MA 226 - Assignment Report 8

Ayush Sharma

150123046

24 March 2017

Q 1. Use the following Monte Carlo estimator to approximate the expected value

$$I = E(exp(\sqrt{U}))$$

where $U \sim \mathcal{U}(0,1)$: $I_M = \frac{1}{M}\sum_{i=1}^{M} Y_i$, where $Y_i = exp(\sqrt{U_i})$ with $U_i \sim \mathcal{U}(0,1)$.
Take all values of M to be 10^2, 10^3, 10^4 and 10^5. Determine the 95% confidence interval for I_M for all the four values of M that you have taken.

Q 2. Repeat the above exercise using antithetic variates via the following estimator and calculate the percentage of variance reduction:

$$\hat{I}_M = \frac{1}{M}\sum_{i=1}^{M} \hat{Y}_i$$

where

$$\hat{Y}_i = \frac{exp(\sqrt{U_i}) + exp(\sqrt{1-U_i})}{2}$$

with $U_i \sim \mathcal{U}(0,1)$.

Q 3. Use $\sqrt{U}$ to construct control variate estimate and repeat the above exercise. Calculate the percentage of variance reduction.

Solution:

Approximating $z_{\frac{\alpha}{2}}$ for 95% confidence interval :

$100(1-\alpha) = 95 \Rightarrow \alpha = 1 - \frac{95}{100} = 0.05$

$\Rightarrow P(X \geq z_{\frac{\alpha}{2}}) = \frac{\alpha}{2}$, where $X \sim \mathcal{N}(0,1)$

$\Rightarrow P(X \leq z_{\frac{\alpha}{2}}) = 1 - \frac{\alpha}{2}$

$\Rightarrow z_{\frac{\alpha}{2}} = quantile_{normal}((1-\frac{\alpha}{2}), \mu = 0, \sigma = 1)$.

The approximate 95% confidence interval is given by

$$(I_M - z_{\frac{\alpha}{2}} \frac{S_M}{\sqrt{M}}, I_M + z_{\frac{\alpha}{2}} \frac{S_M}{\sqrt{M}})$$

where S_M^2 is usual estimate of the variance of Y based on the simulated values $Y_1, \cdots, Y_M$.

Using $\sqrt{U}$ to construct control variate estimate :

$X_i = exp(\sqrt{U_i})$ and $Y_i = \sqrt{U_i}$ with $U_i \sim \mathcal{U}(0,1)$.

Then,

$$W = X + \hat{c}(Y - \mu_Y)$$

is a variance reduced unbiased estimator of $I = E(exp(\sqrt{U}))$ where $\hat{c} = -\frac{Cov(X,Y)}{Var(Y)}$.

Code for R

```
##Question-1:
f1 <- function(m) {
   set.seed(1);
   U <- runif(m,0,1);
   Y <- exp(sqrt(U));
   return (c(mean(Y),sqrt(var(Y))));
}

I1 <- vector(length = 4);
S1 <- vector(length = 4);
Radius1 <- vector(length = 4);

##Same for all questions.
percentage = 95;
alpha = 1 - percentage/100;
p = (1 - (alpha/2));
z = qnorm(p, mean = 0, sd = 1, lower.tail = TRUE, log.p = FALSE);

for (i in 2:5) {
   Out <- f1(10^i);
   I1[i-1] = Out[1]; S1[i-1] = Out[2];
   Radius1[i-1] = (z * (S1[i-1] / sqrt(10^i)));
}

cat("Using Standard Monte Carlo simulation algorithm ::\n");

for (i in 2:5) {
   cat("The 95% confidence interval for", 10^i, "values is (", (I1[i-1] - Radius1[i-1]), ",", (I1[i-1] + Radius1[i-1]), ") .\n");
}

##Question-2:
f2 <- function(m) {
   set.seed(1);
   U <- runif(m,0,1);
   Y <- ((exp(sqrt(U)) + exp(sqrt(1 - U))) / 2);
   return (c(mean(Y),sqrt(var(Y))));
}

I2 <- vector(length = 4);
```

```
S2 <- vector(length = 4);
Radius2 <- vector(length = 4);
for (i in 2:5) {
    Out <- f2(10^i);
    I2[i-1] = Out[1]; S2[i-1] = Out[2];
    Radius2[i-1] = (z * (S2[i-1] / sqrt(10^i)));
}

cat("\nUsing antithetic variates ::\n");

for (i in 2:5) {
    cat("The 95% confidence interval for", 10^i, "values is (", (I2[i-1] - Radius2[i
        -1]), ",", (I2[i-1] + Radius2[i-1]), ") .\n");
}

for (i in 2:5) {
    cat("The percentage of variance rejection from standard method for", 10^i, "
        values is", (((S1[i-1]^2) - (S2[i-1]^2))/(S1[i-1]^2)) * 100, "% .\n");
}

##Question-3:
f3 <- function(m) {
    set.seed(1);
    U <- runif(m,0,1);
    Y <- sqrt(U);
    X <- exp(Y);
    c = -(cov(X,Y)/var(Y));
    mu_y = mean(Y);
    W <- (X + c * (Y - mu_y));
    return (c(mean(W),sqrt(var(W))));
}

I3 <- vector(length = 4);
S3 <- vector(length = 4);
Radius3 <- vector(length = 4);
for (i in 2:5) {
    Out <- f3(10^i);
    I3[i-1] = Out[1]; S3[i-1] = Out[2];
    Radius3[i-1] = (z * (S3[i-1] / sqrt(10^i)));
}

```

```
cat("\nUsing control variates ::\n");

for (i in 2:5) {
    cat("The 95% confidence interval for", 10^i, "values is (", (I3[i-1] - Radius3[i-1]), ",", (I3[i-1] + Radius3[i-1]), ") .\n");
}

for (i in 2:5) {
    cat("The percentage of variance rejection from standard method for", 10^i, "values is", (((S1[i-1]^2) - (S3[i-1]^2))/(S1[i-1]^2)) * 100, "% .\n");
}
```

Results:

Using Standard Monte Carlo simulation algorithm ::

The 95% confidence interval for 100 values is (1.953464 , 2.111522) .

The 95% confidence interval for 1000 values is (1.973193 , 2.027598) .

The 95% confidence interval for 10000 values is (1.990987 , 2.008426) .

The 95% confidence interval for 1e+05 values is (1.996456 , 2.001941) .

Using antithetic variates ::

The 95% confidence interval for 100 values is (2.000865 , 2.011191) .

The 95% confidence interval for 1000 values is (1.998007 , 2.002072) .

The 95% confidence interval for 10000 values is (1.998749 , 2.000047) .

The 95% confidence interval for 1e+05 values is (1.999591 , 1.999999) .

The percentage of variance rejection from standard method for 100 values is 99.57316 % .

The percentage of variance rejection from standard method for 1000 values is 99.44168 % .

The percentage of variance rejection from standard method for 10000 values is 99.44611 % .

The percentage of variance rejection from standard method for 1e+05 values is 99.44644 % .

Using control variates ::

The 95% confidence interval for 100 values is (2.023643 , 2.041343) .

The 95% confidence interval for 1000 values is (1.997195 , 2.003596) .

The 95% confidence interval for 10000 values is (1.99867 , 2.000744) .

The 95% confidence interval for 1e+05 values is (1.998875 , 1.999522) .

The percentage of variance rejection from standard method for 100 values is 98.74582 % .

The percentage of variance rejection from standard method for 1000 values is 98.616 % .

The percentage of variance rejection from standard method for 10000 values is 98.5857 % .

The percentage of variance rejection from standard method for 1e+05 values is 98.60733 % .

The use of **antithetic variates is found to be better than control variates** because of the simple reason that the percentage of variance reduction is more in case of antithetic variates than control variate.