CES Lecture

High-Dimensional Confounding

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References

- Belloni, Chernozhukov, and Hansen (2014): ”High-Dimensional Methods and Inference on Structural and Treatment Effects”, Journal of Economic Perspectives, 28 (2), pp. 29-50, download.
- Chernozhukov, Chetverikov, Demirer, Duflo, Hansen, and Newey (2017): ”Double/Debiased/Neyman Machine Learning of Treatment Effects”, American Economic Review, 107 (5), pp. 261-265, download.

Estimation Target

- Multivariate Linear Regression Model:

$$
\begin{aligned}
Y_i &= D_i\delta + X_i\beta_g + U_i && \text{(structural model)} \\
D_i &= X_i\beta_m + V_i && \text{(selection model)}
\end{aligned}
$$

  with $E[U_i|D_i, X_i] = 0$ and $E[V_i|X_i] = 0$.

- Parameter of interest: δ
- Nuisance parameters: β_g and β_m

- X_i contains $p \gg N$ covariates.
- We assume controlling for $K \ll N$ covariates is sufficient to identify δ.
- Controlling for too many irrelevant covariates may reduce the efficiency of OLS.

Types of Covariates

Relation between covariates and outcome (for some $s_g > 0$):

- $|\beta_{gj}| > s_g$: covariate X_j has a **strong association** with Y_i
- $0 < |\beta_{gj}| \leq s_g$: covariate X_j has a **weak association** with Y_i
- $\beta_{gj} = 0$: covariate X_j has a **no association** with Y_i

Relation between covariates and treatment (for some $s_m > 0$):

- $|\beta_{mj}| > s_m$: covariate X_j has a **strong association** with D_i
- $0 < |\beta_{mj}| \leq s_m$: covariate X_j has a **weak association** with D_i
- $\beta_{mj} = 0$: covariate X_j has a **no association** with D_i

$\rightarrow$ All covariates are standardised

Types of Covariates (cont.)

	$\beta_{gj} = 0$	$0 < \lvert\beta_{gj}\rvert \leq s_g$	$\lvert\beta_{gj}\rvert > s_g$
$\beta_{mj} = 0$	Irrelevant	Irrelevant	Irrelevant
$0 < \lvert\beta_{mj}\rvert \leq s_m$	Irrelevant	Unclear?	Weak Confounder
$\lvert\beta_{mj}\rvert > s_m$	Irrelevant	Weak Confounder	Strong Confounder

- $|\beta_{gj}| > s_g$ and $0 < |\beta_{mj}| \leq s_m$: "Weak Outcome Confounder"
- $|\beta_{mj}| > s_m$ and $0 < |\beta_{gj}| \leq s_g$: "Weak Treatment Confounder"

Naive Approach I: Structural Model

Apply Lasso to the structural model

$$\min_{\beta_g}\{E[(Y_i - D_i\delta - X_i\beta_g)^2] + \lambda\|\beta_g\|_1\}$$

without a penalty on δ and estimate a Post-Lasso model using all covariates with non-zero β_g coefficients.

Covariates that are weakly associated with Y_i could be dropped.

$\rightarrow$ Potentially we drop “weak treatment confounders”

Covariates that are strongly associated with D_i could be dropped.

$\rightarrow$ Potentially we drop “strong confounders”

Naive Approach II: Selection Model

Apply Lasso to the selection model

$$\min_{\beta_m}\{E[(D_i - X_i\beta_m)^2] + \lambda\|\beta_m\|_1\}$$

and estimate a Post-Lasso structural model using all covariates with non-zero β_m coefficients.

Covariates that are weakly associated with D_i could be dropped.

$\rightarrow$ Potentially we drop “weak outcome confounders”

Double Selection Procedure

1. Apply Lasso to the reduced form models

$$\min_{\tilde{\beta}_g} \{ E[(Y_i - X_i \tilde{\beta}_g)^2] + \lambda \| \tilde{\beta}_g \|_1 \}, \tag{1}$$

$$\min_{\beta_m} \{ E[(D_i - X_i \beta_m)^2] + \lambda \| \beta_m \|_1 \}, \tag{2}$$

   with $\tilde{\beta}_g = \delta \beta_m + \beta_g$.

2. Take the union of all covariates $\tilde{X}_i$ with either non-zero β_m or $\tilde{\beta}_g$ coefficients and estimate the Post-Lasso structural model

$$Y_i = D_i \delta + \tilde{X}_i \beta_g^* + u_i.$$

Double Selection Procedure (cont.)

Potentially (2) omits “weak outcome confounders”

$\tilde{\beta}_{gj} \approx \beta_g$ when $0 < |\beta_{mj}| \leq s_m$, such that the missing “weak outcome confounders” are likely selected in (1).

Disadvantages:

$\rightarrow$ Potentially we omit “very weak” confounders with $0 < |\beta_{gj}| \leq s_g$ and $0 < |\beta_{mj}| \leq s_g$.

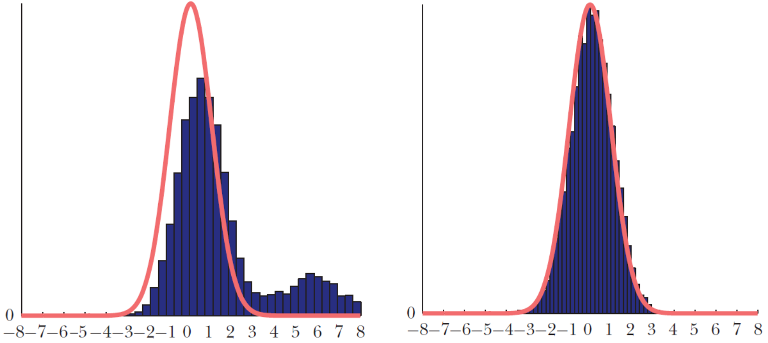
$\rightarrow$ All procedures potentially include irrelevant variables.

Simulation Exercise

Distribution of Estimators

Naive Single-Post-Selection on Structural Model

Double-Post-Selection



Source: Belloni, Chernozhukov, and Hansen (2014)

Asymptotic Results

- Consistency and asymptotic normality

$$\sqrt{N}(\hat{\delta} - \delta) \xrightarrow{d} N(0, \sigma).$$

- Model selection step is asymptotically negligible for building confidence intervals.
- Optimal penalty parameter $\lambda^* = 2c \cdot \Phi^{-1}(1 - \gamma/2p)/\sqrt{N}$ (e.g., $c = 1.1$ and $\gamma \leq 0.05$) for "Feasible LASSO"

$$\min_{\beta} E[(Y_i - X_i\beta)^2] + \lambda^* \|\beta\|_1.$$

Reference: Belloni, Chernozhukov, and Hansen (2014)

Summary Double Selection Procedure

Advantages:

- Standard inference
- Computationally fast
- Packages: `LassoShooting` (for Stata) and `hdm` (for R)

Disadvantages:

- Effect homogeneity
- Potentially irrelevant covariates selected
- Sparsity assumptions required

Potential Outcome Framework

Notation:

- D_i binary treatment dummy (e.g., assignment to training program)
- $Y_i(1)$ potential outcome under treatment (e.g., earnings under participation in training)
- $Y_i(0)$ potential outcome under non-treatment (e.g., earnings under non-participation in training)

Infeasible parameter:

- Individual causal effect: $\delta_i = Y_i(1) - Y_i(0)$

Feasible parameters:

- Average Treatment Effect (ATE): $\delta = E[Y_i(1) - Y_i(0)] = E[\delta_i]$
- Average Treatment Effect on the Treated (ATET): $\rho = E[\delta_i | D_i = 1]$

Identifying Assumptions for ATE

- **Stable Unit Treatment Value Assumption (SUTVA):**

$$Y_i = Y_i(1)D_i + Y_i(0)(1 - D_i)$$

- **Exogeneity of Covariates:**

$$X_i(1) = X_i(0)$$

- **No Support Problems:**

$$\varepsilon < Pr(D_i = 1 | X_i = x) = p(x) < 1 - \varepsilon$$

  for some small $\varepsilon > 0$ and all x in the support of X_i

- **Conditional Independence Assumption (CIA):**

$$Y_i(1), Y_i(0) \perp\!\!\!\perp D_i | X_i = x$$

  for all x in the support of X_i

Modified Outcome Method for ATE

Inverse Probability Weighting:

$$Y^*_{i,IPW} = \frac{D_i}{p(x)} Y_i - \frac{1 - D_i}{1 - p(x)} Y_i = \frac{D_i - p(x)}{p(x)(1 - p(x))} Y_i$$

with the propensity score $p(x) = Pr(D_i = 1 | X_i = x)$.

ATE: $\delta = E[Y^*_{i,IPW}]$ and $\hat{\delta} = \frac{1}{N} \sum_{i=1}^{N} \hat{Y}^*_{i,IPW}$

Proof of Identification

$$
\begin{aligned}
\delta &= E[Y_i(1)] - E[Y_i(0)] \overset{LIE}{=} \int E[Y_i(1)|X_i = x] - E[Y_i(0)|X_i = x]\, f_X(x)dx \\
&\overset{CIA}{=} \int E[Y_i(1)|D_i = 1, X_i = x] - E[Y_i(0)|D_i = 0, X_i = x]\, f_X(x)dx \\
&= \int E[Y_i|D_i = 1, X_i = x] - E[Y_i|D_i = 0, X_i = x]\, f_X(x)dx \\
&= \int E[D_i Y_i|D_i = 1, X_i = x] - E[(1 - D_i) Y_i|D_i = 0, X_i = x]\, f_X(x)dx \\
&\overset{LIE}{=} \int E\left[\frac{D_i Y_i}{p(x)} \middle| X_i = x\right] - E\left[\frac{(1 - D_i) Y_i}{1 - p(x)} \middle| X_i = x\right] f_X(x)dx \\
&= \int E\left[\frac{D_i Y_i}{p(x)} - \frac{(1 - D_i) Y_i}{1 - p(x)} \middle| X_i = x\right] f_X(x)dx \\
&= \int E\left[\frac{D_i - p(x)}{p(x)(1 - p(x))} Y_i \middle| X_i = x\right] f_X(x)dx \overset{LIE}{=} E\left[\frac{D_i - p(x)}{p(x)(1 - p(x))} Y_i\right]
\end{aligned}
$$

Reference: Horvitz and Thompson (1952)

Modified Outcome Method with IPW

Advantages:

- Generic approach
- Sparsity assumptions can be avoided by appropriate choice of estimator for propensity score
- Heterogeneous treatment effects

Disadvantages:

- Potentially omitting “weak outcome confounders”
- Shows weak performance in simulations (Knaus, Lechner, and Strittmatter, 2018)
- Not $\sqrt{N}$-consistent in high-dimensional setting

See comprehensive discussion in Goller, Lechner, Moczall, Wolff (2019).

Conditional Mean Differences

Identification:

$$\begin{aligned}
\delta &= E[Y(1)] - E[Y(0)] \\
&\overset{LIE}{=} \int E[Y(1)|X=x] - E[Y(0)|X=x] f_X(x)dx \\
&\overset{CIA}{=} \int E[Y(1)|D=1,X=x] - E[Y(0)|D=0,X=x] f_X(x)dx \\
&= \int \underbrace{E[Y|D=1,X=x]}_{=\mu_1(x)} - \underbrace{E[Y|D=0,X=x]}_{=\mu_0(x)} f_X(x)dx
\end{aligned}$$

Estimator:

$$\hat{\delta} = \frac{1}{N}\sum_{i=1}^{N}(\hat{\mu}_1(X_i) - \hat{\mu}_0(X_i))$$

with $\hat{\mu}_1(x) = \hat{E}[Y_i|D_i = 1, X_i = x]$ and $\hat{\mu}_0(x) = \hat{E}[Y_i(0)|D_i = 0, X_i = x]$ being the estimated conditional expectations of the potential outcomes.

Double/Debiased Machine Learning (DML)

Efficient Score:

$$
\begin{aligned}
Y^*_{i,DML} &= \mu_1(X_i) - \mu_0(X_i) + \frac{D_i - p(x)}{p(x)(1-p(x))} Y_i - \frac{D_i}{p(x)} \mu_1(X_i) + \frac{1-D_i}{1-p(x)} \mu_0(X_i) \\
&= \mu_1(X_i) - \mu_0(X_i) + \frac{D_i(Y_i - \mu_1(X_i))}{p(X_i)} - \frac{(1-D_i)(Y_i - \mu_0(X_i))}{1-p(X_i)}
\end{aligned}
$$

ATE: $\delta = E[Y^*_{i,DML}]$ and $\hat{\delta} = \frac{1}{N} \sum_{i=1}^{N} \hat{Y}^*_{i,DML}$

We can use standard ML methods to estimate $\hat{\mu}_1(x)$, $\hat{\mu}_0(x)$, and $\hat{p}(x)$.

Reference: Chernozhukov et al., 2017

Proof of Identification

$$
\begin{aligned}
\delta =& E\left[\mu_1(x) - \mu_0(x) + \frac{D_i(Y_i - \mu_1(x))}{p(x)} - \frac{(1-D_i)(Y_i - \mu_0(x))}{1-p(x)}\right] \\
=& E\left[\frac{D_i - p(x)}{p(x)(1-p(x))} Y_i + \frac{(p(x) - D_i)\mu_1(x)}{p(x)} - \frac{(D_i - p(x))\mu_0(x)}{1-p(x)}\right] \\
=& \int E\left[\frac{D_i - p(x)}{p(x)(1-p(x))} Y_i + \frac{(p(x) - D_i)\mu_1(x)}{p(x)} - \frac{(D_i - p(x))\mu_0(x)}{1-p(x)} \middle| X_i = x\right] f_X(x)dx \\
=& \int \left(E\left[\frac{D_i - p(x)}{p(x)(1-p(x))} Y_i \middle| X_i = x\right] + \frac{E[p(x) - D_i | X_i = x]}{p(x)} \mu_1(x) \right. \\
& \left. - \frac{E[D_i - p(x) | X_i = x]}{1-p(x)} \mu_0(x) \right) f_X(x)dx \\
=& \int E\left[\frac{D_i - p(x)}{p(x)(1-p(x))} Y_i \middle| X_i = x\right] f_X(x)dx = E\left[Y_i(1) - Y_i(0)\right]
\end{aligned}
$$

Reference: Robins and Rotnitzki (1995)

DML Cross-Fitting Algorithm

1. Partition the data randomly in samples S^A and S^B
2. Estimate the nuisance parameters $\hat{\mu}_1^A(x)$,$\hat{\mu}_0^A(x)$, and $\hat{p}^A(x)$ in S^A; and $\hat{\mu}_1^B(x)$,$\hat{\mu}_0^B(x)$, and $\hat{p}^B(x)$ in S^B with ML
3. Calculate the efficient scores in samples S^A and S^B, respectively:

$$\hat{Y}_{i,DML}^{A*} = \hat{\mu}_1^B(X_i^A) - \hat{\mu}_0^B(X_i^A) + \frac{D_i^A(Y_i^A - \hat{\mu}_1^B(X_i^A))}{\hat{p}^B(X_i^A)} - \frac{(1-D_i^A)(Y_i^A - \hat{\mu}_0^B(X_i^A))}{1-\hat{p}^B(X_i^A)}$$

$$\hat{Y}_{i,DML}^{B*} = \hat{\mu}_1^A(X_i^B) - \hat{\mu}_0^A(X_i^B) + \frac{D_i^B(Y_i^B - \hat{\mu}_1^A(X_i^B))}{\hat{p}^A(X_i^B)} - \frac{(1-D_i^B)(Y_i^B - \hat{\mu}_0^A(X_i^B))}{1-\hat{p}^A(X_i^B)}$$

4. Calculate ATE,

$$\hat{\delta} = \frac{1}{2}(\underbrace{\hat{E}[\hat{Y}_{i,DML}^{A*}|S^A]}_{=\hat{\delta}_A} + \underbrace{\hat{E}[\hat{Y}_{i,DML}^{B*}|S^B]}_{=\hat{\delta}_B}),$$

Asymptotic Results for ATE

- Main Regularity Condition: Convergence rate of nuisance parameters is at least $\sqrt[4]{N}$.
- ATE can be estimated $\sqrt{N}$-consistently

$$\sqrt{N}(\hat{\delta} - \delta) \xrightarrow{d} N(0, \sigma^2)$$

with $\sigma^2 = Var(Y^*_{i,DML})$ and $Var(\hat{\delta}) = \sigma^2 / N$

- Split sample estimator of σ^2

$$\hat{\sigma}^2 = \frac{1}{2}\left(\hat{\sigma}_A^2 + (\hat{\delta}_A - \hat{\delta})^2\right) + \frac{1}{2}\left(\hat{\sigma}_B^2 + (\hat{\delta}_B - \hat{\delta})^2\right)$$

for $\hat{\delta} = 1/2(\hat{\delta}_A + \hat{\delta}_B)$

Advantages of DML

Advantages compared to IPW and Conditional Mean Differences:

- Treatment and outcome equations are modelled explicitly
- Double robustness property
- $\sqrt{N}$-consistent and asymptotically normal even under high-dimensional confounding

Efficient Score for ATET

$$Y^*_{i,ATET} = \frac{D_i(Y_i - \mu_0(x))}{p} - \frac{p(x)(1-D_i)(Y_i - \mu_0(x))}{p(1-p(x))}$$

with $p = Pr(D_i = 1)$.

ATET: $\rho = E[Y^*_{i,ATET}]$ and $\hat{\rho} = \frac{1}{N}\sum_{i=1}^{N} \hat{Y}^*_{i,ATET}$

Estimator of Asymptotic Variance:

$$\hat{\sigma}^2 = \frac{1}{N}\sum_{i=1}^{N} \left(\hat{Y}^*_{i,ATET} - \hat{\rho}\right)^2$$

and $\hat{Var}(\hat{\rho}) = \hat{\sigma}^2/N$

Reference: Chernozhukov et al., 2017, Farrell, 2015

Proof of Identification for ATET

$$
\begin{aligned}
\rho =& E\left[\frac{D_i(Y_i-\mu_0(x))}{p}-\frac{p(x)(1-D_i)(Y_i-\mu_0(x))}{p(1-p(x))}\right]\\
=&\int E\left[\frac{D_iY_i}{p}-\frac{p(x)(1-D_i)Y_i}{p(1-p(x))}-\frac{(D_i-p(x))\mu_0(x)}{p(1-p(x))}\middle| X_i=x\right]f_X(x)dx\\
=&\int\Bigg(\frac{E[D_iY_i|X_i=x]}{p}-\frac{p(x)E[(1-D_i)Y_i|X_i=x]}{p(1-p(x))}\\
&-\frac{E[D_i-p(x)|X_i=x]}{p(1-p(x))}\mu_0(x)\Bigg)f_X(x)dx\\
=&\int\left(\frac{E[D_iY_i|X_i=x]}{p}-\frac{p(x)E[(1-D_i)Y_i|X_i=x]}{p(1-p(x))}\right)f_X(x)dx\\
=&\int\frac{p(x)}{p}\left(E[D_iY_i|D_i=1,X_i=x]-E[(1-D_i)Y_i|D_i=0,X_i=x]\right)f_X(x)dx\\
=&\int\left(E[Y_i(1)|D_i=1,X_i=x]-E[Y_i(0)|D_i=0,X_i=x]\right)f_{X|D=1}(x)dx\\
=&\int\left(E[Y_i(1)|D_i=1,X_i=x]-E[Y_i(0)|D_i=1,X_i=x]\right)f_{X|D=1}(x)dx\\
=&E[Y_i(1)-Y_i(0)|D_i=1]
\end{aligned}
$$

Other Orthogonal Scores

- LATE (see Chernozhukov et al., 2018).
- Difference-in-differences (see, e.g., Chen, Nie, and Wager, 2018, Zimmert, 2019).
- Multiple treatments (see, e.g., Farrell, 2015).
- Continuous treatments (see, e.g., Graham and Pinto, 2018).
- Mediation analysis (see Tchetgen Tchetgen and Shpitser, 2012).
- Synthetic control group method (see, e.g., Arkhangelsky et al., 2018).

R Exercise

- An interactive version of the exercise is on Binder: https://mybinder.org/v2/gh/AStrittmatter/CES-Lecture/master
- Alternatively, the exercise can be downloaded from the Github repository: https://github.com/AStrittmatter/CES-Lecture