Lecture - 5

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Beta Distribution :

The Beta distribution on $[0,1]$ with parameters $\alpha_1, \alpha_2 > 0$ is given by

$$f(x) = \frac{1}{B(\alpha_1,\alpha_2)} x^{\alpha_1 - 1}(1-x)^{\alpha_2 - 1} \ ; \quad 0 \leq x \leq 1.$$

with

$$B(\alpha_1,\alpha_2) = \int_0^1 x^{\alpha_1 - 1}(1-x)^{\alpha_2 - 1} dx = \frac{\Gamma\alpha_1 \Gamma\alpha_2}{\Gamma\alpha_1 + \alpha_2}$$

Where Γ is the gamma distribution varying the parameters α_1 and α_2 results in a variety of shapes making this a versatile family of distribution. For example, $\alpha_1 = \alpha_2 = \frac{1}{2}$ is the arcsine distribution. If $\alpha_1 > \alpha_2 \geq 1$ and at least one of the parameters exceeds 1, the beta density is unimodel and achieves its maximum at $\frac{(\alpha_1 - 1)}{\alpha_1 + \alpha_2 - 2}$. Let c be the value of the density f at this point. Then $f(x) \leq c, \quad \forall x$. For the purpose of acceptance rejection method, we may choose g to be the uniform density, which is in fact beta density with parameters $\alpha_1 = \alpha_2 = 1$.

Algorithm :

1. Generate $U_1, U_2 \in U[0,1]$ until $cU_2 \leq f(U_1)$.
2. Return U_1

Normal from Double exponential :

Fisher illustrated the use of acceptance-rejection method by generating half-normal samples from an exponential distribution(A half normal random variable has the distribution of the absolute value of a normal random variable). This is important since the method can be used to generate normal random variables that is so critical in financial applications.

1. The double exponential density on $(-\infty, \infty)$ is $g(x) = \frac{1}{2}\exp\big(-|x|\big)$.
2. The normal density is $g(x) = \frac{1}{\sqrt{2\pi}}\exp\big(-\frac{x^2}{2}\big)$.
3. The ratio is $\frac{f(x)}{g(x)} = \sqrt{\frac{2}{\pi}}e^{(-\frac{x^2}{2}+|x|)} \leq \sqrt{\frac{2e}{\pi}} = const$

Then normal distribution is dominates by double exponential density $g(x)$. A sample from the double exponential can be generates (using the formula $X = -\theta\ln(U)$, as already done) to draw a standard exponential random variables and then randomizing the sign. The rejection test $u > \frac{f(x)}{g(x)}$ can be implimented as :

$$u > \frac{e^{-x^2/2}}{\sqrt{2\pi}}\frac{1}{ce^{-|x|}/2} = e^{-\frac{1}{2}(|x|-1)^2}$$

In light of symmetry of f and g it sufficient to generate a positive sample is accepted. Absolute value is unnecessary in the rejection test.

Normal random variables and vectors : The standard univariate normal distribution has density

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\ , \quad -\infty < x < \infty$$

and cumulative distribution function :

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}du$$

The word standard indicates mean 0 and variance 1. The notation $X \sim N(\mu, \sigma^2)$ abbreviates the statement that the random variable X is normaly distributed with mean μ and variance σ^2. If $Z \sim N(0, 1)$ (i.e. Z has standard normal distribution) $\mu + \sigma Z \sim N(\mu, \sigma^2)$ Thus given a method for generating samples $Z_1, Z_2, \cdots$ from the standard normal distribution, we can generate samples $X_1, X_2, \cdots$ from $N(\mu, \sigma^2)$. It therefore suffices to consider methods for sampling from $N(0, 1)$.

Generating half-Normal from exponential :

Suppose $Z \sim N(0, 1)$. $W = |Z| \sim$ half-normal. We know $\Phi(x)$ and $\phi(x)$ are the cdf and pdf of standard normal distribution respectively. Let us explore the probability density function of half-normal distribution.

$$F_W(x) = P(W \leq x) = P(|Z| \leq x) = P(-x \leq Z \leq x) = P(Z \leq x) - P(Z \leq -x) = \Phi(x) - \Phi(-x)$$

$$f_W(x) = \frac{dF_W(x)}{dx} = \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

We can assume it as truncated distribution where truncation is made at $x = 0$.

To generate random numbers of half-normal distribution by acceptance-rejection principal we use exponential distribution with parameter 1. We choose exponential since exponential has similar shape like half-normal and the domain of the pdf for both half-normal and exponential are $[0, \infty)$.

To find the constant c, we need to maximize the function

$$\frac{f(y)}{g(y)} = \frac{\frac{2}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}}{e^{-y}}$$

The maximum value will be attained at $y = 1$ which gives $c = \frac{f(1)}{g(1)} = \sqrt{\frac{2e}{\pi}}$ Therefore The steps of this algorithm can be given as follows :

1. Generate $Y \sim Exp(1)$, $Y = -\log(U_1)$, $U_1 \sim U(0,1)$.

2. Generate another $U_2 \sim U(0,1)$.

3. Test $U_2 \leq \frac{f(Y)}{cg(Y)} = e^{-\frac{1}{2}(Y-1)^2}$, if true set X = Y.

4. Repeat if not.