MATHEMATICS TUTORIAL
03
FOURIER SERIES

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01 D10 A

1) Obtain fourier series of $f(x) = e^{-x}$, in interval $0 < x < 2\pi$

Hence deduce $\sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2}$, further derive series of $\operatorname{cosech} \pi$

$\rightarrow$ let $f(x) = e^{-ax} = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$\therefore \quad a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi} \int_0^{2\pi} e^{-x}\,dx = \frac{1}{2\pi} \left[-e^{-x}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi} (-e^{-2\pi} + 1) = \frac{1 - e^{-2\pi}}{2\pi}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\,dx = \frac{1}{\pi} \int_0^{2\pi} e^{-x} \cos nx\,dx$$

$$= \frac{1}{\pi} \cdot \frac{1}{1+n^2} \left[e^{-x}(-\cos nx + n \sin nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi(1+n^2)} \cdot \left[e^{-2\pi}(-\cos 2n\pi + n \sin 2n\pi) - e^{0}(-\cos 0 + n \sin 0)\right]$$

$$= \frac{1}{\pi(1+n^2)} \cdot \left[e^{-2\pi}(-1) - (-1)\right] = \frac{1 - e^{-2\pi}}{\pi(1+n^2)}.$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\,dx = \frac{1}{\pi} \int_0^{2\pi} e^{-x} \sin nx\,dx$$

$$= \frac{1}{\pi} \cdot \frac{1}{1+n^2} \left[e^{-x}(-\sin nx - n\cos nx)\right]_0^{2\pi}$$

$$= \frac{1}{\pi(1+n^2)} \cdot \left[e^{-2\pi}(-\sin 2n\pi - n\cos 2n\pi) - e^{0}(-\sin 0 - n\cos 0)\right]$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}(-n) - 1(-n)\right] = \frac{n}{\pi(1+n^2)}(1-e^{-2\pi})$$

Substituting values of a_0, a_n, b_n

$$e^{-x} = \frac{1-e^{-2\pi}}{2\pi} + \frac{(1-e^{-2\pi})}{\pi}\left[\sum_{n=1}^{\infty}\frac{1}{1+n^2}\cos nx + \sum_{n=1}^{\infty}\frac{n}{1+n^2}\sin nx\right]$$

$$= \frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{1+n^2}(\cos nx + n\sin nx)\right]$$

putting $x = \pi$,

$$\therefore e^{-\pi} = \frac{1-e^{-2\pi}}{2\pi} + \frac{(1-e^{-2\pi})}{\pi}\left[\sum_{n=1}^{\infty}\frac{1}{1+n^2}(\cos nx + n\sin nx)\right]$$

$$= \frac{1-e^{-2\pi}}{2\pi} + \frac{(1-e^{-2\pi})}{\pi}\left[-\frac{1}{2} + \sum_{n=2}^{\infty}\frac{1}{1+n^2}(-1)^n\right]$$

$$\therefore e^{-\pi} = \frac{1-e^{-2\pi}}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2}$$

$$\therefore \frac{\pi}{e^{\pi}-e^{-\pi}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{1+n^2}$$

$$\therefore \frac{\pi}{2}\cdot\frac{2}{e^{\pi}-e^{-\pi}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(1+n^2)}$$

$$\therefore \operatorname{cosech} \pi = \frac{2}{\pi}\sum_{n=2}^{\infty}\left(\frac{(-1)^n}{1+n^2}\right)$$

$$\therefore \operatorname{cosech}\pi = \frac{2}{\pi}\left[\frac{1}{1+2^2} - \frac{1}{1+3^3} + \frac{1}{1+4^2} - \cdots\right]$$

2) $f(x) = \sqrt{1-\cos x}$ in $(0, 2\pi)$ $\therefore L = \pi$

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \sqrt{1-\cos x} = \sqrt{2\sin^2 x/2} = \sqrt{2}\sin(x/2)$$

$$\therefore a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\,dx = \frac{1}{\sqrt{2}\pi}\left[-2\cos\frac{x}{2}\right]_0^2$$

$$= \frac{1}{\sqrt{2}\pi}(-2-1-1) = \frac{2\sqrt{2}}{\pi}$$

$$\therefore a_n = \frac{1}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\sqrt{2}\sin\frac{x}{2}\cdot\cos nx\,.dx$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\sin\left(\frac{x}{2}+nx\right) + \sin\left(\frac{x}{2}-nx\right)\right]dx$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\sin\left(\frac{1+2x}{2}\right)x + \sin\left(\frac{1-2n}{2}\right)x\right]dx$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{-2}{1+2n}\cos\left(\frac{1+2n}{2}\right)x - \frac{2}{1-2n}\cos\left(\frac{1-2n}{2}\right)x\right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{-1}{2n+1}\cos\left(\frac{2n+1}{2}\right)x + \frac{1}{2n-1}\cos\left(\frac{2n-1}{2}\right)x\right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{2}{2n+1} - \frac{2}{2n-1}\right]$$

$$= \frac{-4\sqrt{2}}{\pi(4n^2-1)}$$

$$\therefore b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\sin nx\,dx$$

$$= \frac{-\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\cos\left(\frac{1}{2}+n\right)x - \cos\left(\frac{1}{2}-n\right)x\right]dx$$

$$= \frac{-\sqrt{2}}{2\pi}\left[\frac{2}{1+2n}\sin\left(\frac{1+2n}{2}\right)x - \frac{2}{1-2n}\sin\left(\frac{1-2n}{2}\right)x\right]_0^{2\pi}$$

$$= \frac{-\sqrt{2}\cdot\cancel{2}}{\cancel{2}\pi}\left[\frac{1}{1+2n}\cdot\sin\left(\frac{2n+1}{2}\right)x - \frac{1}{1-2n}\sin\left(\frac{2n-1}{2}\right)x\right]_0^{2\pi}$$

$$= 0$$

$$\therefore \boxed{f(x) = \sqrt{1-\cos x} \sim \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos nx}$$

put $x = 0$

$$\therefore \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = 0$$

$$\therefore \boxed{\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}}$$

3) $f(x) = \begin{cases} -\pi/4 & -\pi < x < 0 \\ \pi/4 + \sin x & 0 < x < \pi \end{cases}$

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^{0} \frac{-\pi}{4}\, dx + \int_{0}^{\pi} \left(\frac{\pi}{4} + \sin x \right) dx \right]$$

$$= \frac{1}{2\pi} \left[\left[\frac{-\pi}{4} x \right]_{-\pi}^{0} + \left[\frac{\pi}{4} x - \cos x \right]_{0}^{\pi} \right]$$

$$= \frac{1}{2\pi} \left[\frac{-\pi}{4} (\pi) + \left[\frac{\pi^2}{4} + 1 + 1 \right] \right]$$

$$= \frac{1}{2\pi} \left[\frac{-\pi^2}{4} + \frac{\pi^2}{4} + 2 \right]$$

$$= \frac{1}{\pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \left[\int_{0}^{\pi} \left(\frac{\pi}{4} + \sin x \right) \cos nx\, dx + \int_{-\pi}^{0} \frac{-\pi}{4} \cos nx\, dx \right]$$

$$= \frac{1}{\pi} \left[\left[\frac{-\pi}{4} \frac{\sin nx}{n} \right]_{-\pi}^{0} + \int_{0}^{\pi} \frac{\pi}{4} \cos nx\, dx + \int_{0}^{\pi} \sin x \cos nx\, dx \right]$$

$$= \frac{1}{\pi} \left[\left[\frac{-\pi}{4} \frac{\sin nx}{n} \right]_{-\pi}^{0} + \left[\frac{\pi}{4} \frac{\sin nx}{n} \right]_{0}^{\pi} + \int_{0}^{\pi} \sin x \cos nx\, dx \right]$$

$$= \frac{1}{2\pi} \left[\int_{0}^{\pi} [\sin(n+1)x - \sin(n-1)x]\, dx \right]$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)} \right]_0^\pi$$

$$= \frac{1}{2\pi}\left[\frac{(-1)^n - 1}{n-1} + \frac{1-(-1)^n}{n+1} \right]$$

$$= \frac{(-1)^n - 1}{(n^2-1)\pi} \quad \ldots \text{ for } n \neq 1$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x \, dx$$

$$= \frac{1}{\pi}\left[\int_0^\pi \sin x \cos x \, dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^\pi \frac{\sin 2x}{2} \, dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos 2x}{4}\right]_0^\pi$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \sin nx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\frac{\pi}{4} \sin nx \, dx + \int_0^\pi \left(\frac{\pi}{4} + \sin x\right) \sin nx \, dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi}{4}\frac{\cos nx}{x}\right]_{-\pi}^{0} + \left[-\frac{\pi}{4}\frac{\cos nx}{x}\right]_0^\pi + \int_0^\pi \sin x \sin nx \, dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2x} + \frac{1}{2}\int_0^\pi (\cos(n-1)x - \cos(n+1)x) \, dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2k} + \frac{1}{2}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^\pi\right]$$

$$= \frac{1}{2n}$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\frac{\pi}{4}\sin x\,dx + \int_{0}^{\pi}\left(\frac{\pi}{4}+\sin x\right)\sin x\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[\cos x \cdot \frac{\pi}{4}\right]_{-\pi}^{\pi} + \left[-\cos x \cdot \frac{\pi}{4}\right]_{0}^{\pi} + \int_{0}^{\pi}\sin^2 x\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2}\right]$$

$$= \frac{3}{2}$$

$$\boxed{\therefore f(x) \sim \frac{1}{\pi} + \sum_{n=1}^{\infty}\frac{(-1)^n-1}{\pi(n^2-1)}\cos nx + \frac{3}{2} + \sum_{n=2}^{\infty}\frac{1}{2n}\sin nx}$$

4) $$f(x) = \begin{cases} -1+x & -\pi < x < 0 \\ 1+x & 0 < x < \pi \end{cases}$$

$$f(-x) = \begin{cases} -1-x & -\pi < -x < 0 \\ 1-x & 0 < -x < \pi \end{cases} \Rightarrow \begin{cases} -1-x & \pi > x > 0 \\ 1-x & 0 > x > -\pi \end{cases}$$

$$\Rightarrow \begin{cases} -(-1+x) & -\pi < x < 0 \\ -(1+x) & 0 < x < \pi \end{cases}$$

$$= \cancel{f(-x)} \; -f(x)$$

$\therefore f(x)$ is odd

$\therefore a_0 = 0, \quad a_n = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1+x)\sin nx\,dx + \int_{0}^{\pi}(1+x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\sin nx + x\sin nx)\,dx + \int_{0}^{\pi}(\sin nx + x\sin nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[\frac{\cos nx}{n} - \frac{x\cos nx}{n} - \frac{\sin nx}{n^2}\right]_{0}^{-\pi} + \left[-\frac{\cos nx}{n} - \frac{x\cos nx}{n} - \frac{\sin nx}{n^2}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{1}{n} - \frac{\cos n\pi}{n} - \frac{\pi\cos n\pi}{n}\right) + \left(-\frac{\cos n\pi}{n} - \frac{\pi\cos n\pi}{n} - \frac{1}{n}\right)\right]$$

$$= \frac{1}{\pi}\left[-2\left(\frac{\cos n\pi}{n} - \frac{\pi\cos n\pi}{n}\right)\right]$$

$$= \frac{-2}{n\pi}\left[\cos n\pi - \pi\cos n\pi\right]$$

$$= \frac{-2}{n\pi}\left[(-1)^n - \pi(-1)^n\right]$$

$$= \frac{-2(-1)^n(1-\pi)}{n\pi}$$

$$= \frac{2(-1)^{n+1}(1-\pi)}{n\pi}$$

$$\therefore \boxed{f(x) \sim \sum_{n=1}^{\infty}\left[\frac{2(-1)^{n+1}}{n\pi}(1-\pi)\sin n\pi\right]}$$

5) $f(x) = \frac{x(\pi - x)(\pi + x)}{12}$ in $(-\pi, \pi)$

$f(-x) = \frac{-x(\pi + x)(\pi - x)}{12} = -f(x)$

$f(x)$ is an odd function

$a_0 = a_n = 0$

$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$

$= \frac{2}{\pi} \int_0^{\pi} \frac{x(\pi - x)(\pi + x) \sin nx \, dx}{12}$

$= \frac{1}{6\pi} \int_0^{\pi} (\pi^2 x - x^3) \sin nx \, dx$

$= \frac{1}{6\pi} \left[-\frac{(\pi^2 x - x^3) \cos nx}{n} + \frac{(\pi^2 - 3x^2) \sin nx}{n^2} - \frac{6x \cos nx}{x^3} + \frac{6 \sin nx}{n^{5+1}} \right]_0^{\pi}$

$= \frac{1}{6\pi} \left[-\frac{6\pi(-1)^n}{n^3} - 0 \right]$

$\therefore b_n = \frac{(-1)^{n+1}}{n^3}$

$\therefore f(x) \sim \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{x^3} \sin nx$

put $x = \pi/2$

$\therefore \frac{\pi/2 (\pi - \pi/2)(\pi + \pi/2)}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \dots$

$\therefore \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \dots = \frac{\pi^3}{32}$

6) $f(x) = e^{ax} - e^{-ax}$ in $-\pi < x < \pi$

$f(-x) = e^{-ax} - e^{ax} = -f(x)$

$\therefore f(x)$ is odd

$\therefore a_0 = a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} (e^{ax} - e^{-ax}) \sin nx \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi} e^{ax} \sin nx \, dx - \int_0^{\pi} e^{-ax} \sin nx \, dx\right]$$

$$= \frac{2}{\pi}\left[\left[\frac{e^{ax}}{a^2+n^2}(a \sin nx - n \cos nx)\right]_0^{\pi} - \left[\frac{e^{-ax}}{a^2+n^2}(-a \sin x - n \cos nx)\right]_0^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{e^{a\pi}}{a^2+n^2}(-n \cos n\pi) + \frac{n}{a^2+n^2} - \frac{e^{-a\pi}}{a^2+n^2}(-n \cos n\pi) - \frac{n}{a^2+n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{e^{a\pi} - e^{-a\pi}}{a^2+n^2}(-n \cos n\pi)\right]$$

$$\therefore b_n = \frac{2}{\pi}\frac{(e^{a\pi} - e^{-a\pi})}{a^2+n^2}(-n \cos n\pi)$$

$$\therefore \frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{a^2+n^2}(-n)(-1)^n \sin nx$$

$$= \frac{2}{\pi}\left[\frac{1}{a^2+1^2}\sin x - \frac{2}{a^2+2^2}\sin 2x + \frac{3}{a^2 3^2}\sin 3x - \cdots\right]$$

7) $f(x) = \begin{cases} 1 & 0 < x < 1 \\ x & 1 < x < 2 \end{cases}$ $L = 1$

$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} + \sum_{n=1}^{\infty} \sin \frac{n\pi}{L} x$

$\therefore a_0 = \frac{1}{2L} \int_0^{2L} f(x)\, dx$

$= \frac{1}{2} \left[\int_0^1 1 \cdot dx + \int_1^2 x\, dx \right]$

$= \frac{1}{2} \left[1 + \frac{4}{2} - \frac{1}{2} \right]$

$\therefore a_0 = \frac{5}{4}$

$a_n = \frac{1}{L} \int_0^{2L} f(x) \cos \frac{n\pi}{L} x\, dx$

$= \int_0^1 1 \cdot \cos n\pi x\, dx + \int_1^2 x \cos n\pi x\, dx$

$= \left[\frac{\sin n\pi x}{n\pi} \right]_0^1 + \left[\frac{x \sin n\pi x}{n\pi} + \frac{\cos n\pi x}{n^2\pi^2} \right]_1^2$

$= \frac{1}{n^2\pi^2} - \left(\frac{(-1)^n}{n^2\pi^2} \right)$

$= \frac{1 - (-1)^n}{n^2\pi^2}$

$\therefore a_n = \frac{(-1)^{n+1} + 1}{n^2 + \pi^2}$

$\therefore b_n = \frac{1}{L} \int_0^{2L} f(x) \sin \frac{n\pi}{L} x\, dx$

$$= \int_0^1 \sin n\pi x \, dx + \int_1^2 x \sin n\pi x \, dx$$

$$= \left[\frac{-\cos n\pi x}{n\pi} \right]_0^1 + \left[\frac{-x \cos n\pi x}{n\pi} + \frac{\sin n\pi x}{n^2 \pi^2} \right]_1^2$$

$$= \frac{2\cos n\pi - 1}{n\pi}$$

$$= \frac{2(-1)^n - 1}{n\pi}$$

$$\therefore \boxed{f(x) = \frac{5}{4} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 \pi^2} \cos n\pi x + \sum_{n=1}^{\infty} \frac{2(-1)^n - 1}{n\pi} \sin \pi x}$$

8) $f(x) = \begin{cases} \cos x & -\pi < x < 0 \\ -\cos x & 0 < x < \pi \end{cases}$

$$f(-x) = \begin{cases} \cos(-x) & -\pi < -x < 0 \\ -\cos(-x) & 0 < -x < \pi \end{cases} = \begin{cases} \cos x & \pi > x > 0 \\ -\cos x & 0 > x > -\pi \end{cases} = -f(x)$$

$\therefore f(x)$ is odd

$\therefore a_0 = a_n = 0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n x$$

$$\therefore b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} (-\cos x) \sin nx \, dx$$

$$= \frac{-1}{\pi} \int_0^{\pi} 2 \sin nx \cos x \, dx$$

$$= \frac{-1}{\pi} \int_0^{\pi} [\sin(n+1) + \sin(n-1) x] \, dx$$

$$= \frac{-1}{\pi}\left[-\cos(n+1)x + \sin(n-1)x \right] dx$$

$$= \frac{-1}{\pi}\left[-\frac{\cos(n+1)x}{(n+1)} - \frac{\cos(n-1)x}{(n-1)} \right]_0^{\pi}$$

$$= \frac{-1}{\pi}\left[-\frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} + \frac{1}{n-1} \right]$$

$$= \frac{-1}{\pi}\left[\frac{1+\cos n\pi}{n+1} + \frac{1+\cos n\pi}{n-1} \right]$$

$$= \frac{-2n}{\pi} \frac{(1+\cos n\pi)}{n^2-1} \quad \text{for } n \neq 1$$

∴ ~~b_n~~ Putting $n=1$

$$\therefore b_1 = \frac{2}{\pi}\int_0^{\pi} (-\cos x)\sin x \, dx = \frac{-1}{\pi}\int_0^{\pi} \sin 2x \, dx$$

$$= \frac{-1}{\pi}\left[-\frac{\cos 2x}{2} \right]_0^{\pi} = 0$$

$$\therefore f(x) = \frac{-4}{\pi}\left[\frac{2}{3}\sin 2x + \frac{4}{15}\sin 4x + \frac{6}{35}\sin 6x + \cdots \right]$$

$$\therefore \boxed{f(x) = \frac{-4}{\pi}\sum_{n=1}^{\infty} \frac{2n}{4n^2-1}\sin 2nx}$$

9) $f(x) = |\sin 2x|$

$f(-x) = |-\sin 2x| = |\sin 2x|$

$\therefore f(x)$ is even

$\therefore b_n = 0$

$$a_0 = \frac{4}{\pi}\int_0^{\pi/4} |\sin 2x|\, dx$$

$$= \frac{4}{\pi}\left[\int_0^{\pi/4} \sin 2x\right] \quad \because |\sin 2x| = \text{is } \sin 2x \text{ in } [0, \pi/4]$$

$$= \frac{4}{\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi/4}$$

$$= \frac{4}{\pi}\left(\frac{-1}{2}\right)(-1)$$

$$\therefore a_0 = \frac{2}{\pi}$$

$$a_n = \frac{8}{\pi}\int_0^{\frac{\pi}{4}} |\sin 2x| \cos 4x \; dx$$

$$= \frac{8}{\pi}\int_0^{\frac{\pi}{4}} \sin 2x \cos 4x \; dx$$

$$= \frac{8}{\pi}\cdot\frac{1}{2}\int_0^{\frac{\pi}{4}} \left[\sin(2+4n)x - \sin(2-4n)x\right] dx$$

$$= \frac{4}{\pi}\left[\frac{-\cos(2+4n)x}{2+4n} + \frac{\cos(2-4n)x}{2-4n}\right]_0^{\pi/4}$$

$$= \frac{4}{\pi}\left[\frac{1}{2-4n} - \frac{1}{2+4n}\right]$$

$$= \frac{8n}{\pi(1-4n^2)}$$

$$\therefore f(x) \sim \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{8n}{\pi(1-4n^2)} \cos 4nx$$

$$f(x) \sim \frac{2}{\pi} + \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n \cos 4nx}{1-4n^2}$$

$$f(x) \sim \frac{2}{\pi} - \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n \cos 4nx}{4n^2-1}$$

~~$F(x)$~~ $\sim$

10) $$f(x) = \begin{cases} kx & 0 \le x \le L/2 \\ k(L-x) & L/2 \le x \le L \end{cases}$$

Half range cosine series $\Rightarrow$ $f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L}$

$$a_0 = \frac{1}{2 \cdot L/2} \int_0^L f(x)\,dx = \frac{1}{L} \left[\int_0^{L/2} kx\,dx + \int_{L/2}^{L} k(L-x)\,dx \right]$$

$$= \frac{k}{L} \left[\left[\frac{x^2}{2} \right]_0^{L/2} - \left[\frac{(L-x)^2}{2} \right]_{L/2}^{L} \right]$$

$$= \frac{k}{L} \frac{1}{2} \left[\frac{L^2}{4} + \frac{L^2}{4} \right]$$

$$= \frac{kL}{4}$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi}{L} x\,dx$$

$$= \frac{2}{L} \left(\int_0^{L/2} kx \cos \frac{n\pi}{L}\,dx + \int_{L/2}^{L} k(L-x) \cdot \cos \frac{n\pi}{L} x\,dx \right]$$

$$= \frac{2k}{L} \left[\frac{L^2}{2\pi n} \cdot \sin \frac{n\pi}{2} + \frac{L^2}{n^2\pi^2} \left(\cos \frac{n\pi}{2} - \cos 0 \right) \right]$$

$$+ \frac{2k}{L} \left[\frac{L}{n\pi} \left(-\frac{L}{2} \sin \frac{n\pi}{2} \right) - \frac{L^2}{n^2\pi^2} \left(\cos n\pi - \cos n\pi/2 \right) \right]$$

$$= \frac{2k}{L} \frac{L^2}{n^2\pi^2} \left[2 \cos \frac{n\pi}{2} - 1 - (-1)^n \right\}$$

$$\therefore a_n = \frac{2KL}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]$$

$$\therefore f(x) \sim \frac{KL}{4} + \sum_{n=1}^{\infty} \frac{2KL}{n^2\pi^2}\left\{2\cos\frac{n\pi}{2} - 1 - (-1)^n\right\}\cdot \cos nx$$

$$\therefore \boxed{f(x) \sim \frac{KL}{4} + \frac{2KL}{\pi^2}\sum_{n=1}^{\infty} \frac{-1}{n^2}\left(1+(-1)^n\right)\cos nx}$$