Thoughts on the Gauss Method of Constructing the Heptadecagon

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1. Introduction

It was once thought impossible to construct any polygon with more side lengths than that of 5. Along with many other puzzles left by them now solved, the Ancient Greeks left the puzzle of constructing every regular polygon with just ruler and compass. The constructions of many regular polygons are now widely known among many scholars, and many of them spectacular and beautiful. I have selected to analyze one that I found particularly colorful and that is Gauss's method of constructing the heptadecagon.

Gauss never gave a deep proof for his method which made it very hard and all the more fun to analyze his method. I have chosen to go with Nishiyama's paper on Gauss' method of Constructing a Regular Heptadecagon as a deep proof of why Gauss chose some of the rather abstract and seemingly random numbers and equations he did. While some parts to this construction are still fuzzy, it is the only paper that I have found which goes deep enough.

This paper goes over the proof of the constructability of the Heptadecagon, and the holes in the proof, as I have understood them. Again, there are many holes to this paper, which I hope to address.

2. The Written Part

We first understand that once we have an angle, then the Heptadecagon is in essence constructed. Hence we let $360° = 17\theta$. This gives us the following:

$$\theta = \frac{360°}{17}$$

$$\theta = \frac{2\pi}{17}$$

Now comes a tricky part. Gauss defines the seemingly random equations.

$$\cos(\theta) + \cos(4\theta) = a$$

$$\cos(2\theta) + \cos(8\theta) = b$$

$$\cos(3\theta) + \cos(5\theta) = c$$

$$\cos(6\theta) + \cos(7\theta) = d$$

3. The Unwritten Part

The reason Gauss chose these numbers has to do with cyclotomic equations.

For prime p, and primitive root g, let $p - 1 = e * f$. Let $c = g^e$, and k be an arbitrary constant. Then define the p step cycle $(f,\ k,\ c) = [k\ (mod\ p)] + [kc\ (mod\ p)] + [kc^2\ (mod\ p)] + \cdots + [kc^{p-1}\ (mod\ p)]$.

Then observe that $(f, k, c) = (\frac{f}{2}, k\ (mod\ p), c^2\ (mod\ p)) + (\frac{f}{2}, kc\ (mod\ p), c^2\ (mod\ p))$. Take $f = 16, k = 1, c = 3$.

$$(16, 1, 3) = [1\ (mod\ p)] + [3\ (mod\ p)] + [9\ (mod\ p)] + [27\ (mod\ p)] + \ldots + [3^{16}\ (mod\ p)]$$

$$= 1 + 3 + 9 + 10 + 13 + 5 + 15 + 11 + 16 + 14 + 8 + 7 + 4 + 12 + 2 + 6$$

$$= (1 + 9 + 13 + 15 + 16 + 8 + 4 + 2) + (3 + 10 + 5 + 11 + 14 + 7 + 12 + 6)$$

$$= (8, 1, 9) + (8, 3, 9)$$

$$= (\frac{16}{2}, 1\ (mod\ 17), 3^2\ (mod\ 17)) + (\frac{16}{2}, 1 * 3\ (mod\ 17), 3^2\ (mod\ 17))$$

Now let's take the $(8, 1, 9)$. We have preset $c = 9$. Also, $f = 8, k = 1$. Thus we have:

$$(8, 1, 9) = (4, 1\ (mod\ 17), 9\text{^}2\ (mod\ 17)) + (4, 1 * 9\ (mod\ 17), 9\text{^}2\ (mod\ 17))$$
$$= (4, 1, 13) + (4, 9, 13).$$

Similarly,

$$(8, 3, 9) = (4, 3, 13) + (4, 10, 13)$$

Continue this step down to the $f = 2$ level.16

We reach the steps in pairs. For each pair, take the k value, and pair them. Thus for the pair $[(2, 1, 16), (2, 13, 16)]$, we get (1, 13). The pairs we get are (1, 13), (9, 15), (3, 5), (10, 11).

Thus we derive:

$$\cos(\theta) + \cos(13\theta) = a$$

$$\cos(9\theta) + \cos(15\theta) = b$$

$$\cos(3\theta) + \cos(5\theta) = c$$

$$\cos(10\theta) + \cos(11\theta) = d$$

Remember that $\theta = \frac{2\pi}{17}$. Thus: $\cos(n\theta) = \cos([17 - n]\theta)$.

And hence the equations we derived are equivalent to

$$\cos(\theta) + \cos(4\theta) = a$$

$$\cos(2\theta) + \cos(8\theta) = b$$

$$\cos(3\theta) + \cos(5\theta) = c$$

$$\cos(6\theta) + \cos(7\theta) = d$$

(The equations that Gauss defined)

4. Manipulations of the Equations

From here, we work by manipulating the equations.

Let

$$a + b = e$$

$$c + d = f$$

Now we recall that:

$$If\ S_n = \cos(\theta) + \cos(2\theta) + \cdots + \cos(n\theta)$$

$$Then\ S_n = \frac{\sin\left(\frac{2n+1}{2}\theta\right) - \sin\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)}$$

Thus

$$e + f = -\frac{1}{2}$$

We easily calculate:

$$2ab = e + f = -\frac{1}{2}$$

$$2ac = 2a + b + d$$

$$2ad = b + c + 2d$$

$$2bc = a + 2c + d$$

$$2bd = a + 2b + c$$

$$2cd = e + f = -\frac{1}{2}$$

Summing the middle 4 equations we get:

$$2ac + 2ad + 2bc + 2bd = 4a + 4b + 4c + 4d$$

$$2ef = -2$$

$$ef = -1$$

Recall that

$$e + f = -\frac{1}{2}$$

$$ef = -1$$

Then by Vieta's we have e and f to be the roots of

$$x^2 + \frac{1}{2}x - 1 = 0$$

Upon inspection, $e > f$ so

$$e = -\frac{1}{4} + \sqrt{\frac{17}{16}}$$

$$f = -\frac{1}{4} - \sqrt{\frac{17}{16}}$$

Remember that $2ab = e + f = -\frac{1}{2}$

Thus, a and b are the roots of the equation

$$x^2 - e - \frac{1}{4} = 0$$

Upon inspection, $a > b$ so

$$a, b = \frac{1}{2}e \pm \sqrt{\frac{1}{4} + \frac{1}{4}e^2} = -\frac{1}{8} + \frac{1}{8}\sqrt{17} \pm \frac{1}{8}\sqrt{34 - 2\sqrt{17}}$$

Similarly,

$$c, d = \frac{1}{2}f \pm \sqrt{\frac{1}{4} + \frac{1}{4}f^2} = -\frac{1}{8} - \frac{1}{8}\sqrt{17} \pm \frac{1}{8}\sqrt{34 + 2\sqrt{17}}$$

Observe that $\cos(\theta) * \cos(4\theta) = \frac{1}{2}c$

Thus $\cos(\theta)$ and $\cos(4\theta)$ are the roots of the equation

$$x^2 - ax + \frac{1}{2}c$$

Thus

$$\cos(\theta) = \frac{1}{2}a + \sqrt{\frac{1}{4}a^2 - \frac{1}{2}c}$$

Since $2a^2 = 2 + b + 2c$

$$\begin{aligned}\cos(\theta) &= \frac{1}{2}a + \sqrt{\frac{1}{4} + \frac{1}{8}b - \frac{1}{4}c} \\ &= -\frac{1}{16} + \frac{1}{16}\sqrt{17} + \frac{1}{16}\sqrt{34 - 2\sqrt{17}} \\ &\quad + \frac{1}{8}\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}\end{aligned}$$

Thus

$$\begin{aligned}\cos(\theta) = &-\frac{1}{16} + \frac{1}{16}\sqrt{17} + \frac{1}{16}\sqrt{34 - 2\sqrt{17}} \\ &+ \frac{1}{8}\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}\end{aligned}$$

It is very easy now to construct, as the methods of constructing the square root of any number was well known, and all the fractions were powers of two.

5. Remarks

Gauss also conjectured that all polygons with a number of sides equivalent to any of the Fermat Primes could be constructed, and all multiples in between them could be constructed. It turned out that all of them were constructed, even the largest and last Fermat Prime, F(4), the 65537gon was constructed, although no one has yet checked the proof. However, there are many holes even in the proof of the constructability of the Heptadecagon. The most relevant question by far is why Gauss stopped decomposing when $f = 2$ in the cyclotomic equations. It was very easy to finish at $f = 1$, so why stop? Gauss did not explain.

The heptadecagon was a major stepping-stone in mathematics and led way to the discovery of the constructability of the Fermat Primes. As for Gauss, leave, knowing that he thought of this proof at age 19.

6. Sources

“Cyclotomic Field.” Cyclotomic Field - Encyclopedia of Mathematics, www.encyclopediaofmath.org/index.php/Cyclotomic_field.

Nishiyama, Yutaka. “Gauss' Method of Constructing a Regular Heptadecagon.” *International Journal of Pure and Applied Mathematics*, Academic Publications, 5 Nov. 2013, ijpam.eu/contents/2013-82-5/3/3.pdf.