# 2 MAE 701 - Electromagnetism applied to avionics

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Part VII

# Electromagnetism reminds
# Maxwell Equations in matter

- Electrical Field $\vec{E}$
- Magnetic Field $\vec{B}$
- Electric Displacement $\vec{D} = \varepsilon\vec{E}$
- Magnetic Intensity $\vec{H} = \frac{\vec{B}}{\mu}$

In Vacuum $\varepsilon = \varepsilon_0$ , $\mu = \mu_0$
In matter
$\varepsilon$ (permittivity) , $\mu$ (permeability)
$\epsilon \mathbb{R}\ or\ \mathbb{C}$
$\varepsilon = \varepsilon_0\varepsilon_r$ & $\mu = \mu_0\mu_r$
Refractive index $n = \sqrt{\varepsilon_r}$

| | |
|---|---|
| $div\ \vec{D} = \rho$ | Local Gauss Law (electric flux density) |
| $div\ \vec{B} = 0$ | General Magnetism Law |
| $\overrightarrow{rot}\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ | Faraday Law |
| $\overrightarrow{rot}\vec{H} = \left(\vec{j} + \frac{\partial\vec{D}}{\partial t}\right)$ | Ampere law |

From the Ohm local Law : $\vec{j} = \gamma\vec{E}$ where $\gamma$ is the conductivity

# Transversal Waves, Linear Wave and Polarization

**Transversal Wave**

$\vec{\kappa} \cdot \vec{E} = 0$ and $\vec{\kappa} \cdot \vec{B} = 0$

Since $\vec{\kappa} \neq 0 \Rightarrow \vec{E} \ \& \ \vec{B} \perp \vec{\kappa}$

Since $\vec{B}$ proportional to $\vec{\kappa} \times \vec{E} \Rightarrow \vec{B} \perp \vec{E}$

$\Rightarrow \vec{\kappa}, \vec{E} \ \& \ \vec{B}$ form a right-handed orthogonal set

The relative magnitude of $B = \frac{\kappa}{\omega} E$

**Remind from Maxwell :**

$$\vec{\kappa} \times \vec{B} \Rightarrow \vec{\kappa} \times \left(\vec{\kappa} \times \vec{E}\right) = \omega \, \vec{\kappa} \times \vec{B} = -\left(\frac{n\omega}{c}\right)^2 \vec{E}$$

n, is the refractive index, $\vec{\kappa}$ is the wave vector , c is the celerity of the light .

# Transversal Waves, Linear Wave and Polarization

Vector identity

$$\vec{\kappa} \times (\vec{\kappa} \times \vec{E}) = (\vec{\kappa} \cdot \vec{E})\vec{\kappa} - (\vec{\kappa})^2 \vec{E}$$

$$\Downarrow$$

$$\boxed{\vec{\boldsymbol{\kappa}} \cdot \vec{\boldsymbol{E}} = \mathbf{0}}$$

$$-\left(\frac{n\omega}{c}\right)^2 \vec{E} = -\kappa^2 \vec{E}$$

$$\boxed{\boldsymbol{\kappa} = \frac{\boldsymbol{n\omega}}{\boldsymbol{c}}}$$

**Transverse Relation of DISPERSION**

Relation between the Wave vector ($\kappa$) and the Refractive index (n), The pulsation ($\omega$) or Frequency ($\omega = 2\pi f$ ), the celerity of the light (c)

# Transversal Waves, Linear Wave and Polarization

**Monochromatic transverse wave propagated in +$\vec{u}$ direction**

$$\boxed{\overrightarrow{\boldsymbol{E}(\vec{\boldsymbol{r}},\boldsymbol{t})} = \overrightarrow{\boldsymbol{E}_0}\boldsymbol{e}^{-i(\omega t-\vec{\kappa}\cdot\vec{r})}}$$

$$\boxed{\overrightarrow{\boldsymbol{B}(\vec{\boldsymbol{r}},\boldsymbol{t})} = \overrightarrow{\boldsymbol{B}_0}\boldsymbol{e}^{-i(\omega t-\vec{\kappa}\cdot\vec{r})}}$$

$$\boxed{\vec{\kappa} = \kappa\vec{u}}$$

$$\vec{\kappa}//\vec{u}$$
$$\vec{E}\perp\vec{u}$$
$$\vec{u}\cdot\vec{E} = 0$$

As $\kappa = \frac{n\omega}{c}$ $\Rightarrow$ $\boxed{\vec{B} = \frac{n}{c}\vec{u}\times\vec{E}}$

*In Vacuum n=1 ,* $c\vec{B} = \vec{E}$ *and the phase velocity is* $\frac{c}{n}$

# Transversal Waves, Linear Wave and Polarization

If κ is not Real

$$\vec{\kappa} = \overrightarrow{\kappa_r} + i\overrightarrow{\kappa_i}$$

$$|\vec{\kappa}|^2 = \overrightarrow{\kappa_r}\cdot\overrightarrow{\kappa_r} - \overrightarrow{\kappa_i}\cdot\overrightarrow{\kappa_i} + 2i\overrightarrow{\kappa_r}\cdot\overrightarrow{\kappa_i} = \left(\frac{n\omega}{c}\right)^2 = \varepsilon_r\left(\frac{\omega}{c}\right)^2$$

Imaginary part vanish in our case, if $\overrightarrow{\kappa_r}\cdot\overrightarrow{\kappa_r} \gg \overrightarrow{\kappa_i}\cdot\overrightarrow{\kappa_i}$ or $\overrightarrow{\kappa_r}\perp\overrightarrow{\kappa_i}$ .

The wave in which the planes of constant phase ($\boldsymbol{\vec{\kappa}\cdot\vec{r} - \omega t}$) are perpendicular to the planes of constant amplitude.

( exception - metamaterials)

# Transversal Waves, Linear Wave and Polarization

The plane Wave is a restricted class of solution of the Maxwell equations, but the a linear combination of theses wave cover a wide class of solutions.

**Sum of plan Wave**

$$\overrightarrow{\boldsymbol{E}(\vec{\boldsymbol{r}},\boldsymbol{t})} = \sum_{\boldsymbol{i}} \vec{\boldsymbol{E}}(\overrightarrow{\boldsymbol{\kappa_i}},\boldsymbol{\omega_i})\boldsymbol{e}^{-\boldsymbol{i}(\boldsymbol{\omega_i t}-\overrightarrow{\boldsymbol{\kappa_i}}\cdot\vec{\boldsymbol{r}})}$$

$\vec{E}$ depends on $(\overrightarrow{\kappa_i}, \omega_i)$

The superposition of the plane waves forms a **complex FOURIER SERIE** and can represent any solution that was periodic (not necessary sinusoidal)

# Transversal Waves, Linear Wave and Polarization

Each term of the Fourier series satisfy

$$\varepsilon_r \vec{\kappa} \cdot \vec{E} = 0$$

$$\vec{\kappa} \cdot \vec{B} = 0$$

$$\vec{\kappa} \times \vec{E} = \omega \, \vec{B}$$

$$\vec{\kappa} \times \vec{B} = -\omega \left(\frac{n}{c}\right)^2 \vec{E}$$

**For a solution of the wave equation, the sum can be converted into a Fourier Integral.**

$\boldsymbol{\vec{E}\,(\overrightarrow{\kappa\quad}, \omega\quad)}$ **is the Fourier transform of** $\boldsymbol{\overrightarrow{E(\vec{r}, t)}}$

**⇒ n depends on κ and ω DISPERSION EFFECT**

# Oblique incidence reflection & transmission coefficients

We denote

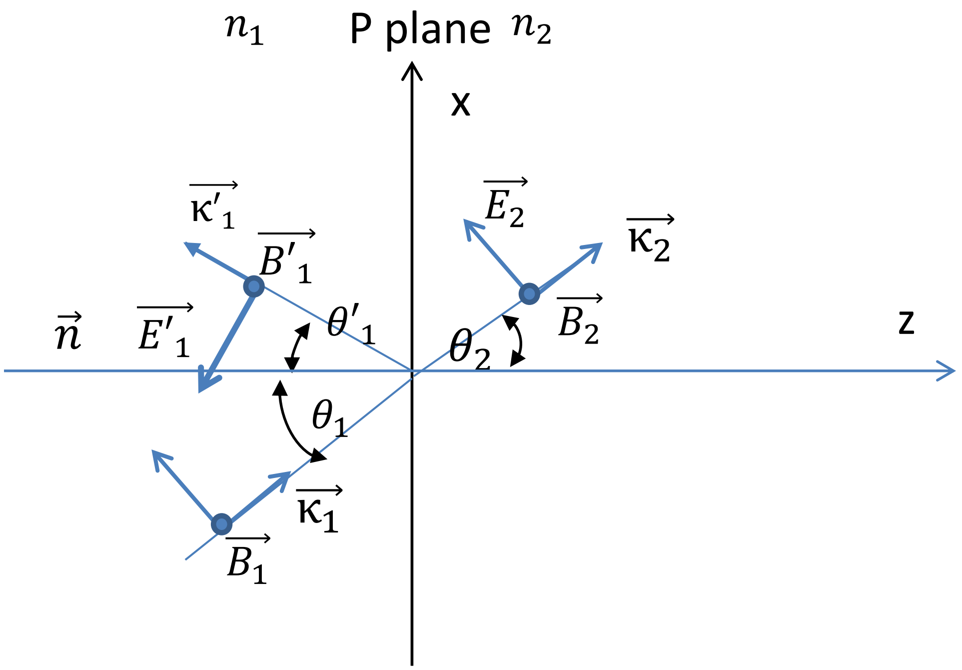
**Simple case, the boundary conditions involve the frequency conservation thus:**

$$\overrightarrow{E'_1} = E'_{10} e^{-i(\omega t - \overrightarrow{\kappa'_1} \cdot \vec{r})} \qquad \overrightarrow{E_2} = E_{20} e^{-i(\omega t - \overrightarrow{\kappa'_2} \cdot \vec{r})}$$

In P plane, the space dependence of the field is $e^{(i\overrightarrow{\kappa_i} \cdot \vec{r})}$ .

To obtain the tangential component we have to do the projection of $\vec{\kappa}$ on P Plane and denote κ this projection.

The normal components are eliminated, $\vec{r}$ is in the P plan.




The continuity at the interface gives:

$\kappa_1 = \kappa'_1 = \kappa_2$

# Oblique incidence reflection & transmission coefficients

- **First Descartes Law**, $\kappa_1$ $and$ $\kappa'_1$, $\kappa_1$ $and$ $\kappa_2$, are coplanar. **The incidence Plan contain the normal of the discontinuity plan and the vectors $\boldsymbol{\kappa_1}, \boldsymbol{\kappa'_1}, \boldsymbol{\kappa_2}$**
- **Second Descartes Law**, The reflection angle $\boldsymbol{\theta'_1}$ **is equal** to the incidence angle $\boldsymbol{\theta_1}$. $\kappa_1$ and $\kappa'_1$ are the wave vector of a propagation with the same pulsation and in the same medium
- **Third Descartes Law** , The refraction angle $\theta_2$ and the incidence angle $\theta_1$ verify the second Descartes Law, such as:

$$\sin\theta_1 = \frac{v_1}{v_2}\sin\theta_2$$

Where $v_1$ and $v_2$ are the Phase velocity of the incident and refracted wave respectively.

As $v_i = \frac{c}{n_i}$, the Third Descartes Law becomes:

$$\boldsymbol{n_1\, sin\,\theta_1 = n_2\, sin\,\theta_2}$$

# Oblique incidence reflection & transmission coefficients

Relationship between incident, refracted and reflected E-Field becomes:

$$n_1\vec{n} \times \left(\overrightarrow{u_1} \times \overrightarrow{E_1} + \overrightarrow{u'_1} \times \overrightarrow{E'_1}\right) = n_2\vec{n} \times \left(\overrightarrow{u_2} \times \overrightarrow{E_2}\right)$$

- For the s-component $\vec{n} \cdot \overrightarrow{E_{1s}} = 0$ (perpendicular to the plane of incidence)

$$\vec{n} \times \left(\overrightarrow{u_1} \times \overrightarrow{E_{1s}}\right) = -\cos(\theta_1)\overrightarrow{E_{1s}}$$

Since $\vec{n} \cdot \overrightarrow{u_1} = \cos(\theta_1)$

$$n_1\left(\cos(\theta_1)\,\overrightarrow{E_{1s}} - \cos(\theta'_1)\overrightarrow{E'_{1s}}\right) = n_2\cos(\theta_2)\,\overrightarrow{E_{2s}}$$

SNELL-DESCARTES LAW $\theta_1 = \theta'_1$

$$n_1\cos(\theta_1)\left(\overrightarrow{E_{1s}} - \overrightarrow{E'_{1s}}\right) = n_2\cos(\theta_2)\,\overrightarrow{E_{2s}}$$

# Oblique incidence reflection & transmission coefficients

From the cross product $\left(\overrightarrow{E_{1s}} + \overrightarrow{E'_{1s}}\right) = \overrightarrow{E_{2s}}$

**For S- polarization**, the FRESNEL coefficient are given by:

$\overrightarrow{E'_{1s}} = r_{12s}\overrightarrow{E_{1s}}$ and
$\overrightarrow{E_{2s}} = t_{12s}\overrightarrow{E_{1s}}$

Reflection

$$\boxed{r_{12s} = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2}}$$

Transmission

$$\boxed{t_{12s} = \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2}}$$

# Oblique incidence reflection & transmission coefficients

- **P-Polarization:** The trace of B-Field is the same than E rotated 90° counterclockwise, that is to say the s-polarization of B-Field correspond to the p-polarization of E-Field.

Since $\vec{n} \cdot \overrightarrow{B_{1s}} = 0 = \vec{n} \cdot \overrightarrow{B_{2s}} = \vec{n} \cdot \overrightarrow{E'_{1s}}$

$$\frac{1}{n_1}\cos(\theta_1)\left(\overrightarrow{B_{1s}} - \overrightarrow{B'_{1s}}\right) = \frac{1}{n_2}\cos(\theta_2)\,\overrightarrow{B_{2s}}$$

And also $$\left(\overrightarrow{B_{1s}} + \overrightarrow{B'_{1s}}\right) = \overrightarrow{B_{2s}}$$

# Oblique incidence reflection & transmission coefficients

As $$\cos(\theta_2) = \sqrt{1 - \left(\frac{n_1}{n_2}\right)^2 sin^2(\theta_1)}$$

- **The REFLECTANCE**

For S-polarisation

$$R_s = \frac{\vec{n}\cdot\overrightarrow{S'_{1s}}}{\vec{n}\cdot\overrightarrow{S_{1s}}} = {r^2}_{12s}$$

For P-polarisation

$$R_p = \frac{\vec{n}\cdot\overrightarrow{S'_{1p}}}{\vec{n}\cdot\overrightarrow{S_{1p}}} = {r^2}_{12p}$$

- **The TRANSMITTANCE**

For S-polarisation

$$T_s = \frac{\vec{n}\cdot\overrightarrow{S_{2s}}}{\vec{n}\cdot\overrightarrow{S_{1s}}} = \frac{n_2\cos(\theta_2)}{n_1\cos(\theta_1)}{t^2}_{12s}$$

For P-polarisation

$$T_p = \frac{\vec{n}\cdot\overrightarrow{S_{2p}}}{\vec{n}\cdot\overrightarrow{S_{1p}}} = \frac{n_2\cos(\theta_2)}{n_1\cos(\theta_1)}{t^2}_{12p}$$

**Identities** $\boldsymbol{R_s + T_s = 1}$ **and** $\boldsymbol{R_p + T_p = 1}$

# Oblique incidence reflection & transmission coefficients

**Reminder**

- Normal incidence - $\theta_1 = 0$, No polarization effect, R ↗ While $\frac{n_2}{n_1} \neq 1$ ↗
- Grazing incidence - $\theta_1 = \frac{\pi}{2}$ $\Rightarrow$ $\cos\theta_1 = 0$

$$\Rightarrow \quad R_s = |-1|^2 = R_p$$

Near Grazing incidence, the reflectance increase *(Ex: a calm lake seems to a mirror)*

- **HOW TO OBTAIN ZERO REFLECTANCE?**

  – $\theta_1 = \theta_2$ such as $\tan(\theta_1 - \theta_2) = 0 = \sin(\theta_1 - \theta_2)$, No reflected wave but SNELL-DESCARTES : $\theta_1 = \theta_2 \Leftrightarrow n_1 = n_2$ NO INTERFACE!!

  – $\boldsymbol{\theta_1 + \theta_2 = \frac{\pi}{2}}$ $\Rightarrow$ $\mathbf{tan}(\boldsymbol{\theta_1 + \theta_2}) \rightarrow \infty$ and a the light could be decomposed in 2 polarized light (s & p polarization along s & p axis), in this case the p polarized reflected light have a zero magnitude.

# Brewster Angle

From 3d Snell-Descartes Law :

$$\mathrm{n}_1 \sin \theta_1 = \mathrm{n}_2 \sin \theta_2$$

By using $\theta_2 = \frac{\pi}{2} - \theta_1$

denote $\theta_1 = \theta_B$ the Brewster Angle

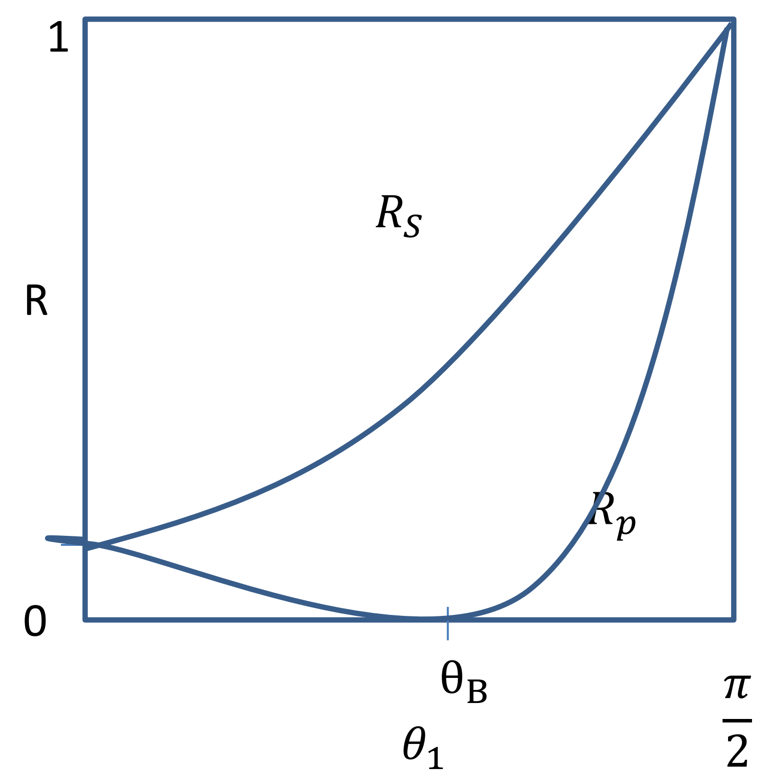
$$\mathrm{n}_1 \sin \theta_\mathrm{B} = \mathrm{n}_2 \sin\left(\frac{\pi}{2} - \theta_\mathrm{B}\right) = \mathrm{n}_2 \cos \theta_\mathrm{B}$$

the Brewster Law:

$$\boxed{\tan \theta_B = \frac{\mathrm{n}_2}{\mathrm{n}_1}}$$

$$\boxed{\theta_1 = \theta_B}$$

Plot $R_S$ and $R_p$ for all values of $\theta_1$



# Brewster Angle

- Interface Air-Glass

Air: $\mathrm{n}_1$=1, Glass: $\mathrm{n}_2$=1,5 Compute $\theta_B = 56°$

Application :

Polaroid Sunglass or Cockpit canopy, The reflectance minimale is for the p-polarized light

# Critical Angle

- If $R_S = R_p = 1$, perfect reflection occurs for $\theta_2 = \frac{\pi}{2}$

The incident angle for $\theta_2 = \frac{\pi}{2}$ is called Critical Angle such as $\theta_1 = \theta_c$:

$\sin \theta_c = \frac{\mathrm{n}_2}{\mathrm{n}_1}$ ,

$\theta_c$ is Real if $\mathrm{n}_2 < \mathrm{n}_1$, $\tan \theta_B = \sin \theta_c$

If $\tan \theta_B > \sin \theta_c$ , $\theta_B > \theta_c$

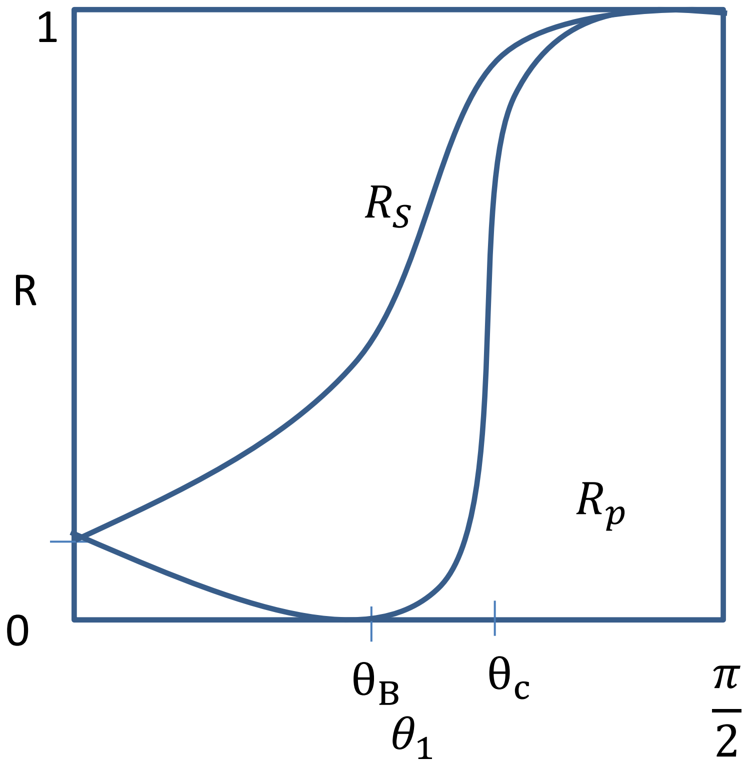
**Numerical application**: Interface Glass ($\mathrm{n}_1$=1,5 ) Air ( $\mathrm{n}_2$=1)

$$\tan \theta_B = 0.667$$

$\theta_B$ = 34° $\theta_c$ = 42°

# Critical Angle

Plot $R_S$ and $R_p$ for all values of $\theta_1$, for the interface Glass Air




$\sin\theta_c = \frac{n_2}{n_1} \Rightarrow \sin\theta_2 > 1$

NO REAL ANGLE PROVIDE $\sin\theta_2 > 1$

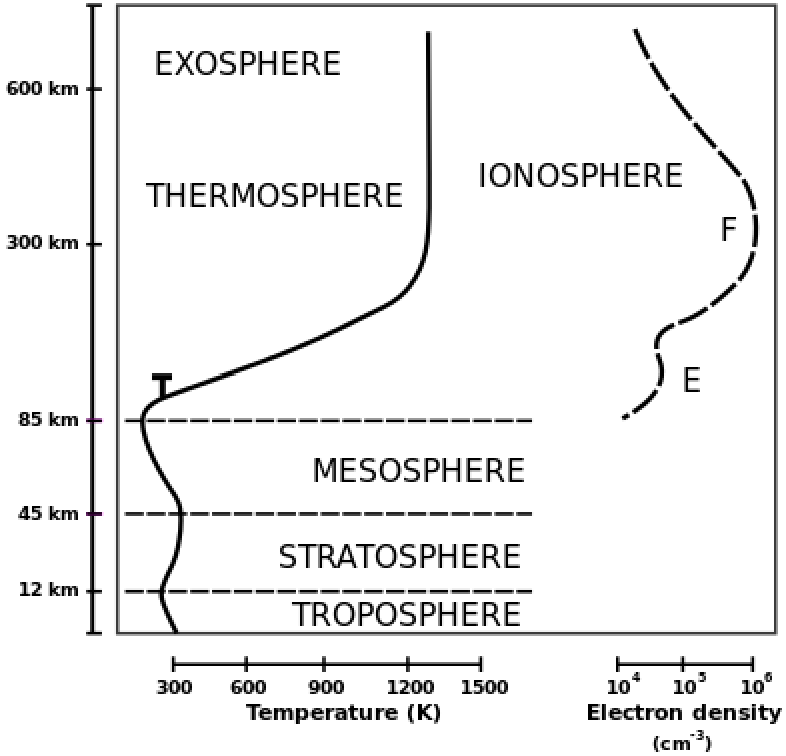
The Result is If $R_S = R_p = 1$, for all $\theta_1 > \theta_c$

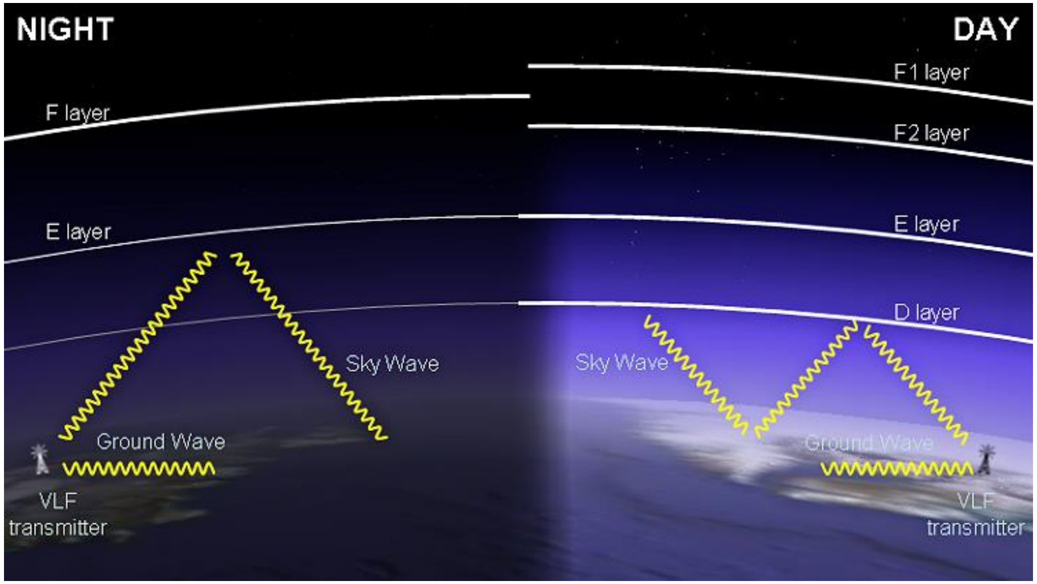
$\Rightarrow$ TOTAL INTERNAL REFLECTION

(for ex: PRISM, AQUARIUM)

Application: Wave Guide in optics and Microwave

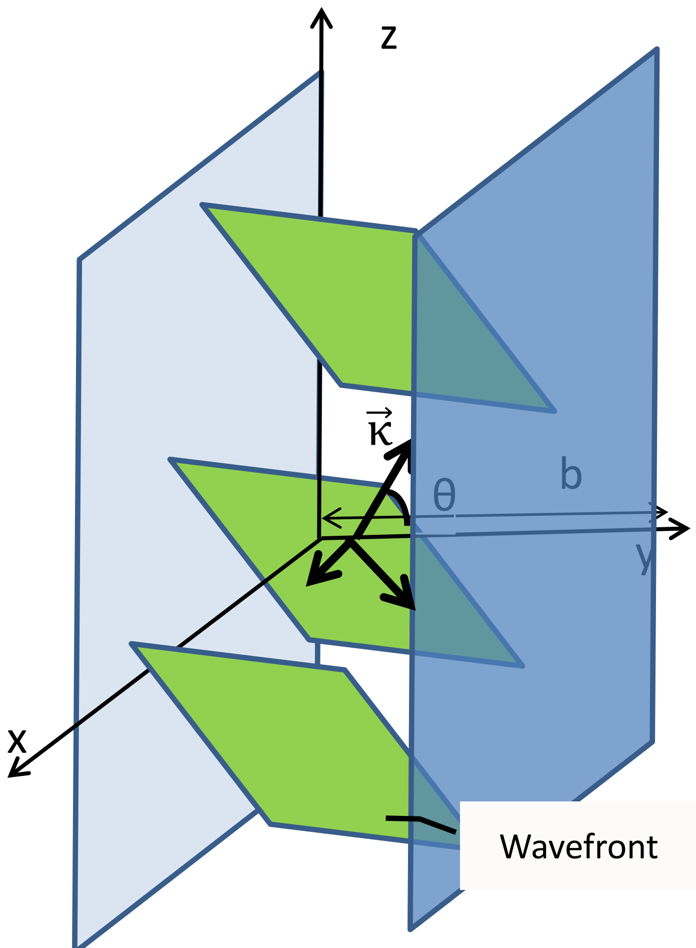
# Applications: Atmospheric transmissions




# Applications: Waveguide

Propagation of a wave in dielectric medium between 2//conducting surfaces.




Assuming :

- Metal with a conductivity ∞

Perfect reflection from conducting plane

$$\Rightarrow \hat{r}_{12s} = -1 \text{ and } \hat{r}_{12p} = +1$$

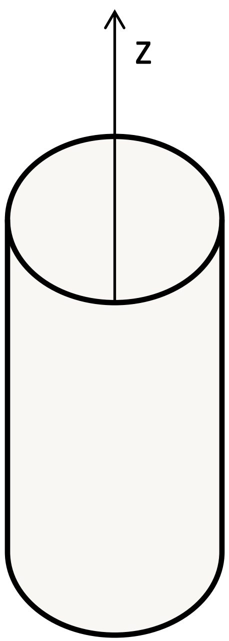
- Dielectric to be vacuum
- Wave Vector is in the plane yz and making an angle θ with the y axis in the plane of incidence

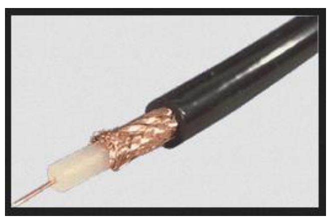
Reflection at y= b and y=0

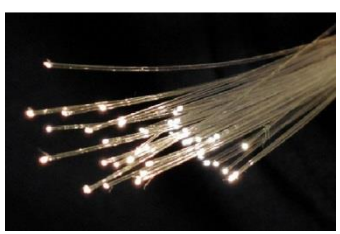
# Applications: Waveguide

E-Field and B-Field satisfied the wave equation in Free space, how to confined the EM wave in a guide?

A cylinder





or

a Rectangular waveguide