2MAE404 MIMO control
Homework report 5
(with later improvements)

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December 2, 2022

1 Exercise 1 – Flying Chardonnay controllability

Recalling the results from homework no. 3, the dynamics of the Flying Chardonnay drone linearised at hover conditions are:

$$\dot{x}(t) = A\Delta x + B\Delta u + E\Delta w = Ax + B(u - u_0) + Ew \tag{1}$$

Where the numerical values of the matrices are:

$$A = \begin{bmatrix} -0.01 & 0 & -10 & 0 & 10 & 0 \\ 0 & -0.005 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -0.01 & 0 & 0 & 0 & 20 & 0 \end{bmatrix} \tag{2}$$

$$B = \begin{bmatrix} 0 & 0 \\ -0.5 & -0.5 \\ 0 & 0 \\ -1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} \tag{3}$$

$$E = \begin{bmatrix} 0.01 & 0 \\ 0 & -0.005 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0.01 & 0 \end{bmatrix} \tag{4}$$

1.1 Stability

The stability of an LTI system is determined by the eigenvalues of its A matrix. The eigen values can be computed using the `eig` function in MATLAB. The results are:

- 3 stable poles: $\lambda_1 = \lambda_2 = -0.0050$, and $\lambda_3 = -4.4746$

- a pair of marginally stable poles: $\lambda_4 = -\lambda_5 = 5.821 \cdot 10^{-8} j$
- 1 unstable pole: $\lambda_6 = 4.4696$

This means that the system is not fully stable.

1.2 Controllability & stabilisability

The controllability of an LTI system can be evaluated by computing the controllability matrix:

$$K = \big([B\ AB\ ...\ A^{n-1}B\big] \tag{5}$$

In MATLAB, this is done using the `ctrb` function. The resulting matrix is of rank 6, therefore the system is fully controllable, and, by extension, stabilisable.

1.3 Degree of actuation

The rank of the system's B matrix is 2, while the system is of rank 6. This means that the system is under-actuated.

2 Excercise 2 – controllability of a linear system

Given is a linear system with the following state-space representation:

$$\dot{x} = \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \tag{6}$$

2.1 Controllable subspace

The controllable subspace of an LTI system can be defined as the column space of the controllability matrix (for simplicity of notation, span$M$ is defined to be the column space of M in this report):

$$\mathcal{C} = \text{span}\left[B\ AB\ ...\ A^{n-1}B\right] \tag{7}$$

Since the studied system has dimension $n = 2$, the controllable subspace becomes:

$$\mathcal{C} = \text{span}[B\ AB] \tag{8}$$

The B and AB matrices in the equation above are:

$$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{9}$$

$$AB = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \tag{10}$$

Hence, the controllable subspace can be written as:

$$\mathcal{C} = \text{span} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \tag{11}$$

The span is, by definition, the set of all possible linear combinations of a set of vectors. In the studied case, this can be written as:

$$\forall x \in \mathcal{C}\ \exists c_1, c_2 \in \mathbb{R} :\ x = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix} \tag{12}$$

This equation could be broken down into two single equations:

$$x_1 = c_1 - c_2 \tag{13}$$

$$x_2 = c_1 - c_2 \tag{14}$$

Since the right hand side of both equations is identical, it can be inferred that all points in the controllable subspace $\mathcal{C}$ follow the equation $x_1 = x_2$, meaning that $\mathcal{C}$ is a straight line in the $\mathbb{R}^2$ space, passing through the origin $x = (0, 0)$ and with a slope of 1. This line is visualised in Figure 1. It can be immediately noted that the state $x = (0, 1)$ does not lay on the controllable subspace.

2.2 Closest reach to point outside of the controllable subspace

By definition, if the initial system state is $x(0) = (0, 0)$, arbitrary actuation $u(t)$ can only make it reach points inside of the controllable subspace $\mathcal{C}$. The smallest achievable distance between the state $x(t)$ and $x = (0, 1)$, assuming $x(0) = (0, 0)$ and any $u(t)$, is therefore the distance between the $x = (0, 1)$ point and the controllable subspace $\mathcal{C}$. This is a very simple geometry problem (distance between a point and a straight line). The point on $\mathcal{C}$ closest to $x = (0, 1)$ is $x = (0.5, 0.5)$, and the minimum distance is $\sqrt{2}/2$. The problem is visualised in Figure 1.

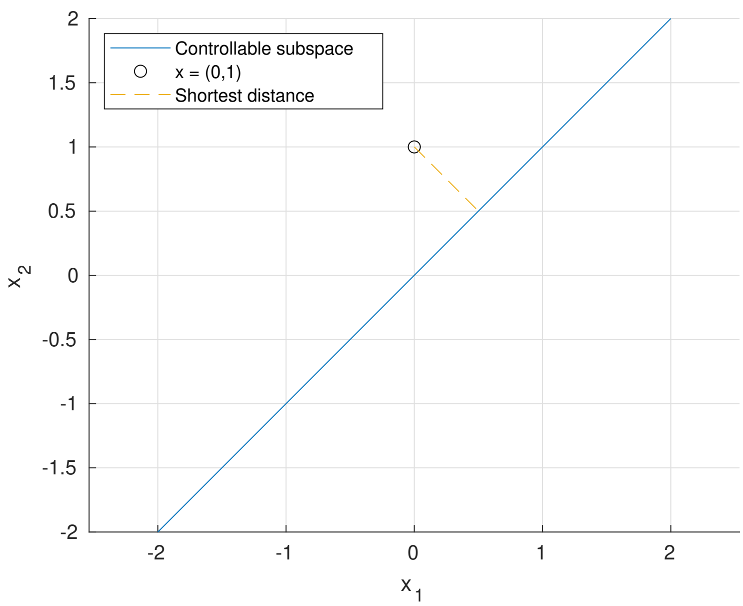

Figure 1: Controllable subspace of the linear system.

3 Excercise 3 – ISS docking

3.1 Nominal operation

In nominal conditions, the dynamics of the spacecraft docking to the ISS are:

$$\dot{x} = \frac{d}{dt}\begin{bmatrix}\Delta r \\ \Delta\theta\end{bmatrix} = Ax + Bu = \begin{bmatrix}10 & -2 \\ 2 & 5\end{bmatrix} x + \begin{bmatrix}0 & 3 & 2 \\ 1 & 5 & 1\end{bmatrix}\begin{bmatrix}u_1 \\ u_2 \\ u_3\end{bmatrix} \tag{15}$$

3.1.1 Controllability

Just as in Excercise 1.2, the controllability matrix of the system can be computed using the MATLAB `ctrb` function. The resulting matrix is of rank 2 (full rank), so the system is, again, fully controllable.

3.1.2 Degree of actuation

Since the rank of B is 2 and the number of inputs is higher than the number of states, the system is overactuated, meaning that some trajectories can be achieved by more than one combination of control inputs.

3.2 Operation under a malfunction

Due to jamming, two control outputs become stuck at the constant values $u_1(t) = 0$ and $u_2(t) = 1$. $u_3(t)$ remains functional. The system can be written as:

$$\dot{x} = \begin{bmatrix}\Delta\dot{r} \\ \Delta\dot{\theta}\end{bmatrix} = Ax + B_3u_3 + B_2u_2 = Ax + B_3u_3 + p \tag{16}$$

where B_2 and B_3 are the 2nd and 3rd column of B matrix, respectively. p denotes the constant perturbation due to u_2. The expanded form of the system is:

$$\dot{x} = \begin{bmatrix}10 & 2 \\ 2 & 5\end{bmatrix} x + \begin{bmatrix}2 \\ 1\end{bmatrix} u_3 + \begin{bmatrix}3 \\ 5\end{bmatrix} \tag{17}$$

3.2.1 Controllable subspace & trajectory

Without any control input, the spacecraft rapidly drifts away from the ISS, as shown in Figure 2. Since the system is no longer fully controllable, it is not sure whether control of u_3 alone can make docking possible.

Since B_2 is independent of x, the control forcing always lies on the same line (but can be of any direction and magnitude, since u_3 can be any real number). This means that the system has a fully controllable direction and a completely uncontrollable one perpendicular to it. It is useful to transform the problem to a coordinate system aligned with these directions.

Since the controllable space is $Bu_3 = \begin{bmatrix}2 & 1\end{bmatrix}^T u_3$, where $u_3 \in \mathbb{R}$, the controllable component of a state x is:

$$x_c = x \cdot \frac{B}{||B||} = \begin{bmatrix}\Delta r \\ \Delta\theta\end{bmatrix} \cdot \begin{bmatrix}2/\sqrt{5} \\ 1/\sqrt{5}\end{bmatrix} \tag{18}$$

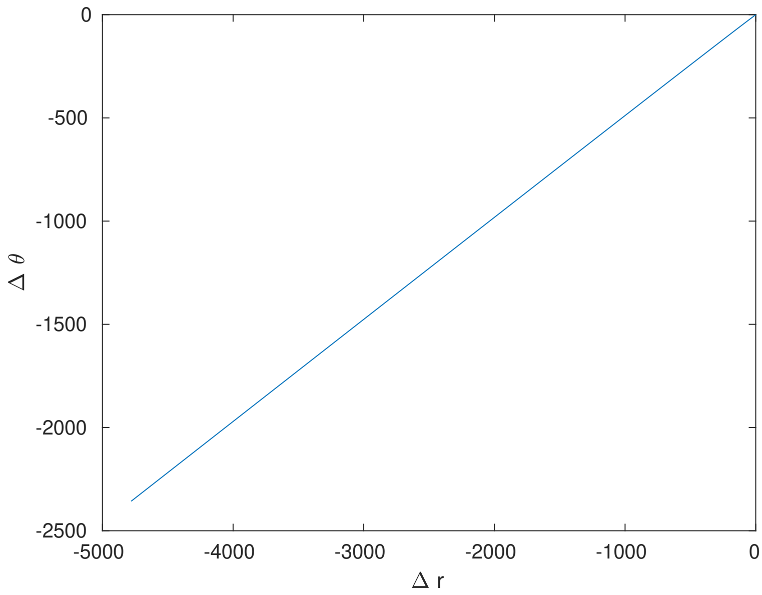

Figure 2: Uncontrolled trajectory of the spacecraft for $t \in [0, 1]$ (computed using `lsim`).

The uncontrollable direction is obtained by rotating B by 90^o.

$$x_u = x \cdot \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} \Delta r \\ \Delta\theta \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \tag{19}$$

Therefore, the transformation matrices between the physical (x) and controllability-defined (x^*) coordinates are:

$$x^* = \begin{bmatrix} x_c \\ x_u \end{bmatrix} = Mx = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} x \tag{20}$$

$$x = M^{-1}x^* = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} x^* \tag{21}$$

The system dynamics rewritten in terms of x^* are:

$$M^{-1}\dot{x}^* = AM^{-1}x^* + Bu_3 + p \tag{22}$$

$$\dot{x}^* = MAM^{-1}x^* + MBu_3 + Mp \tag{23}$$

The numerical values of this new state-space system are:

$$\dot{x}^* = A^*x^* + B^*u_3 + p^* = \begin{bmatrix} 9 & -4 \\ 0 & 6 \end{bmatrix} x^* + \begin{bmatrix} \sqrt{5} \\ 0 \end{bmatrix} u_3 + \begin{bmatrix} 11/\sqrt{5} \\ 7/\sqrt{5} \end{bmatrix} \tag{24}$$

The initial conditions in the transformed coordinates are:

$$x_0^* = Mx_0 = \begin{bmatrix} -3/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} \tag{25}$$

The equation for x_u therefore:

$$\dot{x}_u = 6x_u + 7/\sqrt{5} \tag{26}$$

The computation of the controllable subspace of the systems is less obvious than usually, because the $x = (0, 0)$ point is not a stationary point of the system (because of the p term). The classical way of defining the controllable subspace would be to draw a line following the controllable direction and passing through $x = (0, 0)$. Such a space is easily parametrised in the controllability-defined coordinates as $x^* = [c, 0]^T$, where $c \in \mathbb{R}$. However, this space does not have the typical properties of a controllable subspace, because not only is it impossible to reach any point on it starting from another one, it is not even possible to stay on it at all with any input $u_3(t)$. A different approach is finding a space that satisfies those 2 properties, which make the controllable subspace actually useful. The points on such a space should not have a non-zero uncontrollable component to their dynamics, i.e. satisfy $\dot{x}_u = 0$. This condition assures that trajectories starting on the controllable subspace remain on it. Imposing $\dot{x}_u = 0$ on Equation 26 results in the following space:

$$x_u = -\frac{7}{6\sqrt{5}} \tag{27}$$

Note that since the space is a line parallel to the x_u axis, it is aligned with the controllable direction, meaning that any 2 points on the space are reachable in arbitrary time with appropriate control input u_3. Both subspaces, the classical and affine one (called this way since it does not pass through 0), are plotted on top of a phase portrait of the system in Figure 3. It is important to note that the initial point is quite close to the affine controllable subspace, but it's actually very good that it is not – if that was the case, leaving the subspace would be impossible, causing the ISS to become unreachable.

To analyse the possibility of docking, further simplifications in the system's representation can be made. The equation for x_c is:

$$\dot{x}_c = 9x_c - 4x_u + \sqrt{5}u_3 + \frac{11}{\sqrt{5}} \tag{28}$$

This is still dependent on x_u. However, since x_u is independent and observable, the control input can be rewritten as:

$$u_3 = \frac{4}{\sqrt{5}}x_u(t) + u^*(t) - \frac{11}{5} \tag{29}$$

In the equation above, the $(4/\sqrt{5}) \cdot x_u(t)$ term cancels out the influence of x_u on $\dot{x}_c$, while the $-11/5$ term cancels out the influence of p^*. $u^*(t)$ is the remaining control input, which can be chosen in a way that drives x_c to 0 synchronously with x_u (soon it will be shown how this can be achieved). The x_c dynamics can be therefore rewritten as:

$$\dot{x}_c = 9x_c(t) + \sqrt{5}u^*(t) \tag{30}$$

This is obviously a controllable system. This means that x_c can be driven to 0 independently of x_u. The remaining questions are:

1. Does x_u eventually reach 0 on it's own?
2. If so, can x_c be driven to 0 at exactly the same time?
3. If so, how to design a control law that achieves this?

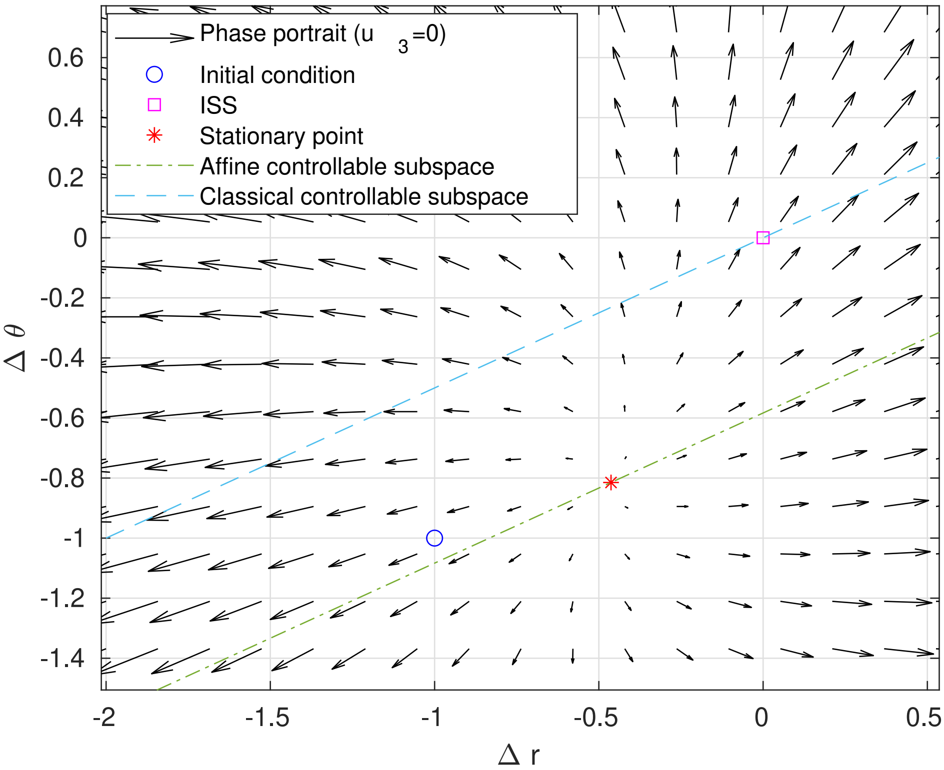

Figure 3: Phase portrait and key spaces and points of the jammed ISS system. Note that the phase portrait vectors are tangent to the affine controllable subspace.

3.2.2 Possibility of docking

No matter the control input u_3, x_u evolves in time according to Equation 26. The solution to this simple linear non-homogenous ODE with an initial condition $x_u(0) = -1\sqrt{5}$ (as computed in Equation 25) is:

$$x_u(t) = \frac{1}{6\sqrt{5}} e^{6t} - \frac{7}{6\sqrt{5}} \tag{31}$$

It can be easily checked that this trajectory does cross 0 at one instant in time, denoted as t_d:

$$x_u(t_d) = 0 \iff e^{6t_d} = 7 \iff t_d = \frac{1}{6} \ln 7 \approx 0.3243 \tag{32}$$

Since x_c is fully controllable, it seems that it should be possible to find a control law $u_3(t)$ such that $x_c(t_d) = 0$. In fact, there are infinitely many ways to achieve this, so the simplest possible idea is considered: finding a constant $u^*(t) = u^*$ such that $x_c(t_d) = 0$. Assuming that it is indeed a constant value, Equation 30 becomes a simple first order nonhomogenous ODE, just like Equation 26. The solution to this equation with the initial condition $x_c(0) = -3/\sqrt{5}$ (as in Equation 25) is:

$$x_c(t) = \frac{5}{9\sqrt{5}} u^* \left(e^{9t} - 1\right) - \frac{3}{\sqrt{5}} e^{9t} \tag{33}$$

The boundary condition $x_c(t_d)$ allows to compute u^*:

$$x_c(t_d) = 0 \iff 27e^{9t_d} = 5u^* \left(e^{9t_d} - 1\right) \iff u^* = \frac{27e^{9t_d}}{5\left(e^{9t} - 1\right)} \approx 5.7082 \tag{34}$$

Substituting Equation 31 and Equation 34 back to Equation 29, the full control law is:

$$u_3(t) = \frac{4}{\sqrt{5}}\left(\frac{1}{6\sqrt{5}}e^{6t} - \frac{7}{6\sqrt{5}}\right) + \frac{27e^{9t_d}}{5\left(e^{9t_d} - 1\right)} - \frac{11}{5} \tag{35}$$

Despite its length, the formula for $u_3(t)$ is essentialy just a sum of an exponential term and a constant one. The correctness of the control law was verified with a simulation (using the original, full system matrices, to avoid computational errors in the derivation of the x^* system getting unnoticed):

```
%% simulation
t_dock = 1/6*log(7); % docking time

N = 1000; % number of time steps
t = linspace(0, t_dock, N); % time array
x0 = [-1, -1]; % initial condition
u(:,1) = zeros(N,1); % u_1 = 0
u(:,2) = ones(N,1); % u_2 = 1
u_star =  27*exp(9*t_dock)/(5*(exp(9*t_dock)-1));  % u*
u(:,3) = 4/sqrt(5)*(1/(6*sqrt(5))*exp(6*t) - 7/(6*sqrt(5))) -11/5 + u_star; % u_3

x_sim = lsim(sys_full, u, t, x0);
delta_r = x_sim(:,1);
delta_theta = x_sim(:,2);

figure;
plot(delta_r, delta_theta, '-', 'DisplayName', 'Trajectory');
hold on;
plot(-1,-1, 'kx', 'DisplayName', 'Initial condition');
plot(0,0, 'k+', 'DisplayName', 'ISS');
xlabel('\Delta r');
ylabel('\Delta \theta');
legend;
grid('on');
hold off;
```

The results of the simulation are shown in Figure 4. It can be seen that the ISS is indeed reached.

3.2.3 Minimum docking time

An important conclusion from Equation 32 is that every successful docking trajectory arrives at the ISS at the same time $t_d = \frac{1}{6}\ln 7 \approx 0.3243$, which is substantially less than 30. Combining with the successful control law demonstrated earlier, this means that docking before $t = 30$ is very much possible.

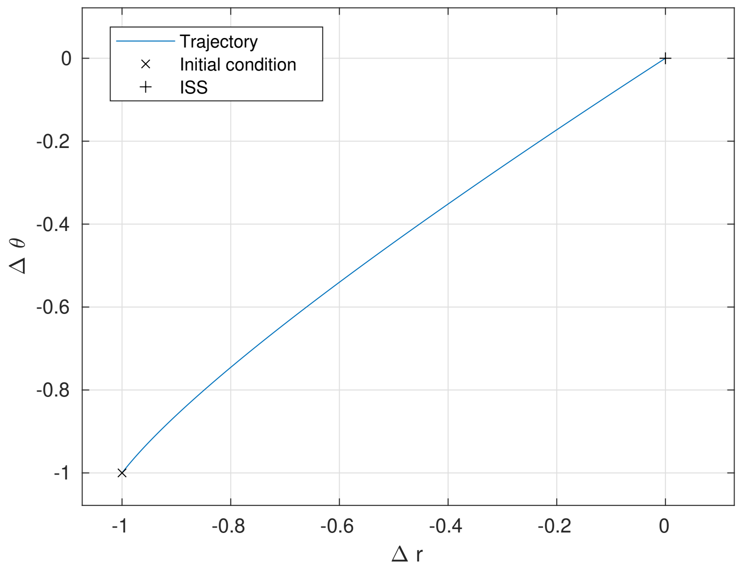

Figure 4: Docking trajectory simulation with a constant u^* as computed in Equation 34.