Chapter 8
Delay-Independent Stability Via Reset Loops

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Abstract In this chapter we propose a control scheme where a pre-designed linear controller in feedback with a time-delay plant is augmented with suitable jump rules that are activated in certain subsets of the state space to ensure closed-loop asymptotic stability. Under suitable feasibility conditions on the data of the linear time-delay plant, we show that the proposed scheme induces delay-independent stability of the closed loop with controller state jumps. Due to the hybrid nature of the proposed scheme, we address stability by proposing a hybrid version of the classical Lyapunov-Krasovskii theorem, relying on a dwell-time condition and on a Lyapunov-Krasovskii function that does not increase accross jumps. The results in the chapter can be seen as preliminary results in the direction of hybrid time-delay dynamical systems, which still remains largely unexplored. A simulation example shows the effectiveness of the proposed hybrid scheme.

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8.1 Introduction

An important aspect that must be taken into account in many control applications is the presence of delay in the loop, which can be the source of performance degradation or even instability (see, for example, [17, 18] and references therein).

A control architecture that received much attention in recent years is that of reset control. Reset control systems can be represented by means of different hybrid systems formulations. Many of the previous work about systems with resets has been done by using the impulsive state dependent state formulation of [10]. Alternatively, the hybrid formulation developed in [7, 8] has been also used for describing reset control systems, and formal stability results have been already obtained (see, e.g., [14, 15, 19]). This last framework has also been used for systems with time delays in the recent work [13], that is inspiring for the notation followed in this chapter. This formulation will be first used for stability analysis of reset control system with time delays, and with a new characterization of flow and jump sets, inspired by [6], that allows us to state a Lyapunov-Krasovskii theorem for delay-independent stability of the reset control system.

From a historical perspective, the idea of reset compensation dates back to the seminal works of Clegg and Horowitz [5, 12], where simple reset compensators such as the Clegg integrator or the First Order Reset Element (FORE) were first proposed with the main motivation of overcoming fundamental limitations of linear time-invariant compensators in control practice. Intuitively, within this quest, since it was observed already in [5] (see also [2]) that reset compensation introduces phase lead without significantly increasing the loop gain, it seems to be appropriate to propose reset compensation for systems with time-delays due to the evident time lag introduced by the delay effect. In [1], a delay-independent stability analysis of reset control system is first performed, that has been extended to the more general delay-dependent case in [2, 4] and in [9].

In this chapter, we are interested in studying the stability of closed loops independently of the delay. In particular, our work is focused on the design of hybrid reset rules based on Lyapunov conditions and applied to time-delay continuous-time plants in order to eliminate possible instability arising from the linear interconnection. We propose a kind of hybrid controller that can guarantee delay-independent stability of the closed loop. The class of systems that we address combines two ingredients: (1) the continuous-time dynamics of a linear time-delay plant and a linear controller, enforced when the overall state belongs to a certain *flow set* $\mathscr{F}$ and (2) a discrete dynamics corresponding to an impulsive action performed on the controller state when the overall state belongs to a certain *jump set* $\mathscr{J}$ [8]. It is important to emphasize that few works are dedicated to the study of hybrid time-delay systems. Notable exceptions comprise the reset systems work in [1, 9] and the framework recently proposed in [13].

The scheme proposed in this chapter arises from adapting the hybrid augmentation paradigm recently proposed in [6, 16]. Moreover, to prove our main result, we also formulate a Lyapunov-Krasovskii theorem that complements the results issued from

[1] and [9], by using a different definition of flow and jumps sets and an additional dwell-time logic.

In the sequel we first introduce the problem under consideration in Sect. 8.2. Then we describe the proposed closed loop with resets in Sect. 8.3. Subsequently we prove suitable stability properties of the control scheme in Sect. 8.4 and finally we discuss a simulation example in Sect. 8.5.

8.2 Problem Data and Standing Assumption

Consider the following strictly proper linear time-delay plant:

$$\mathscr{P}: \quad \begin{aligned} \dot{x}_p(t) &= A_p x_p(t) + A_{pd} x_p(t-\theta) + B_p u_c(t), \\ y_p(t) &= C_p x_p(t), \end{aligned} \tag{8.1}$$

where $x_p \in \mathbb{R}^{n_p}$ is the state of the plant, $\theta \in \mathbb{R}_{\geq 0}$ is a known state delay, $y_p \in \mathbb{R}^{n_y}$ is the output available for measurement and u_c is the control input to be used in the controller design. It is costumary to denote the infinite dimensional state of plant (8.3) as

$$x_{pt} = \{x_p(s), s \in [t-\theta, t]\}, \tag{8.2}$$

and to use the shortcut notation $x_{pd}(t) = x_p(t-\theta)$ so that Eq. (8.3) can be written in compact from as

$$\mathscr{P}: \quad \begin{aligned} \dot{x}_p &= A_p x_p + A_{pd} x_{pd} + B_p u_p, \\ y_p &= C_p x_p. \end{aligned} \tag{8.3}$$

For plant (8.3) we assume that a linear time invariant controller has been designed to ensure suitable closed-loop properties under certain operating conditions:

$$\mathscr{C}: \quad \begin{aligned} \dot{x}_c &= A_c x_p + B_c y_p, \\ u_p &= C_c x_c, \end{aligned} \tag{8.4}$$

where $x_c \in \mathbb{R}^{n_c}$ is the state of the controller. Controller (8.4) has been designed for plant (8.3) disregarding the effect of delay, namely in such a way to stabilize the delay-free dynamics corresponding to the transition matrix $A_p + A_{pd}$, which corresponds to plant (8.3) in the special case $\theta = 0$. Then the goal of this paper is to introduce suitable reset rules on the controller states that ensure recovery of closed-loop asymptotic stability for any known value of the delay θ. Since we do not need the requirement that controller (8.4) stabilizes the plant when $\theta = 0$, we don't make this as an explicit assumption.

In this work we will use Lyapunov-Krasovskii techniques to assess asymptotic stability of the closed loop for any value of the time delay θ, that is, stability independent of the delay. Due to this fact, we will require the following assumption on the plant data.

Assumption 1 *Given the matrices in (8.3) there exist two positive definite matrices P_p and Q, a gain K_p and a scalar $\varepsilon_p > 0$ such that*

$$He \begin{bmatrix} P_p(A_p + B_p C_c K_p) + Q/2 & P_p A_{pd} \\ 0 & -Q/2 \end{bmatrix} \leq -2\varepsilon_p \begin{bmatrix} P_p & 0 \\ 0 & Q \end{bmatrix}. \tag{8.5}$$

Assumption 1 ensures that it is possible to prestabilize plant (8.3) by way of the state feedback gain $u_p = K_p x_p$ and obtain a delay-independent stable closed loop. The following lemma clarifies that the search for parameters P_p, Q and K_p in (8.5) is equivalent to a convex (LMI eigenvalue) problem.

Lemma 1 *Assumption 1 holds if and only if the following LMI in the variables $Q_p = Q_p^T > 0$, $S = S^T > 0$ and X is feasible:*

$$He \begin{bmatrix} A_p Q_p + B_p C_c X + S/2 & A_{pd} Q_p \\ 0 & -S/2 \end{bmatrix} < 0. \tag{8.6}$$

Moreover, whenever (8.6) holds, a solution to Assumption 1 is given by $P_p = Q_p^{-1}$, $Q = P_p S P_p$, $K_p = X P_p$ and a small enough ε_p.

Proof Consider Eq. (8.6) and perform the congruence transformation pre- and post-multiplying by the block-diagonal symmetric matrix $\mathrm{diag}(Q_p^{-1}, S^{-1})$. Then the following equivalent relation to (8.6) is obtained, with the definitions in the lemma:

$$\mathrm{He} \begin{bmatrix} P_p(A_p + B_p C_c K_p) + Q/2 & P_p A_{pd} \\ 0 & -Q/2 \end{bmatrix} < 0. \tag{8.7}$$

Assume now that Assumption 1 holds, then obviously Eq. (8.7) holds too, which is equivalent to (8.6). Viceversa, if (8.6) (therefore (8.7)) holds, then due to the strict inequality in (8.7) there exists a small enough $\varepsilon_p > 0$ such that (8.5) holds with the selections in the statement of the lemma.

Remark 1 Once the feasibility condition (8.6) is verified, it might be of interest to seek for the solution to (8.5) corresponding to maximizing ε_p while imposing that the gain K_p satisfies a prescribed bound $|K_p| \leq \kappa_M$, for some fixed scalar $\kappa_M > 0$. This solution can be computed by solving the following optimization problem:

$$\begin{aligned} \varepsilon_p^* = & \max_{Q_p, X, S, \varepsilon_p} \varepsilon_p \text{ subject to:} \\ & Q_p \geq I, \begin{bmatrix} \kappa_M I & X \\ X^T & \kappa_M I \end{bmatrix} \geq 0, \\ & \mathrm{He} \begin{bmatrix} A_p Q_p + B_p C_c X + S/2 & A_{pd} Q_p \\ 0 & -S/2 \end{bmatrix} \leq -2\varepsilon_p \begin{bmatrix} Q_p & 0 \\ 0 & S \end{bmatrix}, \end{aligned} \tag{8.8}$$

which is a generalized eigenvalue problem (namely a quasi convex optimization problem) for which efficient numerical solution algorithms are available. The corresponding solution to (8.5) can then be computed as $P_p = Q_p^{-1}$, $Q = P_p S P_p$ and $K_p = X P_p$, just as in Lemma 1.

8.3 Hybrid Closed-Loop System

In this section we design a hybrid closed-loop system whose flow dynamics corresponds to the interconnection between (8.3) and (8.4) and whose jump dynamics and jump and flow sets are constructed, based on a solution to (8.5) in Assumption 1, in such a way to guarantee uniform global asymptotic stability of the origin of the plant-controller state space. Note that this property is non-trivial because no assumption is made on the stability properties of the continuous-time interconnection (8.3), (8.4)

Adopting the notation in [7, 8], we propose the following dwell-time hybrid augmentation of the closed loop (8.3), (8.4), where for convenience of notation we denote the aggregated (and transformed) state $\xi = (x_p, x_{pd}, \delta) := (x_p, x_{pd}, x_c - K_p x_p)$:

$$\begin{cases} \dot{x}_p = A_p x_p + A_{pd} x_{pd} + B_p C_c x_c \\ \dot{x}_c = A_c x_c + B_c C_p x_p & (\xi, \tau) \in \mathscr{C} \times [0, 2\rho], \\ \dot{\tau} = 1 - \mathrm{dz}\left(\frac{\tau}{\rho}\right) \end{cases}$$
$$\begin{cases} x_p^+ = x_p \\ x_c^+ = K_p x_p & (\xi, \tau) \in \mathscr{D} \times [\rho, 2\rho], \\ \tau^+ = 0 \end{cases} \tag{8.9a}$$

where $\mathrm{dz}(\cdot)$ denotes the scalar unit deadzone function, the sets $\mathscr{C}$ and $\mathscr{D}$ are defined, based on an arbitrary positive scalar $\varepsilon < \varepsilon_p$, as follows:

$$\mathscr{C} := \left\{ \xi : \ \xi^T \mathrm{He} \begin{bmatrix} P_p A_R + Q/2 & P_p A_{pd} & P_p B_p C_c \\ 0 & -Q/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xi \le -\varepsilon\, \xi^T \begin{bmatrix} P_p & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & I \end{bmatrix} \xi \right\}, \tag{8.9b}$$

$$\mathscr{D} := \left\{ \xi : \ \xi^T \mathrm{He} \begin{bmatrix} P_p A_R + Q/2 & P_p A_{pd} & P_p B_p C_c \\ 0 & -Q/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xi \ge -\varepsilon\, \xi^T \begin{bmatrix} P_p & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & I \end{bmatrix} \xi \right\}. \tag{8.9c}$$

In the sets $\mathscr{C}$ and $\mathscr{D}$, A_R is defined as $A_R := A_p + B_p C_c K_p$ and matrices P_p and Q are defined in Assumption 1.

Following [8] and inspired by the definitions in [13], we introduce the following definitions to suitably characterize solutions to hybrid system (8.9a–8.9c). In particular, the following definitions are slightly different from those in [13] because we are exploiting here the property that the memory of the time-delay system is only in the directions of the plant state, which remains unchanged across jumps.

Definition 1 A subset $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is called a *compact hybrid time domain* if

$$E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j),$$

for some finite sequences of times $0 = t_0 \leq t_1 \leq \cdots \leq t_J$, called "jump times". The set E is called a *hybrid time domain* if for all $(T, J) \in E$, the set $E \cap ([0, T], \{0, 1, \ldots, J\})$ is a compact hybrid time domain.

Given a positive real $\theta > 0$, a subset $E_0 \subset \mathbb{R}_{\geq -\theta} \times \mathbb{Z}_{\leq 0}$ is called a *hybrid time domain with ordinary memory* θ if

$$E_0 = ([-\theta, 0), 0) \cup E,$$

where E is a hybrid time domain.

A *hybrid arc with ordinary plant memory* θ is a triple consisting of a domain $\mathrm{dom}\phi$ that is a hybrid time domain with ordinary memory θ, a continuous function $\phi_{p0}(s),\ s \in [-\theta, 0]$ representing the (infinite dimensional) initial condition of the system in the plant state direction and a function $\phi : \mathrm{dom}_{\geq 0}\phi \to \mathbb{R}^{n_p \times n_c}$, where $\mathrm{dom}_{\geq 0}\phi := \mathrm{dom}\phi \cap (\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0})$, such that $\phi(\cdot, j)$ is locally absolutely continuous on $I^j = \{t :\ (t, j) \in \mathrm{dom}_{\geq 0}\phi\}$.

Similar to [13, Def. 4], but for the special case where the memory of the time-delay system is only in the ordinary time direction, given any hybrid arc $\phi = (\phi_p, \phi_c)$ with ordinary plant memory θ, for each $(t, j) \in \mathrm{dom}_{\geq 0}\phi$, we define the operator $\mu_{[t,j]}\phi_p(s)$ that maps the hybrid arc ϕ into an ordinary memory arc of length θ (this is a function of ordinary time only). In particular, $\mathrm{dom}\mu_{[t,j]}\phi_p = [-\theta, 0]$ and for each $s \in [-\theta, 0]$ we have $\mu_{[t,j]}\phi_p(s) = \phi_p(t + s, i)$ for some $i \in \mathbb{Z}_{\geq 0}$ such that $(t - s, i) \in \mathrm{dom}\phi$. Note that such an i exists because the hybrid arc ϕ has ordinary plant memory θ by assumption. Note also that the definition above has no ambiguity because of the special structure in (8.9a–8.9c) where the p component of the solution remains constant across jumps. Indeed, if there exist multiple values $i_1,\ i_2,\ \in \mathbb{Z}_{\geq 0}$ such that $(t - s, i_1), (t - s, i_2) \in \mathrm{dom}\phi$, then we have $\mu_{[t,j]}\phi_p(s) = \phi_p(t + s, i_1) = \phi_p(t + s, i_2)$. Indeed, projecting the memory of the time-delay system only on the ordinary time domain axis (by way of the operator $\mu_{[t,j]}$) greatly simplifies the forthcoming derivations.

Based on the above characterization, we can formulate a class of hybrid time-delay systems that generalizes the peculiar structure of (8.9a–8.9c) and that can be written in compact form as follows:

$$
\begin{aligned}
&\begin{bmatrix}\dot{x}_p\\ \dot{x}_c\\ \dot{\tau}\end{bmatrix} = \begin{bmatrix} f_p(x_p, x_c, \mu x_p)\\ f_c(x_p, x_c, \mu x_p)\\ 1-\mathrm{dz}\left(\frac{\tau}{\rho}\right)\end{bmatrix}, \quad \begin{bmatrix}x_p\\ x_c\\ \mu x_p\\ \tau\end{bmatrix} \in \widetilde{\mathscr{C}} \times [0, 2\rho], \quad (x_p(0,0), x_c(0,0)) \in \mathbb{R}^{n_p \times n_c}\\
&\begin{bmatrix}x_p^+\\ x_c^+\\ \tau^+\end{bmatrix} = \begin{bmatrix} x_p\\ g_c(x_p, x_c, \mu x_p)\\ 0\end{bmatrix}, \quad \begin{bmatrix}x_p\\ x_c\\ \mu x_p\\ \tau\end{bmatrix} \in \widetilde{\mathscr{D}} \times [\rho, 2\rho], \quad \mu_{[0,0]} x_p \in \mathscr{C}^0([-\theta, 0]),
\end{aligned}
\tag{8.10}
$$

where $\mathscr{C}^0(\mathscr{T})$ denotes the set of continuous functions whose domain is $\mathscr{T} \subset \mathbb{R}$, f_p, f_c and g_c are suitable linear functionals that, for system (8.9a–8.9c), correspond to:

$$
\begin{aligned}
f_p(x_p, x_c, \mu x_p) &= A_p x_p + A_{pd} \mu x_p(-\theta) + B_p C_c x_c,\\
f_c(x_p, x_c, \mu x_p) &= A_c x_c + B_c C_p x_p,\\
g_c(x_p, x_c, \mu x_p) &= K_p x_p.
\end{aligned}
$$

Moreover, the flow and jump sets $\widetilde{\mathscr{C}}$ and $\widetilde{\mathscr{D}}$ are suitable infinite dimensional flow and jump sets.

Based on Definition 1, a solution to hybrid time-delay system (8.10) is a hybrid arc with ordinary plant memory θ satisfying the flow and jump constraints imposed by the hybrid dynamics. A precise formulation of this can be obtained by following the same paradigm as that used in [13, Def. 6].

Remark 2 The extension of the formalism in [7, 8] to the time-delay framework is currently underdeveloped (a notable exception being the recent paper [13]). Therefore, not much can be stated about existence of solutions or even nominal or robust well posedness of solutions to these hybrid systems in the sense of [8, Chap. 6]. Results on well posedness of a particular class of hybrid systems is given in [13], however system (8.10) goes beyond this class because in [13] it is assumed that the flow and jump sets are subsets of the Euclidean spaces where x_p and x_c take values. Here we have a new direction in the jump and flow sets depending on the (infinite dimensional) memory of the plant state x_p and it is not evident how to inherit the results of [13]. However, at least from the existence viewpoint, the architecture (8.10) can still inherit useful properties from the classical results in [11] and from the results in [8, Chap. 6], as long as the initial condition $\mu_{[0,0]} x_p$ of the plant is continuous.

Remark 3 Note that differently from [6], the dwell-time logic is implemented in (8.9a–8.9c) by forcing solutions not to jump unless $\tau \geq \rho$, while also not allowing them to flow if their ξ component does not belong to the set $\mathscr{C}$. Because of this fact, dwell time is artificially enforced on solutions by possibly terminating defective solutions that would jump too often. The natural question that arises is then whether system (8.9a–8.9c) admits complete solutions for all initial conditions starting either in the jump or in the flow set. This question is not addressed here, where we simply limit ourselves to observing that the example treated in Sect. 8.5 exhibits complete solutions. We regard tackling this important aspect as future work.

8.4 Stability Properties of the Reset Control Scheme

8.4.1 A Lyapunov-Krasovskii Theorem

We state in this section a peculiar version of a Lyapunov-Krasovskii theorem for system (8.10). Alternative instances of this type of result have appeared in the recent literature. For example see [1, Prop. 3.1] or [9, Prop. 1]. We state here a different formulation of the result, due to the special definitions and hybrid framework used in the previous section.

Since we are not interested in the evolution of the timer τ within the compact set $[0, 2\rho]$ where it is confined, we will characterize stability properties of the following compact attractor for dynamics (8.10):

$$\mathscr{A} := \{0\} \times [0, 2\rho] \subset \mathbb{R}^{n_p+n_c} \times \mathbb{R}. \tag{8.11}$$

In particular, following standard derivations in the time-delay continuous-time systems framework, given any solution $\phi = (\phi_x, \phi_\tau) = (\phi_p, \phi_c, \phi_\tau)$ to (8.10), we introduce the following notion of distance from the set $\mathscr{A}$ in (8.11) for each $(t, j) \in \mathrm{dom}_{\geq 0}\phi$:

$$\|\phi(t, j)\|_\mu = \max\left\{|\phi_x(t, j)|, \max_{s\in[-\theta,0]} |\mu_{[t,j]}\phi_p(s)|\right\}, \tag{8.12}$$

where $|\cdot|$ denotes the Euclidean norm.

Since we are dealing with a special class of hybrid time-delay systems, we clarify the meaning of stability in the following definition, which is inspired by [13] and [8, Sect. 3.1].

Definition 2 The compact set $\mathscr{A}$ in (8.11) is

1. Globally stable (GS) for (8.10) if there exists a class $\mathscr{K}_\infty$ function α such that any solution $\phi = (\phi_x, \phi_\tau) = (\phi_p, \phi_c, \phi_\tau)$ to (8.10) satisfies $|\phi_x(t, j)| \leq \alpha(\|\phi(0, 0)\|_\mu)$ for all $(t, j) \in \mathrm{dom}_{\geq 0}\phi$.
2. Uniformly globally attractive (UGA) for (8.10) if for each pair r, ε, there exists $T(r, \varepsilon)$ such that any solution ϕ satisfies:
$$\|\phi(0, 0)\|_\mu \leq r \Rightarrow |\phi_x(t, j)| \leq \varepsilon,\ \forall (t, j) \in \mathrm{dom}_{\geq 0}\phi \text{ such that } t+j \geq T(r, \varepsilon). \tag{8.13}$$
3. Uniformly globally asymptotically stable for (8.10) if it is GS and UGA.

Based on Definition 1 we can state a Lyapunov-Krasovskii result that only requires the following mild uniform boundedness assumption on the functions appearing in the flow map of (8.10). This assumption is trivially satisfied by the linear flow dynamics in (8.9a–8.9c).

Assumption 2 *There exists a class $\mathcal{K}_\infty$ function α_M such that for each $r \geq 0$ (where we use $x = (x_p, x_c)$),*

$$\sup_{(x,\mu x_p)\in\mathscr{C} \text{ s.t. } \|x\|_\mu \leq r} \begin{bmatrix} f_p(x, \mu x_p) \\ f_c(x, \mu x_p) \end{bmatrix} \leq \alpha_M(r).$$

Proposition 1 *Under Assumption 2, if there exist a function V, two class $\mathcal{K}_\infty$ functions α_1, α_2 and a positive definite function σ satisfying*

$$\alpha_1(|x|) \leq V((x,\tau), \mu x_p) \leq \alpha_2(\|(x,\tau)\|_\mu), \quad \forall (x, \mu x_p, \tau), \tag{8.14}$$

$$\dot V((x,\tau), \mu x_p) \leq -\rho(|x|), \quad \forall (x, \mu x_p, \tau) \in \widetilde{\mathscr{C}} \times [0, 2\rho], \tag{8.15}$$

$$V((x^+,\tau^+), \mu x_p) - V((x,\tau), \mu x_p) \leq 0, \quad \forall (x, \mu x_p, \tau) \in \widetilde{\mathscr{D}} \times [\rho, 2\rho], \tag{8.16}$$

then the compact attractor $\mathscr{A}$ in (8.11) is uniformly globally asymptotically stable for (8.10).

Proof The proof is omitted due to space constraints but can be found in [3]. □

8.4.2 Main Stability Result

We state next our main stability result for the hybrid dynamics (8.9a–8.9c). To properly state the stability result,

Theorem 1 *Consider a plant (8.3) satisfying Assumption 1 and a controller (8.4). Then for the dwell-time hybrid time-delay dynamics (8.9a–8.9c) there exist a functional V and class $\mathcal{K}_\infty$ functions α_1, α_2 and σ satisfying (8.14)–(8.16). Namely, the set $\mathscr{A}$ in (8.11) is globally asymptotically stable.*

Proof Using P_p and Q of Assumption 1, consider the following Lyapunov-Krasovskii functional for the plant state directions:

$$V_p(x_p, \mu x_p) = x_p^T P_p x_p + \int_{-\theta}^{0} \mu x_p^T(s) Q \mu x_p(s) ds. \tag{8.17}$$

The derivative of V_p along the flow dynamics of (8.9a–8.9c) is (for notational compactness, we use $x_{pd} := \mu x_p(-\theta)$ in the rest of the proof):

$$
\begin{aligned}
\dot{V}_p &= 2x_p^T P_p \left[A_p x_p + B_p C_c \left(K_p x_p + x_c - K_p x_p\right) + A_{pd} x_{pd}\right] + x_p^T Q x_p - x_{pd}^T Q x_{pd} \\
&= 2x_p^T P_p \left[\underbrace{(A_p + B_p C_c K_p)}_{A_R} x_p + A_{pd} x_{pd} + B_p C_c \underbrace{(x_c - K_p x_p)}_{\delta}\right] + x_p^T Q x_p - x_{pd}^T Q x_{pd} \\
&= \begin{bmatrix} x_p \\ x_{pd} \\ \delta \end{bmatrix}^T \mathrm{He} \begin{bmatrix} P_p A_R + Q/2 & P_p A_{pd} & P_p B_p C_c \\ 0 & -Q/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_p \\ x_{pd} \\ \delta \end{bmatrix}.
\end{aligned} \tag{8.18}
$$

Then, from the definition (8.9b) of the flow set $\mathscr{C}$ we have that

$$
\dot{V}_p \leq -\varepsilon \left(|x_p|_{P_p}^2 + |x_{pd}|_Q^2 + |\delta|^2 \right), \quad \forall (x_p, x_{pd}, \delta) \in \mathscr{C} \tag{8.19}
$$

where $|y|_W = \sqrt{y^T W y}$.

Now consider the following Lyapunov-Krasovskii functional for the overall system:

$$
V = V_p(x_p, \mu x_p) + \lambda \delta^2, \tag{8.20}
$$

where λ is a positive scalar selected later. We show next that this function satisfies (8.14)–(8.16), so that the result follows from Proposition 1.

Proof of (8.14) The lower bound on the left comes from

$$
x_p^T P_p x_p + \lambda \delta^2 = \begin{bmatrix} x_p \\ x_c \end{bmatrix}^T \underbrace{\begin{bmatrix} P_p + \lambda K_p^T K_p & -\lambda K_p \\ -\lambda K_p^T & \lambda I \end{bmatrix}}_{\Sigma} \begin{bmatrix} x_p \\ x_c \end{bmatrix} \geq \sigma_m(\Sigma) \left| \begin{bmatrix} x_p \\ x_c \end{bmatrix} \right|^2 =: \alpha_1 \left(\left| \left[\begin{smallmatrix} x_p \\ x_c \end{smallmatrix} \right] \right| \right),
$$

where $\sigma_m(\Sigma) > 0$ because Σ is positive definite.

For the upper bound we get:

$$
\begin{aligned}
V(x, \mu x_p) &\leq \left[\begin{smallmatrix} x_p \\ x_c \end{smallmatrix} \right]^T \Sigma \left[\begin{smallmatrix} x_p \\ x_c \end{smallmatrix} \right] + \int_{-\theta}^{0} \sigma_M(Q) |\mu x_p(s)|^2 ds \\
&\leq \sigma_M(\Sigma) \left| \left[\begin{smallmatrix} x_p \\ x_c \end{smallmatrix} \right] \right|^2 + \theta \sigma_M(Q) \max_{s \in [-\theta, 0]} |\mu x_p(s)|^2 \\
&\leq 2 \max\{\sigma_M(\Sigma),\ \theta \sigma_M(Q)\} \|(x, \tau)\|_\mu^2 =: \alpha_2(\|(x, \tau)\|_\mu).
\end{aligned}
$$

where $\sigma_M(\cdot)$ and $\sigma_m(\cdot)$ denote, respectively, the maximum and minimum singular values of their arguments.

Proof of (8.15) First, we may easily compute:

$$
\dot{V} \leq \dot{V}_p + \lambda \delta ((\underbrace{A_c - K_p B_p C_c}_{A_1}) x_c + (\underbrace{B_c C_p - K_p A_p}_{A_2}) x_p + (\underbrace{-K_p A_{pd}}_{A_3}) x_{pd}). \tag{8.21}
$$

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Since $x_c = \delta + K_p x_p$, then using (8.19), Eq. (8.21) can be rewritten as

$$\begin{aligned}\dot{V} &\leq -\varepsilon|x_p|^2_{P_p} - \varepsilon|x_{pd}|^2_Q - \varepsilon|\delta|^2 + \lambda\delta(A_1\delta + (A_1K_p + A_2)x_p + A_3x_{pd}) \\ &\leq -\varepsilon\underline{c}|x_p|^2 - \varepsilon\underline{c}|x_{pd}|^2 - \varepsilon|\delta|^2 + \lambda\overline{c}(|\delta|^2 + |\delta|\,|x_p| + |\delta|\,|x_{pd}|),\end{aligned}$$

where $\overline{c} = \max\{\sigma_M(A_1),\ \sigma_M(A_1K_p{+}A_2),\ \sigma_M(A_3)\}$ and $\underline{c} = \min\{\sigma_m(P_p), \sigma_m(Q)\}$.
Finally, completing squares and choosing $\lambda = \varepsilon \min\left\{\frac{\underline{c}}{\overline{c}}, \frac{1}{4\overline{c}}\right\}$, we get

$$\begin{aligned}\dot{V} &\leq -\left(\varepsilon\underline{c} - \frac{\lambda\overline{c}}{2}\right)|x_p|^2 - \left(-\varepsilon\underline{c} + \frac{\lambda\overline{c}}{2}\right)|x_{pd}|^2 - (\varepsilon - 2\lambda\overline{c})|\delta|^2 \\ &\leq -\frac{\varepsilon}{2}\left|\begin{bmatrix} I & 0 \\ -K_p & I\end{bmatrix}\right|\left|\begin{bmatrix} x_p \\ x_c\end{bmatrix}\right|^2 =: -\sigma(|(x_p, x_c)|^2),\end{aligned}$$

which implies the flow condition (8.15).
Proof of (8.16) Simply observe that x_p remains constant across jumps, therefore

$$V(x^+, \mu x_p) - V(x, \mu x_p) = \lambda((\delta^+)^2 - \delta^2) = -\delta^2 \leq 0,$$

where we used the fact that δ is reset to zero at each jump. □

8.5 Simulation Example

Consider the following entries for the matrices in (8.3):

$$A_p = \begin{bmatrix} -2 & 0 \\ 0 & -0.9\end{bmatrix}, \quad A_{pd} = \begin{bmatrix} -1 & 0 \\ -1 & -1\end{bmatrix}, \quad B_p = \begin{bmatrix} 1 \\ 1\end{bmatrix}, \quad C_p = \begin{bmatrix} 1 & 1\end{bmatrix}, \tag{8.22}$$

and the selection for controller (8.4):

$$A_c = \begin{bmatrix} -1 & 0 \\ 1 & 0\end{bmatrix}, \quad B_c = \begin{bmatrix} 1 \\ 0\end{bmatrix}, \quad C_c = \begin{bmatrix} 0 & -1\end{bmatrix}, \tag{8.23}$$

which involves an integral action possibly to deal with constant references or disturbances. When interconnecting plant and controller, we obtain a continuous-time dynamics as in the upper equation of (8.9a) with

$$\begin{gathered}A_p = \begin{bmatrix} -2 & 0 \\ 0 & -0.9\end{bmatrix},\ A_{pd} = \begin{bmatrix} -1 & 0 \\ -1 & -1\end{bmatrix},\ B_pC_c = \begin{bmatrix} 0 & -1 \\ 0 & -1\end{bmatrix}, \\ A_c = \begin{bmatrix} -1 & 0 \\ 1 & 0\end{bmatrix},\ B_cC_p = \begin{bmatrix} 1 & 1 \\ 0 & 0\end{bmatrix}.\end{gathered} \tag{8.24}$$

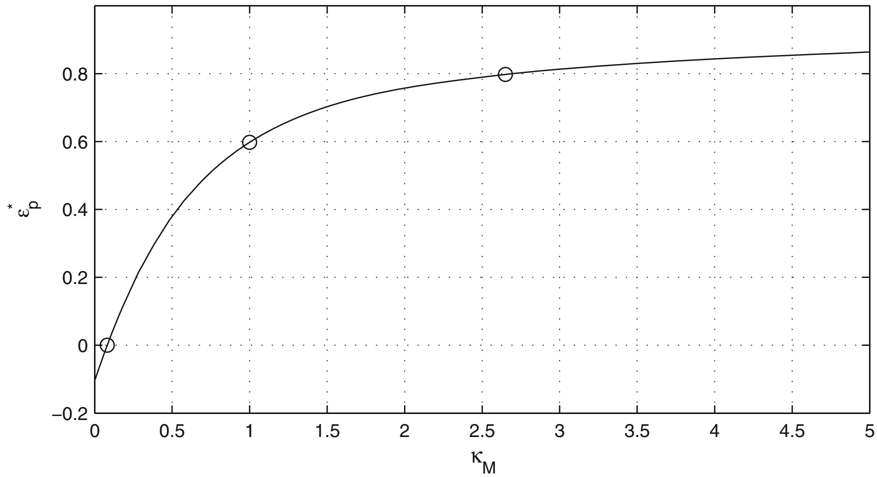

Fig. 8.1 The optimal value ε_p^* in (8.8) as a function of κ_M

Table 8.1 The three cases addressed in the simulation example

	κ_m	ε_p^*	K_p
Case 1	0.081	3.6×10^{-4}	$\begin{bmatrix} 0.0148 & -0.0001 \\ -0.0591 & -0.0542 \end{bmatrix}$
Case 2	1	0.5976	$\begin{bmatrix} 0.4501 & -0.1268 \\ -0.6102 & -0.7634 \end{bmatrix}$
Case 3	2.65	0.7977	$\begin{bmatrix} 1.3365 & -1.2358 \\ -1.3268 & -2.2350 \end{bmatrix}$

Such a continuous-time dynamics is exponentially stable if $\theta = 0$ (case with no delay) but for larger delays, beyond the critical value $\theta_M = 1.6$, the (linear) continuous-time closed loop becomes unstable.

Plant (8.3), (8.22) satisfies Assumption 1, therefore we may follow our hybrid construction to obtain GAS of the origin for any value of the delay, while preserving the continuous-time dynamics induced by (8.4), (8.24). To this aim, we follow the design paradigm in Remark 1 and compute a trade-off curve between the bound $\kappa_M > 0$ and the decrease rate ε_p^*.

Figure 8.1 shows the optimal values of ε_p^* as a function of the bound κ_M on $|K_p|$. Note that if κ_M is too small, then ε_p^* is negative and the design cannot be performed. Clearly, the curve is nondecreasing as increasing κ_M one enlarges the feasible set.

Table 8.1 shows the optimal values corresponding to the three circles reported in Fig. 8.1. The first value is just after the stability limit and the two other ones correspond to different trade-offs between ε_p^* and κ_M. For these three cases we run a time simulation selecting $\theta = 2$.

The simulation results for cases 1, 2, and 3, respectively, are shown in Figs. 8.2 and 8.3 using solid, dashed, and dotted curves, respectively. In the two figures, we

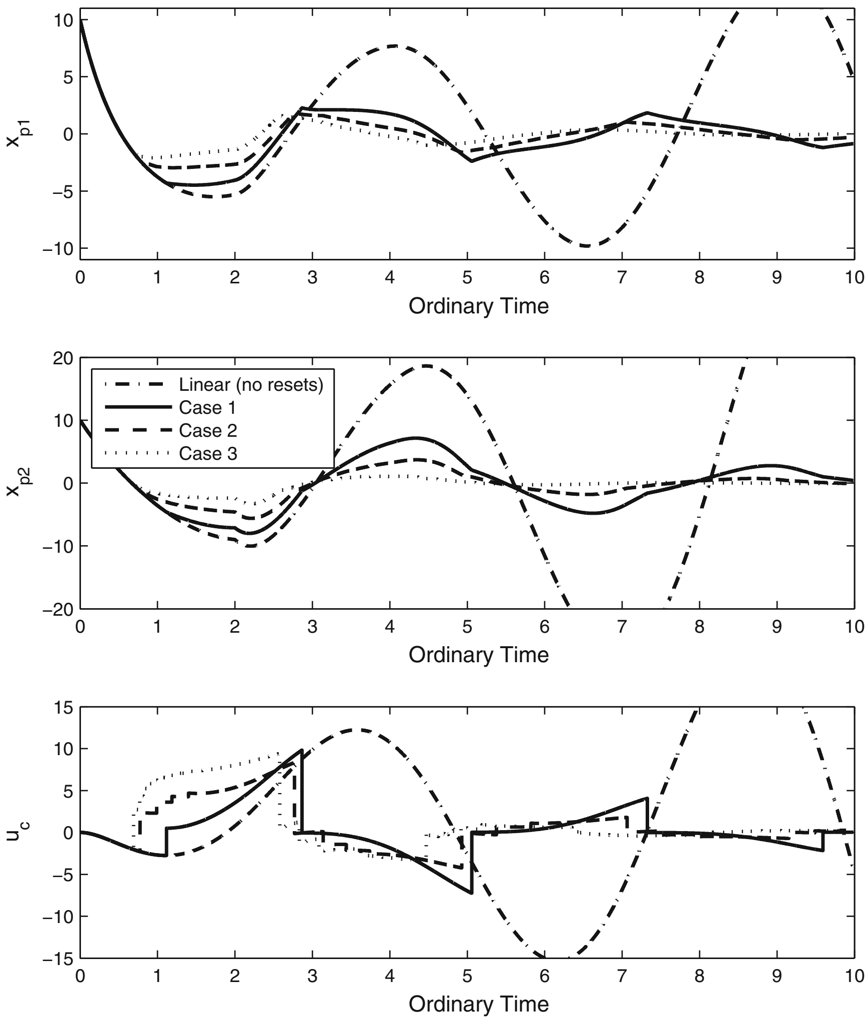

Fig. 8.2 States and input of the plant for the simulation test

also show the linear response (no resets), which is diverging (dash-dotted curves) because $2 > \theta_M \approx 1.6$. For the simulations, we selected $\rho = 0.01$ s that turns out to be well below the time elapsed between any pair of consecutive resets. Therefore, with reference to Remark 3, the dwell-time logic does not prematurely terminate our solutions.

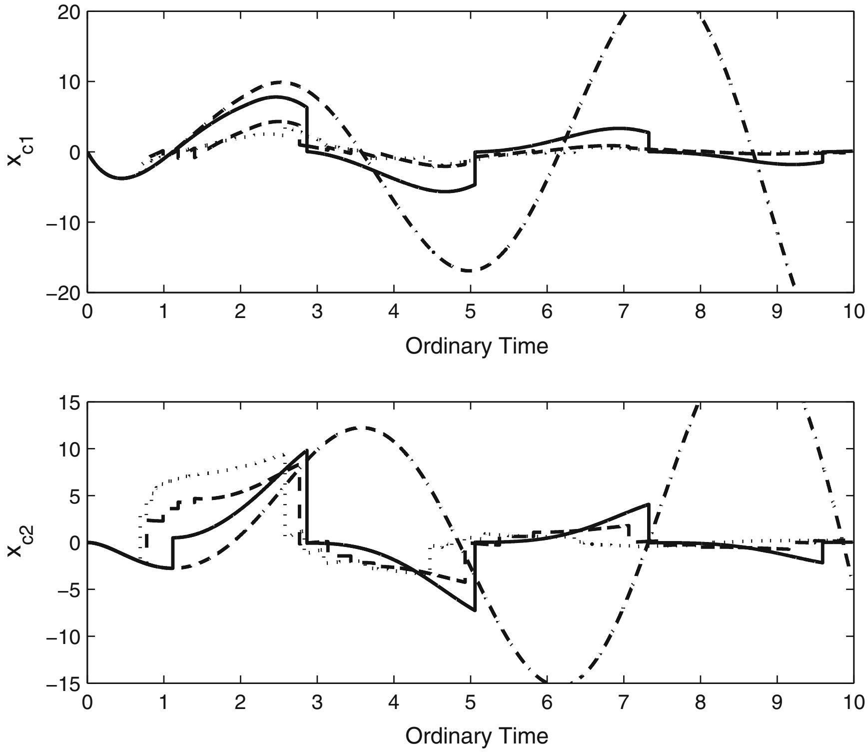

Fig. 8.3 States of the controller for the simulation test

The simulation results confirm the faster convergence rate (larger ε_p^*) envisioned for larger values of κ_M. Quite interestingly, this faster convergence is obtained by resetting earlier, rather than using a larger control input. Indeed, for larger values of κ_M, we observe a reduced amplitude of the control input u_c (see the bottom plot in Fig. 8.2).

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