Lecture - 5

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<u>Normal random variables and vectors :</u> The standard univariate normal distribution has density

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \ , \quad -\infty < x < \infty$$

and cumulative distribution function :

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$$

The word standard indicates mean 0 and variance 1. The notation $X \sim N(\mu, \sigma^2)$ abbreviates the statement that the random variable X is normaly distributed with mean μ and variance σ^2. If $Z \sim N(0,1)$ (i.e. Z has standard normal distribution) $\mu + \sigma Z \sim N(\mu, \sigma^2)$ Thus given a method for generating samples $Z_1, Z_2, \cdots$ from the standard normal distribution, we can generate samples $X_1, X_2, \cdots$ from $N(\mu, \sigma^2)$. It therefore suffices to consider methods for sampling from $N(0,1)$.

Generating Univariate Normal Distribution:

We now discuss algorithms for generating univariate normal distribution. First we generate uniform random variable and try to transform them into normally distributed random variables. Before we proceed on these methods we need a couple of theorems.

Theorem 0.1. *Suppose X is a random variable with density $f(x)$ and distribution $F_X(x)$. Further assume that $h : S \to B$ with $S, B \in \mathcal{R}$, where S is the support of $f(x)$. (i.e. $f(x)$ is zero outside S (S here is not the stock price)) and let h be strictly monotonous.*

1. *Then $Y = h(X)$ is a random variable and its distribution F_Y in case $h' > 0$ is $F_Y(y) = F(h^{-1}(y))$*

2. *If h^{-1} is absolutely continuous then for almost all y the density of $h(X)$ is :*

$$f(h^{-1}(y)) \left| \frac{dh^{-1}(y)}{dy} \right|$$

Theorem 0.2. *Suppose X is a random variable in $\mathcal{R}^n$, with density $f(x) > 0$ on the support S. The transform $h : S \to B$ ($S, B \subset \mathcal{R}^n$) is assumed to be invertible and the inverse be continuously differentiable on B. $Y = h(X)$ is the transformed variable. Then Y has the density :*

$$f^{-1}(h(y)) \left| \frac{\delta(x_1, x_2, \cdots, x_n)}{\delta(y_1, y_2, \cdots, y_n)} \right|, \quad y \in B$$

where, $x = h^{-1}(y)$ and $|\frac{\delta(x_1, x_2, \cdots, x_n)}{\delta(y_1, y_2, \cdots, y_n)}|$ is the determinant of Jacobian matrix of all first order derivatives of $h^{-1}(y)$

Box-Muller Method:

Perhaps the simplest method to implement (though not the fastest or necessarily the most convenient) is the one by Box-Muller. This algorithm generates a sample from a bivariate standard normal, each component of which is thus a univariate standard normal. The algorithm is based on the following two properties of bivariate normal. If Z is $\mathcal{N}(0, 1)$, then

- $R = Z_1^2 + Z_2^2$ is exponentially distributed with mean 2, i.e.,

$$P(R \leq x) = (1 - e^{-\frac{x}{2}})$$

- Given R the point (Z_1, Z_2) is uniformly distributed on the circle of radius $\sqrt{R}$ centered at the origin.

- Given R, the point (Z_1, Z_2) is uniformly dsitributed on the circle of radius $\sqrt{R}$ centered at the origin.

Thus to generate (Z_1, Z_2) we first generate R and then choose a point uniformly from the circle of radius $\sqrt{R}$. To sample from the exponential distribution we may set $R = -2 \ln U_1$ with $U_1 \sim U(0,1)$. To generate a random point on a circle we may generate angle uniformly between 0 and 2π and map the angle to a point on the circle. The random angle may be generated as $V = 2\pi U_2$ with $U_2 \sim \mathcal{U}[0,1]$. The corresponding point on the circle has co-ordinate $(\sqrt{R}cos(V), \sqrt{R}sin(V))$. The complete algorithm is :

1. Generate U_1, U_2 independent on $\mathcal{U}[0,1]$
2. $R = -2\ln(U_1)$
3. $V = 2\pi U_2$
4. $Z_1 = \sqrt{R}cos(V)$ and $Z_2 = \sqrt{R}sin(V)$
5. Return Z_1 and Z_2

We can write

$$y_1 = \sqrt{-2\ln x_1}cos(2\pi x_2) = h(x_1, x_2).$$

$$y_2 = \sqrt{-2\ln x_1}sin(2\pi x_2) = h(x_1, x_2)$$

The function $h(x)$ is defined on $[0,1]^2$ with values in $\mathcal{R}^2$. The inverse function h^{-1} is given by :

$$x_1 = \exp(-\frac{1}{2}(y_1^2 + y_2^2)).$$
$$x_2 = \frac{1}{2\pi}\tan^{-1}(\frac{y_2}{y_1})$$

where we take the main branch of the arctan.

$$\begin{aligned}
\frac{\delta(x_1, x_2)}{\delta(y_1, y_2)} &= det \begin{pmatrix} \frac{\delta x_1}{\delta y_1} & \frac{\delta x_1}{\delta y_2} \\ \frac{\delta x_2}{\delta y_1} & \frac{\delta x_2}{\delta y_2} \end{pmatrix} \\
&= \frac{1}{2\pi}\exp\{-\frac{y_1^2 + y_2^2}{2}\}
\end{aligned}$$

Marsaglia and Bray Method :

Marsaglia and Bray developed a modification of the Box-Muller method that reduces computing time by avoiding evaluation of the cos and sin functions.. The Marsaglia Bray method instead uses accep- tance rejection method to sample paths uniformly in the unit disc and transforms the points to normal variates. The algorithm is as follows :

1. (while $X > 1$) Generate $U_1, U_2 \sim \mathcal{U}[0,1]$

   $U_1 = 2U_1 - 1$

   $U_2 = 2U_2 - 1$

   $X = U_1^2 + U_2^2.$

2. $Y = \sqrt{\frac{-2\log(X)}{X}}$

3. $Z_1 = U_1 Y$ and $Z_2 = U_2 Y$

4. Return $Z_1 = U_1 Y$ and $Z_2 = U_2 Y$.

5. Return Z_1 and Z_2.

The transform $U_i \to 2U_i - 1, \quad i = 1:2$ makes (U_1, U_2) uniformly distributed on the sqaure $[-1,1] \times [-1,1]$. Accepting only those pairs for which $X = U_1^2 + U_2^2$ is less than or equal to 1 produces points uniformly distributed over the disc of radius 1 centered at the origin. Conditional on acceptance, X is uniformly distributed between 0 and 1 so that $\log(X)$ has the same effect as $\log(U_1)$ for Box-Muller. Dividing each accepted U_1 and U_2 by $\sqrt{X}$ projects it from the unit circle, on which it is uniformly distributed.