MAE1- Electromagnetism applied to avionics

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Lesson 4

Wave Equation - In vacuum

(1) $div\,\vec{E} = \frac{\rho}{\epsilon_0}$ Local Gauss Law (electric flux density)

(2) $div\,\vec{B} = 0$ General Magnetism Law

(3) $\overrightarrow{rot}\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ Faraday Law (relationship between Electric and Magnetic Field)

(4) $\overrightarrow{rot}\vec{B} = \mu_0 \left(\vec{j} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$ Ampere law (current flow in a wire creating a magnetic field)

$$\Delta\vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\Delta\vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\boldsymbol{c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}}$$

$$\vec{E} = \overrightarrow{E_0}\cos(\omega t - kz)$$
$$\vec{B} = \vec{B}_0\cos(\omega t - kz)$$

$$\boldsymbol{\vec{k} = \frac{\omega}{c}\vec{u}}$$

Density of Energy : $U = \varepsilon_0 E^2 = \frac{B^2}{2\mu_0}$

Poynting Vector : $\vec{S} = \mathrm{c}U\vec{u}$

$$\lambda = \frac{2\pi}{k}$$

$$v = \frac{c}{\lambda}$$

Field in matter

Material media, at microscopic scale, is essentially composed by vacuum.

Particles Size (electrons, nucleus,…) $10^{-13} cm$	<<	Particles distances 10^{-8}cm

Electromagnetic effects (Lorentz forces, electrostatics,…) without contact.
(contact = choc, collision)

The Maxwell equations, with the contribution of charge ρ and current $\vec{j}$ densities, are valid into the matter at microscopic scale

⇒ Field propagating according to the particles distribution

Field in matter

Laws at Microscopic scale $\neq$ Macroscopic scale

Space and time fluctuations $\Rightarrow$ Mean value of the field while the time sample number is higher than the measurement (according to Heisenberg uncertainty principle)

As this is a course of **classical electromagnetism** (non quantum or statistics physics), we assume $\vec{E}$ & $\vec{B}$ are the resulting mean fields .

Field in matter

The charge and current densities are modified by the matter : Wave Matter interaction

4 field vectors:

- Electrical Field $\vec{E}$
- Magnetic Field $\vec{B}$
- Electric Displacement or induction Field $\vec{D} = \varepsilon\vec{E}$
- Magnetic Intensity or Excitation Field $\vec{H} = \frac{\vec{B}}{\mu}$

$$\vec{\nabla} \cdot \vec{D} = \rho$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \left(\vec{j} + \frac{\partial \vec{D}}{\partial t}\right)$$

In Vacuum $\varepsilon = \varepsilon_0$, μ= μ_0

In matter ε (permittivity) , μ (permeability) $\epsilon \mathbb{R}$ or $\mathbb{C}$

$\varepsilon = \varepsilon_0 \varepsilon_r$ & μ= $\mu_0 \mu_r$

Refractive index $n = \sqrt{\varepsilon_r}$

From the local Ohm Law : $\vec{j} = \gamma \vec{E}$ where γ is the conductivity

Propagation in linear non-conducting, dispersion (scattering) relation

- $\vec{E} = \overrightarrow{E_0}\, e^{-i(\omega t - \vec{k}\cdot\vec{r})}$ idem $\vec{D}, \vec{B}, \vec{H}$ Additionnal information see: https://www.showme.com/sh?h=oAHiyH2
- ω is given
- The relation of transverse dispersion is the relationship between the magnitude of $\vec{k}$ with ω

$$\overrightarrow{D_0} = \varepsilon(\omega)\overrightarrow{E_0} \qquad \& \qquad \overrightarrow{B_0} = \mu(\omega)\overrightarrow{H_0}$$

From Maxwell: $\vec{k}\cdot\vec{B} = 0$

Proof: Compute Maxwell Equation

$$\vec{\nabla}.\vec{B} = 0$$

$$\vec{\nabla}.\vec{B} = \frac{\partial B_{0x} e^{-i(\omega t - \vec{k}\cdot\vec{r})}}{\partial x} + \frac{\partial B_{0y} e^{-i(\omega t - \vec{k}\cdot\vec{r})}}{\partial y} + \frac{\partial B_{0z} e^{-i(\omega t - \vec{k}\cdot\vec{r})}}{\partial z}\ i\left(k_x B_{0x} + k_y B_{0y} + k_z B_{0z}\right) e^{-i(\omega t - \vec{k}\cdot\vec{r})}$$

$$\rightarrow\ \mathrm{i}\,\vec{k}.\vec{B} = 0$$

Propagation in linear non-conducting, dispersion relation

$\vec{k} \cdot \vec{B} = 0$ ⟹ $\vec{B}$ is transversal ⟹ $\vec{H}$ is transversal

From Maxwell $\vec{k} \times \vec{E} = \omega \vec{B}$ MAXWELL FARADAY

Proof: Compute Maxwell Equation: Maxwell Faraday $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = i\omega \vec{B}$

$$\vec{\nabla} \times \vec{E} = \begin{pmatrix} \frac{\partial \mathrm{E_z}}{\partial \mathrm{y}} - \frac{\partial \mathrm{E_y}}{\partial \mathrm{z}} \\ \frac{\partial \mathrm{E_x}}{\partial \mathrm{z}} - \frac{\partial \mathrm{E_z}}{\partial \mathrm{x}} \\ \frac{\partial \mathrm{E_y}}{\partial \mathrm{x}} - \frac{\partial \mathrm{E_x}}{\partial \mathrm{y}} \end{pmatrix} = \begin{pmatrix} -ik_z E_y \\ ik_z E_x \\ 0 \end{pmatrix} = i\vec{k} \times \vec{E} \quad \rightarrow \quad i\vec{k} \times \vec{E} = i\omega \vec{B}$$

$i\vec{k} \times \vec{H} = \gamma \vec{E} - i\omega\varepsilon \vec{E}$ MAXWELL AMPERE

Proof: Compute Maxwell Equation $\vec{\nabla} \times \vec{H} = \left(\vec{j} + \frac{\partial \vec{D}}{\partial t}\right)$,

See https://www.showme.com/sh?h=AWjoW7E

Propagation in linear non-conducting, dispersion relation

$i\vec{k} \times \vec{H} = \gamma\vec{E} \ - i\omega\varepsilon\vec{E} \Longrightarrow \quad \vec{k} \times \vec{B} = -\omega\mu\left(\varepsilon + \frac{i\gamma}{\omega}\right)\vec{E}$

Proof:

$\vec{E}$ is transversal

From Maxwell $i\vec{k} \cdot \vec{D} = \rho$

Proof: Compute Maxwell Equation

Plane Wave Propagation condition: $\boldsymbol{\vec{D}}$ **//** $\boldsymbol{\vec{E}}$ **non conducting media**

$\boldsymbol{\rho = 0}$ No Charge, neutrality of the non conducting media

Propagation in linear non-conducting, non-magnetic media, dispersion relation

Dispersion relation : Remove $\vec{E}$ and $\vec{B}$ in this equation $\vec{k} \times \vec{B} = -\omega\mu\left(\varepsilon + \frac{i\gamma}{\omega}\right)\vec{E}$

We denote : $\tilde{\varepsilon} = \varepsilon + \frac{i\gamma}{\omega}$ as a general dielectric constant

Thus $\vec{k} \times \vec{B} = -\omega\mu\tilde{\varepsilon}\vec{E}$

The time fluctuations of the field are sinusoidal, the fields are complexes, so γ, ε, μ could be complexes according to the phase shift between current density and electromagnetic field.

$$\frac{1}{\omega}\vec{k} \times \left(\vec{k} \times \vec{E}\right) = -\omega\mu\tilde{\varepsilon}\vec{E} = -\frac{k^2}{\omega}\vec{E}$$

Proof:

Propagation in linear non-conducting, dispersion relation

Dispersion equation in a linear non-conducting media : $\boxed{k^2 = \varepsilon\mu\omega^2}$

Comparison with the vacuum

$$k^2 = \varepsilon_0\mu_0\omega^2 = \left(\frac{\omega}{c}\right)^2$$

Example of the dispersion in Fiber Optic

In a Fiber optic, the effect of the dispersion have an impact on the signal transmitted.

For a pulse transmitted the temporal is enhanced of the pulses during the propagation (wavelength λ_0, spectral linewidth $\Delta\lambda$)

According to

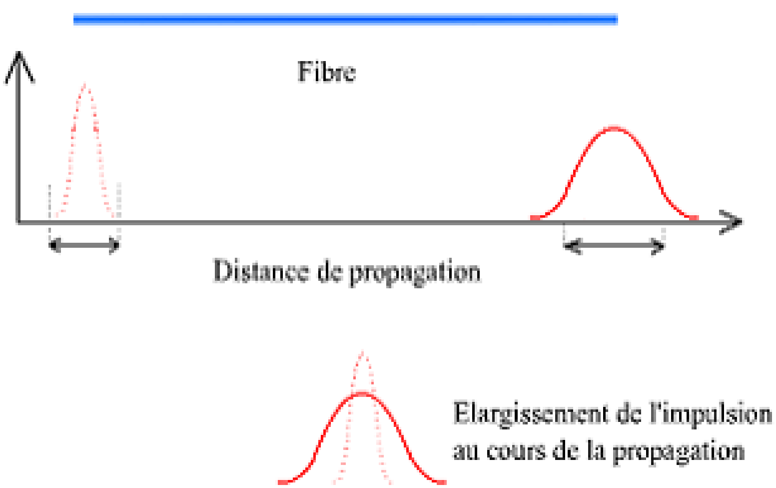
- the injection condition + FO (index profile, core Ø, …)+ curve => the optical power is distributed over ≠ propagation modes.
- Various propagation time
    - From one to another mode
    - For each component of the wave packet

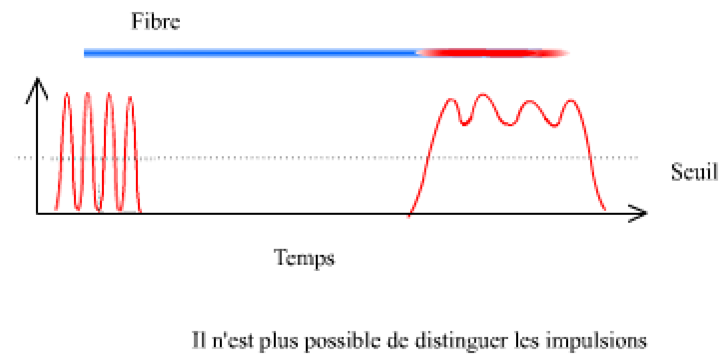
Example of the dispersion in Fiber Optic

Effect of the dispersion on the <u>transmission</u>

of an optical pulse

of an optical pulse packet





=> Temporal enhancement

of the wave envelope $\Delta\tau$

of each mode $\Delta\tau_c$

Resolution of the dispersion equation

The solutions of the dispersion equation given the wave vector k, are distributed in 3 types, associated to 3 kinds of wave

- **Progressive travelling wave - no attenuation: $\tilde{\varepsilon}\mu$ positive real value**

- **Evanescent wave: $\tilde{\varepsilon}\mu$ negative real value: No propagation Of the amplitude and energy**

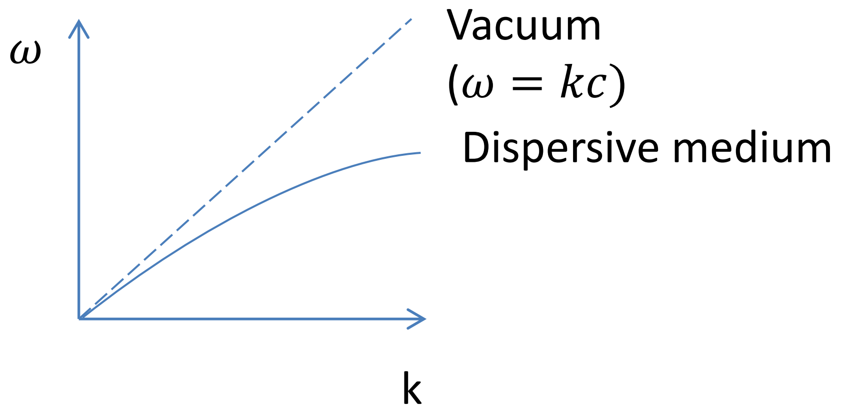
- **Attenuated travelling wave : $\tilde{\varepsilon}\mu$ complexe**

Resolution of the dispersion equation: **NON ATTENUATED Travelling Wave**

$\tilde{\varepsilon}\mu$ positive real value $\Rightarrow k^2$ positive real value (Only the product is real, $\tilde{\varepsilon}$ or μ can be complex)

Quasi-similar to vacuum propagation regarding the magnitude (constant)
But dispersion effect
The phase Velocity is $\boldsymbol{v_{\varphi}} = \boldsymbol{\frac{\omega}{k}} = \boldsymbol{\frac{1}{\sqrt{\tilde{\varepsilon}\mu}}}$ ($\neq c$ and frequency dependent)
The phase is not constant according to the pulsation (or frequency) = dispersive medium

$$k^2 = \tilde{\varepsilon}\mu\omega^2$$



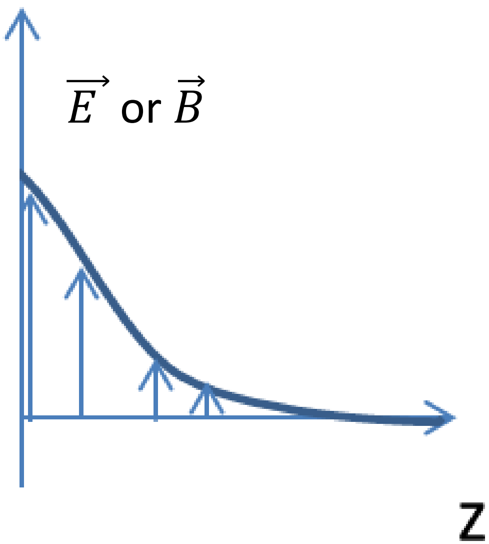
Resolution of the dispersion equation: EVANESCENT WAVE

$\tilde{\varepsilon}\mu$ negative real value $\Rightarrow$ $k = \pm ik"$

Thus $\overrightarrow{E}$ and $\vec{B}$ Field are defined as:

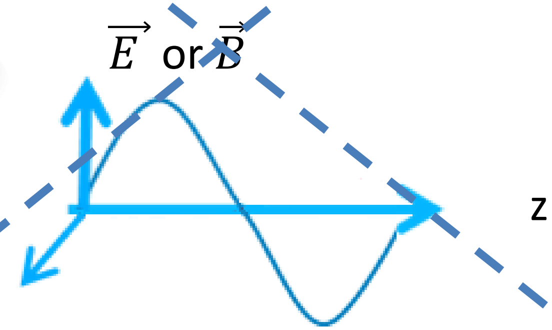
$$\vec{E} = \overrightarrow{E_0}\, e^{-i\omega t} e^{\pm k"z}$$
$$\vec{B} = \overrightarrow{B_0}\, e^{-i\omega t} e^{\pm k"z}$$

E,B Field same Phase, but Magnitude drift according to an exponential law decreasing…



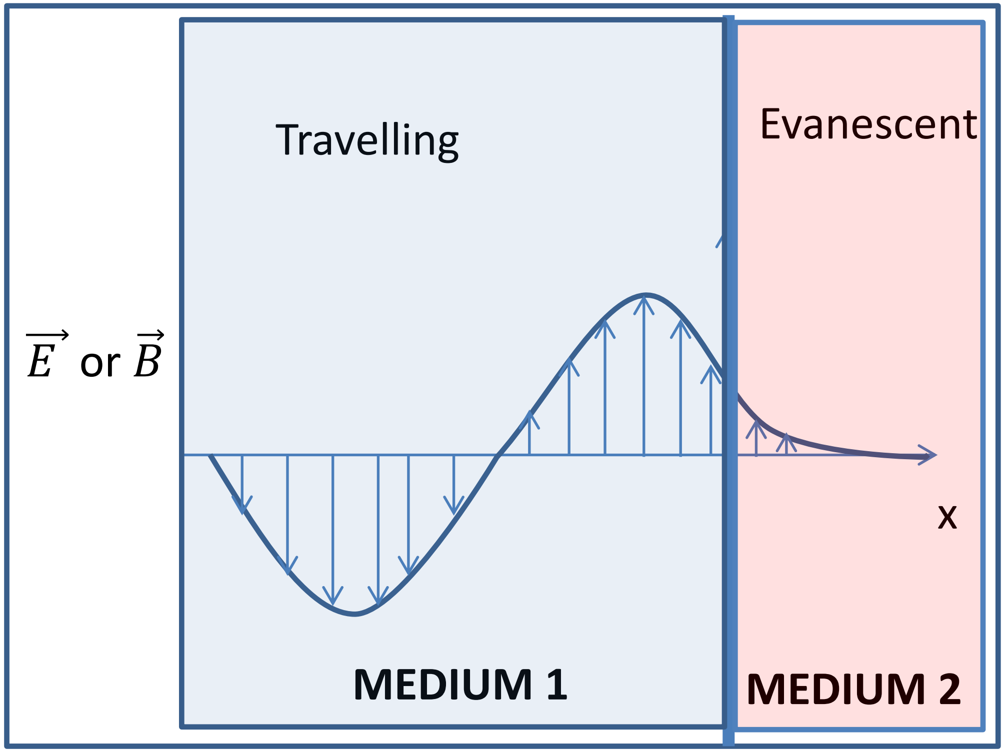
EVANESCENT FIELD!!!

~~$\neq$ of a travelling Plane wave: same magnitude after a period~~



Resolution of the dispersion equation: **EVANESCENT WAVE**

2 media into contact for a given frequency ω
In medium 1 : Travelling plane wave propagating at a frequency ω towards medium 2
In medium 2 : k is pure imaginary



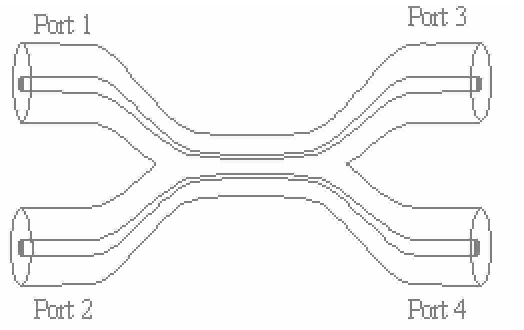
In Medium 2 the Magnitude of the field decreases quickly up to disappears :
This is an EVANESCENT Wave!

Application in optical coupler

Coupling for separating the optical wave into 2 channels or change the optical way.

If 2 fiber optics without protection cladding (tapered) are close, a coupling could obtained by evanescent wave .

The distance between each fiber leads the coupling coefficient.



Resolution of the dispersion equation: TRAVELLING ATTENUATED WAVE

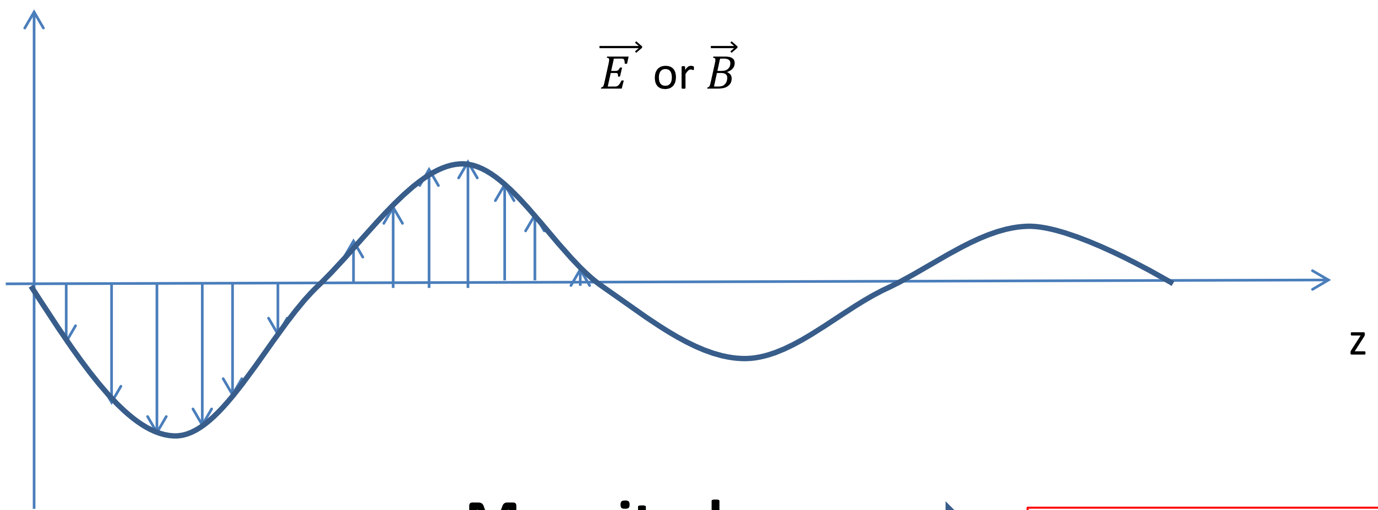
$\tilde{\varepsilon}\mu$ Complex $\Rightarrow$ k² Complex $\Rightarrow$ $k = k' + ik''$

Thus $\overrightarrow{E}$ and $\vec{B}$ Field are defined as:

$$\vec{E} = \overrightarrow{E_0}\, e^{-i(\omega t - k'z)} e^{\pm k''z}$$
$$\vec{B} = \overrightarrow{B_0}\, e^{-i(\omega t - k'z)} e^{\pm k''z}$$

The imaginary part ⇒ Magnitude decreasing

The real Part ⇒ Progressive Wave



Propagation + **Magnitude attenuation** ⇒ **Travelling attenuated Wave**

Resolution of the dispersion equation: Phase Velocity and Refractive index

We denote the phase Velocity :

$$v_p = \frac{\omega}{k'}$$

→ it is not the velocity of the amplitude propagation but velocity of the phase plane

The refractive index is not restricted to the definition : $\frac{c}{v_p} = \frac{ck\prime}{\omega}$, but the information contained into the imaginary part of the wave number is included.

Thus the **refractive index could be complex** such as:

$$n = \frac{ck}{\omega} = \frac{c(k' + ik")}{\omega}$$

It relates the propagation and the attenuation according to the frequency.

APPLICATION: Propagation in a DIELECTRIC

A dielectric contains :

- ***n*** ions per volume unit with static charges (Mass : $m_{ion}, charge$: $+e$)
- ***n*** electrons ($Mass$: m_e, Charge: $-e$).

Each electron are linked to the ions by an elastic force (No Freedom of Motion)

$-\boldsymbol{Kr}$ is a spring force when the electrons deviates of ***r*** from the ions

We assume: $\varepsilon \approx \varepsilon_0$ and $\mu \approx \mu_0$ at the equilibrium

Application: Propagation in a DIELECTRIC

By neglecting the effects of the magnetic force of Lorentz, the electron motion follows :

$$m_e \frac{d^2\vec{r}}{dt^2} = -e\vec{E} - K\vec{r}$$

$\vec{r}$ small $\Rightarrow \frac{d^2\vec{r}}{dt^2} \approx \frac{\partial^2\vec{r}}{\partial t^2}$

Sinusoidal operation, $\vec{r}$ and $\vec{E}$ time variation follows $\boldsymbol{e^{-i\omega t}}$

$$(m_e\omega^2 - K)\vec{r} = e\vec{E} \quad \Rightarrow \quad \vec{r} = \frac{e}{m_e}\left(\omega^2 - \omega_0{}^2\right)\vec{E}$$

$\boldsymbol{\omega_0}$ the Natural pulsation such as $\boldsymbol{\omega_0{}^2 = \frac{K}{m_e}}$

Application: Propagation in a DIELECTRIC

The velocity of the electron is thus: $\vec{v} = \dot{\vec{r}} = (-i\omega)\vec{r} = i\,\frac{e}{m_e}\frac{\omega}{{\omega_0}^2 \quad -\omega^2}\vec{E}$

The current density associated to these electrons (n per volum unit) is:

$$\vec{j} = n(-e)\vec{v} = i\,\frac{n \cdot e^2}{m_e}\frac{\omega}{\omega^2 - {\omega_0}^2}\vec{E}$$

As $\vec{j} = \gamma\vec{E}$ $\Rightarrow$ $\gamma = i\,\frac{n \cdot e^2}{m_e}\frac{\omega}{\omega^2 - {\omega_0}^2}$

As $\tilde{\varepsilon} = \varepsilon_0 + \frac{i\gamma}{\omega} = \varepsilon_0\left[1 - \frac{n \cdot e^2}{m_e\varepsilon_0}\frac{1}{\omega^2 - {\omega_0}^2}\right]$

The plasma pulsation $\omega_p = \left(\frac{n \cdot e^2}{m_e\varepsilon_0}\right)^{1/2}$

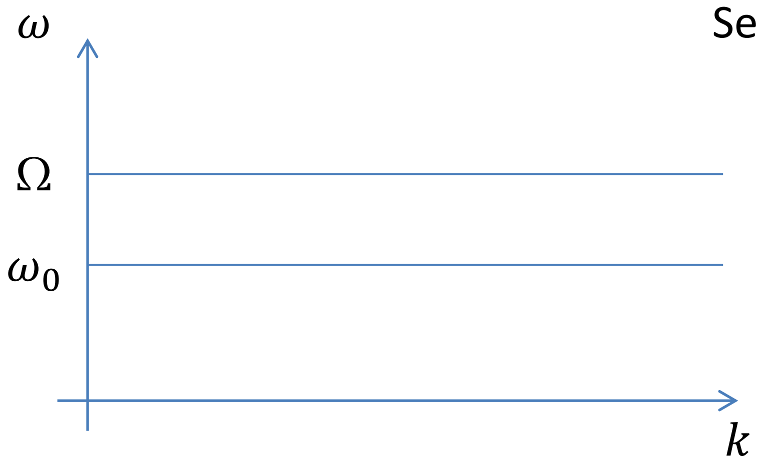
Numerical example: $f_p = 9.0 \cdot 10^9 Hz$ for a particle density $n = 10^{18}\ electrons/m^3$

Application: Propagation in a DIELECTRIC

The plasma frequency is $f_p = \frac{\omega_p}{2\pi} = \frac{1}{2\pi}\left(\frac{n \cdot e^2}{m_e \varepsilon_0}\right)^{1/2}$

$\Omega^2 = {\omega_p}^2 + {\omega_0}^2$ thus $\tilde{\varepsilon} = \varepsilon_0 \frac{\omega^2 - \Omega^2}{\omega^2 - {\omega_0}^2}$

The equation of dispersion is: $k^2 = \frac{\omega^2}{c^2} \frac{\omega^2 - \Omega^2}{\omega^2 - {\omega_0}^2}$



See https://www.showme.com/sh?h=VZMnUDg

Application: Propagation in a METAL

A metallic medium constituted by:

- n ions (charge +e, mass m_{ion})
- n electrons (charge –e, mass m_e)

$\left.\right\}$ NEUTRAL

In a conductor, we consider Free electrons (motion freedom) and a viscous force in opposition to the velocity $(-f\vec{v})$ where f is the friction coefficient.

The electron motion equation (rate equation) is:

$$m\frac{\partial\vec{v}}{\partial t} = -f\vec{v} - e\vec{E}$$

Application: Propagation in a METAL

Small sinusoidal (harmonic) motion assumption with a pulsation ω:

$$\frac{\partial \vec{v}}{\partial t} = -i\omega\vec{v}$$

$$\vec{v} = \frac{e}{m_e}\left(i\omega - \frac{1}{\tau}\right)^{-1}\vec{E}$$

We define $f = \frac{m_e}{\tau}$

The time $\boldsymbol{\tau}$ is a time constant related to the damping of the electron motion .

Application: Propagation in a metal

The current density and conductivity are thus:

$$\vec{j} = -ne\vec{v} = \frac{ne^2}{m_e}\left(\frac{1}{\tau} - i\omega\right)^{-1} \vec{E}$$

$$\gamma = \frac{ne^2\tau}{m_e}(1 - i\omega\tau)^{-1} = \gamma_0(1 - i\omega\tau)^{-1}$$

The dielectric constant is defined:

$\tilde{\varepsilon} = \varepsilon + \frac{i\gamma}{\omega}$ (Slide 11) and $\omega_p = \left(\frac{n \cdot e^2}{m_e \varepsilon_0}\right)^{1/2}$ ⇒ $\tilde{\varepsilon} = \varepsilon_0 \left(1 + \frac{i}{\omega} \cdot \frac{{\omega_p}^2 \tau}{1 - i\omega\tau}\right)$

The dispersion relation becomes:

$k^2 = \tilde{\varepsilon}\mu\omega^2$ Slide 12 ⇒ $k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{{\omega_p}^2/\omega^2}{1 + i/\omega\tau}\right)$

k^2, and k , is complex ∀ the frequency : **Attenuated Travelling Wave !!**

Application: Propagation in a METAL

Attenuated Travelling Wave ⇔ Good agreement with friction forces in opposition to the electron motion.

$$\boxed{k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{{\omega_p}^2/\omega^2}{1 + i/\omega\tau}\right)}$$

$\tau \approx 10^{-14} s$ for metals

(k depends on $\omega\tau$)

$$\omega\tau \ll 1$$
$$\Leftrightarrow$$
$$\omega \ll 10^{14} rad.s^{-1}$$

$$\gamma \cong \gamma_0 = \frac{ne^2\tau}{m_e}$$

$$k^2 \cong \frac{\omega^2}{c^2}\left(1 + i\frac{{\omega_p}^2}{\omega^2}\omega\tau\right) = \frac{\omega^2}{c^2}\left(1 + i\frac{\gamma_0}{\varepsilon_0\omega}\right)$$

$$\omega\tau \gg 1$$
$$\Leftrightarrow$$
$$\omega \gg 10^{14} rad.s^{-1}$$

$$k^2 \cong \frac{\omega^2}{c^2}\left(1 - \frac{{\omega_p}^2}{\omega^2}\left(1 - \frac{i}{\omega\tau}\right)\right)$$
$$= \frac{\omega^2}{c^2}\left(1 - \frac{{\omega_p}^2}{\omega^2}\right)\left(1 + \frac{i}{\omega\tau}\frac{{\omega_p}^2}{\omega^2 - {\omega_p}^2}\right)$$

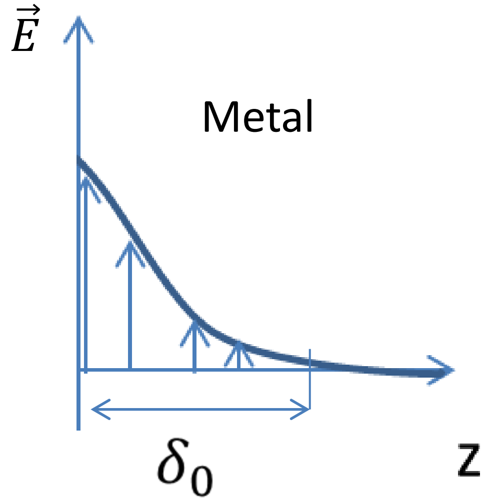
Application: Propagation in a METAL $\omega\tau \ll 1$

For a metal, $\gamma_0 \cong 10^7\ \Omega^{-1} \cdot m^{-1} \Rightarrow$ for a fixed ω , $\frac{\gamma_0}{\varepsilon_0 \omega} \gg 1$

$\Leftrightarrow \quad \vec{j} \gg \frac{\partial \vec{D}}{\partial t} \quad \Leftrightarrow$ From Maxwell Ampere $\quad \overrightarrow{rot}\vec{H} = \gamma \vec{E}$

$$k^2 \cong i\mu_0\gamma_0\omega = \frac{(1+i)^2}{{\delta_0}^2}$$

$$\boxed{\delta_0 = \sqrt{\frac{2}{\mu_0\gamma_0\omega}}}$$



SKIN DEPTH

(**wave attenuation according to the wave pulsation**)

This is the attenuation of the wave inside the metal (evanescent field into the metal)

Application: Propagation in a METAL $\omega\tau \ll 1$

Numerical Values: For the copper, $\omega_p{}^2 = 1.6 \cdot 10^{16} rad \cdot s^{-1}$,
$\tau = 2.5 \cdot 10^{-14} s, \gamma_0 = 6 \cdot 10^7 \Omega^{-1} \cdot m^{-1}$

Value of skin depth for the following frequencies:

$f = 50\ Hz,\quad \delta_0 =$ $9{,}2\ mm$

$f = 1\ kHz,\quad \delta_0 =$ $2{,}1 mm$

$f = 1 MHz,\quad \delta_0 =$ $0{,}064 mm$

$f = 1\ GHz,\quad \delta_0 =$ $6.5\ \cdot 10^{-11} m$

Application: Propagation in a metal $\omega\tau \gg 1$

2 solutions for $k^2 \cong \frac{\omega^2}{c^2}\left(1 - \frac{{\omega_p}^2}{\omega^2}\right)\left(1 + \frac{i}{\omega\tau}\frac{{\omega_p}^2}{\omega^2 - {\omega_p}^2}\right)$

$$\omega < \omega_p$$

$$k \cong i\frac{\omega}{c}\sqrt{\left(\frac{{\omega_p}^2}{\omega^2} - 1\right)\left(1 + \frac{i}{2\omega\tau}\frac{{\omega_p}^2}{\omega^2 - {\omega_p}^2}\right)}$$

The real part of k tends to 0 with $\frac{1}{\omega\tau}$

EVANESCENT WAVE

$$\omega > \omega_p$$

$$k \cong \frac{\omega}{c}\sqrt{\left(1 - \frac{{\omega_p}^2}{\omega^2}\right)\left(1 + \frac{i}{2\omega\tau}\frac{{\omega_p}^2}{\omega^2 - {\omega_p}^2}\right)}$$

The imaginary part of k tends to 0 with $\frac{1}{\omega\tau}$

TRAVELLING WAVE

Example Copper and Sodium

For the Copper and the Sodium $\tau \approx 10^{-14} s^{-1}$

How is $\omega\tau$ in the optical domain $\omega \cong 10^{15} rad. s^{-1}$?

For the copper: $\omega_p \; = 1.6 \cdot 10^{16} rad \cdot s^{-1}$

For the sodium: $\omega_p \; = 9 \cdot 10^{5} rad \cdot s^{-1}$

Conclusion

if $\lambda < 210 nm$,the Frequency $\omega/(2\pi) \sim 10^{15} Hz$ (ultraviolet), $\omega\tau \gg 1$

For copper , $\omega < \omega_p$, no propagation in the metal

For Sodium , $\omega > \omega_p$: The sodium is a transparent metal to the UV light = Alkali metal.

[See showme app for additional informations](https://www.showme.com/sh?h=u4DLQrQ)

https://www.showme.com/sh?h=u4DLQrQ