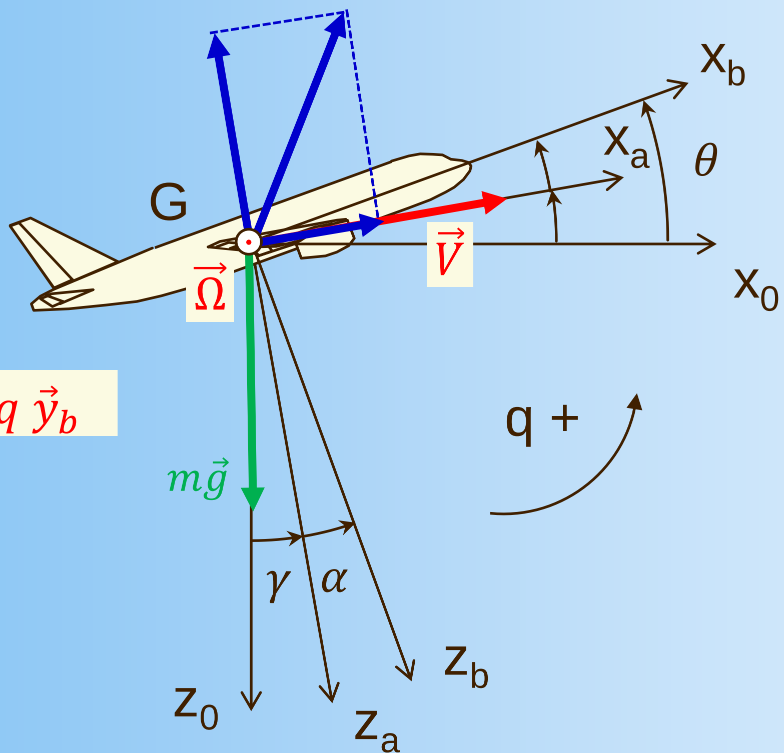
Pure Longitudinal Flight Mechanics equations

Solar Impulse

Pure Longitudinal Flight

$$\frac{d\vec{V}}{dt} = \dot{V}\,\vec{x}_a - V\dot{\gamma}\,\vec{z}_a$$



$$\begin{cases} \theta = \alpha + \gamma \\ \dot{\theta} = q \end{cases}$$

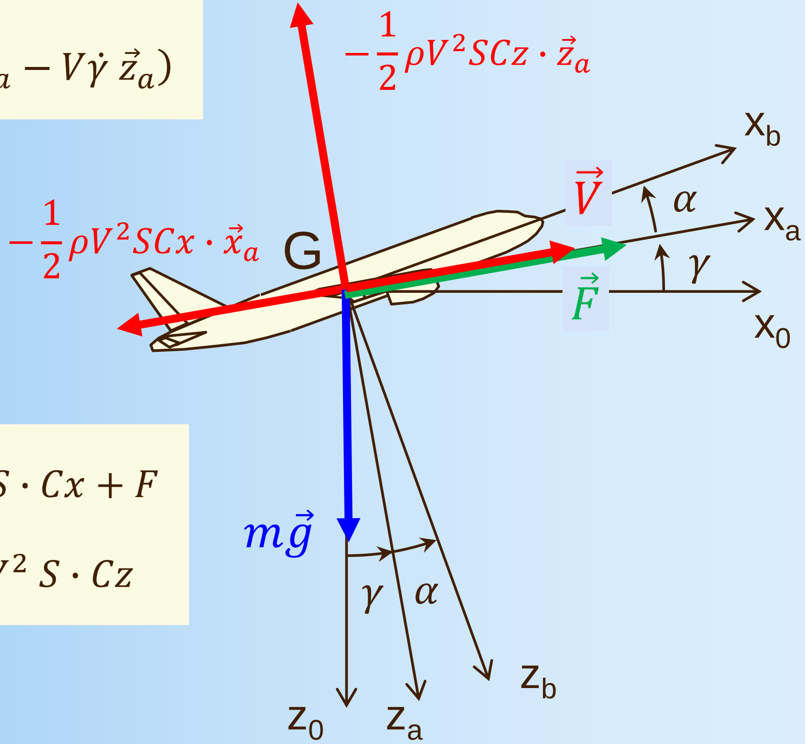
$$\vec{\Omega} = q\,\vec{y}_b$$

Force equations

$$m\vec{g} + \vec{F}_a + \vec{F} = m \cdot \frac{d\vec{V}}{dt} = m \cdot (\dot{V}\,\vec{x}_a - V\dot{\gamma}\,\vec{z}_a)$$

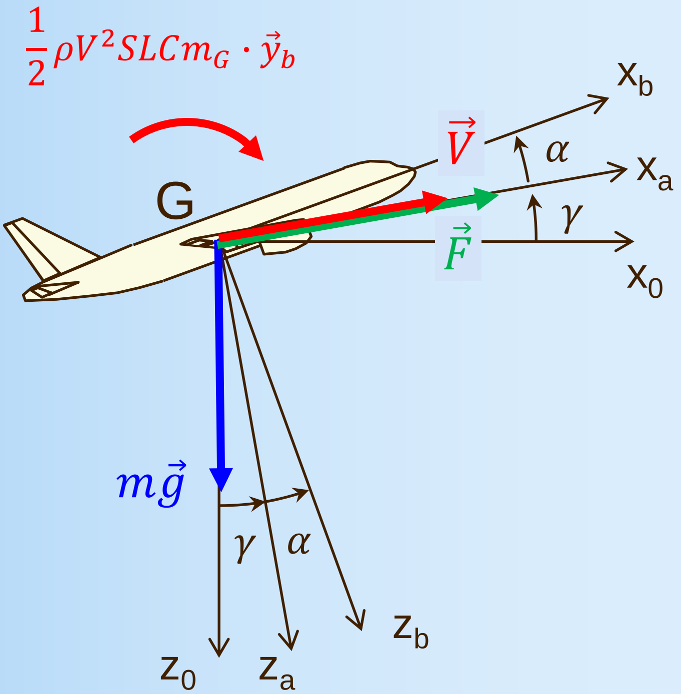
$$x_a \rightarrow m \cdot \dot{V} = -mg \cdot \sin\gamma - \frac{1}{2}\rho V^2\, S \cdot Cx + F$$

$$z_a \rightarrow -mV \cdot \dot{\gamma} = mg \cdot \cos\gamma - \frac{1}{2}\rho V^2\, S \cdot Cz$$



$$\vec{M}_G^a = [I_G] \cdot \frac{d\vec{\Omega}}{dt} = B \cdot \dot{q}\ \vec{y}_b$$

$$y_b \rightarrow B \cdot \dot{q} = \frac{1}{2}\rho V^2\ SL \cdot Cm_G$$

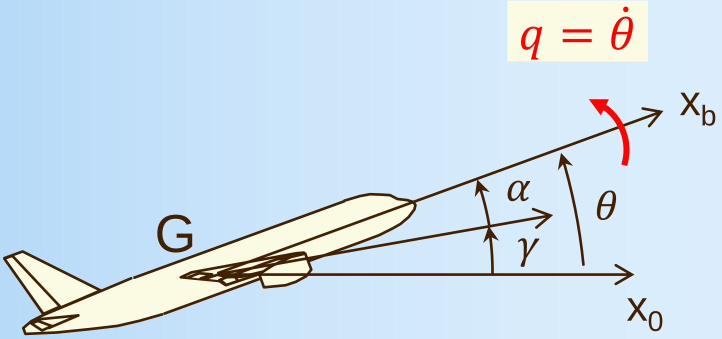


Kinematic equations

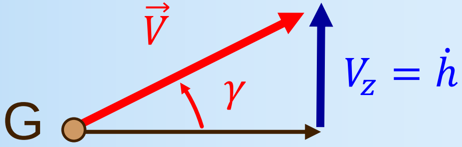
$$\theta = \alpha + \gamma$$

$$q = \dot{\theta} = \dot{\alpha} + \dot{\gamma}$$

$$\rightarrow \dot{\alpha} = q - \dot{\gamma}$$



$$\rightarrow V_z = \dot{h} = V \cdot \sin\gamma$$



Pure Longitudinal Flight equations

$$
\begin{cases}
m \cdot \dot{V} = -mg \cdot \sin\gamma - \frac{1}{2}\rho V^2 S \cdot Cx + F \\
-mV \cdot \dot{\gamma} = mg \cdot \cos\gamma - \frac{1}{2}\rho V^2 S \cdot Cz \\
B \cdot \dot{q} = \frac{1}{2}\rho V^2 SL \cdot Cm_G \\
\dot{\alpha} = q - \dot{\gamma} \\
\dot{h} = V \cdot \sin\gamma
\end{cases}
$$

5 differential equations for 5 state variables $(V, \gamma, q, \alpha, h)$

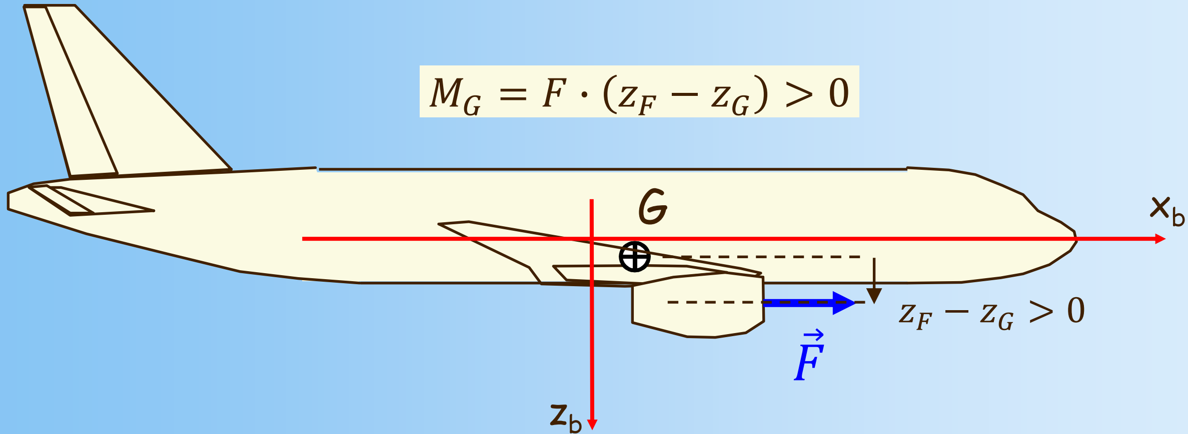
An important remark

The previous equations are obtained, following the assumptions :

- The thrust $\vec{F}$ is aligned with the velocity $\vec{V}$
  - ➢ This is a fair assumption as far as the engine misalignment with the fuselage is small

- The thrust $\vec{F}$ is applied at the centre of gravity G
  - ➢ This is a discutable assumption : commercial aircraft have engines located below the centre of gravity ; any thrust action will create a pitch moment aroud G

$$M_G = F \cdot (z_F - z_G) > 0$$



Pure Longitudinal Statics

Aerospatiale / British Aerospace Concorde

State Vector Formalism

$$\dot{X} = f(X, \delta U)$$

The vector X is called the state vector : the derivative of the X vector is perfectly defined and the state of the System is completely defined with respect to time

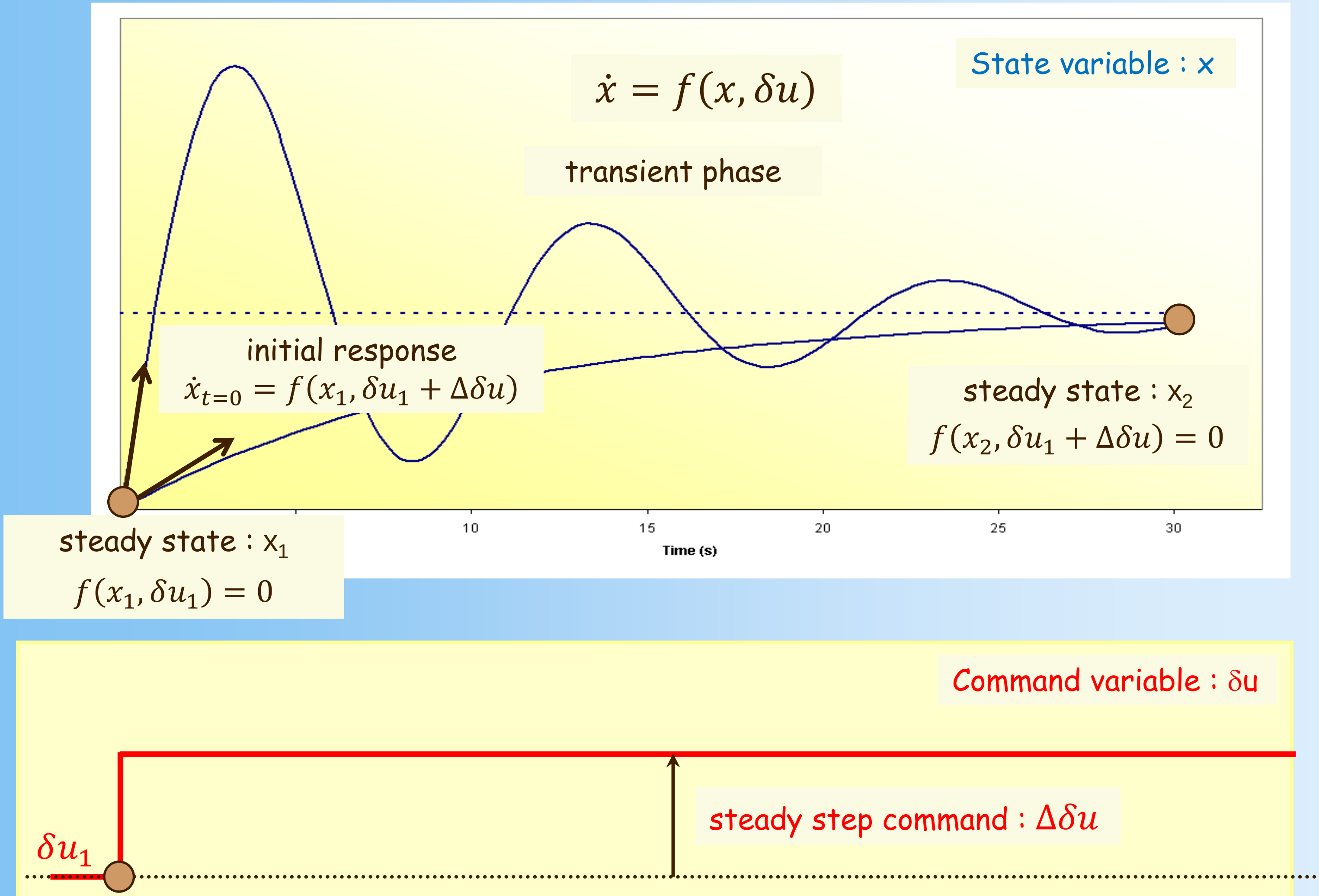
The derivative vector $\dot{X}$ is also function of the command vector δU which is at the disposal of the pilot or the aircraft computer (command law generation).

The steady state of the system is given by : $f(X, \delta U) = 0$

$\rightarrow$ At steady state, a same command δU will generate a same state X

The initial response of the system is given by : $\dot{X}_{t=0} = f(X_{t=0}, \delta U_{t=0})$

Response to Steady Step Command



Pure Longitudinal flight Steady State

$$\begin{cases} m\dot{V} = -\frac{1}{2}\rho V^2 S \cdot Cx + F(\rho, V, \delta x) - mg \cdot \gamma \\ -mV\dot{\gamma} = -\frac{1}{2}\rho V^2 S \cdot Cz_\alpha(\alpha - \alpha_0) + \overset{\text{neglecting } Cz_q \text{ \& } Cz_{\delta m}}{mg} \\ B\dot{q} = \frac{1}{2}\rho V^2 SL \cdot \left[Cm_0 + Cm_{\alpha,G}(\alpha - \alpha_0) + Cm_q \frac{qL}{V} + Cm_{\delta m} \cdot \delta m \right] \\ \dot{\alpha} = q - \dot{\gamma} \end{cases}$$

Steady State : $\dot{\alpha} = \dot{V} = \dot{q} = \dot{\gamma} = 0$

$$\begin{cases} -\frac{1}{2}\rho V^2 S \cdot Cx + F(\rho, V, \delta x) - mg \cdot \gamma = 0 \\ -\frac{1}{2}\rho V^2 S \cdot Cz_\alpha(\alpha - \alpha_0) + mg = 0 \\ Cm_0 + Cm_{\alpha,G}(\alpha - \alpha_0) + Cm_{\delta m} \cdot \delta m = 0 \\ q = 0 \end{cases}$$

5 unknowns $(\alpha, V, \gamma, \delta x, \delta m)$ and 3 equations

Remark on the altitude h

In the previous set, we have omitted the equation describing the altitude evolution $\dot{h} = V \cdot \gamma$

In another term :

- we assume that the altitude h variations are small
- which means that the density ρ can be assumed constant

$$\dot{h} = 0 \rightarrow \rho = Cte$$

Response to Steady Step Command

Starting from an initial flight trim situation,

- If you produce a perturbation by modifying the command state,
- If you maintain this command state,
- If the aircraft is stable,
- The aircraft will go to another flight trim situation after a transient phase which can last a certain time …

- This transient phase depends on the dynamic properties of the aircraft
- The new flight trim situation only depends on the new command state : it doesn't depend on the transient phase

The resolution of the Longitudinal Equations at Steady State leads to :

- 3 equations $(\dot{V} = 0 \,,\ \dot{\gamma} = 0 \,,\ \dot{q} = 0)$
- 5 unknowns $(V \,/\, \alpha \,/\, \gamma \,/\, \delta x \,/\, \delta m\,)$, assuming $\rho = Cte$

  - The knowledge of the 2 Command variables $(\delta x \,/\, \delta m\,)$ leads to the system resolution

Available command variables : δx / δm

The Throttle

$$F(\rho, V, \delta x) = F_0(\rho, V) \cdot \delta x$$

- F_0 is the maximum available thrust
- δx can vary from 0 (idle) to 1 (TOGA)

The Elevator

$$Cm_{\delta m} \cdot \delta m$$

- produces an important pitch moment
- δm can vary from -30° to 30°

Thrust variation : Initial Response

$\Delta\delta x > 0$, fixed / $\Delta\delta m = 0°$

Situation at t=0 : Trim n°1

$$\begin{cases} -\frac{1}{2}\rho V_1^2 SCx + F_0(\rho, V_1)\cdot\delta x_1 - mg\cdot\gamma_1 = 0 \\ -\frac{1}{2}\rho V_1^2 SCz_\alpha(\alpha_1-\alpha_0) + mg = 0 \\ Cm_0 + Cm_{\alpha,G}(\alpha_1-\alpha_0) + Cm_{\delta m}\delta m_1 = 0 \\ q_1 = 0 \end{cases}$$

Variation at t=0

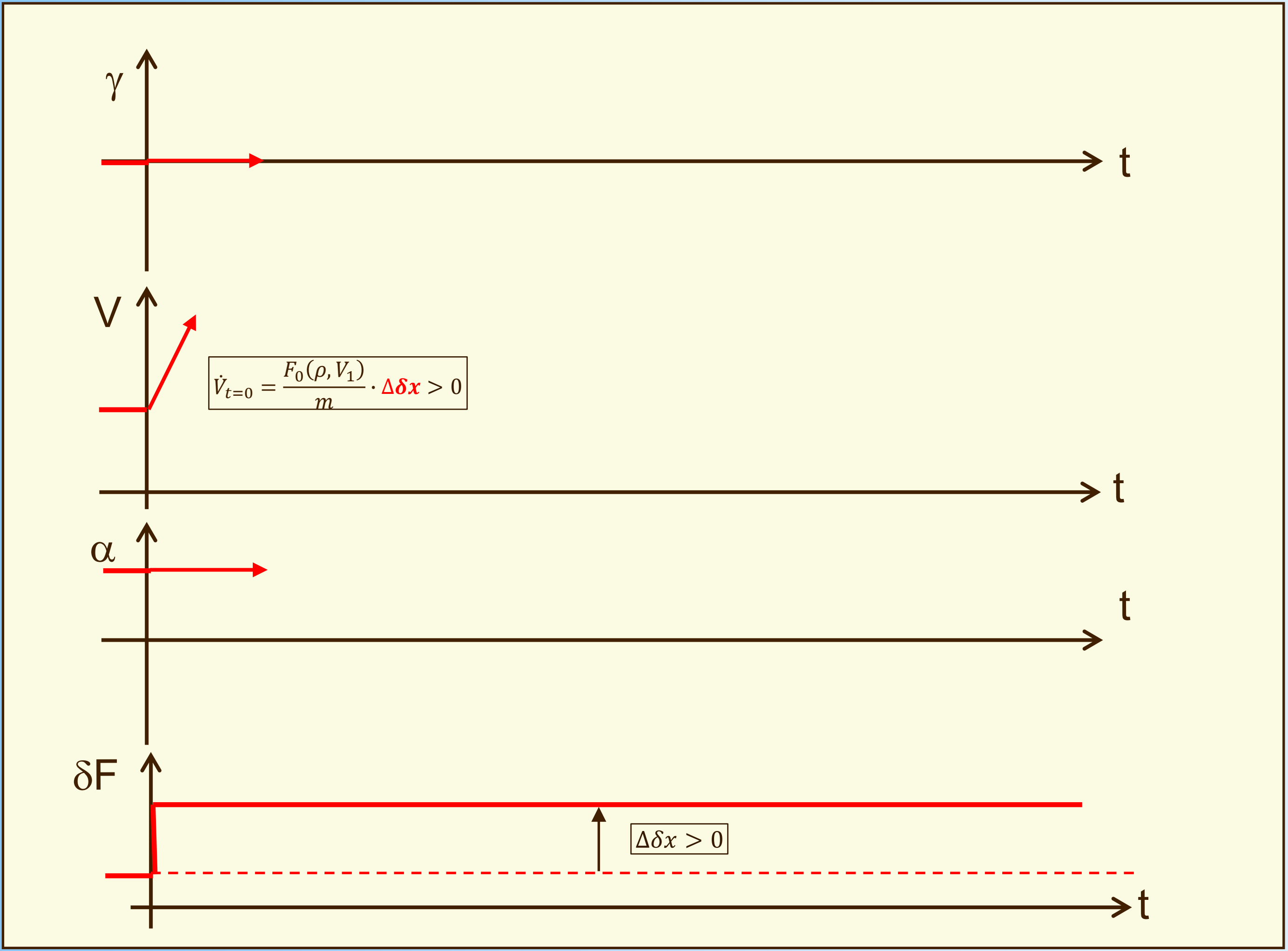
(initial response)

$$\begin{cases} \dot{V}_{t=0} = \frac{F_0(\rho, V_1)}{m}\cdot\boldsymbol{\Delta\delta x} \\ \dot{\gamma}_{t=0} = 0 \\ \dot{q}_{t=0} = 0 \\ \dot{\alpha}_{t=0} = 0 \end{cases}$$

Situation at t=0 + ε

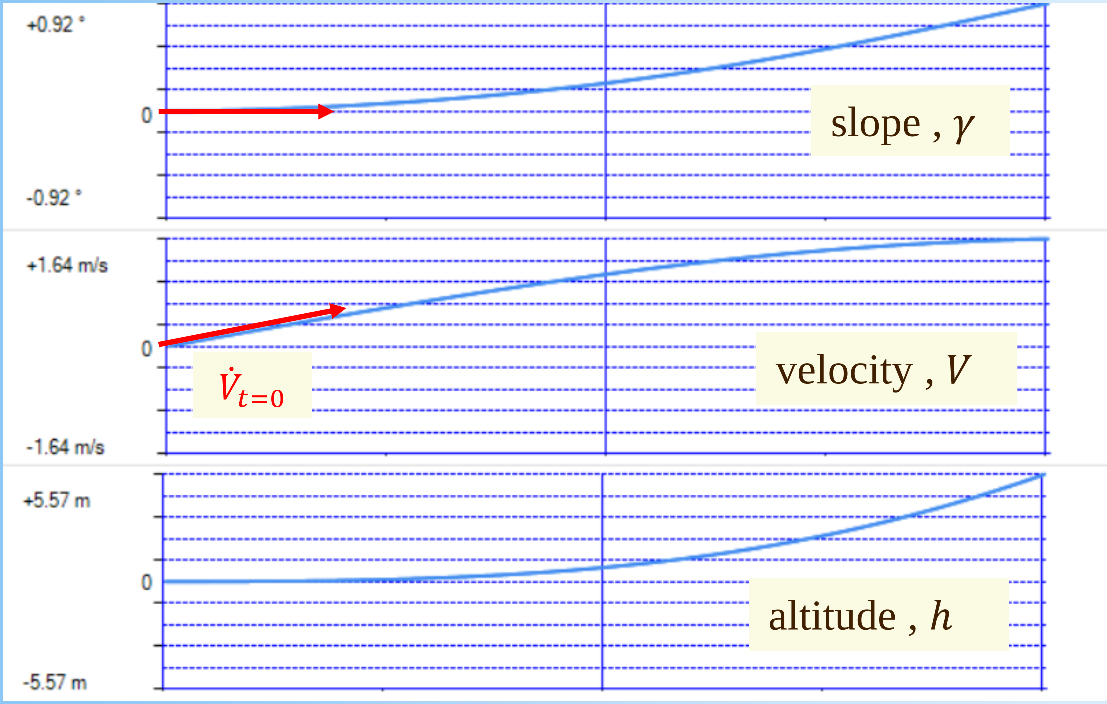
$$\begin{cases} m\dot{V}_{t=0} = -\frac{1}{2}\rho V_1^2 SCx + F_0(\rho, V_1)\cdot(\delta x_1 + \boldsymbol{\Delta\delta x}) - mg\cdot\gamma_1 \\ -mV_1\dot{\gamma}_{t=0} = -\frac{1}{2}\rho V_1^2 SCz_\alpha(\alpha_1-\alpha_0) + mg \\ B\dot{q}_{t=0} = \frac{1}{2}\rho V_1^2 SL\left[Cm_0 + Cm_{\alpha,G}(\alpha_1-\alpha_0) + Cm_{\delta m}\delta m_1\right] \\ \dot{\alpha}_{t=0} = q_1 - \dot{\gamma}_{t=0} \end{cases}$$

General Response from a Δδx variation



Initial effect from a $\Delta\delta x$ variation

Time simulation 10s **Concorde / 5000 ft / 100 m/s / 5% thrust step**



Thrust variation : Final Trim

$\Delta\delta x > 0$, fixed / $\Delta\delta m = 0°$

Situation at t=0 : Trim n°1

$$\begin{cases} -\frac{1}{2}\rho V_1^2 SCx + F_0(\rho, V_1)\cdot\delta x_1 - mg\cdot\gamma_1 = 0 \\ -\frac{1}{2}\rho V_1^2 SCz_\alpha(\alpha_1 - \alpha_0) + mg = 0 \\ Cm_0 + Cm_{\alpha,G}(\alpha_1 - \alpha_0) + Cm_{\delta m}\delta m_1 = 0 \\ q_1 = 0 \end{cases}$$

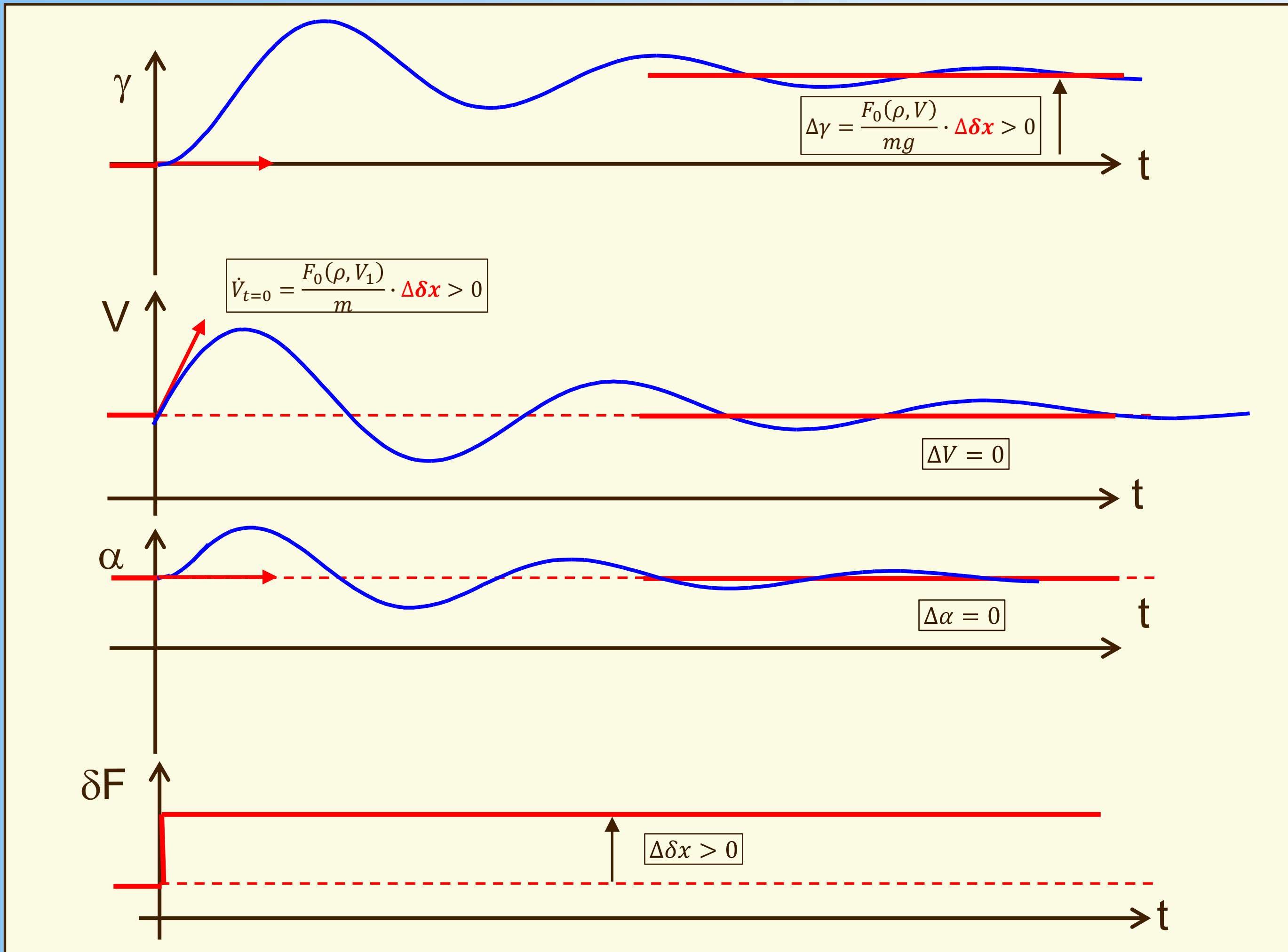
Variation between the 2 successive trims

$$\begin{cases} \Delta\gamma = \gamma_2 - \gamma_1 = \frac{F_0(\rho, V)}{mg}\cdot\boldsymbol{\Delta\delta x} \\ V_2 = V_1 \\ \alpha_2 = \alpha_1 \\ q_2 = q_1 = 0 \end{cases}$$

Situation after a certain time: Trim n°2

$$\begin{cases} -\frac{1}{2}\rho V_2^2 SCx + F_0(\rho, V_2)\cdot(\delta x_1 + \boldsymbol{\Delta\delta x}) - mg\cdot\gamma_2 = 0 \\ -\frac{1}{2}\rho V_2^2 SCz_\alpha(\alpha_2 - \alpha_0) + mg = 0 \\ Cm_0 + Cm_{\alpha,G}(\alpha_2 - \alpha_0) + Cm_{\delta m}\delta m_1 = 0 \\ q_2 = 0 \end{cases}$$

General Response from a ΔδX variation



General Response from a $\Delta\delta x$ variation

Starting from an initial flight trim situation,

- You perform a thrust increase manoeuvre and you maintain it
- What happens ?

There is an immediate initial effect (short term),

- The velocity increases (the kinematic energy increases)

There is final effect (long term),

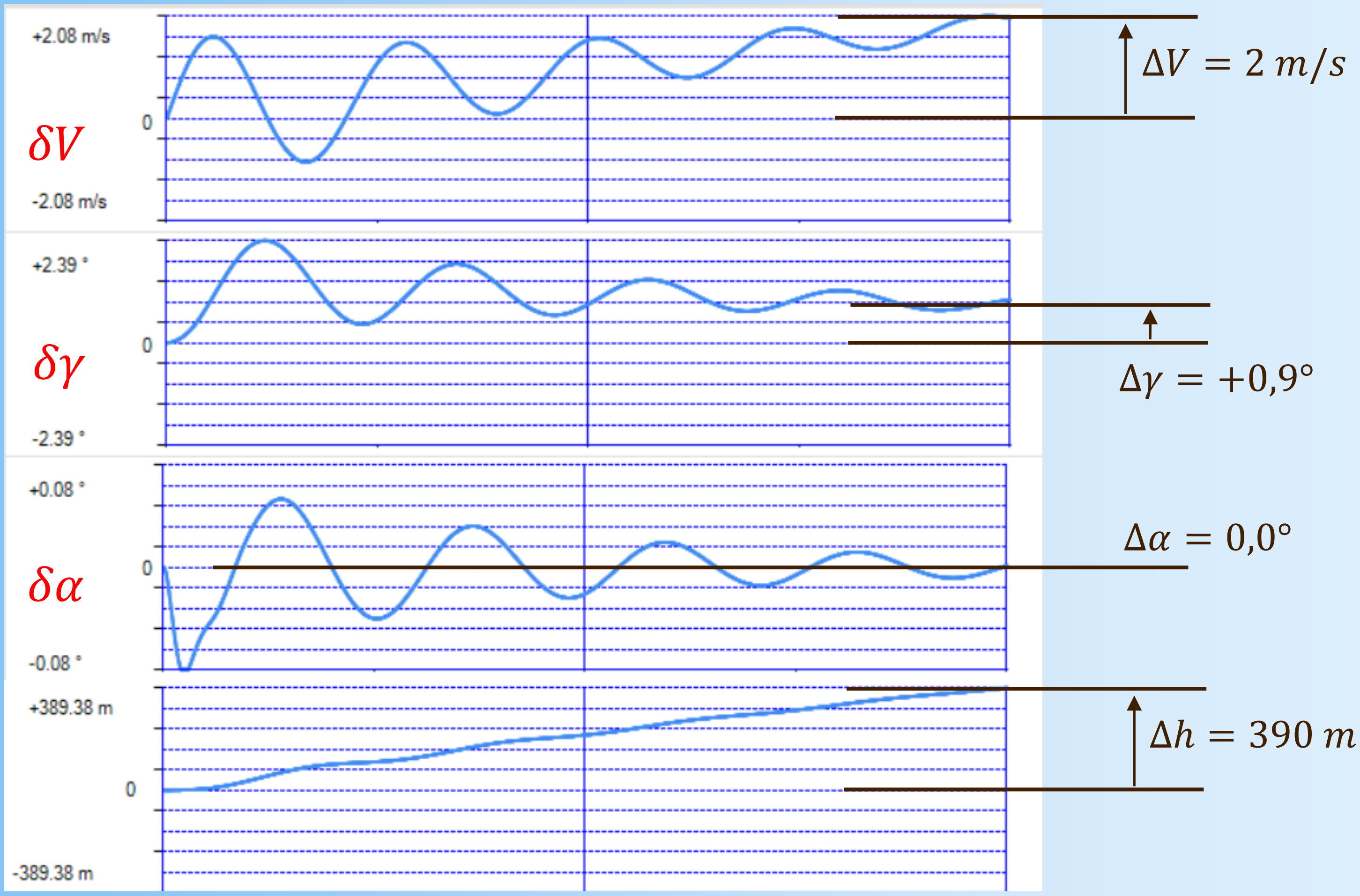
- The slope increases (you gain altitude and the potential energy increases) but keeping the same initial speed

Between these 2 states, there is a transient phase characterized by long period, low damped oscillations called the phugoïde

Remark : we obtain these simplified results assuming that there is no thrust term within the pitch equation (we assume that the engine thrust axis is passing through the CG).

General Response from a $\Delta\delta x$ variation

Time simulation 200 s **Concorde / 5000 ft / 100 m/s / 5% thrust step**



General Response from a $\Delta\delta x$ variation

The flight simulation confirms the results obtained with the 2 successives trim approach except for the variation of V at the final trim.

The variation of V at the final trim is linked to the lift equation

$$mg = \frac{1}{2}\rho V^2 S \cdot Cz$$

Between both trims, there is no variation of α, so no variation of Cz,

so no variation of the dynamic pressure

$$\frac{1}{2}\rho V^2 = Cte$$

As the flight simulation computes the altitude evolution

Between both trims, the altitude h is increasing, so the density ρ is decreasing

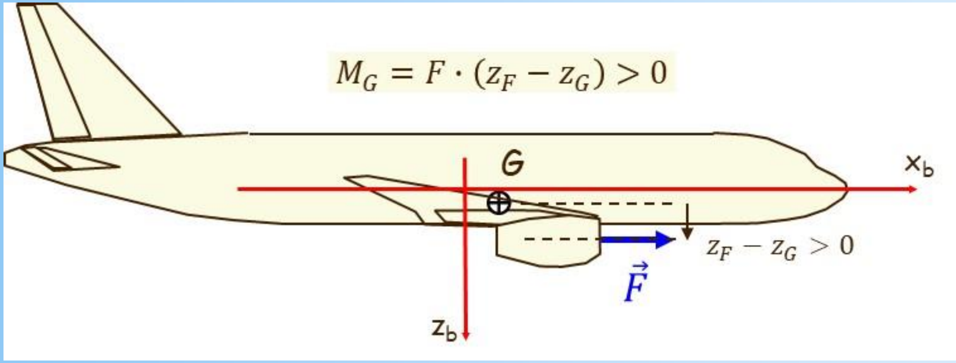
And the velocity V shall increase for keeping the dynamic pressure constant

General Response from a $\Delta\delta x$ variation

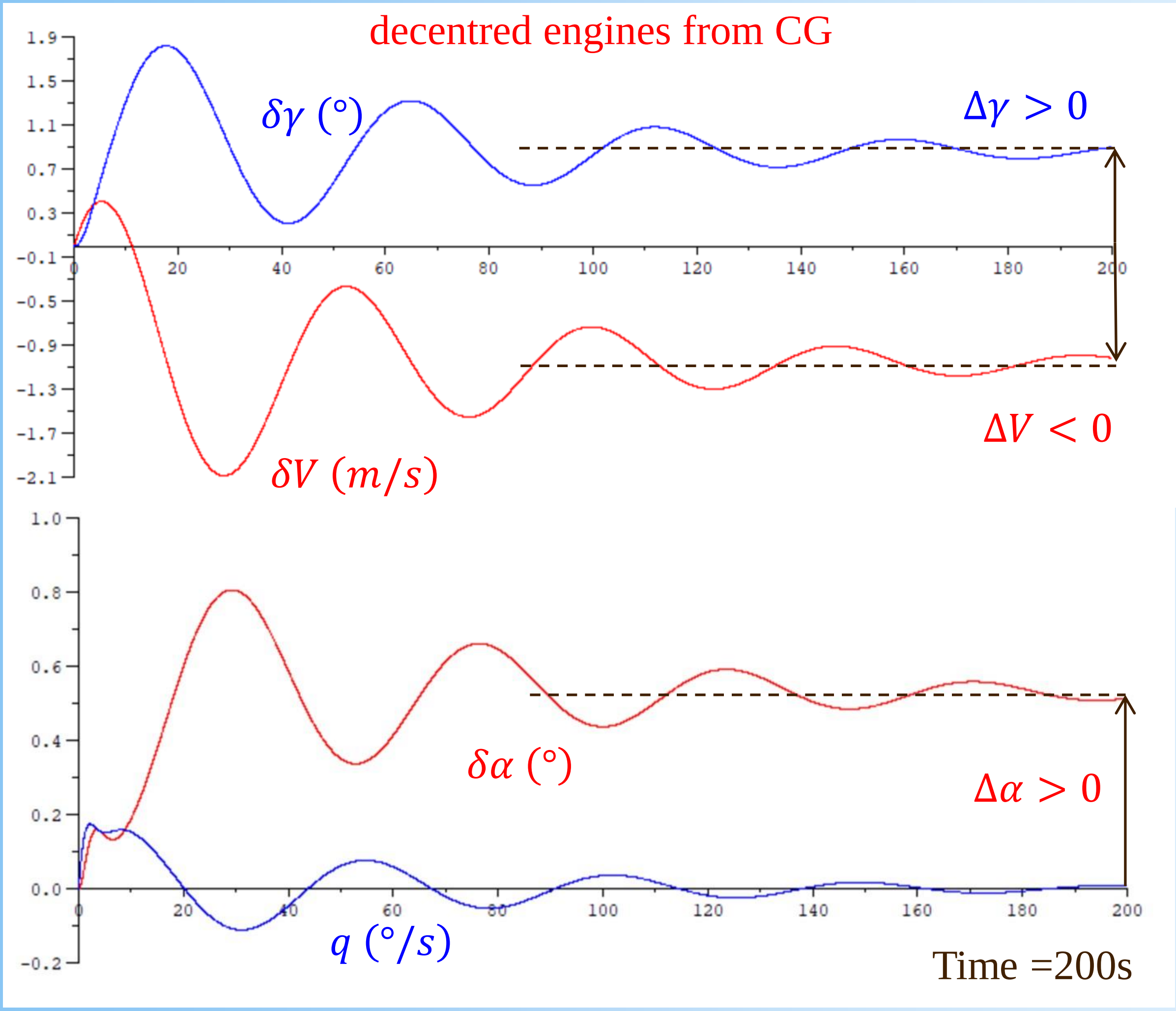
decentred engines from CG

When the engine is decentred (as commercial aircraft), the thrust increase creates a pitch-up motion which modifies the angle of attack , as initial response. The steady solution for the decentred engine case requires the notion of Speed neutral point F_V (see annex 5) ; a simulation on next slide is presented for a nominal situation.

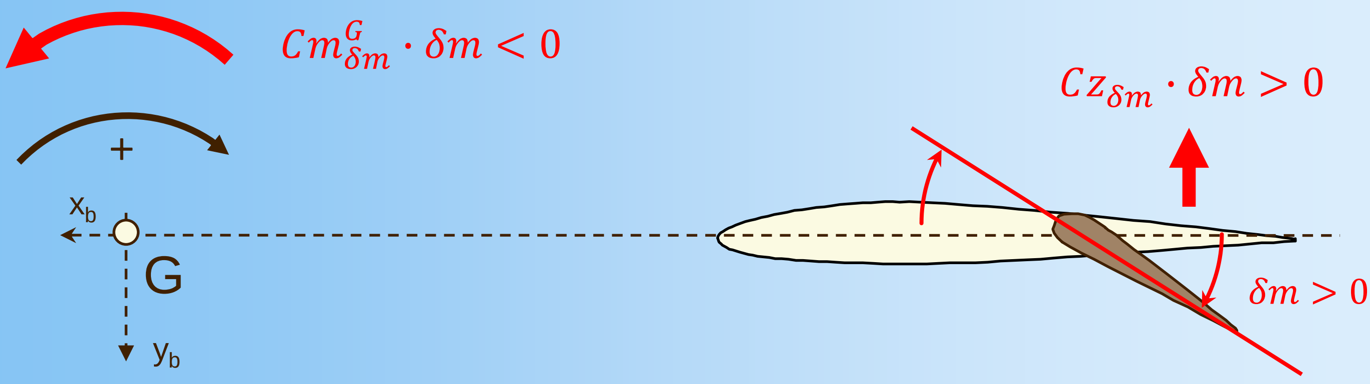
$$\begin{cases} -\frac{1}{2}\rho V^2 S Cx + F - mg\cdot\gamma = 0 \\ -\frac{1}{2}\rho V^2 S Cz_\alpha(\alpha-\alpha_0) + mg = 0 \\ Cm_0 + Cm_{\alpha,G}(\alpha-\alpha_0) + Cm_{\delta m}\delta m + \frac{2F}{\rho V^2 S}\cdot\frac{z_F - z_G}{L} = 0 \end{cases}$$



General Response from a $\Delta\delta x > 0$ variation



Elevator deflection Step



δm positive variation : elevator down motion / $Cz_{\delta m}\delta m$ neglected

We do assume that the pitch response is instantaneous (correct assumption / Short Period mode)

We achieve immediately the final variation for the angle of attack, α

$$\begin{cases} Cm_0 + Cm_{\alpha,G}(\alpha_1 - \alpha_0) + Cm_{\delta m}\delta m_1 = 0 \\ Cm_0 + Cm_{\alpha,G}(\alpha_2 - \alpha_0) + Cm_{\delta m}(\delta m_1 + \Delta\boldsymbol{\delta m}) = 0 \end{cases} \Rightarrow \quad \alpha_2 - \alpha_1 = -\frac{Cm_{\delta m}}{Cm_{\alpha,G}} \cdot \Delta\boldsymbol{\delta m} < 0$$

Elevator deflection : Initial Response

$\Delta\delta m > 0°$, fixed / $\Delta\delta x = 0$

Situation at t=0 : Trim n°1

$$\begin{cases} -\frac{1}{2}\rho V_1^2 SCx + F_0(\rho, V_1)\cdot\delta x_1 - mg\cdot\gamma_1 = 0 \\ -\frac{1}{2}\rho V_1^2 SCz_\alpha(\alpha_1 - \alpha_0) + mg = 0 \\ Cm_0 + Cm_{\alpha,G}(\alpha_1 - \alpha_0) + Cm_{\delta m}\delta m_1 = 0 \\ q_1 = 0 \end{cases}$$

Situation at t=0 + ε

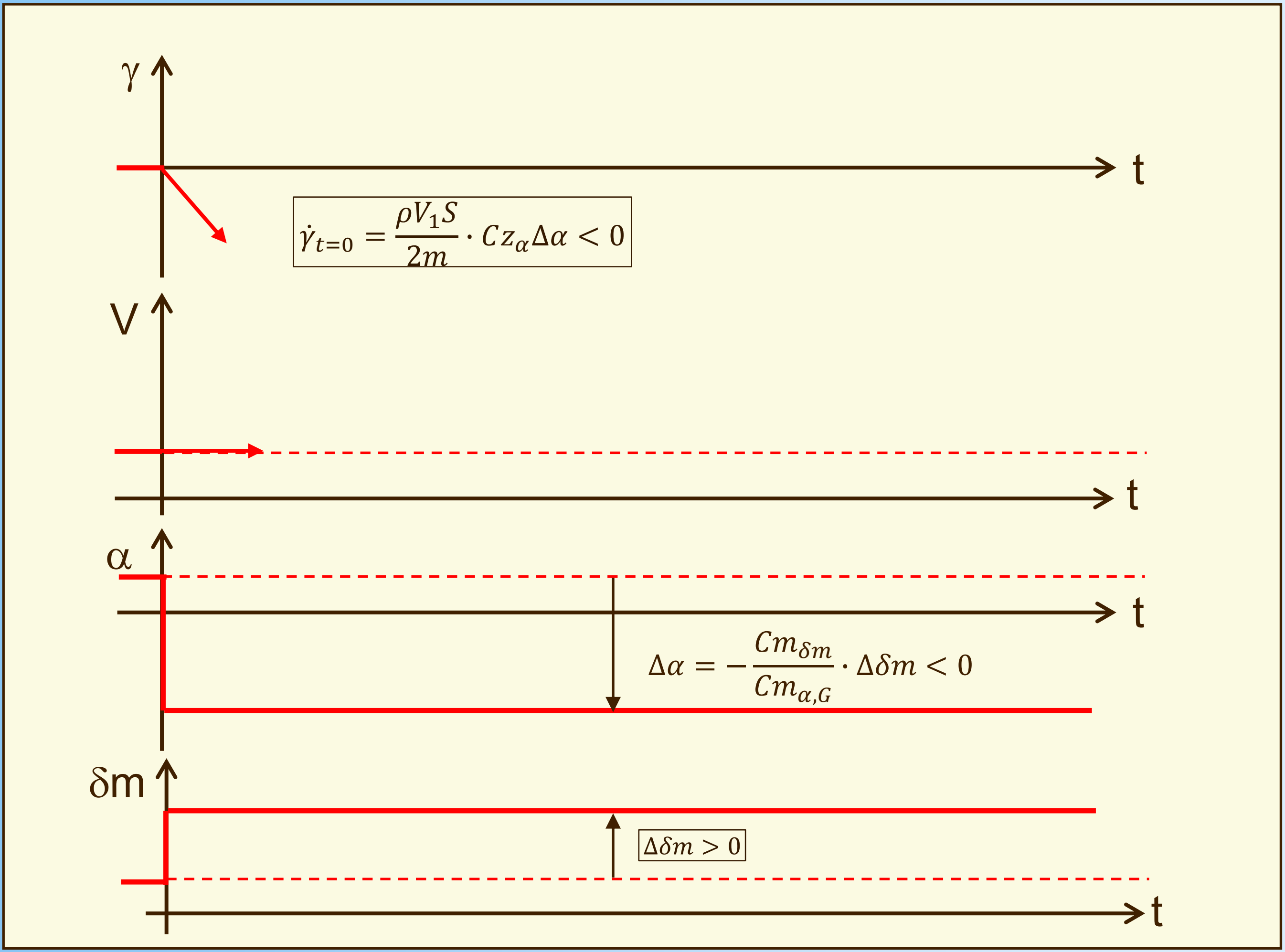
$$\begin{cases} m\dot{V}_{t=0} = -\frac{1}{2}\rho V_1^2 SCx + F_0(\rho, V_1)\cdot\delta x_1 - mg\cdot\gamma_1 \\ -mV_1\dot{\gamma}_{t=0} = -\frac{1}{2}\rho V_1^2 SCz_\alpha(\alpha_2 - \alpha_0) + mg \\ Cm_0 + Cm_{\alpha,G}(\alpha_2 - \alpha_0) + Cm_{\delta m}(\delta m_1 + \boldsymbol{\Delta\delta m}) = 0 \\ q_2 = 0 \end{cases}$$

Variation at t=0

(initial response)

$$\begin{cases} \dot{V}_{t=0} = 0 \\ \dot{\gamma}_{t=0} = \frac{\rho V_1 S}{2m}Cz_\alpha\cdot\boldsymbol{\Delta\alpha} < \mathbf{0} \\ \Delta\alpha = -\frac{Cm_{\delta m}}{Cm_{\alpha,G}}\cdot\boldsymbol{\Delta\delta m} < 0 \end{cases}$$

General Response from a $\Delta\delta m$ variation



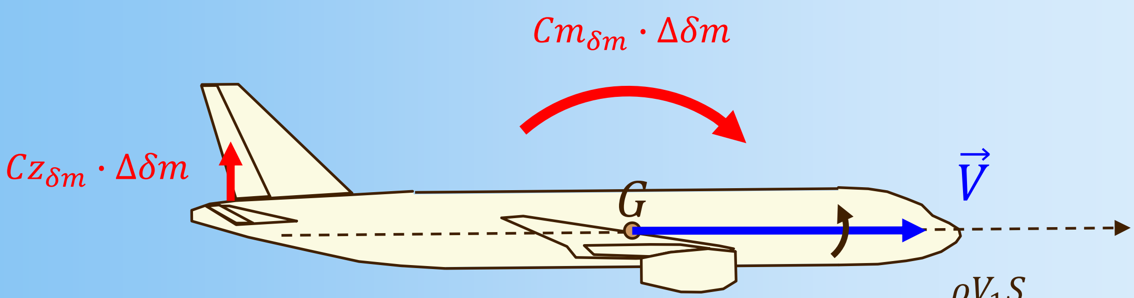
Initial effect from a $\Delta\delta m$ variation

Starting from an initial flight trim situation,

- You perform a pitch down elevator manoeuvre ($\Delta\delta m > 0$) and you maintain it
- What happens ?

There is an immediate initial effect (short term),

- The lift up force produced by the elevator makes rotate up the velocity $\vec{V}$, the slope increases as initial effect (but it is a very small effect …)
- The same lift up force produced a big pitch moment around G

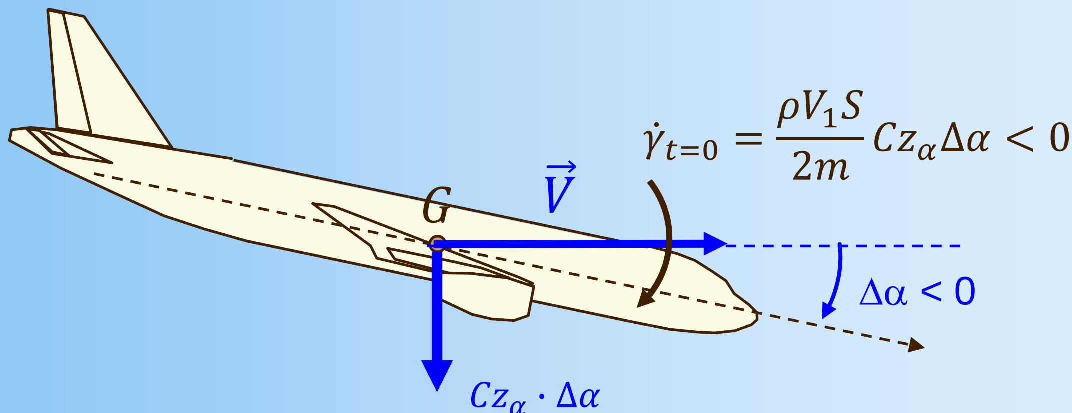


$$\dot{\gamma}_{t=0} = \frac{\rho V_1 S}{2m} Cz_{\delta m} \Delta\delta m > 0$$

Initial effect from a $\Delta\delta m$ variation

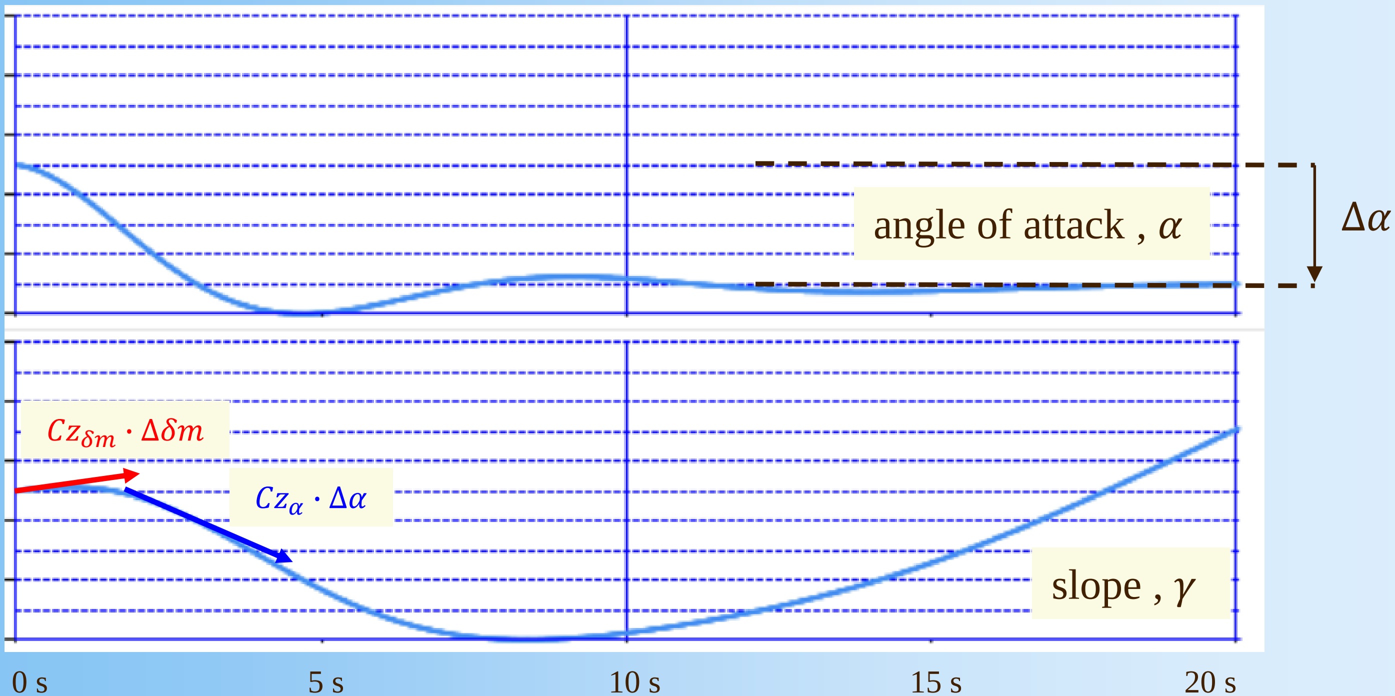
The aircraft is immediately rotating down due to the pitch down moment produced by the elevator move → α are decreasing
As a result, an important down lift is produced, which makes rotate down the velocity : the slope is decreasing strongly

It can be seen also as the main initial effect



Initial effect from a Δδm variation

Time simulation 20s **Concorde / 5000 ft / 100 m/s / 5° elevator step**



Note the quick convergence of the angle of attack, typical of the Short Period mode
The slope γ starts its long term oscillation, typical of the Phugoïd

Elevator deflection : Final Trim

$\Delta\delta m > 0°$, fixed / $\Delta\delta x = 0$

Situation at t=0 : Trim n°1

$$\begin{cases} -\frac{1}{2}\rho V_1^2 SCx + F_0(\rho, V_1)\cdot\delta x_1 - mg\cdot\gamma_1 = 0 \\ -\frac{1}{2}\rho V_1^2 SCz_\alpha(\alpha_1 - \alpha_0) + mg = 0 \\ Cm_0 + Cm_{\alpha,G}(\alpha_1 - \alpha_0) + Cm_{\delta m}\delta m_1 = 0 \\ q_1 = 0 \end{cases}$$

Situation after a certain time: Trim n°2

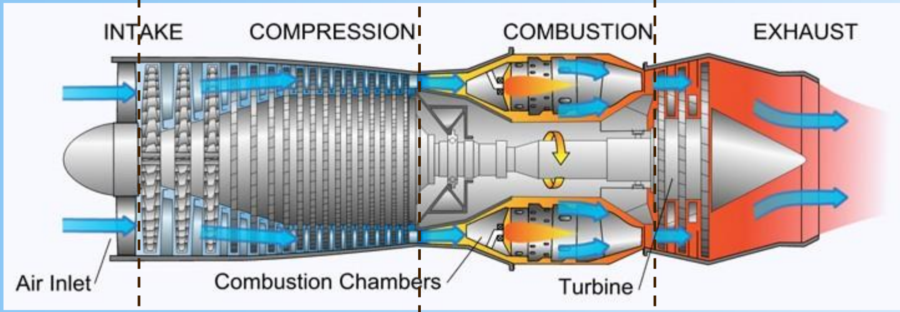
$$\begin{cases} -\frac{1}{2}\rho V_2^2 SCx + F_0(\rho, V_2)\cdot\delta x_1 - mg\cdot\gamma_2 = 0 \\ -\frac{1}{2}\rho V_2^2 SCz_\alpha(\alpha_2 - \alpha_0) + mg = 0 \\ Cm_0 + Cm_{\alpha,G}(\alpha_2 - \alpha_0) + Cm_{\delta m}(\delta m_1 + \boldsymbol{\Delta\delta m}) = 0 \\ q_2 = 0 \end{cases}$$

Variation between the 2 successive trims

$$\begin{cases} \Delta\gamma = \dfrac{\Delta\left\{F_0(\rho, V)\cdot\delta x_1 - \frac{1}{2}\rho V^2 SCx\right\}}{mg} = ? \\ \dfrac{V_2^2}{V_1^2} = \dfrac{Cz_1}{Cz_2} > 1 \rightarrow V_2 > V_1 \\ \Delta\alpha = \alpha_2 - \alpha_1 = -\dfrac{Cm_{\delta m}}{Cm_{\alpha,G}}\cdot\boldsymbol{\Delta\delta m} < 0 \\ q_2 = q_1 = 0 \end{cases}$$

The sign of Δγ stays unknown and needs a graphical resolution (see next chart)

Turbofan (very simple) principle



V_i

V_e

Compressor $V \searrow$

Combustion $V \approx 0$

Turbine $V \nearrow$

$$F \sim \dot{m} \cdot (V_e - V_i)$$

- The intake velocity V_i corresponds to the aircraft velocity V
- The exhaust velocity V_e doesn't depend on V , function on the turbines efficiency

$$F \sim \dot{m} \cdot (\underbrace{V_e}_{\text{Cte}} - V) \Rightarrow V \nearrow \Longrightarrow F \searrow$$

Thrust and Drag evolution versus V

The sign of the quantity $\Delta\gamma$ can be found graphically :

We assume the Steady state and we plot the Thrust and the Drag versus the Velocity V :

- The thrust is a curve decreasing versus V : $\dfrac{\partial F}{\partial V} < 0$
- The Drag is a parabola versus V :

$$\frac{1}{2}\rho V^2 S Cx = \frac{1}{2}\rho V^2 S \cdot (Cx_0 + k_i Cz^2)$$

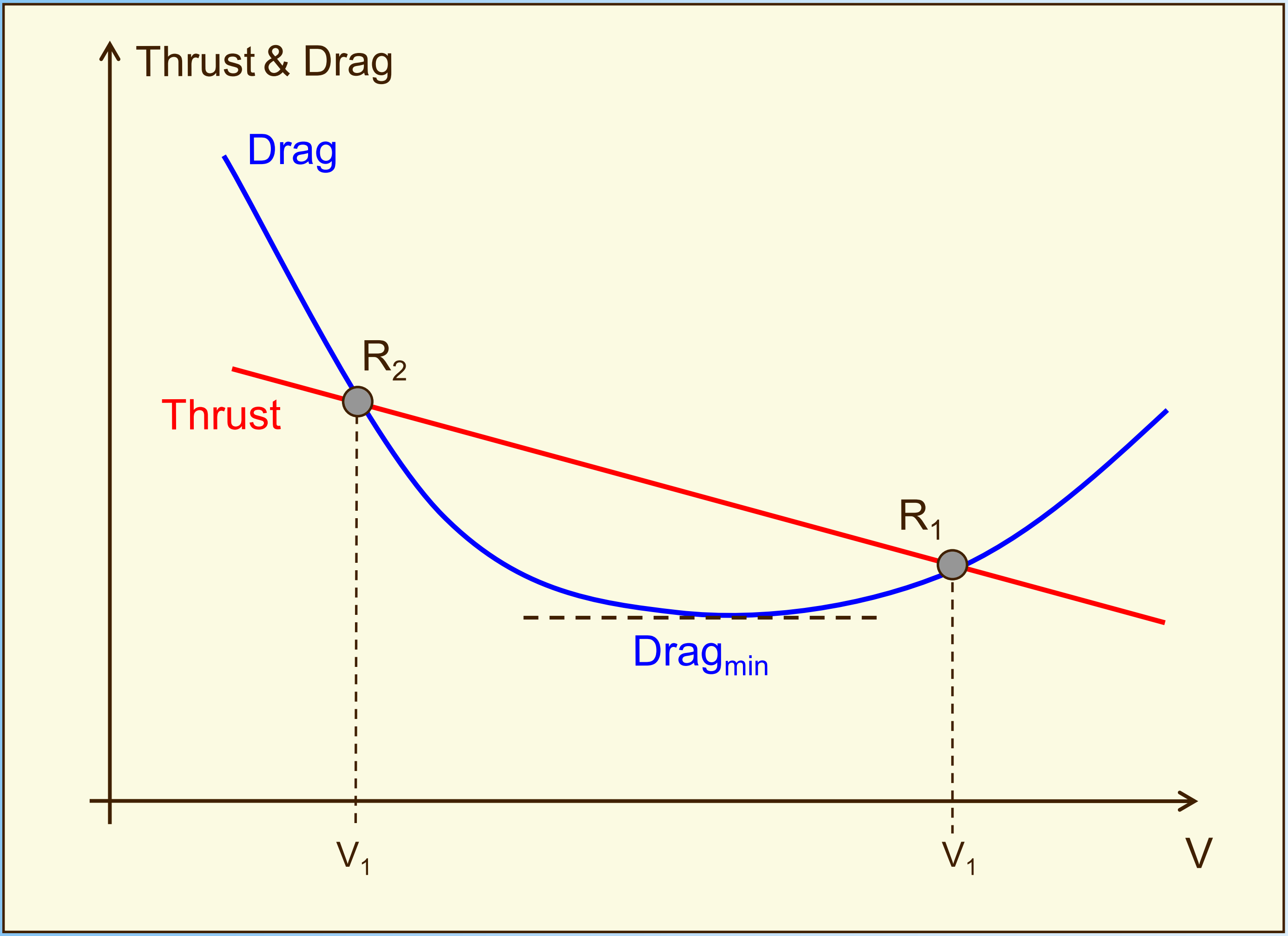
$$\Downarrow$$

$$\frac{1}{2}\rho V^2 S Cx = \frac{1}{2}\rho V^2 S Cx_0 + \frac{1}{2}\rho V^2 S \cdot k_i Cz^2 \longleftarrow Cz = \frac{mg}{\frac{1}{2}\rho V^2 S}$$

$$\Downarrow$$

$$\frac{1}{2}\rho V^2 S Cx = \frac{1}{2}\rho V^2 S Cx_0 + k_i \cdot \frac{2m^2 g^2}{\rho V^2 S} = a \cdot V^2 + \frac{b}{V^2}$$

Thrust and Drag evolution versus V

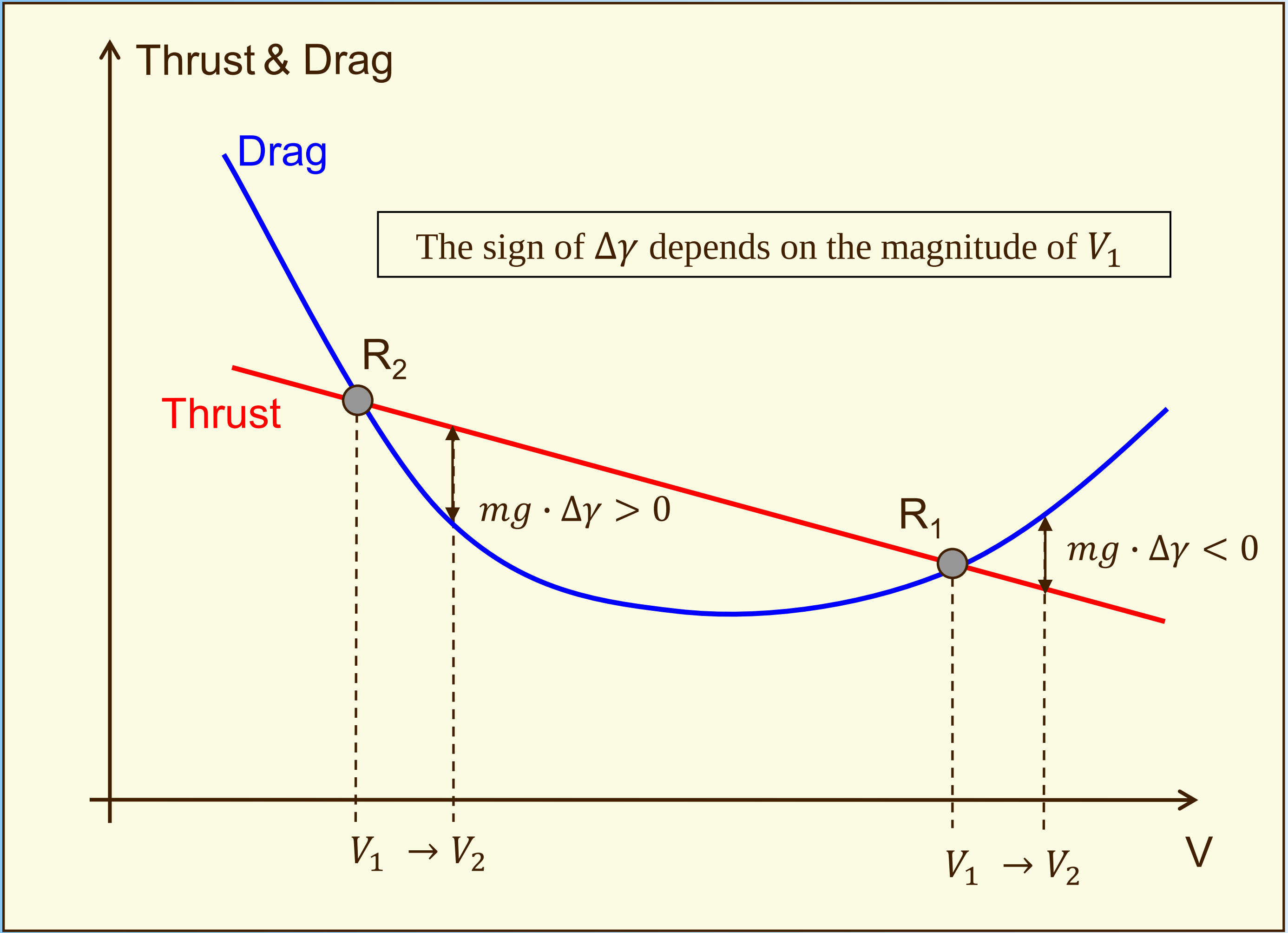


Flight regimes Definition

The Drag evolution versus V is a parabola curve :

- The longitudinal flight at $\gamma = 0°$ is possible when : Thrust > $Drag_{min}$
- There are 2 possible solutions for the initial trim at $\gamma = 0°$ (when Thrust = Drag)
    - The first One at high velocity : the 1^{st} regime
    - The second One at low velocity : the 2^{nd} regime

Graphical Resolution for $\Delta\gamma$



Graphical Resolution for $\Delta\gamma$

You are trimmed at a first equilibrium with V_1
Then, you perform the pitch down elevator manoeuvre,
the velocity reaches a new trimmed value V_2 such as $\Delta V = V_2 - V_1 > 0$

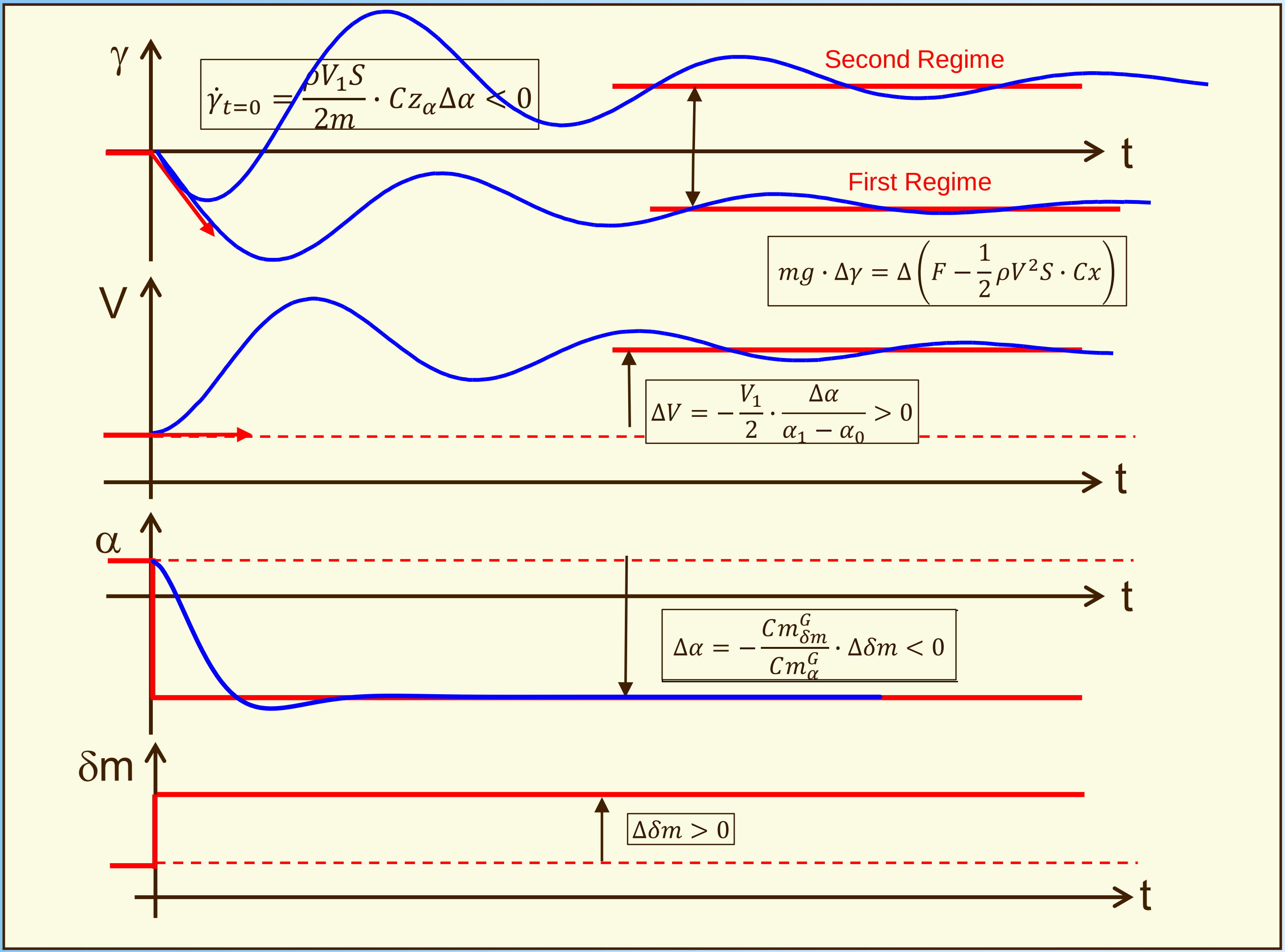
<u>Case n°1</u>
you start from the point R_1 (first regime), the final $\Delta V > 0$ leads to Drag > Thrust

- ➢ your slope decreases and you loose altitude
- ➢ it is the first regime : the usual one

<u>Case n°2</u>
you start from the point R_2 (second), the final $\Delta V > 0$ leads to Thrust > Drag

- ➢ your slope increases and you gain altitude
- ➢ it is the second regime : the unusual one

General Response from a Δδm variation



General Response from a $\Delta\delta m$ variation

Starting from an initial flight trim situation, you perform a pitch down elevator manoeuvre ($\boldsymbol{\Delta\delta m} > 0$) and you maintain it. What happens at Long Term ?

At Long term,

- The angle of attack decreases : $\Delta\alpha = -\frac{Cm_{\delta m}}{Cm_{\alpha,G}} \cdot \boldsymbol{\Delta\delta m} < 0$ (Pitch Equation)
- The velocity increases consequently : $\frac{V_2^2}{V_1^2} = \frac{Cz_1}{Cz_2} > 1$ (Lift Equation)
- The Final variation of the slope depends on the Initial velocity value
  - If the Initial velocity is low (second regime), the slope will increase : you gain altitude
  - If the Initial velocity is high (first regime), the slope will decrease : you loose altitude

Remark : the first regime is the "usual" one. You make a pitch down elevator command, your aircraft rotates down, your angle of attack decreases, your velocity increases and you loose altitude.

As for the thrust variation command, between these 2 states, there is a transient phase characterized by long period, low damped oscillations of the velocity called the phugoïde

General Response from a Δδm variation

Time simulation 200s **Concorde / 5000 ft / 100 m/s / 5° elevator step**