CHAPTER 8

Mathematical Expectation

1. Introduction

Suppose two coins are tossed twenty times. Let X be the number of heads obtained in a toss. Then, takes values 0, 1 and 2. Suppose further that no heads, one head and two heads were obtained 4, 10, 6 times respectively. Then, the average number of heads per toss

$$= \frac{4(0) + 10(1) + 6(2)}{6 + 10 + 4} = 1.1$$

This is the average value and is not necessarily a possible outcome of the toss.

The ratios 4 / 20, 10 / 20, 6 / 20 of 0, 1, 2 heads to the total number of tosses are the relative frequencies of $X = 0, 1, 2$. If the experiment is repeated very large number of times, we know that, these relative frequencies tend to the probabilities 1 / 4, 1 / 2, 1 / 4 of 0, 1, 2 heads because in the toss of two coins we have the following.

Sample space	:	HH	$\underbrace{HT, TH}$	TT
Probability	:	1/4	1/2	1/4

The average calculated with probabilities in place of relative frequencies above is called **expected value** or **mathematical expectation** and is denoted by $E(X)$. Thus,

$$E(X) = \frac{1}{4}(0) + \frac{1}{2}(1) + \frac{1}{4}(2) = 1$$

$$E(X) = \text{sum of the products of the values and their probabilities}$$

$$= p_1 x_1 + p_2 x_2 + p_3 x_3 + \ldots\ldots$$

This means, a person who throws two coins over and over again will get one head per toss on the average. This suggests us that the expected value of X can be obtained by multiplying the values of X by their respective probabilities and taking the sum. This leads us to the following definition of expectation of a discrete random variable X.

2. Expectation of a Random Variable

(a) Definition : If a **discrete random variable X** assumes values $x_1, x_2, \ldots\ldots x_n \ldots\ldots$ with probabilities $p_1, p_2, \ldots\ldots p_n \ldots\ldots$ respectively then the **mathematical expectation of X** denoted by $E(X)$ (if it exists) is defined by

$$\boxed{E(X) = p_1 x_1 + p_2 x_2 + \ldots\ldots + p_n x_n + \ldots\ldots}$$

i.e. $\boxed{E(X) = \Sigma p_i x_i}$ where $\Sigma p_i = 1$.

if $\Sigma p_i x_i$ is absolutely convergent.

This value is also referred to as *mean value* of X. It is also denoted by μ_1'. $\therefore \mu_1' = E(X)$.

Notation : In this chapter we shall slightly deviate from our previous notation. Instead of denoting $P(X = x_i)$ by $p(x_i)$ we shall denote it simply by p_i. This will be found more convenient while dealing with expectations.

(b) Definition : Let X be a **continuous random variable** with probability density function $f(x)$. Then the **mathematical expectation of X**, denoted by $E(X)$ (if it exists), is defined by

$$\boxed{E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx} \qquad \text{where, } \int_{-\infty}^{\infty} f(x)\, dx = 1$$

if the integral is absolutely convergent.

Notes .....

1. If X assumes only a finite number of values then $E(X) = \Sigma\, p_i\, x_i$ and can be considered as "weighted average" of the values $x_1, x_2, \ldots, x_n$ with weights $p_1, p_2 \ldots p_n$.
2. If all values $x_1, x_2, \ldots, x_n$ are equiprobable *i.e.* $p_1 = p_2 = \ldots = p_n = 1/n$ then $E(X) = (1/n)\Sigma x_i$ and can be seen to be simple arithmetic mean of the n values $x_1, x_2, \ldots, x_n$.
3. One should guard oneself from being mislead by the term 'expectation'. $E(X)$ does not give us the value of X, we can expect in a single trial. In the Example 4 below $E(X) = 7/2$ is not even a possible value of X when a die is tossed.

   $E(X)$ denotes *mathematical* expectation of X in the sense that if we toss a die for a fairly large number of times, observe the frequencies of the outcomes 1, 2, 3, 4, 5, 6 then the average of these values will be closer to 7 / 2 the more often the die were tossed.
4. $E(X)$ is expressed in the *same units* as X.
5. **Expectation of a constant is constant**

   (i) $E(c) = \Sigma\, p_i c = c\, \Sigma\, p_i = c \qquad [\because\ \Sigma\, p_i = 1]$

   (ii) $E(c) = \int_{-\infty}^{\infty} c f(x)\, dx = c \int_{-\infty}^{\infty} f(x)\, dx = c \qquad \left[\because\ \int_{-\infty}^{\infty} f(x)\, dx = 1\right]$

Example 1 : A fair coin is tossed 3 times. A person received ₹ X^2 if he gets X heads. Find his expectation. **(M.U. 2004)**

Sol. : When a coin is tossed three times, the sample space is

$$HHH, HHT, HTH, HTT, THH, THT, TTH, TTT$$

The probability distribution of X is

X	:	0	1	2	3
$P(X = x)$	:	1/8	3/8	3/8	1/8

Now, X^2 takes the following values

X^2	:	0	1	2	3
$P(X^2)$	:	1/8	3/8	3/8	1/8

$$\therefore \quad E(X) = \Sigma\, p_i\, x_i = \frac{1}{8} \times 0 + \frac{3}{8} \times 1 + \frac{3}{8} \times 4 + \frac{1}{8} \times 9$$

$$= \frac{3 + 12 + 9}{8} = \frac{24}{8} = 3 \text{ ₹}$$

Example 2 : There are 10 counters in a bag, 6 of which are worth 5 rupees each while the remaining 4 are of equal but unknown value. If the expectation of drawing a single counter at random is 4 rupees, find the unknown value. **(M.U. 2015)**

Sol. : Let x be the value of the remaining 4 counters.

$$P(\text{of counter worth of ₹ 5}) = \frac{6}{10}$$

$$P(\text{of counter of unknown value}) = \frac{4}{10}$$

$$E(X) = \Sigma p_i x_i \quad \therefore \quad 4 = \frac{6}{10}\cdot 5 + \frac{4}{10}\cdot x$$

$$\therefore \quad 40 = 30 + 4x \qquad \therefore \quad 4x = 10 \qquad \therefore \quad x = ₹\ 2{\cdot}5.$$

Example 3 : A fair coin is tossed till a head appears. What is the expectation of the number of tosses required ? **(M.U. 1996, 2010)**

Sol. : Let X denote the order of the toss at which we get the first head. We have

Event	H	TH	TTH	$TTTH$	......
X	1	2	3	4	......
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$	$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{8}$	$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{16}$	......

$$\therefore \quad E(X) = \Sigma p_i x_i = 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{8}\right) + 4\left(\frac{1}{16}\right) + \ldots..$$

Let
$$S = \frac{1}{2} + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{8} + 4\cdot\frac{1}{16} + \ldots..$$

$$\frac{1}{2}S = \frac{1}{4} + 2\cdot\frac{1}{8} + 3\cdot\frac{1}{16} + \ldots..$$

$$\therefore \quad S - \frac{1}{2}S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots..$$

$$\therefore \quad \frac{1}{2}S = \frac{1}{2}\cdot\frac{1}{1-(1/2)} = 1 \qquad \left[\text{G.P. } S_\infty = \frac{a}{1-r}\right]$$

$$\therefore \quad E(X) = 2.$$

Example 4 : Find the expectation of (i) the sum, (ii) the product of the number of points on the throw of n dice. **(M.U. 2004, 06)**

Sol. : Let X_i denote the number of points on the i th dice.

Then if S denotes the sum of the points of n dice then $S = \sum_{i=1}^{n} X_i$.

Now, $E(X_i) = \Sigma p_i x_i = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + \ldots.. + \frac{1}{6}\cdot 6$

$$\therefore \quad E(X_i) = \frac{1}{6}(1+2+3+3+4+5+6) = \frac{21}{6} = \frac{7}{2}$$

$$\therefore \quad S = \sum_{i=1}^{n} X_i = n\cdot\frac{7}{2} = \frac{7n}{2}$$

If x denotes the product of the points

$$E(x) = E(X_1) \cdot E(X_2) \ldots E(X_n) = \left(\frac{7}{2}\right) \cdot \left(\frac{7}{2}\right) \ldots \cdot \left(\frac{7}{2}\right) = \left(\frac{7}{2}\right)^n$$

Example 5 : A box contains n tickets numbered 1, 2, ..., n. If m tickets are drawn at random from the box. What is the expectation of the sum of the numbers on the tickets drawn ? **(M.U. 20..)**

Sol. : Let X_i denote the number on the i th ticket drawn.

Then $S = X_1 + X_2 + \ldots + X_m$

Now $E(X_i) = \Sigma p_i x_i = \frac{1}{n} \cdot 1 + \frac{1}{n} \cdot 2 + \ldots + \frac{1}{n} \cdot n$

$$= \frac{1}{n}(1 + 2 + \ldots + n) = \frac{1}{n} \cdot \frac{n}{2} \cdot (n+1) = \frac{n+1}{2}$$

$$\therefore \quad E(S) = \sum_{i=1}^{m} E(X_i) = \frac{m(n+1)}{2}$$

Example 6 : Three urns contain respectively 3 green and 2 white balls, 5 green and 6 white balls, 2 green and 4 white balls. One ball is drawn from each urn. Find the expected number of white ball drawn. **(M.U. 2007, 08)**

Sol. : If X denotes the number of white balls drawn from an urn then the expectation of X is as follows.

$$E(X) = p_i x_i = p_1 x_1 + p_2 x_2$$

X takes two values. X = 1 if white ball is drawn and X = 0 if greenball is drawn.

$\therefore$ From first urn : $\quad E(X_1) = 1 \times \frac{2}{5} + 0 \times \frac{3}{5} = \frac{2}{5}$

From second urn : $\quad E(X_2) = 1 \cdot \frac{6}{11} + 0 \cdot \frac{5}{11} = \frac{6}{11}$

$$E(X_3) = 1 \cdot \frac{4}{6} + 0 \cdot \frac{2}{6} = \frac{2}{3}$$

$\therefore$ The required expectation $= E(X_1) + E(X_2) + E(X_3)$

$$= \frac{2}{5} + \frac{6}{11} + \frac{2}{3} = \frac{266}{165} = 1.61$$

Example 7 : A box contains 2^n tickets of which ${}^nC_r$ tickets bear the number r ($r = 0, 1, 2, \ldots, n$). A group of m tickets is drawn. What is the expectation of the sum of their numbers ?

Sol. : Let $X_1, X_2, \ldots, X_m$ be the variables denoting the number on the first, second, ... m th ticket.

If S is the sum of the numbers on the tickets drawn then

$$S = \Sigma X_i \text{ and } E(S) = \Sigma E(X_i)$$

Now, X_i is a random variable which can take any one of the value 0, 1, 2, ... n with probabilities ${}^nC_0 / 2^n, {}^nC_1 / 2^n, \ldots, {}^nC_n / 2^n$.

$$\therefore \quad E(X_i) = \Sigma p_i x_i$$

$$= \frac{1}{2^n}\left[0 \cdot {}^nC_0 + 1 \cdot {}^nC_1 + 2 \cdot {}^nC_2 + 3 \cdot {}^nC_3 + \ldots + n \cdot {}^nC_n\right]$$

$$= \frac{1}{2^n}\left[1 \cdot n + 2 \cdot \frac{n(n-1)}{2!} + 3 \cdot \frac{n(n-1)(n-2)}{3!} + \ldots + n \cdot 1\right]$$

$$= \frac{n}{2^n}\left[1 + (n-1) + \frac{(n-1)(n-2)}{2!} + \ldots + 1\right]$$

$$\therefore \; E(X_i) = \frac{n}{2^n}[1+1]^{n-1} = \frac{n}{2^n} \cdot 2^{n-1} = \frac{n}{2}.$$

$$\therefore \; E(S) = \Sigma E(X_i) = m \cdot \frac{n}{2} = \frac{mn}{2}.$$

Cor. 1 : If two tickets are drawn then putting $m = 2$, we get

$$E(\text{Sum}) = 2 \cdot \frac{n}{2} = n.$$

(M.U. 2003)

Example 8 : A box contains 'a' white balls and 'b' black balls. 'c' balls are drawn from the box at random. Find the expected value of the number of white balls. **(M.U. 2005)**

Sol. : Let X_i be the variable denoting the result of the i th draw.

Let $X_i = 1$ if i th ball drawn is white and $X_i = 0$ if i th ball drawn is black.

Since, 'c' balls are drawn the sum of the white ball will be

$$S = X_1 + X_2 + \ldots + X_c = \sum_{i=1}^{c} X_i$$

Now, $P(X_i = 1) = P(\text{drawing a white ball}) = \dfrac{a}{a+b}$

$P(X_i = 0) = P(\text{drawing a black ball}) = \dfrac{b}{a+b}$

$$E(X_i) = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0)$$

$$= 1 \cdot \frac{a}{a+b} + 0 \cdot \frac{b}{a+b} = \frac{a}{a+b}$$

$$\therefore \; E(S) = E(X_1) + E(X_2) + \ldots + E(X_c)$$

$$= c \cdot \frac{a}{a+b} = \frac{ac}{a+b}.$$

Example 9 : A die is thrown until a five is obtained, find the expectation of the number of throws.

Sol. : Probability of getting 5 in the first toss = 1 / 6,

Probability of getting 5 in the second = (5 / 6) · (1 / 6),

Probability of getting 5 in third = (5 / 6) (5 / 6) (1 / 6) and so on,

and X takes values 1, 2, 3, ...

$$\begin{aligned} E(X) &= 1(1/6) + 2(5/6)(1/6) + 3(5/6)^2(1/6) + \ldots \\ &= (1/6)[1 + 2x + 3x^2 + \ldots] \qquad \text{where, } x = 5/6 \\ &= (1/6)(1-x)^{-2} = (1/6)[1-(5/6)]^{-2} \\ &= 6. \end{aligned}$$

Example 10 : A and B throw a fair die for a stake of ₹ 44, which is won by the player who throws 6 first. If A starts first, find their expectations.

Sol. : A can win the game, in the first throw or in the third throw or in the fifth throw and so on.

$$P(\text{A winning}) = \frac{1}{6} + \left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\cdot\frac{1}{6} + \left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\cdot\frac{1}{6} + \ldots$$

$$\therefore \quad P(\text{A winning}) = \frac{1}{6}\left[1 + \left(\frac{25}{36}\right) + \left(\frac{25}{36}\right)^2 + \ldots\right] = \frac{1}{6}\cdot\frac{1}{1-(25/36)} = \frac{6}{11}$$

$$P(\text{B winning}) = 1 - P(\text{A winning}) = \frac{5}{11}$$

$$\therefore \quad \text{Expectation of A} = p \times x = \frac{6}{11} \times 44 = ₹\ 24$$

$$\text{Expectation of B} = p \times x = \frac{5}{11} \times 44 = ₹\ 20$$

Example 11 : A, B, C, D cut a pack of cards successfully in the order mentioned. If a person who cuts a spade first wins ₹ 175. Find their expectations.

Sol. : Probability of cutting a spade $= \frac{13}{52} = \frac{1}{4}$

Let A denote the success of A and $\bar{A}$ denote failure of A and so on.

Probability of A's success

$$= P(A) + P(\bar{A}\,\bar{B}\,\bar{C}\,\bar{D}\,A) + P(\bar{A}\,\bar{B}\,\bar{C}\,\bar{D}\cdot\bar{A}\,\bar{B}\,\bar{C}\,\bar{D}\,A) + \ldots$$

$$= \frac{1}{4} + \left(\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}\right) + \left(\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}\right) + \ldots$$

$$= \frac{1}{4} + \frac{81}{256}\cdot\frac{1}{4} + \left(\frac{81}{256}\right)^2\cdot\frac{1}{4} + \ldots = \frac{1}{4}\cdot\frac{1}{1-(81/256)}$$

$$= \frac{1}{4}\cdot\frac{256}{256-81} = \frac{1}{4}\cdot\frac{256}{175} = \frac{64}{175}$$

Probability of B's success

$$= P(\bar{A}\,B) + P(\bar{A}\,\bar{B}\,\bar{C}\,\bar{D}\cdot\bar{A}\,B) + \ldots$$

$$= \frac{3}{4}\cdot\frac{1}{4} + \left(\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\right)\left(\frac{3}{4}\cdot\frac{1}{4}\right) + \ldots = \frac{3}{16}\left[1 + \frac{81}{256} + \ldots\right]$$

$$= \frac{3}{16}\left[\frac{1}{1-(81/256)}\right] = \frac{3}{16}\left[\frac{256}{175}\right] = \frac{48}{175}$$

Probability of C's success

$$= \frac{3}{4}\cdot\frac{3}{4}\cdot\frac{1}{4} + \ldots = \frac{9}{64}\left[\frac{1}{1-(81/256)}\right] = \frac{9}{64}\cdot\frac{256}{256-81} = \frac{36}{175}$$

Probability of D's success

$$= 1 - [P(A) + P(B) + P(C)] = 1 - \left[\frac{64}{175} + \frac{48}{175} + \frac{36}{175}\right] = \frac{27}{175}$$

Now, the probabilities of A, B, C, D

$\therefore \quad E(A) = p\,x = \dfrac{64}{175} \times 175 = ₹\ 64, \qquad E(B) = p\,x = \dfrac{48}{175} \times 175 = ₹\ 48,$

$E(C) = p\,x = \dfrac{36}{175} \times 175 = ₹\ 36, \qquad E(D) = p\,x = \dfrac{27}{175} \times 175 = ₹\ 27.$

Example 12 : Find the expectation of number of failures preceding the first success in an infinite series of independent trials with constant probabilities p and q of success and failure respectively. **(M.U. 1999, 2003, 17, 19)**

Sol. : We have the following probability distribution

X :	0	1	2	3	....
$P(X = x)$:	p	qp	q^2p	q^3p	

Since, we may get success in the first trial where the number of failures $X = 0$ and the probability is p; we may get success in the second trial when the number of failures $X = 1$ and the probability is $q\,p$ and so on.

$$\therefore \quad E(X) = \Sigma\, p_i\, x_i = p\,(0) + qp\,(1) + q^2p\,(2) + q^3p\,(3) + \ldots.$$

$$= qp\left[1 + 2q + 3q^2 + \ldots..\right] = qp\,(1-q)^{-2} = \frac{qp}{p^2} = \frac{q}{p}.$$

Example 13 : The daily consumption of electric power (in million kwh) is a random variable X with probability distribution function

$$f(x) = \begin{cases} kx\, e^{-x/3} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

Find the value of k, the expectation of x and the probability that on a given day the electric consumption is more than expected value. **(M.U. 2003, 04, 16)**

Sol. : We must have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \quad i.e. \quad k\int_0^{\infty} x\, e^{-x/3}\,dx = 1$$

$$\therefore \quad k\left[(x)\left(\frac{e^{-x/3}}{-1/3}\right) - (1)\left(\frac{e^{-x/3}}{1/9}\right)\right]_0^{\infty} = 1$$

$$\therefore \quad k\left[0 + 9\right] = 1 \quad \therefore \quad 9k = 1 \quad \therefore \quad k = 1/9$$

$$E(X) = \int_{-\infty}^{x} x\, f(x)\,dx = \frac{1}{9}\int_0^{\infty} x^2 \cdot e^{-x/3} \cdot dx$$

$$= \frac{1}{9}\left[x^2\left(\frac{e^{-x/3}}{-1/3}\right) - (2x)\left(\frac{e^{-x/3}}{1/9}\right) + 2\left(\frac{e^{-x/3}}{-1/27}\right)\right]_0^{\infty}$$

$$= \frac{1}{9}\left[0 - 0 + 0 + 54\right] = 6$$

$$\therefore \quad P(X > 6) = \frac{1}{9}\int_6^{\infty} x \cdot e^{-x/3} \cdot dx$$

$$\therefore \quad P(X > 6) = \frac{1}{9}\left[(x)\left(\frac{e^{-x/3}}{-1/3}\right) - (1)\left(\frac{e^{-x/3}}{1/9}\right)\right]_6^\infty$$

$$= \frac{1}{9}\left[(0-0) - (-18\,e^{-2} - 9e^{-2})\right]$$

$$= 3\,e^{-2} = 0.406$$

Example 14 : Find k and then $E(X)$ if X has the p.d.f.

$$f(x) = \begin{cases} k\,x(2-x), & 0 \le x \le 2,\ k > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Sol. : Now $\int_0^2 kx(2-x)\,dx = k\int_0^2 2x - x^2\,dx = k\left[x^2 - \frac{1}{3}x^3\right]_0^2 = k \cdot \frac{4}{3}$

$$\therefore \quad k \cdot \frac{4}{3} = 1 \quad \therefore \quad k = \frac{3}{4}$$

By definition

$$E(X) = \int_0^2 x\,f(x)\,dx = \int_0^2 x \cdot \frac{3}{4}x(2-x)\,dx = \frac{3}{4}\int_0^2 (2x^2 - x^3)\,dx$$

$$\therefore \quad E(X) = \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{3}{4}\left[\frac{16}{3} - \frac{16}{4}\right] = \frac{3}{4} \cdot \frac{16}{12} = 1$$

Example 15 : Find k and then $E(X)$ for the p.d.f.

$$f(x) = \begin{cases} k(x - x^2), & 0 \le x \le 1,\ k > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Sol. : Now $k\int_0^1 (x - x^2)\,dx = k \cdot \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = k\left[\frac{1}{2} - \frac{1}{3}\right] = k \cdot \frac{1}{6}$

But $\quad k \cdot \frac{1}{6} = 1 \quad \therefore \quad k = 6$

By definition

$$E(X) = \int_0^1 x\,f(x)\,dx = \int_0^1 x \cdot 6(x - x^2)\,dx = 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{6}{12} = \frac{1}{2}.$$

EXERCISE - I

1. From an urn containing 3 red balls and 2 white balls, a man is to draw 2 balls at random without replacement. He gets ₹ 20 for each red ball and ₹ 10 for each white ball he draws. Find his expectation. **[Ans. :** ₹ 32**]**
2. Two urns contain respectively 5 white and 3 black balls; 2 white and 3 black balls. One ball is drawn from each urn. Find the expected number of white balls drawn. **[Ans. :** 41 / 40**]**
3. A and B toss a fair coin alternately. One who gets a head first wins ₹ 12. A starts. Find their mathematical expectations. **[Ans. :** ₹ 8, 4**]**

4. A, B, and C toss a fair coin. The first one to throw a head wins the game and gets ₹ 28. If A starts, find their mathematical expectations. [**Ans. :** ₹ 16, 8, 4]

5. A, B and C draw a card in that order from a well shuffled pack of 52 cards. The first to draw a diamond wins ₹ 740. If A starts, find their expectations. [**Ans. :** ₹ 320, 240, 180]

6. Two unbiased dice are thrown. Find the expectation of the sum. **(M.U. 2007)** [**Ans. :** 7]

7. A man with n keys in his pocket wants to open the door of his case by trying the keys independently and randomly one by one. Find the mean and the variance of the number of trials required to open the door if unsuccessful keys are kept aside. **(M.U. 1998)**

   [**Ans. : (i)** $(n + 1) / 2$, **(ii)** $(n^2 - 1) / 12$]

8. A player throwing an ordinary die is to receive ₹ $1/2^n$ where n denotes the number of throws required to get first 3. Find his expectation. **(M.U. 2001, 04)** [**ans. :** ₹ 1/ 7]

9. Three fair coins are tossed. Find the expectation and the variance of number of heads.

   (M.U. 2004) [**Ans. :** $\bar{x} = 3/2$, $\text{Var}(X) = 3/4$]

10. From a box containing n tickets bearing numbers 1, 2, 3, ..., n a ticket is drawn. If X denotes the number on the ticket drawn, find the mean and variance of X.

    [**Ans. :** $\bar{x} = (n+1)/2$, $\text{Var}(X) = (n^2 - 1)/2$]

11. In a game of chance a man is allowed to throw a fair coin indefinitely. He receives rupees 1, 2, 3, ... if he throws a head at the 1st, 2nd, 3rd, ... trial respectively. If the entry fee to participate in the game is ₹ 2, find the expected value of his net gain. [**Ans. :** Zero]

12. A person draws 3 balls from a bag containing 3 white, 4 red and 5 black balls. He is offered ₹ 10, ₹ 5 and ₹ 2 if he draws 3 balls of the same colour, 2 balls of the same colour and 1 ball of each colour respectively. Find his expectation. **(M.U. 2004)**

    [**Ans. :** $p_1 = \frac{3}{44}$, $p_2 = \frac{29}{44}$, $p_3 = \frac{12}{44}$; ₹ 4·52]

13. A continuous random variable X has the density function $f(x) = k(1 + x)$ where $2 \le x \le 5$. Find k, $P(x \le 4)$ and $E(X)$. **(M.U. 2005)** [**Ans. :** $k = \frac{2}{27}$; $\frac{16}{27}$; $\frac{11}{3}$]

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3. Expectation of a Function of a Random Variable X

We can now extend the concept of expectation of a random variable to the function of a random variable.

1. Definition : Let X be a discrete random variable taking values $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$ and $g(X)$ be a function of X then **mathematical expectation of $g(X)$** (if it exists) is defined by

$$\boxed{E[g(X)] = \Sigma p_i g(x_i)}$$

2. Definition : Let X be a continuous random variable with p.d.f. $f(x)$, let $g(X)$ be a function such that $g(X)$ is a random variable then $E[g(X)]$ (if it exists) is defined by,

$$\boxed{E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx}$$

Notes ....

1. If $g(X) = aX^n$, then $E[g(X)] = E[aX^n] = \Sigma p_i a x_i^n = a \Sigma p_i x_i^n = a E(X^n)$

 And $E[g(X)] = E[aX^n] = \int_{-\infty}^{\infty} a x^n f(x)\,dx = a \int_{-\infty}^{\infty} x^n f(x)\,dx = a E(X^n)$

 e.g. $E(aX) = a E(X)$

 e.g. $E(aX^2) = a E(X^2)$, $E(aX^3) = a E(X^3)$

2. If $g(X) = aX + b$, then $E[g(X)] = E[aX + b] = a E(X) + b$.
3. If should be noted that, $E(X^2) \neq [E(X)]^2$ and $E(1/X) \neq 1/E(X)$
4. Putting $a = 1$, $E(X^n) = \Sigma p_i x_i^n$ and $E(X^n) = \int_{-\infty}^{\infty} x^n f(x)\,dx$

 In particular $E(X^2) = \Sigma p_i x_i^2$ and $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

 $E(X^2)$ is denoted by μ_2'

 $\therefore \quad \mu_2' = \Sigma p_i x_i^2 \quad$ or $\quad \mu_2' = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

4. Mean and Variance

If we know the probability density function, discrete or continuous, we can find the mean and variance of the random variable as follows.

$$\mu_1' = \text{Mean} = E(X) = \Sigma p_i x_i \quad \text{or} \quad \mu_1' = \int_{-\infty}^{\infty} x f(x)\,dx$$

We then find $\mu_2' = E(X^2) = \Sigma p_i x_i^2 \quad$ or $\quad \mu_2' = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

Now,

$$\text{Var}(X) = E(X - \overline{X})^2 = E[X - E(X)]^2$$

$$= E\left[X^2 - 2X\,E(X) + \{E(X)\}^2\right]$$

$$= E(X^2) - 2E(X) \cdot E(X) + [E(X)]^2$$

$$\boxed{\textbf{Var.}(X) = E(X^2) - [E(X)]^2}$$

But $E(X^2) = \mu_2'$ and $E(X) = \mu_1'$

$$\therefore \quad \boxed{\textbf{Var.}(X) = \mu_2' - \mu_1'^2}$$

Type I : Examples on Mean and Variance of a Discrete Probability Distribution

Example 1 : If X denotes the smaller of the two numbers that appear when a pair of dice is thrown, find the probability distribution of X, and also the mean and variance of X. **(M.U. 2004)**

Sol. : Refer to the table of Ex. 4 page 7-26.

We see that the number 1 appears as smaller (including equality) of the two numbers in 11 cases out of 36, the number 2 appears in 9 cases, 3 in 7 cases and so on.

The probability distribution of X is as given below.

X	:	1	2	3	4	5	6
$P(X = x)$	:	1/36	9/36	7/36	5/36	3/36	1/36

$$E(X) = \Sigma p_i x_i = \frac{11}{36}(1) + \frac{9}{36}(2) + \frac{7}{36}(3) + \frac{5}{36}(4) + \frac{3}{36}(5) + \frac{1}{36}(6) = 2{\cdot}5278$$

$$E(X^2) = \frac{11}{36}(1^2) + \frac{9}{36}(2^2) + \frac{7}{36}(3^2) + \frac{5}{36}(4^2) + \frac{3}{36}(5^2) + \frac{1}{36}(6^2)$$

$$= \frac{301}{36} = 8{\cdot}3611$$

$$\therefore \quad V(X) = E(X^2) - [E(X)]^2 = 8{\cdot}3611 - (2{\cdot}5278)^2 = 1{\cdot}97$$

Example 2 : A discrete random variable has the probability density function given below.

X	:	−2	−1	0	1	2	3
P(X = x)	:	0·2	k	0·1	$2k$	0·1	$2k$

Find k, the mean and variance. **(M.U. 1997, 2001, 18, 19)**

Solution : We must have $\Sigma p_i = 1$.

$$\therefore \quad 5k + 0{\cdot}4 = 1 \quad \therefore \quad 5k = 0{\cdot}6 \quad \therefore \quad k = \frac{0{\cdot}6}{5} = \frac{3}{25}$$

Hence, the probability distribution is

X	:	−2	−1	0	1	2	3
$P(X = x)$	:	2 / 10	3 / 25	1 / 10	6 / 25	1 / 10	6 / 25

Now, Mean $= E(X) = \Sigma p_i x_i = -\frac{4}{10} - \frac{3}{25} + 0 + \frac{6}{25} + \frac{2}{10} + \frac{18}{25} = \frac{60}{250} = \frac{6}{25}$

$$E(X^2) = \Sigma p_i x_i^2 = \frac{2}{10}(4) + \frac{3}{25}(1) + 0 + \frac{6}{25}(1) + \frac{1}{10}(4) + \frac{6}{25}(9) = \frac{73}{250}$$

$$\therefore \quad \text{Variance} = \sigma^2 = E(X^2) - [E(X)]^2 = \frac{73}{250} - \frac{36}{625} = \frac{293}{625}.$$

Example 3 : Find the value of k from the following data.

X	:	0	10	15
P(x)	:	$\frac{k-6}{5}$	$\frac{2}{k}$	$\frac{14}{5k}$

Also find the distribution function and the expectation of the distribution.

Sol. : Since $\Sigma p_i = 1$.

$$\frac{k-6}{5} + \frac{2}{k} + \frac{14}{5k} = 1 \quad \therefore \quad k^2 - 11k + 14 = 0$$

$$\therefore \quad (k-8)(k-3) = 0 \quad \therefore \quad k = 8 \text{ or } 3$$

But when $k = 3$ $P(x = 0) = \frac{3-6}{5} = -\frac{3}{5}$ which is impossible. $\therefore \quad k = 8$.

$\therefore$ The p.d.f. and distribution function are

X	:	0	10	15
P(x)	:	2 / 5	1 / 4	7 / 20
F(x)	:	2 / 5	13 / 20	1

$$\therefore \quad E(X) = \Sigma p_i x_i = \frac{2}{5}(0) + \frac{1}{4}(10) + \frac{7}{20}(15) = \frac{5}{2} + \frac{21}{4} = \frac{31}{4}.$$

Example 4 : If the mean of the following distribution is 16 find m, n and variance

X	:	8	12	16	20	24
$P(X = x)$	:	1/8	m	n	1/4	1/12

(M.U. 2006, 16, 19)

Sol. : Since $\sum p_i = 1$,

$$\frac{1}{8} + m + n + \frac{1}{4} + \frac{1}{12} = 1 \qquad \therefore \; m + n = \frac{13}{24} \qquad \ldots\ldots (1)$$

Since mean = 16, $\sum p_i x_i = 16$

$$\therefore \; 1 + 12m + 16n + 5 + 2 = 16$$

$$\therefore \; 12m + 16n = 8 \qquad \therefore \; 3m + 4n = 2 \qquad \ldots\ldots (2)$$

Multiply (1) by 3 and subtract from (2),

$$\therefore \; 3m + 4n = 2 \; ; \; 3m + 3n = \frac{13}{8} \qquad \therefore \; n = \frac{3}{8}$$

$$\therefore \; m + n = \frac{13}{24} \text{ gives } \quad m + \frac{3}{8} = \frac{13}{24} \qquad \therefore \; m = \frac{4}{24} = \frac{1}{6}$$

To find variance, consider

$$E(X^2) = \sum p_i x_i^2 = \frac{1}{8}(64) + \frac{1}{6}(144) + \frac{3}{8}(256) + \frac{1}{4}(400) + \frac{1}{12}(576) = 276$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 276 - 16^2 = 20.$$

Example 5 : A woman with n keys with her, wants to open the door of her house by trying keys independently and randomly one by one. Find the mean and the variance of the number of trials required to open the door, if unsuccessful keys are kept aside. **(M.U. 2016)**

Sol. : If unsuccessful keys are kept aside, she will get success in the first trial, or second trial or third trial and so on, the random variable X of the successful trial will take values 1, 2, 3, ..., n.

$$\therefore \; P(\text{Success in the first trial}) = \frac{1}{n}$$

$$P(\text{Failure in the first trial}) = 1 - \frac{1}{n}$$

If there is failure in the first trial, the key is eliminated. There are now $(n - 1)$ keys.

$$\therefore \; P(\text{Success in the second trial}) = \frac{1}{n-1}$$

$\therefore$ P (Failure in the first trial and success in the second trial)

$$= \left(1 - \frac{1}{n}\right)\left(\frac{1}{n-1}\right) = \frac{n-1}{n} \cdot \frac{1}{n-1} = \frac{1}{n}$$

$\therefore$ P (Failure in the first trial, failure in the second trial and success in the third trial)

$$= \left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n-1}\right)\left(\frac{1}{n-2}\right)$$

$$= \frac{(n-1)}{n} \cdot \frac{(n-2)}{(n-1)} \cdot \frac{1}{(n-2)} = \frac{1}{n}$$

Thus, the probability of success at any trial remains constant $= \dfrac{1}{n}$.

Thus, the probability distribution of X is

X	:	1	2	3	4	....	n
$P(X=x)$	:	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	....	$\frac{1}{n}$

$$\therefore \quad E(X) = \Sigma\, x\, p(x) \cdot x = \frac{1}{n} \cdot 1 + \frac{1}{n} \cdot 2 + \frac{1}{n} \cdot 3 + \ldots + \frac{1}{n} \cdot n$$

$$= \frac{1}{n}(1+2+3+\ldots+n) = \frac{1}{n} \cdot \frac{n}{2} \cdot (n+1)$$

$$\therefore \quad E(X) = \frac{n+1}{2} \qquad \left[\because\ 1+2+\ldots+n = \frac{n+1}{2}\right]$$

$$E(X^2) = \Sigma\, p(x)\, x^2 = \frac{1}{n} \cdot 1^2 + \frac{1}{n} \cdot 2^2 + \frac{1}{n} \cdot 3^2 + \ldots + \frac{1}{n} \cdot n^2$$

$$= \frac{1}{n}(1^2+2^2+3^2+\ldots+n^2) = \frac{1}{n} \cdot \frac{n}{6} \cdot (n+1)(2n+1)$$

$$= \frac{(n+1)(2n+1)}{6} \qquad \left[\because\ 1^2+2^2+\ldots+n^2 = \frac{n}{6}(n+1)(2n+1)\right]$$

$$\therefore \quad V(X) = E(X^2) - [E(X)]^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= \frac{2(2n^2+3n+1) - 3(n^2+2n+1)}{12} = \frac{n^2-1}{12}.$$

Type II : Examples on Mean and Variance of a Continuous Probability Distribution

Example 1 : A continuous random variable X has the p.d.f. defined by $f(x) = A + Bx,\ 0 \le x \le 1$. If the mean of the distribution is 1/ 3, find A and B. **(M.U. 2004, 14)**

Sol. : Since $f(x)$ is a probability distribution

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

By data $\int_0^1 (A+Bx)\,dx = 1$

$$\therefore \left[AX + \frac{Bx^2}{2}\right]_0^1 = 1 \quad \therefore\ A + \frac{B}{2} = 1 \qquad \ldots\ldots\ldots\ldots\ (i)$$

Since the mean is $\frac{1}{3}$. $\int_0^1 x\, f(x)\,dx = \frac{1}{3}$ $\quad\therefore\ \int_0^1 (A+Bx)\,x\,dx = \frac{1}{3}$

$$\therefore \int_0^1 (Ax + Bx^2)\,dx = \frac{1}{3} \quad \therefore \left[A\frac{x^2}{2} + \frac{Bx^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\therefore \frac{A}{2} + \frac{B}{3} = \frac{1}{3} \quad \therefore\ 3A + 2B = 2 \qquad \ldots\ldots\ldots\ldots\ (ii)$$

Solving the equations (i) and (ii), we get $A = 2$, $B = -2$.

$\therefore$ The p.d.f. is $f(x) = 2 - 2x,\ \ 0 \le x \le 1$

Example 2 : The distribution function of a r.v. X is given by $F_X(x) = 1-(1+x)e^{-x}$, $x \geq 0$. Find the mean and variance. **(M.U. 2011)**

Sol. : We have

$$f_X(x) = \frac{d\,F_X(x)}{dx} = (1+x)e^{-x} - e^{-x} = x\,e^{-x}, \quad x \geq 0$$

$$\therefore \quad \text{Mean } \overline{X} = \int_0^{\infty} x \cdot f_X(x)\,dx = \int_0^{\infty} x^2 e^{-x}\,dx$$

$$= \left[x^2(-e^{-x}) - 2x(e^{-x}) + 2(1)\cdot(-e^{-x})\right]_0^{\infty} = 2$$

$$E(X^2) = \int_0^{\infty} x^2 f_X(x)\,dx = \int_0^{\infty} x^3 \cdot e^{-x}\,dx$$

$$E(X^2) = \left[x^3(-e^{x}) - 3x^2(e^{-x}) + 6x(-e^{-x}) - 6(e^{-x})\right]_0^{\infty} = 6$$

$$\therefore \quad V(X) = E(X^2) - [E(X)]^2 = 6 - 4 = 2.$$

Example 3 : A continuous random variable X has the p.d.f. $f(x) = kx^2 e^{-x}$, $x \geq 0$. Find k, mean and variance. **(M.U. 2004)**

Solution : We must have $\int_0^{\infty} k\,x^2 e^{-x} \cdot dx = 1$

$$\therefore \quad k\left[x^2(-e^{-x}) - \int -e^{-x}\,2x\,dx\right]_0^{\infty} = 1$$

$$\therefore \quad k\left[-x^2 e^{-x} + 2x(-e^{-x}) - \int -2e^{-x}dx\right]_0^{\infty} = 1$$

[Integrating by parts]

$$k\left[-x^2 e^{-x} - 2x\,e^{-x} - 2e^{-x}\right]_0^{\infty} = 1$$

$$k[0-(-0-0-2)] = 1 \quad \therefore\ 2k = 1 \quad \therefore\ k = \frac{1}{2}.$$

Now, mean $\overline{X} = \int_0^{\infty} x\,f(x)\,dx = \int_0^{\infty} \frac{1}{2} x^3 e^{-x} dx$

$$= \frac{1}{2}\left[x^3(-e^{-x}) - (3x^2)(e^{-x}) + (6x)(-e^{-x}) - (6)(e^{-x})\right]_0^{\infty}$$

(By the generalised rule of integration by parts.

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots..$$

where dashes denote the derivatives and suffixes denote the integrals.)

$$\therefore \quad \overline{X} = \frac{1}{2}[0-(-6)] = \frac{1}{2}\cdot 6 = 3$$

Now, $\mu_2' = \frac{1}{2}\int_0^{\infty} x^2 f(x)\,dx = \frac{1}{2}\int_0^{\infty} x^2 \cdot x^2 e^{-x}\,dx$

$$= \frac{1}{4}\int_0^{\infty} x^4 \cdot e^{-x} dx$$

$$= \frac{1}{2}\left[x^4(-e^{-x}) - (4x^3)(e^{-x}) + (12x^2)(-e^{-x}) - (24x)(e^{-x}) + 24(-e^{-x})\right]_0^{\infty}$$

$$\therefore \quad \mu_2' = \frac{1}{2}[0-(-24)] = \frac{24}{2} = 12$$

$$\therefore \quad \text{Variance} = \mu_2' - \mu_1'^2 = 12 - 9 = 3.$$

Example 4 : If X is a continuous random variable with probability density function given by

$$f(x) = \begin{cases} k(x - x^3), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find (i) k, (ii) mean, (iii) variance. **(M.U. 2015)**

Sol. : (i) We have $\int_0^1 k(x-x^3)\,dx = 1$

$$\therefore \quad k\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = 1 \qquad \therefore \quad k\left[\frac{1}{2} - \frac{1}{4}\right] = 1 \qquad \therefore \quad k \cdot \frac{1}{4} = 1 \qquad \therefore \quad k = 4$$

(ii) Mean, $\bar{X} = \mu_1' = \int_0^1 x\,f(x)\,dx = \int_0^1 x \cdot 4(x - x^3)\,dx$

$$= 4\int_0^1 (x^2 - x^4)\,dx = 4\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = 4\left[\frac{1}{3} - \frac{1}{5}\right] = \frac{8}{15}$$

$$\mu_2' = \int_0^1 x^2 f(x)\,dx = \int_0^1 x^2 \cdot 4(x - x^3)\,dx = 4\int_0^1 (x^3 - x^5)\,dx$$

$$= 4 \cdot \left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1 = 4 \cdot \left[\frac{1}{4} - \frac{1}{6}\right] = 4 \cdot \frac{1}{12} = \frac{1}{3}$$

(iii) Variance $(X) = \mu_2' - \mu_1'^2 = \frac{1}{3} - \frac{64}{225} = \frac{33}{225} = 0.1467.$

Example 5 : Find the value of k, if the function

$$f(x) = kx^2(1 - x^3), \qquad 0 \le x \le 1$$
$$= 0 \qquad \text{otherwise}$$

is a probability density function. Also find $P(0 \le x \le 1/2)$ and the mean and variance. **(M.U. 2017, 19)**

Sol. : We have $\int_0^1 kx^2(1 - x^3)\,dx = 1 \quad \therefore \quad \int_0^1 k(x^2 - x^5)\,dx = 1$

$$\therefore \quad k\left[\frac{x^3}{3} - \frac{x^6}{6}\right]_0^1 = 1 \qquad \therefore \quad k\left[\frac{1}{3} - \frac{1}{6}\right] = 1 \qquad \therefore \quad k \cdot \frac{1}{6} = 1 \qquad \therefore \quad k = 6.$$

Now, $P(0 \le x \le 2) = 6\int_0^{1/2} (x^2 - x^5)\,dx = 6\left[\frac{x^3}{3} - \frac{x^6}{6}\right]_0^{1/2} = 6\left[\frac{1}{8 \cdot 3} - \frac{1}{64 \cdot 6}\right] = \frac{15}{64}.$

Mean $\bar{x} = \mu_1' = \int_0^1 x\,f(x)\,dx = 6\int_0^1 x \cdot x^2(1 - x^3)\,dx = 6\int_0^1 (x^3 - x^6)\,dx$

$$= 6\left[\frac{x^4}{4} - \frac{x^7}{7}\right]_0^1 = 6\left[\frac{1}{4} - \frac{1}{7}\right] = \frac{18}{28} = \frac{9}{14}$$

$$\mu_2' = \int_0^1 x^2 f(x)\,dx = 6\int_0^1 x^2\left[x^2(1-x^3)\right]dx = 6\int_0^1 (x^4 - x^7)\,dx$$

$$= 6\left[\frac{x^5}{5} - \frac{x^8}{8}\right]_0^1 = 6\left[\frac{1}{5} - \frac{1}{8}\right] = \frac{18}{40} = \frac{9}{20}$$

$$\text{Variance} = \mu_2' - \mu_1' = \frac{9}{20} - \frac{91}{196} = \frac{441-405}{980} = \frac{36}{980} = \frac{9}{245}.$$

Example 6 : A continuous random variable X has the following probability density function

$$f(x) = \begin{cases} kx & 0 \le x \le 2 \\ 2k & 2 \le x \le 4 \\ 6k - kx & 4 \le x \le 6 \end{cases}$$

Find k, $P(1 \le x \le 3)$ and the mean. **(M.U. 2019)**

Sol. : For p.d.f. we must have $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\therefore \int_0^2 kx\,dx + \int_2^4 2k\,dx + \int_4^6 (6k - kx)\,dx = 1$$

$$\therefore k\left[\frac{x^2}{2}\right]_0^2 + 2k\left[x\right]_2^4 + k\left[6x - \frac{x^2}{2}\right]_4^6 = 1$$

$$\frac{k}{2}\left[4-0\right] + 2k\left[4-2\right] + k\left[(36-18)-(24-8)\right] = 1$$

$$2k + 4k + 6k = 1 \quad \therefore 12k = 1 \quad \therefore k = \frac{1}{2}$$

$$\therefore P(1 \le x \le 3) = \int_1^2 \frac{x}{12}\,dx + \int_2^3 \frac{1}{6}\,dx = \frac{1}{12}\left[\frac{x^2}{2}\right]_1^2 + \frac{1}{6}\left[x\right]_2^3 = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

$$\text{Mean } \bar{x} = \mu_1' = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \frac{1}{12}\int_0^2 x \cdot x\,dx + \frac{1}{6}\int_2^4 x\,dx + \frac{1}{12}\int_4^6 x(6-x)\,dx$$

$$\text{Mean } \bar{x} = \frac{1}{12}\left[\frac{x^3}{3}\right]_0^2 + \frac{1}{6}\left[\frac{x^2}{2}\right]_2^4 + \frac{1}{12}\left[3x^2 - \frac{x^3}{3}\right]_4^6$$

$$= \frac{8}{36} + \frac{1}{12}\left[16-4\right] + \frac{1}{12}\left[\left(108 - \frac{216}{3}\right) - \left(48 - \frac{64}{3}\right)\right]$$

$$= \frac{2}{9} + 1 + \frac{1}{12} + \frac{28}{3} = \frac{383}{36}.$$

Example 7 : If the distribution function of a random variable is given by

$$F(x) = \begin{cases} 1 - 4/x^2, & x > 2 \\ 0, & x \le 2 \end{cases}$$

find (i) $P(x < 3)$, (ii) $P(4 < x < 5)$, (iii) mean and the variance.

Sol. : The probability density function $f(x)$ is given by

$$f(x) = F'(x) = \begin{cases} 8/x^3, & x > 2 \\ 0, & x \le 2 \end{cases}$$

(i) $P(x < 3) = \int_2^3 \frac{8}{x^3} dx = \left[-\frac{8}{2x^2}\right]_2^3 = \left[-\frac{4}{x^2}\right]_2^3 = -4\left[\frac{1}{9} - \frac{1}{4}\right] = \frac{5}{9}.$

(ii) $P(4 < x < 5) = \int_4^5 \frac{8}{x^3} dx = \left[-\frac{4}{x^2}\right]_4^5 = -4\left[\frac{1}{25} - \frac{1}{16}\right] = 0{\cdot}09$

(iii) $\mu_1' = \int_2^\infty x \cdot \frac{8}{x^3} dx = \int_2^\infty \frac{8}{x^2} dx = 8\left[-\frac{1}{x}\right]_0^\infty = -8\left[0 - \frac{1}{2}\right] = 4$

$\mu_2' = \int_2^\infty x^2 \cdot \frac{8}{x^3} dx = 8\int_2^\infty \frac{1}{x} dx = 8\left[\log x\right]_2^\infty = \infty$

$\therefore$ Var. $(x) = \mu_2' - \mu_1'^2$ $\qquad \therefore$ Var. (x) does not exist.

Example 8 : A continuous random variable has probability density function

$$f(x) = k(x - x^2), \quad 0 \le x \le 1$$

Find (i) k, (ii) mean, (iii) variance. **(M.U. 1997, 2001, 03, 15, 16, 19)**

Sol. : (i) We have

$$\int_0^1 k(x - x^2) dx = 1 \qquad \therefore \quad k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$$

$$\therefore \quad k\left[\frac{1}{2} - \frac{1}{3}\right] = 1 \qquad \therefore \quad k \cdot \frac{1}{6} = 1 \qquad \therefore \quad k = 6.$$

(ii) $\mu_1' = \int_0^1 x \cdot 6(x - x^2) dx = 6\int_0^1 (x^2 - x^3) dx$

$$= 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 6\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{2}$$

(iii) $\mu_2' = \int_0^1 x^2 \cdot 6(x - x^2) dx = 6\int_0^1 (x^3 - x^4) dx$

$$= 6\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = 6\left[\frac{1}{4} - \frac{1}{5}\right] = \frac{6}{20} = \frac{3}{10}$$

$\therefore$ Var. $(x) = \mu_2' - \mu_1'^2 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}$

EXERCISE - II

Type - I

1. The probability distribution of a random variable X is given by

X :	-2	-1	0	1	2	3
$P(X = x)$:	$0{\cdot}1$	k	$0{\cdot}2$	$2k$	$0{\cdot}3$	k

Find k, the mean and variance. **(M.U. 2018)** [**Ans. : (i)** $k = 0{\cdot}1$, **(ii)** $\overline{X} = 0{\cdot}8$, **(iii)** Var. $2{\cdot}16$]

2. If X denotes the larger of the two numbers that appear when a pair of dice is thrown, find the probability distribution of X, and also the mean and variance of X.

[**Ans.** :

X	1	2	3	4	5	6
$P(X = x)$	1/36	3/36	5/36	7/36	9/36	11/36

Mean = 4·47, Var. = 1·97]

3. Given the following distribution

X	-3	-2	-1	0	1	2
$P(X = x)$	0·01	0·1	0·2	0·3	0·2	0·15

Find (i) $P(X \geq 1)$, (ii) $P(X < 0)$, (iii) $E(X)$, (iv) $V(X)$. **(M.U. 1998)**

[**Ans.** : **(i)** 0·35, **(ii)** 0·35, **(iii)** 0·05, **(iv)** 1·8475]

4. Find the value of k from the following data.

X	0	10	15
$P(X = x)$	$(k-6)/5$	$2/k$	$14/5k$

Also find the distribution function and expectation of X. **(M.U. 2003)**

[**Ans.** : $k = 8$ or 3, 3 is impossible since $P(X = 0) = -\dfrac{3}{5}$ for $k = 3$

X	0	10	15
$F(x)$	2/5	13/20	1

$E(X) = 31/4$]

5. A random variable X has the probability law $P(X = x) = 1/n$, $x = 1, 2, \ldots, n$. Find $E(X)$ and $V(X)$. [**Ans.** : **(i)** $(n+1)/2$, **(ii)** $(n^2-1)/12$]

6. A random variable X has the probability distribution

X	-2	-1	0	1	2	3
$p(x)$	0·1	k	0·2	$2k$	0·3	k

Find k and then mean and variance of X. [**Ans.** : $k = 0·1$, 0·8, 2·16]

7. Find the mean and the variance of the following distribution

X	1	3	4	5
$P(X = x)$	0·4	0·1	0·2	0·3

[**Ans.** : **(i)** $\overline{X} = 3$, **(ii)** Var. = 3]

8. A random variable X has the probability distribution $P(X = 0) = P(X = 2) = p$, $P(X = 1) = 1 - 2p$ and $0 \leq p \leq 2/3$.

For what value of p is the Var. (X) maximum ? [**Ans.** : $p = 2/3$]

Type - II

1. Find the mean and the variance of the following distribution

$$f(x) = \begin{cases} 1 - x, & 0 < x < 1 \\ x - 1, & 1 < x < 2 \end{cases}$$

(M.U. 2006) [**Ans.** : **(i)** $\overline{X} = 1$, **(ii)** Var. = 1/2]

2. If the probability density function is given by

$$f(x) = \begin{cases} kx^2(1 - x^3), & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

(i) Find k, (ii) $P(0 < X < 1/2)$, (iii) $\overline{X}$, (iv) σ^2. **(M.U. 1998, 2003, 04)**

[**Ans.** : **(i)** $k = 6$, **(ii)** 15 / 64, **(iii)** 9 / 14, **(iv)** 9 / 245]

3. If the probability density of a random variable is given by

$$f(x) = \begin{cases} kx, & 0 \le x \le 2 \\ 2k, & 2 \le x \le 4 \\ 6k - kx, & 4 \le x \le 6 \end{cases}$$

Find (i) k, (ii) $P(1 \le X \le 3)$, (iii) $\bar{X}$. **(M.U. 2002, 03, 05)** [**Ans. : (i)** 1 / 8, **(ii)** 7 / 16, **(iii)** 7 / 4]

4. Find the mean and the variance of

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \end{cases}$$

(M.U. 2009) [**Ans. :** 1, 5 / 2]

5. If the probability density of a random variable is given by

(a) $f(x) = \begin{cases} kx\,e^{-x/3}, & x > 0 \\ 0, & x \le 0 \end{cases}$ **(M.U. 2005)** (b) $f(x) = \begin{cases} k\,x^2 e^{-x}, & x > 0 \\ 0, & x < 0 \end{cases}$ **(M.U. 2007)**

(i) Find k, (ii) $\bar{X}$, (iii) σ^2. [**Ans. : (a) (i)** 1/ 9, **(ii)** 6, **(iii)** 18 ; **(b) (i)** 1/ 2, **(ii)** 3, **(iii)** 3]

6. A continuous random variable X has p.d.f. $f(x) = k\,x^2\,e^{-x}$, $x \ge 0$. Find k, mean and variance. **(M.U. 2004)** [**Ans. : (i)** 1/ 2, **(ii)** 3, **(iii)** 3]

7. If the probability density of a random variable is given by

$$f(x) = \begin{cases} kx\,e^{-x/3}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

(i) Find k, (ii) $\bar{X}$, (iii) σ^2. **(M.U. 2005)** [**Ans. : (i)** 1/ 9, **(ii)** 6, **(iii)** 18]

8. A continuous random variable has probability density function

$$f(x) = \begin{cases} 2e^{-2x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

Find (i) $E(X)$, (ii) $E(X^2)$, (iii) Var (X), (iv) S.D. of X. **(M.U. 2005)**

[**Ans. : (i)** $\frac{1}{2}$, **(ii)** $\frac{1}{2}$, **(iii)** $\frac{1}{4}$, **(iv)** $\frac{1}{2}$.]

9. The length of time (in minutes) a lady speaks on telephone is found to be a random variable with probability density function

$$f(x) = \begin{cases} Ae^{-x/5} & \text{for } x \ge 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find A and the probability that she will speak for (i) more than 10 minutes, (ii) less than 5 minutes, (iii) between 5 and 10 minutes. **(M.U. 2004)**

[**Ans. : (i)** $A = \frac{1}{5}$, **(ii)** $\frac{1}{e^2}$, **(iii)** $\frac{(e-1)}{e}$, **(iv)** $\frac{(e-1)}{e^2}$.]

10. The probability density function of a random variable is given by

$$f(x) = ke^{-x/\sigma}, \quad 0 < x < \infty$$

Find the mean and standard deviation of X. **(M.U. 2002)** [**Ans. : (i)** σ, **(ii)** σ]

11. A random variable X has the p.d.f. $f(x) = \dfrac{k}{1+x^2}$, $-\infty < X < \infty$. **(M.U. 2004)**

Determine k and the distribution function. Evaluate (i) $P(x \geq 0)$, (ii) Mean, (iii) Variance

[Ans. : $k = \frac{1}{\pi}$, $F(x) = \frac{1}{\pi}\left[\tan^{-1} x + \frac{\pi}{2}\right]$, $P(x \geq 0) = \frac{1}{2}$, $\overline{X} = 0$, variance does not exist]

12. If $f(x) = \begin{cases} x e^{-x^2/2}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

prove that (i) $f(x)$ is a probability density function and (ii) obtain distribution function $F(x)$.

(M.U. 1996) [Ans. : (i) $\int_0^{\infty} x e^{-x^2/2} dx = \int_0^{\infty} e^{-t} dt = 1$ where $t = \frac{x^2}{2}$; **(ii)** $F(x) = 1 - e^{-x^2/2}$]

13. A continuous random variable X takes values between 2 and 5. Its density function is $f(x) = k(1 + x)$. Find k and $P(x < 4)$. **(M.U. 1996) [Ans. :** $k = 2/27$, $16/27$]

14. The distribution function of a continuous random variable X is given by

$$F(x) = 1 - (1 + x)e^{-x}, \; x \geq 0.$$

Find the density function, mean and variance. **[Ans. :** $f(x) = xe^{-x}$, $x \geq 0$; 2 ; 2]

15. A continuous random variable X has a probability density function $f(x) = 3x^2$, $0 \leq x \leq 1$. Find a and b such that (i) $P(X \leq a) = P(X \geq a)$, (ii) $P(X > b) = 0.005$.

[Ans. : (i) $a = \sqrt[3]{1/2}$, **(ii)** $b = \sqrt[3]{.995}$]

16. If $f(x)$ is probability density function of a continuous random variate, find k, mean and variance.

$$f(x) = \begin{cases} kx^2 & 0 \leq x \leq 1 \\ (2 - x)^2 & 1 \leq x \leq 2 \end{cases}$$

(M.U. 2002) [Ans. : $k = 2$, $\overline{x} = \frac{11}{12}$, Var. $= 0{\cdot}626$]

17. A continuous random variable X has the probability density function given by

$$f(x) = \begin{cases} 2ax + b & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

If the mean of the distribution is 3, find the constants a and b. **(M.U. 1996)**

[Ans. : $a = \frac{3}{2}$, $b = -\frac{5}{2}$]

18. The probability density function of a random variable is given by

$$f(x) = \begin{cases} 0 & x < 2 \\ \dfrac{2x + 3}{18} & 2 \leq x \leq 4 \\ 0 & x > 4 \end{cases}$$

Find the mean and variance. **(M.U. 2001, 02) [Ans. :** $\overline{X} = \frac{83}{27}$, Var. $= 0{\cdot}33$]

19. Prove that $f(x) = \begin{cases} 1 - |1 - x| & 0 \leq x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$

is a probability density function. Find its mean and variance. **(M.U. 1998)**

[Ans. : $\overline{X} = 12$, Var. $= \frac{1}{2}$]

20. A continuous random variable X has the following probability density function

$$f(x) = \begin{cases} x^3 & 0 \le x \le 1 \\ (2-x)^3 & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Find $P(0 \cdot 5 \le x \le 1 \cdot 5)$ and the mean of the distribution. **(M.U. 1998)** [**Ans. :** $\frac{15}{32}$, $\overline{X} = \frac{1}{2}$]

21. The probability density function of a continuous random variable X is given by

$$f(x) = kx(2-x), \quad 0 \le x \le 2.$$

Find k, mean and variance. **(M.U. 1997, 99)** [**Ans. :** $k = \frac{3}{4}$, $\overline{X} = 1$, $V(X) = \frac{1}{5}$]

5. Laws of Expectation

We shall now develop some laws or theorems that will simplify the calculations of mathematical expectation of functions of random variables. These rules help us to calculate expectations in terms of known or easily computable expectations.

Theorem 1 : If X is a discrete random variable such that $x_i \ge 0$ for all i, then

$$\boxed{E(X) \ge 0}$$

Proof : Since $x_1, x_2, \ldots\ldots x_n$ are all non-negative and since, $p_1, p_2, \ldots\ldots p_n$ are also non-negative

$\therefore \; \Sigma p_i x_i \ge 0 \qquad \therefore \; E(X) \ge 0$

Theorem 2 : If X is a discrete (or continuous) random variate, a and b are constants then

$$\boxed{E(aX + b) = aE(X) + b}$$

Proof : We shall prove the result for discrete random variable and leave the continuous case to you. Let X take values $x_1, x_2, \ldots\ldots x_n$ with probabilities $p_1, p_2, \ldots\ldots p_n$. Then by definition,

$$\begin{aligned} E(aX + b) &= \Sigma p_i (a x_i + b) \\ &= p_1 (ax_1 + b) + p_2 (ax_2 + b) + \ldots\ldots + p_n (ax_n + b) \\ &= a(p_1 x_1 + p_2 x_2 + \ldots\ldots + p_n x_n) + b(p_1 + p_2 + \ldots\ldots + p_n) \\ &= a E(X) + b \qquad [\because \Sigma p_i = 1] \end{aligned}$$

Cor. 1 : Putting $a = 0$.

$E(b) = b$ *i.e.* **expectation of a constant is the constant itself.**

Cor. 2 : Putting $b = 0$.

$E(ax) = a\,E(X)$ *i.e.* for calculations **the constant can be taken out.**

Cor. 3 : Putting $a = 1$, $b = -\overline{X}$, $E(X - \overline{X}) = 0$.

Theorem 3 : Theorem of Addition (a) Discrete Variates

The expectation of the sum (or difference) of two (discrete) variates is equal to the sum (or difference) of their expectations.

In symbols, $\boxed{E(X \pm Y) = E(X) \pm (Y)}$

We shall accept the result without proof.

(a) Theorem of Addition (Continuous Variates)

The expectation of the sum (or difference) of two continuous variates is equal to the sum (or difference) of their expectations.

In symbols, $\boxed{E(X \pm Y) = E(X) \pm (Y)}$

We shall accept the result without proof.

Generalisation : The mathematical expectation of the sum (or difference) of n random variables is equal **to sum (or difference) of their expectations provided all the expectations exist.**

$$E(X_1 \pm X_2 \pm \ldots\ldots \pm X_n) = E(X_1) \pm E(X_2) \pm \ldots\ldots \pm E(X_n).$$

Theorem 4 : Theorem of Multiplication (a) Discrete variables : The expectation of the product of two **independent** variates is equal to the product of their expectations if the expectations exist.

In symbols, $\boxed{E(XY) = E(X) \cdot E(Y)}$

We shall accept the theorem without proof.

(b) Theorem of Multiplication (continuous variable)

The expectation of the product of two **independent variates** (discrete or continuous) is equal to the product of their expectations if the expectations exist.

In symbols, $\boxed{E(XY) = E(X) \cdot E(Y)}$

We shall accept the theorem without proof.

Generalisation : The mathematical expectation of the product of a number of independent continuous random variates is equal to the product of their expectations, provided all the expectations exist.

$$E(X, Y, Z \ldots\ldots W) = E(X) \cdot E(Y) \cdot E(Z) \ldots\ldots E(W).$$

6. Properties of Variance

We now prove some properties of variance which are useful in calculating variance.

1. Variance of a constant is zero, $V(C) = 0$.

Proof : $V(C) = E(C^2) - [E(C)]^2 = C^2 - C^2 = 0.$

2. If X is a random variate and a, b are constants then

$$\boxed{V(aX + b) = a^2 V(X)}$$ **(M.U. 2006)**

Proof : By definition,

$$\begin{aligned} V(aX + b) &= E[(aX + b) - E(aX + b)]^2 = E[(aX + b) - a\,E(X) - b)]^2 \\ &= E[a\{X - E(X)\}]^2 = E[a^2\{X - E(X)\}^2] \\ &= a^2 E[X - E(X)]^2 = a^2 V(X). \end{aligned}$$

Cor. 1 : $\boxed{V(aX) = a^2 V(X)}$

Putting $b = 0$, we get the result.

$$\boxed{V(X+b) = V(X)}$$

Cor. 2 :

Putting $a = 1$, we get the result.

Note .....

Note that although $E(aX+b) = aE(X) + b$ we do **not** have $V(aX+b) = aV(X) + b$. Instead, we have $V(aX+b) = a^2 V(X)$.

3. $$\boxed{V(a_1X_1 + a_2X_2) = a_1^2\, V(X_1) + a_2^2\, V(X_2)}$$

where X_1 and X_2 are independent random variates.

Proof : Let $Y = a_1X_1 + a_2X_2$ where X_1 and X_2 are independent.

$$\begin{aligned} \therefore \quad Y &= E(a_1X_1 + a_2X_2) \\ &= E(a_1X_1) + E(a_2X_2) \qquad \text{[By Theorem 3, page 8-21]} \\ &= a_1 E(X_1) + a_2 E(X_2) \end{aligned}$$

(This result is true even if X_1, X_2 are not independent.)

$$\begin{aligned} \therefore \quad V(Y) &= E\left[(a_1X_1 + a_2X_2) - \{a_1E(X_1) + a_2E(X_2)\}\right]^2 \\ &= E\left[a_1\{X_1 - E(X_1)\} + a_2\{X_2 - E(X_2)\}\right]^2 \\ &= E\Big[a_1^2\{X_1 - E(X_1)\}^2 + a_2^2\{X_2 - E(X_2)\}^2 \\ &\qquad + 2a_1a_2\{X_1 - E(X_1)\}\{X_2 - E(X_2)\}\Big] \\ &= a_1^2\, E\{X_1 - E(X_1)\}^2 + a_2^2\, E\{X_2 - E(X_2)\}^2 \\ &\qquad + 2a_1a_2\, E\left[\{X_1 - E(X_1)\}\{X_2 - E(X_2)\}\right] \\ &= a_1^2\, E[X_1 - E(X_1)]^2 + a_2\, E[X_2 - E(X_2)]^2 \\ &\qquad + 2a_1a_2\left[E(X_1)E(X_2) - E(X_1)E(X_2) - E(X_1)E(X_2) + E(X_1)E(X_2)\right] \end{aligned}$$

[$\because$ $E(X_1X_2) = E(X_1) \cdot E(X_2)$ as X_1, X_2 are independent.]

$$\therefore \quad V(a_1X_1 + a_2X_2) = a_1^2\, V(X_1) + a_2^2\, V(X_2)$$

Cor. 1 : If $a_1 = 1$, $a_2 = 1$, we get

$$\boxed{V(X_1 + X_2) = V(X_1) + V(X_2)}$$

If $a_1 = 1$, $a_2 = -1$, we get

$$\boxed{V(X_1 - X_2) = V(X_1) + V(X_2)}$$

Example 1 : If $E(X) = 2$ and $V(X) = 5$, find $E(3X+2)$ and $V(3X+2)$.

Sol. : $E(3X+2) = 3E(X) + 2 = 8$

$V(3X+2) = 9V(X) = 45.$

Example 2 : If $E(X) = 1$ and $E(X^2) = 4$, find the mean and variance of $Y = 2X - 3$.

Sol. :
$$\begin{aligned}
E(Y) &= E(2X - 3) = E(2X) - E(3) \\
&= 2E(X) - 3 = 2 \cdot 1 - 3 = -1. \\
E(Y^2) &= E[(2X - 3)^2] = E[4X^2 - 12X + 9] \\
&= E(4X^2) - E(12X) + E(9) \\
&= 4E(X^2) - 12E(X) + 9 \\
&= 4 \cdot 4 - 12 \cdot 1 + 9 = 13 \\
V(Y) &= E(Y^2) - [E(Y)]^2 = 13 - (-1)^2 = 12.
\end{aligned}$$

Alternatively : $V(X) = E(X^2) - [E(X)]^2 = 4 - 1 = 3$

$\therefore V(2X - 3) = 4V(X) = 4 \cdot 3 = 12$ [By Cor. 1, 2, page 8-22, 8-23]

Example 3 : If $E(X) = 10$, $\sigma_x^2 = 1$, and $Y = 2X(X + 20)$, find $E(Y)$.

Sol. :
$$\begin{aligned}
E(Y) &= E[2X^2 + 20X] = E(2X^2) + E(20X) \\
&= 2E(X^2) + 20E(X)
\end{aligned}$$

Now, $E(X) = 10$

and $V(X) = E(X^2) - [E(X)]^2 = 1$

$\therefore E(X^2) - 100 = 1 \qquad \therefore E(X^2) = 101$

$\therefore E(Y) = 2(101) + 20(10) = 202 + 200 = 402.$

Example 4 : Suppose that X is a r.v. with $E(X) = 10$ and $V(X) = 25$. Find the positive values of a and b such that $Y = aX - b$ has expectation 0 and variance 1.

Sol. : We have, $E(Y) = E[aX - b] = aE(X) - b$

$\therefore 0 = a10 - b \qquad \therefore b = 10a$

And $V(Y) = V(aX - b) = a^2 V(X)$

$\therefore 1 = a^2 \cdot 25 \qquad \therefore a = 1/5.$

Thus, $a = 1/5$ and $b = 2$.

Example 5 : If X_1 has mean 4 and variance 9 and X_2 has mean -2 variance 4, and the two are independent, find $E(2X_1 + X_2 - 3)$ and $V(2X_1 + X_2 - 3)$. **(M.U. 2019)**

Sol. : We have $E(X_1) = 4$, $V(X_1) = 9$, $E(X_2) = -2$ and $V(X_2) = 4$.

$$\begin{aligned}
\therefore E(2X_1 + X_2 - 3) &= E(2X_1 + X_2) - 3 = 2E(X_1) + E(X_2) - 3 \\
&= 2(4) + (-2) - 3 = 3. \\
V(2X_1 + X_2 - 3) &= V(2X_1 + X_2) \qquad \text{[By above Cor. 2]} \\
&= 2^2 V(X_1) + V(X_2) \\
&= 4(9) + 5 = 41.
\end{aligned}$$

EXERCISE - III

1. If $E(X) = 3$, $V(X) = 5$ and $Y = 3X + 4$, find $E(Y)$ and $V(Y)$. **[Ans. :** 13 ; 45 **]**
2. Find $V(X + 2)$ and $V(3X + 2)$. **[Ans. :** $V(X)$, $9V(X)$ **]**
3. Let X be a random variate with $E(X) = 12$ and $V(X) = 16$. Find the positive values of a and b such that $Y = aX - b$ has expectations 0 and variance 1. **[Ans. :** $a = 1/4$, $b = 3$ **]**

4. If X and Y are independent random variate, prove that

$$V(X+Y) = V(X) + V(Y)$$

(M.U. 2003)

5. Let X be a random variate with $E(X) = 15$ and $V(X) = 25$. Find the positive values of a and b such that $Y = aX - b$ has expectation 0 and variance 1.

[**Ans.** : 1 / 5, 3]

6. If X_1 has mean 5 and variance 5, X_2 has mean -2 and variance 3, and if X_1, X_2 are independent, find : (i) $E(X_1 + X_2)$, $V(X_1 + X_2)$

(ii) $E(X_1 - X_2)$, $V(X_1 - X_2)$

(iii) $E(2X_1 + 3X_2 - 5)$, $V(2X_1 + 3X_2 - 5)$ [**Ans.** : (i) 3, 8 ; (ii) 7, 8 ; (iii) −1, 52]

7. Covariance and Correlation

Definition : If X and Y are discrete random variates then the covariance between them denoted by cov. (X, Y) is defined by

$$\boxed{\textbf{cov.}\,(X, Y) = E[\{X - E(X)\}\cdot\{Y - E(Y)\}]} \quad \text{.................... (1)}$$

or $$\text{cov.}(X,Y) = \frac{1}{n}\Sigma(x_i - \overline{X})(y_i - \overline{Y})$$

On simplification, we get another expression for cov. (X, Y)

Now, $$\begin{aligned}\text{cov. }(X, Y) &= E[XY - XE(Y) - YE(X) + E(X)E(Y)] \\ &= E(XY) - E(X)E(Y) - E(Y)E(X) + E(X)\cdot E(Y)\end{aligned}$$

$$\boxed{\textbf{cov.}\,(X, Y) = E(XY) - E(X)E(Y)} \quad \text{.................... (2)}$$

Example 1 : Prove that cov. $(aX, bY) = ab$ cov. (X, Y)

Sol. : $$\begin{aligned}\text{Cov. }(aX, bY) &= E(ab\,XY) - E(aX)E(bY) = ab\,E(XY) - ab\,E(X)E(Y) \\ &= ab[E(XY) - E(X)E(Y)] = ab\text{ cov. }(X, Y)\end{aligned}$$

Definition of Correlation Coefficient *r*

Definition : If X, Y are two discrete random variates, then the correlation coefficient between them denoted by r is defined by

$$\boxed{r = \frac{\textbf{cov.}(X,Y)}{\sigma_x\,\sigma_y}}$$

where σ_x, σ_y are the standard deviations of x and y.

Theorem : If X, Y independent random variates, then they are not correlated.

Proof : By definition, coefficient of correlation $r = \dfrac{\text{cov.}(X,Y)}{\sigma_x\,\sigma_y}$.

But cov. $(x, y) = E(X, Y) - E(X)\cdot E(Y)$. [By (2)]

Since x, y are independent $E(X, Y) = E(X)\cdot E(Y)$

$\therefore$ cov. $(X, Y) = E(X)\cdot E(Y) - E(X)\cdot E(Y) = 0$

$\therefore$ $r = 0$

Note ....

However, the converse is not true *i.e.* if $r = 0$, it does not necessarily mean that the variables are not independent.

Consider the following series.

X	Y	$x_i - \bar{X}$	$y_i - \bar{Y}$	$(x_i - \bar{X})(y_i - \bar{Y})$
4	8	4	3	12
2	2	2	−3	−6
−2	2	−2	−3	6
−4	8	−4	3	−12
$\bar{X} = 0$	$\bar{Y} = 5$			Total 0

Since $\Sigma(x_i - \bar{X})(y_i - \bar{y}) = 0$, $\text{cov.}(X, Y) = 0$ $\quad \therefore r = 0$

But it is clear that the above values of x and y satisfy the relation $y = \frac{1}{2}x^2$. This means X and Y are not independent.

Example : Let X and Y be two variates taking values 0 and 1. If $r(X, Y) = 0$, show that $X = 1$ and $Y = 1$ are independent events. **(M.U. 2004)**

Sol. : Let X take values 0, 1 with probability p and q.

Let Y take values 0, 1 with probabilities p' and q'.

Now $\quad r(X, Y) = \dfrac{\text{cov}(X, Y)}{\sigma_x \sigma_y} \quad \because r = 0,\ \text{cov}(X, Y) = 0$

But $\quad \text{cov}(X, Y) = E(X, Y) - E(X) \cdot E(Y)$

$\therefore \quad E(X, Y) - E(X)\, E(Y) = 0$

$\therefore \quad E(X, Y) = E(X) \cdot E(Y)$ .... (A)

Now $\quad E(X) = \Sigma p_i x_i = p \times 0 + q \times 1 = q$

$\quad E(Y) = \Sigma p_i y_i = p' \times 0 + q' \times 1 = q'$

Now $\quad E(XY) = \Sigma p_{ij} \cdot x_i \cdot y_j = p_{00}(0 \times 0) + p_{01}(0 \times 1) + p_{10}(1 \times 0) + p_{11}(1 \times 1) = p_{11}$

$\therefore$ By (A) $\quad p_{11} = qq'$

$\therefore \quad P(X = 1, Y = 1) = p(X = 1) \cdot p(Y = 1)$

Hence, the events $X = 1$ and $Y = 1$ are independent.

8. Raw and Central Moments

The following mathematical expectations have special significance in the study of probability distributions and hence, they are known by special names. They are denoted by special symbols.

(1) ***r*-th moment about the origin (μ_r')**

If we put $g(X) = X^r$, in § 3 (page 8-9), then the expectation is denoted by μ_r'. Thus,

$$\boxed{\mu_r' = E(X^r) = \Sigma p_i x_i^r} \quad \text{or} \quad \boxed{\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx}$$

This is called the r-th moment of the probability distribution of X about the origin, denoted by μ_r'.

Particular cases : (i) If $r = 0$, we get $\boxed{\mu_0' = 1}$

because $\quad \mu_0' = E(X^0) = \Sigma p_i (x_i)^0 = \Sigma p_i = 1$

or $\quad \mu_0' = E(X^0) = \int_{-\infty}^{\infty} x^0 f(x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx = 1$

(ii) If $r = 1$, we get $\boxed{\mu_1' = \overline{X}}$

because $\mu_1' = E(X) = \Sigma p_i x_i = \overline{X}$

or $\mu_1' = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \overline{X}$

(2) **r-th moment about the value a (μ_r')**

If we put $g(X) = (X-a)^r$, in § 3 (page 8-9), then the expectation is denoted by μ_r'. Thus,

$$\boxed{\mu_r' = E(X-a)^r = \Sigma p_i (x_i - a)^r}$$

or $$\boxed{\mu_r' = E(X-a)^r = \int_{-\infty}^{\infty} (x-a)^r f(x)\,dx}$$

This is called the r-th raw moment of the probability distribution of X about a denoted by μ_r'.

Particular cases : (i) If $r = 0$, we get $\boxed{\mu_0' = 0}$

because $\mu_0' = E(X-a)^0 = \Sigma p_i (x_i - a)^0 = \Sigma p_i = 1.$

or $\mu_0' = E(X-a)^0 = \int_{-\infty}^{\infty} (x-a)^0 f(x)\,dx$

$= \int_{-\infty}^{\infty} f(x)\,dx = 1$

(ii) If $r = 1$, we get $\boxed{\mu_1' = \overline{X} - a}$

because $\mu_1' = E(X-a) = \Sigma p_i (x_i - a) = \Sigma p_i x_i - a \Sigma p_i = \overline{X} - a$

or $\mu_1' = E(X-a) = \int_{-\infty}^{\infty} (x-a) f(x)\,dx$

$= \int_{-\infty}^{\infty} x f(x)\,dx - a\int_{-\infty}^{\infty} f(x)\,dx = \overline{X} - a$

(3) **r-th moment about the mean (μ_r)**

If we put $g(X) = (X - \overline{X})^r$, in § 3 (page 8-9), then the expectation is denoted by μ_r'. Thus,

$$\boxed{\mu_r = E[(X-\overline{X})^r] = \Sigma p_i (x_i - \overline{X})^r}$$

or $$\boxed{\mu_r = E(X-\overline{X})^r = \int_{-\infty}^{\infty} (x-\overline{X}) f(x)\,dx}$$

This is called the r-th moment of the probability distribution of X about the mean $\overline{X}$, denoted by μ_r.

Particular cases : (i) If $r = 0$, we get $\boxed{\mu_0 = 1}$

because $\mu_0 = \Sigma p_i (x_i - \overline{X})^0 = \Sigma p_i = 1$

or $\mu_0 = \int_{-\infty}^{\infty} (x-\overline{X})^0 f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 1$

(ii) If $r = 1$, we get $\boxed{\mu_1 = 0}$

because $\mu_1 = \Sigma p_i (x_i - \overline{X}) = \Sigma p_i x_i - \overline{X}\Sigma p_i = \overline{X} - \overline{X} = 0.$

or $\mu_1 = \int_{-\infty}^{\infty} (x - \overline{X}) f(x)\, dx$

$$= \int_{-\infty}^{\infty} x f(x)\, dx - \overline{X} \int_{-\infty}^{\infty} f(x)\, dx$$

$$= \overline{X} - \overline{X} = 0$$

9. Moment Generating Function

(a) Definition (Discrete Random Variable) : The moment generating function (m.g.f.) of a discrete random variate X about a denoted by $M_a(t)$ is defined by

$$M_a(t) = E[e^{t(x-a)}] \qquad \therefore \qquad \boxed{M_a(t) = \Sigma\, p_i\, e^{t(x_i - a)}} \qquad \ldots\ldots (1)$$

The m.g.f. is a function of the real parameter t. The subscript **a** shows the point about which the m.g.f. is taken.

Expanding the exponential in (1), we get

$$M_a(t) = \Sigma\, p_i \left[1 + \frac{t}{1!}(x_i - a) + \frac{t^2}{2!}(x_i - a)^2 + \frac{t^3}{3!}(x_i - a)^3 + \ldots\ldots\right]$$

$$= \Sigma\, p_i + \frac{t}{1!}\Sigma\, p_i\,(x_i - a) + \frac{t^2}{2!}\Sigma\, p_i (x_i - a)^2 + \frac{t^3}{3!}\Sigma\, p_i (x_i - a)^3 + \ldots\ldots \qquad \ldots\ldots (1a)$$

But $\Sigma\, p_i\,(x_i - a)^r$ is the r-th moment of X about a i.e. μ'_r. Hence, we have

$$M_a(t) = 1 + \mu'_1 \cdot \frac{t}{1!} + \mu'_2 \cdot \frac{t^2}{2!} + \mu'_3 \cdot \frac{t^3}{3!} + \ldots\ldots + \mu'_r \cdot \frac{t^r}{r!} + \ldots\ldots \qquad \ldots\ldots (2)$$

Thus, the coefficient of $(t^r / r!)$ is the r-th moment of X about a i.e. μ_r'. In this way $M_a(t)$ generates moments. This is the reason why the function $M_a(t)$ is called the **moment generating function**.

Thus,
$$\boxed{\mu_r' = \text{coefficient of } \frac{t^r}{r!}}$$

(b) Definition (Continuous Random Variable) : The moment generating function (m.g.f.) of a continuous random variate X about a denoted $M_a(t)$ is defined by

$$M_a(t) = E[e^{t(x-a)}] \qquad \therefore \qquad \boxed{M_a(t) = \int_{-\infty}^{\infty} e^{t(x-a)} \cdot f(x)\, dx} \qquad \ldots\ldots (3)$$

where $f(x)$ is the p.d.f. of X.

Expanding the exponential in (3), we get,

$$M_a(t) = \int_{-\infty}^{\infty} f(x) \cdot \left[1 + \frac{t}{1}(x - a) + \frac{t^2}{2!}(x - a)^2 + \frac{t^3}{3!}(x - a)^3 + \ldots\ldots\right] dx$$

$$M_a(t) = \int_{-\infty}^{\infty} f(x)\, dx + \frac{t}{1!}\int_{-\infty}^{\infty} (x - a) f(x)\, dx$$

$$+ \frac{t^2}{2!}\int_{-\infty}^{\infty} (x - a)^2 f(x)\, dx + \frac{t^3}{3!}\int_{-\infty}^{\infty} (x - a)^3 f(x)\, dx + \ldots\ldots \qquad \ldots\ldots (3a)$$

But $\int_{-\infty}^{\infty} (x - a)^r f(x)\, dx$ is the r-th moment μ_r' of X about a. Hence,

$$M_a(t) = 1 + \mu_1' \cdot \frac{t}{1!} + \mu_2' \cdot \frac{t^2}{2!} + \mu_3' \cdot \frac{t^3}{3!} + \ldots + \mu_r' \cdot \frac{t^r}{r!} + \ldots \quad \ldots (4)$$

Thus, the coefficient of $(t^r / r!)$ is the r-th moment of X about a.

(c) To find moments of various orders from m.g.f. : It is clear from (2) and (4) that the moment of order r is the coefficient of $(t^r / r!)$ in the expansion of m.g.f. Hence, one way of obtaining various moments is to obtain expansion of the m.g.f. of X and find the coefficient of $(t^r / r!)$ in the expansion.

However, in practice many a time obtaining the expansion of m.g.f. is not convenient. In such cases we differentiate m.g.f. w.r.t. t for r times and equate it to zero to get μ_r.

Thus, differentiating $M_a(t)$ from (2) (or from 4), successively, we get,

$$\frac{d}{dt}[M_a(t)] = \mu_1' + \mu_2' t + \mu_3' \frac{t^2}{2!} + \ldots$$

Putting $t = 0$, $\quad \frac{d}{dt}[M_a(t)]_{t=0} = \mu_1'$

$$\frac{d^2}{dt^2}[M_a(t)] = \mu_2' + \mu_3' t + \mu_4' \frac{t^2}{2!} + \ldots$$

Putting $t = 0$, $\quad \frac{d^2}{dt^2}[M_a(t)]_{t=0} = \mu_2'$

In general, $\quad \boxed{\frac{d^r}{dt^r}[M_a(t)]_{t=0} = \mu_r'}$

(d) Moment generating function about origin : Putting $a = 0$ in (1), we get,

$$\boxed{M_0 = \Sigma p_i e^{t x_i}} \quad \ldots (5)$$

Putting $a = 0$ in (3), we get,

$$\boxed{M_0(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx} \quad \ldots (6)$$

Putting $a = 0$ in (1a), we get since $\Sigma p_i x_i^r = \mu_r'$ about the origin.

$$M_0(t) = 1 + \mu_1' t + \mu_2' \frac{t^2}{2!} + \ldots + \mu_r' \frac{t^2}{r!} + \ldots \qquad \therefore \quad M_0(t) = \Sigma \mu_r' \frac{t^r}{r!}$$

Note ....

It is assumed that the r.h.s. of (5) and (6) is absolutely convergent.

(e) If $L(t) = \log M(t)$ where $M(t)$ is the moment generating function of a random variable, prove that the mean = $L'(0)$ and variance = $L''(0)$. **(M.U. 2006)**

Proof : We have $\quad M(t) = 1 + \mu_1' t + \mu_2' \frac{t^2}{2} + \ldots$

$$\therefore \quad L(t) = \log M(t) = \log\left[1 + \left(\mu_1' t + \mu_2' \frac{t^2}{2!} + \ldots\right)\right]$$

$$= \left(\mu_1' t + \mu_2' \frac{t^2}{2} + \ldots\right) - \frac{1}{2}\left(\mu_1' t + \mu_2' \frac{t^2}{2!} + \ldots\right)^2 + \ldots$$

$$= \left(\mu_1' t + \mu_2' \frac{t^2}{2!} + \ldots\right) - \frac{1}{2}\left(\mu_1'^2 t^2 + \mu_1' \mu_2' t^3 + \ldots\right) + \ldots$$

$$\therefore \quad L'(t) = \mu_1' + \mu_2' t - \mu_1'^2 \cdot t - \frac{3}{2}\mu_1' \mu_2' t^2 + \text{terms in higher powers of } t$$

Putting $t = 0$, $\therefore \; L'(0) = \mu_1'$ .... (A)

Now, $L''(t) = \mu_2' - \mu_1'^2 - 3\mu_1' \mu_1'^2 t$ + terms in higher powers of t

Putting $t = 0$, $L''(0) = \mu_2' - \mu_1'^2$

Hence, Mean, $\mu = L'(0)$, Variance, $\mu_2 = L''(0)$. .... (B)

(f) Moment generating function of the sum of two independent random variates : "*The* ***moment generating function of the sum of two independent random variates is equal*** *to the* ***product of the m.g.f.s of the two variates.***"

Proof : Let X, Y be two independent random variates then the m.g.f. of their sum $X + Y$ about the origin is given by

$$M_{(X+Y)}(t) = E\left[e^{t(X+Y)}\right] = E\left(e^{tX} \cdot e^{tY}\right)$$

$$= E\left(e^{tX}\right) \cdot E\left(e^{tY}\right) \qquad [\because X \text{ and } Y \text{ are independent}]$$

$$\therefore \quad \boxed{\mathbf{M_{X+Y}(t) = M_X(t) \cdot M_Y(t)}}$$

Generalisation : If $X_1, X_2, \ldots X_n$ are n independent random variates, then the m.g.f. of their sum is equal to the product of their m.g.f.s

$$M_{X_1+X_2+\ldots+X_n}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \ldots\ldots M_{X_n}(t)$$

(g) Uniqueness of Moment Generating Function : This is a very important property of m.g.f. It states that the m.g.f. of a distribution, if it exists, uniquely determines the distribution. In other words it means that for a given probability distribution there is one and only one m.g.f. and corresponding a given m.g.f. there is one and only one probability distribution. Thus, if m.g.f. of X and m.g.f. of Y are equal then X and Y must be identical.

Example 1 : Find the M.G.F. of the following distribution

X	:	-2	3	1
$P(X = x)$	:	1/3	1/2	1/6

Hence, find first four central moments. **(M.U. 2010, 15)**

Sol. : Since we want central moments we shall find m.g.f. about the mean.

Now, $\bar{X} = \sum p_i x_i = -\frac{2}{3} + \frac{3}{2} + \frac{1}{6} = \frac{-4+9+1}{6} = \frac{6}{6} = 1$

$\therefore$ M.G.F. about the mean

$$M_{\bar{x}}(t) = \sum p_i e^{t(x_i - \bar{x})} = \frac{1}{3} \cdot e^{t(-2-1)} + \frac{1}{2} e^{t(3-1)} + \frac{1}{6} e^{t(1-1)}$$

$$= \frac{1}{3}e^{-3t} + \frac{1}{2}e^{2t} + \frac{1}{6}$$

$$M_{\bar{x}}(t) = \frac{1}{3}\left[1 - 3t + \frac{9t^2}{2!} - \frac{27t^3}{3!} + \frac{81t^4}{4!} - \ldots\right] + \frac{1}{2}\left[1 + 2t + \frac{4t^2}{2!} + \frac{8t^3}{3!} + \frac{16t^4}{4!} + \ldots\right] + \frac{1}{6}$$

Now, $\mu_r = \text{Coeff. of } \frac{t^r}{r!}$

$\therefore \mu_0 = \text{Coeff. of } t^0 = \text{constant term} = \frac{1}{3} + \frac{1}{2} + \frac{1}{6} = 1$

$\therefore \mu_1 = \text{Coeff. of } \frac{t}{1!} = \frac{1}{3}(-3) + \frac{1}{2}(2) = 0$

$\mu_2 = \text{Coeff. of } \frac{t^2}{2!} = \frac{9}{3} + \frac{4}{2} = 5$

$\mu_3 = \text{Coeff. of } \frac{t^3}{3!} = \frac{1}{3}(-27) + \frac{1}{2}(8) = -5$

$\mu_4 = \text{Coeff. of } \frac{t^4}{4!} = \frac{1}{3}(81) + \frac{1}{2}(16) = 35.$

Example 2 : Find the m.g.f. of the random variable X about the origin whose p.m.f. is given above in Ex. 1. Also find the first two moments about the ogirion. **(M.U. 2004)**

Sol. : By definition

$$\mu_0(t) = E(e^{tX}) = \Sigma p_i e^{tx_i} = \frac{1}{3} \cdot e^{-2t} + \frac{1}{2}e^{3t} + \frac{1}{6}e^t$$

Now, $\mu_1' = \left[\frac{d}{dt}M_0(t)\right]_{t=0} = \left[-\frac{2}{3}e^{-2t} + \frac{3}{2}e^{3t} + \frac{1}{6}e^t\right]_{t=0}$

$$= -\frac{2}{3} + \frac{3}{2} + \frac{1}{6} = \frac{6}{6} = 1$$

$$\mu_2' = \left[\frac{d^2}{dt^2}M_0(t)\right]_{t=0} = \left[\frac{4}{3}e^{-2t} + \frac{9}{2}e^{3t} + \frac{1}{6}e^t\right]_{t=0}$$

$$= \frac{4}{3} + \frac{9}{2} + \frac{1}{6} = \frac{36}{6} = 6.$$

Example 3 : A random variable X has probability density function $1/(2^x)$, $x = 1, 2, 3 \ldots$. Find the m.g.f. and hence, find the mean and variance.

Sol. : Since, $P(X = x) = \frac{1}{2^x}$, $x = 1, 2, 3 \ldots\ldots$

$$M_0(t) = E(e^{tx}) = \Sigma p e^{tx}$$

$$= \Sigma \frac{1}{2^x}e^{tx} = \sum_{x=1}\left(\frac{e^t}{2}\right)^x = \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \ldots\ldots$$

$$= \frac{e^t}{2}\left[1 + \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^2 + \ldots\ldots\right] = \frac{e^t}{2}\left[1 - \frac{e^t}{2}\right]^{-1}$$

$$\therefore \quad M_0(t) = \frac{e^t}{2} \cdot \frac{1}{1-(e^t/2)} = \frac{e^t}{2} \cdot \frac{2}{2-e^t} = \frac{e^t}{2-e^t}$$

$$\therefore \quad \mu_1' = \left[\frac{d}{dt} M_0(t)\right]_{t=0} = \left[\frac{(2-e^t)e^t - e^t(-e^t)}{(2-e^t)^2}\right]_{t=0}$$

$$= 2\left[\frac{e^t}{(2-e^t)^2}\right]_{t=0} = 2$$

$$\therefore \quad \mu_2' = \left[\frac{d^2}{dt^2} M_0(t)\right]_{t=0} = 2 \cdot \left[\frac{(2-e^t)^2 \cdot e^t - e^t \cdot 2(2-e^t) \cdot (-e^t)}{(2-e^t)^4}\right]_{t=0}$$

$$= 2 \cdot \left[\frac{(2-e^t) \cdot e^t + 2e^{2t}}{(2-e^t)^3}\right]_{t=0} = \frac{2(1+2)}{1} = 6$$

$$\therefore \quad \mu_2 = \mu_2' - \mu_1'^2 = 6 - 4 = 2.$$

$\therefore$ Mean = Variance = 2.

Example 4 : If X denotes the outcome when a fair die is tossed, find M.G.F. of X and hence find the mean and variance of X. **(M.U. 2005, 09, 18, 19)**

Sol. : We have, here X taking values 1, 2, 3, 4, 5, 6 each with probability 1/ 6.

$$M_0(t) = E(e^{tx_i}) = \Sigma p_i e^{tx_i}$$

$$= \frac{1}{6}e^t + \frac{1}{6}e^{2t} + \frac{1}{6}e^{3t} + \ldots\ldots + \frac{1}{6}e^{6t}$$

$$= \frac{1}{6}(e^t + e^{2t} + \ldots\ldots + e^{6t})$$

$$\therefore \quad \mu_1' = \left[\frac{d}{dt} M_0(t)\right]_{t=0} = \frac{1}{6}\left[e^t + 2e^{2t} + \ldots\ldots + 6e^{6t}\right]_{t=0}$$

$$= \frac{1}{6}[1+2+\ldots\ldots+6] = \frac{21}{6} = \frac{7}{2}$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_0(t)\right]_{t=0} = \frac{1}{6}\left[e^t + 4e^{2t} + 9e^{3t} + \ldots\ldots + 36e^{6t}\right]_{t=0}$$

$$= \frac{1}{6}[1+4+9+\ldots\ldots+36] = \frac{91}{6}$$

$$\therefore \quad \mu_2 = \mu_2' - \mu_1'^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}.$$

$$\therefore \quad \text{Mean} = \frac{7}{2} \text{ and Variance} = \frac{35}{12}.$$

Example 5 : Find the m.g.f. of a random variable X if the r-th moment about the origin is given by $\mu_r' = r!$.

Sol. : Be definition, $\mu_r' = E(x^r) = r!$ ......(1)

$$\therefore \quad E(x) = 1, \quad E(x^2) = 2! \qquad E(x^3) = 3! \ldots..$$

Now, $M_0(t) = E(e^{tx})$

$$= E\left[1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \ldots\right]$$

$$= E(1) + t\,E(x) + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \ldots$$

Putting the values of $E(X)$, $E(X)^2$, ... from (1)

$$M_0(t) = 1 + t + \frac{t^2}{2!}\cdot 2! + \frac{t^3}{3!}\cdot 3! + \ldots$$

$$= 1 + t + t^2 + t^3 + \ldots = (1-t)^{-1} = \frac{1}{1-t}.$$

Example 6 : Find the m.g.f. of a random variable whose p.m.f. is

$$P(X = x) = \left(\frac{1}{2}\right)^x, \quad x = 1, 2, 3 \ldots$$
$$= 0 \quad \text{else where}$$

Hence, find the mean and variance of X.

Sol. : By definition

$$M_0(t) = E(e^{tx}) = \Sigma p_i\, e^{tx_i} = \Sigma \frac{1}{2^x} e^{tx} = \Sigma \left(\frac{e^t}{2}\right)^x$$

$$M_0(t) = \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \ldots = \frac{e^t}{2}\left[1 + \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^2 + \ldots\right]$$

$$= \frac{e^t}{2}\,\frac{1}{1-(e^t/2)} = \frac{e^t}{2-e^t}.$$

To find the mean and variance

$$\mu_1' = \left[\frac{d}{dt}M_0(t)\right]_{t=0} = \left[\frac{(2-e^t)\cdot e^t - e^t\cdot(-e^t)}{(2-e^t)^2}\right]_{t=0}$$

$$= \left[\frac{2e^t}{(2-e^t)^2}\right]_{t=0} = 2$$

$$\mu_2' = \left[\frac{d^2}{dt^2}M_0(t)\right]_{t=0} = \left[\frac{(2-e^t)^2\cdot e^t - e^t\cdot 2(2-e^t)(-e^t)}{(2-e^t)^4}\right]_{t=0}$$

$$= \frac{2[1+2]}{1} = 6$$

$\therefore$ Variance $= \mu_2' - \mu_1'^2 = 6 - 4 = 2.$

Remark ....

Mean and variance of the above distribution can be obtained using $E(X)$ and $E(X)^2$. Find them.

Example 7 : A random variable X has the following probability density function

$$f(x)=\begin{cases}1, & 0<x<1\\ 0. & \text{otherwise}\end{cases}$$

Find m.g.f., μ_r', mean, and variance.

Sol. : We have

$$M_0(t)=E(e^{tx})=\int_0^1 e^{tx}(1)\,dx=\left[\frac{e^{tx}}{t}\right]_0^1=\frac{1}{t}\left[e^t-1\right]$$

$$=\frac{1}{t}\left[1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots\ldots-1\right]=\frac{1}{t}\left[t+\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots\ldots\right]$$

$$=1+\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots\ldots+\frac{t^r}{(r+1)!}$$

$$\therefore\quad \mu_r'=\text{Coefficient of } \frac{t^r}{r!}=\frac{1}{r+1},\qquad r=1,2,\ldots\ldots$$

Putting $r = 1, 2$

$$\therefore\quad \text{Mean}=\mu_1'=\frac{1}{2};\ \mu_2'=\frac{1}{3}\qquad \therefore\quad \text{Var. }(X)=\mu_2'-\mu_1'^2=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}.$$

Example 8 : A random variable X has the following probability density function

$$f(x)=\begin{cases}ke^{-kx}, & x>0,\ k>0\\ 0, & \text{elsewhere}\end{cases}$$

Find the m.g.f. and hence, the mean and variance.

Sol. : We have

$$M_0(t)=E(e^{tx})=\int_0^\infty e^{tx}\cdot k\,e^{-kx}\,dx=k\int_0^\infty e^{(t-k)x}dx$$

$$=\frac{k}{t-k}\left[e^{(t-k)x}\right]_0^\infty=\frac{k}{t-k}[0-1]=\frac{k}{k-t}\qquad [t\neq k]$$

Now, $$M_0(t)=\frac{k}{k[1-(t/k)]}=\left[1-\frac{t}{k}\right]^{-1}=1+\frac{t}{k}+\frac{t^2}{k^2}+\frac{t^3}{k^3}+\ldots\ldots$$

$$\therefore\quad \mu_1'=\text{Coefficient of } t=\frac{1}{k};\qquad \mu_2'=\text{Coefficient of } \frac{t^2}{2!}=\frac{2}{k^2}$$

$$\therefore\quad \text{Mean}=\mu_1'=\frac{1}{k};\quad \text{Var. }(X)=\mu_2'-\mu_1'^2=\frac{2}{k^2}-\frac{1}{k^2}=\frac{1}{k^2}.$$

Example 9 : If a random variable has the moment generating function $M_t=\dfrac{3}{3-t}$, obtain the mean and the standard deviation. **(M.U. 2010)**

Sol. : We have

$$M_0(t)=\frac{3}{3-t}=\frac{3}{3[1-(t/3)]}=\left(1-\frac{t}{3}\right)^{-1}=1+\frac{t}{3}+\frac{t^2}{9}+\frac{t^3}{27}+\ldots\ldots\infty.$$

$$\text{Mean} = E(X) = \text{Coefficient of } \frac{t}{1!} = \frac{1}{3}$$

$$\mu_2' = E(X^2) = \text{Coefficient of } \frac{t^2}{2!} = \frac{2}{9}.$$

$$\therefore \ Var(X) = \mu_2' - \mu_1^2 = \frac{2}{9} - \frac{1}{9} = \frac{1}{9} \qquad \therefore \ \text{S.D.} = \frac{1}{3}.$$

EXERCISE - IV

1. A random variable takes values $X = 0, 1$ with probabilities q and p respectively, such that $q + p = 1$. Find the moment generating function of X and show that all moments about the origin are equal to p.

   [**Ans. :** $M_0(t) = qe^0 + pe^t = 1 + (e^t - 1)p = 1 + \left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\right) \ \therefore \ \mu_r' = p$]

2. A random variable X-has the m.g.f. given by $M_0(t) = \frac{2}{2-t}$. Find the standard deviation of X.

   [**Ans. :** $\sigma = 1/2$]

3. A random variable X has the probability distribution

$$P(X = x) = \frac{1}{8}\,{}^3C_x, \quad x = 0, 1, 2, 3.$$

Find the moment generating function of X and then find mean and variance. **(M.U. 2003)**

[**Ans. : (i)** $\frac{1}{8}(1 + e^t)$, **(ii)** $\frac{3}{2}, \frac{3}{4}$.]

4. A random variable X has the following probability distribution.

X	:	0	1	2	3
$P(X = x)$	:	1/6	1/3	1/3	1/6

Compute (i) Moment generating function about the origin, (ii) first two raw moments.

[**Ans. : (i)** $\frac{1}{6} + \frac{1}{3}e^t + \frac{1}{3}e^{2t} + \frac{1}{6}e^{3t}$, **(ii)** $\frac{3}{2}, \frac{19}{6}$.]

5. A random variable X has the following probability distribution.

X	:	0	1	2
$P(X = x)$	:	1/3	1/3	1/3

Find (i) the moment generating function, (ii) find mean and variance.

[**Ans. : (i)** $\frac{1}{3}(1 + e^t + e^{2t})$, **(ii)** 1, $\frac{2}{3}$.]

6. Find the m.g.f. of the random variable having the following probability density function. Also find the mean and variance.

(i) $f(x) = \begin{cases} 1/2, & -1 \le x < 1 \\ 0, & \text{elsewhere} \end{cases}$ **(ii)** $f(x) = \begin{cases} e^{-(x-5)}, & x \ge 5 \\ 0, & \text{elsewhere} \end{cases}$

[**Ans. : (i)** $\frac{(e^t - e^{-t})}{2t}$, 0, $\frac{1}{3}$; **(ii)** $\frac{e^{5t}}{1-t}$, 6, 1]

7. A random variable X has the probability density function

$$f(x) = \begin{cases} 1/3, & -1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

Find the m.g.f. of X.

$$\left[\textbf{Ans. : } M_0(t) = \frac{e^{2t} - e^{-t}}{3t},\ t \neq 0\right]$$

8. A random variable X has the following density function

$$f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

Find the m.g.f. and hence, its mean and variance.

$$\left[\textbf{Ans. : } M_0(t) = \frac{2}{2-t}\ ;\ t \neq 2,\ \ \mu_1 = \frac{1}{2},\ \ \mu_2 = \frac{1}{4}\right]$$

9. Find the m.g.f. of the random variable having probability density function

$$f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 2-x, & 1 \leq x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

$$\left[\textbf{Ans. : } M_0(t) = \left(\frac{e^t - 1}{t}\right)^2 ;\ t \neq 0\right]$$

EXERCISE - V

Theory

1. Define mathematical expectation.
2. Define expectation of a function of a random variable X.
   (a) when X is a discrete r.v., (b) when X is a continuous r.v.
3. Explain the following terms :
   (i) Expectation of a random variable.
   (ii) Variance of a random variable. **(M.U. 1998)**
4. Explain the following terms
   (i) Moments about origin. (ii) Moment Generating Function. **(M.U. 2002)**
5. Prove that $V(X) = E(X^2) - \{E(X)\}^2$.
6. If X is a random variable and $Y = aX + b$ show that
   (i) $\mu_Y = a\mu_X + b$, (ii) $\sigma_Y^2 = a^2 \sigma_X^2$
   where μ denotes the mean and σ denotes the standard deviation.
7. Prove that $V(aX + b) = a^2 V(x)$.
8. Define Moment Generating Function of a random variable X about a and explain how you will get moments from it.
9. Derive the formulae for rth moment about any point and rth moment about the mean. **(M.U. 1999)**
10. If $L(t) = \log M(t)$ where $M(t)$ is the m.g.f. of a discrete random variable, then prove that mean $= L'(0)$ and variance $= L''(0)$. **(M.U. 2006)**