Generating continuous Random variables

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The Inverse transform algorithm

- *Proposition*: Let $U \sim U(0,1)$. For any continuous *CDF* F , the random variable X defined by
  $X = F^{-1}(U)$
  has distribution F.
- *proof* in class.
- So long as we can derive the explicit form for F^{-1} , we can use this result to generate from a continuous distribution with CDF F:
  step 1 : generate a random number U.
  step 2 : set $X = F^{-1}(U)$ and you are done.

proof

$X = F^{-1}(U)$ has a cdf of F.
Look at cdf

$$\begin{aligned} P(X \leq x) = F_x(X) &= P(F(F^{-1}(U)) \leq F(x)) \\ &= P(U \leq F(x)) \\ &= F(x) \end{aligned}$$

therefore $F(x)$ is the cdf of $F^{-1}(U)$.

Some examples

- Let X come from a distribution with CDF
  $F(x) = x^n, 0 < x < 1.$
  We can generate from this distribution by setting $X = U^{1/n}$ after generating U.
- *Exponential* Suppose X comes from an $Exp(\lambda)$ with PDF
  $f(x) = \lambda e^{-\lambda x}, x > 0.$
  We can generate from this distribution by setting $X = -\frac{1}{\lambda} log U$ where U is the generated random number.
- *Weibull* A random variable X is said to have a Weibull(α, β) distribution if its CDF has the form
  $F(x) = 1 - exp(-\alpha x^{\beta}), x > 0.$
  Describe a method to generate from this distribution.

Generating a Gamma random variable

- For a Gamma(n, λ) random variable, note that because we cannot write an explicit form for the expression of the F^{-1} , it is diffcult to directly apply inverse transform method.
- However, recall that the sum of independent Exponentials leads us to a Gamma distribution.
- We can generate from a Gamma distribution by generating n random numbers $U_1, U_2, \cdots, U_n$ and then setting
$$\begin{aligned} X &= -\frac{1}{\lambda}\log U_1 - \cdots - \frac{1}{\lambda}\log U_n \\ &= -\frac{1}{\lambda}\log(U_1 \cdots U_n). \end{aligned}$$
- This works provided n is a positive integer.

The rejection method

- Suppose we wish to generate X from a distribution with PDF $f(x)$.
- Assume that we are able to generate Y from a distribution with PDF $g(y)$ and that there is a constant c such that
  $\frac{f(y)}{g(y)} \leq c$, for all y.
- According to the rejection method, we can generate X using the following steps:

  step 1 : generate Y from distribution with density g.

  step 2 : generate a random number U.

  step 3 : if $U \leq \frac{f(Y)}{cg(Y)}$, set $X = Y$.

  step 4 : else return to step 1.

Important results

$X \sim U(a,b)\ , \quad a < b$

use inverse transform method

$F(x) = \int_a^x \frac{1}{b-a} dz = \frac{x-a}{b-a}$

- Generate a random number U
- $\frac{x-a}{b-a} = U \Longrightarrow X = a + U(b-a).$

  $$\begin{aligned} Examples: \quad & X \sim U(0, 2\pi) \Longrightarrow X = 2\pi U \\ & X \sim U(-1, 1) \Longrightarrow X = 2U - 1. \end{aligned}$$

–continued

Theorem :

- The random variable X generated by the rejection method has density f .
- The number of iterations required for the rejection algorithm has a geometric distribution with mean c.

proof : to be discussed in class.

Some examples

- $Example5d$: Use the rejection method to generate from $f(x) = 20x(1-x)^3$, $0 < x < 1$.
- $Example5e$: Use the rejection method to generate from a Gamma(shape para=3/2, scale = 1) density with
  $f(x) = Kx^{1/2}e^{-x}$, $x > 0$.
  where $K = 1/\Gamma(3/2) = 2/\sqrt{\pi}$.