MA 226 - Assignment Report 6

Ayush Sharma
150123046

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Q 1 Use the Box-Muller method and Marsaglia-Bray method to do the following :

(a) Generate a sample of 100, 500 and 10000 values from $\mathcal{N}(0,1)$. Hence find the sample mean and variance.

(b) Draw histogram in all cases.

Q 2 Now use the above generated values to generated samples from $\mathcal{N}(\mu = 0, \sigma^2 = 5)$ and $\mathcal{N}(\mu = 5, \sigma^2 = 5)$. Hence plot the empirical (from sample with size 500) distribution function and theoretical distribution function in the same plot.

(Use R / you should also try making the step function in C).

Q 3 Keep a track of the computational time required for both the methods. Which method is faster ?

Q 4 For the Marsaglia-Bray method keep track of the proportional of values rejected. How does it compare with $1 - \frac{\pi}{4}$?

Solution:

Generating random number from the standard normal distribution by the **Box-Muller** method :
This algorithm generates a sample from a bivariate standard normal, each component of which is thus a univariate standard normal.
The algorithm is based on the following two properties of bivariate normal. If Z is $N(0,1)$, then

- $R = Z_1^2 + Z_2^2$ is exponentially distributed with mean 2, i.e., $P(R \leq x) = (1 - e^{-\frac{x}{2}})$.
- Given R, the point (Z_1, Z_2) is uniformly distributed on the circle of radius $\sqrt{R}$ centered at the origin.

Thus to generate (Z_1, Z_2) we first generate R and then choose a point uniformly from the circle of radius $\sqrt{R}$. To sample from the exponential distribution we may set $R = 2\log U_1$ with $U_1 \sim \mathcal{U}(0,1)$.
To generate a random point on a circle we may generate angle uniformly between 0 and 2π and map the angle to a point on the circle. The random angle may be generated as $V = 2\pi U_2$ with $U_2 \sim \mathcal{U}(0,1)$.
The corresponding point on the circle has co-ordinate $(\sqrt{R}\cos V, \sqrt{R}\sin V)$.
The complete algorithm is :

Algorithm 1 Generating Random number from the standard normal distribution by the Box-Muller method

1: Generate U_1, U_2 from $\mathcal{U}(0,1)$.
2: Generate R and V from the relations $R = 2\log U_1$ and $V = 2\pi U_2$.
3: Generate Z_1 and Z_2 from the relations $Z_1 = \sqrt{R}\cos V$ and $Z_2 = \sqrt{R}\sin V$.
4: Return Z_1 and Z_2.

Generating random number from the standard normal distribution by the **Marsaglia and Bray** method :
Marsaglia and Bray developed a modification of the Box-Muller method that reduces computing time by avoiding evaluation of the cos and sin functions. The Marsaglia-Bray method instead uses acceptance-rejection method to sample paths uniformly in the unit disc and transforms the points to normal variates.
The transform $U_i \rightarrow 2U_i - 1$, $i = 1:2$ makes (U_1, U_2) uniformly distributed on the square $[1,1][1,1]$.
Accepting only those pairs for which $X = U_1^2 + U_2^2$ is less than or equal to 1 produces points

uniformly distributed over the disc of radius 1 centered at the origin.

Conditional on acceptance, X is uniformly distributed between 0 and 1 so that $\log(X)$ has the same effect as $\log(U_1)$ for Box-Muller.

Dividing each accepted U_1 and U_2 by $\sqrt{X}$ projects it from the unit circle, on which it is uniformly distributed.

The algorithm is as follows :

Algorithm 2 Generating Random number from the standard normal distribution by the Marsaglia and Bray method

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1: Generate U_1, U_2 from $\mathcal{U}(0,1)$.

2: Transform U_1 and U_2 using the relation $U_i \to 2U_i - 1, \quad i = 1:2$.

3: **if** $X \leq 1$ **then**

4: Generate Y from the relation $Y = \sqrt{\frac{-2\log X}{X}}$.

5: Generate Z_1 and Z_2 from the relation $Z_i = U_i Y, \quad i = 1:2$.

6: Return Z_1 and Z_2.

7: **else**

8: return to step 1.

9: **end if**

Code for R

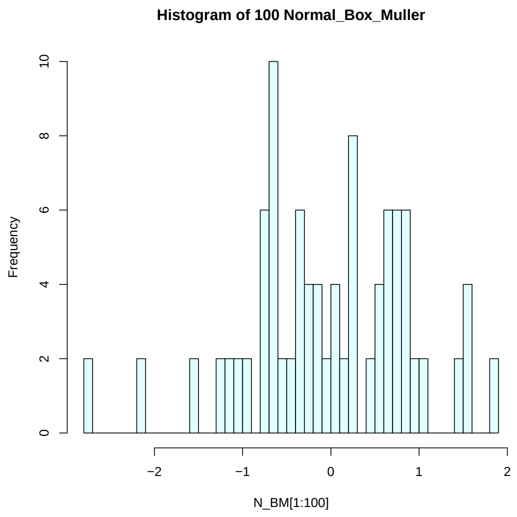
```
genNormal_Box_Muller<-function(sample) {
   N<-vector(length = sample);
   set.seed(1);
   for (i in seq(1, sample, 2)) {
   u<-runif(2, 0, 1);
   R = -2 * log(u[1]);
   V = 2 * pi * u[2];
   N[i] = sqrt(R) * cos(V);
   N[i + 1] = sqrt(R) * cos(V);
   }
   return(N);
}

genNormal_Marsaglia_Bray<-function(sample) {
   N<-vector(length = sample);
   set.seed(1);
   j = 0;
   for (i in seq(1, sample, 2)) {
      repeat {
         j = j + 2;
         u<-runif(2, 0, 1);
         u = (2 * u) - 1;
         X = (u[1]^2) + (u[2]^2);
         if (X < 1) {
            Y = sqrt((-2 * log(X))/X);
            N[i] = u[1] * Y;
            N[i + 1] = u[2] * Y;
            break;
         }
      }
   }
   return(c((1 - (sample / j)), N));
}

time_BM = proc.time()[3];
N_BM <- genNormal_Box_Muller(10000);
time_BM = proc.time()[3] - time_BM;

time_MB = proc.time()[3];
N_MB <- genNormal_Marsaglia_Bray(10000);
```

```
time_MB = proc.time()[3] - time_MB;

r = N_MB[1];
N_MB <- N_MB[2:10001];

cat("Using Box-Muller method, the sample mean, and the sample variance, for
    different values of sample size, are calculated to be:\n");
cat("Sample size = 100 :: Mean = ", mean(N_BM[1:100]), "\t;\tVariance = ", var(N_BM
    [1:100]), "\n");
cat("Sample size = 500 :: Mean = ", mean(N_BM[1:500]), "\t;\tVariance = ", var(N_BM
    [1:500]), "\n");
cat("Sample size = 10000 :: Mean = ", mean(N_BM), "\t;\tVariance = ", var(N_BM), ".\
    n");

cat("\nUsing Marsaglia-Bray method, the sample mean, and the sample variance, for
    different values of sample size, are calculated to be:\n");
cat("Sample size = 100 :: Mean = ", mean(N_MB[1:100]), "\t;\tVariance = ", var(N_MB
    [1:100]), "\n");
cat("Sample size = 500 :: Mean = ", mean(N_MB[1:500]), "\t;\tVariance = ", var(N_MB
    [1:500]), "\n");
cat("Sample size = 10000 :: Mean = ", mean(N_MB), "\t;\tVariance = ", var(N_MB), ".\
    n");

pdf("N_BM100.pdf");
hist(N_BM[1:100], breaks = 50, col = "light cyan", plot = TRUE, main = "Histogram of
     100 Normal_Box_Muller");
pdf("N_BM500.pdf");
hist(N_BM[1:500], breaks = 50, col = "light cyan", plot = TRUE, main = "Histogram of
     500 Normal_Box_Muller");
pdf("N_BM10000.pdf");
hist(N_BM[1:10000], breaks = 50, col = "light cyan", plot = TRUE, main = "Histogram
    of 10000 Normal_Box_Muller");

pdf("N_MB100.pdf");
hist(N_MB[1:100], breaks = 50, col = "light cyan", plot = TRUE, main = "Histogram of
     100 Normal_Marsaglia_Bray");
pdf("N_MB500.pdf");
hist(N_MB[1:500], breaks = 50, col = "light cyan", plot = TRUE, main = "Histogram of
     500 Normal_Marsaglia_Bray");
pdf("N_MB10000.pdf");
```

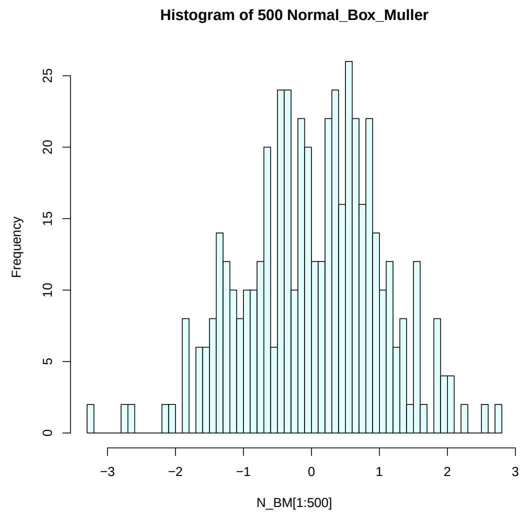
```
hist(N_MB[1:10000], breaks = 50, col = "light cyan", plot = TRUE, main = "Histogram
    of 10000 Normal_Marsaglia_Bray");

##For N(0,5)
sN_BM <- sqrt(5) * sort(N_BM[1:500]);
sN_MB <- sqrt(5) * sort(N_MB[1:500]);
sN_T <- sort(rnorm(500, mean = 0, sd = sqrt(5)));
pdf("N(0,5).pdf");
#plot.ecdf(sN_BM);
#plot.ecdf(sN_MB);
#plot.ecdf(sN_T);
plot(ecdf(sN_BM), do.points = FALSE, main = "", col = "red")
par(new = TRUE)
plot(ecdf(sN_MB), do.points = FALSE, main = "", axes = FALSE, col = "green")
#plot(ecdf(sN_T), do.points = FALSE, main = "")
lines(sN_T, pnorm(sN_T, mean = 0, sd = sqrt(5)), type='l', col = "blue")
legend('topleft', legend = c('Experimental (Box-Muller method)', 'Experimental (
    Marsaglia-Bray method)', 'Theoretical'), lty = 1, col = c("red", "green", "blue"
    ), bty = 'n')
title("Cumulative Distribution Function for N(0,5)");

##For N(5,5)
sN_BM <- (sqrt(5) * sort(N_BM[1:500])) + 5;
sN_MB <- (sqrt(5) * sort(N_MB[1:500])) + 5;
sN_T <- sort(rnorm(500, mean = 5, sd = sqrt(5)));
pdf("N(5,5).pdf");
#plot.ecdf(sN_BM);
#plot.ecdf(sN_MB);
#plot.ecdf(sN_T);
plot(ecdf(sN_BM), do.points = FALSE, main = "", col = "red")
par(new = TRUE)
plot(ecdf(sN_MB), do.points = FALSE, main = "", axes = FALSE, col = "green")
#plot(ecdf(sN_T), do.points = FALSE, main = "")
lines(sN_T, pnorm(sN_T, mean = 5, sd = sqrt(5)), type='l', col = "blue")
legend('topleft', legend = c('Experimental (Box-Muller method)', 'Experimental (
    Marsaglia-Bray method)', 'Theoretical'), lty = 1, col = c("red", "green", "blue"
    ), bty = 'n')
title("Cumulative Distribution Function for N(5,5)");

cat("\nComputional time (elapsed time) for Box-Muller method and Marsaglia-Bray
    method are ", time_BM, ", and ", time_MB, "respectively.\n");
```

```
if (time_BM < time_MB) {
    cat("Box-Muller method is faster than Marsaglia-Bray method.\n");
} else {
    cat("Marsaglia-Bray method is faster than Box-Muller method.\n");
}

cat("\nFor the Marsaglia-Bray method the proportion of values rejected (in
    generating 10000 sample values) is ", r, ".\n");
```

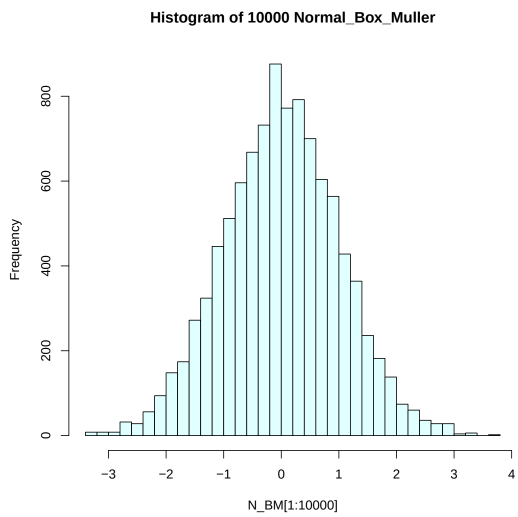
Results:



(**a**) *100 samples*



(**b**) *500 samples*



(**c**) *10000 samples*

Figure 1: *Histogram for Box-Muller method*

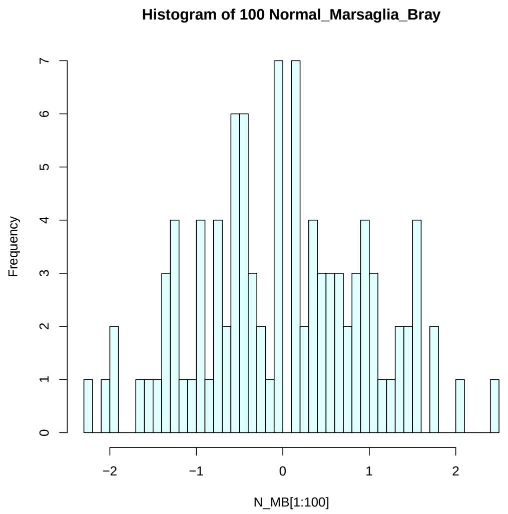
Using Box-Muller method, the sample mean, and the sample variance, for different values of sample size, are calculated to be:

Sample size = 100 :: Mean = -0.01703602 ; Variance = 0.866979

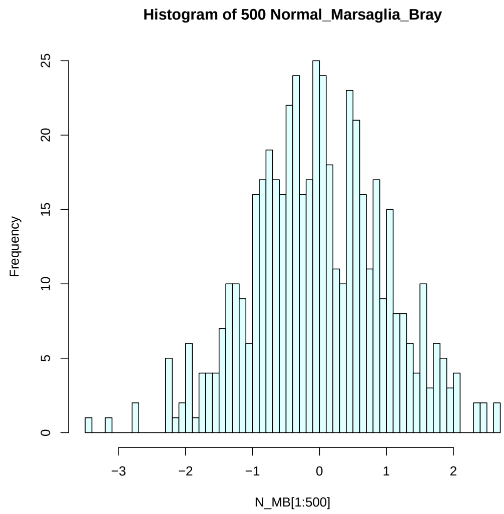
Sample size = 500 :: Mean = 0.03307071 ; Variance = 0.9975302

Sample size = 10000 :: Mean = 0.008148474 ; Variance = 1.011222.

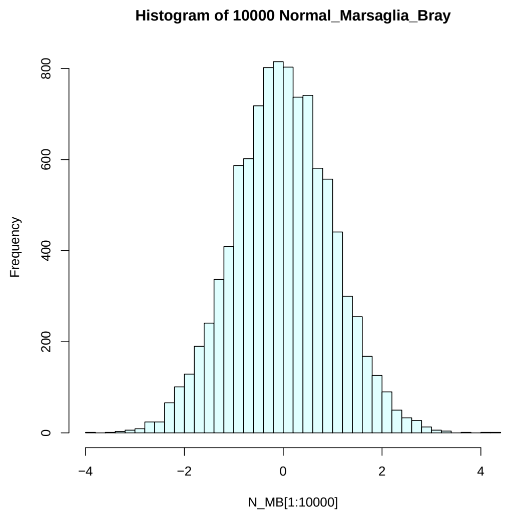
These values are close to the theoretical ones, 0, and 1.



(a) *100 samples*



(b) *500 samples*



(c) *10000 samples*

Figure 2: *Histogram for Marsaglia-Bray method*

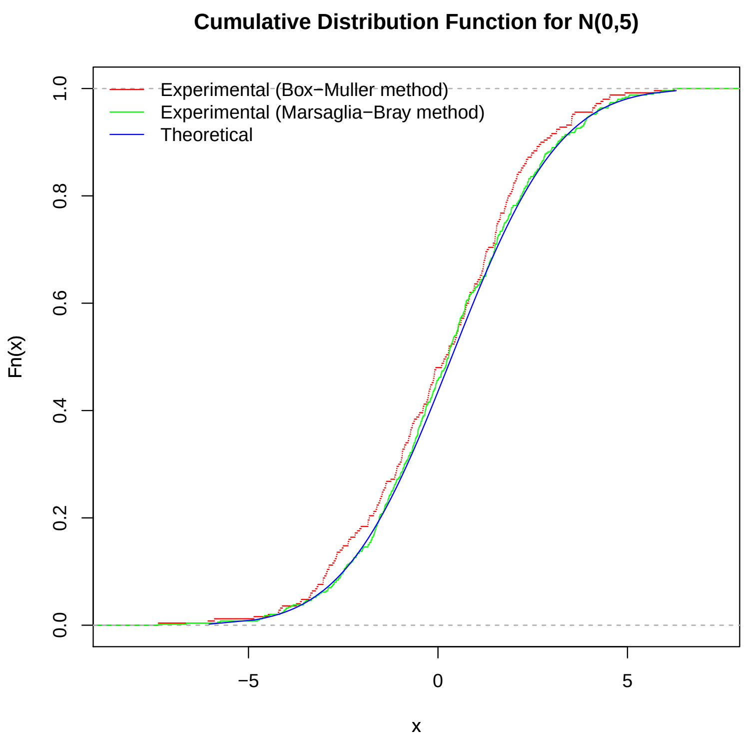
Using Marsaglia-Bray method, the sample mean, and the sample variance, for different values of sample size, are calculated to be:

Sample size = 100 :: Mean = -0.001358307 ; Variance = 1.009955

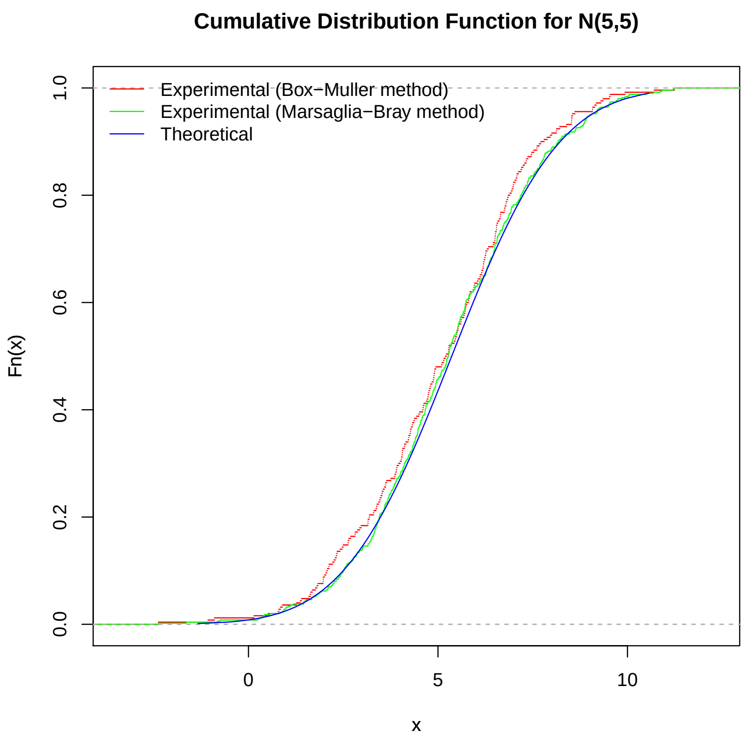
Sample size = 500 :: Mean = -0.02857793 ; Variance = 0.9973459

Sample size = 10000 :: Mean = -0.009924773 ; Variance = 0.9804145.

These values are close to the theoretical ones, 0, and 1.



(a) $\mathcal{N}(0,5)$



(b) $\mathcal{N}(5,5)$

Figure 3: *Plot of Cumulative Distribution Function*

Computional time (elapsed time) for Box-Muller method and Marsaglia-Bray method are 0.041 , and 0.04 respectively. Marsaglia-Bray method is faster than Box-Muller method.

For the Marsaglia-Bray method the proportion of values rejected (in generating 10000 sample values) is 0.2181392.
The value is close to theoretical one, $1 - \frac{\pi}{4} = 0.2146018$.