Bayesian Dictionary Learning for EEG Source Identification

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Preface

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Contents

Introduction

The problem addressed throughout this thesis arise from the increasing use of electroencephalographic measurements for a wide range of scientific purposes, especially within the medical field. An electroencephalography captures electric signals caused by activity within the brain. The signals from the brain is recorded over time by multiple sensors placed on the scalp. One essential issue concerning an electroencephalography is to extract the exact sources of the captured brain activity. This is of interest when studying correlation among activities in different parts of the brain, referred to as functional integration. The recorded signal from one sensor is basically a mixture of electric signals released from a various number of active neurons within the brain, forming one or several sources. Furthermore, this mixture is distorted as it travels through the scalp. The need for source extraction is confirmed by studies showing how analysis performed on electroencephalographic measurements differs significantly from similar analysis performed directly on the original source[12].

Considering this issue of source extraction from a mathematical perspective the electroencephalographic measurements can be modelled by a linear system of equations, from which it is possible to extract a limited number of sources under certain conditions. However, it is a general acknowledged issue that the true number of sources is unknown. The task complexity of extracting the sources from the linear system is increased in cases where the number of sources exceeds the number of sensors providing measurements.

This thesis explores a state of the art mathematical method for source extraction, embracing the case of more sources than sensors. Overall this method, published in 2015, consist of two steps, that is finding receptively the mixture the signals have undergone and then extracting the source signals. The two steps originates from two different approaches considering the mathematical orientation. The main goal of the thesis is to explore and unite the necessary theory into one algorithm. The practical aspect will include an implementation of the algorithm to be tested on new electroencephalographic measurements with the purpose of supporting(?) the current results. Furthermore the problem is connected to an current application within the hearing aid industry. Here the intention is to reduce the amount of energy spent by the hearing aid user. Basically, this is attempted by identifying the listening direction

intended by the user, from analysis of the active sources measured on the user. In this thesis the number of active sources are sought related to the amount of energy used by the hearing aid user. This includes considerations upon the issue of the true number of active sources being unknown.

skal opdateres

(følgende kan skrives bedre hvis ikke det skal være et andet sted)The thesis consist of a motivational part introducing electroencephalography and the potential use within research especially in the hearing aid industry. Furthermore, existing literature considering different mathematical approaches for source extraction are examined. The Motivational part is concluded by the problem statement specifying the objective of the thesis. Next is the theoretical part. The system model is specified and the solution approach are presented. The necessary theory are introduced leading to the state of the art algorithms for source extraction. The theoretical part is followed by implementation and test of the algorithm for verification. Next is.. Finally discussion and conclusion upon the achieved results are presented followed by a consideration upon further studies.

Chapter 1

Motivation

This chapter accounts for the motivation behind source extraction from an Electroencephalography (EEG). The concept of EEG is introduced along with current applications. The potential and importance of source extraction are considered and related to the hearing aid industry. The commonly applied mathematical model for EEG measurements is presented. Currently applied methods for source extraction are considered leading to a presentation of the current state of the art methods which succeeds to overcome the limitations of previous methods. Lastly the objective of this thesis is specified.

1.1 Introduction to EEG Measurements

EEG is an imaging technique used within the medical field. EEG is measuring electric signals on the scalp, caused by brain activity. The human central nerve system consist of various nerve cells connecting the neurons within the brain. Nerve cells respond to certain stimuli, for instance a physical stimuli, and transmit informations between neurons. Generally speaking these activities induce local currents that are transferred throughout the nerve system. Several nearby simultaneous activations result in local potential fields, referred to as one signal *source*[18]. EEG measurements are provided by a number of metal electrodes, referred to as sensors, carefully placed on the human scalp. Each sensor reads the present electrical signals over time. For the source signal to reach a sensor it has to penetrate the skull, skin and several other thin layers of biological tissue. This causes an unknown distortion and reduction of a signal. It is most likely that the measurement of one sensor is a sum of multiple signals from different sources. Nor is the range of a single sensor separated from the other sensors. Thus the same signal can easily be measured by two or more sensors. The process of distorsion and mixing of signals is called volume conduction [18, p. 68] [19]. From this it is clarified that EEG measurements is a mixture of fluctuating electrical signals originating from brain activities. Due to the mixing and the nature of the signals the

true number of sources is generally considered unknown[18]. Furthermore, EEG is a subject for interfering noise. Noise signals can occur in the measurements resulting from physical movement of e.g. eyes and jawbone [21]. The concept of volume conduction is sought illustrated on figure 1.1.

The source signals are classified within four groups according to the dominant frequency. The delta wave $(0.5 - 4 \text{ Hz})$ is observed from infants and sleeping adults, the theta wave $(4 - 8 \text{ Hz})$ is observed from children and sleeping adults, the alpha wave $(8 - 13 \text{ Hz})$ is the most extensively studied brain rhythm, which is induced by an adult laying down with closed eyes. Lastly, the beta wave $(13 - 30 \text{ Hz})$ is considered the normal brain wave for adults, associated with active thinking, active attention or solving concrete problems [18, p. 11]. An example of EEG measurements within the four categories is illustrated by figure 1.2.

Generally, the distribution of EEG measurements of multiple sensors are considered multivariant Gaussian [18, p. 50]. Though the mean and covariance properties generally changes over time. Therefore EEG measurements are considered quasistationary i.e. stationary only within small intervals. This motivates the need for segmentation of the EEG measurements to achieve signals with similar characteristics.

does this comply to the non-gaussian assumption of ICA?
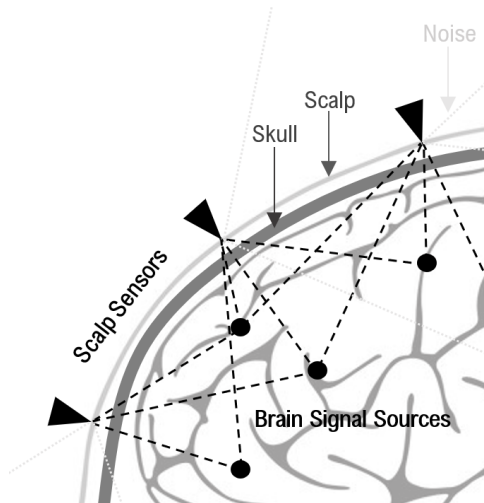
Figure 1.1: Illustration of volume conduction
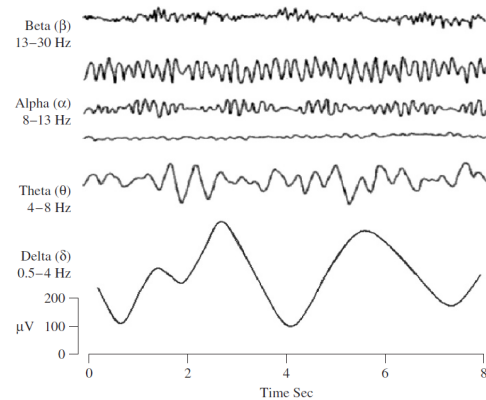
Figure 1.2: Example of time dependent EEG measurements within the four defined categories, source: [18]

1.1.1 Application

EEG performed on humans and animals have a great number of applications with both clinical and research purposes. Examples of clinical applications covers diagnosis and management of neurological disorders such as epilepsy and monitor alertness regarding coma or brain death. EEG capitalizes on the procedure being non-invasive

and fast. Neural activity can be measured within fractions of a second after a stimuli has been provided. These advantages contributes to the wide range of applications within research of the neural processes involved in or resulting from actions, emotions or cognition. Today such neural research are use in many different fields[21, p. 4]. The hearing aid industry is one example where this research is highly prioritized. At Eriksholm research center, which is a part of the hearing aid manufacturer Oticon, cognitive hearing science is a research area within fast development [20]. One main purpose at Eriksholm is to make it possible for a hearing aid to identify the user-intended sound source from real time EEG measurements and thereby exclude noise from elsewhere [2] [6]. It is essentially the well known but unsolved cocktail problem which is sought improved by use of EEG. This is where EEG and occasionally so called in-ear EEG is interesting. In conjunction with the technology of beamforming it is possible for a hearing aid to receive only signals from a specific direction.

Over the past two decades, functional integration has become an area of interest regarding EEG research [11]. Within neurobiology functional integration refers to the study of the correlation among activities in different regions of the brain. In other words, how do different parts of the brain work together to process information and conduct a response [12]. For this purpose separation and localization of the original sources which contribute to the EEG measurement is of interest. An article from 2016 [19] points out the importance of performing analysis regarding functional integration at source level rather than at EEG level. It is argued through experiments that analysis at EEG level does not allow interpretations about the interaction between sources. This emphasize a potential for improving results within a wide range of EEG research, if the original active sources can be extracted from a specific EEG measurements.

1.1.2 Modelling

Consider the issue of extracting the activated sources from EEG measurements. A known approach is to model the observed data by a linear system

$$\mathbf{y} = \mathbf{A}\mathbf{x}.$$

The vector $\mathbf{y} \in \mathbb{R}^M$ is the EEG measurement of one time sample containing M sensor measurements. $\mathbf{x} \in \mathbb{R}^N$ is the corresponding N sources within the brain. The non-zero entries of $\mathbf{x}$ represent the active sources at the time of the measurement. $\mathbf{A} \in \mathbb{R}^{M \times N}$ is an unknown projection/transformation(?) matrix, also referred to as the mixing matrix resembling the volume conduction. The ith column of $\mathbf{A}$ represents the relative projection weights from the ith source to every sensor [5]. Representing one time sample the linear system is in general referred to as a single measurement vector model. It it only the measurement vector $\mathbf{y}$ that is known hence it is not possible to solve the linear system with respect to $\mathbf{x}$ using basic linear algebra. The

task in this case is to identify both **A** and then **x**, given the measurement vector **y**. This problem is referred to as the inverse problem of EEG. Finding **x** from the inverse problem is referred to as source separation and localization. Separation is to find the signal of each active source and localization is to place each active source signal at the right position within the source vector of dimension N, where N is the maximum number of sources to be active.

Independent Component Analysis (ICA) is one commonly applied method to solve the inverse problem of EEG [15], [14]. ICA is a technique to find the matrix A such that the column wise elements of X is statistically independent. Thus statistical independence between the active sources is the essential assumption, which in the case of EEG are considered valid due to the volume conduction being effectively instantaneous[14, p. 3]. Application of ICA has shown great results regarding source separation of high-density EEG. However, a significant flaw to this method is that the EEG measurements are only separated into a number of sources that is equal to or less than the number of sensors [3]. Meaning that the EEG inverse problem can not be solved when it forms an under-determined system, which is the case when the maximum number of unknown sources N exceeds the number of sensors M. Such assumption undermines the reliability and usability of ICA, as the number of active sources easily exceed the number of sensors [5]. This is especially a drawback when low-density EEG are considered. Low-density EEG measurements are collected from equipment with less than 32 sensors, increasing the changes of M being less that N. However, improved capabilities of low-density EEG devices are desirable due to their relative low cost, mobility and ease to use.

This argues the importance of considering the inverse problem of EEG in the under-determined case where $N > M$. In the next section existing work considering the under-determined inverse problem of EEG is investigated further.

1.2   Related Work and Our Objective

As mentioned above ICA is a solid method for source separation in the case where separation into a number of sources equal to the number of sensors is adequate. The issue occurs in cases where the number of sources N exceeds the number of sensors M. To overcome this issue an extension of ICA was suggested, referred to as the ICA mixture model [3]. Instead of identifying one overcomplete mixing matrix $\mathbf{A} \in \mathbb{R}^{M \times N}$ this approach learns N_{model} different mixing matrices $\mathbf{A}_i \in \mathbb{R}^{M \times M}$, to make computations more tractable. This method was further adapted into the Adaptive Mixture ICA (AMICA) which showed successful results regarding identification of more sources than sensors [17]. However, these successful results relies on the assumption that no more than M out of N possible sources is simultaneously active. That is explicit that the source vector of dimension N has at most M non-zero entries. This assumption is still an essential limitation to the frame work,

especially when considering low-density EEG. Other types of ICA algorithms for under-determined systems have been proposed, without overcoming the limitation of jointly active sources exceeding the number of sensors.

In 2015 O. Balkan et. al. suggested a new approach also targeting the identification of more active sources than sensors regarding EEG measurements. One method is proposed for learning $\mathbf{A}$ from $\mathbf{y}$[3] and a different method is proposed for finding $\mathbf{x}$ given $\mathbf{y}$ and $\mathbf{A}$[4].

To learn $\mathbf{A}$ the suggested method, referred to as Cov-DL, is a covariance-domain based dictionary learning algorithm. The method is based upon theory of dictionary learning and compressive sensing. Which dictates a framework for solving an under-determined system when $\mathbf{x}$ contains a sufficiently amount of zeros. This is similar to the constraint of ICA. However, to overcome this the point is to transfer the EEG measurements into the covariance domain. In the covariance domain a higher dimensionality can be achieved compared to the original EEG sensor domain with dimension M. The transformation can be done when assuming a linear volume conduction and uncorrelated sources. As a result the theory of compressive sensing is found to apply to the covariance domain, allowing to learn $\mathbf{A}$ by dictionary learning – even in the case where the active sources exceeds the number of measurements.

The Cov-DL algorithm stands out from other straight forward dictionary learning methods as it does not relay on the sparsity of active sources. This is an essential advantage when low-density EEG is considered. Cov-DL was tested and found to outperform AMICA [3]. As mentioned, the Cov-DL algorithm only learns the mixing matrix $\mathbf{A}$, resembling the volume conduction.

For the purpose of recovering $\mathbf{x}$, from $\mathbf{y}$ and $\mathbf{A}$, a multiple measurement sparse Bayesian learning (M-SBL) algorithm is proposed. This method is also targeting the case of more active sources than sensors. The method was proven to outperform the previously used algorithms, even when the defined recovery conditions regarding the found mixing matrix $\mathbf{A}$ was not fulfilled[4].

One drawback, which is not fully covered in the referred literature, is that the two methods rely on the number of active sources being known. In practise this is not the case. Hence an estimation of the number of active sources has to be considered for the algorithm to be useful in practice. To address this issue a simple approach is to optimise the result with respect to the number active source, provided that some prior assumption of the expected result can be made.

The two state of the art methods resulting in source separation and localization will make the foundation of this thesis. Our aim is to investigate and fully understand the two methods in order to implement and test a joint algorithm – recovering the original sources $\mathbf{x}$ from the measurements $\mathbf{y}$, when the number of active sources exceeds the number of measurements. Secondary it is of interest to consider the practical application of the algorithm, for instance within a hearing aid as described

[margin note: is it okay to mention sparsity here for the first time?]

[margin note: yderligere beskrivelse nødvedig? eller fjerne lidt fra Cov måske, da det samme lidt kommer i kap 3?]

in section 1.1. As mentioned, the number of active sources is in general unknown in practise thus it is first of all an estimation of the number of active sources which is of interest for practical use of the algorithm. For this we want to investigate whether it is possible to estimate the number of active sources, through optimization.

Chapter 2

Problem Statement

Through chapter 1 the potential of EEG measurement and especially low-density EEG measurements have been established. Furthermore, this potential is found to be increased through recovery of the original brain sources given the measurements. This involves solving the EEG inverse problem, in the case of the problem being under-determined. Two state of the art methods are seen to solve the issue with success, but the use of a parameter unknown in practise limits the potential of the methods for practical use.

This motivates the following problem statement.

Based on state of the art methods, how can the original sources of brain activity be recovered from the EEG inverse problem, in the under-determined case, and how can this be modified to increase the potential of practical use?

From the problem statement the following sub-questions is established for clarification.

- How can Cov-DL be used to estimate the mixing matrix **A** from the overcomplete EEG inverse problem?

- How can M-SBL be used to estimate the source matrix **X** from the overcomplete EEG inverse problem?

- How can the above methods be implemented as one application performing source recovery from EEG measurements in real-time?

- How can the number of active sources be estimated, based only on the EEG measurement?

Chapter 3

System Model

Through this chapter a model representing the EEG measurements is specified. Along with the model different terminologies is introduced and described for further use in this thesis. At last the solution approach for estimating the model variables(rather than parameters right?) is described, setting the outline of the remaining chapters of the thesis.

3.1   System of Linear Equations

Let $\mathbf{y} \in \mathbb{R}^M$ be some vector. By basic linear algebra $\mathbf{y}$ can be described as a linear combination of a coefficient matrix $\mathbf{A} \in \mathbb{R}^{M \times N}$ and some scalar vector $\mathbf{x} \in \mathbb{R}^N$ such that

$$\mathbf{y} = \mathbf{A}\mathbf{x}, \tag{3.1}$$

Let $\mathbf{y}$ and $\mathbf{A}$ be known, then 3.1 makes a system of M linear equations with N unknowns, referred to as a linear system.

To solve the linear system 3.1 with respect to $\mathbf{x}$ one must look at the three different cases that can occur, depending on the relation between the number of equations M and the number of unknowns N. For $M = N$, the system has one unique solution, provided that a solution exist. If the square coefficient matrix $\mathbf{A}$ has full rank the solution can be found by inverting $\mathbf{A}$.

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}.$$

For $M > N$ the system is over-determined, having more equations than unknown. In general there is no solution to an over-determined system. An/The exception occur when the system contains a sufficient amount of linearly dependent equations. For $M < N$ the system is under-determined, having fewer equations than unknowns. There exist infinitely many solutions to an under-determined system, provided that one solution exist [8, p. ix].

> note: skal vi nævne løsnings metoder for under-determined system her?

13

Consider now $\mathbf{y} \in \mathbb{R}^M$ as the observed measurements provided by M EEG sensors at time t. The linear system 3.1 is then considered as a single measurement vector (SMV) model. Modelling the EEG measurements by the SMV model embody the following interpretations, based on chapter 1. Remember from chapter 1 that EEG measurements basically is a mixture of original brain signals affected by volume conduction. $\mathbf{x}$ is seen as the original brain signal sources, each entry representing the signal of one source. Thus, $\mathbf{x} \in \mathbb{R}^N$ is referred to as the source vector. N is considered the maximum number of sources, however zero-entries may occur. Let k denote the number of non-zero entries in $\mathbf{x}$, referred to as the active sources at time t. The projection matrix $\mathbf{A}$, referred to as the mixing matrix, models the volume conduction by mapping the source vector from $\mathbb{R}^N$ to $\mathbb{R}^M$, where M is the number of sensors hence the dimension of the measurement vector $\mathbf{y}$.

3.2   Multiple Measurement Vector Model of EEG

In practise EEG measurements are sampled over time by a certain sample frequency. Thus multiple EEG measurement vectors are achieved. Let L be the total number of samples. Now the the SMV model is expanded to include L measurement vectors:

$$\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{E}, \tag{3.2}$$

now $\mathbf{Y} \in \mathbb{R}^{M \times L}$ is the observed measurement matrix, $\mathbf{X} \in \mathbb{R}^{N \times L}$ is the source matrix, and $\mathbf{A} \in \mathbb{R}^{M \times N}$ is the mixing matrix. Furthermore $\mathbf{E} \in \mathbb{R}^{M \times L}$ is consider an additional noise matrix, to be expected from psychical measurements. The model is now referred to as the multiple measurement vector (MMV) model. As for (3.1) the solution set of the linear system (3.2) depends on the relation between N and M [8, p. 42].

In chapter 1 it is specified that the case of more sources than sensors, $N > M$, is the case of interest in this thesis.

3.2.1   Segmentation

In chapter 1 it is argued that EEG measurements are only stationary within small segments. Hence the following segmentation is considered.

Let f be the sample frequency of the observed EEG measurements $\mathbf{Y}$ and let t_s be the length of a segment. Here s is the segment index. As such the observed EEG measurements can be divided into stationary segments $\mathbf{Y}_s \in \mathbb{R}^{M \times L_s}$, possibly overlapping, where $L_s = t_s f$. For each segment the MMV model (3.2) holds and is rewritten into

her er ts ikke konstant med afhængig af index s, ikke?

$$\mathbf{Y}_s = \mathbf{A}\mathbf{X}_s + \mathbf{E}_s, \quad \forall s. \tag{3.3}$$

Due to a segment being stationary it is assumed that each source remains either active or non-active throughout the segment. Thus, $\mathbf{X}_s$, consists of k non-zero rows – the active sources.

In order to characterise the source matrix with respect the amount of non-zero rows the term row sparseness is considered. By common definition the support of the segmented source matrix $\text{supp}(\mathbf{X}_s)$ denotes the index set of non-zero rows of $\mathbf{X}_s$. To count the non-zeros row of a matrix the ℓ_0-norm is defined:

$$\|\mathbf{X}\|_0 := \text{card}(\text{supp}(\mathbf{X})),$$

where the function $\text{card}(\cdot)$ gives the cardinality of the input set. $\mathbf{X}_s$ is said to be k-sparse if it contains at most k non-zeros rows:

$$\|\mathbf{X}_s\|_0 \leq k$$

A model for the EEG measurements is now established. From the model the aim is to recover the source matrix $\mathbf{X}_s \forall s$, which gives us the separated original brain signals as intended by the problem statement. In the next section the solution method is presented and discussed – outlining the remaining chapters of the thesis.

3.3   Solution Method

It is now justified that the EEG measurements can be modelled by the multiple measurement vector model defined by the system of linear equations (3.3), including an additional noise. By the problem statement cf. chapter 2 the aim is to recover the source vector $\mathbf{X}$, in the case where the number of sensors is less than the number of sources, $M < N$. That is recovering $\mathbf{X}$ from an under-determined linear system. Therefore, the solution must be found in the infinite solution space, provided that one solution exists, thus simple linear algebra can not be used. However, by considering numerical methods such as mathematical optimization it is possible restrict the solution by some constraint and then find the unique optimal solution which respect to a defined cost relative to the solution. The theory of compressive sensing dictates a framework for solving an under-determined system when $\mathbf{X}$ is known to have non-zeros entries. Specifically a unique solution $\mathbf{X}$ can be found when $\mathbf{X}$ is M-sparse, cf. theorem B.1.1 in appendix B.1. When $\mathbf{A}$ is unknown, as it is in the current case, the concept of dictionary learning can be used to determine $\mathbf{A}$, again under the assumption that $\mathbf{X}$ is M-sparse.

As discussed in chapter 1 the aim of this thesis is to overcome the limitation of fewer sources than measurements, which is the limitation of compressive sensing.

A method to overcome this limitation, with respect to learning $\mathbf{A}$, is the Covariance-domain dictionary learning (Cov-DL) method[3], introduced in chapter 1. The method manage to leverage the increased dimensionality of the covariance domain in order to the allow the theory of compressive sensing to apply to an under-determined system. However, this method does only apply to the process of learning $\mathbf{A}$, hence a different approach is necessary to recover $\mathbf{X}$.

as the relation between $\mathbf{X}$ and the corresponding δ is unknown(?) det kan vi ikke skrive her kan vi?

For recovering $\mathbf{X}$, given both $\mathbf{Y}$ and $\mathbf{A}$, the method Multiple sparse Bayesian learning (M-SBL), introduced in chapter 1, is considered. A method which ensure that the sparsity holds(?). O. Balkan [4] did also, in 2014, proposed a method which could identify the sources, in the time-domain, by creating a likelihood which ensure the wanted sparsity of the source matrix $\mathbf{X}$ and controlled by some variance. This method is called multiple sparse Bayesian learning (M-SBL) and takes advantage of a Bayesian approach. In [4] a variance dependent log-likelihood which has been induce by a empirical prior that ensure sparsity of the likelihood has been constructed to be minimised with respect to the variance. From the log-likelihood an estimate for the source matrix $\mathbf{X}$ is drawn with respect to the support set S which has been influence by variance used in the minimization.

evt. tilføj afslutning der siger hvad der kommer i kapitlerne?

Chapter 4

Covariance-Domain Dictionary Learning

Through this chapter the method Covariance-domain dictionary learning (Cov-DL) is presented in details. Along the presentation of the general method, necessary computational details are derived for the practical solution. The purpose is to recover the mixing matrix $\mathbf{A}$ from the MMV model, derived in chapter 3, in the under-determined case. In the context of compressive sensing the matrix $\mathbf{A}$ in the MMV model is referred to as the dictionary matrix. That is the true mixing matrix is estimated as a dictionary matrix, which can take different forms. This will elaborated further in the section of dictionary learning.

obs! stemmer dette på tværs a kapitlerne, vi siger A bliver estimeret som en dictionary.

Cov-DL is an algorithm proposed by O. Balkan [3], leveraging the increased dimensionality of the covariance domain. The method have shown successful recovering of the mixing matrix $\mathbf{A}$, even in the non-sparse under-determined case with more active sources k than available measurements M, $k \geq M$. In short the algorithm consist of three steps. First the segmented MMV model of the EEG measurements is transformed into the covariance domain. Then, by the increased dimensionality of the covariance domain, it is possible to learn the mixing matrix of the covariance domain, denoted by $\mathbf{D}$, based on the theory of compressive sensing. Here two different cases will appear dependent on the relation between the number of sources N and the found dimension of the covariance domain, which of course depends on the number of measurements M. Lastly, an inverse transformation is performed on the found mixing matrix of the covariance domain $\mathbf{D}$, in order to obtain the wanted mixing matrix $\mathbf{A}$. An important aspect of this method is the prior assumption that the sources within one segment are uncorrelated, that is the rows of $\mathbf{X}_s$ being mutually uncorrelated.

The section is inspired by chapter 3 in [5] and the article [3]. Selected general theory supporting essential parts of the method is elaborated in appendix B.

4.1   Covariances Domain Representation

Consider a single sample vector $\mathbf{y}_i$, containing EEG measurements. The covariance of $\mathbf{y}_i$ is to be defined by

$$\boldsymbol{\Sigma}_{\mathbf{y}_i} = \mathbb{E}[(\mathbf{y}_i - \mathbb{E}[\mathbf{y}_i])(\mathbf{y}_i - \mathbb{E}[\mathbf{y}_i])],$$

where $\mathbb{E}[\cdot]$ is the expected value operator. Assume that all samples vectors $\mathbf{y}_i$ within one segment has zero mean and the same distribution. Then, the observed segmented EEG measurements matrix $\mathbf{Y}_s \in \mathbb{R}^{M \times L_s}$ is to be described in the covariance domain by the sample covariance $\widehat{\boldsymbol{\Sigma}}$ which is defined as the covariance among the M measurements across the L_s samples. That is a $M \times M$ matrix $\boldsymbol{\Sigma}_{\mathbf{Y}_s} = [\sigma_{jk}]$ with entries

$$\sigma_{jk} = \frac{1}{L_s} \sum_{i=1}^{L_s} y_{ji} y_{ki}.$$

Using matrix notation the sample covariance of $\mathbf{Y}_s$ can be written as

$$\widehat{\boldsymbol{\Sigma}}_{\mathbf{Y}_s} = \frac{1}{L_s} \mathbf{Y}_s \mathbf{Y}_s^T.$$

Similar the source matrix $\mathbf{X}_s$ can be described in the covariance domain by the sample covariance matrix.

$$\widehat{\boldsymbol{\Sigma}}_{\mathbf{X}_s} = \frac{1}{L_s} \mathbf{X}_s \mathbf{X}_s^T = \boldsymbol{\Lambda}_s + \varepsilon$$

From the assumption of uncorrelated sources within $\mathbf{X}_s$ the sample covariance matrix is expected to be nearly diagonal, thus it can be written as $\boldsymbol{\Lambda}_s + \varepsilon$ where $\boldsymbol{\Lambda}_s$ is a diagonal matrix consisting of the diagonal entries of $\widehat{\boldsymbol{\Sigma}}_{\mathbf{X}_s}$ and ε is the estimation error[3]. Each segment is then modelled in the covariance domain as

$$\begin{aligned}
\widehat{\boldsymbol{\Sigma}}_{\mathbf{Y}_s} = \frac{1}{L_s} \mathbf{Y}_s \mathbf{Y}_s^T &= \frac{1}{L_s} \left(\mathbf{A}\mathbf{X}_s + \mathbf{E}_s\right)\left(\mathbf{A}\mathbf{X}_s + \mathbf{E}_s\right)^T \\
\mathbf{Y}_s \mathbf{Y}_s^T &= (\mathbf{A}\mathbf{X}_s)(\mathbf{A}\mathbf{X}_s)^T + \mathbf{E}_s \mathbf{E}_s^T + \mathbf{E}_s(\mathbf{A}\mathbf{X}_s)^T + \mathbf{A}\mathbf{X}_s \mathbf{E}_s^T \\
&= \mathbf{A}\mathbf{X}_s \mathbf{X}_s^T \mathbf{A}^T + \mathbf{E}_s \mathbf{E}_s^T + \mathbf{E}_s \mathbf{X}_s^T \mathbf{A}^T + \mathbf{A}\mathbf{X}_s \mathbf{E}_s^T \\
&= \mathbf{A}(\boldsymbol{\Lambda}_s + \varepsilon)\mathbf{A}^T + \mathbf{E}_s \mathbf{E}_s^T + \mathbf{E}_s \mathbf{X}_s^T \mathbf{A}^T + \mathbf{A}\mathbf{X}_s \mathbf{E}_s^T \\
&= \mathbf{A}\boldsymbol{\Lambda}_s \mathbf{A}^T + \mathbf{A}\varepsilon\mathbf{A}^T + \mathbf{E}_s \mathbf{E}_s^T + \mathbf{E}_s \mathbf{X}_s^T \mathbf{A}^T + \mathbf{A}\mathbf{X}_s \mathbf{E}_s^T \quad (4.1) \\
&= \mathbf{A}\boldsymbol{\Lambda}_s \mathbf{A}^T + \widetilde{\mathbf{E}} \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (4.2)
\end{aligned}$$

From (4.1) to (4.2) all terms where noise is included are defined as a united noise term $\widetilde{\mathbf{E}}$. By vector notation (4.2) is rewritten to be vectorized. Because the covariance matrix $\widehat{\boldsymbol{\Sigma}}_{\mathbf{Y}_s}$ is symmetric it is sufficient to vectorize only the lower triangular parts,

including the diagonal. For this the function $\text{vec}(\cdot)$ is defined to map a symmetric $M \times M$ matrix into a vector of size $\frac{M(M+1)}{2} := \widetilde{M}$ making a row-wise vectorization of its upper triangular part. Furthermore, let $\text{vec}^{-1}(\cdot)$ be the inverse function for devectorisation. This results in the following model

$$\widehat{\boldsymbol{\Sigma}}_{\mathbf{Y}_s} = \sum_{i=1}^{N} \mathbf{a}_i \boldsymbol{\Lambda}_{s_{ii}} \mathbf{a}_i^T + \widetilde{\mathbf{E}}$$

$$
\begin{aligned}
\text{vec}(\widehat{\boldsymbol{\Sigma}}_{\mathbf{Y}_s}) &= \sum_{i=1}^{N} \text{vec}(\mathbf{a}_i \mathbf{a}_i^T) \boldsymbol{\Lambda}_{s_{ii}} + \text{vec}(\widetilde{\mathbf{E}}) \\
&= \sum_{i=1}^{N} \mathbf{d}_i \boldsymbol{\Lambda}_{s_{ii}} + \text{vec}(\widetilde{\mathbf{E}}) \\
&= \mathbf{D}\boldsymbol{\delta}_s + \text{vec}(\widetilde{\mathbf{E}}), \quad \forall s. \quad (4.3)
\end{aligned}
$$

Here $\boldsymbol{\delta}_s \in \mathbb{R}^N$ contains the diagonal entries of the source sample-covariance matrix $\boldsymbol{\Lambda}_s$ and the matrix $\mathbf{D} \in \mathbb{R}^{\widetilde{M} \times N}$ consists of the columns $\mathbf{d}_i = \text{vec}(\mathbf{a}_i \mathbf{a}_i^T)$. Note that $\mathbf{D}$ and $\boldsymbol{\delta}_s$ are unknown while $\text{vec}(\widehat{\boldsymbol{\Sigma}}_{\mathbf{Y}_s})$ is known from the observed data. By this transformation to the covariance domain one segments is now represented by the single measurement model with $\widetilde{M}$ "measurements". It has been shown that this transformed model allows for identification of $k \leq \widetilde{M}$ active sources [16], which is a much weaker sparsity constraint than the original sparsity constraint $k \leq M$. The purpose of the Cov-DL algorithm is to leverage this model to find the dictionary $\mathbf{A}$ from $\mathbf{D}$ and then still allow for $k \leq \widetilde{M}$ active sources to be identified. That is the number of active sources are allowed to exceed the number of observations as intended.

4.2 Determination of the Mixing Matrix

The goal is now to learn first $\mathbf{D}$ and then the associated mixing matrix $\mathbf{A}$. Two methods are considered relying on the relation of M and N. For now the noise vector is ignored.

4.2.1 Under-determined system

When $N > \widetilde{M}$ the transformed model (4.3) makes an under-determined system. This is similar to the original MMV model (3.2) being under-determined when $N > M$. Thus, it is from the theory of compressive sensing again possible to solve the under-determined system if a certain sparsity is withhold. Namely $\boldsymbol{\delta}_s$ being $\widetilde{M}$-sparse. Assuming the sufficient sparsity on $\boldsymbol{\delta}_s$ is withhold it is possible to learn the dictionary matrix of the covariance domain $\mathbf{D}$ by traditional dictionary learning methods applied

to the observations represented in the covariance domain $\mathrm{vec}(\widehat{\boldsymbol{\Sigma}}_{\mathbf{Y}_s})$ for all segments s.

Dictionary Learning

Within the theory of compressive sensing the matrix $\mathbf{A}$ is referred to as a dictionary matrix, as it determines how a sparse vector $\mathbf{x}$ is transformed to the original non-sparse signal. When the dictionary is not known i prior it is essential how to choose the the dictionary matrix in order to achieve the best recovery, of the sparse vector $\mathbf{x}$ from the measurements $\mathbf{y}$. This is clarified from the proof of theorem B.1.1 in appendix B.1. One choice is a pre-constructed dictionary. In many cases the use of a pre-constructed dictionary results in simple and fast algorithms for reconstruction of $\mathbf{x}$ [9]. However, a pre-constructed dictionary is typically fitted to a specific kind of data. For instance the discrete Fourier transform or the discrete wavelet transform are used especially for sparse representation of images [9]. Hence the results of using such dictionaries depend on how well they fit the data of interest, which is creating a certain limitation.

The alternative option is to consider an adaptive dictionary based on a set of training data that resembles the data of interest. For this purpose learning methods are considered to empirically construct a fixed dictionary which can take part in the application. There exist several dictionary learning algorithms. One is the K-SVD algorithm which was presented in 2006 by Elad et al. and found to outperform pre-constructed dictionaries, when computational cost is of secondary interest [1]. The concept of the K-SVD algorithm is introduced, and the more detailed algorithm is to be found in appendix B.2. Consider the measurement matrix $\mathbf{Y} \in \mathbb{R}^M$ consisting of measurement vectors $\{\mathbf{y}_j\}_j^L$ making a set of L training examples forming a linear system

$$\mathbf{y}_j = \mathbf{A}\mathbf{x}_j.$$

from which one can learn a suitable dictionary $\hat{\mathbf{A}}$, and the sparse representation of the source matrix $\hat{\mathbf{X}} \in \mathbb{R}^N$ with the source vectors $\{\hat{\mathbf{x}}_j\}_j^L$. For a known sparsity constraint k the dictionary learning can be defined by the following optimisation problem.

$$\min_{\mathbf{A},\mathbf{X}} \sum_{j=1}^{L} \|\mathbf{y}_j - \mathbf{A}\mathbf{x}_j\|_2^2 \quad \text{subject to} \quad \|\mathbf{x}_j\|_1 \le k, \ 1 \le j \le L. \tag{4.4}$$

where both $\mathbf{A}$ and $\mathbf{x}_j$ are variables to be determined. Learning the dictionary by the K-SVD algorithm constitute joint solving of the optimization problem with respect to $\mathbf{A}$ and $\mathbf{X}$ respectively. An initial $\mathbf{A}_0 = [\mathbf{a}_0, \cdots, \mathbf{a}_N]$ and the corresponding $\mathbf{X}_0$ is determined. Then, for each iteration an update rule is applied to each column of $\mathbf{A}_0$, that is updating first $\mathbf{a}_j$ and then the corresponding row $\mathbf{x}_i$. More details on

comparison to l1 problem is removed here, ok?

the K-SVD algorithm is found in appendix B.2. The uniqueness of $\mathbf{A}$ depends on the recovery sparsity condition. As clarified earlier in 3.3 the recovery of a unique solution $\mathbf{X}^*$ is only possible if $k < M$ [5].

Application of dictionary learning

By the establishments of an dictionary learning algorithm it is now used to learn the transformed dictionary matrix $\mathbf{D}$ in (4.3). Here the transformed and vectorised measurements $\left\{\text{vec}\left(\hat{\Sigma}_{\mathbf{Y}}\right), \forall s\right\}$ makes the training dataset. By this note that each segment the original measurement sample constitute only one sample in the covariance domain. Thus the number of training samples depends on the length of a segment. When K-SVD is applied and $\mathbf{D}$ is found it is possible to estimate the mixing matrix $\mathbf{A}$ that generated found $\mathbf{D}$ through the relation

$$\mathbf{d}_j = \text{vec}(\mathbf{a}_j \mathbf{a}_j^T).$$

Here each column is found from the optimisation problem

$$\min_{\mathbf{a}_j} \|\text{vec}^{-1}(\mathbf{d}_j) - \mathbf{a}_j \mathbf{a}_j^T\|_2^2,$$

for which the global minimizer is $\mathbf{a}_j^* = \sqrt{\lambda_j} \mathbf{b}_j$. Here λ_j is the largest eigenvalue of $\text{vec}^{-1}(\mathbf{d}_j)$,

> redegørelse for resultatet her skal laves

$$\text{vec}^{-1}(\mathbf{d}_j) = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1N} \\ d_{21} & d_{22} & \cdots & d_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ d_{N1} & d_{N2} & \cdots & d_{NN} \end{bmatrix}, \quad j \in [N]$$

and $\mathbf{b}_j$ is the corresponding eigenvector.

By this each column of the mixing matrix $\mathbf{A}$ can be estimated hence it is possible to determine the mixing matrix in the case where the measurements transformed into the covariance domain makes an under-determined system, but the necessary sparsity constraint, $\boldsymbol{\delta}_s$ being $\widetilde{M}$-sparse (instead of M-sparse), is withhold.

4.2.2 Over-determined system

Consider again the measurements represented in the covariance domain (4.3). In the case of $N < \widetilde{M}$ an over-determined system is achieved where $\mathbf{D}$ is high and thin. In general such system is inconsistent. Thus it is not possible to find $\mathbf{D}$ by traditionally dictionary learning methods and different methods most be considered.

> tjek, har vi tidligerer nævnt at der teoretisk godt kan være en løsning?

When $N < \widetilde{M}$ it is certain from the model (4.3) that the transformed measurements $\text{vec}(\widehat{\boldsymbol{\Sigma}}_{\mathbf{Y}_s})$ will live on or near a subspace of dimension N. This subspace is spanned by the columns of $\mathbf{D}$, and is denoted as $\mathcal{R}(\mathbf{D})$. To learn $\mathcal{R}(\mathbf{D})$ without having to impose any sparsity constraint on $\boldsymbol{\delta}_s$ it is possible to use Principal Component

Analysis(PCA). When PCA is applied to the set $\left\{\mathrm{vec}(\widehat{\boldsymbol{\Sigma}}_{\mathbf{Y}_s}), \forall s\right\}$ a set of N principal components are found. The principal components forms a set of basis vectors $\mathbf{U}$ such that $\mathcal{R}(\mathbf{U}) = \mathcal{R}(\mathbf{D})$. However, this do not imply that $\mathbf{D} = \mathbf{U}$. In the case of two sets of basis vectors spanning the same space, namely $\mathcal{R}(\mathbf{U}) = \mathcal{R}(\mathbf{D})$, the projection operator of the given subset must be unique(need prove here? or is there just one $P : V \to V$ where $V = \mathcal{R}(U) = \mathcal{R}(D)$). The projection matrix $P : \mathcal{R}(U) \to \mathcal{R}(U)(?)$ can be find by considering the projection of an vector $\mathbf{b} \in \mathbb{R}^{\widetilde{M}}$ onto $\mathcal{R}(U)$, that is solving the least squares problem $\|\mathbf{A}\mathbf{x} - b\|$ where $\mathbf{A}\mathbf{x} \subset \mathcal{R}(U)$ and $P = Ax$. The solution is given by the normal equation

$$\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$$
$$\mathbf{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$$

resulting in

$$P\mathbf{b} = \mathbf{A}\mathbf{x} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$$
$$P = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$$

Thus $\mathcal{R}(U)$ and $\mathcal{R}(D)$ having the same projection matrix is true if and only if $\mathbf{D}(\mathbf{D}^T\mathbf{D})^{-1}\mathbf{D}^T = \mathbf{U}(\mathbf{U}^T\mathbf{U})^{-1}\mathbf{U}^T$. Now, remember from the relation between $\mathbf{A}$ and $\mathbf{D}$ that $\mathbf{d}_i = \mathrm{vec}(\mathbf{a}_i\mathbf{a}_i^T)$. From this it is possible to obtain $\mathbf{A}$ through the following optimisation problem

$$\min_{\{\mathbf{a}_i\}_{i=1}^N} \|\mathbf{D}(\mathbf{D}^T\mathbf{D})^{-1}\mathbf{D}^T - \mathbf{U}(\mathbf{U}^T\mathbf{U})^{-1}\mathbf{U}^T\|_F^2$$
$$\text{s.t. } \mathbf{d}_i = \mathrm{vec}(\mathbf{a}_i\mathbf{a}_i^T) \tag{4.5}$$

where $\mathbf{U}$ results from PCA performed on $\mathrm{vec}(\widehat{\boldsymbol{\Sigma}}_{\mathbf{Y}_s})$. In the following section the optimization problem is analysed and processed in order to determine a suitable solution method. Additional optimization theory to support the analysis is found in appendix **??**

4.2.3   Solution to optimization problem

The optimization problem (4.5) consist of an objective function forming a least-square problem with respect to the frobenius norm. That is a convex quadratic objective function. The constraints is a set of quadratic equality constraints. In general it is a thumb rule that non-linear equality constraint are not convex. Due to the constraints not being considered convex the optimization problem does not meet the requirements of a convex optimization problem. Hence the numerical solution methods for convex optimization problems, for which convergence is ensured, does not imply directly.

> fact: any column in D must be a lin comb. of columns in U and visa versa.
> Any basis has the same dim(dimension theorem).

Due to the nature of the constraints it should be possible to reformulate the objective function to include the constraints into the objective function. That is construction an unconstrained least-squares problem, which is a special subclass of convex optimization[7].

Let $\mathbf{D} = f(\mathbf{a}_0, \cdots, \mathbf{a}_N)$ then... an optimization problem is without constraints is achieved... and it can be solved be use of the gradient decent method or something

4.2.4 Pseudo Code of the Cov-DL Algorithm

Algorithm 1 Cov-DL

1: **procedure** Cov-DL($\mathbf{Y}_s$)
2:     **for** $s \leftarrow 1, \cdots, \text{n\_seg}$ **do**
3:         compute sample covariance matrix $\widehat{\mathbf{\Sigma}}_{\mathbf{Y}_s}$
4:         $\mathbf{y}_{\text{cov}_s} = \text{vec}(\widehat{\mathbf{\Sigma}}_{\mathbf{Y}_s})$
5:     **end for**
6:     $\mathbf{Y}_{\text{cov}} = \{\mathbf{y}_{\text{cov}_s}\}_{s=1}^{\text{n\_seg}}$
7:     **if** $N > \frac{M(M+1)}{2}$ **then**
8:         **procedure** K-SVD($\mathbf{Y}_{\text{cov}}$)
9:             returns $\mathbf{D} \in \mathbb{R}^{(M(M+1))/2 \times N}$
10:         **end procedure**
11:         **for** $j \leftarrow 1, \cdots, N$ **do**
12:             $\mathbf{T} = \text{vec}^{-1}(d_j)$
13:             $\lambda_j \leftarrow \max\{\text{eigenvalue}(\mathbf{T})\}$
14:             $\mathbf{b}_j \leftarrow \text{eigenvector}(\lambda_j)$
15:             $\mathbf{a}_j \leftarrow \sqrt{\lambda_j}\mathbf{b}_j$
16:         **end for**
17:         $\mathbf{A} = \{\mathbf{a}_j\}_{j=1}^N$
18:     **end if**
19:
20:     **if** $N < \frac{M(M+1)}{2}$ **then**
21:         **procedure** PCA($\text{vec}(\mathbf{\Sigma}_{\mathbf{Y}_s})$)
22:             returns $\mathbf{U} \in \mathbb{R}^{(M(M+1))/2 \times N}$
23:         **end procedure**
24:         **procedure** Quasi-Newton(problem (4.5))
25:             returns $\mathbf{A} = \{\mathbf{a}_j\}_{j=1}^N$
26:         **end procedure**
27:     **end if**
28: **end procedure**

4.3   Considerations and Remarks

Through this chapter different theory aspects haven been investigated to create a foundation to present one method to be use in the localisation of the sources from EEG measurements – the recovering of the mixing matrix $\mathbf{A}$ – yet one method is still to be presented. Before the method to recover the source matrix $\mathbf{X}$ from the found mixing matrix $\mathbf{A}$ and EEG measurements $\mathbf{Y}$ will be introduced some considerations and remarks regarding the Cov-DL algorithm must be taken – this will be used in the implementation of the algorithm which will be described in chapter 6.

The length of each segment determined whenever the covariance of the source matrix $\mathbf{X}$ can be described as a diagonal matrix $\mathbf{\Lambda}$. That is a segment of L_s samples becomes stationary and therefore the sources within that segment becomes uncorrelated – the covariance of the source can be described by a diagonal matrix. The number of samples L_s used in one segment affect whenever the segment is stationary or not. This must be taken into account in the preprocessing part of the baseline algorithm when the EEG measurements are divided into segments.

For the Cov-DL algorithm when $\mathbf{D}$ is under-determined a dictionary learning algorithm K-SVD is used to learn the matrix $\mathbf{D}$ and by that an estimate for the mixing matrix $\hat{\mathbf{A}}$. Because of the segmentation the number of samples used in the dictionary learning are reduced remarkably and will affect the learning process. This is another point which must be taken into account in the preprocessing part of the code. To improved the dictionary learning the overlapping of the segments can be look into as each segment will have some similarity and therefore learn towards one direction.

For the Cov-DL algorithm when $\mathbf{D}$ is over-determined the solution tends to be unique when $M < N < \widetilde{M}$ from testing the solution. That is the cost function tends toward a local minima and therefore an unique solution occur in first run of one trial. For the baseline algorithm it would therefore be necessary to include several random initial points when finding the mixing matrix $\mathbf{A}$ for $\mathbf{D}$ being over-determined.

For a general perspective the sources within the source matrix $\mathbf{X}$ must not be constant over time when using the MMV model (3.2) ...

Find lige kilde på dette argument

Chapter 5

Multiple Sparse Bayesian Learning

In this chapter the multiple sparse Bayesian learning (M-SBL) method is described in details. As the method leverage a Bayesian framework the generel concept of Bayesian inference is shortly introduced prior to the the M-SBL method, with respect to the model of interest (5.1). The chapter is inspired by [22] and the articles [23], [4].

Consider again the multiple measurement vector (MMV) model for a non-segmented case of EEG measurements

$$\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{E}, \tag{5.1}$$

with measurement matrix $\mathbf{Y} \in \mathbb{R}^{M \times L}$, sources matrix $\mathbf{X} \in \mathbb{R}^{N \times L}$ mixing matrix $\mathbf{A} \in \mathbb{R}^{M \times N}$ and noise matrix $\mathbf{E} \in \mathbb{R}^{M \times L}$. Note that $\mathbf{A}$ is known throughout the chapter, as it was found by Cov-DL in chapter 4.

The aim is to recover the source matrix $\mathbf{X}$ in the case of fewer measurements than active sources, $k > M$. In [4] is proven that exact localization of the active sources can be achieved with M-SBL for $k > M$, when two sufficient conditions satisfied. With M-SBL the basic approach is to find the support set S providing the non-zeros rows of the source matrix $\mathbf{X}$ which corresponds to localization of the active sources.

5.1   Bayesian Inference

Bayesian probability differs from the classical probability (*the frequentist, relative frequency with which* $\mathbf{A}$ *occurs*) by interpreting probability as reasonable expectation. In general the Bayesian framework builds upon the task of inferring what the model parameters must be, given the model and data. This is centred around Bayes'

theorem, a posterior distribution of some unobserved variable $\mathbf{C}$ given some observed variable $\mathbf{B}$

$$p(\mathbf{C}|\mathbf{B}) = \frac{p(\mathbf{B}|\mathbf{C})p(\mathbf{C})}{p(\mathbf{B})},$$

where $p(\mathbf{B}|\mathbf{C})$ is the probability density function of $\mathbf{B}$ given $\mathbf{C}$, also referred to as the likelihood function of $\mathbf{C}(?)$, $p(\mathbf{C})$ is a prior distribution of $\mathbf{C}$ and $p(\mathbf{B})$ is the distribution of $\mathbf{B}$ serving as a normalizing parameter. By maximizing the posterior distribution $p(\mathbf{C}|\mathbf{B})$ with respect to $\mathbf{C}$ the maximum a posteriori (MAP) estimate is achieved

$$\mathbf{C}_{\mathrm{MAP}} = \arg\max_{\mathbf{C}} \frac{p(\mathbf{B}|\mathbf{C})p(\mathbf{C})}{p(\mathbf{B})}.$$

That is the estimate of $\mathbf{C}$ with the highest probability of causing the given variable $\mathbf{B}$.

With the basic concept of the framework settled, consider now the current non-segmented MMV model (5.1), where we want to estimate the model parameter – the source matrix $\mathbf{X}$ – given the measurement matrix $\mathbf{Y}$. In the desired case where $M < N$ the MMV model (5.1) makes an under-determined system and potentially an infinitely number of solutions exist with equal likelihoods. Within this Bayesian framework the source matrix $\mathbf{X}$ can be seen as a variable which is drawn from some distribution $p(\mathbf{X})$, as such it is possible to narrow down the infinitely solution space. Assuming a prior belief that $\mathbf{Y}$ is generated from a sparse source matrix, gives a so-called sparsity inducing prior. That is $\mathbf{X}$ is drawn from a distribution which has a sharp, possibly infinite, spike at zero surrounded by fat tails. Now the estimation of $\mathbf{X}$ from the MMV model (5.1) can be view as the following MAP estimation task

$$\hat{\mathbf{X}} = \arg\max_{\mathbf{X}} \frac{p(\mathbf{Y}|\mathbf{X})p(\mathbf{X})}{p(\mathbf{Y})},$$

with a prior distribution $p(\mathbf{X}) \propto \exp\left(-\|\mathbf{X}\|_0\right)$ [22, p. 14]. Different MAP estimation approaches exists separated by the choice of sparsity inducing prior and optimization method. However, regardless of the approach some problems have shown to occur when using a fixed and algorithm-dependent prior. One issue is the posterior not being sparse enough if a prior is not as sparse, leading to non-recovery. Another issue is that a combinatorial number of suboptimal local solutions can occur. By use of automatic relevance determination (ARD) the problems related to the fixed sparse prior can be avoided [22, p. 20]. The main asset of this alternative approach is the use of an empirical prior. That is an flexible prior distribution which depends on an unknown set of hyperparameters, which is to be learned from the data.

5.1.1   Empirical Bayesian Estimation

Assume the likelihood function $p(\mathbf{Y}|\mathbf{X})$ is Gaussian, with known noise variance σ^2.

Then for each column in $\mathbf{Y}$ and $\mathbf{X}$ the likelihood is written as

$$p(\mathbf{y}_{\cdot j}|\mathbf{x}_{\cdot j}) = \mathcal{N}(\mathbf{A}\mathbf{x}_{\cdot j}, \sigma^2\mathbf{I})$$
$$= (2\pi)^{-\frac{M}{2}}|\sigma^2\mathbf{I}|^{-\frac{1}{2}}\exp\left(-\frac{1}{2\sigma^2\mathbf{I}}\|\mathbf{y}_{\cdot j} - \mathbf{A}\mathbf{x}_{\cdot j}\|_2^2\right)$$

The empirical prior is now defined by application of ARD. The i-th row of the source matrix $\mathbf{x}_{i\cdot}$, has an L-dimensional independent Gaussian prior with zero mean and a variance controlled by an unknown hyperparameter γ_i:

$$p(\mathbf{x}_{i\cdot}; \gamma_i) = \mathcal{N}(0, \gamma_i\mathbf{I}).$$

By combing the row priors

$$p(\mathbf{X}; \boldsymbol{\gamma}) = \prod_{i=1}^{N} p(\mathbf{x}_{i\cdot}; \gamma_i),$$

a full prior of $\mathbf{X}$ is achieved modulated by the hyperparameter vector $\boldsymbol{\gamma} = [\gamma_1, \ldots, \gamma_N]^T$. By combining the full prior and the likelihood $p(\mathbf{y}_{\cdot j}|\mathbf{x}_{\cdot j})$ the posterior of the j-th column of the source matrix $\mathbf{X}$ becomes

$$p(\mathbf{x}_{\cdot j}|\mathbf{y}_{\cdot j}; \boldsymbol{\gamma}) = \frac{p(\mathbf{y}_{\cdot j}|\mathbf{x}_{\cdot j}; \boldsymbol{\gamma})p(\mathbf{x}_{\cdot j}; \boldsymbol{\gamma})}{p(\mathbf{y}_{\cdot j}|\boldsymbol{\gamma})}$$
$$= \frac{p(\mathbf{y}_{\cdot j}|\mathbf{x}_{\cdot j}; \boldsymbol{\gamma})p(\mathbf{x}_{\cdot j}; \boldsymbol{\gamma})}{\int p(\mathbf{y}_{\cdot j}|\mathbf{x}_{\cdot j})p(\mathbf{x}_{\cdot j}; \boldsymbol{\gamma})\ d\mathbf{x}_{\cdot j}}$$
$$= \frac{p(\mathbf{x}_{\cdot j}, \mathbf{y}_{\cdot j}; \boldsymbol{\gamma})}{\int p(\mathbf{x}_{\cdot j}, \mathbf{y}_{\cdot j}; \boldsymbol{\gamma})\ d\mathbf{x}_{\cdot j}}$$
$$= \mathcal{N}(\boldsymbol{\mu}_{\cdot j}, \boldsymbol{\Sigma}), \tag{5.2}$$

where the denominator is the marginal likelihood of $\mathbf{y}_{\cdot j}$ also referred to as the evidence, this is elaborated in the next section. The mean and covariance of (5.2) is given as

$$\boldsymbol{\Sigma} = \text{Cov}(\mathbf{x}_{\cdot j}|\mathbf{y}_{\cdot j}; \boldsymbol{\gamma}) = \boldsymbol{\Gamma} - \boldsymbol{\Gamma}\mathbf{A}^T\boldsymbol{\Sigma}_y^{-1}\mathbf{A}\boldsymbol{\Gamma}, \quad \forall j = 1, \ldots, L \tag{5.3}$$
$$\mathcal{M} = [\boldsymbol{\mu}_{\cdot 1}, \ldots, \boldsymbol{\mu}_{\cdot L}] = \mathbb{E}[\mathbf{X}|\mathbf{Y}; \boldsymbol{\gamma}] = \boldsymbol{\Gamma}\mathbf{A}^T\boldsymbol{\Sigma}_y^{-1}\mathbf{Y}, \tag{5.4}$$

where $\boldsymbol{\Gamma} = \text{diag}(\boldsymbol{\gamma})$ and $\boldsymbol{\Sigma}_y = \sigma^2\mathbf{I} + \mathbf{A}\boldsymbol{\Gamma}\mathbf{A}^T$.

Let the posterior mean $\mathcal{M}$ serve as the point estimate for the source matrix $\mathbf{X}$. It is clear that row sparsity is achieved whenever $\gamma_i = 0$. From this the posterior must satisfy the following

$$\mathbb{P}(\mathbf{x}_{i\cdot} = \mathbf{0}|\mathbf{Y}; \gamma_i = 0) = 1,$$

this ensures that the posterior mean $\mathcal{M}$ of the i-th row, $\boldsymbol{\mu}_{i\cdot}$, become zero, whenever $\gamma_i = 0$ as desired.

note one whole row is controlled by the same hyperparameter, leading to the same support over time

uddyb udledning af udtrykkende her

From this it is evident that for estimating the support set of $\mathbf{X}$ it is sufficient to estimate the hyperparameter $\boldsymbol{\gamma}$, from which the support set S can be extracted. Furthermore, the point estimate of $\mathbf{X}$, providing the source signal estimate, is given by $\mathcal{M}$ [22, p. 147]. This leads to the actual M-SBL algorithm for which the aim is to estimate $\boldsymbol{\gamma}$ and the corresponding $\mathcal{M}$.

5.2   M-SBL for estimation of X

Different hyperparameters $\boldsymbol{\gamma}$ correspond to different hypothesis for the prior distribution of the underlying generation of $\mathbf{Y}$. Therefore the determination of $\boldsymbol{\gamma}$ is seen as a model selection. Due to the empirical Bayesian strategy the unknown variables, making the source matrix $\mathbf{X}$ are integrated out as follows. By integrating the likelihood of $\mathbf{Y}$ with respect to the unknown sources $\mathbf{X}$ the marginal likelihood of the observed mixed data $\mathbf{Y}$ is achieved as $p(\mathbf{Y}; \boldsymbol{\gamma})$ [22, p. 146]. The resulting marginal likelihood is to be maximised with respect to $\boldsymbol{\gamma}$, that is the resulting ARD based M-SBL cost function. The $-2\log(\cdot)$ transformation is applied in order to the cost function to be minimized

$$
\begin{aligned}
\ell(\boldsymbol{\gamma}) &= -2\log\left(\int p(\mathbf{Y}|\mathbf{X})p(\mathbf{X};\boldsymbol{\gamma})\,d\mathbf{X}\right) \\
&= -2\log(p(\mathbf{Y};\boldsymbol{\gamma})) \\
&= \log(|\boldsymbol{\Sigma}_y|) + \frac{1}{L}\sum_{j=1}^{L}\mathbf{y}_{\cdot j}^T\boldsymbol{\Sigma}_y^{-1}\mathbf{y}_{\cdot j}.
\end{aligned}
$$

To minimise the marginal log likelihood $\ell(\boldsymbol{\gamma})$ with respect to $\boldsymbol{\gamma}$ the expectation maximisation (EM) algorithm can be used. The E-step of the EM algorithm is to compute the posterior moments using (5.3) and (5.4) while the M-step is the following update rule of γ_i [22, p.147]

$$
\gamma_i^{(k+1)} = \frac{1}{L}\|\boldsymbol{\mu}_{i\cdot}\|_2^2 + \boldsymbol{\Sigma}_{ii}, \quad \forall i = 1,\ldots,N.
$$

The M-step is very slow on large data. Instead one could use a fixed point update to fasten the convergence on large data, however convergence is no longer ensured. The fixed point updating step is achieved by taking the derivative of the marginal log likelihood $\ell(\boldsymbol{\gamma})$ with respect to $\boldsymbol{\gamma}$ and equating it with zero. This lead to the following update equation which can replace the above M-step in the EM-algorithm:

$$
\gamma_i^{(k+1)} = \frac{\frac{1}{L}\|\boldsymbol{\mu}_{i\cdot}\|_2^2}{1 - \gamma_i^{-1(k)}\boldsymbol{\Sigma}_{ii}}, \quad \forall i = 1,\ldots,N.
$$

Empirically this alternative update rule have shown use full in highly under-determined large scale cases by driving many hyper parameters toward zero allowing for the corresponding weight in the source matrix to be discarded. For simultaneous sparse

approximation problems this is the process referred to as multiple sparse Bayesian learning, M-SBL.

From the resulting $\boldsymbol{\gamma}^*$ the support set S of the source matrix $\mathbf{X}$ can be extracted,

$$S = \{i | \hat{\gamma}_i \neq 0\},$$

concluding the localization of active sources within $\mathbf{X}$. In practise some arbitrary small threshold can be used such that that any sufficiently small hyperparameter is discarded. For identification of the active sources the estimate of the source matrix $\mathbf{X}$ is given as $\mathbf{X}^* = \mathcal{M}^* \approx \mathbf{X}$, with $\mathcal{M}^* = \mathbb{E}[\mathbf{X} | \mathbf{Y}; \boldsymbol{\gamma}^*]$. This leads to the following estimate

$$\mathbf{X}^* = \begin{cases} \mathbf{x}_{i\cdot} = \boldsymbol{\mu}_{i\cdot}^*, & i \in S \\ \mathbf{x}_{i\cdot} = \mathbf{0}, & i \notin S \end{cases}$$

5.2.1   Pseudo Code for the M-SBL Algorithm

Algorithm 2 M-SBL

1: **procedure** M-SBL($\mathbf{Y}, \mathbf{A}$, iterations)
2:    $\boldsymbol{\gamma} = \mathbf{1} \in \mathbb{R}^{\text{iterations}+2 \times N \times 1}$
3:    iter = 0
4:    **while** $\boldsymbol{\gamma} \geq 10^{-16}$ **do**
5:       $\boldsymbol{\Gamma} = \text{diag}(\boldsymbol{\gamma}^{\text{iter}})$
6:       **for** $i = 1, \ldots, N$ **do**
7:          $\boldsymbol{\Sigma} = \boldsymbol{\Gamma} - \boldsymbol{\Gamma}\mathbf{A}^T\boldsymbol{\Sigma}_y^{-1}\mathbf{A}\boldsymbol{\Gamma}$
8:          $\mathcal{M} = \boldsymbol{\Gamma}\mathbf{A}^T\boldsymbol{\Sigma}_y^{-1}\mathbf{Y}$
9:          $\gamma_i^{(\text{iter}+1)} = \dfrac{\frac{1}{L}\|\boldsymbol{\mu}_{i\cdot}\|_2^2}{1 - \gamma_i^{-1(\text{iter})}\Sigma_{ii}}$
10:      **end for**
11:      **if** iter = iterations **then**
12:         Break
13:      **end if**
14:      iter += 1
15:   **end while**
16:   Return $\mathcal{M}^*, \boldsymbol{\gamma}^*$
17: **end procedure**
18: **procedure** SUPPORT($\mathcal{M}^*, \boldsymbol{\gamma}^*, k$)
19:    Support $= \mathbf{0} \in \mathbb{R}^k$
20:    $\boldsymbol{\gamma}_{\text{value}} = \boldsymbol{\gamma}^*(-2)$
21:    **for** j in range(k) **do**
22:       **if** $\boldsymbol{\gamma}_{\text{value}}(\arg\max(\boldsymbol{\gamma}_{\text{value}})) \mathrel{!}= 0$ **then**
23:          Support(j) $= \arg\max(\boldsymbol{\gamma}_{\text{value}})$
24:          $\boldsymbol{\gamma}_{\text{value}}(\arg\max(\boldsymbol{\gamma}_{\text{value}})) = 0$
25:       **end if**
26:    **end for**
27:    $\mathbf{X} = \mathbf{0} \in \mathbb{R}^{N \times L-2}$
28:    **for** i in Support **do**
29:       $\mathbf{X}(i) = \mathcal{M}^*(-1)(i)$
30:    **end for**
31:    Return $\mathbf{X}$
32: **end procedure**

5.2.2   Sufficient Conditions for Exact Source Localization

In [4] it is proven that exact source localization is guaranteed in the under-determined case, $k > M$ when the following conditions on $\mathbf{A}$ is fulfilled. The theorem is based on

a theoretical analysis of the minima where noise-free conditions are considered, that is letting $\sigma^2 \to 0$. Thus the following theorem applies to the noise less case. First, defined a function $f : \mathbb{R}^{M \times N} \to \mathbb{R}^{\frac{M(M+1)}{2} \times N}$, such that for $B = f(\mathbf{A})$ the j-th column is given as $\mathbf{b}_{\cdot j} = \text{vec}(\mathbf{a}_{\cdot j}\mathbf{a}_{\cdot j}^T)$. Here the function $\text{vec}(\cdot)$ corresponds to the function defined in section 4.1, being a vectorization of the lower triangular part of a matrix.

Theorem 5.2.1

Given a dictionary matrix $\mathbf{A}$ and a set of observed measurement $\mathbf{Y}$, M-SBL recovers the support set of any size k exactly in the noise-free case, if the following conditions are satisfied.

1. The active sources $\mathbf{X}_S$ are orthogonal. That is, $\mathbf{X}_S\mathbf{X}_S^T = \mathbf{\Lambda}$, where $\mathbf{\Lambda}$ is a diagonal matrix and S the support set.

2. $\text{Rank}(f(\mathbf{A})) = N$.

The proof can be found in [4, p. 16].

Chapter 6

Baseline Algorithm

From the start of this thesis to the start of this chapter a motivation behind investigation the identification of sources from EEG measurement with more sources than sensors as been exploit. From a linear multiple measurement vector (MMV) model

$$\mathbf{Y} = \mathbf{AX},$$

theory and methods behind recovering the mixing matrix $\mathbf{A}$ and the source matrix $\mathbf{X}$ have been investigated and researched. This lead to two algorithms – covariance-domain dictionary learning (Cov-DL) algorithm and multiple sparse Bayesian learning (M-SBL) algorithm – which recover a mixing matrix and a source matrix from the EEG measurements with more sources than sensors, $N > M$.

The goal of this chapter is to describe how the two found algorithms can be implemented into one algorithm – the baseline algorithm – and what one must have in mind when combining the two algorithms. Furthermore, some tests of the baseline algorithm, with different data sets of different type of data, will be performed with the purpose to investigate how good the recovering process is.

The chapter will begin with a discussion of the choice for the parameters M, N and k as in the realistic case with real EEG measurements the number of active sources within the brain k are unknown – which parameters lead to a good recovering?

6.1   Parameter Choice

1. Describe why we want to set $N = k$, logical discussion. No results yet to back up the assuming.

   - The amount of N is unknown as it change for every brain
   - Therefore, we do not know the amount of activation inside the brain, but it those there are of interest in this thesis.

2. Talk about the parameter choice for M and N – a low-density system have
   what? $M = 8$ and $N = 16$ or $N = 32$.

6.2   Tests of Parameters Performance

With the implemented algorithms – Cov-DL and M-SBL – the goal is now to investi-
gate the performance of each algorithm. The performance will be measure by a error
measure – the mean square error (MSE) – this will be presented in the next section.

Each test will be looking at one parameter of the algorithm where it is the
performance of that parameter there will be measured. The parameters of interest
are the initial $\mathbf{A}_{\mathrm{ini}}$ using in the Cov-DL for the optimisation part of finding the matrix
$\mathbf{D}$, and the segmentations in Cov-DL as well.

First at all, we will run a test, with random setting, to see how the error margin
is for both algorithms.

6.2.1   Error

To evaluate performance of the algorithms we will be looking at the differences
between the real and estimated matrices, mixing matrix $\mathbf{A}$ and source matrix $\mathbf{X}$.
For this task a mean squared error (MSE) method has been chosen. The MSE
method measure the average squared difference between some estimated value and
the actual value. The MSE can be written as

$$\mathrm{MSE} = \frac{1}{T} \sum_{i=1}^{T} (G_i - \hat{G}_i)^2,$$

with $T = \{M, k\}$ the numbers of rows of $\mathbf{A}$ or $\mathbf{X}$, $G_i = \{\mathbf{A}_i, \mathbf{X}_i\}$ is a measurement row
of the actual matrix and $\hat{G}_i = \{\hat{\mathbf{A}}_i, \hat{\mathbf{X}}_i\}$ is an estimated row of the estimated matrix.
The MSE is viewed as a measure of the quality of an estimator, in this case of how
M-SBL and Cov-DL perform. For a large MSE value the estimated matrix/values
are dispersed widely around its mean while for a small MSE value the estimated
matrix/values is closely dispersed around the mean. Usually, a small MSE value
indicate a good estimator but the value cannot be to small as this would indicate

**Vi skal lige finde en god
kilde som siger dette**

that the data has been overfitted . Therefore, a good MSE and therefore a good
performance would be depending on how the data is scattered as widely scattered
data may lead to a MSE value not close to zero but it would still be the a good
measure for the estimator.

6.2.2   Simulation of Data Sets

For the test of the performance of the algorithms some simulated data sets will be
used as we then know the real mixing matrix and source matrix which will be using
when measuring the performance with MSE.

Two data sets has been made, one which is just a simple data set with simple measurement, and one which is more random and fluctuated – more like real EEG measurements.

Simple Data Set

The simple data set is a date set which do not represent realistic data, EEG measurements, but is only used to confirm that the two algorithm works and to test the performance. The simple data set is constructed from four different signals representing the sources of the source matrix $\mathbf{X}$:

- a sinus signal $\sin(2t)$

- a sinus signal $\sin(4t)$

- a sawtooth signal with period $2\pi t$

- a sign signal of $\sin(3t)$

with t being a time index defined in the interval $[0, 10]$ with L samples. Each of the four signal are randomly drawn and used to construct a source matrix $\mathbf{X}$ of size $k \times L$. Therefore some of the signals may occur two or more time as sources. The mixing matrix $\mathbf{A}$ of size $M \times k$ is randomly generated from a Gaussian distribution. Its rows are then normalised. By multiplying the source matrix and the mixing matrix the measurement matrix $\mathbf{Y}$ is achieved. The simple data set then consist of $\{\mathbf{Y}, \mathbf{X}, \mathbf{A}\}$ which are the true values of the MMV model.

An illustration of the measurement matrix $\mathbf{Y}$ from the simple data set can be seen in figure **??**. The data set is constructed for $M = 3$, $k = 4$ and $L = 1000$.
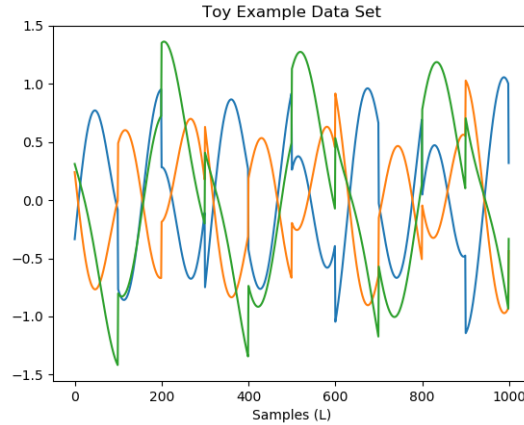

Figure 6.1: All the signal of the simple data set for $M = 3$, $k = 4$ and $L = 1000$.

Autoregressive

The second data set illustrate a more realistic data set. The data set is constructed from different autoregressive processes:

- $\mathbf{x}^t = \mathbf{a1}^{t-1} \cdot \mathbf{x}^{t-1} + \mathbf{a1}^{t-2} \cdot \mathbf{x}^{t-2} + \mathbf{w1}_t$

- $\mathbf{x}^{t+1} = \mathbf{a2}_t \cdot \mathbf{x}_{t-1} + \mathbf{a}_t \cdot \mathbf{x}_t + \mathbf{w}_t$

- $\mathbf{x}_{t+1} = \mathbf{a}_t \cdot \mathbf{x}_{t-2} + \mathbf{a}_t \cdot \mathbf{x}_t + +\mathbf{a}_t \cdot \mathbf{x}_{t-1}\mathbf{w}_t$

- $\mathbf{x}_{t+1} = \mathbf{a}_t \cdot \mathbf{x}_t + \mathbf{a}_t \cdot \mathbf{x}_{t-3} + \mathbf{w}_t$

with t being a time index defined in the interval $[0, 10]$ with L samples. Each of the four signal are randomly drawn and used to construct a source matrix $\mathbf{X}$ of size $k \times L$. With a source matrix $\mathbf{X}$ a mixing matrix $\mathbf{A}$ of size $M \times k$ is randomly generated from a Gaussian distribution and then normalised. By multiplying the source matrix and the mixing matrix the measurement matrix $\mathbf{Y}$ is achieved and this complete the toy example data set.

An illustration of the data set can be seen in figure **??**. The data set is constructed for $M = 3$, $k = 4$ and $L = 1000$.
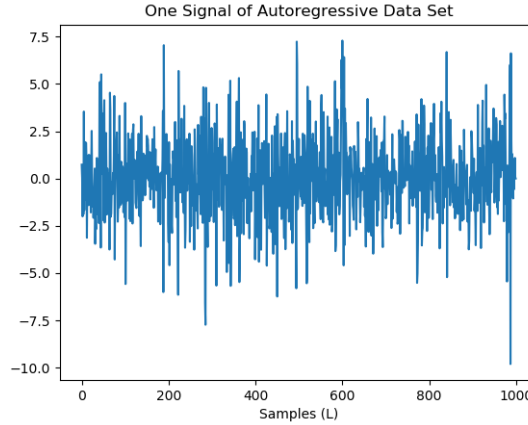

Figure 6.2: All the signal of the toy example data set for $M = 3$, $k = 4$ and $L = 1000$.

6.2.3   Tests

INTRODUCTION...

Before any testing of the performance of the algorithms one must test if the algorithm works. The Cov-DL and M-SBL has been tested with the toy example data to see if the algorithms give us a results.

In the following figure we see the recovered sources aligned with the real source of a system with $M = 3$ and $k = 4$ for $L = 100$ – a small system.
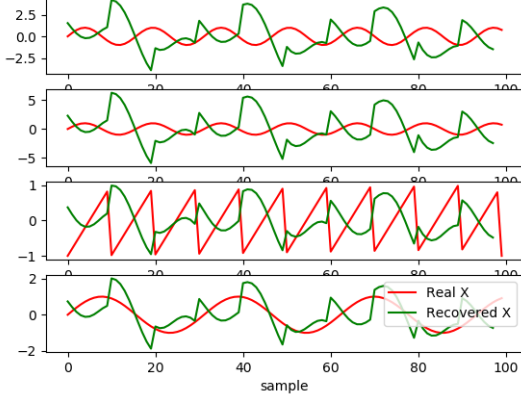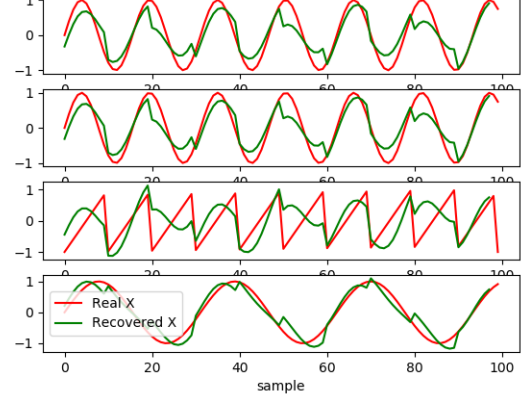

Figure 6.3: Estimated A

Figure 6.4: Real A

Furthermore, if we look at the at the mse then we have a representation error of

	Estimated A	Real A
MSE of A	1.76	×
MSE of X	4.28	0.14

Table 6.1: Toy Example Data Set

Initial A in Cov-DL

In the Cov-DL algorithm described in XX the over-determined system use the principal component analysis to make an optimization problem from which the matrix **D** can be found. To solve the optimization problem and finding **D** an initial $\mathbf{A}_{\mathrm{ini}}$ is used as a starting point in the optimization process. The choice of this initial $\mathbf{A}_{\mathrm{ini}}$ will effect how the good an estimate our found mixing matrix $\hat{\mathbf{A}}$ is.

In this section we will be testing three different choice for $\mathbf{A}_{\mathrm{ini}}$:

- A matrix **A1** drawn from a continuous uniform distribution in the half-open interval $[0.0, 1.0)$

- A matrix **A2** drawn from a uniform distribution in the half-open interval $[-1.0, 1.0)$

- A matrix **A3** drawn from a Gaussian distribution with mean 0 and variance 1

The test of different initial A's will be performed on the AR data set as this resemble the real data, EEG measurements, at most. The goal with this test is to find the best initial A, with lowest error, such that when the baseline algorithm is used on realistic data, the parameters as been chosen with the best performance in mind – leading to the best scenario of finding the true mixing matrix and source matrix from EEG measurements.

As the initial A will be used in finding an estimate for the mixing matrix which then be use to finding an estimate of the source matrix, we will look at the performance/error of both Cov-DL and M-SBL to find the best choice in both algorithms.

For the Cov-DL and M-SBL algorithm we are looking at a system of size $M = 8$, $k = 16$ and $L = 1000$. Furthermore, for the Cov-DL the data has been divided in segments of 10 samples.

	A1	**A2**	**A3**
MSE of A	1.09	1.12	1.13
MSE of X	0.72	0.74	0.72

Table 6.2: Toy Example Data Set

	A1	**A2**	**A3**
MSE of A	2.04	2.14	1.99
MSE of X	9.81	9.09	9.73

Table 6.3: AR Data Set

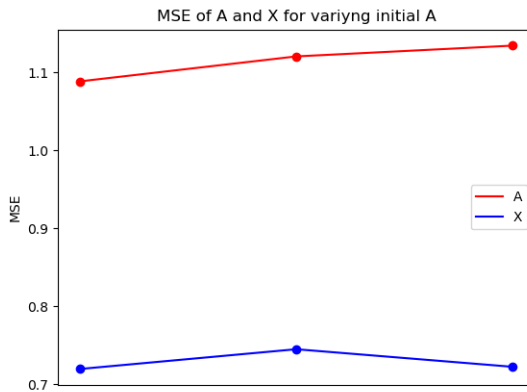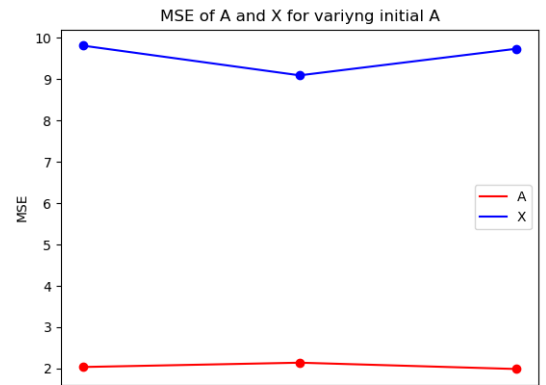**Figure 6.5:** Toy Example Data Set   **Figure 6.6:** AR Data Set

Segmentation in Cov-DL

<div style="display: flex;">
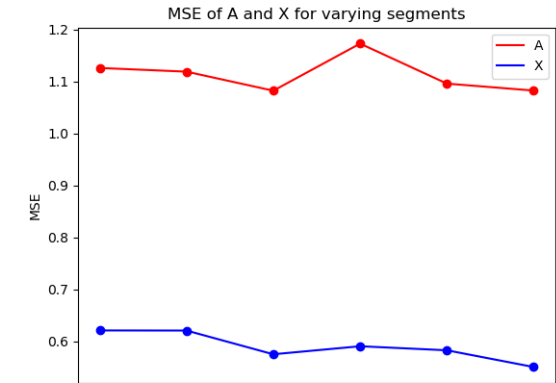
Figure 6.7: Toy Example Data Set   **Figure 6.8:** AR Data Set
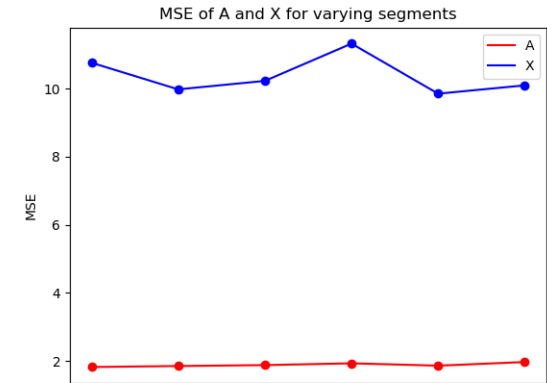
</div>

Toy Example Data Set						
	10	20	30	40	50	60
MSE of A	1.13	1.12	1.08	1.17	1.10	1.08
MSE of X	0.62	0.62	0.58	0.59	0.58	0.55
AR Data Set						
	10	20	30	40	50	60
MSE of A	1.82	1.85	1.88	1.93	1.86	1.97
MSE of X	10.76	9.98	10.23	11.33	9.85	10.10

Table 6.4: Toy and AR Data Set

6.3 Set-Up of the Baseline Algorithm

1. Introduction to the section

2. Insert the flow diagram and describe how the two algorithm are connected.

3. Perhaps end the section with some description of the complexity of the code.

6.4 Test of the Baseline Algorithm

1. Test the algorithm for the toy example

- Test for $M = 8$ and varying $N = 8, 16, 32$. Plot the errors in the same plot.
- Test for $M = 16$ and varying $N = 16, 32$. Plot the errors in the same plot.
- Compare the errors with the errors achieved with the real A

2. Test the algorithm for the AR data

- Test for $M = 8$ and varying $N = 8, 16, 32$. Plot the errors in the same plot.
- Test for $M = 16$ and varying $N = 16, 32$. Plot the errors in the same plot.
- Compare the errors with the errors achieved with the real A

array([2.06408178, 2.15138486, 2.11953172])
    array([2.74595238e+04, 8.69452640e+00, 6.81332825e+00])
    A = array([2.00624627, 1.98957772]) X = array([1577.47059511, 8.12263095])

6.5   Conclusion

Opsumering/konklusion af kapitel – henvisning/led hen til test af rigtig data

Chapter 7

Estimation of Active Sources

Chapter 8

Test on EEG measurement

Chapter 9

Discussion

Chapter 10

Conclusion

Chapter 11

Further Studies

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Appendix A

Extended ICA Algorithms

This appendix provide an extension to the basic algorithm for ICA regarding the measure of non-Gaussianity and the computation method. This extended algorithm is referred to as fast ICA and is more commonly used for source separation. This is the algorithm used to apply ICA on EEG measurements for comparison within the thesis.

A.1   Fixed-Point Algorithm - FastICA

An advantage of gradient algorithms is the possibility of fast adoption in non-stationary environments due the use of all input, $\mathbf{y}$, at once. A disadvantage of the gradient algorithm is the resulting slow convergence, depending on the choice of γ for which a bad choice in practise can disable convergence. A fixed-point iteration algorithm to maximise the non-Gaussianity is an alternative that could be used.

Consider the gradient step derived in section **??**. In the fixed point iteration the sequence of γ is omitted and replaced by a constant. This builds upon the fact that for a stable point of the gradient algorithm the gradient must point in the direction of $\mathbf{b}_j$, hence be equal to $\mathbf{b}_j$. In this case adding the gradient to $\mathbf{b}_j$ does not change the direction and convergence is achieved.

Letting the gradient given in (**??**) be equal to $\mathbf{w}$ and considering the same simplifications again suggests the new update step as [13, p. 179]

$$\mathbf{b}_j \leftarrow \mathbb{E}[\mathbf{y}(\mathbf{b}_j^T \mathbf{y})^3] - 3\mathbf{b}_j.$$

After the fixed point iteration $\mathbf{b}_j$ is again divided by its norm to withhold the constraint $\|\mathbf{b}_j\| = 1$. Instead of γ the fixed-point algorithm compute $\mathbf{b}_j$ directly from previous $\mathbf{b}_j$.

The fixed-point algorithm is referred to as FastICA. The algorithm has shown to converge fast and reliably, then the current and previous $\mathbf{w}$ laid in the same direction [13, p. 179].

wiki: The fixed point is stable if the absolute value of the derivative of $\mathbf{w}$ at the point is strictly less than 1?

A.1.1   Negentropy

An alternative measure of non-Gaussianity is the negentropy, which is based on the differential entropy. The differential entropy H of a random vector $\mathbf{y}$ with density $p_y(\boldsymbol{\eta})$ is defined as

$$H(\mathbf{y}) = - \int p_y(\boldsymbol{\eta}) \log(p_y(\boldsymbol{\eta})) \, d\boldsymbol{\eta}.$$

The entropy describes the information that a random variable gives. The more unpredictable and unstructured a random variable is higher is the entropy, e.g. Gaussian random variables have a high entropy, in fact the highest entropy among the random variables of the same variance [13, p. 182].

Negentropy is a normalised version of the differential entropy such that the measure of non-Gaussianity is zero when the random variable is Gaussian and non-negative otherwise. The negentropy J of a random vector $\mathbf{y}$ is defined as

$$J(\mathbf{y}) = H(\mathbf{y}_{\text{gaus}}) - H(\mathbf{y}),$$

with $\mathbf{y}_{\text{gaus}}$ being a Gaussian random variable of the same covariance and correlation as $\mathbf{y}$ [13, p. 182].

As the kurtosis is sensitive for outliers the negentropy is instead difficult to compute computationally as the negentropy require a estimate of the pdf. As such an approximation of the negentropy is needed.

To approximate the negentropy it is common to use the higher order comulants including the kurtosis. The following approximation is stated without further elaboration, the derivation can be found in [13, p. 182].

A.1.2   Fixed-Point Algorithm with Negentropy

Maximization of negentropy by use of the fixed-point algorithm is now presented, for derivation of the fixed point iteration see [13, p. 188]. Algorithm 3 show Fast ICA using negentropy, this is the algorithm which is implemented for comparison with the source separation methods which are tested in this thesis.

Algorithm 3 Fast ICA – with negentropy

1: **procedure** PRE-PROCESSING($\mathbf{y}$)
2:     Center measurements $\mathbf{y} \leftarrow \mathbf{y} - \bar{\mathbf{y}}$
3:     Whitening $\mathbf{y} \leftarrow \mathbf{y}_{white}$
4: **end procedure**
5:
6: **procedure** FASTICA($\mathbf{y}$)
7:     $k = 0$
8:     Initialise random vector $\mathbf{b}_{j(k)}$ ▷ unit norm
9:     **for** $j \leftarrow 1, 2, \cdots, N$ **do**
10:         **while** convergance critia not meet **do**
11:             $k = k + 1$
12:             $\mathbf{b}_{j(k)} \leftarrow \mathbb{E}[\mathbf{y}(\mathbf{b}_j^T \mathbf{y})] - \mathbb{E}[g'(\mathbf{b}_j^T \mathbf{y})]\mathbf{b}_j$ ▷ g defined in [13, p. 190]
13:             $\mathbf{b}_{j(k)} \leftarrow \mathbf{b}_j / \|\mathbf{b}_j\|$
14:         **end while**
15:         $x_j = \mathbf{b}_j^T \mathbf{y}$
16:     **end for**
17: **end procedure**

Appendix B

Supplementary theory for chapter 4

description of content of the chapter.

B.1 Introduction to Compressive Sensing

Compressive sensing is the theory of efficient recovery of a signal from a minimal number of observed measurements. It is build upon empirical observations assuring that many signals can be approximated by remarkably sparser signals. Assume linear acquisition of the observed measurements, then the relation between the measurements and the signal to be recovered can be modelled by the multiple measurement vector (MMV) model (3.2) [10].

Through this section the introduction of the theory behind compressive sensing will be presented for one measurement vector of (3.2), $\mathbf{y}$, such that the theory is based on the linear system (3.1). This will be done for simplicity but the theory will still apply for the extend linear system (3.2).

In compressive sensing terminology, $\mathbf{x} \in \mathbb{R}^N$ is the signal of interest which is sought recovered from the EEG measurement $\mathbf{y} \in \mathbb{R}^M$ by solving the linear system (3.1). In the typical compressive sensing case the system is under-determined, $M < N$, and there will therefore exist infinitely many solutions, provided that one solution exist. However, by enforcing certain sparsity constraints it is possible to recover the wanted signal, hence the term sparse signal recovery [10]. The sparsity constraints are the ones presented in 3.1 where the ℓ_0 is introduced to count the non-zeros of the signal of interest, the source vector $\mathbf{x}$. The number of non-zeros (active sources) k describe how sparse the source vector is.

To find a k-sparse solution to the linear system (3.1) it can be viewed as the

following optimisation problem.

$$\mathbf{x}^* = \arg\min_{\mathbf{x} \in \mathbb{C}} \|\mathbf{x}\|_0 \quad \text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{y}.$$

Unfortunately, this optimisation problem is non-convex due to the definition of the ℓ_0-norm and is therefore difficult to solve – it is an NP-hard problem. Instead, by replacing the ℓ_0-norm with the ℓ_1-norm, the optimisation problem can be approximated and hence becomes computationally feasible [8, p. 27]

$$\mathbf{x}^* = \arg\min_{\mathbf{x} \in \mathbb{C}} \|\mathbf{x}\|_1 \quad \text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{y}. \tag{B.1}$$

> 4.1: skal vi indfør z som en approximation til x. og så et nyt omega eller? eller kan vi lade x* være løslingen til både P0 og P1

With this optimisation problem the best k-sparse solution $\mathbf{x}^*$ can be found. The optimisation problem is referred to as ℓ_1 optimisation problem or Basis Pursuit. The following theorem justifies that the ℓ_1 optimisation problem finds a sparse solution [10, p. 62-63].

Theorem B.1.1
A mixing matrix $\mathbf{A} \in \mathbb{R}^{M \times N}$ is defined with columns $\mathbf{A} = [\mathbf{a}_1, \ldots, \mathbf{a}_N]$. By assuming uniqueness of a solution $\mathbf{x}^*$ to

$$\min_{\mathbf{x} \in \mathbb{R}^N} \|\mathbf{x}\|_1 \quad \text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{y},$$

the system $\{\mathbf{a}_j, j \in \text{supp}(\mathbf{x}^*)\}$ is linearly independent, and in particular

$$\|\mathbf{x}^*\|_0 = \text{card}(\text{supp}(\mathbf{x}^*)) \le M.$$

Proof
Assume that the set $\{\mathbf{a}_l, l \in S\}$ of l columns from matrix $\mathbf{A} \in \mathbb{R}^{M \times N}$ is linearly dependent with the support $S = \text{supp}(\mathbf{x}^*)$. Thus a non-zero vector $\mathbf{v} \in \mathbb{R}^N$ supported on S exists such that $\mathbf{A}\mathbf{v} = \mathbf{0}$ – the system is linear dependent. The unique solution $\mathbf{x}^*$ can then be written as, for any $t \ne 0$,

$$\|\mathbf{x}^*\|_1 < \|\mathbf{x}^* + t\mathbf{v}\|_1 = \sum_{l \in S} |x_l^* + tv_l| = \sum_{l \in S} \text{sgn}(x_l^* + tv_l)(x_l^* + tv_l). \tag{B.2}$$

For a small $|t|$

$$|t| < \min_{l \in S} \frac{|x_l^*|}{\|\mathbf{v}\|_\infty},$$

then the sign function become

$$\text{sgn}(x_l^* + tv_l) = \text{sgn}(x_l^*), \quad \forall l \in S.$$

By including this result in (B.2) and remembering $t \neq 0$:

$$\|\mathbf{x}^*\|_1 < \sum_{l \in S} \mathrm{sgn}(x_l^*)(x_l^* + t v_l) = \sum_{l \in S} \mathrm{sgn}(x_l^*)x_l^* + t \sum_{l \in S} \mathrm{sgn}(x_l^*)v_l = \|\mathbf{x}^*\|_1 + t \sum_{l \in S} \mathrm{sgn}(x_l^*)v_l.$$

From this it can be seen that it is always possible to choose $t \neq 0$ small enough such that

$$t \sum_{l \in S} \mathrm{sgn}(x_l^*)v_l \leq 0,$$

which contradicts that $\mathbf{v}$ make the columns of $\mathbf{A}$ linear dependent. Therefore, the set $\{\mathbf{a}_l, l \in S\}$ must be linearly independent. ■

From the theorem is must be conclude that the choice of the mixing matrix $\mathbf{A}$ has a significant impact on whenever a unique solution $\mathbf{x}^*$ exist for the ℓ_1 optimisation problem (B.1). Therefore, when recovering $\mathbf{A}$, some considerations regarding the recovering process of $\mathbf{A}$ must be taken into account. A method for the recovering of $\mathbf{A}$ could be to use a dictionary. This will be explain in the following section 4.2.1.

An alternative solution method to the ℓ_1 optimisation includes greedy algorithms such as the Orthogonal Matching Pursuit (OMP) [10, P. 65]. The OMP algorithm is an iteration process where an index set S is updated – at each iteration – by adding indices corresponding to the columns of $\mathbf{A}$ which describe the residual best possible, hence greedy. The vector $\mathbf{x}$ is then updated by a vector supported on S which minimise the residual, that is the orthogonal projection of $\mathbf{y}$ onto the span$\{\mathbf{a}_l \mid l \in S\}$.

B.2 K-SVD Algorithm

The dictionary learning algorithm K-SVD provides an updating rule which is applied to each column of $\mathbf{A}_0 = [\mathbf{a}_0, \ldots, \mathbf{a}_N]$ where $\mathbf{A}_0$ being a random initial dictionary matrix. Updating first $\mathbf{a}_j$ and then the corresponding coefficients in $\mathbf{X}$ which it is multiplied with the i-th row in $\mathbf{X}$ denoted by $\mathbf{x}_{i\cdot}$. Let $\mathbf{a}_{j_0}$ be the column to be updated and let the remaining columns be fixed. By rewriting the objective function in (4.4) using matrix notation it is possible to isolate the contribution from $\mathbf{a}_{j_0}$.

> Tjek nedenstående udledning. a og x er ikke lige lange da a_j er M lang mens x_i. er L lang

$$
\begin{aligned}
\|\mathbf{Y} - \mathbf{A}\mathbf{X}\|_F^2 &= \left\| \mathbf{Y} - \sum_{j=1}^{N} \mathbf{a}_j \mathbf{x}_{i\cdot} \right\|_F^2 \\
&= \left\| \left(\mathbf{Y} - \sum_{j \neq j_0}^{N} \mathbf{a}_j \mathbf{x}_{i\cdot} \right) - \mathbf{a}_{j_0} \mathbf{x}_{i_0 \cdot} \right\|_F^2,
\end{aligned} \tag{B.3}
$$

where $i = j$, $i_0 = j_0$ and where F is the Frobenius norm that works on matrices

$$\|\mathbf{A}\|_F = \sqrt{\sum_{i=1}^{M} \sum_{j=1}^{N} |a_{ij}|^2}.$$

In (B.3) the term in the parenthesis is denoted by $\mathbf{E}_{j_0}$, an error matrix, and hence by minimising (B.3) with respect to $\mathbf{a}_{j_0}$ and $\mathbf{x}_{i_0.}$ leads to the optimal contribution from j_0

$$\min_{\mathbf{a}_{j_0}, \mathbf{x}_{i_0.}} \| \mathbf{E}_{j_0} - \mathbf{a}_{j_0} \mathbf{x}_{i_0.} \|_F^2 . \tag{B.4}$$

The optimal solution to (B.4) is known to be the rank-1 approximation of $\mathbf{E}_{j_0}$. This comes from the Eckart–Young–Mirsky theorem [?] saying that a partial single value decomposition (SVD) makes the best low-rank approximation of a matrix such as $\mathbf{E}_{j_0}$. The SVD is given as

$$\mathbf{E}_{j_0} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \in \mathbb{R}^{M \times N},$$

with $\mathbf{U} \in \mathbb{R}^{M \times M}$ and $\mathbf{V} \in \mathbb{R}^{N \times N}$ being unitary matrices[1] and $\mathbf{\Sigma} = \mathrm{diag}\left[\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_M\right] \in \mathbb{R}^{M \times N}$ a diagonal matrix. $\boldsymbol{\sigma}_j$ are the non-negative singular values of $\mathbf{E}_{j_0}$. The best k-rank approximation to $\mathbf{E}_{j_0}$, with $k < \mathrm{rank}(\mathbf{E}_{j_0})$ is then given by :

$$\mathbf{E}_{j_0}^{(k)} = \sum_{j=1}^{k} \boldsymbol{\sigma}_j \mathbf{u}_j \mathbf{v}_j^T .$$

Since the outer product always have rank-1 letting $\mathbf{a}_{j_0} = \mathbf{u}_1$ and $\mathbf{x}_{i_0.} = \boldsymbol{\sigma}_j \mathbf{v}_1^T$ solves the optimisation problem (B.4). However in order to preserve the sparsity in $\mathbf{X}$ while optimising, only the non-zero entries in $\mathbf{x}_{i_0.}$ are allowed to vary. For this purpose only a subset of columns in $\mathbf{E}_{j_0}$ is considered, those which correspond to the non-zero entries of $\mathbf{x}_{i_0.}$. A matrix $\mathbf{P}_{i_0}$ is defined to restrict $\mathbf{x}_{i_0.}$ to only contain the non-zero-rows corresponding to N_{j_0} non-zero rows:

$$\mathbf{x}_{i_0.}^{(R)} = \mathbf{x}_{i_0.} \mathbf{P}_{i_0}$$

where R denoted the restriction. By applying the SVD to the error matrix which has been restricted $\mathbf{E}_{j_0}^{(R)} = \mathbf{E}_{j_0} \mathbf{P}_{i_0}$ and updating $\mathbf{a}_{j_0}$ and $\mathbf{x}_{i_0.}^{(R)}$ the rank-1 approximation is found and the original representation vector is updated as $\mathbf{x}_{i_0.} = \mathbf{x}_{i_0.}^{(R)} \mathbf{P}_{i_0}^T$.

The main steps of K-SVD is described in algorithm 4.

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[1]Unitary matrix: $\mathbf{U}^T\mathbf{U} = \mathbf{U}\mathbf{U}^T = \mathbf{I}$

Algorithm 4 K-SVD

1: $k = 0$
2: Initialize random $\quad\mathbf{A}_{(0)}$
3: Initialize $\quad\mathbf{X}_{(0)} = \mathbf{0}$
4:
5: **procedure** K-SVD($\mathbf{A}_{(0)}$)
6: $\quad$ Normalize columns of $\mathbf{A}_{(0)}$
7: $\quad$ **while** error $\geq$ limit **do**
8: $\quad\quad$ $j = j + 1$
9: $\quad\quad$ **for** $j \leftarrow 1, 2, \ldots, L$ **do** $\qquad\qquad\qquad\qquad$ $\triangleright$ updating each col. in $\mathbf{X}_{(k)}$
10: $\quad\quad\quad$ $\hat{\mathbf{x}}_j = \min_{\mathbf{x}} \|\mathbf{y}_j - \mathbf{A}_{(k-1)}\mathbf{x}_j\|$ $\quad$ subject to $\quad \|\mathbf{x}_j\| \leq k$ $\;$ $\triangleright$ use Basis Pursuit
11: $\quad\quad$ **end for**
12: $\quad\quad$ $\mathbf{X}_{(k)} = \{\hat{\mathbf{x}}_j\}_{j=1}^{L}$
13: $\quad\quad$ **for** $j_0 \leftarrow 1, 2, \cdots, N$ **do**
14: $\quad\quad\quad$ $\Omega_{j_0} = \{j \mid 1 \leq j \leq L, \mathbf{X}_{(k)}[j_0, j] \neq 0\}$
15: $\quad\quad\quad$ From Ω_{j_0} define $\mathbf{P}_{i_0}$
16: $\quad\quad\quad$ $\mathbf{E}_{j_0} = \mathbf{Y} - \sum_{j \neq j_0}^{N} \mathbf{a}_j \mathbf{x}_i.$
17: $\quad\quad\quad$ $\mathbf{E}_{j_0}^{(R)} = \mathbf{E}_{j_0}\mathbf{P}_{i_0}$
18: $\quad\quad\quad$ $\mathbf{E}_{j_0}^{(R)} = \mathbf{U\Sigma V}^T$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\triangleright$ perform SVD
19: $\quad\quad\quad$ $\mathbf{a}_{j_0} \leftarrow \mathbf{u}_1$ $\qquad\qquad\qquad\qquad$ $\triangleright$ update the j_0 col. in $\mathbf{A}_{(k)}$
20: $\quad\quad\quad$ $(\mathbf{x}_{i_0\cdot})^{(R)} \leftarrow \boldsymbol{\sigma}_1 \mathbf{v}_1$
21: $\quad\quad\quad$ $\mathbf{x}_{i_0\cdot} \leftarrow (\mathbf{x}_{i_0\cdot})^{(R)} \mathbf{P}_{i_0}^T$ $\qquad\qquad$ $\triangleright$ update the i_0 row in $\mathbf{X}_{(k)}$
22: $\quad\quad$ **end for**
23: $\quad\quad$ error $= \|\mathbf{Y} - \mathbf{A}_{(k)}\mathbf{X}_{(k)}\|_F^2$
24: $\quad$ **end while**
25: **end procedure**

B.3 Principal Component Analysis

B.4 General Optimation Theory...or more specific what?