# MA 226 - Assignment Report 9

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Consider the multivariate normal,

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathcal{N}(\mu, \Sigma),$$

$$\text{where } \mu = \begin{pmatrix} 5 \\ 8 \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} 1 & 2a \\ 2a & 4 \end{pmatrix}.$$

Q 1. For the cases a = -0.25, 0, 0.25, generate 1000 values of $X$ and calculate sample means, sample variances and sample correlations. Make empirical contour plots based on above generated samples.

Q 2. Also, plot the actual and empirical marginal cdfs of $X_1$ and $X_2$.

Q 3. Let us recall generating a bivariate normal with the help of conditional distributions. Suppose that $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2), X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ and the conditional distribution of $X_2$ given $X_1 = x$ is $\mathcal{N}(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1), \sigma_2^2(1 - \rho^2))$ where $|\rho| < 1$ is the correlation coefficient between $X_1$ and $X_2$. The vector $(X_1, X_2)$ is said to have a bivariate normal distribution. Simulate the vector for a particular set of parameter values, using this idea of conditional distributions. Estimate the sample quantities (mean, etc.) and compare with actual values.
Take same $\mu_1$, $\mu_2$, $\sigma_1$, $\sigma_2$ and $\rho$.

**Solution**:

Generating the multivariate normal,

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathcal{N}(\mu, \Sigma),$$
$$\text{where } \mu = \begin{pmatrix} 5 \\ 8 \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} 1 & 2a \\ 2a & 4 \end{pmatrix}.$$

Generation by Cholesky's decomposition :

$$A = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{(1-\rho^2)}\sigma_2 \end{pmatrix}, \text{ where } AA^T = \Sigma.$$
$$\Rightarrow X = \mu + AZ$$

**Algorithm 1** Generation by Cholesky's decomposition.

1: Generate two independent $Z_1, Z_2 \sim \mathcal{N}(0,1)$.
2: First generate $X_1 = \mu_1 + \sigma_1 Z_1$.
3: Then generate $X_2 = \mu_2 + \rho\sigma_2 Z_1 + \sqrt{(1-\rho^2)}\sigma_2 Z_2$.

Generation from conditional distribution :

In bivariate set-up,

$$(X_2|X_1 = x) \sim \mathcal{N}(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1), \sigma_2^2(1-\rho^2)), \text{ where } X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2).$$

**Algorithm 2** Generation from conditional distribution.

1: Generate two independent $Z_1, Z_2 \sim \mathcal{N}(0,1)$.
2: First generate $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$,
   i.e. set $X_1 = \mu_1 + \sigma_1 Z_1$.
3: Then generate $(X_2|X_1 = x) \sim \mathcal{N}(\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1), \sigma_2^2(1-\rho^2))$,
   i.e. set $X_2 = \mu^* + \sigma^* Z_2$, where $\mu^* = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)$ and $\sigma^* = \sigma_2\sqrt{(1-\rho^2)}$.

Code for R

```r
library(MASS)  #For kde2d and mvrnorm

#Question-1:
gen_cholesky <- function(mu, SIGMA, sample) {
   U <- matrix(runif((2 * sample), 0, 1), nrow = sample, ncol = 2);
#  u1 <- runif(sample, 0, 1);
#  u2 <- runif(sample, 0, 1);
   R <- -2 * log(U[,1]);
   V <- 2 * pi * U[,2];
#  R <- -2 * log(u1);
#  V <- 2 * pi * u2;
   Z <- matrix(c((sqrt(R) * cos(V)), (sqrt(R) * sin(V))), nrow = sample, ncol = 2);
#  Z1 <- sqrt(R) * cos(V);
#  Z2 <- sqrt(R) * sin(V);

## Z <- mvrnorm(n = sample, c(0, 0), matrix(c(1,0,0,1), 2, 2));

   rho = (SIGMA[1,2] / sqrt(SIGMA[1,1] * SIGMA[2,2]));   #Correlation
   sigma = c(sqrt(SIGMA[1,1]), sqrt(SIGMA[2,2]));  #Standard Deviations
   X <- matrix(c((mu[1] + (sigma[1] * Z[,1])), (mu[2] + (rho * sigma[2] * Z[,1]) + (
       sqrt(1 - (rho^2)) * sigma[2] * Z[,2]))), nrow = sample, ncol = 2);

   cat("The sample means, variances, and covariance are :","\nmean(X1) =", mean(X
       [,1]), "\nmean(X2) =", mean(X[,2]), "\nvariance(X1) =", var(X[,1]), "\
       nvariance(X2) =", var(X[,2]), "\ncorrelation(X1, X2) =", cor(X[,1], X[,2]), "
       \n");
   cat("\nWhile, the actual means, variances, and covariance are :","\nmean(X1) =",
       mu[1], "\nmean(X2) =", mu[2], "\nvariance(X1) =", SIGMA[1,1], "\nvariance(X2)
        =", SIGMA[2,2], "\ncorrelation(X1, X2) =", rho, "\n");

   f <- kde2d(X[,1], X[,2], n = sample);  # Two dimensional kernel density
       approximation
   pdf(paste("1",100*rho,".pdf"));
   contour(f, xlab = "X1", ylab = "X2", main = "")
#  legend('topright', legend = paste("a =", rho), lty = 0, bty = 'n');

##Question-2 ::
   X_T <- mvrnorm(n = sample*100, mu, SIGMA);

   pdf(paste("2_X1",100*rho,".pdf"));
```

```
    plot(ecdf(sort(X[,1])), do.points = FALSE, main = "", col = "red", xlab = "", ylab = "")
    par(new = TRUE)
    plot(ecdf(sort(X_T[,1])), do.points = FALSE, main = "", col = "green", xlab = "", ylab = "", axes = FALSE)
    legend('topleft', legend = c('Experimental (Empirical)', 'Theoretical (Actual)'), lty = 1, col = c("red", "green"), bty = 'n')
#   title("Cumulative Distribution Function for X1");
    title(xlab = "x", ylab = "F(x)");

    pdf(paste("2_X2",100*rho,".pdf"));
    plot(ecdf(sort(X[,2])), do.points = FALSE, main = "", col = "red", xlab = "", ylab = "")
    par(new = TRUE)
    plot(ecdf(sort(X_T[,2])), do.points = FALSE, main = "", col = "green", xlab = "", ylab = "", axes = FALSE)
    legend('topleft', legend = c('Experimental (Empirical)', 'Theoretical (Actual)'), lty = 1, col = c("red", "green"), bty = 'n')
#   title("Cumulative Distribution Function for X2");
    title(xlab = "x", ylab = "F(x)");

}

#Question-3:
gen_conditional <- function(mu, SIGMA, sample) {
   U <- matrix(runif((2 * sample), 0, 1), nrow = sample, ncol = 2);
#  u1 <- runif(sample, 0, 1);
#  u2 <- runif(sample, 0, 1);
   R <- -2 * log(U[,1]);
   V <- 2 * pi * U[,2];
#  R <- -2 * log(u1);
#  V <- 2 * pi * u2;
   Z <- matrix(c((sqrt(R) * cos(V)), (sqrt(R) * sin(V))), nrow = sample, ncol = 2);
#  Z1 <- sqrt(R) * cos(V);
#  Z2 <- sqrt(R) * sin(V);

## Z <- mvrnorm(n = sample, c(0, 0), matrix(c(1,0,0,1), 2, 2));

    rho = (SIGMA[1,2] / sqrt(SIGMA[1,1] * SIGMA[2,2]));    #Correlation
    sigma = c(sqrt(SIGMA[1,1]), sqrt(SIGMA[2,2]));  #Standard Deviations

```

```r
  X <- matrix(0, nrow = sample, ncol = 2);
  X[,1] <- (mu[1] + (sigma[1] * Z[,1]));
  X[,2] <- ((mu[2] + (rho * (sigma[2] / sigma[1]) * (X[,1] - mu[1]))) + (sqrt(1 - (
      rho^2)) * sigma[2] * Z[,2]));

  cat("The sample means, variances, and covariance are :","\nmean(X1) =", mean(X
      [,1]), "\nmean(X2) =", mean(X[,2]), "\nvariance(X1) =", var(X[,1]), "\
      nvariance(X2) =", var(X[,2]), "\ncorrelation(X1, X2) =", cor(X[,1], X[,2]), "
      \n");
  cat("\nWhile, the actual means, variances, and covariance are :","\nmean(X1) =",
      mu[1], "\nmean(X2) =", mu[2], "\nvariance(X1) =", SIGMA[1,1], "\nvariance(X2)
       =", SIGMA[2,2], "\ncorrelation(X1, X2) =", rho, "\n");

}

### EXECUTION :::
set.seed(1);

a = c(-0.25, 0, 0.25);
mu <- c(5, 8);

cat("Generation by Cholesky s decomposition :-")
for (i in 1:3) {
  cat("\n\nCase -", i, ":: a =", a[i], "::\n");
  SIGMA <- matrix(c(1, (2 * a[i]), (2 * a[i]), 4), nrow = 2, ncol = 2);
  gen_cholesky(mu, SIGMA, 1000);
}

cat("\n\n###    ###    ###    ###    ###    ###    ###\n\n\n");

cat("Generation from conditional distribution :-")
for (i in 1:3) {
  cat("\n\nCase -", i, ":: a =", a[i], "::\n");
  SIGMA <- matrix(c(1, (2 * a[i]), (2 * a[i]), 4), nrow = 2, ncol = 2);
  gen_conditional(mu, SIGMA, 1000);
}
```

**Results**:

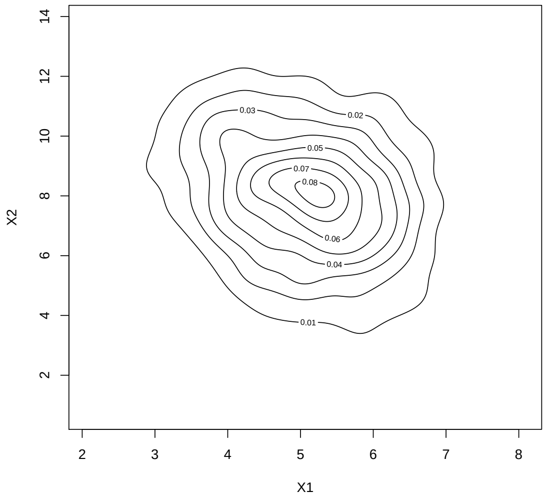
Generation by Cholesky's decomposition :-

| a | -0.25 | | 0 | | 0.25 | |
|---|---|---|---|---|---|---|
| Values | Sample | Actual | Sample | Actual | Sample | Actual |
| mean($X_1$) | 5.042667 | 5 | 5.010323 | 5 | 5.02068 | 5 |
| mean($X_2$) | 7.984635 | 8 | 8.075212 | 8 | 7.937095 | 8 |
| variance($X_1$) | 0.9423585 | 1 | 0.9465603 | 1 | 0.9758604 | 1 |
| variance($X_2$) | 4.137109 | 4 | 3.835417 | 4 | 4.158121 | 4 |
| correlation($X_1$, $X_2$) | -0.2542622 | -0.25 | -0.0217607 | 0 | 0.2661262 | 0.25 |

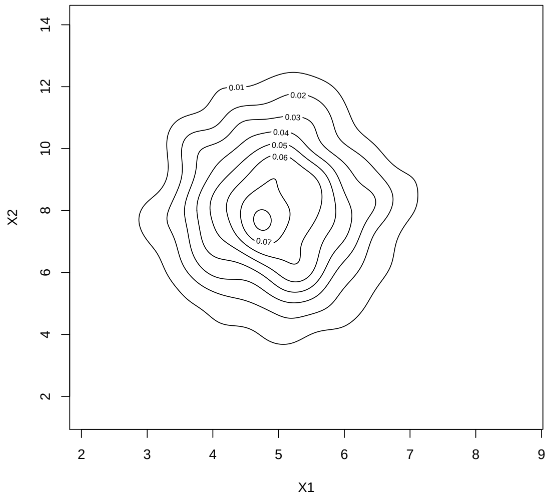
Generation from conditional distribution :-

| a | -0.25 | | 0 | | 0.25 | |
|---|---|---|---|---|---|---|
| Values | Sample | Actual | Sample | Actual | Sample | Actual |
| mean($X_1$) | 5.020026 | 5 | 4.98154 | 5 | 5.031342 | 5 |
| mean($X_2$) | 7.9408 | 8 | 7.97478 | 8 | 8.078837 | 8 |
| variance($X_1$) | 0.9632567 | 1 | 0.9585365 | 1 | 1.031422 | 1 |
| variance($X_2$) | 3.82001 | 4 | 4.102417 | 4 | 4.321624 | 4 |
| correlation($X_1$, $X_2$) | -0.2120682 | -0.25 | -0.01003784 | 0 | 0.2554145 | 0.25 |

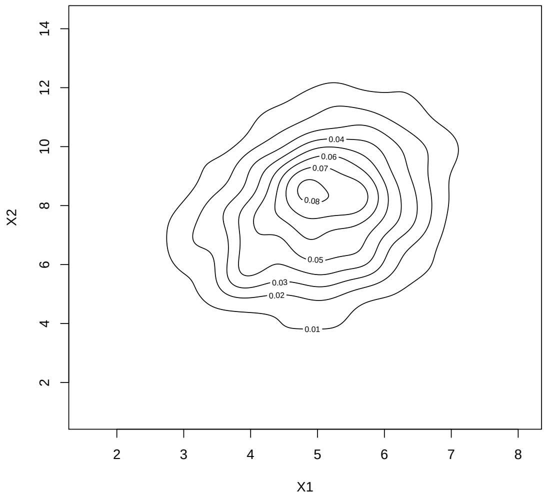
The sample values are close to the theoretical ones, in all cases of both the methods (i.e. generation by Cholesky's decomposition and generation from conditional distribution).




(**a**) *a = -0.25*

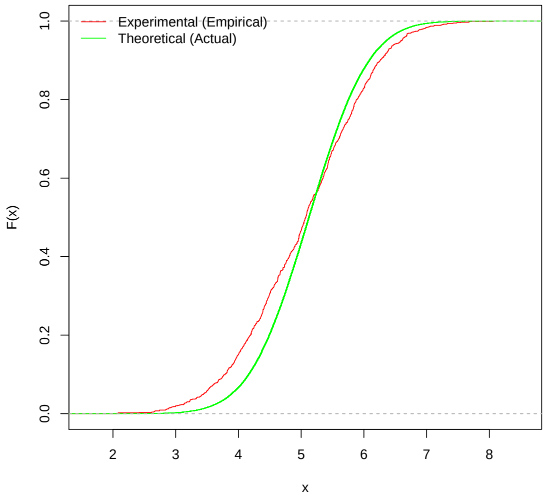



(**b**) *a = 0*

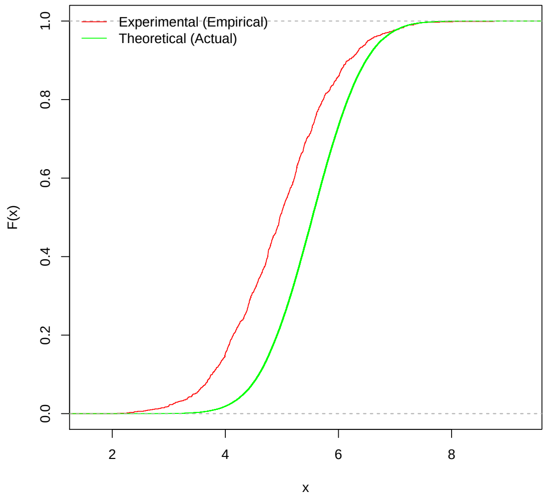



(**c**) *a = 0.25*

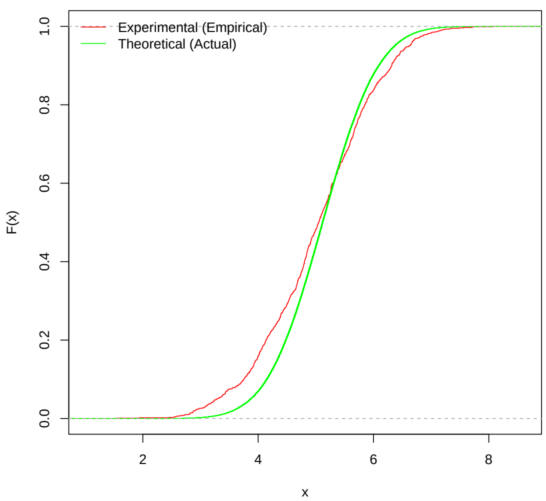
**Figure 1:** *Empirical contour plots*




**(a)** *a = -0.25*

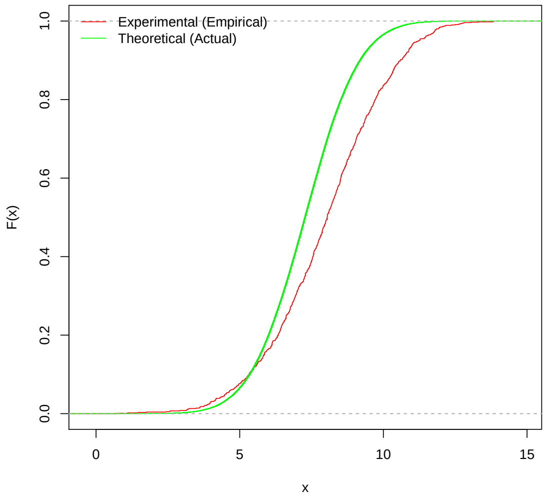



**(b)** *a = 0*

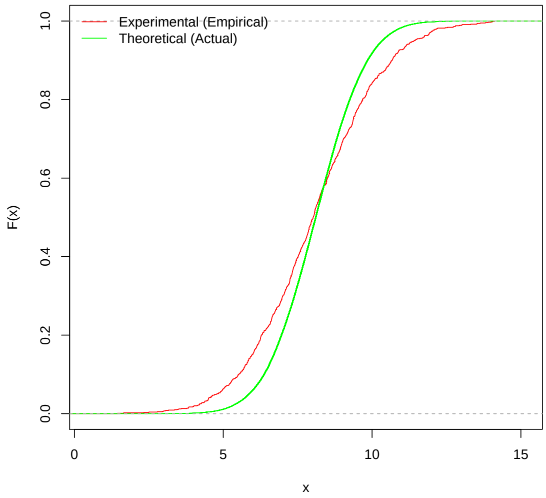



**(c)** *a = 0.25*

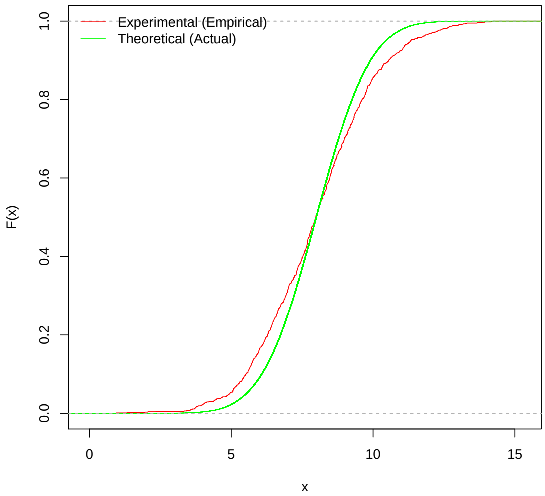
**Figure 2:** *Plots of marginal Cumulative Distribution Function of* $X_1$




**(a)** *a = -0.25*




**(b)** *a = 0*




**(c)** *a = 0.25*

**Figure 3:** *Plots of marginal Cumulative Distribution Function of* $X_2$