Transfer Matrices

Northrop Grumman RQ-4 Global Hawk

Transfer Matrices

We consider 2 referentials, R_1 and R_2 ; let's consider a vector $\vec{V}$ and express it with respect to both referentials

$$\vec{V} = x_1\vec{i}_1 + y_1\vec{j}_1 + z_1\vec{k}_1 = x_2\vec{i}_2 + y_2\vec{j}_2 + z_2\vec{k}_2$$

which gives,

$$\rightarrow \begin{cases} x_2 = (\vec{i}_1 \cdot \vec{i}_2)x_1 + (\vec{j}_1 \cdot \vec{i}_2)y_1 + \left(\vec{k}_1 \cdot \vec{i}_2\right) z_1 \\ y_2 = (\vec{i}_1 \cdot \vec{j}_2)x_1 + (\vec{j}_1 \cdot \vec{j}_2)y_1 + \left(\vec{k}_1 \cdot \vec{j}_2\right) z_1 \\ z_2 = \left(\vec{i}_1 \cdot \vec{k}_2\right) x_1 + \left(\vec{j}_1 \cdot \vec{k}_2\right) y_1 + \left(\vec{k}_1 \cdot \vec{k}_2\right) z_1 \end{cases}$$

$$\rightarrow \left[\begin{matrix} x_2 \\ y_2 \\ z_2 \end{matrix}\right. = \begin{pmatrix} \vec{i}_1 \cdot \vec{i}_2 & \vec{j}_1 \cdot \vec{i}_2 & \vec{k}_1 \cdot \vec{i}_2 \\ \vec{i}_1 \cdot \vec{j}_2 & \vec{j}_1 \cdot \vec{j}_2 & \vec{k}_1 \cdot \vec{j}_2 \\ \vec{i}_1 \cdot \vec{k}_2 & \vec{j}_1 \cdot \vec{k}_2 & \vec{k}_1 \cdot \vec{k}_2 \end{pmatrix} \cdot \left[\begin{matrix} x_1 \\ y_1 \\ z_1 \end{matrix}\right. = T_{R_2/R_1} \cdot \left[\begin{matrix} x_1 \\ y_1 \\ z_1 \end{matrix}\right.$$

which defines the Transfer matrix T_{R_2/R_1} from R_1 to R_2

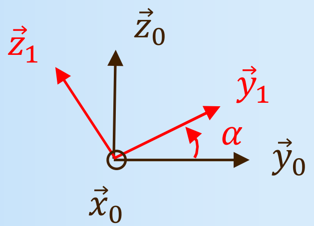
$$\rightarrow \vec{V} = T_{R_2/R_1} \; {}_{R_1}\!\left[\begin{matrix} x_1 \\ y_1 \\ z_1 \end{matrix}\right. = {}_{R_2}\!\left[\begin{matrix} x_2 \\ y_2 \\ z_2 \end{matrix}\right.$$

Elementary Transfer Matrices

There are 3 elementary rotations which define 3 elementary transfer matrices

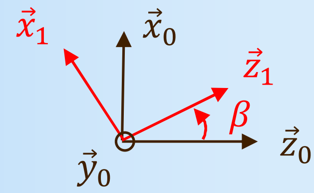
Rotation around the x-axis by an angle α

$$T_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$



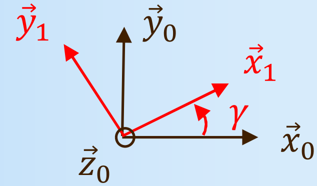
Rotation around the y-axis by an angle β

$$T_y(\beta) = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}$$



Rotation around the z-axis by an angle γ

$$T_z(\gamma) = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



General Properties

Any transformation from a Referential R_0 to R can be achieved by a maximum of 3 elementary rotations (Euler rotations). The Tranfer matrix T_{R/R_0} is the product of the 3 ones

$$R_0 \underset{(z,\gamma)}{\rightarrow} R_1 \underset{(y,\beta)}{\rightarrow} R_2 \underset{(x,\alpha)}{\rightarrow} R$$

$$T_{R/R_0} = T_{R/R_2} \otimes T_{R_2/R_1} \otimes T_{R_1/R_0} = T_x(\alpha) \otimes T_y(\beta) \otimes T_z(\gamma)$$

The inverse transformation (from R to R_0) is associated to the Transposed matrix

$$T_{R_0/R} = T_{R/R_0}^{-1} = T_{R/R_0}^{t}$$

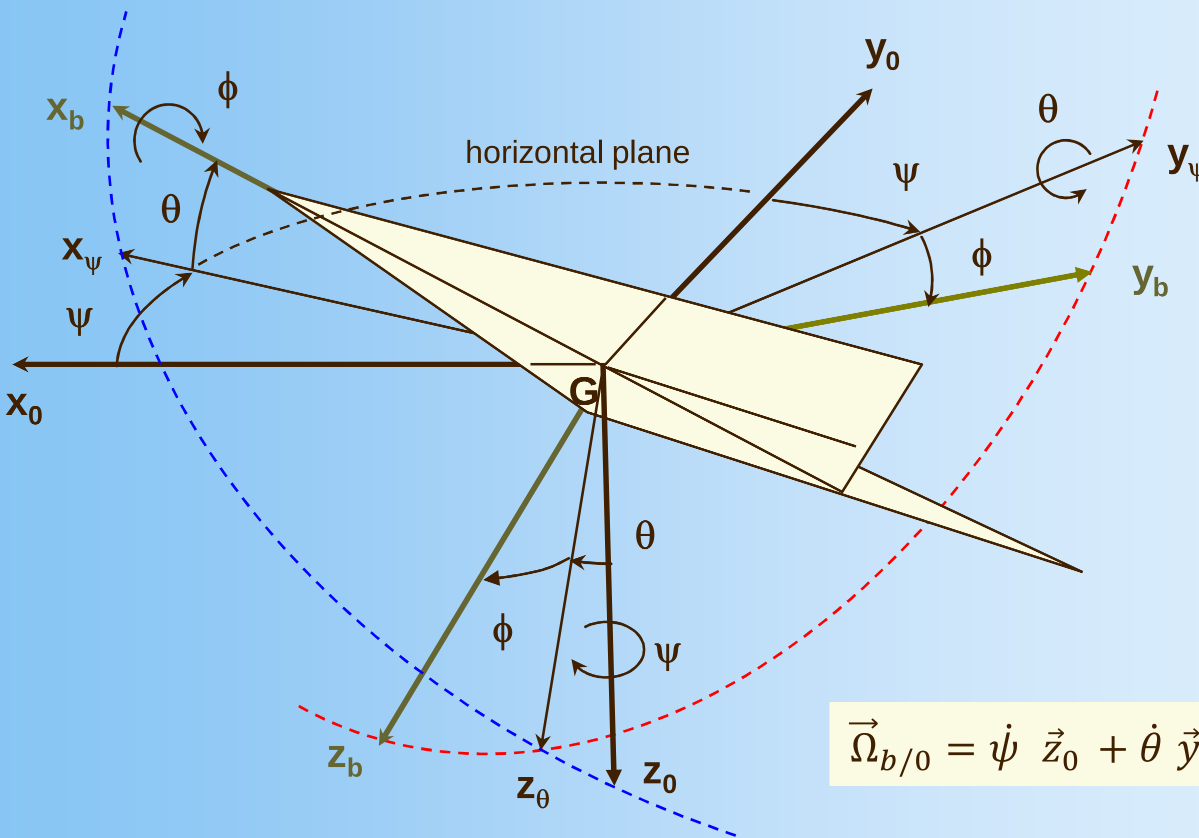
The coordinates of any vector $\vec{X}$ are transformed from R_0 to R according to

$$\vec{X} = T_{R/R_0} \cdot \left|\begin{matrix} x_0 \\ y_0 \\ z_0 \end{matrix}\right._{R_0} = \left|\begin{matrix} x \\ y \\ z \end{matrix}\right._{R}$$

with

$$T_{R/R_0} = \begin{array}{l} \begin{pmatrix} \vec{i}\cdot\vec{i}_0 & \vec{i}\cdot\vec{j}_0 & \vec{i}\cdot\vec{k}_0 \\ \vec{j}\cdot\vec{i}_0 & \vec{j}\cdot\vec{j}_0 & \vec{j}\cdot\vec{k}_0 \\ \vec{k}\cdot\vec{i}_0 & \vec{k}\cdot\vec{j}_0 & \vec{k}\cdot\vec{k}_0 \end{pmatrix} \end{array} \begin{array}{l} \rightarrow \vec{i} \text{ composants wrt } R_0 \\ \rightarrow \vec{j} \text{ composants wrt } R_0 \\ \rightarrow \vec{k} \text{ composants wrt } R_0 \end{array}$$

Rotation Angles from R_0 to R_b



$$\vec{\Omega}_{b/0} = \dot{\psi}\ \vec{z}_0 + \dot{\theta}\ \vec{y}_\psi + \dot{\phi}\ \vec{x}_b$$

Transfer Matrix from R_0 to R_b

We need 3 rotations for moving from R_0 to R_b :

$$R_0 \underset{(z,\psi)}{\rightarrow} R_{1b} \underset{(y,\theta)}{\rightarrow} R_{2b} \underset{(x,\phi)}{\rightarrow} R_b$$

The transfer matrix from R_0 to R_b is given by the product of the 3 elementary transfer matrices

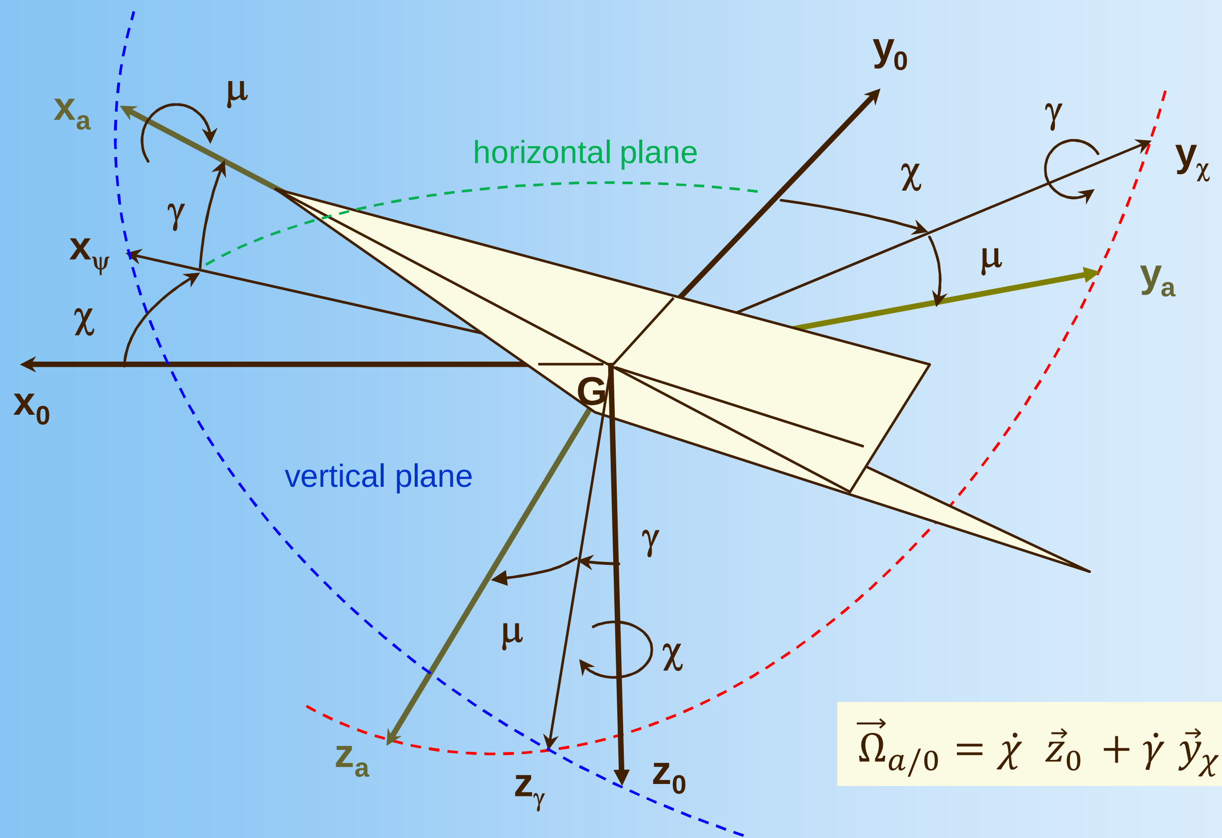
$$T_{b/0} = T_x(\phi)\otimes T_y(\theta)\otimes T_z(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \otimes \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \otimes \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_{b/0} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \cos\theta\sin\phi \\ \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi & \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

Coordinates transformation : we want to express the vector $\vec{z}_0$ within R_b

$$\vec{z}_0 = T_{b/1b}\cdot \left|\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}\right._{R_{1b}} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ \sin\theta\sin\phi & \cos\phi & \cos\theta\sin\phi \\ \sin\theta\cos\phi & \sin\phi & \cos\theta\cos\phi \end{bmatrix} \cdot \left|\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}\right._{R_{1b}} = \left|\begin{matrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{matrix}\right._{R_b}$$

Rotation Angles from R_0 to R_a



$$\vec{\Omega}_{a/0} = \dot{\chi}\ \vec{z}_0 + \dot{\gamma}\ \vec{y}_\chi + \dot{\mu}\ \vec{x}_b$$

Transfer Matrix from R_0 to R_a

We need 3 rotations for moving from R_0 to R_a :

$$R_0 \underset{(z,\chi)}{\rightarrow} R_{1a} \underset{(y,\gamma)}{\rightarrow} R_{2a} \underset{(x,\mu)}{\rightarrow} R_a$$

The transfer matrix from R_0 to R_a is given by the product of the 3 elementary transfer matrices

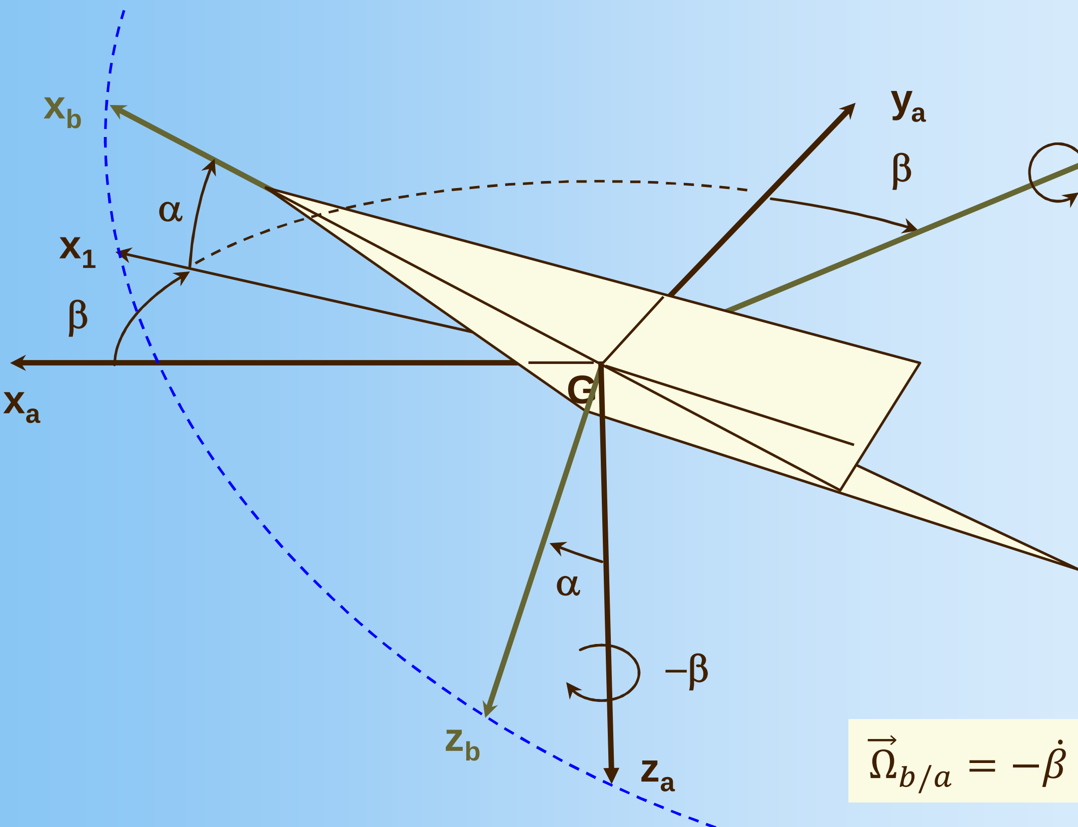
$$T_{a/0} = T_x(\mu) \otimes T_y(\gamma) \otimes T_z(\chi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\mu & \sin\mu \\ 0 & -\sin\mu & \cos\mu \end{bmatrix} \otimes \begin{bmatrix} \cos\gamma & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ \sin\gamma & 0 & \cos\gamma \end{bmatrix} \otimes \begin{bmatrix} \cos\chi & \sin\chi & 0 \\ -\sin\chi & \cos\chi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_{a/0} = \begin{bmatrix} \cos\gamma\cos\chi & \cos\gamma\sin\chi & -\sin\gamma \\ \cos\chi\sin\gamma\sin\mu - \sin\chi\cos\mu & \sin\chi\sin\gamma\sin\mu + \cos\chi\cos\mu & \cos\gamma\sin\mu \\ \cos\chi\sin\gamma\cos\mu + \sin\chi\sin\mu & \sin\chi\sin\gamma\cos\mu - \cos\chi\sin\mu & \cos\gamma\cos\mu \end{bmatrix}$$

Coordinates transformation : we want to express the vector $\vec{x}_a$ within R_0

$$\vec{x}_a = T_{0/2a} \cdot {}_{R_{2a}}\left|\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right. = \begin{bmatrix} \cos\gamma\cos\chi & \sin\chi & \cos\chi\sin\gamma \\ \cos\gamma\sin\chi & \cos\chi & \sin\chi\sin\gamma \\ -\sin\gamma & 0 & \cos\gamma \end{bmatrix} \cdot {}_{R_{2a}}\left|\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right. = {}_{R_0}\left|\begin{matrix} \cos\gamma\cos\chi \\ \cos\gamma\sin\chi \\ -\sin\gamma \end{matrix}\right.$$

Rotation Angles from R_a to R_b



$$\vec{\Omega}_{b/a} = -\dot{\beta}\ \vec{z}_a + \dot{\alpha}\ \vec{y}_b$$

The side slip is normally positive on that picture ; however, in our convention, β is counted negative (the air flowfield is coming from the left) so the (positive) rotation around z_a corresponds to a –β variation

Transfer Matrix from R_a to R_b

We need 2 rotations for moving from R_a to R_b :

$$R_a \underset{(z,-\beta)}{\rightarrow} R_i \underset{(y,\alpha)}{\rightarrow} R_b$$

The transfer matrix from R_a to R_b is given by the product of the 2 elementary transfer matrices

$$T_{b/a} = T_y(\alpha) \otimes T_z(-\beta) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \otimes \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

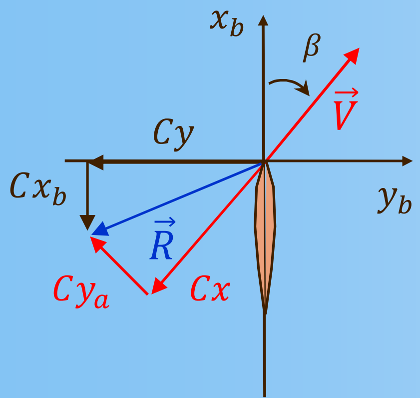
$$T_{b/a} = \begin{bmatrix} \cos\alpha\cos\beta & -\cos\alpha\sin\beta & -\sin\alpha \\ \sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & -\sin\alpha\sin\beta & \cos\alpha \end{bmatrix}$$

Coordinates transformation : we want to express the vector $\vec{x}_a$ within R_b

$$\vec{x}_a = T_{b/a} \cdot \left| \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \right._{R_a} = \left| \begin{matrix} \cos\alpha\cos\beta \\ \sin\beta \\ \sin\alpha\cos\beta \end{matrix} \right._{R_b}$$

Lateral force, Cy and lateral lift ,Cy_a

We consider a fin submitted to a side slip β. Compute the drag, Cx which is generated ?



We assume no angle of attack α=0

The fin produces a lateral force (along y_b) : $Cy = Cy_\beta \cdot \beta$

The fin produces a drag force (along $\vec{V}$) : $Cx = Cx_0 + k \cdot Cy_a^2$
where Cy_a is the lateral lift produced by the fin

The resulting force $\vec{R}$ is given by :

$$\vec{R} = \left|\begin{matrix} -Cx \\ Cy_a \\ 0 \end{matrix}\right._{R_a} = T_{b/a} \cdot \left|\begin{matrix} -Cx \\ Cy_a \\ 0 \end{matrix}\right._{R_a} = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \left|\begin{matrix} -Cx \\ Cy_a \\ 0 \end{matrix}\right._{R_a} = \left|\begin{matrix} -Cx_b \\ Cy \\ 0 \end{matrix}\right._{R_b}$$

$$\rightarrow Cy = -Cx \sin\beta + Cy_a \cos\beta = -(Cx_0 + k \cdot Cy_a^2\ ) \sin\beta + Cy_a \cos\beta$$

$$k \cdot Cy_a^2 \sin\beta - Cy_a \cos\beta + Cy_\beta \cdot \beta + Cx_0 \sin\beta = 0$$

Cy_a is a solution of a second-order equation, then calculation of Cx

General expressions for $\vec{\Omega}$

Douglas X-3 Stiletto

General expression for $\vec{\Omega}_b$ and $\vec{\Omega}_a$

$$\vec{\Omega}_{b/0} = \left.\begin{matrix} p \\ q \\ r \end{matrix}\right|_{R_b} = \dot{\psi}\ \vec{z}_0 + \dot{\theta}\ \vec{y}_\psi + \dot{\phi}\ \vec{x}_b$$

We express all vectors with respect to R_b

$$R_0 \underset{(z,\psi)}{\rightarrow} R_{1b} \underset{(y,\theta)}{\rightarrow} R_{2b} \underset{(x,\phi)}{\rightarrow} R_b$$

$$\vec{\Omega}_{b/0} = \dot{\psi}\ \cdot T_{b/1b} \left.\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}\right|_{R_{1b}} + \dot{\theta} \cdot T_{b/2b} \left.\begin{matrix} 0 \\ 1 \\ 0 \end{matrix}\right|_{R_{2b}} + \dot{\phi}\ \vec{x}_b$$

$$\vec{\Omega}_{b/0} = \dot{\psi}\ \cdot T_x(\phi) \otimes T_y(\theta) \left.\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}\right|_{R_{1b}} + \dot{\theta} \cdot T_x(\phi) \left.\begin{matrix} 0 \\ 1 \\ 0 \end{matrix}\right|_{R_{2b}} + \dot{\phi}\ \vec{x}_b$$

$$\vec{\Omega}_{b/0} = \dot{\psi}\ \cdot \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ \sin\phi\sin\theta & \cos\phi & \sin\phi\cos\theta \\ \cos\phi\sin\theta & -\sin\phi & \cos\phi\cos\theta \end{bmatrix}_{R_{1b}} \left|\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}\right. + \dot{\theta} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}_{R_2 b} \left|\begin{matrix} 0 \\ 1 \\ 0 \end{matrix}\right. + \dot{\phi}\ \vec{x}_b$$

$$\overrightarrow{\Omega}_{b/0} = \left|\begin{array}{l} p = \dot{\phi} - \dot{\psi}\ \sin\theta \\ q = \dot{\psi}\ \sin\phi\cos\theta + \dot{\theta}\cos\phi \\ r = \dot{\psi}\ \cos\phi\cos\theta - \dot{\theta}\sin\phi \end{array}\right._{R_b} \Rightarrow \begin{cases} \dot{\psi}\cos\theta = q\sin\phi + r\cos\phi \\ \dot{\theta} = q\cos\phi - r\sin\phi \\ \dot{\phi} = p + \dot{\psi}\ \sin\theta \end{cases}$$

In a similar way

$$\overrightarrow{\Omega}_{a/0} = \left|\begin{array}{l} p_a = \dot{\mu} - \dot{\chi}\ \sin\gamma \\ q_a = \dot{\chi}\ \sin\mu\cos\gamma + \dot{\gamma}\cos\mu \\ r_a = \dot{\chi}\ \cos\mu\cos\gamma - \dot{\gamma}\sin\mu \end{array}\right._{R_a} \Rightarrow \begin{cases} \dot{\chi}\cos\gamma = q_a\sin\mu + r_a\cos\mu \\ \dot{\gamma} = q_a\cos\mu - r_a\sin\mu \\ \dot{\mu} = p_a + \dot{\chi}\ \sin\gamma \end{cases}$$

Relation between $\vec{\Omega}_{a/0}$ and $\vec{\Omega}_{b/0}$

$$\vec{\Omega}_{a/b} = \dot{\beta}\,\vec{z}_a - \dot{\alpha}\,\vec{y}_b$$

$$\boxed{\vec{\Omega}_{a/0} = \vec{\Omega}_{a/b} + \vec{\Omega}_{b/0}} \quad \rightarrow \quad \begin{vmatrix} p_a \\ q_a \\ r_a \end{vmatrix}_{R_a} = \begin{vmatrix} -\dot{\alpha}\sin\beta \\ -\dot{\alpha}\cos\beta \\ \dot{\beta} \end{vmatrix}_{R_a} + \vec{\Omega}_{b/0}$$

We express $\vec{\Omega}_{b/0}$ within the Referential R_a using the transfer matrix from R_b to R_a (= $T_{a/b}$)

$$\vec{\Omega}_{b/0} = T_{a/b} \cdot \begin{vmatrix} p \\ q \\ r \end{vmatrix}_{R_b} = \begin{bmatrix} \cos\alpha\cos\beta & \sin\beta & \sin\alpha\cos\beta \\ -\cos\alpha\sin\beta & \cos\beta & -\sin\alpha\sin\beta \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \cdot \begin{vmatrix} p \\ q \\ r \end{vmatrix}_{R_b} = \begin{vmatrix} (p\cos\alpha + r\sin\alpha)\cos\beta + q\sin\beta \\ -(p\cos\alpha + r\sin\alpha)\sin\beta + q\cos\beta \\ r\cos\alpha - p\sin\alpha \end{vmatrix}_{R_a}$$

which gives (assuming the usual small angle approximation) :

$$\begin{cases} p_a = (p\cos\alpha + r\sin\alpha)\cos\beta + (q - \dot{\alpha})\sin\beta \\ q_a = -(p\cos\alpha + r\sin\alpha)\sin\beta + (q - \dot{\alpha})\cos\beta \\ r_a = \dot{\beta} + r\cos\alpha - p\sin\alpha \end{cases}$$

$$\begin{aligned} p_a &\approx p + r\cdot\alpha \\ q_a &\approx q - \dot{\alpha} \\ r_a &\approx r - p\cdot\alpha + \dot{\beta} \end{aligned}$$

Approximate expression for (χ,γ,μ)

Schempp-Hirth Arcus

General expression for χ

We express the $\vec{x}_a$ vector wrt R_b and R_0

$$\vec{x}_a = \left|\begin{matrix} \cos\alpha\cos\beta \\ \sin\beta \\ \sin\alpha\cos\beta \end{matrix}\right._{R_b}$$

and

$$\vec{x}_a = T_{0/b} \cdot \left|\begin{matrix} \cos\alpha\cos\beta \\ \sin\beta \\ \sin\alpha\cos\beta \end{matrix}\right._{R_b} = \left|\begin{matrix} \cos\gamma\cos\chi \\ \cos\gamma\sin\chi \\ -\sin\gamma \end{matrix}\right._{R_0}$$

by extracting the y-compound and using the usual small angle approximation :

$$\sin\chi \simeq \sin\psi + \sin\beta\cos\phi - \sin\alpha\sin\phi$$

$$\boxed{\chi \simeq \psi + \beta\cos\phi - \alpha\sin\phi}$$

General expression for γ

We express the $\vec{z}_0$ vector wrt R_a and R_b

$$\vec{z}_0 = \left|\begin{matrix} -\sin\gamma \\ \cos\gamma\sin\mu \\ \cos\gamma\cos\mu \end{matrix}\right._{R_a} = \left|\begin{matrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{matrix}\right._{R_b}$$

and $$\vec{z}_0 = T_{a/b} \cdot \left|\begin{matrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{matrix}\right._{R_b} = \left|\begin{matrix} -\sin\gamma \\ \cos\gamma\sin\mu \\ \cos\gamma\cos\mu \end{matrix}\right._{R_a}$$

by extracting the x-compound :

$$\sin\gamma = \cos\alpha\cos\beta\sin\theta - \sin\beta\cos\theta\sin\phi - \sin\alpha\cos\beta\cos\theta\cos\phi \;(*)$$

Using, the usual small angle approximation : $$\boxed{\gamma \simeq \theta - \alpha\cos\phi - \beta\sin\phi}$$

(*) for pure longitudinal flight $(\beta = \phi = 0)$:

$$\sin\gamma = \cos\alpha\sin\theta - \sin\alpha\cos\theta = \sin(\theta - \alpha) \rightarrow \theta = \alpha + \gamma$$

General expression for μ

We express the $\vec{z}_0$ vector wrt R_a and R_b

$$\vec{z}_0 = \left|\begin{matrix} -\sin\gamma \\ \cos\gamma\sin\mu \\ \cos\gamma\cos\mu \end{matrix}\right._{R_a} = \left|\begin{matrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{matrix}\right._{R_b}$$

and $$\vec{z}_0 = T_{a/b} \cdot \left|\begin{matrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{matrix}\right._{R_b} = \left|\begin{matrix} -\sin\gamma \\ \cos\gamma\sin\mu \\ \cos\gamma\cos\mu \end{matrix}\right._{R_a}$$

by extracting the y-compound :

$$\cos\gamma\sin\mu = \cos\beta\cos\theta\sin\phi + \cos\alpha\sin\beta\sin\theta - \sin\alpha\sin\beta\cos\theta\cos\phi$$

Using, the usual small angle approximation :

$$\boxed{\sin\mu \simeq \sin\phi + \beta\,(\theta - \alpha\cos\phi)}$$