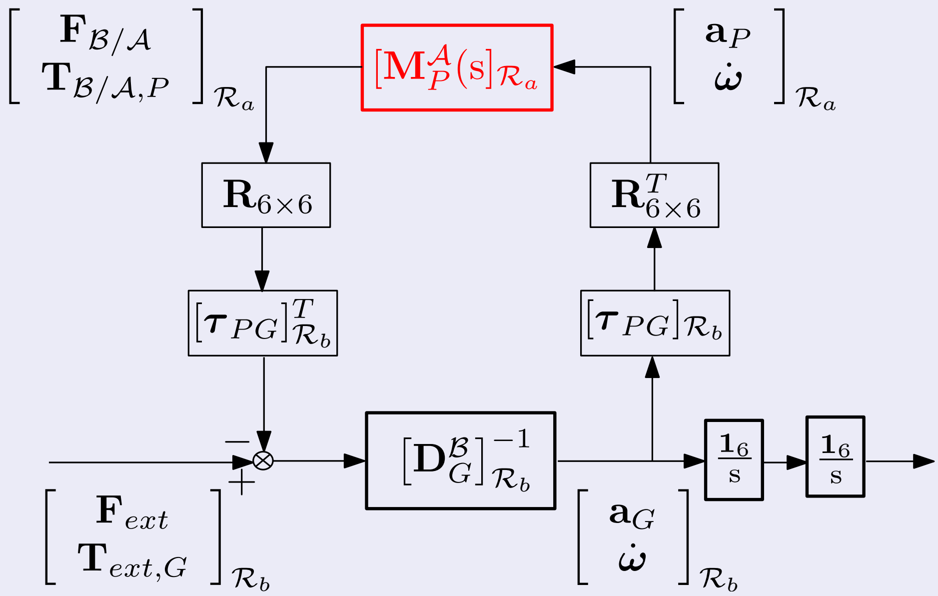
To build the dynamic model of the whole system (rigid base + flexible appendage), we only have to remove $\mathbf{D}_P^{\mathcal{A}}$ by $\mathbf{M}_P^{\mathcal{A}}(\mathrm{s})$ in the block diagram depicted in slide 37.

Block-diagram representation of the appendage dynamic model $\mathbf{M}_P^{\mathcal{A}}(\mathrm{s})$



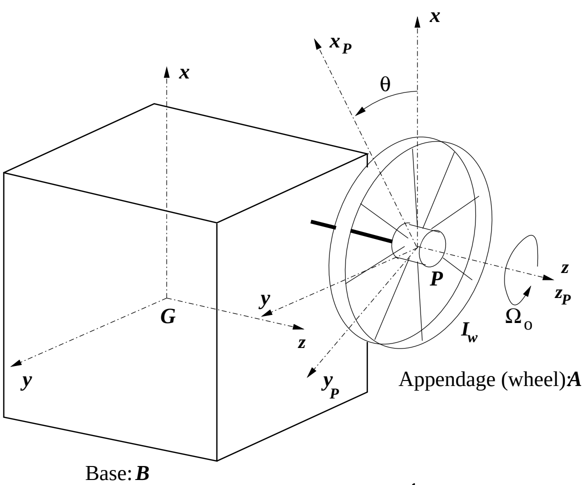
Block-diagram representations

Composite : main body + appendage : $\frac{\mathbf{1}_6}{s^2}\,[\mathbf{M}_G^{\mathcal{B}+\mathcal{A}}(s)]^{-1}_{\mathcal{R}_b}$.



Angular momentum modeling

An onboard angular momentum (reaction wheels for instance) can be modelize as a dynamic appendage and connected to the main body (base) in the same way.



Indeed : Assuming :
- $\mathbf{z}$ the spin axis,
- $\boldsymbol{\omega}$ is small,
- Ω_0 (the spin angular rate) is constant,

then :

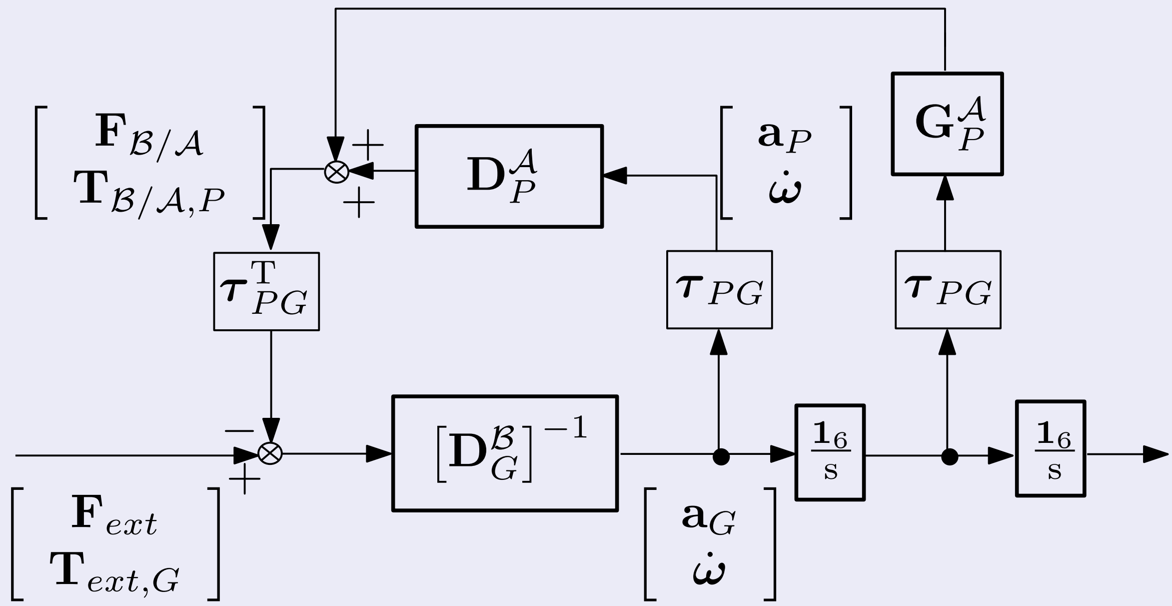
$$\mathbf{M}_P^{\mathcal{A}}(s) = \left(\mathbf{D}_P^{\mathcal{A}} + \begin{bmatrix}\mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3}\\ \mathbf{0}_{3\times 3} & -\frac{1}{s} I_w \Omega_0\,({}^*\mathbf{z})\end{bmatrix}\right)$$

where $\mathbf{D}_P^{\mathcal{A}} = \begin{bmatrix} m^{\mathcal{A}}\mathbf{1}_3 & \mathbf{0}\\ \mathbf{0} & \mathbf{J}_P^{\mathcal{A}}\end{bmatrix}$

Extra integrators in $\mathbf{M}_P^{\mathcal{A}}(s)$ can be removed considering the following minimal block-diagram where $\mathbf{G}_P^{\mathcal{A}} = \begin{bmatrix}\mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3}\\ \mathbf{0}_{3\times 3} & -I_w\Omega_0\,({}^*\mathbf{z})\end{bmatrix}$.

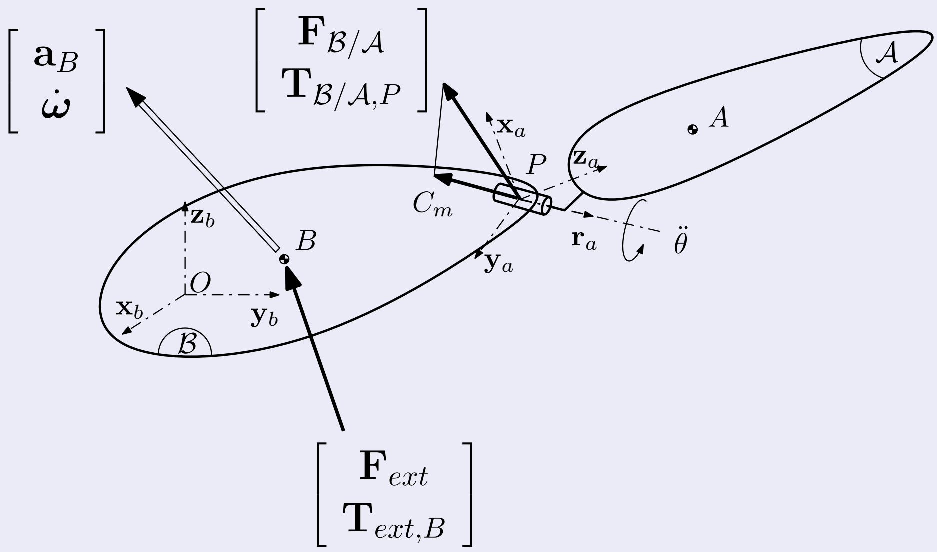
Angular momentum modeling

Block Diagram of the inverse Dynamic Model with a spinning wheel appendage : $\frac{\mathbb{1}_6}{s^2}\,[\mathrm{M}_G^{\mathcal{B}+\mathcal{A}}(s)]^{-1}_{\mathcal{R}_b}$.



Pivot joint between base and appendage

Assembly of the base $\mathcal{B}$ and the appendage $\mathcal{A}$ linked with a revolute joint along r_a



$\mathbf{r}_a$: Unit vector along the revolute joint axis : $\mathbf{r}_a = [x_{r_a}\ y_{r_a}\ z_{r_a}]^T_{\mathcal{R}_a}$.

$\ddot{\theta}$: Revolute joint's angular acceleration.

C_m : Revolute joint's torque.

Pivot joint between base and appendage

The objective is to compute the **augmented direct** model $\mathbf{P}_B^{\mathcal{A}+\mathcal{B}}(\mathrm{s})$ (7×7) of the assembly $\mathcal{A}+\mathcal{B}$ and its inverse $[\mathbf{P}_B^{\mathcal{A}+\mathcal{B}}(\mathrm{s})]^{-1}$, such that :

$$\begin{bmatrix} \mathbf{F}_{ext} \\ \mathbf{T}_{ext,B} \\ C_m \end{bmatrix} = \mathbf{P}_B^{\mathcal{A}+\mathcal{B}}(\mathrm{s}) \begin{bmatrix} \mathbf{a}_B \\ \dot{\boldsymbol{\omega}} \\ \ddot{\theta} \end{bmatrix} ; \begin{bmatrix} \mathbf{a}_B \\ \dot{\boldsymbol{\omega}} \\ \ddot{\theta} \end{bmatrix} = [\mathbf{P}_B^{\mathcal{A}+\mathcal{B}}(\mathrm{s})]^{-1} \begin{bmatrix} \mathbf{F}_{ext} \\ \mathbf{T}_{ext,B} \\ C_m \end{bmatrix}$$

Applying the **direct** dynamic model $\mathbf{M}_P^{\mathcal{A}}(\mathrm{s})$ of the appendage at point P :

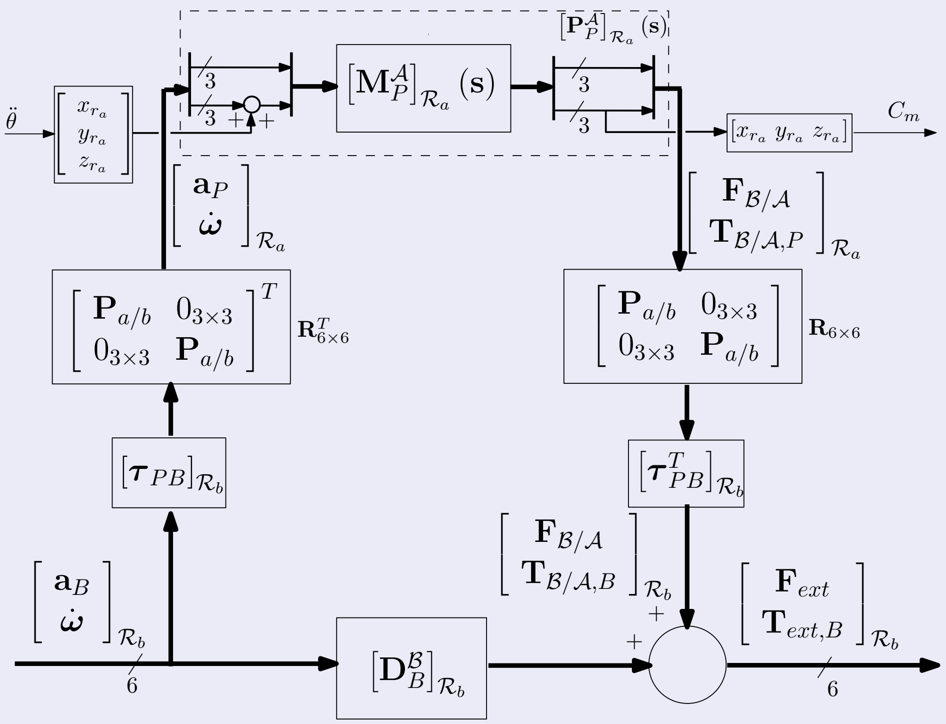
$$\begin{bmatrix} \mathbf{F}_{\mathcal{B}/\mathcal{A}} \\ \mathbf{T}_{\mathcal{B}/\mathcal{A},P} \end{bmatrix} = \mathbf{M}_P^{\mathcal{A}}(\mathrm{s}) \begin{bmatrix} \mathbf{a}_P \\ \dot{\boldsymbol{\omega}} + \ddot{\theta}\mathbf{r}_a \end{bmatrix} \quad (\text{indeed : } \dot{\boldsymbol{\omega}}_{\mathcal{A}/\mathcal{B}} = \ddot{\theta}\,\mathbf{r}_a)$$

Because of the revolute joint, the projection of the torque $\mathbf{T}_{\mathcal{B}/\mathcal{A},P}$, applied by the base to the appendage at point P, along $\mathbf{r}_a$ axis is either : null in case of a free revolute joint or equal to C_m in case of an actuated joint.

$$C_m = \mathbf{T}_{\mathcal{B}/\mathcal{A},P}.\mathbf{r}_a$$

Pivot joint between base and appendage

Direct dynamic model with revolute joint : $[\mathbf{P}_B^{\mathcal{A}+\mathcal{B}}(\mathrm{s})]_{\mathcal{R}_b}$.



Conclusions and perspectives

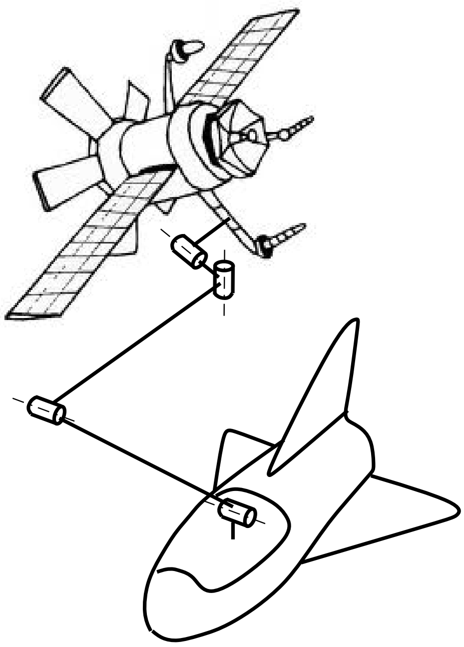
The 6 d.o.f. linear dynamic model of a spacecraft involving one or several dynamic appendages can be simply represented by a *well-posed* block-diagram sketch. That is a diagram where :

- number of integrators is minimized,
- dynamic parameters are directly link to sub-structure dynamic models ($\mathbf{D}_G^{\mathcal{B}}$, $\boldsymbol{\tau}_{PG}$, $\mathbf{R}$ for main body and ω_i, ξ_i, $\mathbf{l}_P^i$, $\mathbf{D}_{P_0}^{\mathcal{A}}$ for each appendage),
- occurence of gain-blocks involving these dynamic parameters is minimized.

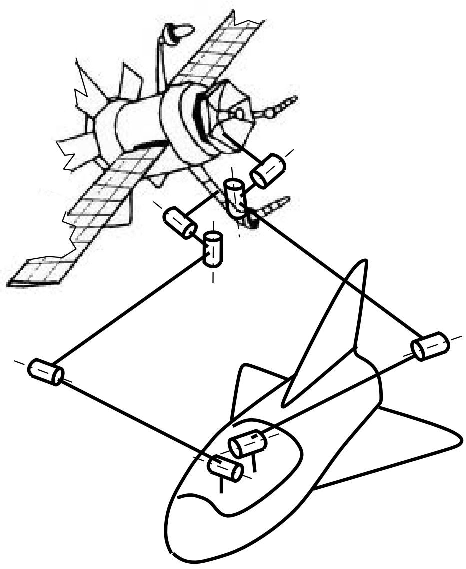
It is thus possible to model any open-chain mechanical system (tree chain) where terminal bodies are flexible and to take into account efficiently parametric uncertainties.

Perspectives

To model any open-chain mechanical systems with intermediate flexible links.



To model any closed-chain mechanical systems with rigid and/or flexible bodies.



Outline

Dynamic modeling of flexible mechanical systems | Positivity and actuator/sensor collocation | Control of flexible systems

Outline

Positivity : a short survey

Positivity of a system can be seen as the frequency-domain interpretation of the **asymptotic hyperstability**.

- **Global stability :** practical approach for SISO system (introduced for non-linear system) : *a SISO system S is hyperstable if S is stabilized by any positive gain in negative feedback*.
- **Hyperstability :** mathematical formulation for MIMO system in the time-domain : *a system S is hyperstable if its state $\mathbf{x}(t)$ is bounded for any input $\mathbf{u}(t)$ such that the scalar product of $\mathbf{u}(t)$ with the output $\mathbf{y}(t)$ is bounded by $\sup_t \mathbf{x}(t)$*.

  Furthermore, if $\lim_{t\to\infty} \mathbf{x}(t) = 0$, S is **asymptotically hyperstable**.

  .

Dynamic modeling of flexible mechanical systems | Positivity and actuator/sensor collocation | Control of flexible systems

Positivity : a short survey

Let us consider a square system ($m = p$ same number of inputs and outputs), strictly proper, defined by its transfer matrix $\mathbf{Z}(\mathrm{s})$ ($\mathbf{Z}(\infty) = 0$) or its minimal realization $\mathbf{Z}(\mathrm{s}) = \mathbf{C}(\mathrm{s}\mathbf{1}_n - \mathbf{A})^{-1}\mathbf{B}$, ($\mathbf{D} = 0$).
Frequency-domain definition of positivity : $Z(\mathrm{s})$ is strictly positive real if :

- the components of $\mathbf{Z}(\mathrm{s})$ are real $\forall$ $\mathrm{s} \in \mathbb{R}$,
- the components of $(\mathbf{Z}\mathrm{s})$ have no instable poles,
- $\mathbf{Z}(j\omega) + \mathbf{Z}^*(j\omega)$ is hermitian definite positive $\forall \omega \in \mathbb{R}$.

For SISO system the last condition can be directly interpreted on the frequency-domain response of $\mathbf{Z}(j\omega)$:

- $\forall \omega \quad$ Real$(\mathbf{Z}(j\omega)) > 0$ (on the NYQUIST plot),
- $\forall \omega \quad -90° <$ Arg$(\mathbf{Z}(j\omega)) < 90°$ (on the BODE or NICHOLS plots).

Example :$\mathbf{Z}(\mathrm{s}) = \frac{s+10}{s+1}$, $\mathbf{Z}(\mathrm{s}) = \frac{s(\mathrm{s}^2+s+4)}{(\mathrm{s}^2+s+1)(\mathrm{s}^2+s+9)}$.

Theorem (time-domain interpretation of positivity) : $\mathbf{Z}(\mathrm{s})$ is positive real iff :

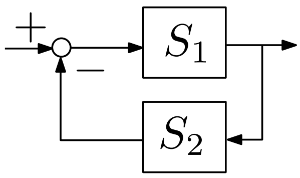
- $\mathbf{A}$ has no eigenvalue with a positive real part,
- the eigenvalues of $\mathbf{A}$ along the imaginary axis are simple,
- a symmetric positive definite matrix $\mathbf{P}$ and a matrix $\mathbf{L}$ exist such that :

$$\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} = -\mathbf{L}\mathbf{L}^T \quad (20)$$
$$\mathbf{C}^T = \mathbf{P}\mathbf{B} \quad (21)$$

Rk : $\mathbf{Z}(\mathrm{s})$ is strictly positive real iff L is non singular.

Feedback connection of positive systems :

Considering two positive real systems S_1 and S_2 in negative feedback, than the closed-loop system is asymptoticaly stable.



Outline

Application to flexible structures

Let us consider now that S_1 is a flexible system defined by (9) (see slide # 12) and S_2 is static output feedback (**positive**) gain. The positivity of S_1 is a very interesting property since the closed loop system will be stable for any (positive) value of S_2. This property has a direct link with the location of actuators $\mathbf{u}$ and velocity sensors $\mathbf{y}^V$ in the structure.
Indeed : applying the theorem of slide # 55 to the modal coordinates representation of the flexible structure (9) and partionning $\mathbf{P}$ with 2×2 blocks $\mathbf{P}_i$, $i = 1, \cdots, n$ on the diagonal, the condition (20) can be rewritten :

$$\exists \quad \mathbf{P}_i > 0 \quad / \quad \mathbf{P}_i \mathbf{A}_i + \mathbf{A}_i^T \mathbf{P}_i \leq \mathbf{0} \quad \forall\, i = 1, \cdots, n$$

Let $\mathbf{P}_i = \begin{bmatrix} p_{1i} & p_{2i} \\ p_{2i} & p_{3i} \end{bmatrix}$ with $p_{1i} > 0$, $p_{3i} > 0$ and $p_{1i}p_{3i} - p_{2i}^2 > 0$, then :

$$\mathbf{P}_i \mathbf{A}_i + \mathbf{A}_i^T \mathbf{P}_i = \begin{bmatrix} -2\omega_i^2 p_{2i} & p_{1i} - 2\xi\omega_i p_{2i} - \omega_i^2 p_{3i} \\ \text{sym} & 2(p_{2i} - 2\xi\omega_i p_{3i}) \end{bmatrix} \leq \mathbf{0}$$

- Case 1 : $\xi = 0$ (conservative structure), then

$$\mathbf{P}_i \mathbf{A}_i + \mathbf{A}_i^T \mathbf{P}_i = \begin{bmatrix} -2\omega_i^2 p_{2i} & p_{1i} - \omega_i^2 p_{3i} \\ \text{sym} & 2p_{2i} \end{bmatrix} \leq \mathbf{0} \;\Rightarrow\; p_{2i} = 0$$

(due to diagonal terms) and then $\mathbf{P}_i \mathbf{A}_i + \mathbf{A}_i^T \mathbf{P}_i \leq \mathbf{0}$ iff $p_{1i} - \omega_i^2 p_{3i} = 0$. $\mathbf{P}_i$ must be $> \mathbf{0}$, then the set of solutions is :

$$\mathbf{P}_i = \alpha_i \begin{bmatrix} \omega_i^2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{with : } \alpha_i > 0.$$

Then $\mathbf{P}_i \mathbf{A}_i + \mathbf{A}_i^T \mathbf{P}_i = \mathbf{0}$ exactly, it is just a sufficient condition.

- Case 2 : $\xi > 0$ (dissipative structure), the previous condition still works, then

$$\mathbf{P}_i \mathbf{A}_i + \mathbf{A}_i^T \mathbf{P}_i = \alpha_i \begin{bmatrix} 0 & 0 \\ 0 & -4\xi\omega_i \end{bmatrix} \quad \text{which is SDN } (\Rightarrow\; L_i = [0\ 2\sqrt{\alpha_i \xi \omega_i}])$$

Rk : $\mathbf{A}_i$ being stable and thanks to LYAPUNOV theorem, it is possible to find a positive solution $\widetilde{\mathbf{P}_i}$ to $\widetilde{\mathbf{P}_i}\mathbf{A}_i + \mathbf{A}_i^T \widetilde{\mathbf{P}_i} + \mathbf{Q} = \mathbf{0}$ with $\mathbf{Q} > \mathbf{0}$ (strictly positive). But the interest of the solution $\mathbf{P}_i$ is the independance w.r.t ξ (and the BASILE assumption).

Furthermore the condition (21) becomes (see slide # 12) :

$$\forall i \quad \mathbf{C}_i^T = \mathbf{P}_i \mathbf{B}_i = \alpha_i \begin{bmatrix} \omega_i^2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{0}_{1\times m} \\ 1/m_i \boldsymbol{\phi}_i^T \boldsymbol{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{1\times m} \\ \frac{\alpha_i}{m_i} \boldsymbol{\phi}_i^T \boldsymbol{\mathcal{B}} \end{bmatrix} .$$

Choosing $\alpha_i = m_i \Rightarrow \boxed{\mathbf{C}_i = \begin{bmatrix} \mathbf{0}_{m\times 1} & \boldsymbol{\mathcal{B}}^T \boldsymbol{\phi}_i \end{bmatrix} \quad \forall i}$.

Regarding the expressions of $\mathbf{C}_i^P$ and $\mathbf{C}_i^V$ in slide # 12, a **sufficient** condition for the system to be positive is to have m velocity sensors ($p_V = m$) such that $\boldsymbol{\mathcal{C}}^V = \boldsymbol{\mathcal{B}}^T$, i.e. the m velocity sensors are at the same location in the structure than the m actuators.

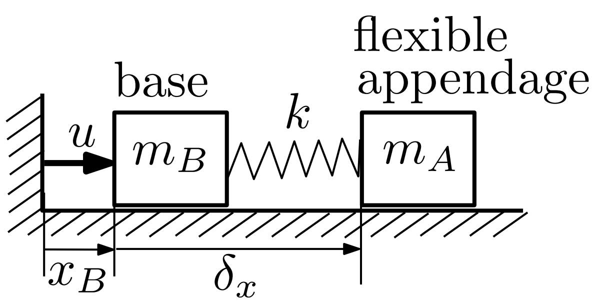
Actuator/sensor collocation $\Rightarrow$ the flexible system is positive.

Remarks :

- this a **sufficient but not necessary** condition. But this condition depends neither on the frequencies ω_i, damping ratios ξ_i and modal shapes $\boldsymbol{\phi}_i$ of flexible modes, nor on the number of flexible modes considered in the model $\Rightarrow$ a positive feedback on a flexible structure with actuator/velocity sensor collocation is robust to parametric uncertainties on ω_i, $\xi_{(i)}$ and $\boldsymbol{\phi}_i$.

- in the case of a rigid mode : $\omega_i = 0$, the condition is no more sufficient since $\mathbf{P}_i$ becomes just semi-definite positive. An interpretation is that the position of the rigid mode is no longer observable from the velocity sensors. A position sensor is required to stabilize the system.

  Example : the spring mass system :

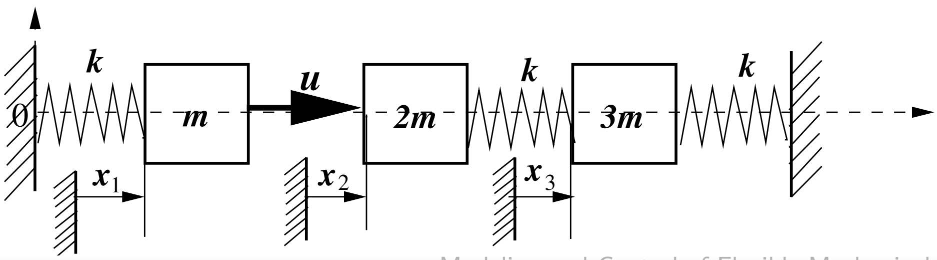


- in the case where several flexible modes have the same frequency $\omega_{i_1} = \omega_{i_2} = \cdots = \omega_{i_r}$ (but different modal shapes ϕ_{i_j}, $j = 1, \cdots, r$), the condition is no more sufficient (2nd condition of theorem in slide # 55 is not met). A necessary condition is : the number m of actuators must be greater or equal than the frequency multiplicity order r and

$$\text{Rank}\begin{bmatrix} \phi_{i_1}\boldsymbol{\mathcal{B}} \\ \vdots \\ \phi_{i_r}\boldsymbol{\mathcal{B}} \end{bmatrix} \geq r\ .$$

Condition for the system to be controllable (and observable if sensor/actuator collocation condition is met).

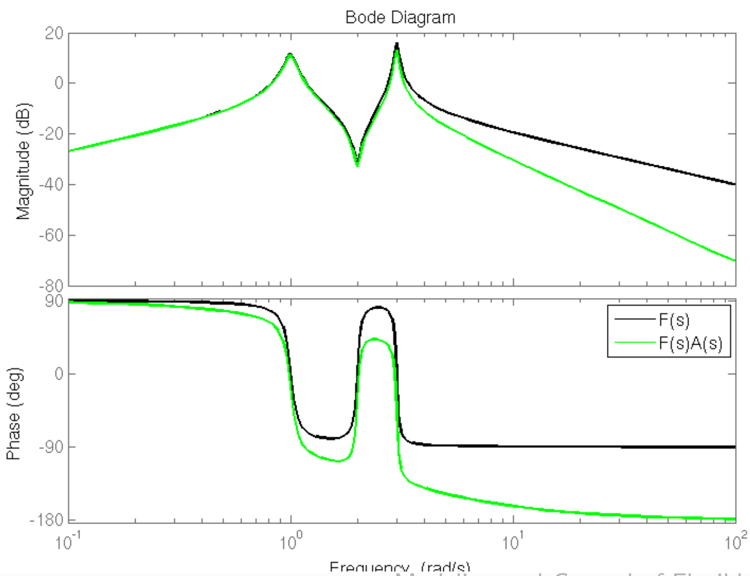
Example (of uncontrollable flexible system) :



Collocation : limit of the theorem

The positivity of a flexible system with actuator/velocity sensors collocation is limited, on practical application, by the own dynamics of sensors and actuators. These dynamics brings phase lags and the positivity of high frequency flexible modes is lost.

Illustration : $F(\mathrm{s}) = \frac{s(\mathrm{s}^2+0.1s+4)}{(\mathrm{s}^2+0.1s+1)(\mathrm{s}^2+0.1s+9)}$ and $A(\mathrm{s}) = \frac{3}{s+3}$:



Outline

SISO flexible systems

The transfer function $F(\mathrm{s})$ of a Single Input Single Output conservative ($\xi = 0$) flexible system has some interesting properties :

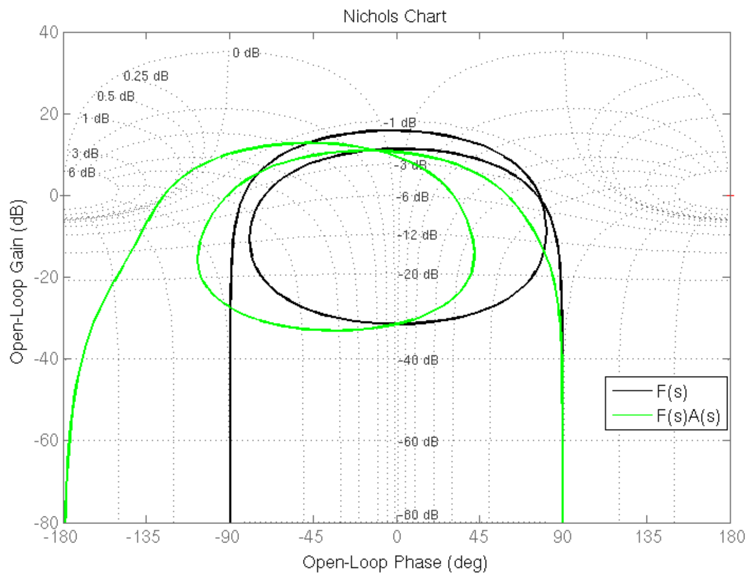
- $F(\mathrm{s})$ is a function of s^2 (assuming u is a force and y is a position or an acceleration) :

$$F(\mathrm{s}) = G\frac{\Pi_{i=1}^{m}(\mathrm{s}^2 + z_i^2)}{\Pi_{i=1}^{n}(\mathrm{s}^2 + \omega_i^2)} \ . \quad (22)$$

  Then $F(j\omega)$ is real (positive or negative) :
  - if $\omega = \omega_i \;\Rightarrow$ a resonance on the magnitude response and a 180° phase lag on the phase response,
  - if $\omega = z_i \;\Rightarrow$ an anti-resonance (notch) on the magnitude response and a 180° phase lead on the phase response.

  NICHOLS plot is not easily legible if $\xi = 0$. If $0 < \xi << 1$, then resonances and anti resonances magnitudes are bounbed and 180° phase variations are smoother (see slide # 62). The succession of a "flexible zero" $(\mathrm{s}^2 + z_i^2)$ and a flexible pole $(\mathrm{s}^2 + \omega_i^2)$ appears as a loop on the NICHOLS plot.

Illustration : $F(\mathrm{s}) = \frac{s(\mathrm{s}^2+0.1s+4)}{(\mathrm{s}^2+0.1s+1)(\mathrm{s}^2+0.1s+9)}$ and $A(\mathrm{s}) = \frac{3}{s+3}$:



- if the actuator/velocity sensor collocation is met then $F(\mathrm{s})$ is positive and there is an alternance of poles and zeros along the imaginary axis (required for the phase response to be between $-90°$ and $+90°$) :

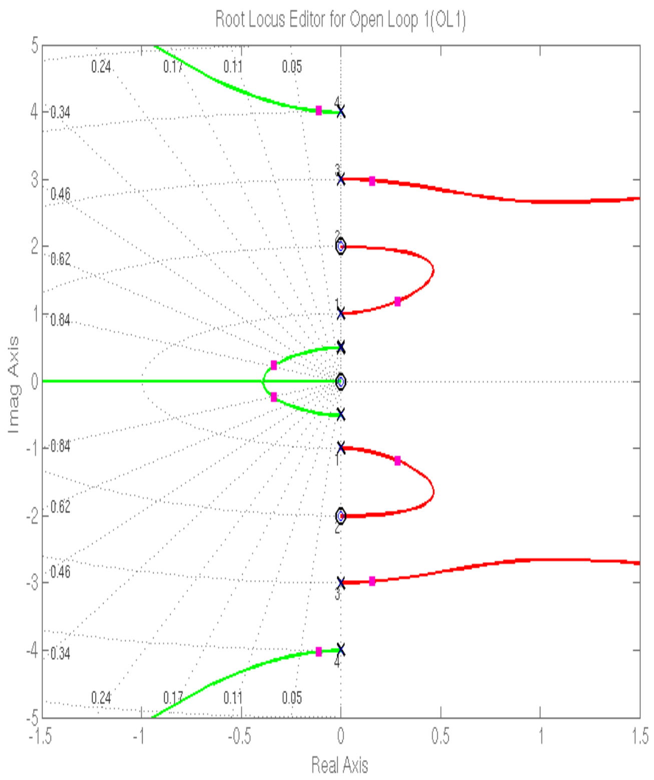
$$F(\mathrm{s}) = G\, s \frac{\Pi_{i=1}^{n-1}(\mathrm{s}^2 + z_i^2)}{\Pi_{i=1}^{n}(\mathrm{s}^2 + \omega_i^2)} .$$

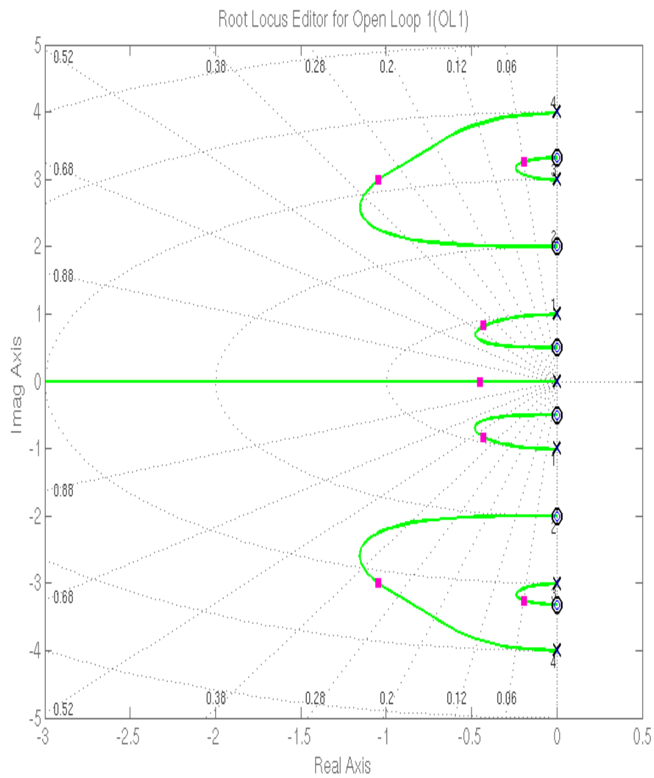
  with $\omega_i < z_1 < \omega_2 < z_2 < \cdots < z_{n-1} < \omega_n$ (and with a pole/zero cancellation at the origin if $\omega_1 = 0$, i.e. if there is a rigid mode).

- Root locus : if the branch starting from a flexible mode ($\pm j\omega_i$) is stabilizing then the branch starting from the next mode ($\pm j\omega_{i+1}$) is un-stabilizing (and reciprocally) **except** if there is a “flexible” zero z_i ($\pm jz_i$) between ω_i and ω_{i+1}. That is why the alternation of poles and zeros of a positive transfer guarantees that any positive feedback stabilizes all the modes.

Illustration :

$$F(s) = \frac{s(s^2+4)}{(s^2+\frac{1}{4})(s^2+1)(s^2+9)(s^2+16)} \qquad F'(s) = \frac{s(s^2+0.25)(s^2+4)(s^2+11)}{s^2(s^2+1)(s^2+9)(s^2+16)}$$





Outline

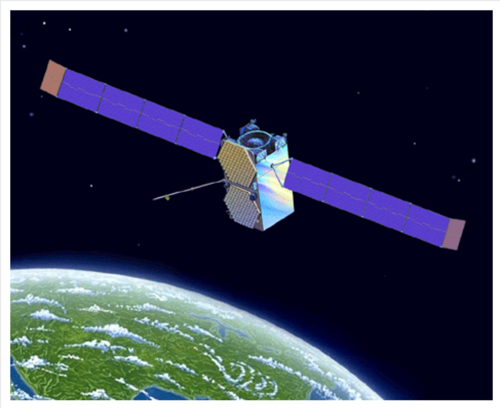
Outline

“Classical” control design in SISO case.

If the system is positive, a simple gain allows to damp all flexible modes (see example $F'(\mathrm{s})$ in slide # 67 or example $F(\mathrm{s})$ in slide # 65, even with actuator dynamics $F(\mathrm{s})A(\mathrm{s})$).
The tuning of the gain can be done interactively (using `sisotool`) to maximize the damping ; keeping in mind that :

- a flexible system is also badly damped for a high loop gain,
- it is not possible to master the damping of all the modes using a simple gain : the closed loop damping depends on the distribution of poles and zeros along the imaginary (i.e. : the residues of the flexible modes),
- the delays and dynamics of the avionics can perturb the positivity of the system : it is recommended to tune the gain including a model of this avionics,
- the first cantelivered frequency can limit the bandwitdh for the rigid mode control (if the first flexible mode has a high residue).

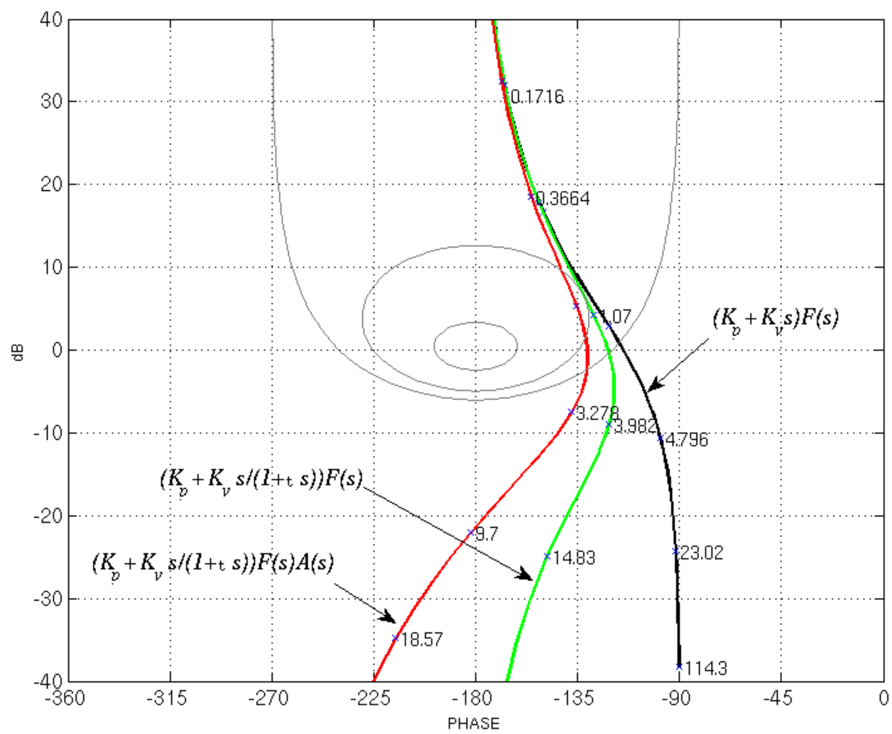
A typical example : single axis attitude control of a spacecraft.



Control of the rigid mode :

Considering $F_r(\mathrm{s}) = \frac{\Theta}{U}(\mathrm{s}) = \frac{1}{Js^2}$, the rigid model for a satellite attitude control (single axis) and ω_r and ξ_r the required attitude servo-loop bandwidth and damping ratio, then $K_p = J\omega_r^2$ and $K_v = 2J\xi_r\omega_r$:

- state-feedback ($\mathbf{x} = [\theta, \ \dot{\theta}]^T$) : $u = K_p(\theta_{ref} - \theta) - K_v\dot{\theta}$,
- output-feedback : $u = K_p(\theta_{ref} - \theta) - \frac{K_v s}{1+\tau s}\theta$ (ex : $\tau = 0.1/\omega_r$),
- output-feedback and additional dynamics (filter or avionics) : $A(\mathrm{s}) = \frac{\omega_a}{s+\omega_a}$ (ex : $\omega_a = 10\omega_r = 1/\tau$).

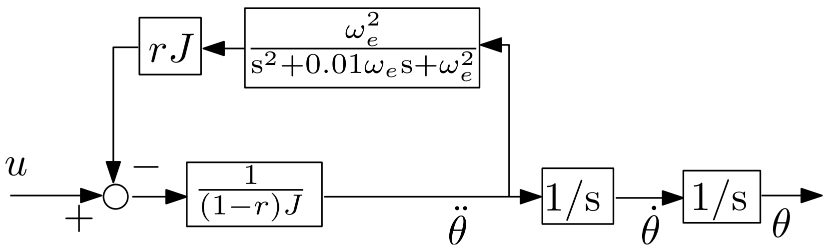


NICHOLS plot of PD control on the double integration (N.A. : $J = 1\,Kg\,m^2$, $\omega_r = 1\,rd/s$, $\xi_r = 0.7$).

Considering now the first flexible mode of an appendage :

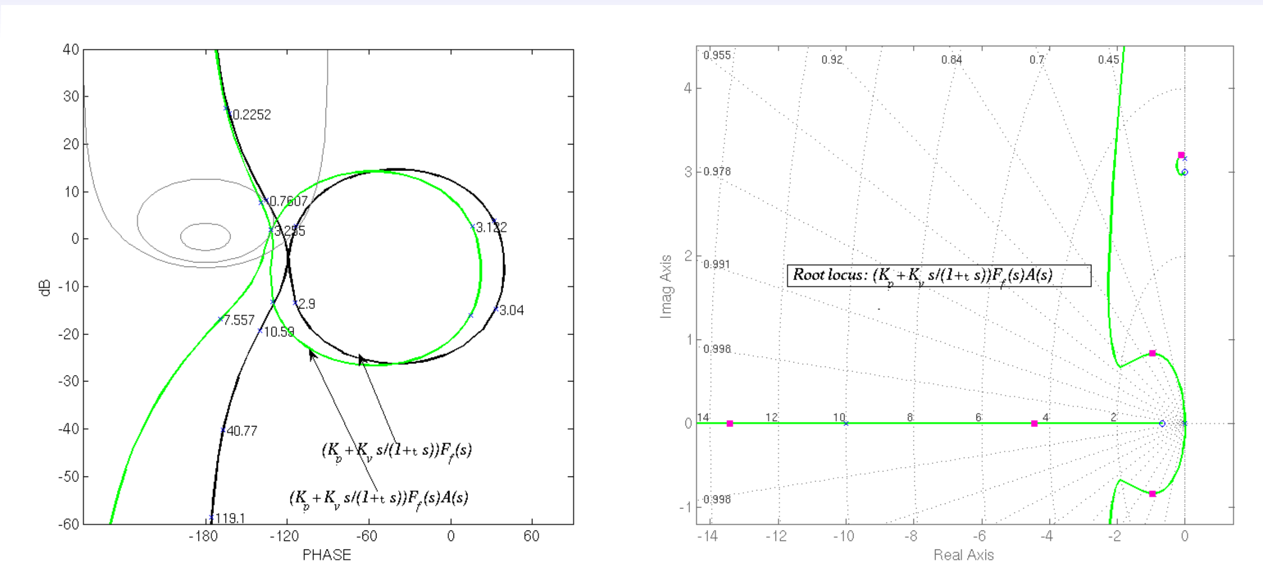
$$M^A(\mathrm{s}) = J_A \frac{\omega_e^2}{\mathrm{s}^2 + 0.01\omega_e s + \omega_e^2}.$$

$J_A = rJ$ and $J_s = (1-r)J$ are the appendage and main body inertias such that $J_S + J_A = J(= 1\,Kg\,m^2)$; defined by the appendage to total inertia ratio r. ω_e is the appendage cantilevered frequency.
The model $F_r(\mathrm{s})$ becomes now $F_f(\mathrm{s})$:

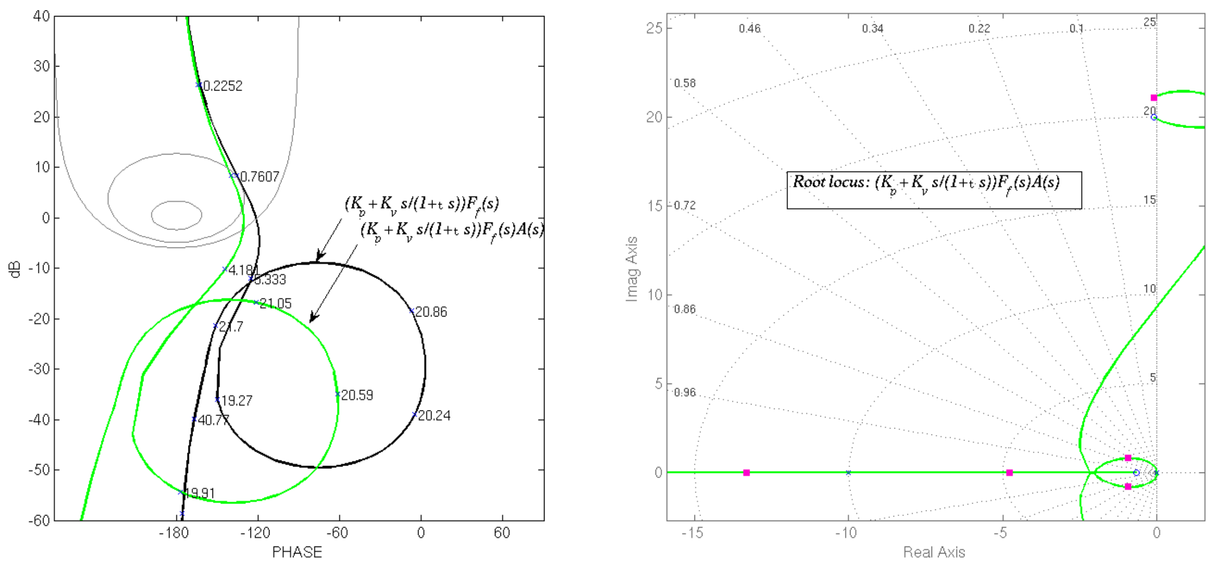


Model of a satellite with one flexible mode.

Various behaviors and control problems according to the values of r and ω_e (w.r.t. ω_r and ω_a).

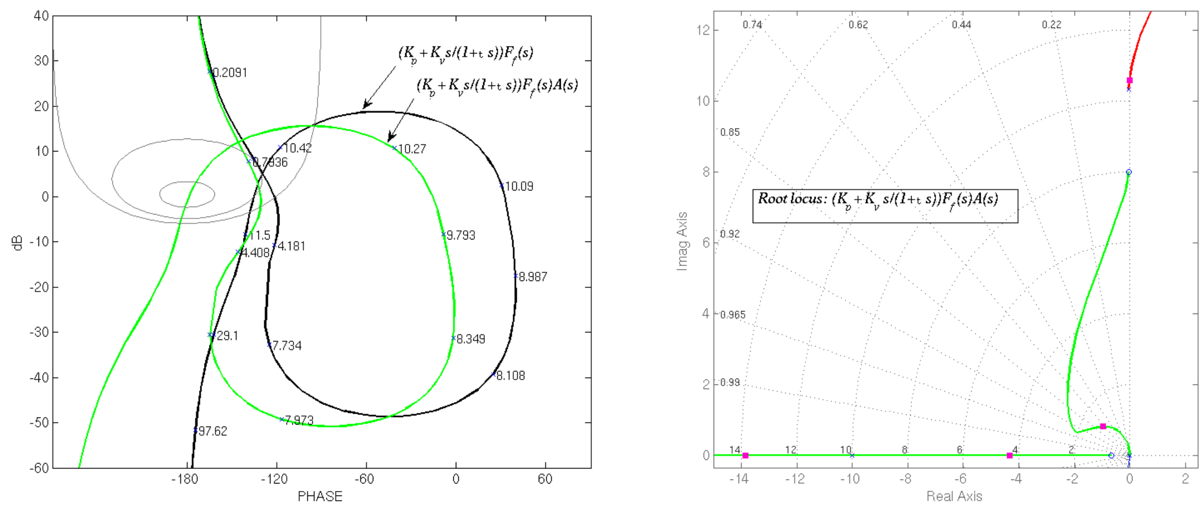


$r = 10\%$ (low residue flexible mode), $\omega_e = 5\,rd/s < \omega_a$: the flexible mode is “phase controlled” (no problem...).

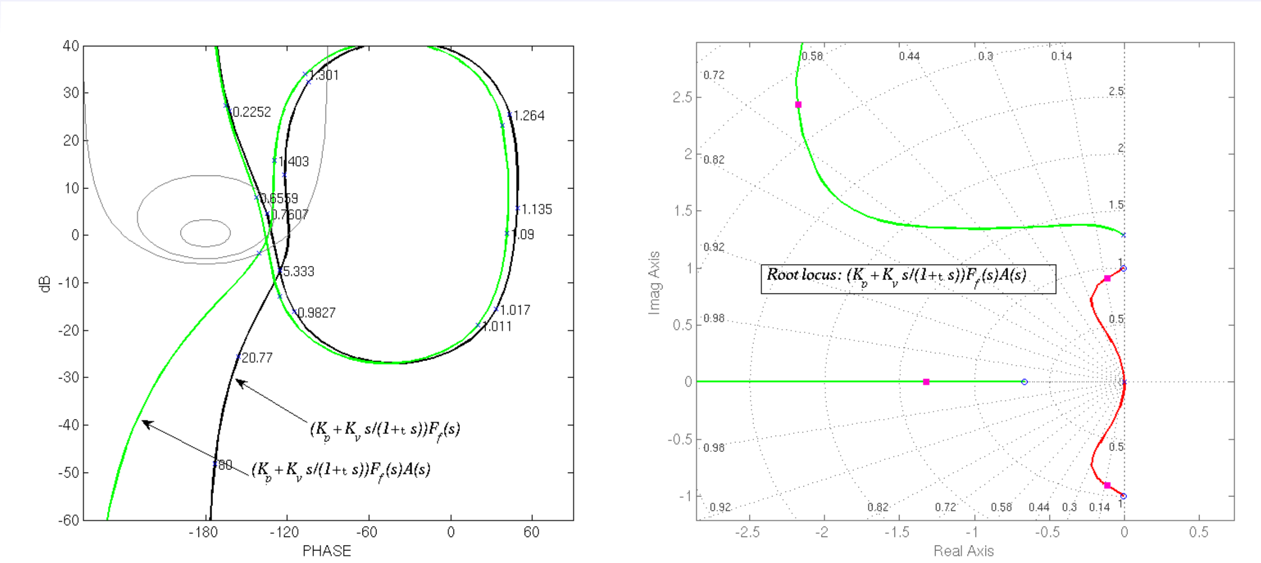


$r = 10\%$ (low residue flexible mode), $\omega_e = 20\,rd/s > \omega_a$: the flexible mode is "gain controlled" (no problem...).

Dynamic modeling of flexible mechanical systems Positivity and actuator/sensor collocation Control of flexible systems



$r = 40\%$ (high residue flexible mode), $\omega_e = 8\,rd/s$: the phase lag due to $A(\mathrm{s})$ destablizes the flexible mode.



$r = 40\%$ (high residue flexible mode), $\omega_e = 1\, rd/s \approx \omega_r$: the cantilevered pulsation limits the closed-loop dynamics of the rigid mode.

“Classical” control design in SISO case.

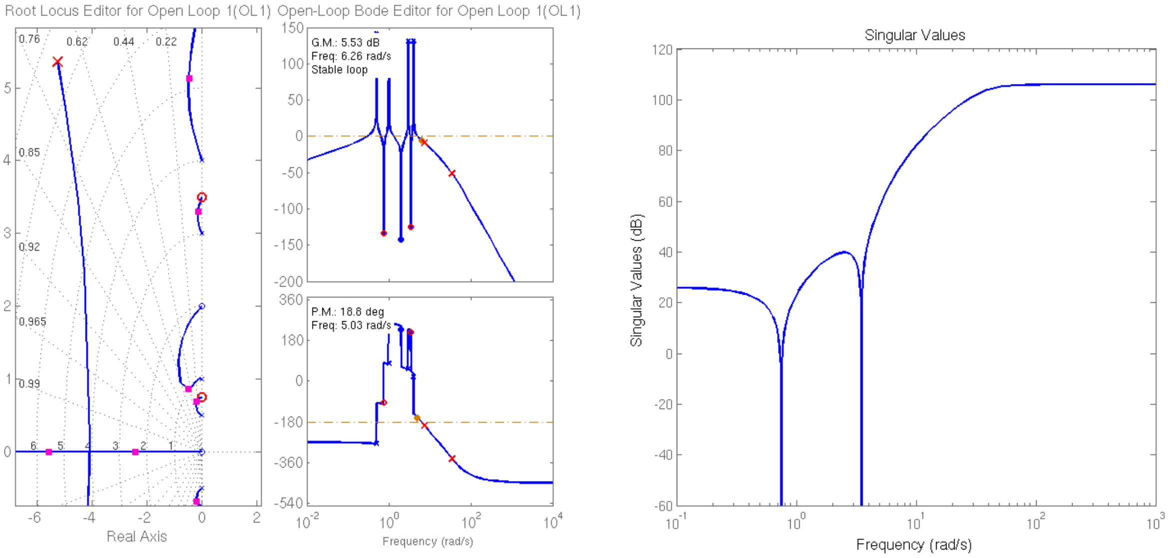
If the system is not positive, one can use the property on the alternance of “flexibles” poles and zeros to restore the positivity of the loop transfer $L(\mathrm{s})$ assigning poles and zeros in the controller dynamics along the imaginary axis.

- the lack of a “flexible zero” between 2 consecutive flexible modes (ω_i and ω_{i+1}) requires $(\mathrm{s}/z)^2 + 1$ in the controller (with : $\omega_i < z < \omega_{i+1}$), $\Rightarrow$ 2 poles $(1/((\mathrm{s}/p)^2 + 2\xi \mathrm{s}/p + 1))$ must be introduced in the controller (for properness) :
  - $\Rightarrow$ phase lag if these poles are too slow $(p < z)$! !
  - $\Rightarrow$ very high pass controller if these poles are too fast $(p >> z)$: $\Rightarrow$ measurement noise amplification and **spill-over**
  - Notch filter : $N(\mathrm{s}) = \frac{((\mathrm{s}/z)^2+1)}{((\mathrm{s}/z)^2+2\xi\mathrm{s}/z+1)}$ $(p = z)$ to filter a flexible mode $(z = \omega_i)$.
- if the alternance is inverted due to uncertainties or variations on system frequencies $\Rightarrow$ **loss of stability robustness ! !**.
- if there is a frequency decoupling between modes, one can also use a low pass filter in the controller (gain control).

Exemple : Considering the transfer $F(\mathrm{s})$ of slide # 67, one can propose :

$$K(\mathrm{s}) = 20\left(\frac{(\mathrm{s}/0.75)^2+1}{(\mathrm{s}/7.5)^2+1.4\mathrm{s}/7.5+1}\right)\left(\frac{(\mathrm{s}/3.5)^2+1}{(\mathrm{s}/35)^2+1.4\mathrm{s}/35+1}\right)$$

to damp all flexible modes but the frequency-domain response of $K(\mathrm{s})$ is too much high pass ! !

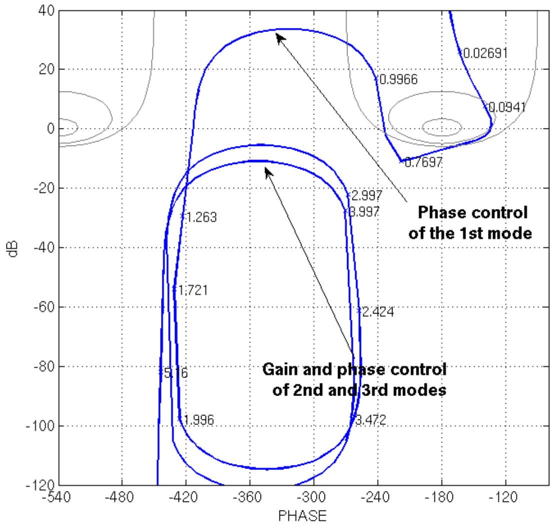


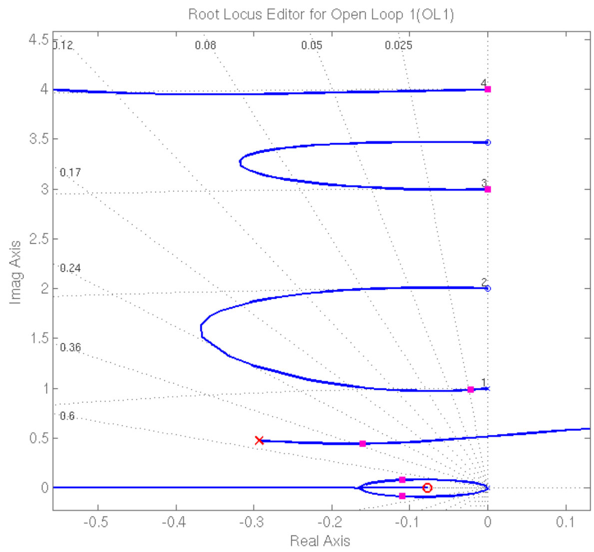
Phase control/Gain control

Phase control :
A flexible mode is "phase controlled" if the controller phase response brings the required phase lead or lag at the flexible mode frequency for this mode to be "locally" positive. The damping of this mode will increase in closed-loop. The resonnance magnitude of the flexible mode is higher than 1 ($0\,dB$) and its phase between $-90°$ and $90°$ ($\pm 360°$) on the loop transfer frequency-domain response (Nichols plot). The controller can be defined by a template on its phase response.

Gain control :
A flexible mode is "gain controlled" if the controller magnitude response brings the required attenuation at the flexible mode frequency for the resonance to be filtered. This mode will not be more damped in closed-loop than in open-loop. The resonnance magnitude of the flexible mode is very lower than 1 ($0\,dB$) on the loop transfer frequency-domain response (Nichols plot). The controller can be defined by a template on its magnitude response.

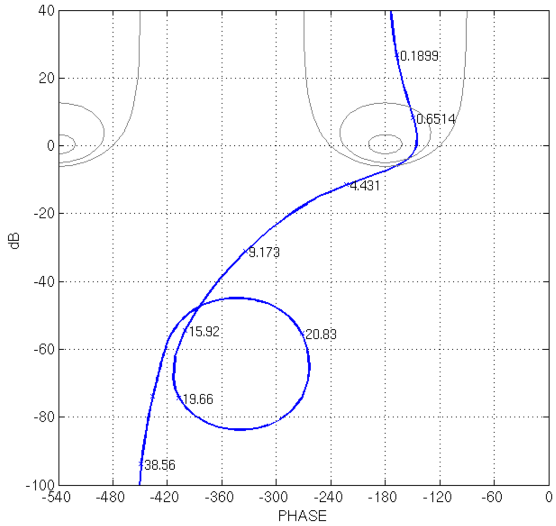
Phase and gain control can be mixed. Example : use a second order (instead of a first or third order) low pass filter to roff-off the controller behavior and bring $180°$ of phase lag for high-frequency non-collocated flexible modes : see exemple of transfer $F_P(\mathrm{s})$ in slide # 16.
$\Rightarrow K(\mathrm{s}) = 0.04\frac{13s+1}{1.8\mathrm{s}^2+1.9\mathrm{s}+1}$ to control the rigid mode with a $0.14\,rd/s$ bandwidth.

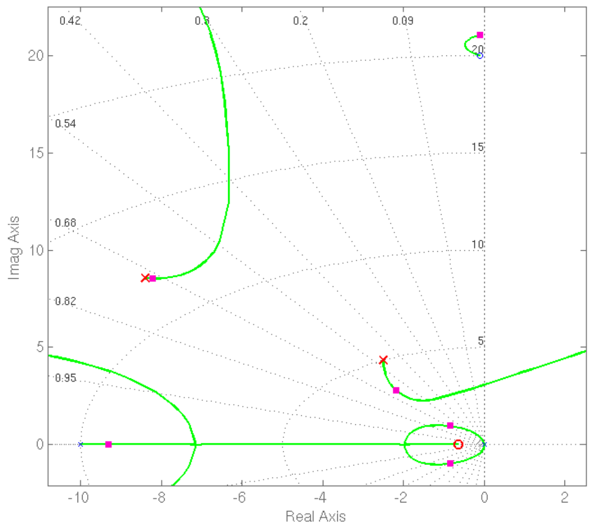




Dynamic modeling of flexible mechanical systems Positivity and actuator/sensor collocation Control of flexible systems

Other example : see slide # 75.





$r = 10\%$ (low residue flexible mode), $\omega_e = 20\,rd/s$.

$K(\mathrm{s}) = 0.75\left(K_p + K_v\frac{\mathrm{s}}{1+\mathrm{s}/5+\mathrm{s}^2/25}\right)\left(\frac{1}{1+1.4\mathrm{s}/12+(\mathrm{s}/12)^2}\right)$ to control the gain and the phase of flexible mode (but the phase margin is reduced ! !).

Outline

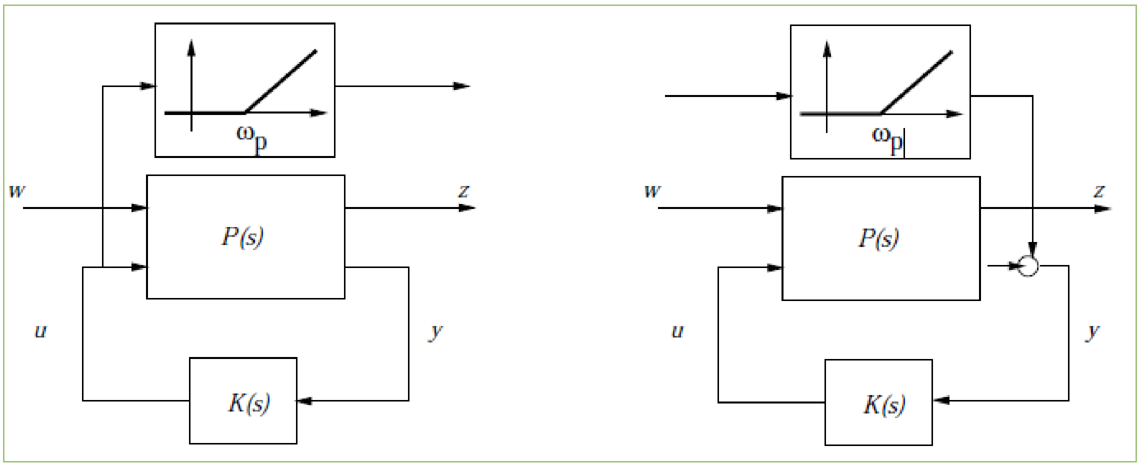
Optimal approach in the general case

Some problems encountoured applying (H_∞, H_2, LQG) optimal control designs to flexible structures and some elements of solutions :

- "spill-over" problem : $\Rightarrow$ *Roll-off specification*,
- local inversion of the plant in the controller : $\Rightarrow$ *desensitization by the PR-LQG method, multi-model approaches*,
- uncertain model $\mathbf{M}-\mathbf{\Delta}$ formulation : $\Rightarrow$ *rough modeling of uncertainties on flexible modes frequencies and damping ratios*,
- multi-objective control problem : $\Rightarrow$ *Cross Standard Form, Fixed-structure (controller and/or cost functions)* H_∞ *design*.

Spill-over ⇒ Roff-off spec.

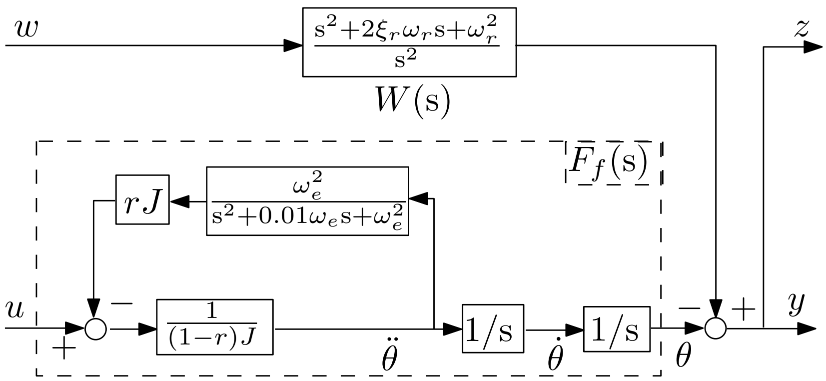
Spill-over : closed-loop unstability due to high frequency flexible modes neglected in the design model.
Prevention : Weighting the control signal (or the measurement) in high frequency to cut-off the controller action in the frequnecy range where the design model is not representative enough.



Roll-off specification on u. Roll-off specification on y.

Plant local inversion : illustration

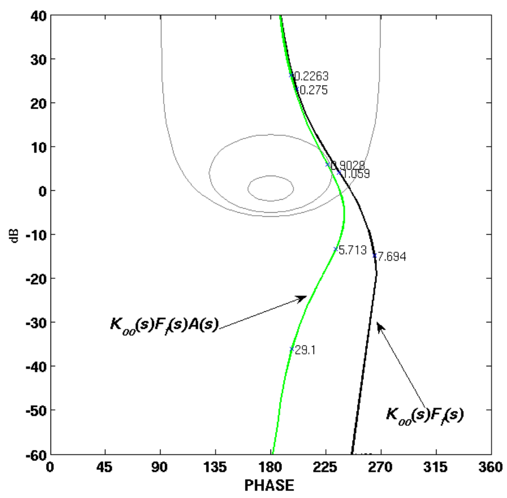
Considering the single-axis single-flexible-mode model $F_f(\mathrm{s})$ of a S/C (see slide # 73 with : $J = 1\,Kgm^2$, $r = 0.4$ and $\omega_e = 8\,rd/s$) and the H_∞ standard problem $P_{SW}(\mathrm{s})$:

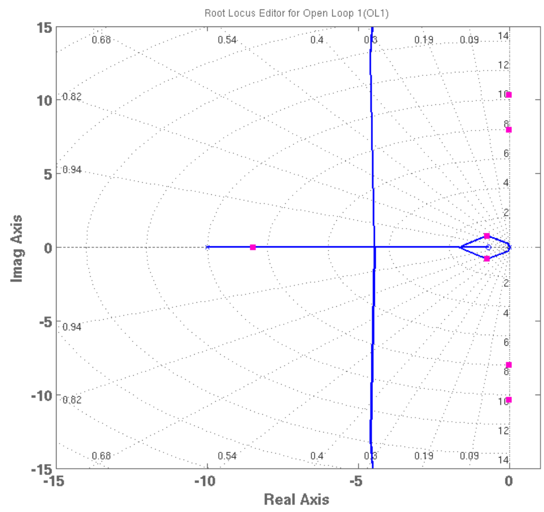


H_∞ standard problem P_{SW} to weight the sensitivity function

$$S(\mathrm{s}) = (1 + F_f(\mathrm{s})K(\mathrm{s}))^{-1} \quad \Rightarrow \quad K_\infty(\mathrm{s}) = \arg\min_K \|S(\mathrm{s})W(\mathrm{s})\|_\infty = \arg\min_K \|F_l(\text{minreal}(P_{SW}), K)\|_\infty .$$

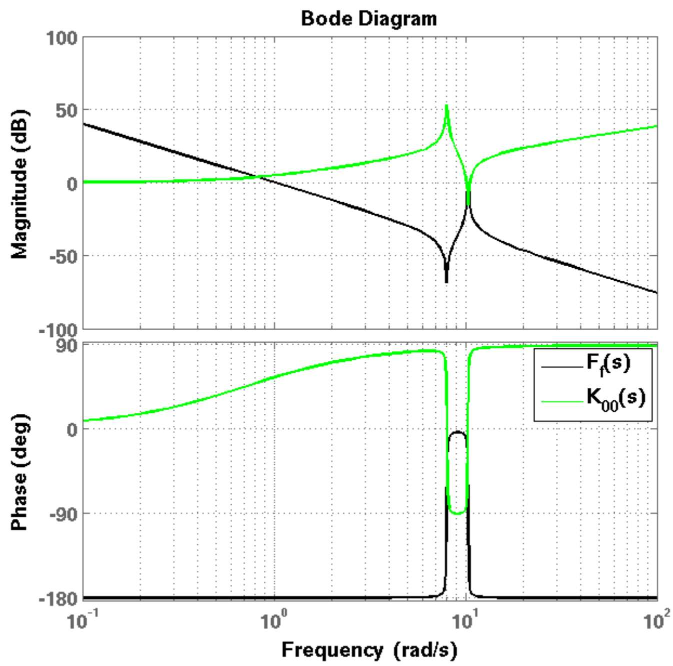
where $W(\mathrm{s}) = 1/S_{obj}(\mathrm{s})$ and $S_{obj}(\mathrm{s})$ is the sensitivity function obtained with the PD control ($K(\mathrm{s}) = K_p + \mathrm{s}K_v$) on the rigid model $F_r(\mathrm{s})$ (see slide # 71, N.A. $\xi_r = 0.7$, $\omega_r = 1\,rd/s$).





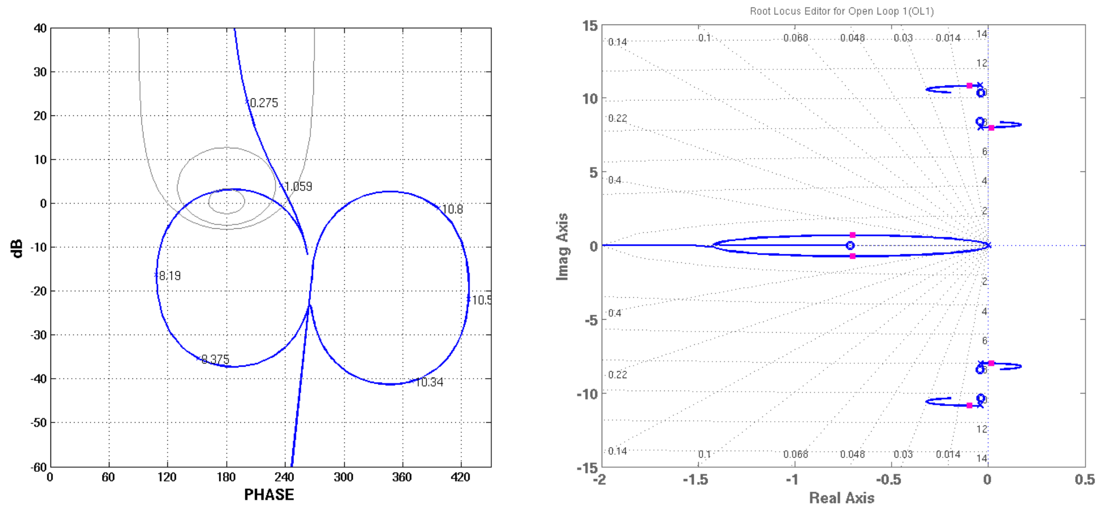
K_∞ reveals good stability margins in comparison with PD control (see slide # 76), **BUT** ...

$$K_\infty(\mathrm{s}) = \frac{2.1099e07(\mathrm{s}+0.7141)(\mathrm{s}^2+0.08\mathrm{s}+106.7)}{(\mathrm{s}^2+0.08\mathrm{s}+64)(\mathrm{s}^2+7150\mathrm{s}+2.512e07)}$$



... poles/zeros cancellations along imaginary axis $\Rightarrow$ local inversion of the flexible mode in $K_\infty(\mathrm{s}) = \frac{\mathrm{s}^2+0.01\omega_e\mathrm{s}+\omega_l^2}{\mathrm{s}^2+0.01\omega_e\mathrm{s}+\omega_e^2} K_{fast}(\mathrm{s})$ with $\omega_l^2 = \omega_e^2/(1-r)$ (free frequency).

Although stability margins are good, such a local inversion is very sensitive to parametric uncertainties. Indeed : considering $\omega_e \rightarrow 1.05\omega_e$, then $F_f(\mathrm{s}) \rightarrow F_{f,+5\%}(\mathrm{s})$:



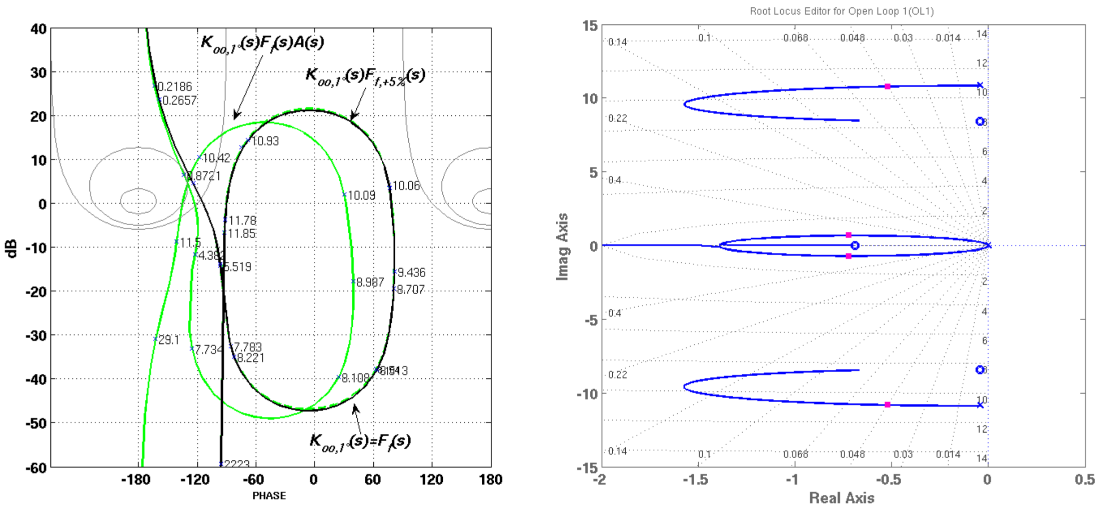
NICHOLS plot and root locus of $K_\infty(\mathrm{s})F_{f,+5\%}(\mathrm{s})$: closed-loop is unstable ! !.

Full order optimal design requires parametric uncertainties

to be taken into account.

Fixed-structure H_∞ design.

Pole/zero cancellation (local inversion) can be avoided imposing a low order controller and using fixed-structure H_∞ design. Indeed : considering the problem $P_{SW}(\mathrm{s})$ and fixing to 1 the order of the controller $\Rightarrow K_{\infty,1^\circ}(\mathrm{s})$



NICHOLS plots and root locus of $K_\infty(\mathrm{s})F_{f,+5\%}(\mathrm{s})$: closed-loop is stable ! !.

MATLAB command lines for $K_\infty(\mathrm{s})$ and $K_{\infty,1^\circ}(\mathrm{s})$.

```
% Model :
we=8 ;Ja=40/100 ;Js=1-Ja ;
G=feedback(1/Js,Ja*tf(we*we,[1 0.01*we we*we]))*tf(1,[1 0 0]) ;
% Standard Problem :
W=tf([1 1.4 1],[1 0 0]) ;
P=minreal(ss([1 ;1]*[W -G])) ;
% Full-order H_infty design :
K=hinfsyn(P,1,1) ;
% First order H_infty design :
order1=ltiblock.ss('order1',1,1,1) ;
CL=lft(P,order1) ;
[CLopt,gam]=hinfstruct(CL) ;
Kr=CLopt.Blocks.order1 ;
```


Modelling of parametric uncertainties

Robust stability and performances analysis using μ-analysis requires a $\mathbf{M}-\boldsymbol{\Delta}$ formulation of the uncertain model with the minimal size for Δ. Not always possible ! !
$\Rightarrow$ uncertainties must be taken into account ASAP in the modeling process : for instance on the generalized second order model :
$\mathbf{M}\ddot{\mathbf{q}}+\mathbf{D}\dot{\mathbf{q}}+\mathbf{K}\mathbf{q}=\mathbf{F}$ rather that the state-space representation (particularly for incertainties in the matrix $\mathbf{M}$).
$\Rightarrow$ using the substructure of the system,
In some cases, the analytic expression of the uncertain parameters cannot be done : $\Rightarrow$ a set of models sampled in the parametric space (ex : how the pulsations of the aircraft flexible modes depends on MCI ?).
Alternatives : using high level uncertain parameters :

- variations on main flexibles modes damping ratios and frequencies,
- frequency-domain templates to frame high frequency flexible modes and their variations.

Modelling of parametric uncertainties

$\mathbf{M}-\mathbf{\Delta}$ form for relative uncertainties on flexible mode damping ratios :
One can always an internal state mapping such that $M(\mathrm{s})$ reads :

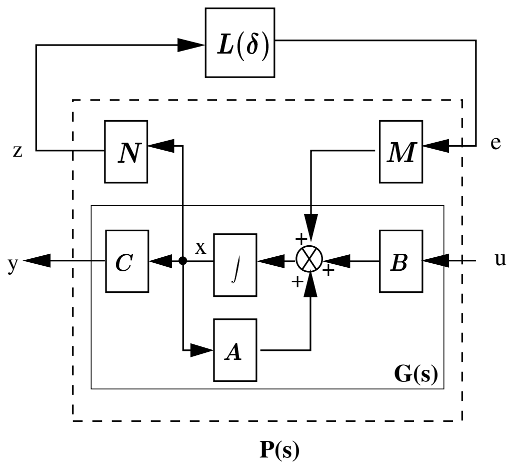
$$\left[\begin{array}{c} \vdots \\ \vdots \\ \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \vdots \\ \hline z_i \\ \hline y \end{array}\right] = \left[\begin{array}{cccccc|c|c} \ddots & & & & & & \vdots & \vdots \\ & \ddots & & & & & 0 & \vdots \\ & & 0 & 1 & & & 0 & B_i^1 \\ & & -\omega_i^2 & -2\xi_i\omega_i & & & 1 & B_i^2 \\ & & & & \ddots & & 0 & \vdots \\ & & & & & \ddots & \vdots & \vdots \\ \hline \cdots & 0 & 0 & 2\omega_i & 0 & \cdots & 0 & 0 \\ \hline \cdots & \cdots & C_i^2 & C_i^2 & \cdots & \cdots & 0 & 0 \end{array}\right] \left[\begin{array}{c} \vdots \\ \vdots \\ x_1 \\ x_2 \\ \vdots \\ \vdots \\ \hline w_i \\ \hline u \end{array}\right]$$

$$w_i = \delta_{\xi_i} z_i \quad \text{or} \quad F_u(\mathbf{M}(\mathrm{s}), \delta_{\xi_i}) \quad \Rightarrow \quad \xi_i \rightarrow \xi_i(1+\delta_{\xi_i})\ .$$

For several flexible modes : $\mathbf{\Delta} = \mathrm{diag}(\delta_{\xi_i},\ i = 1, \cdots, n)$.

Parametric Robust LQG

PRINCIPLE: closed–loop dynamics desensitization w.r.t. uncertains parameters by a judicious choice of weighting matrices



Nominal plant: $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$

Perturbed plant: $\begin{cases} \dot{x} = \hat{A}x + \hat{B}u \\ y = \hat{C}x \end{cases}$

Assumption: $\Delta A(\delta) = \hat{A} - A = ML(\delta)N$

Asymptotic robustness properties:

$F_l(P, K_{PRLQG}) \rightarrow 0$ si : (desensitization)

- $Q = N^T N$ et $R \rightarrow 0$,
- $\dim(u) \geq \dim(z)$,
- $P_{zu}(s) = N(sI - A)^{-1}B$ is minimun phase.

or (dual)

- $V = MM^T$ et $W \rightarrow 0$,
- $\dim(y) \geq \dim(e)$,
- $P_{ye}(s) = C(sI - A)^{-1}M$ is minimun phase.

Parametric Robust LQG

Practical use:

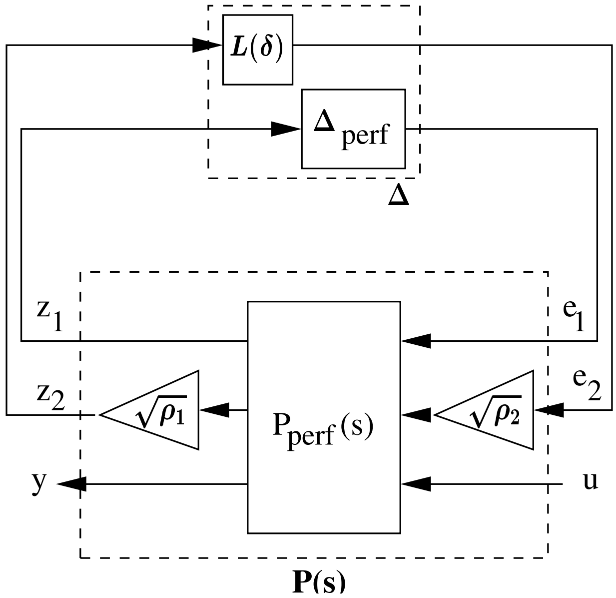
Pure performance problem P_{perf} is augmented with the uncertain block $L(\delta)$

⇓

trade–off tuning:
performance / parametric robustness

$$Q \leftarrow Q_{\text{perf}} + \rho_1 N^T N$$

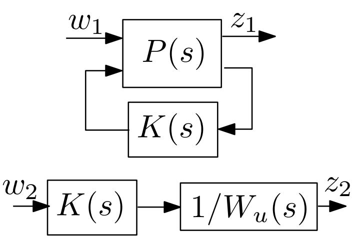
and/or $V \leftarrow V_{\text{perf}} + \rho_2 M M^T$



Multi-objective fixed-structure H_∞ control

Considering an H_∞ standard problem $P(\mathrm{s})$ to meet the performance on the nominal model $G(\mathrm{s})$, roll-off and **strong stabilization** can be handled using a multi-objective control problem involving the template $W_u(\mathrm{s})$ to be met by the controller $K(\mathrm{s})$:

$$\widehat{K}(\mathrm{s}) = \arg\min_{K(\mathrm{s})} \max(\|F_l(P(\mathrm{s}), K(\mathrm{s}))\|_\infty, \|1/W_u\, K(\mathrm{s})\|_\infty)$$



Matlab syntax (following slide # 91) :

```
Wu=10000*tf([1/1000000 1.4/1000 1],[0.01 0.14 1]);
order3=ltiblock.ss('order3',3,1,1);
CL=lft(P,order3);
[CLopt,gam]=hinfstruct(blkdiag(CL,1/Wu*order3));
K3rd=CLopt.Blocks.order3;
```

Synthesis-analysis iterative procedure

Considering the LFT representation of an uncertain model $G(\mathrm{s}, \Delta) = F_u(M(\mathrm{s}), \Delta)$ and an H_∞ standard problem $P(\mathrm{s}, \Delta)$ to meet performence requirement on the model $G(\mathrm{s}, \Delta)$, then a multi-model procedure can be used to meet robust performance, roll-off and strong stabilization :

- Initialization : $i = 0$, $\Delta_{WC,0} = 0$, $\gamma_{i-1} = 0$;
- Design :

$$\begin{aligned} \widehat{K}(\mathrm{s}) &= \arg\min_{K(\mathrm{s})} \max(\gamma_{i-1}, \|F_l(P(\mathrm{s}, \Delta_{WC,i}), K(\mathrm{s}))\|_\infty, \cdots \\ &\quad \cdots \|1/W_u\, K(\mathrm{s})\|_\infty) \\ &= \arg\min_{K(\mathrm{s})} \gamma_i(K(\mathrm{s})) \ . \end{aligned}$$

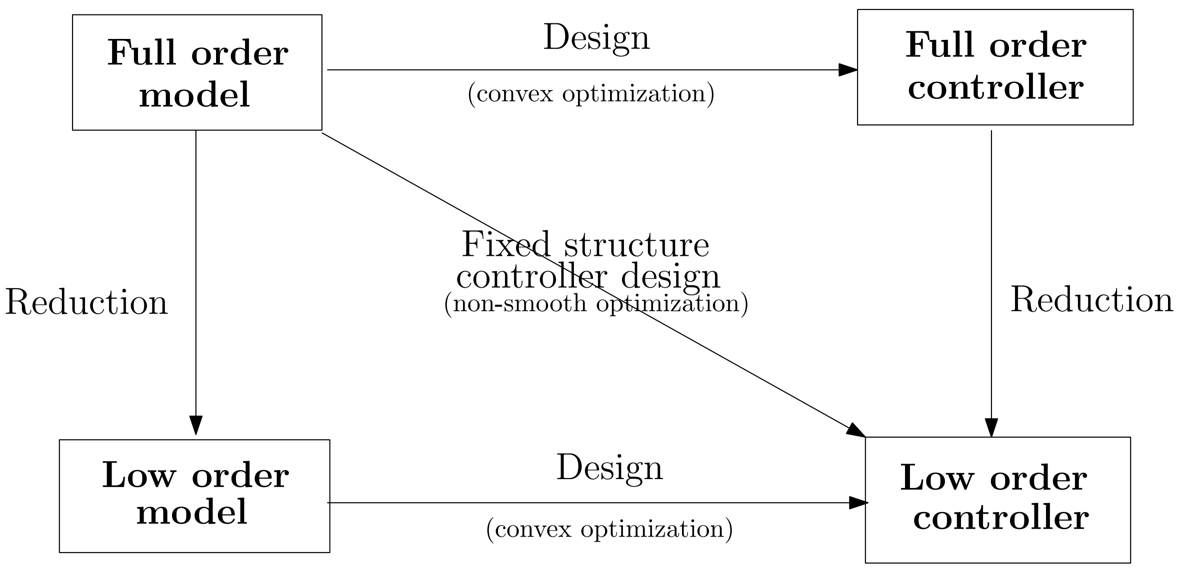
- Analysis :
  if $\underline{\mu}_\Delta(F_l(P(\mathrm{s}, \Delta), \widehat{K}(\mathrm{s}))) < 1$: end,
  else : find $\Delta_{WC,i}$, then : $i = i + 1$, go to design step
  endif.

Outline

Outline

Model (or controller) reduction

General problem



Paths from a high order model to a reduced order controller.

State space operations

Change of state coordinates (`help ss2ss`)
Let us consider a state space relization :

$$\begin{cases} \dot{x}_1 &= A_1 x_1 + B_1 u \\ y &= C_1 x_1 + D_1 u \end{cases}$$

the change of variable $x_1 = M x_2$ (with M^{-1} $\exists$) leads to :

$$\begin{cases} \dot{x}_2 &= M^{-1} A_1 M x_2 + M^{-1} B_1 u \\ y &= C_1 M x_2 + D_1 u \end{cases}$$

That is :

$$\begin{array}{rclrcl} A_2 &=& M^{-1} A_1 M, & B_2 &=& M^{-1} B_1, \\ C_2 &=& C_1 M, & D_2 &=& D_1. \end{array}$$

The reduction consists to eliminate some state variables is a particular state coordinate system.

Dynamic modeling of flexible mechanical systems Positivity and actuator/sensor collocation Control of flexible systems

Truncation : (`see modred`)
The full state vector x (length n) is partitioned as $x = [x_1^T \quad x_2^T]^T$ where x_2 is to be eliminated, and the reduced state is set to $x_r = x_1$ (length r)

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\ y = [C_1 \quad C_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D \, u \end{cases}$$

The reduced model $(A_r, B_r, C_r, D_r)_r$ is defined by :

$$\begin{array}{rclcl} A_r &=& A_{11} &=& \Gamma_r A \Gamma_r^T \\ B_r &=& B_1 &=& \Gamma_r B \\ C_r &=& C_1 &=& C \Gamma_r^T \end{array} \quad \text{with} \quad \Gamma_r = [I_r \quad 0_{r \times n-r}]_{r \times n}; \quad D_r = D\,.$$

Match DC gain : The full state vector x (length n) is partitioned as x_1 (to gather slow state variables) and x_2 (to gather fast state variables) :

$$\left\{ \begin{array}{rcl} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} & = & \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\ y & = & [C_1 \quad C_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Du \end{array} \right. \tag{23}$$

Assuming $\dot{x}_2 = 0$ (that is "x_2 is fast enough to reach its steady state during the rise-time of x_1"), then :

$$x_2 = -A_{22}^{-1}[A_{21}x_1 + B_2 u] \ ,$$

then, removing x_2 in (23) :

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & (A_{11} - A_{12}A_{22}^{-1}A_{21})x_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u \\ y & = & (C_1 - C_2A_{22}^{-1}A_{21})x_1 + (D - C_2A_{22}^{-1}B_2)u \end{array} \right.$$

Outline

Algebraic reduction methods

The partial fraction expansion of the transfer between input u_j and output y_k reads :

$$y_k = [\sum_{i=1}^{n} G_{kj}(\mathrm{s})^{(i)}]u_j \quad \text{with} \quad G_{kj}^{(i)}(\mathrm{s}) = \frac{A_{kj}^{(i)}}{s+\lambda_i} \text{ (or : } \frac{A_{kj}^{(i)} + B_{kj}^{(i)} s}{\mathrm{s}^2 + 2s\xi_i\omega_i + \omega_i^2})$$

Modal reduction : state variables are eliminated in the modal (or diagonal) realization (`see canon`). Modes or eigenvalues are sorted by :

- increasing pulsation order (λ_i, ω_i). High frequency modes are eliminated (case of flexible structure models for instance),
- decreasing eigenvalue real part order ($-\lambda_i$, $-\xi\omega_i$). Very fast modes are eliminated,
- decreasing **modal predominance** order. Modes with a weak participation on the system step or impuse responses are eliminated (time-domain approach).

Modal predominance (cont) :

- For the response of y_k to an impulse on u_j :

$$\lim_{t\to 0} y_k(t) = \lim_{s\to\infty} \sum_{i=1}^{n} sG_{kj}^{(i)}(\mathrm{s}) = \sum_{i=1}^{n} R_{kj}^{(i)} \quad \text{with} \quad R_{kj}^{(i)} = A_{kj}^{(i)} \text{ (or : } B_{kj}^{(i)}).$$

- For the response of y_k to a step on u_j :

$$\lim_{t\to\infty} y_k(t) = \lim_{s\to 0} \sum_{i=1}^{n} G_{kj}^{(i)}(\mathrm{s}) = \sum_{i=1}^{n} R_{kj}^{(i)} \quad \text{avec} \quad R_{kj}^{(i)} = \frac{A_{kj}^{(i)}}{\lambda_i} \text{ (ou : } \frac{A_{kj}^{(i)}}{\omega_i^2}).$$

Then the modal predominance is :

$$D_i = \max_{k,j} |R_{kj}^{(i)}|\ .$$

- Rk : works only for stable system with distinct eigenvalues.

DeVillemagne and Skelton projection :

That consists to approximate the transfer function (or matrix) by a TAYLOR series :

- around low frequencies : $G(\mathrm{s}) = -\sum_{i=1}^{\infty} s^{i-1} CA^{-i}B$.
  Terms $M_i(\mathrm{s}=0) = CA^{-i}B$ are called *null frequency moments*.
- around high frequencies : $G(\mathrm{s}) = \sum_{i=0}^{\infty} \frac{1}{s^{i+1}} CA^{i}B$.
  $M(\mathrm{s}=\infty) = CA^{i}B$ are *infinite frequency moments* also called *Markov parameters*.

The method proposed by DEVILLEMAGNE and SKELTON aims to keep the first $2r$ moments in the reduced r-order model.

DeVillemagne and Skelton projection (cont) : Example : r-order model with the first $2r$ null frequency moments of the system defined by $(A, B, C, D)_n$:

$$\begin{array}{rcl} A_r & = & LAR \\ B_r & = & LB \\ C_r & = & CR \end{array} \quad \text{avec} \quad \begin{array}{rcl} L & = & \begin{bmatrix} CA^{-r} \\ \vdots \\ CA^{-1} \end{bmatrix} \\ Z & = & [A^{-r}B \cdots A^{-1}B] \\ R & = & Z(LZ)^{-1} \end{array}$$

Remarks :

- This method supports only stable systems.
- Initial eigenvalues are lost.
- The physical meaning of states variable is lost.
- The approximation of null frequency moments ($s = 0$) provides a reduced model with the correct DC gain.
- The reduced order model stability is not guaranted.

Outline

Balanced reduction

Summary

- Introduction. A practical approach to controllabilty and observability
- Definitions : controllability and observability grammians,
- Balanced reduction.

A practical approach to controllabilty and observability

General criteria : binary information
Practical need to quantify (to measure) controllability and observability.
Example :

$$\left[\begin{array}{c}\dot{x}\\ \hline y\end{array}\right]=\left[\begin{array}{cc|c}\lambda & 0 & 1\\ 0 & \lambda+\epsilon & 1\\ \hline 1 & 1 & 0\end{array}\right]\left[\begin{array}{c}x\\ \hline u\end{array}\right]\quad \text{with} \quad \epsilon \quad \text{small .}$$

$\Rightarrow$ Controlability and observability grammians.

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Controlability grammian : X_c

(`see gram`) Let us consider a stable system :

$$\left[\begin{array}{c}\dot{x}\\ \hline y\end{array}\right]=\left[\begin{array}{c|c}A & B\\ \hline C & D\end{array}\right]\left[\begin{array}{c}x\\ \hline u\end{array}\right]$$

Definition : $\boxed{X_c=\int_0^{\infty}\left(e^{At}BB^Te^{A^Tt}\right)dt}$

Computation : X_c is solution of a LYAPUNOV equation :
$AX_c+X_cA^T+BB^T=0$
Properties :

- X_c is symmetric, semi-definite positive. Its SVD reads :
  $X_c=V_c\Sigma_c^2V_c^T$ avec $\Sigma_c^2=\text{diag}(\sigma_i)$
- The system is controllable iff X_c is definite positive.

Observability grammian : X_o

Definition : $\boxed{X_o = \int_0^\infty \left(e^{A^Tt}C^TCe^{At}\right)dt}$

Computation : X_o is solution of a LYAPUNOV equation :
$A^TX_o + X_oA + C^TC = 0$

Properties :

- X_o is symmetric, semi-definite positive.
- System is observable iff X_o is positive definite ($\sigma_i > 0 \quad \forall i$).

Grammians depends on the state coordinate system.
Indeed : if $x = M\tilde{x}$, then :
$\boxed{\widetilde{X_c} = M^{-1}X_cM^{-T}}$ and $\boxed{\widetilde{X_o} = M^TX_oM}$.

Remark : $\widetilde{X_c}\widetilde{X_o} = M^{-1}X_cX_oM$ the eigenvalues of X_cX_o are invariant by M.

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Balanced realization

$$\exists\, M \;/\; \widetilde{X_c} = \widetilde{X_o} = \Sigma^2 = \text{diag}(\sigma_i) \quad i = 1, \cdots, n \text{ with :}$$

- $\sigma_1 > \sigma_2 > \cdots \sigma_n$ (HANKEL singular values),
- $\sigma_i = \sqrt{\lambda_i(X_cX_o)}$,
- the realization $(\tilde{A}, \tilde{B}, \tilde{C})$ associated with M is said **balanced**.

Computation of M :

a) compute X_c and X_o using LYAPUNOV equations,

b) singular value decomposition (`svd`) of X_c and X_o :

$$X_c = V_c\Sigma_c^2V_c^T, \quad X_o = V_o\Sigma_o^2V_o^T,$$

c) then $P = \Sigma_oV_o^TV_c\Sigma_c$ (root square of X_oX_c) and P is decomposed (`svd`) :

$$P = U\Sigma^2V^T \quad (\text{rk} : \Sigma^2 = \Sigma_c\Sigma_o \quad \text{since } V_o^T \text{ and } V_c \text{ is unitary}),$$

d) then $\boxed{M = V_c\Sigma_cV\Sigma^{-1}}$.

Balanced reduction

Remarks :

- the frequency-domain error between G and G_r is bounded : $||G(\mathrm{s}) - G_r(\mathrm{s})||_\infty \leq 2(\sigma_{r+1} + \sigma_{r+2} + \cdots + \sigma_n)$,
- the reduced model is stable and balanced.
- Initial eigenvalues are lost.
- The physical meaning of states variable is lost.

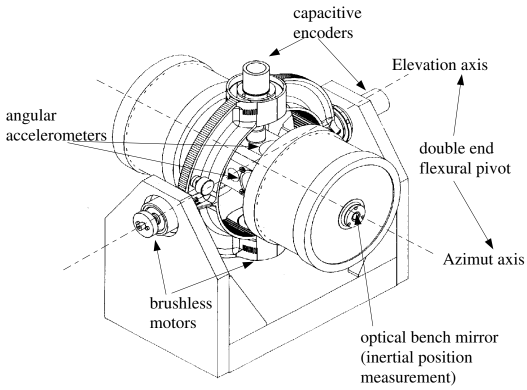
Outline

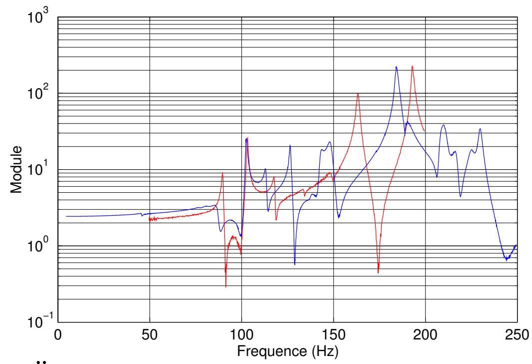
Dynamic isolation of a space telescope.

Objective : isolation of the line of sight θ_p from the holder vehicle distrubances θ_s using inertial position servo-loop and fictitious inertia feedback.

Problems :

- dynamic couplings between rigid and flexible modes,
- parametric robustess to the mechanical impedance of the support.





CAT : Telescope Attitude Control. $\frac{\ddot{\theta}_{p,elev}}{U}(\mathrm{s})$ for 2 different supports.

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Dynamic isolation of a space telescope.

Principle :

- "Rigid" open-loop (elevation-axis) model :

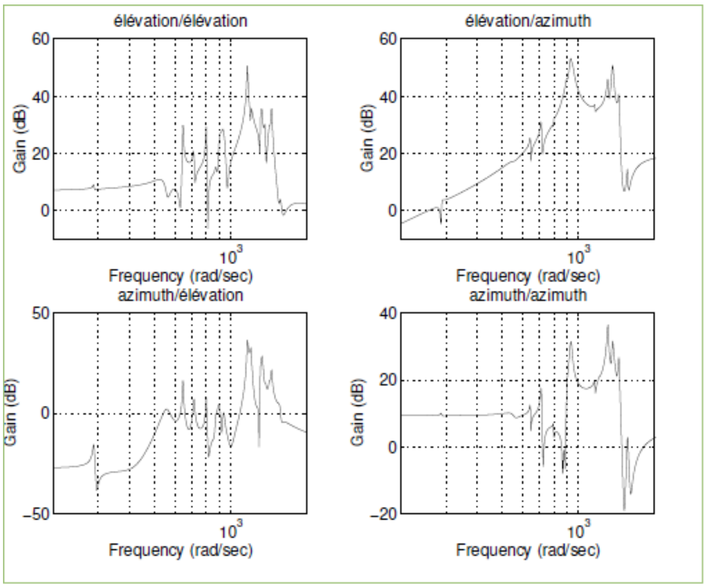
$$I\ddot{\theta}_p = u - k_B(\theta_p - \theta_s) - f_B(\dot{\theta}_p - \dot{\theta}_s)\,.$$

  where k_B and f_b are the stiffness and damping of the BENDIX flexural pivot.

- Inertial servo loop : $u = -K_p\theta_p - K_v\dot{\theta}_p \Rightarrow$ low-frequency disturbance rejection,
- fictitious inertia feedback : $u = -J_{fict}\ddot{\theta}_p \Rightarrow$ high-frequency disturbance rejection. But acceleremoter feedback very sensitive to phase lag (actuators and sensors) and flexible modes.

Dynamic isolation of a space telescope.

Identification of the 2×2 (elevation and azimut axes) inertia model $\frac{1}{I}(\mathrm{s})$.

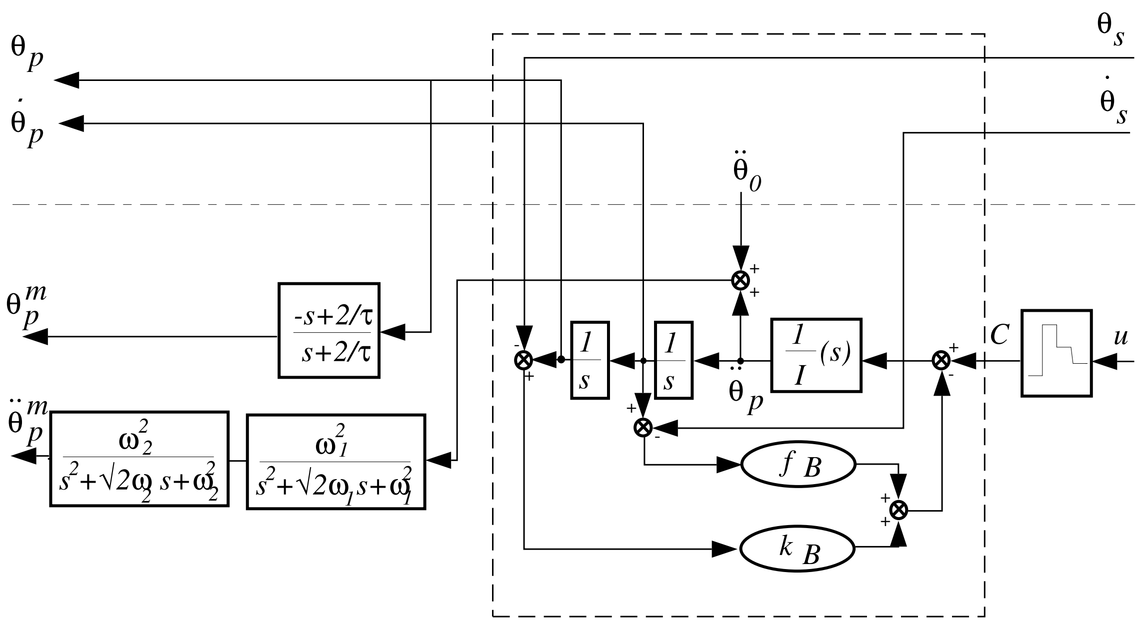


40-th order identified model $1/I(\mathrm{s})$.

Only the elevation axis is considered hereafter.

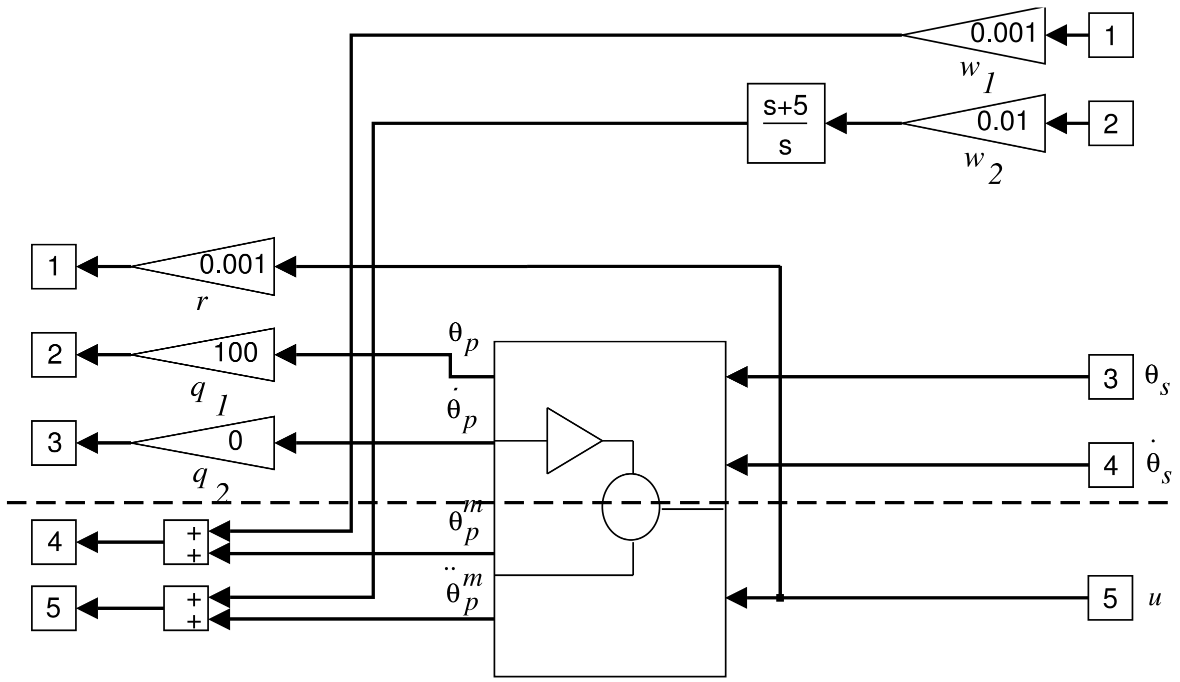
Dynamic isolation of a space telescope.

Full model with flexural pivot, avionics and disturbances :



Dynamic isolation of a space telescope.

H_2 design for pure disturbances rejection performance (and accelerometer bias wash-out).



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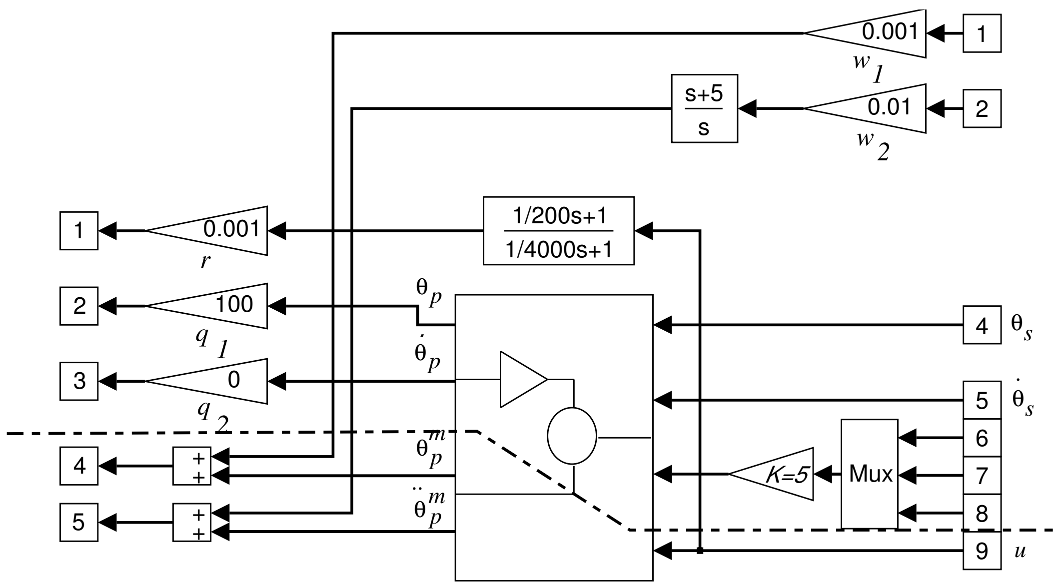
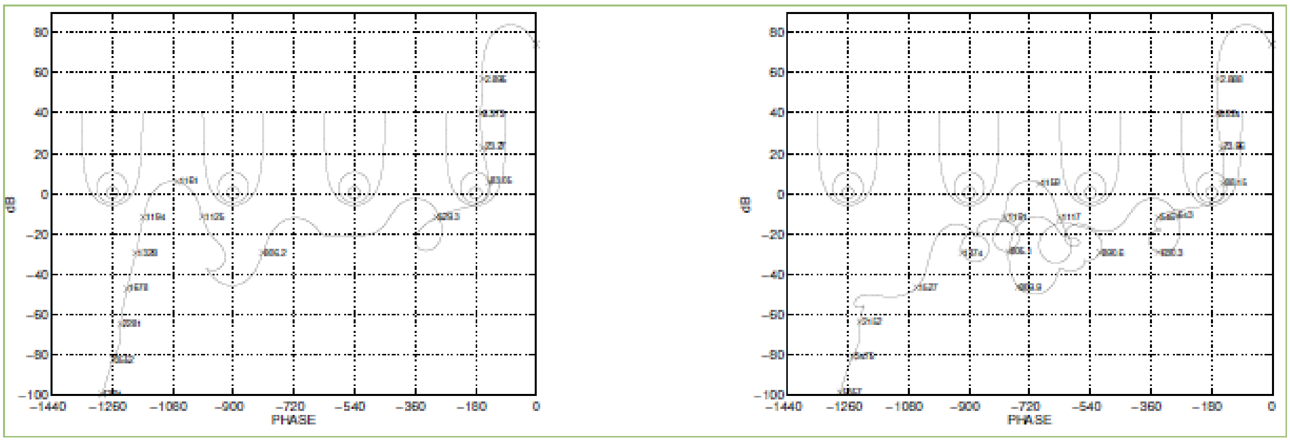
Dynamic isolation of a space telescope.

PR-H_2 (or PRLQG) design for pure disturbances rejection performance, roll-off and robustness to flexible mode damping ratios variations (the inertia model $1/I(\mathrm{s})$ is reduced to the first 3 critical flexible modes) :



$\Rightarrow$ 22-th order controller.

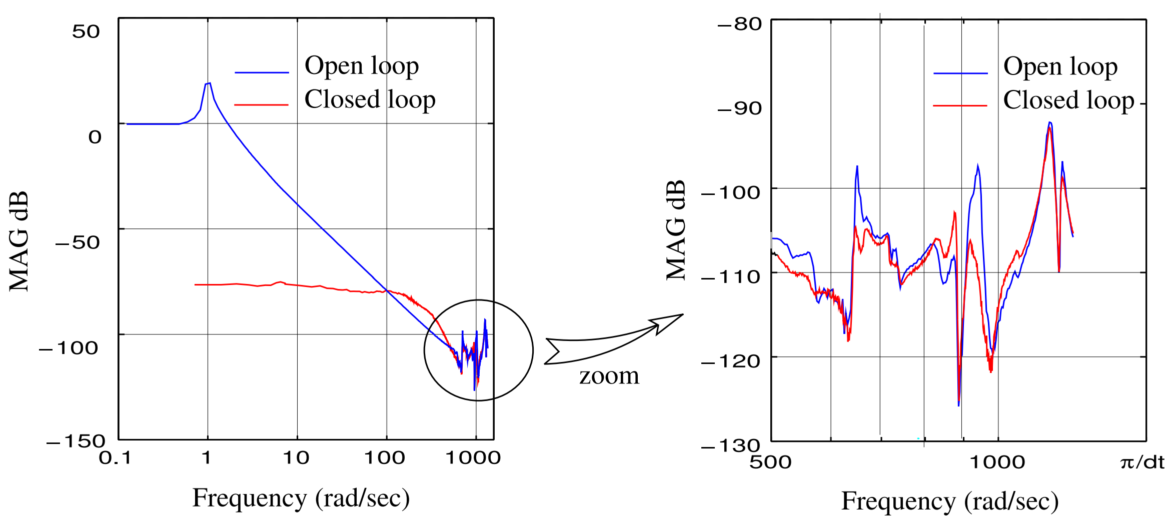
Dynamic isolation of a space telescope.



NICHOLS plots of the controlled open-loop with the design model (right) and the full-order model (left).

Dynamic isolation of a space telescope.

Experimental results : open-loop and closed disturbances rejection functions $|\frac{\theta_p}{\theta_s}(j\omega)|$ measured using a shaker :



$\Rightarrow$ disturbance rejection $> \ 75\,dB\ \forall\,\omega$ and attenuation of the first resonnances.

Outline

Chapter 1

MATLAB labworks

1.1 Bamoss: Flexible structure modeling, analysis and control

The experimental testbed (developped in ISAE/DMIA) studied in this practical labwork is depicted in the Figure 1.1. A video on the dynamic (open-loop and closed-loop) behavior of this testbed can be downloaded at `http://personnel.supaero.fr/alazard-daniel/demos/film_bamoss_ve.mpg`.

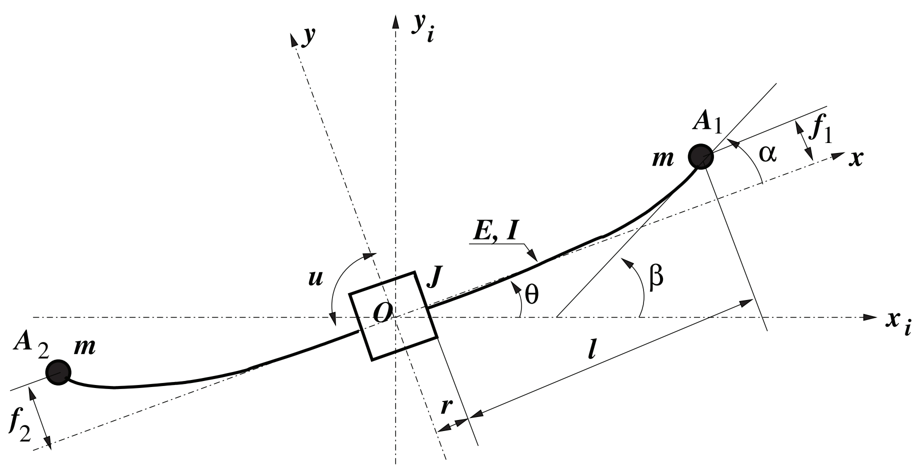


Figure 1.1: BAMOSS simplified sketch.

This testbed works in the horizontal plane $R_i = (0,\ x_i,\ y_i)$ (r_i is an inertial frame) and is composed of:

- a hub with a square shape ($R = (0,\ x,\ y)$ is the hub-body frame), articulated w.r.t. the inertial frame by a pin-join around a vertical axis $(0,\ z_i)$. The half-side and the inertia (around $(0, z_i)$) of this hub are denoted r and J, respectively,

- a torque motor, driving the hub in rotation around the vertical axis $(0, z_i)$,
- 2 identical flexible beams (in the horizontal plane) cantilevered on the hub. The sizes of each beam are l (length), b (width) and h (thickness in the horizontal plane) and the young modulus of the beam material is denoted E,
- 2 local masses m fitted at the tip of each beam,
- an optical encoder to measure the angular position θ of the hub w.r.t. the inertial frame,
- a tachometer to measure the angular rate $\dot{\theta}$ of the hub w.r.t. the inertial frame,
- a gyrometer to measure the inertial angular rate $\dot{\beta}$ at one tip mass (at point a_1).

Let us denote :

- $u\,(Nm)$: the driving torque (control signal),
- $\theta\,(rd)$: angular position of the hub,
- f_1 and $f_2\,(m)$: lateral deflections at the free end of each beam,
- $\alpha\,(rd)$: the angular deviation (slope, w.r.t. to equilibrium position) at the free-end of the beam fitted with the gyrometer,
- $\beta = \theta + \alpha$,
- $S(s)$ the LAPLACE transformation of a signal $s(t)$.

The objective of the lab-work is to model the dynamic behavior of this system (applying structural dynamics theory) and to use MATLAB tools (applying automatic control theory) to analyse this model and to design a controller for the regulation of this system (servo-loop) around the position $\theta = 0$. the servo-loop performances will be analyzed by comparison of the open-loop and closed-loop responses to initial conditions. a main goal consists in simulating (using MATLAB) the experimental response presented in figure 1.2, obtained in the following conditions:

- at time $t = 0$, the following initial conditions are manually prescribed (see video):

$$\theta(t=0) = 0.18 \text{ deg}, \quad f_1(t=0) = 4\,mm, \quad f_2(t=0) = 0.5\,mm, \ldots$$

$$\ldots \dot{\theta}(t=0) = 20 \text{ deg}/s, \quad \dot{f}_1(t=0) = -7\,cm/s, \quad \dot{f}_2(t=0) = 18\,cm/s\ ,$$

- at time $t = 5\,s$, the feedback-loop is closed (on a proportional-derivative controller).

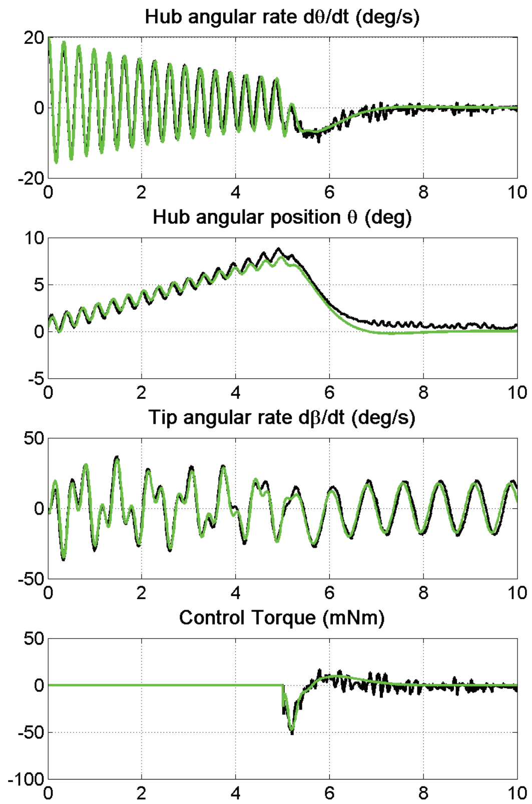


Figure 1.2: Experimental (black) and simulated (green) responses to initial conditions.

Assumptions:

- only motions in the horizontal plane are considered,
- masses and inertia of beams are neglected (inertia of local masses at each beam tip are also neglected),
- only small motions are considered.

Numerical application: $J = 0.015\,kg\,m^2$, $l = 0.286\,m$, $E = 200\,10^9\,N/m^2$, $m = 0.30\,kg$, $r = 0.05\,m$, $h = 0.64\,mm$, $b = 4\,cm$.

1.1.1 Dynamic modeling

Static deflection of a beam

Let us consider a single cantilever beam with a tip load P at its free-end (see following figure 1.3).

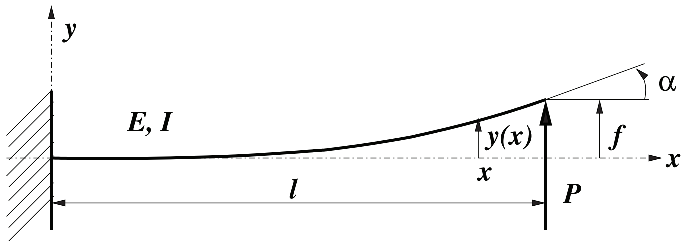


Figure 1.3: Static deformation of a cantilever beam with a tip load.

Question A.1: compute the lateral deflection f at the free-end and deduce the equivalent stiffness k of the beam (i.e. such that : $P = kf$) as a function of EI (I is the section quadratic moment) and l. Express the slope α at the free-end as a function of f and l.

The flexible beam can be now considered as a string with stiffness k acting between the mass m and the tip of a fictitious rigid beam of the hub body (see Figure 1.4. In the sequel, it is assumed that strings and iniertials forces of masses

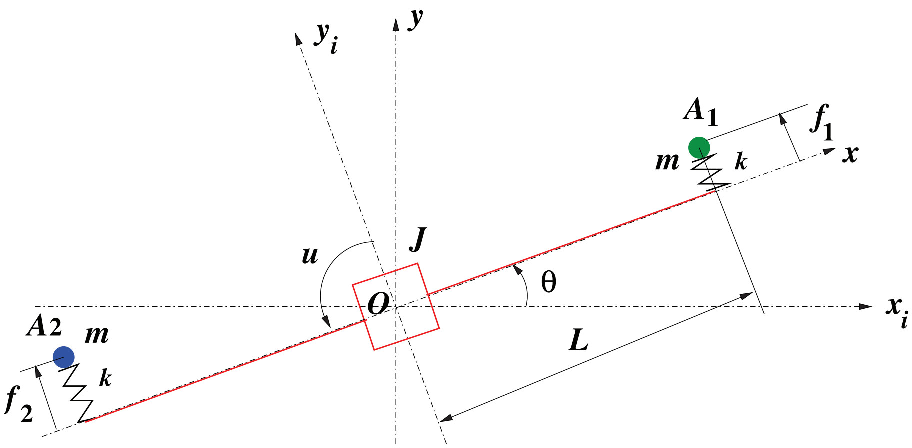


Figure 1.4: BAMOSS very simplified sketch.

(m) work on ly the normal direction y of the beam (i.e. CORIOLIS and centrifugal accelerations are negleted. To derive the dynamic model, 2 options are proposed:

- Option 1: dynamic modeleling and analysis using Lagrange derivation. Questions A.2, A.3, A.4, A.5, ...
- Option 2: dynamic modeling using Euler, Newton approach. Questions A.2bis, A.3bis, A.5, ...

Option 1: dynamic modeling and analysis using Lagrange derivation

Let us denote : $q = [\theta \quad f_1 \quad f_2]^T$ the vector of generalized coordinates (that is the set of variables allowing the geometric configuration of the system to be entirely defined).
Question A.2: express the kinetic energy $\mathcal{T}$, the potential energy $\mathcal{V}$ and the dynamic model of this system under the generalized second-order form:

$$M\ddot{q} + Kq = Fu \; . \tag{1.1}$$

Open-loop modal analysis

Question A.3:

- compute eigen-pulsation $\omega_i : \; i = 1, \cdots, 3$ of the free system ($u = 0$),
- draw qualitatively the modal shape of each eigen-mode,
- study the system controllability directly from the second-order form.

Closed-loop modal analysis

Question A.4: compute the transmission zeros of the transfer function $\frac{\Theta}{U}(s)$ using a closed-loop modal analysis of system (1.1) with $u = -K_\theta \theta$ and $K_\theta \rightarrow \infty$. Do the same question with the transmission zeros of the transfer function $\frac{\beta}{U}(s)$.

State-space representation

An arbitrary damping matrix is introduced in the dynamic model:

$$D = \begin{bmatrix} 0.007 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$

and the dynamic model reads now:

$$M\ddot{q} + D\dot{q} + Kq = Fu \; . \tag{1.2}$$

Option 2: dynamic modeling using Euler, Newton approach

Question A.2bis: applying the Dynamic fundamental principle on the 3 bodies of Figure 1.4 (the green mass, the blue mass and the red main body), show that the dynamic behavior is described by the following set of 2-nd order differential equations:

$$J\ddot{\theta} = u + kLf_1 - kLf_2 - f_v\dot{\theta} \tag{1.3}$$
$$m(\ddot{f}_1 + L\ddot{\theta}) = -kf_1 \tag{1.4}$$
$$m(\ddot{f}_2 - L\ddot{\theta}) = -kf_2 \tag{1.5}$$

where $f_v = 0.007\,Ns/m$ is the viscous friction coefficient acting inside the hub pivot joint.

Question A.3bis: describe this set of equations by a functional block-diagram involving only integrators, sum and gains. Build the associated SIMULINK file.

Question A.5: create, using MATLAB macro-function `ss` (option 1) or `linmod` (option 2), the model of the system between the input u and the 3 outputs θ, $\dot{\theta}$ and $\dot{\beta}$ and analyze:

- system eigen-values (macro-function `eig` or `damp`),
- system controllability (macro-function `ctrbf`),
- poles and zeros of the 3 transfers using function `zpk` (pole/zero cancelations and non-minimum phase response on the transfer between u and $\dot{\beta}$ should be highlighted and must match analytical results obtained in questions A.3 and A.4).

Question A.6: plot and comment the frequency-domain responses of the 3 transfers (macro-function `bode`).

Question A.7: plot output time-domain responses to initial conditions x_0 (macro-function `initial`) and comment observability property of each mode from each output.

1.1.2 Control design

Question B.1: using root loci and NICHOLS plots (macro-function `rltool`), analyze the control law :

$$u = K_p(\theta_{ref} - \theta) - K_d\dot{\theta}$$

(the loop must be " opened " on the input u). θ_{ref} is the hub position input reference and is null during regulation. Proportional and derivative gains K_p and K_d are tuned on the rigid-assumed model (which can be obtained when the Young modulus E tends toward infinity) in order to have a second order closed-loop behavior with a pulsation $\omega = 2\,rd/s$ and a damping ratio $\xi = 0.707$.

Question B.2: simulate the closed-loop time-domain responses to initial conditions (macro functions `feedback`, `initial`) and the step response (macro function `step`). Comment these responses.

Question B.3: convert the continuous-time model to a discrete time-model taking into account a zero-order hold on the input u (macro-function `c2d`, sampling period $T_s = 0.1\,s$) and analyze, on root locus and Nichols plots, the previous proportional-derivative control law. Choose a sampling period to have at least a 6 dB gain margin.

Question B.4: let us consider one more time the continuous-time domain model and assume that only θ et $\dot{\beta}$ are measured: analyze, on root locus and Nichols plot, the control law:

$$u = K_p(\theta_{ref} - \theta) - K_d\dot{\beta}\ .$$

Design a filter $F(s)$ such that the new control law :

$$u = F(s)\Big(K_p(\theta_{ref} - \theta) - K_d\dot{\beta}\Big)\ .$$

is stabilizing.

Representation, analysis and reduction of multi-variable systems
Matlab/Simulink Lab-work
D. Alazard

Model analysis and reduction

The objective of this labwork is to implement multivariable system analysis tools and to use reduction techniques (balanced reduction) to get a **design model** simple and representative of the dynamical behavior of the system (or the full order **validation model**). The objective is also to highlight that reduction criteria depend also on the required closed-loop behavior, i.e. on the control law (which depend itself on the design model, i.e. on the reduction (which depends on the control law, i.e....)). Brief, model reduction and control design are strongly linked.

The minimal **design model** G to describe the lateral flight of a rigid aircraft is given by a state space realization:

$$\left[\begin{array}{c}\dot{\beta}\\ \dot{p}\\ \dot{r}\\ \dot{\phi}\\ \hline n_y\\ p\\ r\\ \phi\end{array}\right]=\left[\begin{array}{cccc|cc}-0.080 & 0.104 & -0.995 & 0.045 & 0 & 0.016\\ -2.016 & -0.843 & 0.280 & 0 & 0.342 & 0.192\\ 1.711 & -0.179 & -0.293 & 0 & 0.010 & -0.782\\ 0 & 1.000 & 0.104 & 0 & 0 & 0\\ \hline 0.0224 & 0.0013 & 0.0015 & 0 & -0.0002 & -0.0023\\ 0 & 1.0000 & 0 & 0 & 0 & 0\\ 0 & 0 & 1.0000 & 0 & 0 & 0\\ 0 & 0 & 0 & 1.0000 & 0 & 0\end{array}\right]\left[\begin{array}{c}\beta\\ p\\ r\\ \phi\\ \hline dp\\ dr\end{array}\right]. \tag{1}$$

Two static output feedbacks were designed in order to decouple the yaw and the roll channels:

$$K_{lent}=\left[\begin{array}{cccc}-249. & 3.26 & 1.23 & 2.46\\ 7.57 & 0.32 & -2.02 & 0.034\end{array}\right]. \tag{2}$$

$$K_{rapide}=\left[\begin{array}{cccc}-215.93 & 12.913 & 1.9431 & 18.194\\ 147.69 & -0.3517 & -3.3046 & -0.7355\end{array}\right]. \tag{3}$$

The **validation model** G_f, taking into account the actuators dynamics, anti-aliasing filters (FAR), delays, ... is given in Figure 1.

These data (G, G_f, K_{lent} and K_{rapide} can be downloaded from the file `data_avionlat.mat` (function `load`)).

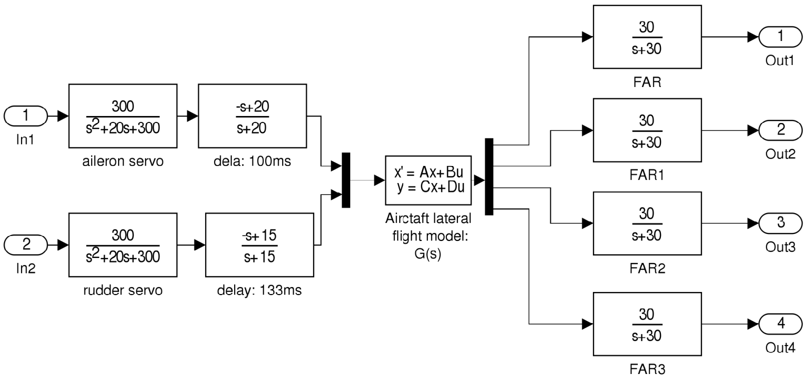


Figure 1: Block diagram description of the full model.

1. Comparison criteria

The objective is to compare the design model G and the validation model G_f using several criteria: time-domain responses, frequency-domain responses, eigen-structures, in open-loop or in closed-loop.

1.a Compare and comment the frequency-domain responses of G and G_f (function `sigma`),

1.b compare and comment the frequency-domain responses of $I+K_{lent}G$ and $I+K_{lent}G_f$,

1.c analyse the dynamics of G and G_f in closed-loop on the controller K_{lent} (it is recommended to plot the root-loci of $K_{lent}G$ and $K_{lent}G_f$ using the function `rlocusp`[1]),

1.d the responses of β et ϕ to initial conditions on β and ϕ are now considered. Compare the reponses obtained from the design model G and from the validation modele G_f in closed loop with the controller K_{lent} (a `SIMULINK` file can be used),

1.e same questions (from 1.b ro 1.d) substituting K_{lent} by K_{rapide}.

2. Balanced reduction

From the previous analyzes, one can conclude that:

[1]This function is in the toolbox `bib1` available at:
`http://personnel.isae.fr/daniel-alazard/matlab-packages/`

- if the closed-loop performances are not challenging, the dynamic behaviour obtained with K_{lent} on the validation model G_f is close to the predicted behavior obtained on the design model G: G is thus sufficiently representative of the full model dynamic behavior and can be used to design a controller,

- if the closed-loop performances are challenging, the dynamic behaviour obtained with K_{rapide} on the validation model G_f is not at all consistent with the behavior predicted using the design model G (oscillation problems, ...): G is not representative enough when performances require to be pushed up (settling time, ...) and connot be used to design a high performance controller,

A new reduced-order design model G_r is thus required to re-design the high performance controller (the controller K_{rapide} designed on G is not satisfaying).

2.1 Compute a balanced realization of the full order model G_f (function `balreal`). Plot and comment the Hankel singular values on a bar graph.

2.2 Reduce the balanced realization till you get a design model G_r with minimal order and satisfaying the comparison criteria used in question 1 with K_{rapide}, substituting G by G_r (i.e. a reduced model G_r representative of the dynamic behavior obtained with G_f and K_{rapide}).

ENSICA: Flexible Structure Modelling and Control (2008/2009)
Written examiniation - 2h30 Open book

D. Alazard

1 Exercise

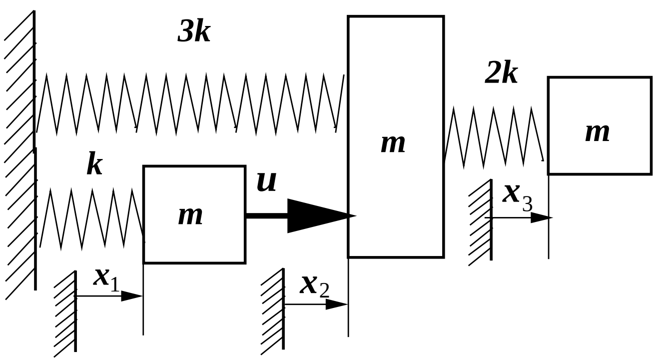
The transfer function $G(s)$ of flexible system between the applied force u and the velocity measurement y is:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s(s^2+4)}{(s^2+1)(s^2+25)} \ .$$

- plot qualitatively the BODE response (magnitude and phase) of $G(s)$,
- plot qualitatively the root locus of the loop transfer obtained with the controller $u = -\frac{s^2+9}{s^2+16}y$. Is it a stabilizing controller?
- same questions with the controller $u = -\frac{s^2+16}{s^2+9}y$.

2 Problem

The system depicted in the following Figure involves 3 identical masses (value: $m\,(Kg)$) and 3 stiffnesses (values: k, $2k$ and $3k\,(N/m)$):



x_1, x_2, x_3 represent "small" displacements of the 3 masses around their equilibrium positions w.r.t. the inertial frame. The control signal u is a force applied by a cylinder (actuator) located between the first and the second masses.

1. Open loop modal analysis

1.1: express the kinetic energy T of the system and the mass matrix M_0 such that:

$$T = \frac{1}{2}\dot{X}^T M_0 \dot{X} \quad \text{with} \quad X = [x_1,\ x_2,\ x_3]^T.$$

1.2: express the potential energy V of the system and the stiffness matrix K_0 such that:

$$V = \frac{1}{2} X^T K_0 X.$$

1.3: express the dynamic model of the system:

$$M_0 \ddot{X} + K_0 X = F_0 u$$

and perform its modal analysis (pulsations and modal shapes).

2. Closed-loop modal analysis

2.1: the position x_2 is feedback to u through an infinite gain: ($u = -Kx_2$ with $K \to +\infty$):

- compute litteraly the finite frequencies of the closed-loop system,
- draw, in the s-plane, the pole and zero map of the transfer $G(s) = \frac{X_2}{U}(s)$ and the root-locus of the loop transfer obtained with a derivative control law: $u = -K_v \dot{x}_2$, Comment your results,

2.2: same question substituting x_2 by x_1,

2.3: same question substituting x_2 by x_3,

2.4: same question substituting x_2 by $x_2 - x_1$,

2.5: What do you think about the system controllability ?