Hybrid Dynamical Systems - Part II: Notions of solution

Sophie Tarbouriech

LAAS-CNRS, Toulouse, France

January 30, 2023 - ISAE - Toulouse

Outline

1. Modeling framework
2. Classical solutions
3. Solutions for hybrid dynamical systems
4. Concluding remarks

- The hybrid system can be described as follows:

$$\begin{aligned} \dot{x} &= f(x)\ ,\ x \in \mathcal{C} \\ x^+ &= g(x)\ ,\ x \in \mathcal{D} \end{aligned} \qquad (1)$$

  where

  - ▷ f describes the continuous evolution (flow) of the state
  - ▷ g describes the discrete evolution (jumps) of the state
  - ▷ $\mathcal{C}$ describes the set where the flow can occur (continuous evolution)
  - ▷ $\mathcal{D}$ describes the set where the jumps can occur (discrete evolution)

- The sets $\mathcal{C}$ and $\mathcal{D}$ may represent

  - ▷ physical constraints,
  - ▷ model for logical modes by constraining them to some discrete set (for example $0, 1$ for "off" and "on"),
  - ▷ switching law, ...

- System (1) can be considered as a set of constraints satisfied by a solution but for which multiple solutions may occur, in particular if $\mathcal{C} \cap \mathcal{D} \neq \emptyset$.
- Then, in this case, it is useful to look at a more general description with

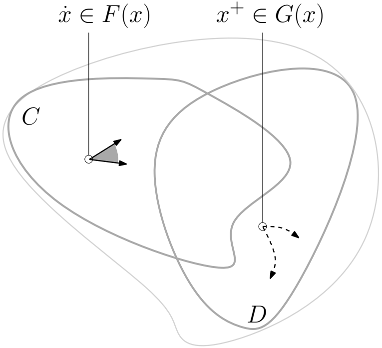
$$\begin{array}{c} \dot{x} \in F(x) \ , \ x \in \mathcal{C} \\ x^+ \in G(x) \ , \ x \in \mathcal{D} \end{array} \tag{2}$$

  ▷ Both F and G may be set-valued to represent effects of perturbations, uncertainty, lack of determinism, decision-making capabilities, overlapping guards and different resets, ...

Data of a Hybrid dynamical system: $\mathcal{H} = (C, D, F, G)$

$\mathcal{H} = (C, D, F, G)$

- $n \in \mathbb{N}$ (state dimension)
- $C \subseteq \Re^n$ (flow set)
- $D \subseteq \Re^n$ (jump set)
- $F : C \rightrightarrows \Re^n$ (flow map)
- $G : D \rightrightarrows \Re^n$ (jump map)



$$\mathcal{H} : \begin{cases} \dot{x} \in F(x) \quad , \quad x \in C \\ x^+ \in G(x) \quad , \quad x \in D \end{cases}$$

- From the hybrid formalism of the following equations

$$\begin{aligned} \dot{x} = f(x)\ ,\ x \in \mathcal{C} \\ x^{+} = g(x)\ ,\ x \in \mathcal{D} \end{aligned} \qquad (3)$$

we can describe pure:

- ▷ continuous-time dynamical systems if $\mathcal{C} = \Re^n$ and $\mathcal{D} = \emptyset$
- ▷ or discrete-time dynamical systems if $\mathcal{C} = \emptyset$ and $\mathcal{D} = \Re^n$.

- Let us first recall the definitions of solution for discrete-time and continuous-time systems.
- Discrete-time systems.
  - ▷ A solution ϕ to the discrete-time system

$$x^+ = g(x)$$

    is a sequence defined for

$$j \in \mathbb{N} = \{0, 1, 2 \ldots\}, j \to \phi(j) \in \mathbb{R}^n$$

    such that

$$\phi(j+1) = g(\phi(j)) \text{ for all } j \in \mathbb{N}$$

  - ▷ The domain of ϕ is $\mathrm{dom}\phi = \mathbb{N}$.

- Simple example for discrete-time systems: the linear case.
  - ▷ A solution ϕ to the discrete-time system

$$x^+ = Ax$$

is defined by

$$\phi(j) = A^j \phi(0) \text{ for all } j \in \mathbb{N}$$

- Continuous-time systems:
  - ▷ A solution ϕ to the continuous-time system

$$\dot{x} = f(x)$$

is a (locally absolutely) continuous function of

$$t \in \mathbb{R}_{\geq 0} = [0, +\infty), t \to \phi(t) \in \mathbb{R}^n$$

such that

$$\frac{d\phi(t)}{dt} = f(\phi(t)) \text{ for almost all } t \in \mathbb{R}_{\geq 0}$$

  - ▷ The domain of ϕ is $\mathrm{dom}\phi = [0, \mathrm{T})$, with T possibly being $+\infty$

- Why for almost t?
- Note that a solution should be absolutely continuous so that, according to Lebesgue integrability theory, the function itself coincides with the integral of its derivative (the so-called fundamental theorem of Lebesgue integral calculus - a fundamental property for a solution of a classical differential equation to make sense)
- Important remark. If the solution ϕ is locally absolutely continuous (LAS) we can retrieve ϕ after integrating its derivative.
  - ▷ As a consequence, solutions to differential equations are always LAS functions.
  - ▷ The "almost all" notion comes from the fact that LAS functions are differentiable everywhere except in sets of measure zero (isolated points).

- Simple example for continuous-time systems: the linear case.
  - ▷ A solution ϕ to the continuous-time system

$$\dot{x} = Ax$$

    is defined by

$$\phi(t) = e^{tA}\phi(0) \text{ for all } t \in \mathbb{R}_{\geq 0}$$

  - ▷ The domain of ϕ is $\mathrm{dom}\phi = [0, +\infty)$

- Another example:
  - ▷ A solution ϕ to the continuous-time system

$$\dot{x} = -x^2, x(0) = -1$$

is defined by

$$\phi(t) = \frac{1}{t + 1/x(0)} = \frac{1}{t-1} \text{ for almots all } t \in \mathbb{R}_{\geq 0}$$

  - ▷ This solution cannot be extended beyond $t = 1$!
  - ▷ The domain of ϕ is $\mathrm{dom}\phi = [0, 1)$

Remarks

- The concept of solution for a discrete-time system is simple because for any initial condition $\phi(0) \in \mathbb{R}^n$ a solution can be indefinitely extended by iterating evaluations of the continuous function g.
- The continuous-time solutions are not always defined for all positive times (namely they do not necessarily evolve forever) because they may exhibit the so-called finite escape time phenomenon where a solution blows up to infinity in finite time
  - ▷ previous example: $\dot{x} = -x^2, x(0) = -1 : t_f = 1$
- Solutions defined forever (namely for arbitrarily large times) are called complete as defined and illustrated in the following of this course.

- Hybrid systems blend together continuous-time and discrete-time.
  - ▷ This suggests that a solution to a hybrid system should experience such a combined behavior.
  - ▷ As a consequence, solutions exist on a time domain which is parametrized both by ordinary time $t \in \mathbb{R}_{\geq 0}$ and by the number of jumps $j \in \mathbb{N}$.

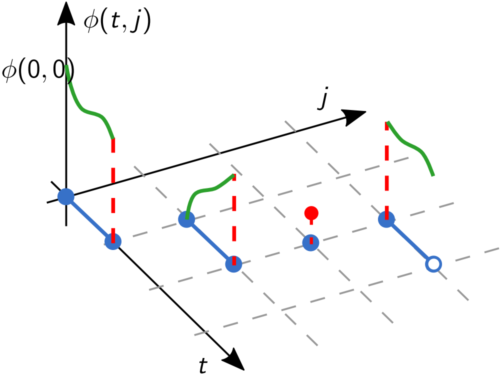
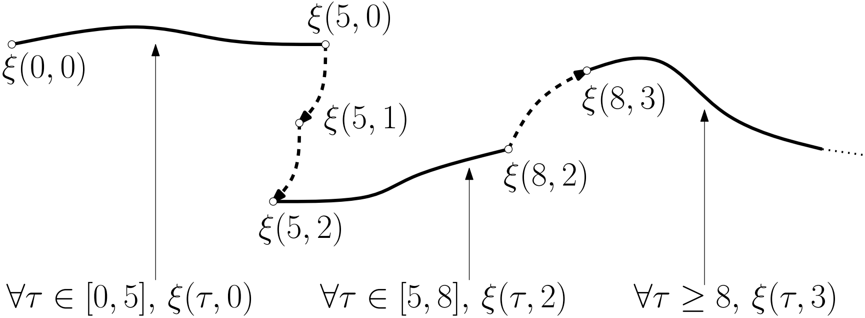


Figure: Solution ϕ to a hybrid system.

- The motion of the state is parameterized by two parameters:
  - $t \in \Re_{\geq 0}$, takes into account the elapse of time during the continuous motion of the state;
  - $j \in \mathbb{N}_{\geq 0}$, takes into account the number of jumps during the discrete motion of the state.



- Of course, it is impossible to parameterize a particular evolution of a hybrid system with all $(t, j) \in \Re_{\geq 0} \times \mathbb{N}_{\geq 0}$.
- Example. Suppose an evolution in which 3 jumps occur before the total time of flow reaches 2 seconds.
  - ▷ It makes no sense to ask what is happening after 4 seconds of flow and before any jumps.
- Hence, only certain subsets of $\Re_{\geq 0} \times \mathbb{N}_{\geq 0}$ can correspond to evolutions of hybrid systems. Such sets are called hybrid time domains.

- Intuitively, a hybrid time domain is an ordered collection of time intervals on which each solution is defined.
- Every solution has its own time domain, $\mathrm{dom}\phi$ constructed to match the jumping and flowing properties of that solution, depending on its evolution.
- Solutions to hybrid systems may be non-unique.
  - ▷ A single initial condition may lead to multiple solutions, each of them having its own hybrid time domain

Definition 1

A compact hybrid time domain

$$\mathbb{E}$$

is a subset of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ such that

$$\mathbb{E} = \bigcup_{j=0}^{J-1} (\underbrace{[t_j, t_{j+1}]}_{I^j} \times \{j\})$$

for a given finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_J$.

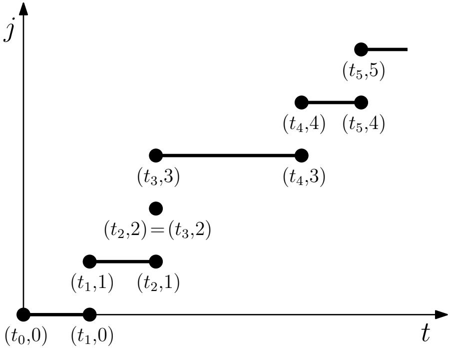
- $t_1, t_2 \ldots t_J$ are called jump times
- Possibly with $J = \infty$ or J is finite possibly with $t_J = \infty$

- However, such a definition is not enough to completely characterize a (possibly unbounded) hybrid time domain because it defines a bounded set
  - ▷ Recall that compact means closed and bounded

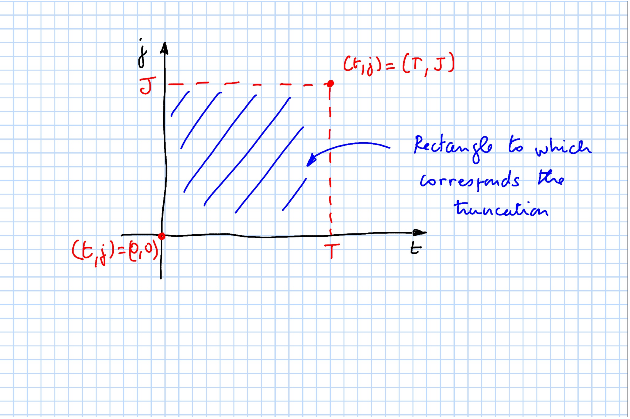
Definition 2

A hybrid time domain $\mathbb{E}$ is a subset of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ such that any truncation is a compact hybrid time domain.

- ▷ For any $(T, J) \in \mathbb{E}$, the set $\mathbb{E} \cap ([0, T] \times \{0, 1, \dots, J\})$ is a compact hybrid time domain



- Note that the truncation mentioned in Definition 2 corresponds to take only the points in the set $\mathbb{E}$ belonging to the rectangle having vertices at $(t, j) = (0, 0)$ and $(t, j) = (T, J)$.
- That can be depicted as follows.

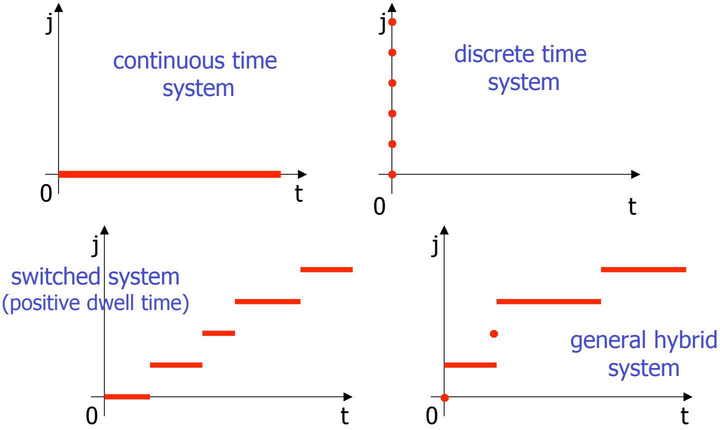


- Recall that the times t_i are called the jump times.
- There is a natural ordering in the points in a specific time domain ($\mathbb{E}$).
- If $(t_1, j_1) \in \mathbb{E}$ and $(t_2, j_2) \in \mathbb{E}$ then
  - ▷ $t_1 < t_2 \;\Rightarrow\; j_1 \leq j_2$
  - ▷ $j_1 < j_2 \;\Rightarrow\; t_1 \leq t_2$
- Then one can say that

$$(t_1, j_1) \leq (t_2, j_2) \text{ if } \; t_1 + j_1 \leq t_2 + j_2$$

- Of course, it makes no sense to order (or compare) elements of different hybrid time domains.

- Some examples of hybrid time domains (in red)



- Given any hybrid time domain $\mathbb{E} = \mathtt{dom}\phi$
  - ▷ it is natural to associate with each $(t, j) \in \mathbb{E}$ the one-dimensional variable $t + j$,
  - ▷ $t + j$ is strictly increasing along any (continuous or discrete) evolution of $\phi(t, j)$.
  - ▷ $t + j$ is the elapsed hybrid time.
- From the hybrid time (t, j) we can define the concept of solution.
- Let us propose a rigorous definition of solution to a hybrid system with differential and difference equations.

Definition 3

If $\mathcal{C}, \mathcal{D}$ are closed, f, g are continuous functions, a solution to the hybrid system

$$\begin{cases} \dot{x} &= f(x), \quad x \in \mathcal{C} \\ x^+ &= g(x), \quad x \in \mathcal{D}. \end{cases}$$

is a function ϕ such that

1. $\mathrm{dom}\phi$ is a hybrid time domain;
2. if $(t, j) \in \mathrm{dom}\phi$ and $(t, j+1) \in \mathrm{dom}\phi$, then
$$\begin{aligned} &\phi(t,j) \in \mathcal{D}, \\ &\phi(t,j+1) = g(\phi(t,j)); \end{aligned}$$
3. if $(t_a, j) \in \mathrm{dom}\phi$ and $(t_b, j) \in \mathrm{dom}\phi$, $t_a < t_b$, then $\phi(t,j)$ is locally absolutely continuous $\forall t \in I^j$, $\forall j \in \mathbb{N}$ and
$$\begin{aligned} &\phi(t,j) \in \mathcal{C} && \text{for almost all } t \in [t_a, t_b], \\ &\frac{d}{dt}(\phi(t,j)) = f(\phi(t,j)) && \text{for almost all } t \in [t_a, t_b]. \end{aligned}$$

Some comments on Definition 3

- This solution concept is inherited by the continuous-time and discrete-time solutions previous recalled.
  - ▷ Note that absolute continuity of flowing solutions is only required for values of t in the corresponding flowing interval I^j.
  - ▷ Even though for compact notation we require that $\phi(t, j) \in \mathcal{C}$ for almost all $t \in [t_a, t_b]$ (that is for all $t \in [t_a, t_b]$ except for a number of isolated points).
  - ▷ Since $\mathcal{C}$ is closed, then it holds that $\phi(t, j) \in \mathcal{C}$ for all $t \in [t_a, t_b]$.
  - ▷ In the case where $\mathcal{C}$ is not closed, deciding whether or not a solution should flow outside $\mathcal{C}$ in a number of isolated points is a delicate matter because it could undermine jumping from thin jump sets and other similar issues.
- The basic assumptions that we impose in this course (closedness of $\mathcal{C}$ and $\mathcal{D}$ and continuity of f and g) rule out those difficulties and make our definitions more natural.

- The solution concept in Definition 3 is easily extended to the case of hybrid inclusions

$$\begin{cases} \dot{x} \in F(x), & x \in \mathcal{C} \\ x^+ \in F(x), & x \in \mathcal{D}. \end{cases}$$

  - ▷ The inclusions are based on set-valued maps F and G, rather than single valued maps (functions) f and g.
- One has to generalize the definition of solution:
  - ▷ Condition 2: instead of asking that $\phi(t, j+1) = g(\phi(t, j))$, we may ask the generalized inclusion

$$\phi(t, j+1) \in G(\phi(t, j))$$

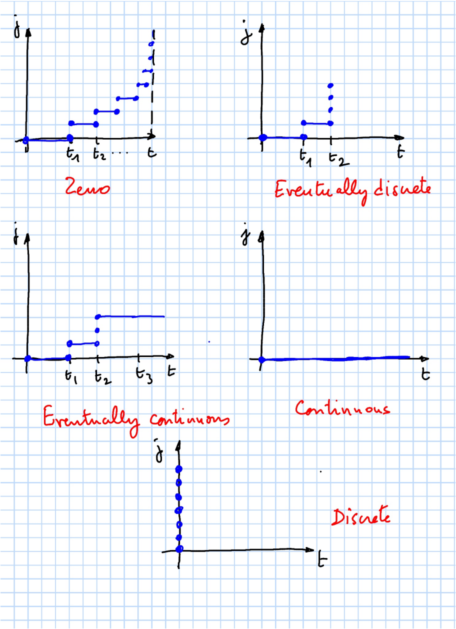
  - ▷ Condition 3: that follows the same logic: instead of $\frac{d}{dt}(\phi(t, j)) = f(\phi(t, j))$, we require

$$\frac{d}{dt}(\phi(t, j)) \in F(\phi(t, j)) \text{ for almost all } t \in I^j$$

Definition 4

A solution (or a hybrid arc) ϕ to the hybrid system is called:

- nontrivial if $\mathtt{dom}\phi$ contains at least two points.
- complete if $\mathtt{dom}\phi$ is unbounded (i.e. $\mathit{length}(\mathtt{dom}\phi) = \infty$)
- Zeno if it is complete and $\sup_t \mathtt{dom}\phi < \infty$
  - ▷ the solution jumps infinitely many times in a bounded ordinary time interval
  - ▷ the hybrid time domain is unbounded in the jump direction (j) while it is remaining bounded in the flow direction (t).
- Eventually discrete if $T = \sup_t \mathtt{dom}\phi < \infty$ and $\mathtt{dom}\phi \cap (\{T\} \times \mathbb{N})$ contains at least 2 points.
- Discrete if it is nontrivial and $\mathtt{dom}\phi \subseteq (\{0\} \times \mathbb{N})$.
- Eventually continuous if $J = \sup_j \mathtt{dom}\phi < \infty$ and $\mathtt{dom}\phi \cap (\mathbb{R}_{\geq 0} \times \{J\})$ contains at least 2 points.
- Continuous if it is nontrivial and $\mathtt{dom}\phi \subseteq (\mathbb{R}_{\geq 0} \times \{0\})$.
- Compact if $\mathtt{dom}\phi$ is compact.


j
t₁ t₂ ... t
Zeno
j
t₁ t₂ t
Eventually discrete
j
t₁ t₂ t₃ t
Eventually continuous
j
t
Continuous
j
t
Discrete

- Consider the following hybrid system with state $x \in \mathbb{R}^2$:

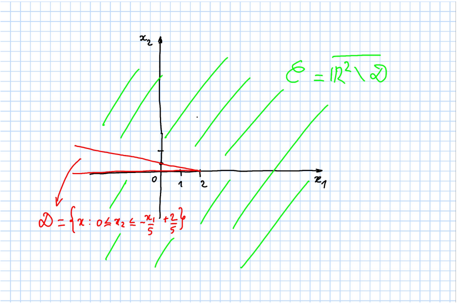
$$\begin{aligned} \dot{x} &= f(x) := \begin{bmatrix} 1 \\ 1 \end{bmatrix}, & x \in \mathcal{C} &:= \overline{\mathbb{R}^2 \setminus \mathcal{D}}, \\ x^+ &= g(x) := \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix} x_1, & x \in \mathcal{D} &:= \left\{ x \in \mathbb{R}^2 : 0 \leq x_2 \leq -\frac{1}{5} x_1 + \frac{2}{5} \right\}. \end{aligned} \tag{4}$$

- $\mathcal{C}$ and $\mathcal{D}$ are closed sets.
- Note that $\mathcal{C}$ is the closure of the complement of $\mathcal{D}$. Then

$$\mathcal{C} \cup \mathcal{D} = \mathbb{R}^2$$

  - ▷ Solutions can evolve (continuously, discretely, or both) from any point $x \in \mathbb{R}^2$

- Then, the objective is to pick some initial conditions
  - ▷ then solve the differential equation $\dot{x} = f(x)$ during flows
  - ▷ then solve the difference equation $x^+ = g(x)$ at jumps,
  - ▷ and check that the solution belongs to the flow or jump set, respectively.
- This allows us to illustrate the different nature of solutions to system (4), and to hybrid systems in general.



- Consider the following 6 initial conditions $\phi(t=0, j=0) \in \mathbb{R}^2$:

$$\phi_a(0,0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}; \phi_b(0,0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \phi_c(0,0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix};$$
$$\phi_d(0,0) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}; \phi_e(0,0) = \begin{bmatrix} -2 \\ -1 \end{bmatrix}; \phi_f(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- One can then compute the solutions $\phi(t,j) = \begin{bmatrix} \phi_1(t,j) \\ \phi_2(t,j) \end{bmatrix}$ and the domain $\mathrm{dom}\phi$

$$\begin{aligned} \phi(t,0) &= \phi(0,0) + \begin{bmatrix} t \\ t \end{bmatrix} \\ \phi(0,j) &= g(\phi(0,j-1)) = \begin{bmatrix} (3/4)^j \\ (1/4)^j \end{bmatrix} \phi_1(0,0) \end{aligned} \tag{5}$$

Solution from $\phi_a(0,0)$

- $\phi_a(0,0) = (2,-1)$ is in $\mathcal{C}$
- The solution from $\phi_a(0,0) = (2,-1)$ is defined as:

$$
\begin{aligned}
\phi_a(t,0) &= \phi_a(0,0) + \begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t+2 \\ t-1 \end{bmatrix}, \quad \forall t \in I^0 = [0,+\infty) \\
\phi_a(0,j) &= \begin{bmatrix} (3/4)^j \\ (1/4)^j \end{bmatrix} \phi_{a1}(0,0) = \begin{bmatrix} (3/4)^j \\ (1/4)^j \end{bmatrix} 2 \longrightarrow \text{never belongs to } \mathcal{D}
\end{aligned}
$$

▷ The solution from $\phi_a(0,0) = (2,-1)$ is in $\mathcal{C}$ and never jumps.

▷ Therefore, this solution only evolves in continuous time according to the flow dynamics, in the domain

$$\mathrm{dom}\phi_{\mathrm{a}} = \mathbb{R}_{\geq 0} \subset (\mathbb{R}_{\geq 0} \times \{0\})$$

▷ For this reason, it is referred to as continuous solution

$\boxed{\text{Solution from } \phi_b(0,0)}$

- $\phi_b(0,0) = (1,-1)$ is in $\mathcal{C}$
- The solution from $\phi_b(0,0) = (1,-1)$ is defined as:

$$
\begin{aligned}
\phi_b(t,0) &= \phi_b(0,0) + \begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t+1 \\ t-1 \end{bmatrix}, \quad \forall t \in I^0 = [0,+\infty) \\
\phi_b(0,j) &= \begin{bmatrix} (3/4)^j \\ (1/4)^j \end{bmatrix} \longrightarrow \text{may belong to } \mathcal{D}
\end{aligned}
$$

- ▷ The solution from $\phi_b(0,0) = (1,-1)$ is in $\mathcal{C}$ (and flows) and reaches the point $(2,0)$ for $t = t_1 = 1$ and $j = 0$.
- ▷ The point $(2,0)$ is both in $\mathcal{C}$ and $\mathcal{D}$. That means that from $(2,0)$, one can flow or jump.
- ▷ Then, we can consider two cases:

- Case 1: continuous: the solution $\phi_b(t,j)$ keeps flowing forever

$$\phi_b(t,j) = \phi_b(0,0) + \begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t+1 \\ t-1 \end{bmatrix}, \quad \forall t \in I^0 = [0,+\infty).$$

- Case 2: eventually continuous: the solution $\overline{\phi}_b(t,j)$ jumps at $(t,j) = (1,0)$ to reach the point $(2,0)$ and then flows:

$$\overline{\phi}_b(t,0) = \begin{bmatrix} t+1 \\ t-1 \end{bmatrix}, \quad \forall t \in I^0 = [0,1],$$

$$\overline{\phi}_b(1,0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \in \mathcal{D}, \; g(\overline{\phi}_b(1,0)) = \overline{\phi}_b(1,1) = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix} 2 = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} \notin \mathcal{D},$$

$$\overline{\phi}_b(t,1) = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} t-t_1 \\ t-t_1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} t-1 \\ t-1 \end{bmatrix}, \quad \forall t \in I^1 = [1,+\infty).$$

- This kind of solution is called eventually continuous because the time domain is unbounded in the t direction only, i.e. the solution is continuous after a finite number of jumps.

- One gets

$$\mathrm{dom}\phi_\mathrm{b} = [0,1] \times \{0\} \cup [1,+\infty) \times \{1\}$$

Solution from $\phi_c(0,0)$

- $\phi_c(0,0) = (0,-1)$ is in $\mathcal{C}$
- The solution from $\phi_c(0,0) = (0,-1)$ is defined as:

$$\begin{aligned}
\phi_c(t,0) &= \phi_c(0,0) + \begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t-1 \end{bmatrix}, \quad \forall t \in I^0 = [0,1] \\
\phi_c(1,0) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathcal{D}, \\
\phi_c(1,1) &= g(\phi_c(1,0)) = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix} \in \mathcal{C} \cap \mathcal{D}
\end{aligned}$$

- ▷ The solution from $\phi_c(0,0) = (0,-1)$ is in $\mathcal{C}$ (and flows) and reaches the point $(1,0)$ for $t = t_1 = 1$ and $j = 0$.
- ▷ The point $(1,0)$ is both in $\mathcal{C}$ and $\mathcal{D}$. That means that from $(1,0)$, one can flow or jump.
- ▷ Then, we can consider two cases:

- Case 1: Eventually continuous: the solution $\phi_c(t,j)$ keeps flowing forever after a finite amount of jumps (i.e after $(t,j)=(1,0)$)

$$\phi_c(t,1) = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix} + \begin{bmatrix} t-1 \\ t-1 \end{bmatrix}, \quad \forall t \in I^1 = [1,+\infty).$$

  ▷ One gets

$$\mathrm{dom}\phi_c = [0,1]\times\{0\} \cup [1,+\infty)\times\{1\}$$

- Case 2: Eventually discrete: the solution $\overline{\phi}_c(t,j)$ keeps jumping forever after a finite amount of flow (i.e after $(t,j)=(1,0)$)

$$\overline{\phi}_c(1,2) = g(\overline{\phi}_c(1,1)) = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix} 3/4 \in \mathcal{D},$$

$$\overline{\phi}_c(1,3) = g(\overline{\phi}_c(1,2)) = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix} (3/4)^2 \in \mathcal{D},$$

- Case 2 (continued):

$$\overline{\phi}_c(1,j) = g(\overline{\phi}_c(1,j-1)) = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix} (3/4)^{j-1} \in \mathcal{D}, \ \forall j \geq 1.$$

  - ▷ Moreover, it can be proved that $\lim_{j\to\infty} \overline{\phi}_c(1,j) = 0$, since $\frac{3}{4} < 1$. In this case, the hybrid time domain is unbounded in the j direction.
  - ▷ One gets

$$\mathrm{dom}\phi_{\mathrm{c}} = [0,1] \times \{0\} \cup \{1\} \times \mathbb{N}$$

Solution from $\phi_d(0,0)$

- $\phi_d(0,0) = (-1,-1)$ is in $\mathcal{C}$
- The solution from $\phi_d(0,0) = (-1,-1)$ is defined as:

$$
\begin{aligned}
\phi_d(t,0) &= \phi_d(0,0) + \begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t-1 \\ t-1 \end{bmatrix}, \quad \forall t \in I^0 = [0,1] \\
\phi_d(1,0) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathcal{C} \cap \mathcal{D} \\
\phi_d(1,j) &= g(\phi_d(1,j-1)) = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix} 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \forall j \geq 1
\end{aligned}
$$

▷ The solution from $\phi_d(0,0) = (-1,-1)$ is in $\mathcal{C}$ (and flows) and reaches the point $(0,0)$ for $t = t_1 = 1$ and $j = 0$.

▷ The point $(0,0)$ is both in $\mathcal{C}$ and $\mathcal{D}$.

▷ However, the solution is not allowed to flow and keeps jumping forever: solution eventually discrete.

▷ One gets

$$\mathrm{dom}\phi_\mathrm{d} = [0,1] \times \{0\} \cup \{1\} \times \mathbb{N}$$

Solution from $\phi_e(0,0)$

- $\phi_e(0,0) = (-2,-1)$ is in $\mathcal{C}$
- The solution from $\phi_e(0,0) = (-2,-1)$ is defined as:

$$
\begin{aligned}
\phi_e(t,0) &= \phi_e(0,0) + \begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t-2 \\ t-1 \end{bmatrix}, \quad \forall t \in I^0 = [0,1], \\
\phi_e(1,1) &= g(\phi_e(1,0)) = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix} (-1) = \begin{bmatrix} -3/4 \\ -1/4 \end{bmatrix} \in \mathcal{C}, \\
\phi_e(t,1) &= \phi_e(1,1) + \begin{bmatrix} t-t_1 \\ t-t_1 \end{bmatrix} = \begin{bmatrix} -3/4 \\ -1/4 \end{bmatrix} + \begin{bmatrix} t-1 \\ t-1 \end{bmatrix} \\
&= \begin{bmatrix} t-7/4 \\ t-5/4 \end{bmatrix}, \\
&\quad \forall t \in I^1 = [t_1, t_2] = [t_1, t_1 + 1/4] = [1, 5/4] = [1, 1.25], \\
\phi_e(1.25,2) &= g(\phi_e(1.25,1)) = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix} (-1/2) = \begin{bmatrix} -3/8 \\ -1/8 \end{bmatrix} \in \mathcal{C}. \\
&\vdots
\end{aligned}
$$

Solution from $\phi_e(0,0)$

$$
\begin{aligned}
& \vdots \\
\phi_e(t,2) &= \phi_e(t_2,2) + \begin{bmatrix} t-t_2 \\ t-t_2 \end{bmatrix} = \begin{bmatrix} -3/8 \\ -1/8 \end{bmatrix} + \begin{bmatrix} t-t_2 \\ t-t_2 \end{bmatrix} \\
&= \begin{bmatrix} t-t_2-3/8 \\ t-t_2-1/8 \end{bmatrix} \\
&\qquad \forall t \in I^2 = [t_2, t_3] = [t_2, t_2+1/8] = [1.25, 1.375], \\
\phi_e(t_3,2) &= \phi_e(t_2+1/8, 2) = \begin{bmatrix} -1/4 \\ 0 \end{bmatrix} \\
\phi_e(t_3,3) &= g(\phi_e(t_3,2)) = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix} (-1/4) = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix} (-1/2^2) \in \mathcal{C}, \\
& \vdots \\
\phi_e(t_j,j) &= g(\phi_e(t_j,j-1)) = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix} (-1/2^{j-1}) \in \mathcal{C}, \\
\phi_e(t,j) &= \begin{bmatrix} -3/2^{j+1} \\ -1/2^{j+1} \end{bmatrix} + \begin{bmatrix} t-t_j \\ t-t_j \end{bmatrix}, \quad \forall t \in I^j = [t_j, t_{j+1}] = [t_j, t_j + 1/2^{j+1}]
\end{aligned}
$$

- The solution from $\phi_e(0,0) = (-2,-1)$ is in $\mathcal{C}$ (and flows) and reaches the point $\phi_e(1,0) = (-1,0)$ for $t = t_1 = 1$ and $j = 0$.
- From the point $\phi_e(1,0) = (-1,0)$ the solution jumps to $\phi_e(1,1)$ and will flow again (for a short amount of time $t \in [1, 5/4] = [1, 1.25]$) and then jumps.
  - ▷ Then, the solution is constituted by an alternation of flows and jumps until the time interval of flow gradually shrinks to zero and the solution jumps forever.
- One gets

$$\mathrm{dom}\phi_{\mathrm{e}} = [0,1] \times \{0\} \cup [1, 1.25] \times \{1\} \cup [1.25, 11/8] \times \{2\} \cup \ldots$$

- One can compute the amount of time t^* taken by the solution to reach the origin, which is the sum of all the small intervals of flow in between jumps and it is the result of the geometrical series

$$t^* = \sum_{i=0}^{+\infty} \frac{1}{2^i}$$

  - ▷ It can be proved that t^* is finite
  - ▷ t^* represents the time after which the solution is not allowed to flow anymore and keeps jumping forever.
  - ▷ No times $t \geq t^*$ are found in $\mathtt{dom}\phi_e$
- This kind of solutions is called Zeno solution.

Solution from $\phi_f(0,0)$

- $\phi_f(0,0) = (0,0)$ is in $\mathcal{C} \cap \mathcal{D}$
- The solution from $\phi_f(0,0) = (0,0)$ is defined as:

$$
\begin{aligned}
\phi_f(t,0) &= \phi_f(0,0) + \begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}, \quad \forall t \in [0, +\infty) \\
\phi_f(0,0) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathcal{C} \cap \mathcal{D} \\
\phi_f(0,j) &= g(\phi_f(0,j-1)) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; \forall j \geq 1
\end{aligned}
$$

- ▷ The point $(0,0)$ is both in $\mathcal{C}$ and $\mathcal{D}$.
- ▷ The point $(0,0)$ is an equilibrium point
- ▷ However, the solution is not allowed to flow and keeps jumping forever: Discrete solution
- ▷ One gets

$$\mathrm{dom}\phi_{\mathrm{f}} = \{0\} \times \mathbb{N}$$

- Solutions from different initial conditions. The red region represents the jump set $\mathcal{D}$, while the green one is the flow set $\mathcal{C}$. The dashed gray line is the set $g(\mathcal{D})$.

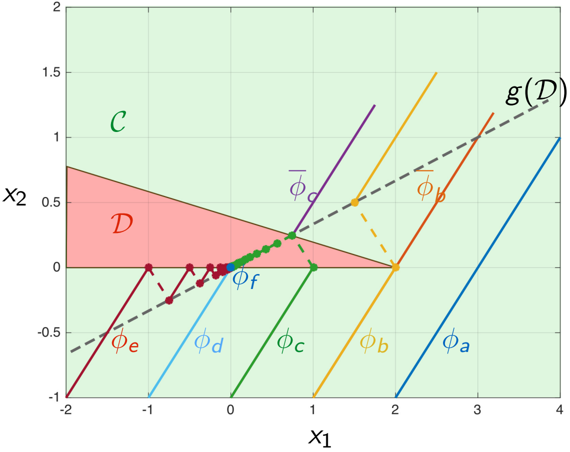


Figure: Solutions from different initial conditions in the phase plane.

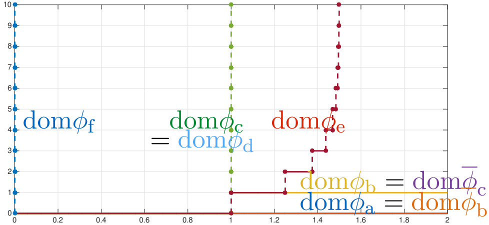


Figure: Hybrid time domains.

Some comments on this example

- Note that all the solutions in the previous example evolve forever, namely they keep indefinitely jumping or flowing (or both).
- This feature is clearly associated to the fact that their domain is unbounded (in the t direction or in the j direction, or both, as in the sample-and-hold example).
- A solution with unbounded domain will be called forever solution, the classical name for this feature is completeness (item in Definition 4), as recalled below.

Definition 5

A solution ϕ is complete if $\mathrm{dom}\phi$ is unbounded.

- In other words, a solution is complete if it evolves forever.
- Let us recall that in Definition 3 we defined a solution as a function ϕ that satisfies some properties on a specific hybrid time domain, without any restriction on the shape of such a domain.
  - ▷ This allows to take into account solutions with bounded time domains, i.e. non complete ones.

- An important concept is that one of maximality of a solution, stemming from the intention to propagate further a solution as long as this is possible.

Definition 6

A solution ϕ is *maximal* if there is no other solution ψ which is a continuation of ϕ, such that

$$\mathrm{dom}\phi \subset \mathrm{dom}\psi \quad \text{(strictly)}$$

and

$$\psi(t,j) = \phi(t,j) \ , \ \forall (t,j) \in \mathrm{dom}\phi.$$

- A complete solution is of course maximal, but the converse implication does not hold.
- Example: Consider a solution ϕ that jumps outside $\mathcal{C} \cup \mathcal{D}$:
  - ▷ This means that ϕ cannot be extended, therefore it is maximal,
  - ▷ But at the same time it is not allowed to evolve forever,
  - ▷ Hence it is not complete.

Example for which all maximal solutions are not complete

- Consider the following hybrid dynamical system with scalar state $x \in \mathbb{R}$:

$$\begin{aligned} \dot{x} &= f(x) := x^2, & x \in \mathcal{C} &:= \{x \in \mathbb{R} : |x| \geq 1\}, \\ x^+ &= g(x) := x - 1, & x \in \mathcal{D} &:= [0, 1]. \end{aligned} \tag{6}$$

- Consider the initial condition $\phi_a(0, 0) = 1$, which is in $\mathcal{C} \cap \mathcal{D}$.
- Clearly two solutions start from $\phi_a(0, 0) = 1$
  - ▷ A flowing solution
  - ▷ A jumping solution

- Case 1. Flowing solution. Compute the solution from $\phi_a(0,0) = 1$:

$$\phi_a(t,0) = (\phi_a(0,0)^{-1} - t)^{-1} = (1-t)^{-1} = \frac{1}{1-t}, \quad \forall t \in [0,1).$$

  ▷ This is a purely continuous solution, but its time domain is finite, since it explodes to $+\infty$ at $t = 1$.

  ▷ One gets:

$$\mathrm{dom}\phi_{\mathrm{a}} = [0,1) \times \{0\}$$

  ▷ Such a phenomenon is called *finite escape time* and causes the solution to be maximal but not complete.

  - Note that this solution is not compact, because its time domain is not compact: it is bounded but not closed.

- Case 2. Jumping solution. Now consider the same initial condition $\phi_b(0,0) = 1$, which is in $\mathcal{C} \cap \mathcal{D}$.
- Assume that the solution jumps:

$$\phi_b(0,1) = 0 \in \mathcal{D},$$
$$\phi_b(0,2) = -1 \in \mathcal{C}.$$

- One gets

$$\begin{array}{rcl} \phi_b(t,0) & = & \frac{1}{1-t}, \quad \forall t \in [0,1) \\ \phi_b(0,0) & = & 1 \in \mathcal{D} \\ \phi_b(0,1) & = & 0 \in \mathcal{D} \\ \phi_b(0,2) & = & -1 \in \mathcal{C} \end{array}$$

  and

$$\mathrm{dom}\phi_\mathrm{b} = \{0\} \times \{0,1,2\}$$

- At $x = -1$, the vector field $f(-1)$ points to the right and the solution is not allowed to flow anymore, otherwise it would escape the set $\mathcal{C}$.
- Again, ϕ_b is maximal but not complete.
- Note also that this solution is compact because it has a compact time domain. In addition, this solution is also discrete.

- Consider the initial condition $\phi_c(0,0) = -2$, which is in $\mathcal{C}$.
- One gets

$$\begin{array}{rcl} \phi_c(t,0) & = & (\phi_c(0,0)^{-1} - t)^{-1} = (-\frac{1}{2} - t)^{-1} = -\frac{2}{1+2t}, \\ & \Rightarrow & \phi_c(t,0) \in \mathcal{C} \quad \forall t \in [0, \frac{1}{2}] \\ \phi_c(\frac{1}{2},0) & = & -1 \notin \mathcal{D} \end{array}$$

- One gets:

$$\mathrm{dom}\phi_{\mathrm{c}} = [0, \frac{1}{2}] \times \{0\}$$

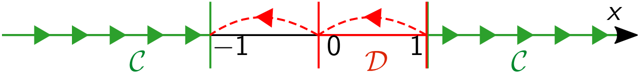


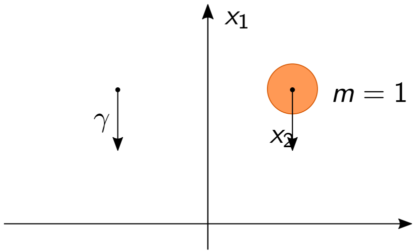
Figure: Flow and jump sets with direction of the flow map $f(x)$ and the jump map $g(x)$.

Suggestion - Example 3: bouncing ball

- Study of the solutions of the bouncing ball (cf Lecture Part I - Introduction)
- Recall that the system is defined as follows:

$$\dot{x} = f(x) = \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix} \text{ if } x \in \mathcal{C} = \{x \in \Re^2; x_1 \geq 0\}$$

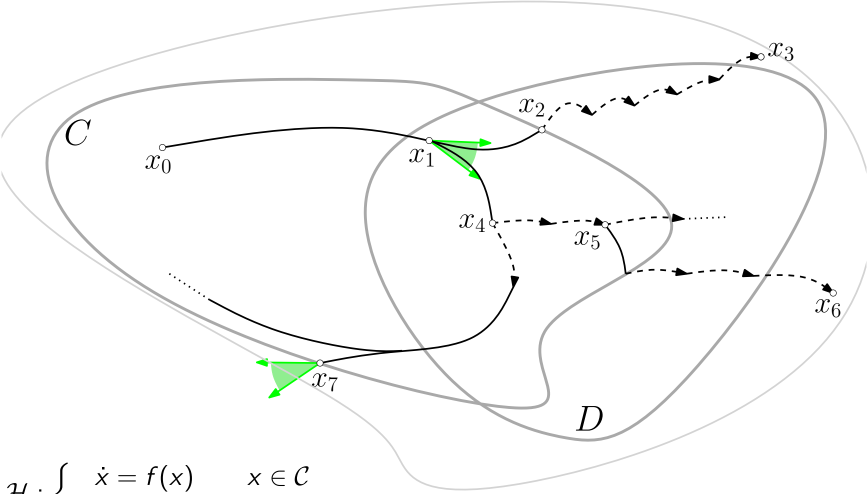
$$x^+ = g(x) = \begin{bmatrix} 0 \\ -\lambda x_2 \end{bmatrix} \text{ if } x \in \mathcal{D} = \{x \in \Re^2; x_1 = 0 \text{ and } x_2 \leq 0\}$$



- Consider the hybrid dynamical system

$$\begin{cases} \dot{x} &= f(x), \quad x \in \mathcal{C} \\ x^+ &= g(x), \quad x \in \mathcal{D}. \end{cases} \tag{7}$$

- In some cases a solution to (7) is not allowed to neither flow nor jump:
  - ▷ For example when the state flows or jumps outside $\mathcal{C} \cup \mathcal{D}$.



$$\mathcal{H}: \begin{cases} \dot{x} = f(x) & x \in \mathcal{C} \\ x^{+} = g(x) & x \in \mathcal{D} \end{cases}$$

- One can note the following remarks:
  - ▷ An useful property of the corresponding description is when the union of the flow and the jump sets coincides with the whole state space:

$$\mathcal{C} \cup \mathcal{D} = \mathbb{R}^n$$

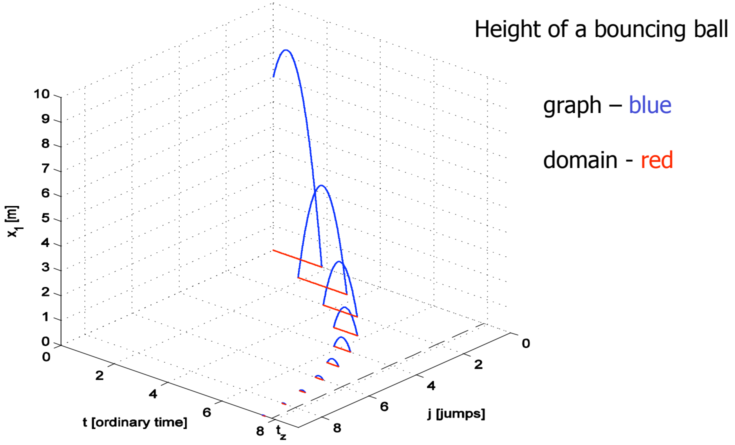
  - ▷ So that a jump or a flow rule will be available for any initial condition (existence of solution is guaranteed), even though this is not yet sufficient to guarantee completeness.
  - ▷ Solutions may terminate prematurely if $\mathcal{C} \cup \mathcal{D} \neq \mathbb{R}^n$
  - ▷ When $\mathcal{C}$ and $\mathcal{D}$ overlap, one has to define priority (Lab-class): jump, flow or random between flow and jump

Existence of solutions

- Important issue is the existence of solutions.
- Recall, from Definition 4, that a nontrivial solution is a solution whose domain has at least two points, namely some evolution happens, continuous or discrete, or both.
  - ▷ All of the solutions discussed in the previous examples are nontrivial.
- Importance of the selection of $\mathcal{C}$ and $\mathcal{D}$. A selection of closed jump and flow sets $\mathcal{C}$ and $\mathcal{D}$ and continuous functions f and g is that nontrivial solutions exist for any initial condition in $\mathcal{C} \cup \mathcal{D}$ as long as the so-called viability condition holds at points in $\mathcal{C} \setminus \mathcal{D}$.
  - ▷ This viability condition essentially means that the flow map does not point outside the flow set.
  - ▷ Such a viability condition does not hold in the last example at point $x = -1 \in \mathcal{C}$, but not in $\mathcal{D}$.

- We have studied an important notions (see Chapter 2 in the book of Goebel et al, for more details and examples):
  - ▷ Hybrid time: (t, j)
  - ▷ Solution: The existence of solutions for hybrid dynamical systems has been addressed in many different ways in the recent literature.
  - ▷ Well-posedness: A system is said to be well-posed if a solution of the system exists and is unique given an initial condition
  - ▷ One of the main issues is to avoid Zeno solutions
    - ▷ correspond to solutions that jump infinitely many times in a bounded ordinary time interval
    - ▷ and characterize a peculiar situation where the hybrid time domain is unbounded in the jump direction (j) while it is thereby remaining bounded in the flow direction (t).

- Example of Zeno solutions : bouncing ball.
- Zeno solutions correspond to t finite and j infinite



- To avoid the presence of Zeno solutions, dwell-time logics are enforced on the solutions: conditions preventing consecutive jumps that are too close to each other.
  - ▷ A dwell time is a duration of time without any jump.
  - ▷ Each pair of jump times will then be not closer than ρ to each other (namely, $|t_i - t_j| > \rho$ for all $i \neq j$), so that it will be impossible to have Zeno solutions as they would require infinitely many jumps in a bounded time interval.
  - ▷ To enforce a desirable dwell time between each pair of consecutive jumps of the hybrid control systems, we augment the state with a timer (or dwell-time logic): $\tau \in \mathbb{R}$.
- Main sources for this lecture: Chapter 2 of the book of Teel, notes of Luca Zaccarian, Christophe Prieur, Francesco Ferrante, Ricardo Sanfelice, ST