Hybrid Dynamical Systems - Part IV: Lyapunov

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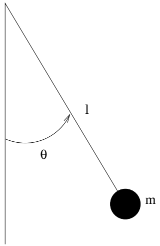
Outline

- In the previous course, we have studied the notion of stability and attractivity of a equilibrium point (the origin) or of a compact set ($\mathcal{A}$)
  - ▷ Several definitions and theorems have been proposed to deal with the notions of Lyapunov stability, global attractivity, uniform global attractivity, uniform global asymptotic stability, ..
  - ▷ However, these definitions and theorems proposed are based on the knowledge of the solution or equivalently of the trajectory behavior of the system.
  - ▷ The ability to compute the solution of a systems is strongly dependent of its structure: For example it is easy if the system is linear!

- In this course, we will study the way to prove stability and attractivity of a equilibrium point (the origin) or of a compact set ($\mathcal{A}$) without describing the trajectories of the system
  - ▷ Lyapunov theory
  - ▷ Lyapunov functions
- The lecture is structured as follows:
  - ▷ Some backgrounds on Lyapunov theory for linear and nonlinear systems is provided
  - ▷ The case of hybrid dynamical systems is then addressed.

- Consider the pendulum plotted in the following figure. It evolves in the vertical plane.
- One assumes that there exists a friction force on the mass proportional to the speed.



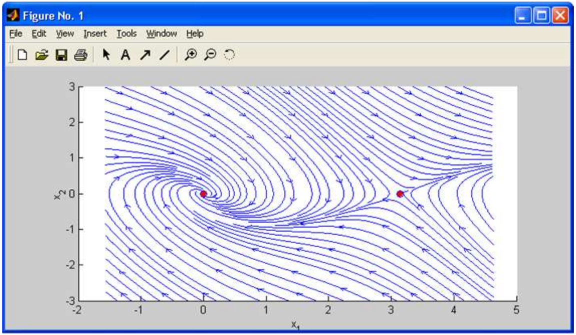
- The fundamental principe of mechanics gives:

$$mg\ddot{\theta}(t) = -mg\sin(\theta(t)) - kl\dot{\theta}(t)$$

- One considers the state vector $x(t) = \begin{bmatrix} \theta(t) & \dot{\theta}(t) \end{bmatrix}^\top$, which gives the nonlinear model:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\frac{g}{l}\sin(x_1(t)) - \frac{k}{m}x_2(t) \end{cases}$$

- There is two position of equilibrium of interest.
  - ▷ $x_1 = 0$, $x_2 = 0$: the pendulum is in the bottom position. This equilibrium is stable.
  - ▷ $x_1 = \pi$, $x_2 = 0$: the pendulum is in the up position. This equilibrium point is unstable.



- To conclude on the stability, one uses the phase plane to check the evolution of the trajectories of the system,
  - ▷ That is not obvious.
- Can we use other approaches?

- One studies the characteristics of the equilibrium point $[0\ 0]^\top$.
- Compute the energy function of the system energy.
  - ▷ This function is constituted by the sum of the potential energy and the kinetic energy, considering $E(0) = 0$.

$$E(x) = \int_0^{x_1} \frac{g}{l}\sin(y)dy + \frac{1}{2}x_2^2 = \frac{g}{l}(1-\cos(x_1)) + \frac{1}{2}x_2^2.$$

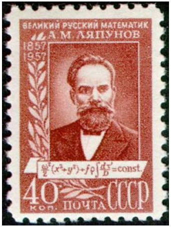
- Compute the time-derivative of $E(x)$:

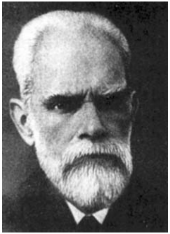
$$\begin{aligned}
\frac{dE(x)}{dt} &= \dot{E}(x) = \frac{g}{l}\sin(x_1)\dot{x}_1 + \frac{1}{2}2x_2\dot{x}_2 \\
&= \frac{g}{l}\sin(x_1)x_2 + \left(-\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2\right)x_2 \\
&= -\frac{k}{m}x_2^2
\end{aligned}$$

- If $k = 0$, one gets that $\dot{E}(x) = 0$ along the system trajectories.
  - ▷ The system is conservative $\rightarrow$ Stable equilibrium point.
- If $k \neq 0$, one can show that $\dot{E}(x) \leq 0$ along the system trajectories.
  - ▷ Then, locally, the energy is decreasing until, eventually, reaching $E = 0$, and by the way reaching the equilibrium point.
  - ▷ The system looses energy $\rightarrow$ Asymptotically stable equilibrium point.
- Question. Is it possible to provide an extension of this notion with more general functions than the energy functions:

Yes $\rightarrow$ Lyapunov functions

- The main tool we use is the Lyapunov theory.
- Aleksandr Mikhailovich Lyapunov (1857–1918), defended his PhD thesis *On Stability of Elliptic Equilibrium Forms of a Rotating Liquid*, in 1885, At the University of St Petersburg, with the advisor Pafnuti Lvovich Tchebychev (1821–1894). Tchebychev who also supervised the PhD of Andrei Andreyevich Markov (1856–1922).





- Lyapunov formalized the idea (illustrated on the pendulum example):

  If the energy is dissipated then the system is stable

- By looking at an energy-like function we can conclude about the stability (asymptotic stability) of a system without solving the nonlinear equations
- But the new difficulty is: how find such a function?

- The second method of Lyapunov is a major tool to deal with complex dynamical systems (as nonlinear systems, hybrid dynamical systems, ...)
- The objective: check some properties of the system as stability, safety, reachability, convergence of the system trajectories towards the origin or a set $\mathcal{A}$ (equilibrium point or attractor of interest)

  without an explicit description of the system trajectories

- The solution: use

  functions satisfying non-negativity constraints

  in a convex region Ω of the state space including the origin.

- Let us define functions with definite sign in a convex region Ω of the state space including the origin.

Definition of signed functions

- ▷ $V(x)$ is positive definite in Ω if $V(x) > 0$ for any $x \in \Omega$, $x \neq 0$ and $V(0) = 0$.
- ▷ $V(x)$ is positive semi-definite in Ω if $V(x) \geq 0$ for any $x \in \Omega$, $x \neq 0$ and $V(0) = 0$.
- ▷ $V(x)$ is negative definite in Ω if $V(x) < 0$ for any $x \in \Omega$, $x \neq 0$ and $V(0) = 0$.
- ▷ $V(x)$ is negative semi-definite in Ω if $V(x) \leq 0$ for any $x \in \Omega$, $x \neq 0$ and $V(0) = 0$.

Remark

$V(x)$ negative semi-definite $\Leftrightarrow$ -$V(x)$ positive semi-definite.

- Examples:
  - ▷ $V(x) = (x_1 + x_2)^2$ is positive semi-definite in $\mathbb{R}^2$.
  - ▷ $V(x) = x_1^2 + x_2^2$ is positive definite in $\mathbb{R}^2$.
  - ▷ $V(x) = x_1^2 + x_2^2 - 4$ is negative definite in the circle in $\mathbb{R}^2$ with the ray equal to 2.

- Now we can state the first theorem of Lyapunov in the local case.
- Consider the nonlinear continuous-time system:

$$\dot{x} = f(x) \tag{1}$$

with $x \in \mathbb{R}^n$ and f a regular function, $f(0) = 0$.

Theorem 1

Let us consider the equilibrium point $x_e = 0$ and a domain Ω containing 0. Let $V : \Omega \to \mathbb{R}$, be a C^1 function such that:

$$V(0) = 0 \text{ and } V(x) > 0 \text{ for } x \in \Omega - \{0\}, \tag{2}$$

$$\dot{V}(x) \leq 0 \text{ in } \Omega, \tag{3}$$

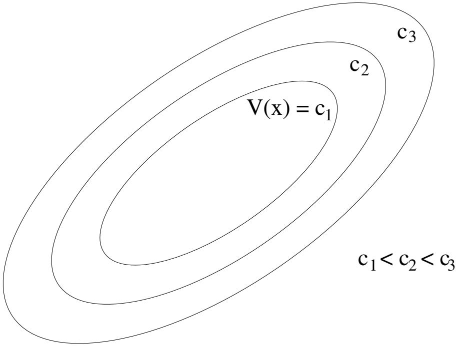
then $x = 0$ is a stable equilibrium point. Moreover, if

$$\dot{V}(x) < 0 \text{ in } \Omega - \{0\}, \tag{4}$$

then $x = 0$ is asymptotically stable.

▷ $\dot{V}(x) = \frac{\partial V(x)}{\partial x}\dot{x} = \frac{\partial V(x)}{\partial x} f(x) = \left[\frac{\partial V}{\partial x_1} \cdots \frac{\partial V}{\partial x_n} \right] f(x).$

- A function V satisfying (2)-(3) or (2)-(4) is called a Lyapunov function.
- The surface $V(x) = c$ is called a level (or surface) set of the Lyapunov function.



Example 1: linear system

- Consider the following example:

$$\dot{x}(t) = ax(t),\ a < 0.$$

- This is a linear first-order system.
- Let us consider $V(x(t)) = \frac{1}{2}x^2(t)$.
- The time-derivative of V along the trajectories of the system gives:

$$\dot{V}(x) = x\dot{x} = ax^2.$$

- Since $a < 0$, it follows that $\dot{V}(x) < 0$ for $x \in \mathbb{R} - \{0\}$, then, the system is asymptotically stable.

Application to linear systems

- Extension to the matrix case of the function of definite sign
  - ▷ A symmetric matrix $P \in \Re^{n\times n}$ is positive definite if and only if all its eigenvalues are strictly positive, or iff all its main minors are strictly positive.
  - ▷ P is positive semi-definite iff all its eigenvalues are positive or null.
  - ▷ By the same way, P is negative definite if $\lambda_i(P) < 0$, or if $-P$ is positive definite.
- Main property. If P is positive definite then

$$\forall\ x \neq 0,\ x \in \mathbb{R}^n,\ x^\top P x > 0$$

- Examples. Are the following matrices positive definite?
  - ▷ $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $P = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$; $P = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

Theorem 2 - Linear case

Given the system $\dot{x}(t) = Ax(t)$, $x_e = 0$ is asymptotically stable if and only if, $\forall\ Q = Q^\top$ positive definite, the symmetric matrix $P = P^\top$ solution to the Lyapunov equation

$$A^\top P + PA + Q = 0$$

is positive definite.

- **Test.** If there exists a matrix P solution to the set of linear matrix inequalities

$$\left\{ \begin{array}{c} P > 0 \\ A^\top P + PA < 0 \end{array} \right. \qquad \Rightarrow \text{Linear Conditions (LMI)}$$

Example 2: inverse pendulum (1)

- Consider the example of the inverse pendulum.

$$\left\{ \begin{array}{l} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -a\sin(x_1(t)) - bx_2(t) \end{array} \right.$$

  and the function

$$V(x) = a(1 - \cos(x_1)) + \frac{1}{2}x_2^2$$

- Questions.
  - ▷ Determine Ω.
  - ▷ Is $V(x)$ a Lyapunov function?
  - ▷ Let us consider another function:

$$V(x) = a(1 - \cos(x_1)) + \frac{1}{2}x^\top Px$$

    Is it a Lyapunov function?

Example 2: inverse pendulum (2)

- Determine Ω. Recall $V(x) = a(1 - \cos(x_1)) + \frac{1}{2}x_2^2$
  - ▷ We have $V(0) = 0$ for $x_1 = 0$ and $x_2 = 0$.
  - ▷ $V(x)$ is positive definite ($V(x) > 0$) over the domain $-2\pi < x_1 < 2\pi$ and for any x_2.
- Is $V(x)$ a Lyapunov function?
  - ▷ Compute $\dot{V}(x)$:

$$\begin{aligned}
\dot{V}(x) &= a\sin(x_1)\dot{x}_1 + x_2\dot{x}_2 \\
&= a\sin(x_1)x_2 + x_2(-a\sin(x_1) - bx_2) \\
&= -bx_2^2
\end{aligned}$$

  - ▷ One gets : $\dot{V}(x) \leq 0$, since $\dot{V}(x) = 0$ for $x_2 = 0$ and $\forall x_1$.
  - ▷ The function V satisfies (2)-(3) and therefore is a Lyapunov function.
  - ▷ The origin is stable.

Example 2: inverse pendulum (3)

- Try another function. Consider $V(x) = a(1-\cos(x_1)) + \frac{1}{2}x^T P x$ with $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$.
  - ▷ P is chosen to be positive definite: $p_{11} > 0$, $p_{22} > 0$ and $det(P) = p_{11}p_{22} - p_{12}^2 > 0$. Then $V(x)$ is positive definite.
  - ▷ Compute $\dot{V}(x)$:

$$
\begin{aligned}
\dot{V}(x) &= a\sin(x_1)\dot{x}_1 + \tfrac{1}{2}(\dot{x}^\top P x + x^\top P \dot{x}) \\
&= a\sin(x_1)\dot{x}_1 + (x_1 p_{11} + x_2 p_{12})\dot{x}_1 + (x_1 p_{12} + x_2 p_{22})\dot{x}_2 \\
&= a\sin(x_1)x_2 + (x_1 p_{11} + x_2 p_{12})x_2 + (x_1 p_{12} + x_2 p_{22})(-a\sin(x_1) \\
&= a\sin(x_1)x_2(1-p_{22}) + x_1x_2(p_{11} - bp_{12}) + x_2^2(p_{12} - bp_{22}) \\
&\quad -a\sin(x_1)x_1p_{12}
\end{aligned}
$$

  - ▷ Possible choice for P:

$$
\begin{aligned}
&1 - p_{22} = 0 \text{ then } p_{22} = 1 \\
&p_{11} - bp_{12} = 0 \text{ then } p_{11} = bp_{12} > 0 \\
&p_{12} - bp_{22} = p_{12} - b < 0 \text{ then } 0 < p_{12} < b \\
&\text{ and one can choose } p_{12} = \tfrac{b}{2}
\end{aligned}
$$

Example 2: inverse pendulum (4)

- With such a matrix $P = \begin{bmatrix} \frac{b^2}{2} & \frac{b}{2} \\ \frac{b}{2} & 1 \end{bmatrix}$
  - ▷ one gets for $\dot{V}(x)$:

$$\dot{V}(x) = -\frac{b}{2}x_2^2 - \frac{ab}{2}\sin(x_1)x_1$$

  - ▷ $V(x)$ is positive definite ($V(x) > 0$) over the domain $-\pi < x_1 < \pi$ and for any x_2.
  - ▷ V satisfies (2)-(4).
  - ▷ The origin is asymptotically stable.
- Suggestion. Do the same study regarding the other equilibrium point $x_e = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$.

- We have thanks to the notion of Lyapunov function studied the local stability of the equilibrium point. In particular to simplify, we have studied the stability of the origin ($x_e = 0$).
  - ▷ Recall that to prove this we use the conditions (2)-(4):

$$V(x_e) = 0 \text{ and } V(x) > 0 \text{ for } x \in \Omega - \{0\},$$
$$\dot{V}(x) < 0 \text{ in } \Omega - \{0\},$$

- By definition, when we consider an asymptotically stable equilibrium point $x_e = 0$, that signifies that it is stable and attractor.
- A nature question arises: Is it possible to estimate the attraction region of the equilibrium point, i.e. the set of initial conditions such that the solutions of (1) initiated at this initial conditions converge toward $x_e = 0$?

- Find such a set is very complicated, or even often impossible.
- On the other hand, an estimate of the set may be obtained thanks to the Lyapunov theory.
- The set

$$\Omega_c = \{x \in \mathcal{D}, V(x) \leq c\}$$

  is such that
  - ▷ any trajectory initialized inside Ω_c remains in Ω_c as $\dot{V}(x(t)) \leq 0 \Rightarrow V(x(t)) \leq V(x(0)) \leq c, \forall t \geq 0$ (invariance)
  - ▷ Moreover, if $\dot{V}(x) < 0$ then $\lim\limits_{t \to +\infty} x(t) = 0$ by definition (stability)
  - ▷ Ω_c is an estimate of the domain of attraction (also called the basin of attraction).
- The idea is to find the largest $c > 0$, but note that this may be very conservative.

- Example. Consider the following example (balancing pointer):

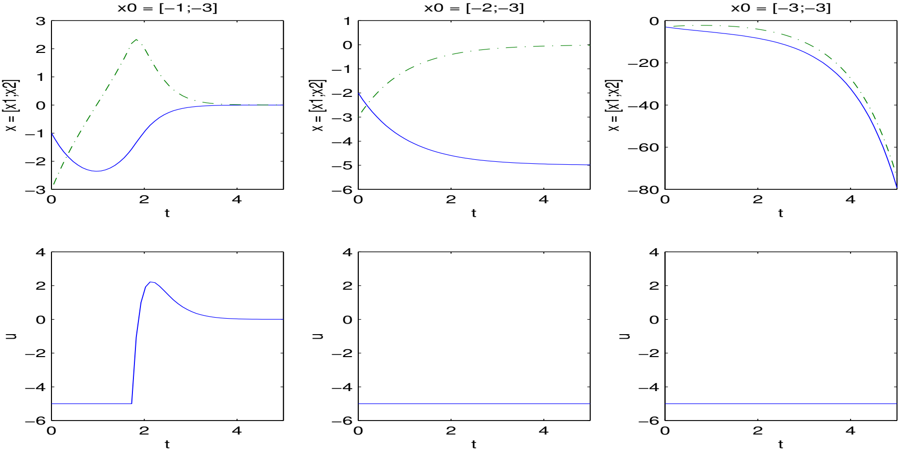
$$\dot{x} = Ax + Bsat(Kx) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} sat(\begin{bmatrix} 13 & 7 \end{bmatrix} x)$$

with $sat(Kx(t)) = \begin{cases} 5 & \text{if} \quad Kx(t) > 5 \\ Kx(t) & \text{if} \quad |Kx(t)| \leq 5 \\ -5 & \text{if} \quad Kx(t) < -5 \end{cases}$

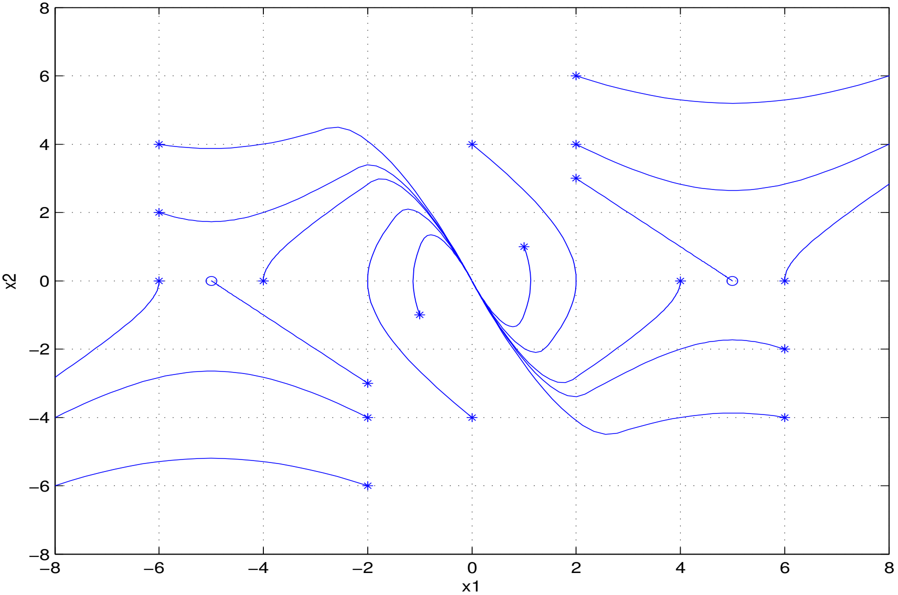
  - ▷ Matrix A is unstable : $\sigma(A) = \{1; -1\}$.
  - ▷ Matrix $A + BK$ is asymptotically stable : $\sigma(A + BK) = \{-3; -4\}$
  - ▷ There are two equilibrium points in plus of $x = 0$ when saturation is active:

$$x_{e1} = \begin{bmatrix} -5 \\ 0 \end{bmatrix} \quad \text{and} \quad x_{e2} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

- What happens for the system trajectories?



- To summarize the previous picture:
  - $x_1(0) = [-1 \ \ -3]^\top$ - in spite of the fact that the control signal u is saturated during the first instants, the trajectory converges asymptotically to the origin.
  - $x_2(0) = [-2 \ \ -3]^\top$ - in spite of the fact that $(A + BK)$ is Hurwitz, the trajectory converges to an equilibrium point x_e different from the origin ($x_e = [-5 \ 0]^\top$). Note that in this case the control signal remains saturated all the time.
  - $x_3(0) = [-3 \ \ -3]^\top$ - in spite of the fact that $(A + BK)$ is Hurwitz, the trajectory diverges. Note that in this case the control signal also remains saturated all the time.


8
6
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2
0
−2
−4
−6
−8
x2
−8
−6
−4
−2
0
2
4
6
8
x1

- Until now, we have manipulated a theorem of local asymptotic stability.
- A natural questions arises: Under which conditions is obtained a attraction domain equal to $\mathbb{R}^n$?

Theorem 3 - GAS

Let us consider an equilibrium point $x = 0$ for system (1). Let $V : \mathbb{R}^n \to \mathbb{R}$, be a C^1 function such that:

$$\|x(t)\| \to +\infty \;\Rightarrow V(x) \to +\infty \tag{5}$$

$$V(x_e) = 0 \;\text{ and }\; V(x) > 0 \;\forall x \neq 0, \tag{6}$$

$$\dot{V}(x) < 0 \;\forall x \neq 0, \tag{7}$$

then $x = 0$ is globally asymptotically stable.

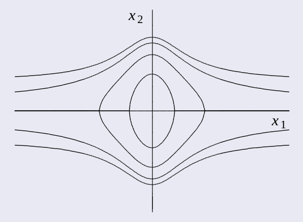
- The condition (5)

$$\|x(t)\| \to +\infty \ \Rightarrow V(x) \to +\infty$$

expresses that the function V is radially unbounded.

  - ▷ This property is needed to prove the global asymptotic stability.
  - ▷ Thanks to this property one can prove that the level set $\{x \in \mathbb{R}^n; V(x) \leq c\}$ is bounded for every $c > 0$.

- Example. Consider $V(x) = x_2^2 + \frac{x_1^2}{1+x_1^2}$. For $c = 5$ and $x_2 = 2$, the set is unbounded in the x_1-direction.



- In Part III - stability, we have seen that we can check stability and attractivity properties by using the shape of the solutions, namely their explicit expressions as functions of time.
  - ▷ Such a an expression of the solutions was possibly by exploiting the linearity of some vector fields (flow and/or jump map),
  - ▷ We lost this if the system is a nonlinear system, where no explicit expression of the solutions can be easily determined.
- In the previous backgrounds proposed for linear or nonlinear systems we have seen that we can use some particular functions to guarantee the stability (and attractivity): Lyapunov functions
  - ▷ We introduce Lyapunov theorems, which provide sufficient conditions for checking UGAS for (nonlinear) hybrid systems.

- Consider the hybrid dynamical system

$$\begin{cases} \dot{x} &= f(x), \quad x \in \mathcal{C} \\ x^+ &= g(x), \quad x \in \mathcal{D}. \end{cases} \tag{8}$$

where

- f describes the continuous evolution (flow) of the state
- g describes the discrete evolution (jumps) of the state
- $\mathcal{C}$ describes the set where the flow can occur (continuous evolution)
- $\mathcal{D}$ describes the set where the jumps can occur (discrete evolution)

- We have the first general result.

Theorem 4 - Lyapunov Theorem

Given a closed set $\mathcal{A}$ and system (8), if there exist a function $x \mapsto V(x)$ continuous and differentiable in an open set containing $\mathcal{C}$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a continuous positive definite function ρ (namely a function such that $\rho(0) = 0$ and $\rho(s) > 0$ for all $s > 0$) such that

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}), \ \forall x \in \mathcal{C} \cup \mathcal{D} \cup g(\mathcal{D}), \tag{S}$$

$$\dot{V}(x) := \frac{\partial V(x)}{\partial x} f(x) = \langle \nabla V(x), f(x) \rangle \leq -\rho(|x|_{\mathcal{A}}), \ \forall x \in \mathcal{C}, \tag{F}$$

$$\Delta V(x) := V(g(x)) - V(x) \leq -\rho(|x|_{\mathcal{A}}), \ \forall x \in \mathcal{D}, \tag{J}$$

then $\mathcal{A}$ is UGAS for (8).

Some remarks on Theorem 4

- Let us comment the three conditions in Theorem 4:
  - ▷ Condition (S), is called the sandwich condition, due to the fact that the condition requires that V be "sandwiched" between two class $\mathcal{K}_\infty$ functions of the distance to $\mathcal{A}$;
  - ▷ Condition (F) is the flow condition, because it imposes conditions on the flow map and in the flow set;
  - ▷ Condition (J) is the jump condition, because it imposes conditions on the jump map and in the jump set.

Part III-Definition 5

A continuous function α of $[0, a]$ valued in $[0, +\infty]$ is said to be of class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to be of class $\mathcal{K}_\infty$ if $a = \infty$ et $\lim_{r \to +\infty} \alpha(r) = +\infty$.

- Condition (F) can be written with two different notations, both expressing the directional derivative of V at x, in the direction $f(x)$.
  - ▷ This corresponds to the scalar product between the gradient $\nabla V(x)$ and $f(x)$
  - ▷ that can also be expressed in terms of $\dfrac{\partial V(x)}{\partial x} = (\nabla V(x))^\top$.
- Both notations are found in the literature.

Example

- Let us look at what happens with the conditions of Theorem 4 with the quadratic Lyapunov function $V(x) = x^\top P x$
- Condition $\alpha_1(|x|_\mathcal{A}) \leq V(x) \leq \alpha_2(|x|_\mathcal{A}), \ \forall x \in \mathcal{C} \cup \mathcal{D} \cup g(\mathcal{D})$
  - ▷ One can remark that $\lambda_{min}(P)|x|^2 \leq x^\top P x \leq \lambda_{max}(P)|x|^2$
- Condition $\dot{V}(x) := \dfrac{\partial V(x)}{\partial x} f(x) = \langle \nabla V(x), f(x) \rangle \leq -\rho(|x|_\mathcal{A}), \ \forall x \in \mathcal{C}$
  - ▷ 2-order case. Compute the time-derivative of $V(x)$ with

$$V(x) = x^\top P x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 p_1 + 2 x_1 x_2 p_2 + x_2^2 p_3$$

  - ▷ $\dfrac{\partial V(x)}{\partial x} = (\nabla V(x))^\top = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \cdots & \frac{\partial V}{\partial x_n} \end{bmatrix} = 2x^\top P.$
  - ▷ $\dot{V}(x) = x^\top P \dot{x} + \dot{x}^\top P x = 2x^\top P \dot{x}$
- Condition $\Delta V(x) := V(g(x)) - V(x) \leq -\rho(|x|_\mathcal{A}), \ \forall x \in \mathcal{D}$
  - ▷ $\Delta V(x) = V(x^+) - V(x) = (x^+)^\top P x^+ - x^\top P x = g(x)^\top P g(x) - x^\top P x$

Example (1)

- Consider the hybrid system (8) defined with state $x = (z, \tau)$ and with purely continuous dynamics

$$\dot{x} = f(x) := \begin{bmatrix} z \\ 1 \end{bmatrix}, \quad \forall x \in \mathcal{C} := \{(z, \tau) : \tau \in [0, M]\} = \mathbb{R} \times [0, M], \tag{9}$$

  - ▷ The jump set $\mathcal{D} = \emptyset$ is empty, so that the jump dynamics plays no role.
  - ▷ Maximal solutions are not complete, since the state τ is only allowed to flow in the bounded set $[0, M]$ with dynamics $\dot{\tau} = 1$. In particular, the flow time is bounded by $t \leq M$ (See Part III).

- Objective. We want to study the stability of the set $\mathcal{A} = \{(z, \tau) : z = 0, \tau \in [0, M]\} = \{0\} \times [0, M]$.

Example (2)

- Consider the Lyapunov function $V(x) = z^2 e^{-2\sigma\tau}$, with $\sigma > 1$ (recall the state is $x = (z, \tau)$).
- Note that $|x|_{\mathcal{A}} = |z|$ and $e^{-2\sigma M} \leq e^{-2\sigma\tau} \leq 1$, $\forall x \in \mathcal{C}$
- For $x \in \mathcal{C}$ one gets:

$$e^{-2\sigma M}|x|_{\mathcal{A}}^2 \leq V(x) \leq |x|_{\mathcal{A}}^2$$

- $\dot{V}(x) := \dfrac{\partial V(x)}{\partial x} f(x) = \langle \nabla V(x), f(x) \rangle$ reads:

$$\begin{aligned}
\dot{V}(x) &= \begin{bmatrix} \dfrac{\partial V(x)}{\partial z} & \dfrac{\partial V(x)}{\partial \tau} \end{bmatrix} f(x) \\
&= \begin{bmatrix} 2ze^{-2\sigma\tau} & -2\sigma z^2 e^{-2\sigma\tau} \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} \\
&= 2(1-\sigma)V(x)
\end{aligned}$$

$\boxed{\text{Example (3)}}$

- Since $\sigma > 1$ $(1 - \sigma < 0)$ one can write

$$2(1-\sigma)|x|^2_{\mathcal{A}} \leq 2(1-\sigma)V(x) \leq 2(1-\sigma)e^{-2\sigma M}|x|^2_{\mathcal{A}}$$

- Then conditions (S) and (F) are satisfied with

$$\begin{aligned} \alpha_1(s) &= e^{-2\sigma M}s^2 \\ \alpha_2(s) &= s^2 \\ \rho(s) &= 2(1-\sigma)\alpha_1(s) \end{aligned}$$

Example (4)

- Consider that the system is modified as follows to consider jump condition

$$
\begin{aligned}
\dot{x} &= f(x) := \begin{bmatrix} z \\ 1 \end{bmatrix}, \forall x \in \mathcal{C} := \{(z,\tau) : \tau \in [0, M]\} = \mathbb{R} \times [0, M], \\
x^{+} &= g(x) := \begin{bmatrix} \lambda z \\ 0 \end{bmatrix}, \forall x \in \mathcal{D} := \{(z,\tau) : \tau = M\} = \mathbb{R} \times \{M\}.
\end{aligned}
$$

  - ▷ with the state $x = \begin{bmatrix} z \\ \tau \end{bmatrix} \in \mathbb{R}^2$,
  - ▷ λ is a positive scalar such that $0 \leq \lambda < e^{-M}$
  - ▷ Note that $\mathcal{C}$ and $\mathcal{D}$ read:

$$\mathcal{C} = \mathbb{R} \times [0, M] \text{ and } \mathcal{D} = \mathbb{R} \times \{M\}$$

- Objective. We want to study the stability of the set $\mathcal{A} = \{(z,\tau) : z = 0, \tau \in [0, M]\} = \{0\} \times [0, M]$.

Example (5)

- We consider the same Lyapunov function $V(x) = z^2 e^{-2\sigma\tau}$,
  - ▷ We impose to σ to satisfy $\sigma > 1$ and $\lambda e^{\sigma M} < 1$
- Consider the condition (J):
  $\Delta V(x) := V(g(x)) - V(x) \leq -\rho(|x|_{\mathcal{A}}), \ \forall x \in \mathcal{D}$
  - ▷ For $x \in \mathcal{D}$ one gets $x_2 = \tau = M$ and therefore $\Delta V(x) := V(g(x)) - V(x)$ reads:

$$
\begin{aligned}
\Delta V(x) &= V(g(x)) - V(x) \\
&= \lambda^2 z^2 - z^2 e^{-2\sigma M} \\
&= (\lambda^2 e^{2\sigma M} - 1) z^2 e^{-2\sigma M} \\
&= ((\lambda e^{\sigma M})^2 - 1) z^2 e^{-2\sigma M} \\
&= ((\lambda e^{\sigma M})^2 - 1) V(x)
\end{aligned}
$$

  - ▷ $((\lambda e^{\sigma M})^2 - 1) V(x) < ((\lambda e^{\sigma M})^2 - 1)\alpha_1(|x|_{\mathcal{A}}) < 0$
  - ▷ Therefore $\rho(s) = min\{2(1-\sigma)\alpha_1(s), (1-(\lambda e^{\sigma M})^2)\alpha_1(s)\}$

Extension to the set-valued case

$$\begin{aligned} \dot{x} &\in F(x) \ , \ x \in \mathcal{C} \\ x^+ &\in G(x) \ , \ x \in \mathcal{D} \end{aligned} \tag{10}$$

- Condition $\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}), \ \forall x \in \mathcal{C} \cup \mathcal{D} \cup g(\mathcal{D})$
  - ▷ Almost unchanged: $\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}), \ \forall x \in \mathcal{C} \cup \mathcal{D} \cup G(\mathcal{D})$
- Condition $\dot{V}(x) := \dfrac{\partial V(x)}{\partial x} f(x) = \langle \nabla V(x), f(x) \rangle \leq -\rho(|x|_{\mathcal{A}}), \ \forall x \in \mathcal{C}$
  - ▷ $\dot{V}(x) := \dfrac{\partial V(x)}{\partial x} f(x) = \langle \nabla V(x), f(x) \rangle \leq -\rho(|x|_{\mathcal{A}}), \forall x \in \mathcal{C}, \ \forall f \in F(x)$
- Condition $\Delta V(x) := V(g(x)) - V(x) \leq -\rho(|x|_{\mathcal{A}}), \ \forall x \in \mathcal{D}$
  - ▷ $\Delta V(x) := V(g(x)) - V(x) \leq -\rho(|x|_{\mathcal{A}}), \forall x \in \mathcal{D}, \ \forall g \in G(x).$
- $\rho(|x|_{\mathcal{A}})$ is a positive definite function: $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $\rho(s) > 0$ for all $s > 0$ and $\rho(0) = 0$.

Final remarks on Theorem 4

- Theorem 4 provides a sufficient certificate of UGAS
- The main problem resides in the characterization of a Lyapunov function.
  - ▷ it might be quite difficult for general nonlinear/hybrid dynamical systems
- An important result states that whenever the GAS property holds, then a function V satisfying (S), (F) and (J) is guaranteed to exist.
  - ▷ The next theorem generalizes the well-known fact that for exponentially stable linear systems $\dot{x} = Ax$ (or $x^+ = Ax$) there always exists a Lyapunov function of the form $V(x) = x^\top Px$, with P symmetric and positive definite.

Part III- Assumption 1 - Basic assumptions

1. $\mathcal{C}$ and $\mathcal{D}$ are closed;
2. $\mathcal{A}$ is compact;
3. f and g are continuous.

Part III - Assumption 2 - Basic assumptions

The hybrid inclusion (10) satisfies the hybrid basic assumptions if sets $\mathcal{C}$ and $\mathcal{D}$ are closed and maps F and G satisfy the following properties:

1. maps F and G are locally bounded, namely for each compact set $\mathcal{K} \subset C \cup D$, both $F(\mathcal{K})$ and $G(\mathcal{K})$ are bounded;
2. the graph of F (namely the set of all pairs (y, x) such that $y \in F(x)$) and the graph of G are closed (this property corresponds to a generalization of continuity called *outer semicontinuity*);
3. $F(x)$ is nonempty for each $x \in C$ and $G(x)$ is nonempty for each $x \in D$;
4. $F(x)$ is convex for each $x \in C$.

Recall system (8):

$$\begin{aligned} \dot{x} &= f(x), \quad x \in \mathcal{C} \\ x^+ &= g(x), \quad x \in \mathcal{D}. \end{aligned}$$

Recall system (10):

$$\begin{aligned} \dot{x} &\in F(x)\ ,\ x \in \mathcal{C} \\ x^+ &\in G(x)\ ,\ x \in \mathcal{D} \end{aligned}$$

Theorem 5 - Converse Lyapunov function

If hybrid equation (8) or hybrid inclusion (10) satisfies Assumption 1 or 2 then

GAS of a compact attractor $\mathcal{A}$ implies the existence of a function V satisfying (S), (F) and (J)

for some functions α_1, α_2 and ρ.

- Given a Lyapunov function candidate V, it can be difficult to find the finding functions α_1, α_2 and ρ in Theorem 5, even in relatively simple cases.
- To simplify this work, we can transform conditions (S), (F), (J) into apparently less restrictive conditions, which are actually equivalent
  - ▷ To do this, we consider some standing assumptions on the data of the hybrid system.

- First we can relax the sandwich requirement (S) in the case where $\mathcal{A}$ is compact.

Proposition 1

Assume that $\mathcal{A}$ is compact and that function V is

- *Positive definite with respect to $\mathcal{A}$ relative to $\mathcal{C} \cup \mathcal{D} \cup g(\mathcal{D})$*, namely

$$
\begin{aligned}
V(x) &= 0, \;\; \forall x \in \mathcal{A}, \\
V(x) &> 0, \;\; \forall x \in \mathcal{C} \cup \mathcal{D} \cup g(\mathcal{D}) \setminus \mathcal{A};
\end{aligned}
$$

- *Radially Unbounded relative to $\mathcal{C} \cup \mathcal{D} \cup g(\mathcal{D})$*, namely for any sequence such that $x_i \in \mathcal{C} \cup \mathcal{D} \cup g(\mathcal{D})$ for all $i \in \mathbb{Z}_{\geq 0}$ and $\lim\limits_{i \to +\infty} |x_i| = +\infty$, then $\lim\limits_{i \to +\infty} V(x_i) = +\infty$.

Then the sandwich condition (S) holds for some class $\mathcal{K}_\infty$ functions α_1 and α_2.

- The meaning of Proposition 1 is that the following condition

$$V \quad \text{Positive definite and radially unbounded with resp. to } \mathcal{A} \text{ relative to } \mathcal{C} \cup \mathcal{D} \cup g(\mathcal{D}) \tag{S'}$$

  implies (S).

- Positive definiteness and radial unboundedness are simpler to check than (S) because they do not require determining an explicit expression of the class $\mathcal{K}_\infty$ functions α_1 and α_2.

- Now consider relaxing version of flow and jump conditions

Proposition 2

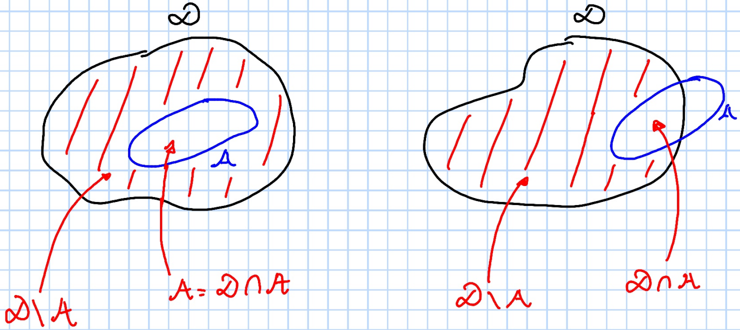
If system (8) or (10) satisfies Assumption 1 or 2 , then

- The flow condition (F) is satisfied for some function ρ if the following condition holds:

$$\dot{V}(x) := \frac{\partial V(x)}{\partial x} f(x) = \langle \nabla V(x), f(x) \rangle < 0, \qquad \forall x \in \mathcal{C} \setminus \mathcal{A}; \tag{F'}$$

- The jump condition (J) is satisfied for some function ρ if the following condition holds:

$$\begin{array}{ll} \Delta V(x) := V(g(x)) - V(x) < 0, & \forall x \in \mathcal{D} \setminus \mathcal{A}, \\ g(x) \in \mathcal{A}, & \forall x \in \mathcal{D} \cap \mathcal{A}. \end{array} \tag{J'}$$



$x \in \mathcal{D} \setminus \mathcal{A} \iff x \in \mathcal{D}$ and $x \notin \mathcal{A}$

$x \in \mathcal{D} \cap \mathcal{A} \iff x \in \mathcal{D}$ and $x \in \mathcal{A}$

- Conditions (F') and (J') are simpler to check that the original conditions (F) and (J), because there is no need to determine function ρ.
- Combining the result of Propositions 1 and 2, we can state a more classical formulation of Lyapunov's theorem
  - ▷ That corresponds to the extension of the classical case addressed at the beginning of the course
  - ▷ Of course, such a theorem is true under the hybrid basic condition.
- Recall that the classical case corresponds to systems of the form $\dot{x} = f(x), x \in \mathbb{R}^n$, with f continuous (or even more, often differentiable) or $x^+ = g(x), x \in \mathbb{R}^n$, with g continuous and attractors corresponding to an equilibrium point.

- Recall the conditions (S'), (F') and (J'):

V Positive definite and radially unbounded with resp. to $\mathcal{A}$ relative to $\mathcal{C} \cup \mathcal{D} \cup g(\mathcal{D})$ (S')

$$\dot{V}(x) := \frac{\partial V(x)}{\partial x} f(x) = \langle \nabla V(x), f(x) \rangle < 0, \quad \forall x \in \mathcal{C} \setminus \mathcal{A}; \tag{F'}$$

$$\begin{array}{ll} \Delta V(x) := V(g(x)) - V(x) < 0, & \forall x \in \mathcal{D} \setminus \mathcal{A}, \\ g(x) \in \mathcal{A}, & \forall x \in \mathcal{D} \cap \mathcal{A}. \end{array} \tag{J'}$$

Theorem 6 -Classical Lyapunov Theorem

If system (8) or (10) satisfies Assumption 1 or 2, and $\mathcal{A}$ is compact, then the satisfaction of conditions (S'), (F') and (J') imply UGAS of $\mathcal{A}$.
In other words,
V positive definite, $\dot{V}$ negative definite (in $\mathcal{C}$), and ΔV negative definite (in $\mathcal{D}$) with respect to $\mathcal{A}$, implies UGAS of $\mathcal{A}$.

- Consider the following class of homogeneous hybrid systems:

$$\begin{aligned} \dot{x} &= A_F x, \quad x \in \mathcal{C} := \{x \in \mathbb{R}^n; x^\top M_F x \geq 0\}, \\ x^+ &= A_J x, \quad x \in \mathcal{D} := \{x \in \mathbb{R}^n; x^\top M_J x \geq 0\}, \end{aligned}$$

  where
  - ▷ Matrices M_F and M_J are symmetric.
- This class of systems corresponds to the family of linear reset control systems.
- In particular, we can select

$$M_J = -M_F$$

  - ▷ Then one gets $\mathcal{C} := \{x \in \mathbb{R}^n; x^\top M_F x \geq 0\}$ and $\mathcal{D} := \{x \in \mathbb{R}^n; x^\top M_F x \leq 0\}$
  - ▷ Then in any point of the state space solutions can either flow or jump.

Remarks on the notion of homogenous systems

- The system under consideration is close to standard linear time-invariant dynamics.
- It is important to note that linearity comprises:
  - ▷ homogeneity: a scaled version of any solution is also a solution.
  - ▷ and additivity: the sum of two solutions is also a solution.
- Clearly additivity is hard to obtain in hybrid dynamics, because two solutions generally have different domains, therefore their sum is ill defined.
- An interesting point related to the system studied in this example (homogeneous hybrid system) is that the only compact attractor that can be asymptotically stable is the origin and its asymptotic stability coincides with global exponential stability.

- If matrices M_F and M_J are not positive semi-definite or negative semi-definite, the sets $\mathcal{C}, \mathcal{D}$ are nonempty symmetric cones.
  - ▷ Recall that a cone is a set containing any positive scaling of any of its points.
- Note that the flow and jump dynamics are both linear
  - ▷ Therefore, we can take inspiration from the classical Lyapunov results for linear systems (Theorem 2 of this course)
  - ▷ Then a universal Lyapunov function is the quadratic one

$$V(x) = x^\top P x$$

  - ▷ We can study GAS (therefore GES) of the origin (i.e. $\mathcal{A} = \{0\}$) with a quadratic Lyapunov function $V(x) = x^\top P x$.

- Assume there exists a matrix P is symmetric positive definite: $P = P^\top > 0$
  - ▷ Then $V(x)$ is positive definite and radially unbounded,
  - ▷ Hence condition (S') is satisfied
- To prove

$$\dot{V}(x) := \frac{\partial V(x)}{\partial x} f(x) = \langle \nabla V(x), f(x) \rangle < 0, \qquad \forall x \in \mathcal{C} \setminus \mathcal{A}; \qquad \text{(F')}$$

  - ▷ We compute the time derivative of V along solutions using $\nabla V(x) = 2Px$, which gives

$$\frac{\partial V(x)}{\partial x} f(x) = \langle \nabla V(x), A_F x \rangle = 2x^\top P A_F x = x^\top (A_F^\top P + P A_F) x$$

  - ▷ If we assume that P is such that $A_F^\top P + P A_F < 0$ then

$$\frac{\partial V(x)}{\partial x} f(x) = 2x^\top P A_F x = x^\top (A_F^\top P + P A_F) x < 0, \forall x \neq 0,$$

  - ▷ Hence, condition (F') is satisfied.

- To prove

$$\begin{array}{ll} \Delta V(x) := V(g(x)) - V(x) < 0, & \forall x \in \mathcal{D} \setminus \mathcal{A}, \\ g(x) \in \mathcal{A}, & \forall x \in \mathcal{D} \cap \mathcal{A}. \end{array} \tag{J'}$$

  - We compute the ΔV across jumps:

$$V(g(x)) - V(x) = V(A_J x) - V(x) = x^\top (A_J^\top P A_J - P)x$$

  - If we assume that P is such that $A_J^\top P A_J - P < 0$ then

$$V(g(x)) - V(x) = x^\top (A_J^\top P A_J - P)x < 0, \forall x \neq 0,$$

  - Hence, condition (J') is satisfied.

- Hence, we can formulate the following result regarding the stability of the system:

Result 1

If there exists a matrix P such that

$$\begin{aligned} P = P^\top > 0 \\ A_F^\top P + PA_F < 0 \\ A_J^\top PA_J - P < 0 \end{aligned} \tag{11}$$

then conditions (S'), (F'), (J') hold, and therefore the origin is UGAS.

- Note that implies the stronger GES property due to homogeneity.

- Let us provide some remarks about this results.
- Condition (11) is interesting because it is linear in the decision variable P and therefore one can say that it is a linear matrix inequality (LMI).
- This condition suffers from a main drawback:
  - ▷ (11) is quite conservative because they require that the Lyapunov function satisfies both (F') and (J') for any x.
- Solution: we would like to enforce (F') only in the flow set and (J') only in the jump set.
- Is it possible? YES
  - ▷ To this end, we exploit the particular structure of $\mathcal{C}$ and $\mathcal{D}$
  - ▷ We can use a particular property: S-procedure

- Let us give the definition of S-Procedure

S-procedure

Consider M_0, ..., $M_p \in \mathbb{R}^{n\times n}$ be symmetrical matrices. The condition:

$$\xi^\top M_0 \xi > \text{ for all } \xi \neq 0 \text{ such that } \xi^\top M_i \xi > 0, i = 1..., p$$

is satisfied if there exist $\tau_1 \geq 0$, ..., $\tau_p \geq 0$ such that $M_0 - \sum_{i=1}^{p} \tau_i M_i > 0$

- Remark. In the case $p = 1$, there is equivalence of both conditions.
- Example. There exists $x \neq 0$ such that

$$x^\top M_1 x \leq 0 \text{ and } x^\top M_2 x \leq 0$$

holds if there exists $\lambda \geq 0$ such that $M_1 - \lambda M_2 \leq 0$.

- To prove

$$\dot{V}(x) := \frac{\partial V(x)}{\partial x} f(x) = \langle \nabla V(x), f(x) \rangle < 0, \qquad \forall x \in \mathcal{C} \setminus \mathcal{A}; \qquad \text{(F')}$$

  - ▷ We compute the time derivative of V along solutions using $\nabla V(x) = 2Px$, which gives

$$\frac{\partial V(x)}{\partial x} f(x) = \langle \nabla V(x), A_F x \rangle = 2x^\top P A_F x = x^\top (A_F^\top P + P A_F) x$$

  - ▷ We want to verify $x^\top (A_F^\top P + P A_F) x < 0$ for all x such that $x^\top M_F x \geq 0$
  - ▷ By using the S-procedure, we want to find P, $\tau_F \geq 0$ such that

$$x^\top (A_F^\top P + P A_F) x + \tau_F x^\top M_F x < 0$$

- To prove

$$\begin{array}{ll} \Delta V(x) := V(g(x)) - V(x) < 0, & \forall x \in \mathcal{D} \setminus \mathcal{A}, \\ g(x) \in \mathcal{A}, & \forall x \in \mathcal{D} \cap \mathcal{A}. \end{array} \tag{J'}$$

  - ▷ We compute the ΔV across jumps:

$$V(g(x)) - V(x) = V(A_J x) - V(x) = x^\top (A_J^\top P A_J - P) x$$

  - ▷ We want to verify $x^\top (A_J^\top P A_J - P) x < 0$ for all x such that $x^\top M_J x \geq 0$
  - ▷ By using the S-procedure, we want to find P, $\tau_J \geq 0$ such that

$$x^\top (A_J^\top P A_J - P) x + \tau_J x^\top M_J x < 0$$

- Hence, we can formulate the second result regarding the stability of the system:

Result 2

If there exist a matrix P and two scalars τ_F, τ_J such that

$$\begin{aligned} P = P^\top > 0, \tau_F \geq 0, \tau_J \geq 0 \\ A_F^\top P + PA_F + \tau_F M_F < 0 \\ A_J^\top PA_J - P + \tau_J M_J < 0 \end{aligned} \tag{12}$$

then conditions (S'), (F'), (J') hold, and therefore the origin is UGAS.

- Note that one retrieves condition (11) when $\tau_F = \tau_J = 0$.

- Remark. Let us compare Result 1 with Result 2.
  - ▷ When A_F is not Hurwitz (all eigenvalues have negative real part), then the second condition in (11) cannot be satisfied,
  - ▷ When A_F is not Hurwitz, second condition in (12) may be feasible for some suitable $\tau_F > 0$ if matrix M_F allows obtaining enough negativity.

- Let us come back to the more general Theorem 4, recalled below.

Theorem 4 - Lyapunov Theorem

Given a closed set $\mathcal{A}$ and system (8), if there exist a function $x \mapsto V(x)$ continuous and differentiable in an open set containing $\mathcal{C}$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a continuous positive definite function ρ (namely a function such that $\rho(0) = 0$ and $\rho(s) > 0$ for all $s > 0$) such that

$$\alpha_1(|x|_\mathcal{A}) \leq V(x) \leq \alpha_2(|x|_\mathcal{A}), \ \forall x \in \mathcal{C} \cup \mathcal{D} \cup g(\mathcal{D}), \tag{S}$$

$$\dot{V}(x) := \frac{\partial V(x)}{\partial x} f(x) = \langle \nabla V(x), f(x) \rangle \leq -\rho(|x|_\mathcal{A}), \ \forall x \in \mathcal{C}, \tag{F}$$

$$\Delta V(x) := V(g(x)) - V(x) \leq -\rho(|x|_\mathcal{A}), \ \forall x \in \mathcal{D}, \tag{J}$$

then $\mathcal{A}$ is UGAS for (8).

- Question: is it possible to check properties (F) and (J)? YES

- Indeed, we can impose $\forall x \neq 0$

$$\frac{\partial V(x)}{\partial x} f(x) = x^\top (A_F^\top P + P A_F) x < -\varepsilon x^\top x - \tau_F x^\top M_F x \leq -\varepsilon |x|^2$$

and

$$V(g(x)) - V(x) = x^\top (A_J^\top P A_J - P) x < -\varepsilon x^\top x - \tau_J x^\top M_J x \leq -\varepsilon |x|^2$$

- That corresponds to choose the positive definite function $\rho(s) = \varepsilon s^2$, which is a class $\mathcal{K}_\infty$ function.

- Hence, we can formulate the third result regarding the stability of the system:

Result 3

If there exist a matrix P and three scalars τ_F, τ_J, ϵ such that

$$\begin{aligned} P = P^\top > 0,\ \varepsilon > 0,\ \tau_F \geq 0, \tau_J \geq 0 \\ A_F^\top P + PA_F + \tau_F M_F < -\varepsilon I_n, \\ A_J^\top PA_J - P + \tau_J M_J < -\varepsilon I_n. \end{aligned} \tag{13}$$

then conditions (S), (F), (J) hold, and therefore the origin is UGAS.

- The construction of Lyapunov functions is often a challenging task due to the three positivity and negativity conditions (sandwich, flow, jump) to satisfy.
- It can then be convenient to construct Lyapunov functions that remain constant along some of the system evolutions,
  - ▷ We will then seek for the following weakened flow and jump conditions

$$\dot{V}(x) := \frac{\partial V(x)}{\partial x} f(x) = \langle \nabla V(x), f(x) \rangle \leq 0, \qquad \forall x \in \mathcal{C}, \qquad \text{(F")}$$
$$\Delta V(x) := V(g(x)) - V(x) \leq 0, \qquad \forall x \in \mathcal{D}. \qquad \text{(J")}$$

  - ▷ Observe that the conditions (F") and (J") are non strict.
- Example. The bouncing ball. wherein it is reasonable to use the mechanical energy (kinetic plus potential) as a Lyapunov function
  - ▷ This kind of Lyapunov function is a weak Lyapunov function due to the non strict inequalities in (F"), (J").

- Let us use these weak flow and jump conditions to prove UGS but not UGA.
- First recall the definition of UGS

Part III - Definition 12 (UGS, UGA, UGAS)

For system (8), a compact set $\mathcal{A}$ is

1. Uniformly Globally Stable (UGS) if there exists a function α of class $\mathcal{K}_\infty$ such that all maximal solutions ϕ satisfy

$$|\phi(t,j)|_{\mathcal{A}} \leq \alpha(|\phi(0,0)|_{\mathcal{A}}), \quad \forall (t,j) \in \mathrm{dom}\phi$$

- One gets the following results

Theorem 7 - Weak Lyapunov theorem

Given a closed set $\mathcal{A}$ and hybrid equation (8) or hybrid inclusion (10). If there exist a function $x \mapsto V(x)$ continuous and differentiable in an open set containing $\mathcal{C}$, and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ satisfying (S), (F") and (J"), then $\mathcal{A}$ is uniformly globally stable (UGS).

Some ingredients of the proof

- Consider any solution $\phi \in \mathcal{S}_{\mathcal{H}}$ and the function $V(x)$.
- Using (F") we obtain: $\frac{d}{dt}V(\phi(t,j)) \leq 0$ for almost all $t \in I^j$ and $(t,j) \in \mathtt{dom}\phi$.
- Using (J") we obtain: $V(\phi(t_j,j)) \leq V(\phi(t_j,j-1))$ for all jump times t_j, $j \geq 1$.
- Combining these inequalities, we obtain $V(\phi(t,j)) \leq V(\phi(0,0))$, for all $(t,j) \in \mathtt{dom}\phi$.
- Using (S) one can write

$$\alpha_1(|x(t,j)|_{\mathcal{A}}) \leq V(\phi(t,j)) \leq V(\phi(0,0)) \leq \alpha_2(|x(t,j)|_{\mathcal{A}}) \tag{14}$$

- Let us know recall that a class $\mathcal{K}_\infty$ function is invertible and its inverse is also a class $\mathcal{K}_\infty$ function, then we obtain

$$|x(t,j)|_{\mathcal{A}} \leq \alpha_1^{-1}\left(\alpha_2(|x(t,j)|_{\mathcal{A}})\right), \tag{15}$$

which proves the UGS bound

$$|\phi(t,j)|_{\mathcal{A}} \leq \alpha(|\phi(0,0)|_{\mathcal{A}}), \quad \forall (t,j) \in \mathrm{dom}\phi$$

with $\alpha(s) = \alpha_1^{-1}(\alpha_2(s))$ of class $\mathcal{K}_\infty$.

Bouncing ball (1)

- Let us consider again the bouncing ball described by the following data ($x_1 = p$ and $x_2 = v$):

$$f(x) = \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix}, \qquad \mathcal{C} = \{(x_1, x_2) \in \mathbb{R}^2; x_1 \geq 0\},$$
$$g(x) = \begin{bmatrix} x_1 \\ -\lambda x_2 \end{bmatrix}, \qquad \mathcal{D} = \{(x_1, x_2) \in \mathbb{R}^2; x_1 = 0 \text{ and } x_2 \leq 0\}.$$

- Recall that we are assuming unit mass and that the gravity constant is denoted by γ.
- Objective. We want to study the stability of $\mathcal{A} = \{0\}$

Bouncing ball (2)

- Taking into account that this is a physical system, a choice for the Lyapunov function is the mechanical energy of the system, defined as

$$V(x) = \frac{1}{2}v^2 + \gamma p = \frac{1}{2}x_2^2 + \gamma x_1 \tag{16}$$

- We have to check the sandwich condition (S)
  - ▷ The sandwich condition (S)
  - ▷ The fact that the function V is also radially unbounded
  - ▷ The jump and flow conditions

Bouncing ball (3)

- The sandwich condition (S) is satisfied, because

$$V(0) = 0 \text{ and } V(x) > 0, \quad \forall x \in \mathcal{C} \cup \mathcal{D} \setminus \{0\}$$

  - ▷ In fact, the set $\mathcal{C} \cup \mathcal{D}$ only contains non-negative values of x_1
  - ▷ for $x_2 \neq 0$ the function is clearly positive,
  - ▷ for $x_2 = 0$ the only possible points $x \notin \mathcal{A}$ correspond to $x_1 > 0$, so that V is positive again.

- The function V is also radially unbounded
  - ▷ When $|x| \to \infty$, either $x_1 \to \infty$ or $x_2 \to \infty$
  - ▷ That means that at least one of the two terms in V go to infinity, while the other one is never negative.

Bouncing ball (4)

- Let us now verify the jump and flow conditions.
- For the flow condition we obtain

$$\frac{\partial V(x)}{\partial x} f(x) = \begin{bmatrix} \gamma & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix} = \gamma x_2 - \gamma x_2 = 0, \tag{17}$$

  - ▷ That proves (F") (indeed we know that without friction the total energy remains constant along the motion).
- For the jump condition, instead, we have

$$V(x^+) - V(x) = \frac{1}{2}(\lambda x_2)^2 - \frac{1}{2}x_2^2 = -(1-\lambda^2)\frac{x_2^2}{2} \leq 0, \quad \forall x \in \mathcal{D}. \tag{18}$$

  - ▷ That proves again (J") for any $\lambda \in [0, 1]$.
- As a consequence, applying Theorem 7 the UGS of the origin is shown.

Remarks (1)

- The UGS conclusion drawn in the previous example for any $\lambda \in [0, 1]$ is not surprising,
  - for $\lambda = 1$ (fully elastic impact), we cannot expect solutions to converge to the origin.
  - for $\lambda = 1$ we have $\frac{\partial V(x)}{\partial x} f(x) = 0$ and $V(x^+) - V(x) = 0$
  - Indeed the ball keeps bouncing forever through a hybrid periodic motion never converging to zero.
- The corresponding hybrid periodic orbits are not much different from those depicted in the Example 2 of Part III (see the figure in slide 76).

Remarks (2)

- For $\lambda < 1$, we have $\frac{\partial V(x)}{\partial x} f(x) = 0$ and $V(x^+) - V(x) = -(1-\lambda^2)\frac{x_2^2}{2} < 0$
  - ▷ One knows that some dissipation is expected at each jump, so that it seems intuitively reasonable to state that the ball should converge to the origin (it should eventually stop bouncing).
  - ▷ This is indeed true, but proving this fact with a weak Lyapunov function is far from being trivial.
  - ▷ Fortunately, in the case when the hybrid system satisfies the hybrid basic condition of Part III - Assumption 1, it is possible to apply a powerful result called "Invariance Principle" or "La Salle theorem".

- We have studied the way to prove the stability of the origin or a set $\mathcal{A}$ for a hybrid dynamical system but without computing explicitly its solutions (as required in Part III).
- In particular, we addressed the major tool to prove stability of complex systems: the theory of Lyapunov.
  - ▷ Lyapunov function
- We have proposed different results to address GAS, UGS and UGAS, based on three main conditions: sandwich conditions, flow condition and jump condition
- We have also developed relaxed conditions

- Main sources for this lecture: Chapter 3 of the book of Teel, notes of Luca Zaccarian, Christophe Prieur, Francesco Ferrante, Ricardo Sanfelice, Germain Garcia, ST
- Part V: LaSalle principle
  - ▷ In the following lecture, we will recall some results dealing with LaSalle invariance principle for nonlinear systems
  - ▷ We will see how it can be extended to the hybrid systems case and allows to prove stability of the origin or a set $\mathcal{A}$ in particular cases.