CHAPTER

4 Complex Variables

1. Introduction

In this chapter we are going to study functions of a complex variable $z = x + iy$ where x, y are real. We shall consider how to differentiate a function of a complex variable, the meaning of analytic functions of a complex variable and the conditions of analyticity viz. Cauchy-Riemann equations.

2. Definition of A Complex Function

If by a rule or set of rules we can find one or more complex numbers w for every z $(= x + iy)$ in a given domain, we say that **w is a function of z** and denote it as

$$w = f(z)$$

Since, both z and w are complex quantities the function is called a complex function.

If for a given z there corresponds one and only one w then the function is called **single valued function.** For example $w = z^2$ is a single valued function. If on the other hand if for a given z there correspond two or more values of w, then the function is called **multiple valued function.** For example $w = \sqrt[6]{z}$ is a multiple valued function. We shall consider single valued functions only.

Since, $z = x + iy$, $w = f(z)$ can be put in the form $w = u(x, y) + i\,v(x, y)$ where, u and v are functions of x and y. Thus, we can write

$$\mathbf{w = u(x, y) + iv(x, y)}$$

For example, if $w = z^2 + 2z + 3$ then

$$w = (x + iy)^2 + 2(x + iy) + 3 = x^2 + 2i\,xy - y^2 + 2x + 2iy + 3$$
$$= (x^2 - y^2 + 2x + 3) + i(2xy + 2y) = u(x, y) + iv(x, y)$$

3. Z-plane and W-plane

We know that real function $y = f(x)$ can be represented by a curve on x-y plane. But $w = f(z)$ i.e. $w = u + iv$ where, x and y are real, involves four variables x, y, u and v. Hence, we cannot represent $w = f(z)$ on a single plane. However, we can have sufficient idea of a function $w = f(z)$ if we use two planes, one z-plane on which we plot a point (x, y) and another w-plane on which we plot the corresponding point (u, v).

For example, consider a simple function $w = z^2$ i.e. $w = (x + iy)^2 = (x^2 - y^2) + 2ixy$. Here $u = x^2 - y^2$, $v = 2xy$. If we have a point $P(2, 1)$ on the z-plane then since, $u = 4 - 1 = 3$ and $v = 4$, the corresponding point on the w-plane will be $P'(3, 4)$. As the point P traces a curve C in the z-plane the corresponding point P' will trace another curve C' on the w-plane. Thus, a curve C in the z-plane is transformed into another curve C' in the w-plane. We shall consider in more details this type of transformation of curves in z-plane into the curves in w-plane in the next chapter.

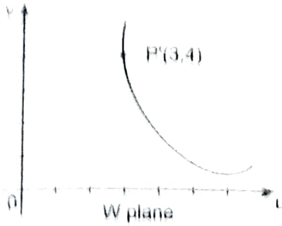


Fig. 4.1 (a)

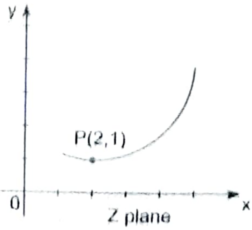


Fig. 4.1 (b)

Neighbourhood of A point $P(z_0)$

Consider the inequality $|z - z_0| < \epsilon$ *i.e.* consider a circle with centre and radius ϵ. The inequality clearly defines a region of all points within the circle $|z - z_0| = \epsilon$ including the point z_0 but excluding the s on the boundary of the circle. The circular region $|z - z_0| < \epsilon$ is d a **neighbourhood** of the point z_0. It is clear that as the number ϵ mes smaller and smaller, the neighbourhood also becomes smaller smaller.

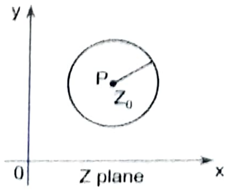


Fig. 4.2

Limit of A Function

The limit of a function of a complex variable $w = f(z)$ is defined on the lines of the definition of mit of a function of a real variable.

Definition : Let $w = f(z)$ be a single valued function of z defined in a bounded and closed in D and let z approach z_0 along any path in D. Given a positive number ϵ, however small (but ro) if we can find another small positive number δ such that $|f(z) - w_0| < \epsilon$ for all z for which $|z - z_0| < \delta$ then we say that w_0 is the limit of $f(z)$ as z tends to z_0 and denote it is

$$\lim_{z \to z_0} f(z) = w_0$$

(Note that the point z may approach z_0, along any path and the limit does not depend upon ath.)

Limits in terms of Real and Imaginary Parts : Let $f(z) = u + iv$ where, u and v are functions l variables x, y. Let $z = x + iy$, $w_0 = u_0 + iv_0$, $z_0 = x_0 + iy_0$. Let $\lim\limits_{z \to z_0} f(z) = w_0$.

Then by the above definition of the limit

$$|f(z) - w_0| < \epsilon \qquad \text{for } 0 < |z - z_0| < \delta$$

$$\therefore \quad |(u + iv) - (u_0 + iv_0)| < \epsilon \qquad \text{for } 0 < |(x + iy) - (x_0 + iy_0)| < \delta.$$

$$\therefore \quad |(u - u_0) + i(v - v_0)| < \epsilon \qquad \text{for } |(x - x_0) + i(y - y_0)| < \delta.$$

This means, in terms of real functions

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} u = u_0 \quad \text{and} \quad \lim_{\substack{x \to x_0 \\ y \to y_0}} v = v_0$$

$$\therefore \quad \lim_{z \to z_0} f(z) = w_0 = u_0 + iv_0 = \lim_{\substack{x \to x_0 \\ y \to y_0}} u + i \lim_{\substack{x \to x_0 \\ y \to y_0}} v$$

6. Continuity

Continuity of a function of a complex variable is defined exactly as the continuity of a function of a real variable.

Let $w = f(z)$ be a single valued function defined in a bounded and closed domain D. $w = f(z)$ is said to be continuous at $z = z_0$ if $\lim_{z \to z_0} f(z) = f(z_0)$.

Continuity in terms of Real and Imaginary Parts : Using the results obtained in § 5, it can be easily shown that if $w = f(z)$ is continuous at $z = z_0$ then the real and imaginary parts u and v are separately continuous at $z_0 = x_0 + i\, y_0$

i.e.
$$\lim_{\substack{x \to x_0 \\ y \to y_0}} u = u_0 \quad \text{and} \quad \lim_{\substack{x \to x_0 \\ y \to y_0}} v = v_0$$

Example 1 : Discuss the continuity of the following functions at the given points

(i) $\dfrac{\bar{z}}{z}$ at $z = 0$ (ii) $\dfrac{z^2}{z^4 + 3z^2 + 1}$ at $z = e^{i\pi/4}$

Sol. : (i) Let $f(z) = \dfrac{\bar{z}}{z} = \dfrac{x - i\,y}{x + i\,y}$

$$\therefore \quad f(z) = \frac{x - iy}{x + iy} \cdot \frac{x - iy}{x - iy} = \frac{x^2 - y^2 - 2i\,xy}{x^2 + y^2} = \frac{x^2 - y^2}{x^2 + y^2} - 2i\,\frac{xy}{x^2 + y^2}$$

Let $u = \dfrac{x^2 - y^2}{x^2 + y^2}, \quad v = -\dfrac{2xy}{x^2 + y^2}$

Let $y = kx, \quad \therefore \quad u = \dfrac{1 - k^2}{1 + k^2}, \quad v = -\dfrac{2k}{1 + k^2}$

$$\lim_{\substack{x \to 0 \\ y \to 0}} u = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x \to 0} \frac{1 - k^2}{1 + k^2} = \frac{1 - k^2}{1 + k^2}$$ which depends upon k *i.e.* on the path.

$$\lim_{\substack{x \to 0 \\ y \to 0}} v = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{-2xy}{x^2 + y^2} = \lim_{x \to 0} -\frac{2k}{1 + k^2} = -\frac{2k}{1 + k^2}$$ which depends upon k *i.e.* on the path.

Hence, $f(z)$ is not continuous at $z = 0$.

Alternative Method

We shall find the limits along the x-axis and along the y-axis.

Along the x-axis, $z = x + iy = x + i\,(0) = x$

$$\therefore \quad \lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{x \to 0} \frac{x}{x} = 1 \qquad \text{from (i)}$$

Along the y-axis, $z = x + iy = 0 + iy = iy$

$$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{y \to 0} -\frac{i\,y}{i\,y} = -1 \qquad \text{from (ii)}$$

From (i) and (ii), we see that the two limits along two different paths are different.

Hence, the limit does not exist. Hence, $f(z)$ is not continuous at $z = 0$.

Let $f(z) = \dfrac{z^2}{z^4 + 3z^2 + 1}$

When $z = e^{i\pi/4}$, $z^2 = e^{i\pi/2} = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2} = i$

$z^4 = e^{i\pi} = \cos\pi + i\sin\pi = -1$

$\therefore$ At $z = e^{i\pi/4}$, $f(z) = \dfrac{i}{-1+3i+1} = \dfrac{i}{3i} = \dfrac{1}{3}$

Now $\lim\limits_{z \to e^{i\pi/4}} = \dfrac{i}{-1+3i+1} = \dfrac{i}{3i} = \dfrac{1}{3}$. Since, $\lim\limits_{z \to e^{i\pi/4}} f(z) = f(e^{i\pi/4})$

The function is continuous at $z = e^{i\pi/4}$.

Example 2 : Find the following limits

(i) $\lim\limits_{z \to i} \dfrac{z^2+1}{z^6+1}$ **(ii)** $\lim\limits_{z \to (1+i)} \left(\dfrac{z-(1+i)}{z^2-2z+z}\right)^3$

: (i) As $z \to i$, $z^2 \to -1$

$$\therefore \lim_{z \to i} \frac{z^2+1}{(z^2+1)(z^4-z^2+1)} = \lim_{z^2 \to -1} \frac{1}{z^4-z^2+1} = \frac{1}{3}.$$

Now, $z^2 - 2z + 2 = (z^2 - 2z + 1) + 1 = (z-1)^2 - (i)^2$

$= (z - 1 + i)(z - 1 - i)$

$$\therefore \lim_{z \to (1+i)} \left\{\frac{z-(1+i)}{[z-(1-i)][z-(1+i)]}\right\}^3 = \lim_{z \to (1+i)} \left\{\frac{1}{z-(1-i)}\right\}^3$$

$$= \left\{\frac{1}{1+i-1+i}\right\}^2 = \left\{\frac{1}{2i}\right\}^3 = -\frac{1}{8i}.$$

EXERCISE - I

1. Find $\lim\limits_{z \to 3i} (3x + iy^2)$.
2. Find $\lim\limits_{z \to 1} \dfrac{z^3-1}{z-1}$.
3. Find $\lim\limits_{z \to i} \dfrac{z^2+1}{z-i}$.
4. Find $\lim\limits_{z \to i} \dfrac{z^3+i}{z-i}$.
5. Show that $\lim\limits_{z \to 0} \dfrac{xy}{x^2+y^2}$ does not exist.
6. Show that $\lim\limits_{z \to 0} \dfrac{x^2 y}{x^4+y^2}$ does not exist. (**Hint :** Put $y = kx^2$)

[**Ans. : (1)** $9i$, **(2)** 3, **(3)** $2i$, **(4)** -3]

Differentiability

Definition : Let $w = f(z)$ be a single valued function of z defined in domain D. $f(z)$ is said to be ntiable at any point z if

$$\lim_{\delta z \to 0} \frac{\delta w}{\delta z} = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

is unique as $\delta z \to 0$ along any path of the domain D.

Example 1 : Find whether the following functions are differentiable.

(i) z^3 at $z = i$ **(ii)** $\cos z$ at $z = i$

Sol. : (i) Let $f(z) = z^3$ and $z_0 = i$

$$\therefore \quad f(z_0) = i^3 = -i$$

$$\therefore \quad f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to i} \frac{z^3 - i^3}{z - i} = \lim_{z \to i} \frac{(z-i)(z^2 + iz + i^2)}{(z-i)}$$

$$= \lim_{z \to i} z^2 + iz + i^2 = 3i^2 = -3$$

$\therefore$ $f(z)$ is differentiable at $z = i$.

(ii) Let $f(z) = \cos z$ and $z_0 = i$.

$$\therefore \quad f(z_0) = \cos i$$

$$\therefore \quad f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to i} \frac{\cos z - \cos i}{z - i}$$

$$= \lim_{z \to i} \frac{-2\sin\left(\frac{z+i}{2}\right)\sin\left(\frac{z-i}{2}\right)}{z - i} = \lim_{z \to i} -2\sin\left(\frac{z+i}{2}\right) \frac{\sin\left(\frac{z-i}{2}\right)}{\left(\frac{z-i}{2}\right)} \cdot \frac{1}{2}$$

$$= -\sin i.$$

$\therefore$ $f(z)$ is differentiable at $z = i$.

Example 2 : Prove that the function $|z|^2$ is continuous everywhere but nowhere differentiable except at the origin.

Sol. : Let $f(z) = |z|^2 = |x + iy|^2 = x^2 + y^2$

Since, $x^2 + y^2$ is continuous everywhere, $f(z)$ is also continuous everywhere.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z} = \lim_{\delta z \to 0} \frac{|z_0 + \delta z|^2 - |z_0|^2}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{(z_0 + \delta z)(\bar{z}_0 + \delta \bar{z}) - z_0 \bar{z}_0}{\delta z} = \lim_{\delta z \to 0} \frac{z_0\,\delta \bar{z} + \bar{z}_0\,\delta z + \delta z\,\delta \bar{z}}{\delta z}$$

$$= \lim_{\delta z \to 0} z_0 \frac{\delta \bar{z}}{\delta z} + \bar{z}_0 + \delta \bar{z} \quad \text{.................... (i)}$$

(i) When δz is real : Then $\delta y = 0$ and $\delta \bar{z} = \delta z = \delta x$. As $\delta z \to 0$, $\delta x \to 0$.

$$\therefore \quad f'(z_0) = \lim_{\delta z \to 0} z_0 \frac{\delta \bar{z}}{\delta z} + \bar{z}_0 + \delta \bar{z} = \lim_{\delta x \to 0} (z_0 + \bar{z}_0 + \delta x) = z_0 + \bar{z}_0$$

(ii) When dz is imaginary : Then $dx = 0$ and $\delta z = i\,\delta y$, $\delta \bar{z} = -i\,\delta y$. As $\delta z \to 0$, $\delta y \to 0$.

$$\therefore \quad f'(z_0) = \lim_{\delta z \to 0} z_0 \frac{\delta \bar{z}}{\delta z} + \bar{z}_0 + \delta \bar{z} = \lim_{\delta x \to 0} z_0\left(-\frac{i\,\delta y}{i\,\delta y}\right) + \bar{z}_0 - i\,\delta y$$

$$= -z_0 + \bar{z}_0$$

Since, the two limits are different along two different paths except at $z = 0$, $f'(z_0)$ does not exist anywhere except at $z = 0$.

Hence, $f(z)$ is not differentiable anywhere except at $z = 0$.

8. Analytic Functions

If a single valued function $w = f(z)$ is defined and differentiable at each point of a domain D then it is called analytic or regular or holomorphic function of z in the domain D.

A function is said to be analytic at a point if it has a derivative at that point and in some neighbourhood of that point.

If a function ceases to be analytic at a point of the domain then the point is called a **singular point.**

9. Cauchy-Riemann Equations in Cartesian Coordinates

Theorem : The necessary and sufficient conditions for a continuous one valued function

$$w = f(z) = u(x, y) + i\, v(x, y)$$

to be analytic in a region R are (i) $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ are continuous functions of x and y in a region R and

(ii) $$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (i.e.\ u_x = v_y \text{ and } u_y = -v_x)}$$

at each point of R.

The conditions (ii) are known as **Cauchy-Riemann equations** or briefly **C-R equations.**

Augustin Louis (Baron de) Cauchy (1789 - 1857)

A French mathematician of great repute who contributed to various branches of mathematics. He wanted to be an engineer but because of poor health he was advised to pursue mathematics. His mathematical work began in 1811 when he gave briliant solutions to some difficult problems of that time. In the next 35 years he published 700 papers in various branches of mathematics. He is supposed to have initiated the era of modern analysis.

Bernhard Riemann (1826 - 1866)

Bernhard Riemann was a great German mathematician who made lasting contributions to analysis and differential geometry. Riemann exhibited exceptional mathematical skills at early age. He joined University of Göttingen to study mathematics and at this university he first met Carl Friedrich Gauss. In 1847 Riemann moved to Berlin where Jacobi, Dirichlet and Steiner were teaching. He founded the field of Riemannian geometry which was used by Einstein in general theory of relativity. He was the first to suggest dimensions higher than three or four. Riemann made major contributions to real analysis. He defined Riemann integral by means of Riemann sums. He introduced Riemann-zeta function. He is also known for Riemannian metric, Riemannian geometry, Riemannian curvature tensor.

Proof : (a) The conditions are necessary

Let $w = f(z) = u(x, y) + iv(x, y)$ be analytic at every point of a region R. Then dw / dz exists uniquely at every point of R.

Let δx and δy be the increments in x, y. Let δu, δv, δw be the corresponding increments in u, v, w respectively. Now,

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{(u+\delta u) + i(v+\delta v) - (u+iv)}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{\delta u + i\,\delta v}{\delta z} = \lim_{\delta z \to 0} \left(\frac{\delta u}{\delta z} + i \frac{\delta v}{\delta z} \right)$$

As $w = f(z)$ is analytic in R, the above limit exists independent of the path along which $\delta z \to 0$. Since, $\delta z = \delta x + i\,\delta y$, the limit is independent of the path along which $\delta x \to 0$ and $\delta y \to 0$.

First consider the path ($QP'P$) first parallel to the x-axis *i.e.* $\delta z \to 0$ such that $\delta y = 0$ and $\delta z = \delta x$ and then $\delta x \to 0$.

$$\therefore\ f'(z) = \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} + i \frac{\delta v}{\delta x} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \qquad \text{.................... (i)}$$

Now let $\delta z \to 0$ along the path ($QP''P$) first parallel to the y-axis *i.e.* $\delta z \to 0$ such that $\delta x = 0$ and $\delta z = i\,\delta y$ and then $\delta y \to 0$.

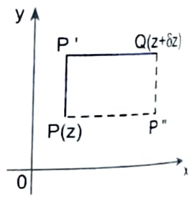


Fig. 4.3

$$\therefore\ f'(z) = \lim_{\delta y \to 0} \left(\frac{\delta u}{i\,\delta y} + i \frac{\delta v}{i\,\delta y} \right)$$

$$\therefore\ f'(z) = \frac{1}{i} \cdot \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \left[\because \frac{1}{i} = \frac{i}{i^2} = -i \right]$$

$$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \text{.................... (ii)}$$

For existence of $f'(z)$ (i) and (ii) must be equal.

From (i) and (ii) we get,

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating the real and imaginary parts

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

This proves that for a function to be analytic, the **Cauchy-Riemann equations** are necessary.

(b) The conditions are sufficient

Let $f(z) = u(x, y) + i\,v(x, y)$ be a single valued function possessing continuous partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ at every point in the region R and satisfying the conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

We have to show that $f'(z)$ exists at every point of R.

By Taylor's theorem for functions of two variables, omitting the second and higher degree terms in x and y, we get,

$$f(z+\delta z) = u(x+\delta x,\ y+\delta y) + iv(x+\delta x,\ y+\delta y)$$

$$= \left[u(x,y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right)\right] + i\left[v(x,y) + \left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right)\right]$$

$$= [u(x,y) + iv(x,y)] + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

$$= f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

$$\therefore\ f(z+\delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

Using the C-R equations (in the second bracket)

$$f(z+\delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)i\,\delta y \qquad [\because i^2 = -1]$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\delta x + i\,\delta y)$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta z \qquad [\because \delta x + i\,\delta y = \delta z]$$

$$\therefore\ \frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\therefore\ f'(z) = \lim_{\delta z \to 0}\frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Hence, $f'(z)$ exists as $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ exist.

$\therefore$ $f(z)$ is analytic.

$\therefore$ The conditions are sufficient.

Notes ....

1. The Cauchy-Riemann equations are only **necessary conditions** for a function to be analytic. This means even if Cauchy-Riemann equations are satisfied the function need not be analytic at that point (see Examples 10 and 11, page 4-14)

2. When $f(z)$ is analytic, its derivative is given by any one of the following expressions

$$\boxed{\begin{aligned} f'(z) = u_x + i\,v_x\ ;\ f'(z) = v_y + i\,v_x \\ f'(z) = u_x - i\,u_y\ ;\ f'(z) = v_y - i\,u_y \end{aligned}}$$

3. **If f (z) is analytic then it can be differentiated in usual manner.**

e.g. if $f(z) = z^2$ then $f'(z) = 2z$

if $f(z) = \sin z$ then $f'(z) = \cos z$

4. If $f(z) = f(x+iy) = u + iv$ and $f(z)$ is analytic then the functions u and v of real variables x and y are called **conjugate functions.**

Example 1 : Prove that an analytic function with its derivative zero is constant.

Sol. : Let $f(z) = u + iv$ be the given analytic function whose derivative is zero.

$\therefore \quad f'(z) = u_x + iv_x = 0 \qquad \therefore \quad u_x = 0,\ v_x = 0$

But $f(z)$ is analytic. Hence, C-R equations are satisfied

$\therefore \quad u_x = v_y$ and $u_y = -v_x \qquad \therefore \quad v_y = 0,\ u_y = 0$

As $u_x = 0,\ u_y = 0 \ \therefore u$ = a constant and $v_x = 0,\ v_y = 0 \ \therefore v$ = a constant.

$\therefore \quad f(z) = u + iv$ = a constant.

Example 2 : If $f(z)$ and $\overline{f(z)}$ are both analytic, prove that $f(z)$ is constant. **(M.U. 1993, 2003, 06)**

Sol. : Let $f(z) = u + iv$ then $\overline{f(z)} = u - iv = u + i(-v)$

Since, $f(z)$ is analytic $u_x = v_y$ and $u_y = -v_x$ C-R equations.

Since, $\overline{f(z)}$ is analytic $u_x = (-v_y)$ and $u_y = -(-v_x)$ C-R equation.

Adding $u_x = v_y$ and $u_x = -v_y$, we get $u_x = 0$.

Adding $u_y = -v_x$ and $u_y = v_x$, we get $u_y = 0$.

Since, $u_x = 0$ and $u_y = 0$, u = a constant.

Similarly by subtraction we can prove that $v_x = 0$ and $v_y = 0$. $\therefore \quad v$ = a constant.

Hence, $f(z) = u + iv$ = a constant.

Example 3 : If $f(z)$ is an analytic function, show that $\dfrac{\partial f}{\partial \bar{z}} = 0$. **(M.U. 1996)**

Sol. : Since, $z = x + iy$, $\bar{z} = x - iy$.

$$\therefore \quad x = \frac{1}{2}(z + \bar{z}) \quad \text{and} \quad y = \frac{1}{2i}(z - \bar{z}).$$

Let $f(z) = u + iv$

$$\therefore \quad \frac{\partial f}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}(u + iv) = \left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right) + i\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right)$$

$$= \left[\frac{\partial u}{\partial x}\cdot\frac{1}{2} + \frac{\partial u}{\partial y}\left(-\frac{1}{2i}\right)\right] + i\left[\frac{\partial v}{\partial x}\cdot\frac{1}{2} + \frac{\partial v}{\partial y}\left(-\frac{1}{2i}\right)\right]$$

$$= \frac{1}{2}u_x - \frac{1}{2i}u_y + \frac{i}{2}v_x - \frac{1}{2}v_y$$

$$= \frac{1}{2}u_x + \frac{i}{2}u_y + \frac{i}{2}v_x - \frac{1}{2}v_y$$

But since, $f(z)$ is analytic, $u_x = v_y$ and $u_y = -v_x$.

$$\therefore \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}v_y - \frac{i}{2}v_x + \frac{i}{2}v_x - \frac{1}{2}v_y = 0$$

Restatement

The above theorem can also be stated as follows.

"Every analytic function $w = u + iv$ can be expressed as a function of z only".

Notes ....

1. We have proved that if $f(z)$ is analytic then $\partial f / \partial \bar{z} = 0$. This means if $f(z)$ is analytic, $f(z)$ is free from $\bar{z}$. In other words if $f(z)$ is analytic, it can be reconstructed in z *i.e.* in $x + iy$ only.

2. It can be easily shown that usual rules of differentiating sums, products, quotients of real functions are applicable to analytic functions of complex variable.

The formulae for differentiating elementary complex functions are identical with the corresponding formulae in calculus of real variables.

For example, $\dfrac{d}{dz}(z^n) = n\,z^{n-1}$, $\dfrac{d}{dz}(e^z) = e^z$, $\dfrac{d}{dz}(\log z) = \dfrac{1}{z}$,

$\dfrac{d}{dz}(\sin z) = \cos z$, $\dfrac{d}{dz}(\sin hz) = \cos hz$.

In general, if the differential coefficient of $f(x)$ is $\phi(x)$, we can assume that if $f(z)$ is analytic, the differential coefficient of $f(z)$ is $\phi(z)$.

3. Since integration is inverse of differentiation a result similar to the one quoted in note 2 holds for integration also.

e.g. $\int z^n dz = \dfrac{z^{n+1}}{n+1}$, $\int e^z dz = e^z$ etc.

Derivatives of Elementary Functions

1. $\dfrac{d}{dz}(c) = 0$
2. $\dfrac{d}{dz}(z^n) = n\,z^{n-1}$
3. $\dfrac{d}{dz}(e^z) = e^z$
4. $\dfrac{d}{dz}(a^z) = a^z \log a$
5. $\dfrac{d}{dz}(\sin z) = \cos z$
6. $\dfrac{d}{dz}(\cos z) = -\sin z$
7. $\dfrac{d}{dz}(\tan z) = \sec^2 z$
8. $\dfrac{d}{dz}(\cot z) = -\operatorname{cosec}^2 z$
9. $\dfrac{d}{dz}(\sec z) = \sec z \tan z$
10. $\dfrac{d}{dz}(\operatorname{cosec} z) = -\operatorname{cosec} z \cot z$
11. $\dfrac{d}{dz}(\log z) = \dfrac{1}{z}$
12. $\dfrac{d}{dz}(\log_a z) = \dfrac{1}{z \log_e a}$
13. $\dfrac{d}{dz}(\sin^{-1} z) = \dfrac{1}{\sqrt{1-z^2}}$
14. $\dfrac{d}{dz}(\cos^{-1} z) = -\dfrac{1}{\sqrt{1-z^2}}$
15. $\dfrac{d}{dz}(\tan^{-1} z) = \dfrac{1}{1+z^2}$
16. $\dfrac{d}{dz}(\cot^{-1} z) = -\dfrac{1}{1+z^2}$
17. $\dfrac{d}{dz}(\sec^{-1} z) = \dfrac{1}{z\sqrt{z^2-1}}$
18. $\dfrac{d}{dz}(\operatorname{cosec}^{-1} z) = \dfrac{-1}{z\sqrt{z^2-1}}$
19. $\dfrac{d}{dz}(\sin hz) = \cos hz$
20. $\dfrac{d}{dz}(\cos hz) = \sin hz$
21. $\dfrac{d}{dz}(\tan hz) = \sec h^2 z$
22. $\dfrac{d}{dz}(\cot hz) = -\operatorname{cosec} h^2 z$
23. $\dfrac{d}{dz}(\sec hz) = -\sec hz \tan hz$
24. $\dfrac{d}{dz}(\operatorname{cosec} hz) = -\operatorname{cosec} hz \cot h$

25. $\frac{d}{dz}(\sin h^{-1}z) = \frac{1}{\sqrt{1+z^2}}$

26. $\frac{d}{dz}(\cos h^{-1}z) = \frac{1}{\sqrt{z^2-1}}$

27. $\frac{d}{dz}(\tan h^{-1}z) = \frac{1}{1-z^2}$

28. $\frac{d}{dz}(\cot h^{-1}z) = \frac{1}{z^2-1}$

29. $\frac{d}{dz}(\sec h^{-1}z) = \frac{-1}{z\sqrt{1-z^2}}$

30. $\frac{d}{dz}(\text{cosec}\, h^{-1}z) = \frac{-1}{z\sqrt{z^2+1}}$

Example 4 : If $f(z)$ is an analytic and $|f(z)|$ is constant, prove that $f(z)$ is constant.
(M.U. 1999, 2002, 03, 05, 08, 09)

Sol. : Let $f(z) = u + iv$ but $|f(z)| = C$. $\quad \therefore \; u^2 + v^2 = C^2$

Differentiating it partially w.r.t. x, $uu_x + vv_x = 0$

Differentiating it partially w.r.t. y, $uu_y + vv_y = 0$

Since, $f(z)$ is analytic $u_x = v_y$ and $u_y = -v_x$

$\therefore \; uu_x - vu_y = 0$ and $uu_y + vu_x = 0$

Eliminating u_y, $(u^2+v^2)u_x \quad \therefore \; C^2 u_x = 0 \quad \therefore \; u_x = 0$

Similarly, we can show that $u_y = 0,\ v_x = 0,\ v_y = 0$

Since, $f(z)$ is analytic. $\quad f'(z) = u_x + iv_x = 0 \quad \therefore \; f(z) = \text{constant}$.

Restatement

(i) The above theorem can also be restated as

"If $f(z)$ is an analytic function with constant modulus then, prove that $f(z)$ is constant."
(M.U. 1994, 99)

(ii) A regular function of constant magnitude is constant. **(M.U. 2005)**

Example 5 : If $f(z)$ is analytic and if the amplitude of $f(z)$ is constant, prove that $f(z)$ is constant.
(M.U. 2003)

Sol. : Let $f(z) = u + iv$. Since its amplitude $= \tan^{-1}(v/u)$ is constant c say, we have

$$\tan^{-1}\frac{v}{u} = c \quad \therefore \; \frac{v}{u} = \tan c$$

Differentiating this w.r.t. x and y,

$$\frac{uv_x - vu_x}{u^2} = 0 \text{ and } \frac{uv_y - vu_y}{u} = 0$$

$\therefore \; uv_x - vu_x = 0$ and $uv_y - vu_y = 0$

Since $f(z)$ is analytic, $u_x = v_y$ and $u_y = -v_x$.

$\therefore \; -uu_y - vu_x = 0$ .... (1) and $uu_x - vu_y = 0$ .... (2)

Multiply the first by u and second by v and add.

$\therefore \; (-u^2 - v^2)u_y = 0 \quad \therefore \; u_y = 0$

Multiply the first by v and second by u and subtract

$\therefore \; (-v^2 - u^2)u_x = 0 \quad \therefore \; u_x = 0$

But $u_x = v_y$ and $u_y = -v_x \quad \therefore \; v_y = 0$ and $v_x = 0$

Since, all four partial derivatives of u, v are zero, u and v are constants.

$\therefore$ $f(z)$ is constant.

Example 6 : If $f(z) = u + iv$ is an analytic function and (i) u = constant or (ii) v = constant then $f(z)$ is constant

Sol. : If u is constant $u_x = 0$, $u_y = 0$.

But $f'(z) = u_x + iv_x$

$= u_x - i u_y$ (By C-R equations)

$= 0$ (By data)

$\therefore$ $f(z)$ = constant.

Similarly we can prove the other part.

Note ....

From Examples 1, 2, 4, 5 and 6, we find that an **analytic function** $f(z)$ is constant if (i) $f'(z) = 0$ or (ii) $\overline{f(z)}$ is also analytic or (iii) modulus of $f(z)$ is constant or (iv) amplitude of $f(z)$ is constant or (v) real part is constant or (vi) imaginary part is constant.

Example 7 : Show that the following functions are analytic and find their derivatives.

(i) e^z, **(ii)** $\sin hz$ **(M.U. 1999, 2015)**

Sol. : (i) $f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$

$\therefore$ $u = e^x \cos y$, $v = e^x \sin y$

$u_x = e^x \cos y$, $u_y = -e^x \sin y$; $v_x = e^x \sin y$, $v_y = e^x \cos y$

$\therefore$ $u_x = v_y$ and $u_y = -v_x$

Further u_x, u_y, v_x, v_y are continuous and Cauchy-Riemann equations are satisfied.

Hence, e^z is analytic.

Now, $f'(z) = u_x + iv_x = e^x \cos y + i e^x \sin y = e^x(\cos y + i \sin y)$

$= e^x \cdot e^{iy} = e^{x+iy} = e^z$.

(ii) $f(z) = \sin h\, z = \sin h\,(x + iy)$

$= \sin h\, x \cos h\, iy + \cos h\, x \sin h\, iy$

$= \sin h\, x \cos y + i \cos h\, x \sin y$

$\therefore$ $u = \sin h\, x \cos y$, $v = \cos h\, x \sin y$

$u_x = \cos h\, x \cos y$, $u_y = -\sin h\, x \sin y$

$v_x = \sin h\, x \sin y$, $v_y = \cos h\, x \cos y$

$\therefore$ $u_x = v_y$ and $u_y = -v_x$

Further, u_x, u_y, v_x, v_y are continuous and Cauchy-Riemann equations are satisfied.

Hence, $\sin h\, z$ is analytic.

Now, $f'(z) = u_x + iv_x$

$= \cos hx \cos y + i \sin hx \sin y$

$= \cos hx \cos hiy + \sin hx \sin hiy$

$= \cos h\,(x + iy) = \cos hz$.

Example 8 : Show that the following functions are analytic and find their derivatives.

(i) $f(z) = z^3$ (ii) $f(z) = ze^z$ **(M.U. 2016)** (iii) $f(z) = \sin z$. **(M.U. 2003)**

Sol : (i) We have $f(z) = z^3 = (x + iy)^3$

$\therefore\ f(z) = x^3 + 3ix^2y - 3xy^2 - iy^3$

$\therefore\ u = x^3 - 3xy^2,\ v = 3x^2y - y^3$

$\therefore\ \dfrac{\partial u}{\partial x} = 3x^2 - 3y^2,\ \dfrac{\partial v}{\partial x} = 6xy,\ \dfrac{\partial u}{\partial y} = -6xy,\ \dfrac{\partial v}{\partial y} = 3x^2 - 3y^2$

$\therefore\ u_x = v_y$ and $u_y = -v_x$ $\therefore\ f(z) = z^3$ is analytic and can be differentiated as u

$\therefore\ f(z) = 3z^2$.

(ii) We have $f(z) = ze^z = (x + iy)\, e^{x+iy}$

$\therefore\ f(z) = (x + iy)\, e^x (\cos y + i \sin y)$

$\therefore\ u = e^x (x \cos y - y \sin y),\ v = e^x (x \sin y + y \cos y)$

$\dfrac{\partial u}{\partial x} = e^x(x \cos y - y \sin y) + e^x \cos y,\quad \dfrac{\partial u}{\partial y} = e^x(-x \sin y - y \cos y - \sin y),$

$\dfrac{\partial v}{\partial x} = e^x(x \sin y + y \cos y) + e^x \sin y,\quad \dfrac{\partial v}{\partial y} = e^x(x \cos y + \cos y - y \sin y).$

$\therefore\ u_x = v_y$ and $u_y = -v_x$

$\therefore\ f(z) = ze^z$ is analytic and can be differentiated as usual.

$\therefore\ f'(z) = ze^z + e^z = e^z (z + 1).$

(iii) $f(z) = \sin z = \sin (x + iy) = \sin x \cos iy + \cos x \sin iy$

$= \sin x \cos hy + i \cos x \sin hy$

$\therefore\ u = \sin x \cos hy,\ v = \cos x \sin hy$

$\dfrac{\partial u}{\partial x} = \cos x \cos hy,\ \dfrac{\partial v}{\partial x} = -\sin x \sin hy,\ \dfrac{\partial u}{\partial y} = \sin x \sin hy,\ \dfrac{\partial v}{\partial y} = \cos x \cos hy$

$\therefore\ u_x = v_y$ and $u_y = -v_x$

$\therefore\ f(z) = \sin z$ is analytic and can be differentiated as usual.

$\therefore\ f'(z) = \cos z.$

Example 9 : If $f(z)$ is equal to (a) $\bar{z}$ (b) $2x + i\,xy^2$, show that $f'(z)$ does not exist. **(M.U. 2002)**

Sol. : (a) $f(z) = \bar{z} = x - iy$ $\therefore\ u = x, v = -y$ $\therefore\ u_x = 1, u_y = 0$; $v_y = -1, v_x = 0$

Since, $u_x \neq v_y$ Cauchy-Riemann equations are not satisfied and $f'(z)$ does not exist.

Alternatively

$$f'(z) = \lim_{\delta z \to 0} \frac{\overline{z + \delta z} - \bar{z}}{\delta z}$$

$$\therefore\ f'(z) = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{\overline{(x + iy + \delta x + i\,\delta y)} - \overline{(x + iy)}}{\delta x + i\,\delta y}$$

$$= \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{x - iy + \delta x - i\,\delta y - x + iy}{\delta x + i\,\delta y} = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{\delta x - i\,\delta y}{\delta x + i\,\delta y}$$

If $\delta y = 0$, the required limit $= \lim\limits_{\delta x \to 0} \dfrac{\delta x}{\delta x} = 1$

If $\delta x = 0$, the required limit $= \lim\limits_{\delta y \to 0} -\dfrac{\delta y}{\delta y} = -1$

Since, the two limits are different $f'(z)$ does not exist.

(b) $f(z) = 2x + i\,xy^2 \quad \therefore\ u = 2x,\ v = xy^2$

$u_x = 2,\ u_y = 0,\ v_x = y^2,\ v_y = 2xy$

Since, $u_x \neq v_y$ and $u_y \neq v_x$. Cauchy-Riemann equations are not satisfied and hence, $f'(z)$ does not exist.

Example 10 : Show that $f(z) = z\,\bar{z} = |z|^2$ satisfies Cauchy-Riemann equations at $z = 0$ and yet is not analytic anywhere. **(M.U. 2004)**

Sol. : $f(z) = |z|^2 = x^2 + y^2 \quad \therefore\ u = x^2 + y^2,\ v = 0$

$$\therefore \quad \frac{\partial u}{\partial x} = 2x,\ \frac{\partial u}{\partial y} = 2y,\ \frac{\partial v}{\partial x} = 0 \text{ and } \frac{\partial v}{\partial y} = 0$$

Hence, $u_x = v_y = 0$ and $u_y = -v_x = 0$ when $x = 0$ and $y = 0$.

Thus, C-R equations are satisfied at $z = 0$.

The partial derivatives u_x, u_y, v_x, v_y are also continuous everywhere.

Thus, $f'(z) = |z|^2$ is differentiable **only** at $z = 0$ but at no other point. There is no neighbourhood of $z = 0$ in which the conditions of analyticity are satisfied. Hence, $f(z)$ is **not analytic** anywhere.

Example 11 : Show that $f(z) = \sqrt{|xy|}$ is not analytic at the origin although Cauchy-Riemann equations are satisfied at that point. **(M.U. 2005, 08)**

Sol. : Let $f(z) = u + iv$ so that $u = \sqrt{|xy|}$ and $v = 0$

$$\text{Now, } \frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \frac{0 - 0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = \frac{0 - 0}{y} = 0$$

Clearly since, $v = 0$, $\dfrac{\partial v}{\partial x} = 0$, $\dfrac{\partial v}{\partial y} = 0$ at zero.

$\therefore$ Cauchy-Riemann equations are satisfied at $z = 0$.

$$\text{Now } f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{z \to 0} \frac{\sqrt{|xy|} - 0}{x + iy}$$

Let $z \to 0$ along $y = mx$

$$\therefore\ f'(0) = \lim_{x \to 0} \frac{\sqrt{|mx^2|}}{x(1 + im)} = \lim_{x \to 0} \frac{x\sqrt{|m|}}{x(1 + im)} = \frac{\sqrt{m}}{1 + im}$$

Since, the limit depends upon m, $f'(0)$ does not exist.

Note ....

The above function satisfies Cauchy-Riemann equations and yet is not analytic at $z = 0$. This is because for analyticity in addition to C-R equations its four partial derivatives $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$

must be continuous at that point. The value of u_x at $x = 0$ is zero. We shall now find lim of u_x as $x \to 0$.

Now, if $x > 0$, $\delta x > 0$ $\dfrac{\partial u}{\partial x} = \dfrac{\sqrt{|y|}}{2\sqrt{|x|}}$, let $y = mx$

$$\therefore \lim_{x \to 0} u_x = \lim_{x \to 0} \frac{\sqrt{|mx|}}{2\sqrt{|x|}} = \frac{|m|}{2}$$

$\therefore$ Thus, the limit depends upon the path and hence, u_x is not continuous at $x = 0$. Similarly u_y, v_x, v_y are not continuous at $x = 0$. Hence, $f(x)$ is **not** analytic even though C-R equations are satisfied.

Example 12 : Prove that $f(z) = (x^3 - 3xy^2 + 2xy) + i\,(3x^2y - x^2 + y^2 - y^3)$ is analytic and find $f'(z)$ and $f(z)$ in terms of z. **(M.U. 2004)**

Sol. : We have $u = x^3 - 3xy^2 + 2xy$, $v = 3x^2y - x^2 + y^2 - y^3$

$$\therefore \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 2y, \quad \frac{\partial u}{\partial y} = -6xy + 2x$$

$$\frac{\partial v}{\partial x} = 6xy - 2x, \quad \frac{\partial v}{\partial y} = 3x^2 + 2y - 3y^2$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Also partial derivatives are continuous. $\therefore$ $f(z)$ is analytic.

Now, $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = (3x^2 - 3y^2 + 2y) + i\,(6xy - 2x)$

$$\therefore f'(z) = 3\,(x^2 + 2i\,xy - y^2) - 2i\,(x + iy)$$
$$= 3\,(x^2 + 2i\,xy + i^2 y^2) - 2i\,(x + iy) = 3z^2 - 2iz.$$

Or by Milne-Thompson method (See page 4-29), putting $x = z$, $y = 0$, we get $f'(z) = 3z^2 - 2iz$.

$\therefore$ By integration, $f(z) = z^3 - iz^2 + k$.

Example 13 : Show that $w = \dfrac{x}{x^2 + y^2} - \dfrac{iy}{x^2 + y^2}$ is an analytic function and find $\dfrac{dw}{dz}$ in terms of z. **(M.U. 2003, 07)**

Sol. : Since, $u = \dfrac{x}{x^2 + y^2}$, $\dfrac{\partial u}{\partial x} = u_x = \dfrac{(x^2 + y^2) \cdot 1 - x \cdot 2x}{(x^2 + y^2)^2} = \dfrac{-x^2 + y^2}{(x^2 + y^2)^2}$

$$\frac{\partial u}{\partial y} = u_y = -\frac{x \cdot 2y}{(x^2 + y^2)^2}$$

$$v = -\frac{y}{x^2 + y^2} \quad \therefore \frac{\partial v}{\partial x} = v_x = +\frac{y \cdot 2x}{(x^2 + y^2)^2}$$

$$\frac{\partial v}{\partial y} = v_y = -\frac{(x^2 + y^2) \cdot 1 - y \cdot 2y}{(x^2 + y^2)^2} = -\frac{x^2 - y^2}{(x^2 + y^2)^2} = \frac{-x^2 + y^2}{(x^2 + y^2)^2}$$

$\therefore$ $u_x = v_y$ and $u_y = -v_x$

Further u_x, u_y, v_x and v_y are continuous functions except at $z = x + iy = 0$ *i.e.* $(x = 0, y = 0)$ w is analytic everywhere except at $z = 0$.

$$\therefore \quad \frac{dw}{dz} = u_x + iv_x = \frac{-x^2 + y^2}{(x^2+y^2)^2} + i \cdot \frac{2xy}{(x^2+y^2)^2} \qquad \text{.................... (i)}$$

$$= -\frac{(x^2 - 2ixy - y^2)}{(x^2+y^2)^2} = -\frac{(x^2 - 2ixy + i^2y^2)}{(x^2 - i^2y^2)^2}$$

$$= -\frac{(x-iy)^2}{(x-iy)^2(x+iy)^2} = -\frac{1}{(x+iy)^2} = -\frac{1}{z^2}$$

[Or to find $\frac{dw}{dz}$ in terms of z, use Milne-Thomson method (See page 4-29) and put $x = z$, $y = 0$ in (i) $\therefore \ \frac{dw}{dz} = -\frac{1}{z^2}$.]

Example 14 : Show that $f(z) = \frac{\bar{z}}{|\bar{z}|^2}$, $|z| \neq 0$ is analytic and find $f'(z)$. **(M.U. 2013)**

Sol. : We have $\quad f(z) = \frac{x - iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

Now, proceed as in the above example.

Example 15 : Is $f(z) = \frac{z}{\bar{z}}$ analytic? **(M.U. 2016)**

Sol. : We have

$$f(z) = \frac{z}{\bar{z}} = \frac{x+iy}{x-iy} = \frac{(x+iy)}{(x-iy)} \cdot \frac{(x+iy)}{(x+iy)}$$

$$= \frac{(x+iy)^2}{x^2+y^2} = \frac{x^2 + 2ixy + y^2}{x^2+y^2}$$

$$= \frac{x^2+y^2}{x^2+y^2} + i \cdot \frac{2xy}{x^2+y^2}$$

$$\therefore \quad u = 1 \ \text{ and } \ v = \frac{2xy}{x^2+y^2}.$$

Now, $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = 0$. But $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are not zero.

Hence, $f(z)$ is not analytic.

Example 16 : Show that $\quad f(z) = \frac{xy^2(x+iy)}{x^2+y^4}$ when $z \neq 0$

$= 0$ when $z = 0$ is not differentiable at $z = 0$.

Sol. : By definition,

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z\to 0} \frac{xy^2(x+iy) - 0}{(x^2+y^4)(x+iy)} = \lim_{z\to 0} \frac{xy^2}{x^2+y^4}$$

Now, let $z \to 0$ along path $y^2 = mx$

$$\therefore \quad f'(0) = \lim_{x\to 0} \frac{x \cdot mx}{x^2 + m^2x^2} = \lim_{x\to 0} \frac{x^2}{x^2} \cdot \left(\frac{m}{1+m^2}\right) = \frac{m}{1+m^2}$$

Since, the derivative depends upon the path, $f'(0)$ does not exist and hence, the fun not differentiable at $z = 0$.

Example 17 : Prove that the function defined by

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} \quad \text{when } z \neq 0$$
$$= 0 \quad \text{when } z = 0$$

is (i) continuous ; (ii) Cauchy-Riemann equations are satisfied at the origin and yet (iii) $f'(0$ not exist. **(M.U. 200**

Sol. : We have $u = \dfrac{x^3 - y^3}{x^2+y^2}$, $v = \dfrac{x^3+y^3}{x^2+y^2}$

(i) When $z \neq 0$, $x \neq 0$, $y \neq 0$

Hence, u, v being rational functions of x and y are continuous and consequently f continuous when $z \neq 0$

To test the continuity at $z = 0$ we put $x = r\cos\theta$, $y = r\sin\theta$

$$\therefore \quad u = r(\cos^3\theta - \sin^3\theta), \quad v = r(\cos^3\theta + \sin^3\theta)$$

When $z \to 0$, $r \to 0$

$$\therefore \quad \lim_{z \to 0} u = \lim_{r \to 0} r(\cos^3\theta - \sin^3\theta) = 0$$

$$\lim_{z \to 0} v = \lim_{r \to 0} r(\cos^3\theta + \sin^3\theta) = 0$$

$$\therefore \quad \lim_{z \to 0} f(z) = 0 = f(0) \text{ by data}$$

$\therefore$ $f(z)$ is continuous at $z = 0$. $\quad\therefore$ $f(z)$ is continuous everywhere.

(ii) Now
$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0} \frac{x - 0}{x} = 1$$
$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y \to 0} \frac{-y - 0}{y} = -1$$
$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x \to 0} \frac{x - 0}{x} = 1$$
$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y \to 0} \frac{y - 0}{y} = 1$$
$$\therefore \quad u_x = v_y = 1 \text{ and } u_y = -v_x = -1$$

Hence, C-R equations are satisfied at the origin.

(iii) Now, $f'(0) = \lim_{z \to 0} \dfrac{f(z) - f(0)}{z} = \lim_{z \to 0} \dfrac{(x^3 - y^3) + i(x^3+y^3) - 0}{(x^2+y^2)(x+iy)}$

Let $z \to 0$ along the line $y = mx$

$$\therefore \quad f'(0) = \lim_{x \to 0} \frac{x^3(1-m^3) + ix^3(1+m^3)}{x^2(1+m^2)x(1+im)} = \frac{(1-m^3) + i(1+m^3)}{(1+m^2)(1+im)}$$

The limit depends upon the path,

$\therefore$ $f'(0)$ does not exist.

Note ....

The above function is continuous at $z = 0$; Cauchy-Riemann equations are satisfied at $z = 0$ and yet the function is not analytic at $z = 0$. This is so because for analyticity in addition to C-R equations the four partial derivatives must be continuous at that point.

Now we have proved above that the value of u_x at $x = 0$ is one, we prove below that $\partial u / \partial x$ is not continuous at $z = 0$

$$u = \frac{x^3 - y^3}{x^2 + y^2},\ u_x = \frac{(x^2 + y^2)3x^2 - (x^3 - y^3)\cdot 2x}{(x^2 + y^2)^2} = \frac{x^4 + 3x^2y^2 + 2xy^3}{(x^2 + y^2)^2}$$

Now, let $y = mx$, $\displaystyle \lim_{x \to 0} u_x = \lim_{x \to 0} \frac{x^4 + 3m^2x^4 + 2m^3x^4}{x^4(1 + m^2)^2} = \frac{1 + 3m^2 + 2m^3}{(1 + m^2)^2}$

The limit depends upon the path chosen and hence, is not continuous. Similarly other partial derivatives can be shown to be not continuous. Hence, the function is **not** analytic even though C-R equations are satisfied.

Example 18 : Show that the following function

$$f(z) = \frac{x^2y^5(x + iy)}{x^4 + y^{10}}, \quad z \neq 0$$

$$= 0 \qquad\qquad z = 0$$

is not analytic at the origin although Cauchy-Riemann equations are satisfied.

Sol. : We have $u = \dfrac{x^3y^5}{x^4 + y^{10}}$, $v = \dfrac{x^2y^6}{x^4 + y^{10}}$

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0} \frac{0 - 0}{x} = 0$$

Similarly, $\dfrac{\partial u}{\partial y} = 0,\ \dfrac{\partial v}{\partial x} = 0,\ \dfrac{\partial v}{\partial y} = 0$

Hence, Cauchy-Riemann equations are satisfied.

Now $\displaystyle f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \left[\frac{x^2y^5(x + iy)}{x^4 + y^{10}} - 0\right] \Big/ (x + iy)$

$$= \lim_{z \to 0} \frac{x^2y^5}{x^4 + y^{10}} \qquad \text{.................... (A)}$$

Let $z \to 0$ along the line $y = mx$

$$\therefore\ f'(0) = \lim_{x \to 0} \frac{x^7m^5}{x^4[1 + m^{10}x^6]} = \lim_{x \to 0} \frac{x^3m^5}{1 + m^{10}x^6} = 0$$

Now let $z \to 0$ along $y^5 = x^2$ then from (A), we get,

$$\therefore\ f'(0) = \lim_{x \to 0} \frac{x^4}{2x^4} = \frac{1}{2}$$

Since, the limit depends upon the path, $f(z)$ is not analytic at $z = 0$.

Example 19 : Find k such that $\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\frac{kx}{y}$ is analytic. **(M.U. 2004, 07, 16, 19)**

Sol. : Let $f(z)=\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\frac{kx}{y}$

$$\therefore\ u=\frac{1}{2}\log(x^2+y^2),\ v=\tan^{-1}\frac{kx}{y}$$

$$\therefore\ u_x=\frac{x}{x^2+y^2},\quad u_y=\frac{y}{x^2+y^2}$$

$$v_x=\frac{1}{1+\frac{k^2x^2}{y^2}}\cdot\frac{k}{y}=\frac{ky}{k^2x^2+y^2};\qquad v_y=\frac{1}{1+\frac{k^2x^2}{y^2}}\cdot\left(-\frac{kx}{y^2}\right)=-\frac{kx}{k^2x^2+y^2}$$

Since, the function is given to be analytic C-R equations are satisfied.

$$\therefore\ u_x=v_y \text{ and } u_y=-v_x$$

$$\therefore\ \frac{x}{x^2+y^2}=-\frac{kx}{k^2x^2+y^2},\quad \frac{y}{x^2+y^2}=-\frac{ky}{k^2x^2+y^2}$$

which are satisfied when $k=-1$.

Example 20 : Determine the constants a, b, c, d if

$$f(z)=x^2+2axy+by^2+i(cx^2+2dxy+y^2) \text{ is analytic.}$$ **(M.U. 1998, 2010, 11, 13, 19)**

Sol. : We have $f(z)=u+iv$

and $u=x^2+2axy+by^2$; $v=cx^2+2dxy+y^2$

$$\therefore\ u_x=2x+2ay,\ u_y=2ax+2by$$

$$v_x=2cx+2dy,\ v_y=2dx+2y$$

Since, $f(z)$ is analytic, Cauchy-Riemann Equations are satisfied.

$$\therefore\ u_x=v_y \text{ and } u_y=-v_x$$

$$\therefore\ 2x+2ay=2dx+2y \text{ and } 2ax+2by=-2cx-2dy$$

Equating the coefficients of x and y, we get $a=1$, $d=1$ and $a=-c$, $b=-d$

$$\therefore\ a=1,\ b=-1,\ c=-1,\ d=1.$$

Example 21 : Find the constants a, b, c, d, e, if

$$f(z)=(ax^4+bx^2y^2+cy^4+dx^2-2y^2)+i(4x^3y-exy^3+4xy) \text{ is analytic.}$$

(M.U. 2002, 03, 07, 08, 09, 12, 13, 16)

Sol. : We have $f(z)=u+iv$

and $u=ax^4+bx^2y^2+cy^4+dx^2-2y^2$

and $v=4x^3y-exy^3+4xy$

$$\therefore\ u_x=4ax^3+2bxy^2+2dx;\qquad u_y=2bx^2y+4cy^3-4y.$$

And $v_x=12x^2y-ey^3+4y$; $\qquad v_y=4x^3-3exy^2+4x.$

Since $f(z)$ is analytic.

Cauchy-Riemann equations are satisfied.

$$\therefore\ u_x=v_y \quad\text{and}\quad u_y=-v_x$$

$\therefore\ 4ax^3 + 2bxy^2 + 2dx = 4x^3 - 3exy^2 + 4x$

and $2bx^2y + 4cy^3 - 4y = -12x^2y + ey^3 - 4y$

$\therefore$ Equating the coefficients of like powers of x and y, we get

$4a = 4,\quad 2b = -3e \quad \therefore\ 2d = 4 \quad$ and $\quad 2b = -12, \quad 4c = e.$

Hence, we get $\quad a = 1,\ d = 2,\ b = -6$

$\therefore\ -12 = -3e \quad \therefore\ e = 4 \quad$ and $\quad 4c = 4 \quad \therefore\ c = 1.$

Thus, we have $\quad a = 1,\ b = -6,\ c = 1,\ d = 2,\ e = 4.$

Example 22 : Find the values of z for which the following functions are not analytic.

(i) $z = e^{-v}(\cos u + i \sin u)$ **(ii)** $z = \sin hu \cos v + i \cos hu \sin v.$

Sol. : (i) We have $\quad z = e^{-v}(\cos u + i \sin u) = e^{-v} e^{iu}$

$\therefore\ z = e^{-v+iu} = e^{i^2 v + iu} = e^{i(u+iv)} = e^{iw}$ where $w = u + iv.$

$\therefore\ iw = \log z \quad \therefore\ w = \dfrac{1}{i}\log z \quad \therefore\ \dfrac{dw}{dz} = \dfrac{1}{i}\cdot\dfrac{1}{z}$

$\therefore\ w$ is not analytic at $z = 0.$

(ii) We have $\quad z = \sin hu \cos v + i \cos hu \sin v$

But $\cos v = \cos h\, iv$ and $i \sin v = \sin h\, iv$

$\therefore\ z = \sin hu \cos h\, iv + \cos hu \sin h\, iv$

$= \sin h\,(u + iv)$

$= \sin hw$ where $w + iv$

$\therefore\ w = \sin h^{-1} z = \log(z + \sqrt{z^2+1})$

$$\therefore\ \frac{dw}{dz} = \frac{1}{z + \sqrt{z^2+1}}\left(1 + \frac{1}{\sqrt{z^2+1}}\right) = \frac{1}{\sqrt{z^2+1}}$$

$\therefore\ w$ is not analytic when $\sqrt{z^2+1} = 0$ i.e. $z^2 = -1$ i.e. $z = \pm i.$

Example 23 : If $f(z) = u + iv$ is analytic in R show that

(i) $\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = |f'(z)|^2$ **(M.U. 2004, 06, 07, 09)**

(ii) $\left[\dfrac{\partial |f(z)|}{\partial x}\right]^2 + \left[\dfrac{\partial |f(z)|}{\partial y}\right]^2 = |f'(z)|^2$ **(M.U. 1995, 97, 2009, 10)**

Sol. : First we note that [see note (2) page 4-8]

if $f(z)\ u + iv,\quad f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$ .... (i)

(i) $\quad$ l.h.s. $= \dfrac{\partial u}{\partial x}\cdot\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\cdot\dfrac{\partial v}{\partial x}$

But by C-R equations $u_x = v_y,\ u_y = -v_x$

$\therefore$ l.h.s. $= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$

From (i) r.h.s $= |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$

Hence, the result.

(ii) Now, $|f(z)| = \sqrt{u^2+v^2}$

$$\therefore \frac{\partial}{\partial x}|f(z)| = \frac{1}{2\sqrt{u^2+v^2}}\left(2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right)$$

$$\frac{\partial}{\partial y}|f(z)| = \frac{1}{2\sqrt{u^2+v^2}}\left(2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}\right)$$

$$\therefore \text{l.h.s.} = \frac{1}{u^2+v^2}\left[u^2\left(\frac{\partial u}{\partial x}\right)^2 + v^2\left(\frac{\partial v}{\partial x}\right)^2 + 2uv\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + u^2\left(\frac{\partial u}{\partial y}\right)^2 + v^2\left(\frac{\partial v}{\partial y}\right)^2 + 2uv\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}\right]$$

Using C-R equations

$$\text{l.h.s.} = \frac{1}{u^2+v^2}\left[u^2\left(\frac{\partial u}{\partial x}\right)^2 + v^2\left(\frac{\partial v}{\partial x}\right)^2 + 2uv\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + u^2\left(\frac{\partial v}{\partial x}\right)^2 + v^2\left(\frac{\partial u}{\partial x}\right)^2 - 2uv\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial x}\right]$$

$$= \frac{1}{(u^2+v^2)}(u^2+v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = |f'(z)|^2 = \text{r.h.s.}$$

EXERCISE - II

1. Show that the following functions are not analytic.

   (i) $f(z) = \bar{z}$ (ii) $f(z) = z\,|\,z\,|$.

2. Determine whether the following functions are analytic and if so find their derivatives

   (i) $\cos h\, z$ (ii) $\cos z$ (iiii) $\frac{1}{z}$ (iv) $z^2 + z$ (v) $z^2 - \bar{z}$

   (vi) $e^x(\cos y - i\sin y)$ (vii) $\frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}$

   (viii) $e^{-x}(\cos y - i\sin y)$ (ix) $x^2 - y^2 + 2i\,xy$ **(M.U. 2004, 14)**

   (x) $(x^3 - 3xy^2 + 3x) + i(3x^2y - y^3 + 3y)$ **(M.U. 2004)**

   (xi) $z\,e^{2z}$ **(M.U. 2003, 04, 05, 11)**

[**Ans. :** (i) Yes, (ii) Yes, (iii) Yes except at $z = 0$, **(iv)** Yes, **(v)** No, **(vi)** No, **(vii)** Yes ; $\log z + C$, **(viii)** Yes, **(ix)** Yes, **(x)** Yes, **(xi)** Yes.]

3. If $f(z) = \dfrac{x^3 y(y - ix)}{x^6 + y^2}$, $z \neq 0$, $f(0) = 0$, prove that $\dfrac{f(z) - f(0)}{z - 0} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in **any manner.**

(**Hint :** Putting $y = mx$ $\lim\limits_{x \to 0} \dfrac{f(z) - f(0)}{z - 0} = 0$. But putting $y = x^3$ lim is $-i/2$)

4. Prove that
$$f(z) = \frac{x^2(1+i) - y^2(1-i)}{x + y}, \quad z \neq 0$$
$$f(0) = 0$$
is not analytic although Cauchy-Riemann equations are satisfied at $z = 0$.

5. Prove that
$$f(z) = e^{-z^{-4}}, \quad z \neq 0$$
$$f(0) = 0$$
is not analytic although Cauchy-Riemann equations are satisfied at $z = 0$.

6. Prove that the following function
$$f(z) = \frac{x^3 y^5 (x + iy)}{x^6 + y^{10}}, \quad z \neq 0$$
$$f(0) = 0$$
is not analytic at the origin although Cauchy-Riemann equations are satisfied.

(**Hint :** Put $x^3 = y^5$)

7. Find the values of z for which the following function is not analytic
$$z = \sin u \cos hv + i \cos u \sin hv$$
[**Ans. :** $z = \pm 1$]

8. If $f(z) = \dfrac{xy^2(x + iy)}{x^2 + y^4}$, $z \neq 0$, $f(0) = 0$ prove that $\dfrac{f(z) - f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in **any manner**.

(**Hint :** Put $y = mx$ and then put $y^2 = x$)

9. Prove that the function $f(z) = e^{2z}$ is analytic and find $f'(z)$.

(**Hint :** $u = e^{2x} \cos 2y$, $v = e^{2x} \sin 2y$, $f'(z) = 2e^{2z}$)

10. Show that $u + iv = \dfrac{x - iy}{x - iy + a}$, $a \neq 0$ is not analytic but $u - iv$ is analytic.

(**Hint :** Let $f(z) = u + iv = \dfrac{\bar{z}}{\bar{z} + a} = \dfrac{\bar{z} + a - a}{\bar{z} + a} = 1 - \dfrac{a}{\bar{z} + a}$
$$\Phi(z) = u - iv = \frac{z}{z + a} = \frac{z + a - a}{z + a} = 1 - \frac{a}{z + a}$$
Now try to find $f'(z)$ and $\Phi'(z)$ on the lines of alternative method of Ex. (9) (a), page 4-13.)

11. If $u + iv$ is analytic prove that $v - iu$ and $-v + iu$ are analytic. Show further that $v + iu$ will be analytic if $u + iv$ is a constant.

12. Find the constants a, b, c, d, e if

(i) $f(z) = (ax^3 + bxy^2 + 3x^2 + cy^2 + x) + i(dx^2 y - 2y^3 + exy + y)$ **(M.U. 2006, 08, 19)**

(ii) $f(z) = (ax^4 + bx^2 y^2 + cy^4 + dx^2 - 2y^2) + i(4x^3 y - exy^3 + 4xy)$

are analytic.

(M.U. 2002, 03, 07, 08, 09, 12, 13)

[**Ans. :** (i) $a = 2, b = -6, c = -3, d = 6, e = 6$; **(ii)** $a = 1, b = -6, c = 1, d = 2, e = 4$]

13. Show that the following function is not analytic at the origin although C
equations are satisfied

$$f(z) = \frac{xy\,(y - ix)}{x^2 + y^2}, \quad z \neq 0$$

$$f(0) = 0$$

(**Hint :** $f'(0) = \dfrac{f(z) - f(0)}{z - 0} = \lim\limits_{z \to 0} \dfrac{xy\,(y - ix)/(x^2 + y^2) - 0}{(x + iy) - 0}$

$$= \lim_{z \to 0} \frac{xy\,(y - ix)}{(x^2 + y^2)(x + iy)} = \lim_{z \to 0} - \frac{i\,xy\,(x + iy)}{(x^2 + y^2)(x + iy)}$$

$$= \lim_{z \to 0} - \frac{i\,xy}{x^2 + y^2}.$$

Now put $y = mx$.)

10. Cauchy-Riemann Equations in Polar Coordinates (M

Let (r, θ) be the polar coordinates of a point whose Cartesian coordinates are (x, y)

$$\therefore \quad x = r\cos\theta, \quad y = r\sin\theta,$$

$$z = x + iy = r(\cos\theta + i\sin\theta) = r\,e^{i\theta}$$

Let $f(z) = u + iv$ be the given function

$$\therefore \quad f(z) = u + iv = f(r\,e^{i\theta}) \qquad \ldots\ldots$$

Differentiating (1) partially w.r.t. r,

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(r\,e^{i\theta}) \cdot e^{i\theta} \qquad \ldots\ldots$$

Differentiating (1) partially w.r.t. θ,

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(r\,e^{i\theta}) \cdot r\,e^{i\theta} \cdot i = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) \qquad [\text{By (2)}]$$

$$= -r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}$$

Equating real and imaginary parts $\dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r}$ and $\dfrac{\partial v}{\partial \theta} = r\dfrac{\partial u}{\partial r}$

or $\boxed{\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} \text{ and } \dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r} \text{ i.e. } u_r = \dfrac{1}{r}v_\theta \text{ and } u_\theta = -r\,v_r}$

Note ....

From (2), we get an important result.

$$f'(r\,e^{i\theta}) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) \qquad \therefore \quad \boxed{f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)}$$

dw

$$\therefore \quad \frac{\partial u}{\partial r} = \frac{1}{r}, \ \frac{\partial u}{\partial \theta} = 0 \ ; \ \frac{\partial v}{\partial r} = 0, \ \frac{\partial v}{\partial \theta} = 1.$$

$$\therefore \quad u_r = \frac{1}{r} \cdot 1 = \frac{1}{r} v_\theta \ \text{ and } \ u_\theta = 0 = -r\, v_r$$

Hence, C-R equations are satisfied when $r \neq 0$.

Partial derivatives are continuous except at $r = 0$ *i.e.* at $z = 0$

Now $f'(z) = \dfrac{d}{dz}(\log z) = e^{-i\theta}\left(\dfrac{\partial u}{\partial r} + i\dfrac{\partial u}{\partial r}\right) = e^{-i\theta}\left(\dfrac{1}{r}\right) = \dfrac{1}{r\,e^{i\theta}} = \dfrac{1}{z}.$

Example 2 : Find p if $f(z) = r^2 \cos 2\theta + i\, r^2 \sin p\,\theta$ is analytic. **(M.U. 1998, 08, 10)**

Sol. : Let $w = f(z) = u + iv = r^2 \cos 2\theta + i\, r^2 \sin p\,\theta$

$$\therefore \quad u = r^2 \cos 2\theta, \quad v = r^2 \sin p\,\theta$$

$$\frac{\partial u}{\partial r} = 2r\cos 2\theta, \quad \frac{\partial u}{\partial \theta} = -2r^2 \sin 2\theta; \qquad \frac{\partial v}{\partial r} = 2r\sin p\,\theta, \quad \frac{\partial v}{\partial \theta} = p\,r^2 \cos p\,\theta$$

Since, $f(z)$ is analytic. $\quad \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$ and $u_\theta = -r\,v_r$

The first relation gives, $\quad 2r\cos 2\theta = \dfrac{1}{r}\cdot p\,r^2 \cos p\,\theta \quad \therefore\ p = 2$

And the second realtion also gives, $\ -2r^2 \sin 2\theta = -2r^2 \sin p\,\theta \quad \therefore\ p = 2.$

Example 3 : Find the value of k if $f(z) = r^3 \cos k\,\theta + i\, r^k \sin 3\theta$ is analytic.

[**Ans. :** $k = 3$]

Sol. : Left to you.

(M.U. 2004)

Example 4 : Is $f(z) = \dfrac{z}{\bar{z}}$ analytic ?

Sol. : Put $z = r\,e^{i\theta}$ and use C-R equations in polar form. No.

Example 5 : If $w = z^n$ find $\dfrac{dw}{dz}$.

Sol. : Let $z = r\,e^{i\theta}$ $\ \therefore\ z^n = r^n e^{in\theta}$

$$\therefore \quad z^n = r^n(\cos n\,\theta + i\sin n\,\theta) \qquad \therefore \quad u = r^n \cos n\,\theta, \ v = r^n \sin n\,\theta.$$

$$\frac{\partial u}{\partial r} = n\,r^{n-1}\cos n\,\theta, \quad \frac{\partial u}{\partial \theta} = -r^n \cdot n \cdot \sin n\,\theta$$

$$\frac{\partial v}{\partial r} = n\,r^{n-1}\sin n\,\theta, \quad \frac{\partial v}{\partial \theta} = n\,r^n \cos n\,\theta$$

$$\therefore \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \ \text{ and } \ \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$

Also partial derivatives are continuous. Hence, w is analytic.

$$\therefore \quad \frac{dw}{dz} = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = e^{-i\theta}(n\,r^{n-1}\cos n\,\theta + i\,n\,r^{n-1}\sin n\,\theta)$$

$$= n\,r^{n-1}\cdot e^{-i\theta}(\cos n\,\theta + i\sin n\,\theta) = n\,r^{n-1}\cdot e^{-i\theta}\cdot e^{in\theta}$$

$$= n\,r^{n-1}\cdot e^{i(n-1)\theta} = n\,(r\,e^{i\theta})^{n-1} = n\,z^{n-1}.$$

Example 6 : Show that $f(z) = \frac{1}{r^2}[\cos 2\theta - i \sin 2\theta]$ is analytic. **(M.U. 2019)**

Sol. : Left to you.

Example 7 : Using Cauchy-Riemann equations in polar form prove that

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Sol. : We know that Cauchy-Riemann equations in polar form are

$$u_r = \frac{1}{r} v_\theta \quad \text{.................. (i)}$$

and $$u_\theta = -r v_r \quad \text{.................. (ii)}$$

Differentiating (i) w.r.t. r, we get,

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta \, \partial r} \quad \text{.................. (iii)}$$

Differentiating (ii) w.r.t. θ, we get,

$$\frac{\partial^2 u}{\partial \theta^2} = -r \frac{\partial^2 v}{\partial \theta \, \partial r} \quad \text{.................. (iv)}$$

Now, using (iii) and (iv), we get,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \left(-\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta \, \partial r}\right) + \frac{1}{r} \cdot \frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{1}{r^2} \cdot r \frac{\partial^2 v}{\partial \theta \, \partial r}$$

$$= -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta \, \partial r} + \frac{1}{r^2}\frac{\partial v}{\partial \theta} - \frac{1}{r}\frac{\partial^2 v}{\partial \theta \, \partial r} = 0$$

Note ....

The equation $\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$ is called Laplace's equation in **Cartesian Form** and the equation $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$ is called Laplace's equation in **Polar Form.**

11. Harmonic Functions

Any function of x, y which has continuous partial derivatives of the first and second order and satisfies Laplace's equation $\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$ is called a **Harmonic Function.**

Theorem : The real and imaginary parts u, v of an analytic function $f(z) = u + iv$ are harmonic functions. **(M.U. 2003, 13)**

Proof : Since, $f(z)$ is an analytic function in some region of the z-plane, u, v satisfy C-R equations.

$$\therefore \quad u_x = v_y \text{ and } u_y = -v_x \quad \text{.................. (A)}$$

Differentiating the first w.r.t. x and second w.r.t. y, we get,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \, \partial y} \text{ and } \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \, \partial x}$$

Assuming $\dfrac{\partial^2 v}{\partial x\,\partial y} = \dfrac{\partial^2 v}{\partial x\,\partial y}$ and adding the above results we get,

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

Similarly differentiating the equations in (A) with respect to y and x respectively, we can show that

$$\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$$

Hence, the result.

Notes ....

1. In other words the above theorem states that if $f(z) = u + iv$ is analytic, then its real and imaginary parts u, v satisfy Laplace equation.

$$\boxed{\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0}$$

2. The above theorem states that if $f(z) = u + iv$ is analytic then u and v satisfy Laplace's equation *i.e.* u and v are harmonic functions. But, the converse is **not** true. If u and v are any two functions satisfying Laplace's equation then $u + iv$ need not be analytic. [See Ex. 7 of § 13 page 4-47].

Theorem : If $f(r e^{i\theta}) = u(r, \theta) + iv(r, \theta)$ is analytic then the real and imaginary parts u, v are harmonic.

Proof : You can prove it easily by using Cauchy-Riemann equations in polar form. Prove it.

Note ....

3. In other words the above theorem states that if $f(r e^{i\theta}) = u(r, \theta) + iv(r, \theta)$ is analytic then the real and imaginary parts u, v satisfy Laplace equation in polar form

$$\boxed{\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0}$$

Example 1 : Show that a harmonic function satisfies the differential equation $\dfrac{\partial^2 u}{\partial z\,\partial \bar{z}} = 0$. .... (i)

Sol. : If u is a harmonic function then $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

Now, $x = \dfrac{1}{2}(z + \bar{z})$ and $y = \dfrac{1}{2i}(z - \bar{z})$

$$\therefore \quad \frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2i}\cdot\frac{\partial u}{\partial y}$$

$$\therefore \quad \frac{\partial^2 u}{\partial z\,\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial x}{\partial z} + \frac{\partial^2 u}{\partial x\,\partial y}\cdot\frac{\partial y}{\partial z}\right] - \frac{1}{2i}\left[\frac{\partial^2 u}{\partial y\,\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial^2 u}{\partial y^2}\cdot\frac{\partial y}{\partial z}\right]$$

$$= \frac{1}{2}\left[\frac{\partial^2 u}{\partial x^2}\cdot\frac{1}{2} + \frac{\partial^2 u}{\partial x\,\partial y}\cdot\frac{1}{2i}\right] - \frac{1}{2i}\left[\frac{\partial^2 u}{\partial x\,\partial y}\cdot\frac{1}{2} + \frac{\partial^2 u}{\partial y^2}\cdot\frac{1}{2i}\right]$$

$$\therefore \quad \frac{\partial^2 u}{\partial z\,\partial \bar{z}} = \frac{1}{4}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] = 0 \qquad [\text{ By (i) }]$$

Hence, the required result.

Example 2 : If $u(x, y)$ is a harmonic function then prove that $f(z) = u_x - i\,u_y$ is an analytic function. **(M.U. 2004, 08)**

Sol. : Since u is harmonic $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ .... (1)

By data $f(z) = u_x - i\,u_y$.

Let $u_x = U$ and $-u_y = V$, so that $f(z) = U + i\,V$.

We have to show that $f(z)$ is analytic.

$$\text{Now,} \quad U_x = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \quad [\text{ By (1) }] \quad \text{and} \quad U_y = \frac{\partial^2 u}{\partial x\,\partial y}$$

$$V_x = -\frac{\partial^2 u}{\partial y\,\partial x} \text{ and } V_y = -\frac{\partial^2 u}{\partial y^2} \qquad \therefore \quad U_x = V_y \text{ and } U_y = -V_x.$$

$\therefore \quad f(z) = U + i\,V$ is analytic *i.e.* $f(z) = u_x - i\,u_y$ is analytic.

Example 3 : If u, v are harmonic conjugate functions, show that uv is a harmonic function. **(M.U. 2003)**

Sol. : Let $f(z) = u + iv$ be the analytic function. [See note (2), page 4-26]

$$\therefore \quad u_x = v_y \text{ and } u_y = -v_x$$

And u, v are harmonic $\quad \therefore \quad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ and $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$ .... (1)

$$\text{Now,} \quad \frac{\partial}{\partial x}(uv) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}$$

$$\therefore \quad \frac{\partial^2}{\partial x^2}(uv) = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + v\frac{\partial^2 u}{\partial x^2}$$

$$\therefore \quad \frac{\partial^2}{\partial x^2}(uv) = u\frac{\partial^2 v}{\partial x^2} + v\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} \qquad \text{..................... (2)}$$

Similarly, we can prove that

$$\frac{\partial^2}{\partial y^2}(uv) = u\frac{\partial^2 v}{\partial y^2} + v\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}$$

But $u_x = v_y$ and $u_y = -v_x$

$$\therefore \quad \frac{\partial^2}{\partial y^2}(uv) = u\frac{\partial^2 v}{\partial y^2} + v\frac{\partial^2 u}{\partial y^2} - 2\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} \qquad \text{..................... (3)}$$

Adding (2) and (3), we get

$$\frac{\partial^2}{\partial x^2}(uv) + \frac{\partial^2}{\partial y^2}(uv) = u\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$= 0 \qquad [\text{ By (1) }]$$

$\therefore \quad uv$ is harmonic.

Example 4 : If Φ and Ψ are functions of x and y satisfying Laplace equation and if $u = \Phi_y - \Psi_x$ and $v = \Phi_x + \Psi_y$, prove that $u + iv$ is analytic (holomorphic). **(M.U. 2003)**

Sol. : Since Φ and Ψ satisfy Laplace equation, we have

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \quad \text{.......... (1)} \quad \text{and} \quad \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0 \quad \text{.......... (2)}$$

Now $u_x = \Phi_{yx} - \Psi_{xx} = \Phi_{xy} + \Psi_{yy}$ [By (2)]

And $u_y = \Phi_{yy} - \Psi_{xy} = -(\Phi_{xx} + \Psi_{xy})$ [By (1)]

Similarly, $v_x = \Phi_{xx} + \Psi_{xy}$ and $v_y = \Phi_{xy} + \Psi_{yy}$

Hence, $u_x = v_y$ and $u_y = -v_x$

Hence, $u + iv$ is analytic.

12. To Find an Analytic Function whose Real or Imaginary Part is given

Method 1 : Let $f(z) = u + iv$ and let u be given. Since, u is given we can find u_x and u_y.

As $f(z)$ is analytic, by C-R equations $u_x = v_y$ and $u_y = -v_x$.

$\therefore \; f'(z) = u_x + iv_x = u_x - iu_y = \Phi(z)$ say.

Hence, by mere integration $f(z)$ can be obtained. (See Ex. 1, page 4-31)

Note ....

The method can be used only when we are able to express $u_x - iu_y$ as a function of z, say $\Phi(z)$.

Method 2 : Let $f(z) = u + iv$ and let u be given. This means $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are known.

Now, $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

But by C-R equations $u_x = v_y$ and $u_y = -v_x$

$$\therefore \; dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \quad \text{.................... (1)}$$

Further $\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}$

Since, u is harmonic $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

i.e. $-\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2}$ i.e. $\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$

$\therefore$ R.H.S. of (1) is an exact differential.

(Recall that if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then $M\,dx + N\,dy = du$)

$\therefore$ Integrating (i), $v = \int \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy\right) + c$

From this we get v then $f(z) = u + iv$.

Notes ....

1. Recall how we obtain the solution of an exact differential equation $M\,dx + N\,dy = 0$. In this case we have $M = -\dfrac{\partial u}{\partial y}$ and $N = \dfrac{\partial u}{\partial x}$. Hence, to obtain v, **integrate** $\left(-\dfrac{\partial u}{\partial y}\right)$ **w.r.t. x treating y constant and integrate only those terms in $\dfrac{\partial u}{\partial x}$ which are free from x. Their sum plus a constant is equal to v.**

2. If $f(z) = u + iv$ is an analytic function so that u and v both are harmonic functions then u and v are called **harmonic conjugate functions**. Each is called the **harmonic conjugate function** of the other.

3. Instead of u if v is given we can reason exactly as above to find $f(z)$. In this case **integrate** $\dfrac{\partial v}{\partial y}$ **treating x constant and integrate those terms in $-\dfrac{\partial v}{\partial x}$ which are free from x. Their sum plus a constant is equal to u.**

Method 3 : Milne-Thompson's Method

Since, $z = x + iy$, $\bar{z} = x - iy$ $\quad\therefore\quad x = \dfrac{z+\bar{z}}{2},\ y = \dfrac{z-\bar{z}}{2i}$

$$\therefore\ f(z) = u(x,y) + iv(x,y) = u\left[\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right] + iv\left[\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right]$$

This can be regarded as an identity in two independent variables z and $\bar{z}$. We can, therefore, put $\bar{z} = z$ and get

$$f(z) = u(z,0) + iv(z,0)$$

Thus, **$f(z)$ can be obtained in terms of z by putting $x = z$ and $y = 0$ in $f(z) = u(x, y) + iv(x, y)$ when $f(z)$ is analytic.**

Now $f'(z) = u_x + iv_x = u_x - iu_y$ [$\because$ C-R equations]

Let $u_x = \Phi_1(x,y)$ and $u_y = \Phi_2(x,y)$

$$\therefore\ f'(z) = \Phi_1(x,y) - i\,\Phi_2(x,y) = \Phi_1(z,0) - i\,\Phi_2(z,0)$$

Integrating, we get,

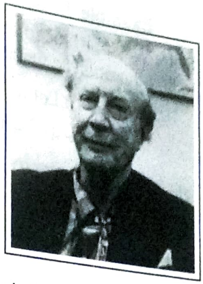
$$\boxed{f(z) = \int \Phi_1(z,0)\,dz - i\int \Phi_2(z,0)\,dz + c}$$

Similarly if v is given arguing on the above lines we can show that

$$\boxed{f(z) = \int \psi_1(z,0)\,dz + i\int \psi_2(z,0)\,dz + c}$$

where, $v_y = \psi_1(x,y)$, $v_x = \psi_2(x,y)$

Louis Melville Milne-Thompson (1891-1974)



He became Wrangler in 1913 and joined Winchester College as an assistant mathematics master in 1914. He left Winechester College and joined Royal Naval College, Greenwich in 1921. His earlier research was related to tables. He published in 1931 "Standard Four Figure Mathematical Tables" and in the next year he published "Standard Table of Square Root And Jacobian Elliptic Functions". He published his first book on "The Calculus Of Finite Differences" in 1933. In 1938, he published his second book but on different subject "Theoretical Hydrodynamics". He retried from the Royal Naval College in 1956 at the age of 65 and then took various posts as visiting professor at various institutions throughout the world. He was elected a fellow of the Royal Society of Edinburg, Royal Astronomical Society and Cambridge Philosophical Society. We know him for Milne-Thomson method for finding analytic function.

Method 4 : Let $f(z) = f(z+iy) = u(x,y) + iv(x,y)$

Then $\overline{f(z)} = \overline{f(x+iy)} = u(x,y) - i\,v(x,y) \quad \therefore\ f(z) + \overline{f(z)} = 2u(x,y)$

We can consider $\overline{f(z)}$ as a function of $\bar{z}$ and denote it as $\bar{f}(\bar{z})$.

$$u(x,y) = \frac{1}{2}[f(z) + \bar{f}(\bar{z})] = \frac{1}{2}[f(x+iy) - \bar{f}(x-iy)]$$

This can be regarded as an identity and holds even if x and y are complex. Putting $x = \frac{z}{2}$ and $y = \frac{z}{2i}$, we get, from above

$$u(z/2,\ z/2i) = \frac{1}{2}\left[f\left(\frac{z}{2} + i\frac{z}{2i}\right) + \bar{f}\left(\frac{z}{2} - i\frac{z}{2i}\right)\right] = \frac{1}{2}[f(z) + \bar{f}(0)]$$

$$\therefore\ f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - \bar{f}(0)$$

We can assume that $\bar{f}(0)$ is purely real and write $\bar{f}(0) = u\ (0,\ 0)$

$$\therefore\ f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0)$$

We may add a purely imaginary constant ci and get

$$\boxed{f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + ci}$$

Notes ....

1. Note that this method is purely algebraic and does not involve integration.
2. Of these four methods the third method *i.e.* **Milne-Thompson method is commonly used.**

Type I : To find the analytic function whose real part u is given

Example 1 : Construct an analytic function whose real part is $x^4 - 6x^2y^2 + y^4$.

Sol. : We shall first solve this example by the above four methods and then lay down the procedure to find the analytic function $f(z)$.

Method 1 : Let $u = x^4 - 6x^2y^2 + y^4$ and let $f(z) = u + iv$ be the required function.

$\therefore \quad u_x = 4x^3 - 12xy^2 \; ; \; u_y = -12x^2y + 4y^3$

As seen above (or see note (2), page 4-8)

$$f'(z) = u_x - iu_y$$
$$= 4x^3 - 12xy^2 + 12ix^2y - 4iy^3$$
$$= 4[x^3 + 3x(iy)^2 + 3x^2(iy) + (iy)^3]$$
$$= 4(x+iy)^3 = 4z^3$$

$$\therefore \quad f(z) = \int f'(z)\,dz = \int 4z^3\,dz = z^4 + c.$$

Method 2 : As before $\dfrac{\partial u}{\partial x} = 4x^3 - 12xy^2 \; ; \; \dfrac{\partial u}{\partial y} = -12x^2y + 4y^3$.

Since, $f(z)$ is given to be analytic we can use the note 1 given on page 4-29.

$$\int -\frac{\partial u}{\partial y}\,dx = \int -(-12x^2y + 4y^3)\,dx = 4x^3y - 4y^3x$$

And $\quad \displaystyle\int (\text{terms in } \frac{\partial u}{\partial x} \text{ free from } x)\,dy = \int 0\,dy = 0$

$$\therefore \quad v = 4x^3y - 4xy^3 + 0 = 4x^3y - 4xy^3$$

$$\therefore \quad f(z) = u + iv + c = x^4 - 6x^2y^2 + y^4 + 4ix^3y - 4ixy^3 + c$$

Setting $x = z,\; y = 0$, we get $f(z) = z^4 + c$.

Method 3 : *Milne-Thompson Method* : We have as above

$$\Phi_1 = u_x = 4x^3 - 12xy^2 \; ; \; \Phi_2 = u_y = -12xy^2 + 4y^3$$

$$\therefore \quad f'(z) = \Phi_1(z,0) - i\,\Phi_2(z,0)$$
$$= 4z^3 - i(0) \qquad [\text{ Putting } x = z,\; y = 0 \text{ in } \Phi_1 \text{ and } \Phi_2]$$

$\therefore \quad f(z) = \int 4z^3\,dz = z^4 + c$ as before.

Method 4 : We have

$$u\left(\frac{z}{2}, \frac{z}{2i}\right) = \frac{z^4}{16} - 6\left(\frac{z^2}{4}\right)\left(\frac{z^2}{4i^2}\right) + \left(\frac{z^4}{16i^4}\right) = \frac{z^4}{16} + \frac{6z^4}{16} + \frac{z^4}{16} = \frac{8z^4}{16}$$

Further $u(0, 0) = 0$

$$\therefore \quad f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + c_i = 2 \cdot \frac{8z^4}{16} - 0 + c_i = z^4 + c_i$$

Procedure to find $f(z)$ when real part u is given.

1. From u, first find u_x and u_y.
2. Then put $\Phi_1 = u_x$ and $\Phi_2 = u_y$.

3. Putting $x = z$ and $y = 0$, find $\Phi_1(z, 0)$ and $\Phi_2(z, 0)$
4. Then $f'(z) = \Phi_1(z, 0) - i\Phi_2(z, 0)$
5. $f(z) = \int f'(z)dz + c$

Example 2 : Construct an analytic function whose real part is $e^x \cos y$. **(M.U. 2019)**

Sol. : Let $u = e^x \cos y$

$\therefore u_x = e^x \cos y$ and $u_y = -e^x \sin y$

$\therefore \Phi_1 = u_x = e^x \cos y,\ \Phi_2 = u_y = -e^x \sin y$

By Milne-Thompson method

$\therefore f'(z) = \Phi_1(z, 0) - i\Phi_2(z, 0) = e^z - i(0)$

$\therefore f(z) = \int e^z dz = e^z + c$ which is the required analytic function.

Example 3 : Find an analytic function whose real part is

$e^{-x}\left\{(x^2 - y^2)\cos y + 2xy \sin y\right\}$ **(M.U. 2017, 18, 19)**

Sol. : Let $u = e^{-x}\left\{(x^2 - y^2)\cos y + 2xy \sin y\right\}$

$$\therefore u_x = -e^{-x}\left\{(x^2 - y^2)\cos y + 2xy \sin y\right\} + e^{-x}\left\{2x\cos y + 2y \sin y\right\}$$

$$= e^{-x}\left[-(x^2 - y^2)\cos y + 2x\cos y + 2y \sin y - 2xy \sin y\right]$$

$$u_y = e^{-x}\left[-(x^2 - y^2)\sin y - 2y\cos y + 2x \sin y + 2xy \cos y\right]$$

$\therefore \Phi_1 = u_x$ and $\Phi_2 = u_y$

By Milne-Thompson method

$$f'(z) = \Phi_1(z, 0) - i\Phi_2(z, 0) = e^{-z}[-z^2 + 2z]$$

$$\therefore f(z) = \int e^{-z}(-z^2 + 2z)\,dz$$

Integrating by parts,

$$f(z) = (-z^2 + 2z)(-e^{-z}) - \int(-e^{-z})(-2z + 2)\,dz$$

$$= e^{-z}(z^2 - 2z) + \int e^{-z}(2 - 2z)\,dz$$

Integrating by parts again,

$$\therefore f(z) = e^{-z}(z^2 - 2z) + (2 - 2z)(-e^{-z}) - \int(-e^{-z})(-2)\,dz$$

$$= e^{-z}(z^2 - 2z) - e^{-z}(2 - 2z) + 2e^{-z}$$

$$= z^2 e^{-z} + c.$$

Example 4 : Find the imaginary part of the analytic function whose real part is

$e^{2x}(x\cos 2y - y \sin 2y)$. **(M.U. 1993, 2000, 03, 14)**

Also verify that v is harmonic. **(M.U. 2004, 05, 10, 19)**

Sol. : Let $u = e^{2x}(x\cos 2y - y\sin 2y)$

$$\therefore \Phi_1 = u_x = e^{2x} \cdot 2(x\cos 2y - y\sin 2y) + e^{2x}(\cos 2y)$$

$$= e^{2x}(2x\cos 2y - 2y\sin 2y + \cos 2y)$$

$$\Phi_2 = u_y = e^{2x}(-2x \sin 2y - \sin 2y - 2y \cos 2y)$$

By Milne-Thompson Method

$$\therefore \quad f'(z) = \Phi_1(z, 0) - i\,\Phi_2(z, 0) = e^{2z}(2z+1) - i\,e^{2z}(0) = e^{2z}(2z+1)$$

Integrating by parts,

$$\therefore \quad f(z) = \int e^{2z}(2z+1)\,dz = (2z+1)\frac{e^{2z}}{2} - \int \frac{e^{2z}}{2} \cdot 2 \cdot dz$$

$$= (2z+1)\frac{e^{2z}}{2} - \int e^{2z} dz = (2z+1)\frac{e^{2z}}{2} - \frac{e^{2z}}{2} = e^{2z} z + c.$$

Now, $\quad f(z) = e^{2(x+iy)} \cdot (x+iy) = e^{2x} \cdot e^{2iy}(x+iy)$

$$= e^{2x}[\cos 2y + i \sin 2y](x+iy)$$

$$\therefore \quad v = e^{2x}(y \cos 2y + x \sin 2y)$$

$$\therefore \quad \frac{\partial v}{\partial x} = 2e^{2x}(y \cos 2y + x \sin 2y) + e^{2x}(\sin 2y)$$

$$\frac{\partial^2 v}{\partial x^2} = 4\,e^{2x}(y \cos 2y + x \sin 2y) + 2\,e^{2x} \sin 2y + 2e^{2x} \sin 2y$$

$$\frac{\partial^2 v}{\partial x^2} = 4\,e^{2x}(y \cos 2y + x \sin 2y) + 4\,e^{2x} \sin 2y$$

$$\frac{\partial v}{\partial y} = e^{2x}(\cos 2y - 2y \sin 2y + 2x \cos 2y)$$

$$\frac{\partial^2 v}{\partial y^2} = e^{2x}(-2 \sin 2y - 2 \sin 2y - 4y \cos 2y - 4x \sin 2y)$$

$$= e^{2x}(-4 \sin 2y - 4y \cos 2y - 4x \sin 2y)$$

$$\therefore \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad \therefore \; v \text{ is harmonic.}$$

Example 5 : Find an analytic function $f(z) = u + iv$ where

$$u = \frac{x}{2}\log(x^2+y^2) - y \tan^{-1}\frac{y}{x} + \sin x \cos hy.$$

(M.U. 2003, 08, 16)

Sol. : We have $\quad u = \frac{x}{2}\log(x^2+y^2) - y \tan^{-1}\frac{y}{x} + \sin x \cos hy$

$$\therefore \quad \Phi_1 = u_x = \frac{1}{2}\log(x^2+y^2) + \frac{x^2}{x^2+y^2} - y \cdot \frac{1}{1+(y^2/x^2)}\left(-\frac{y}{x^2}\right) + \cos x \cos hy$$

$$= \frac{1}{2}\log(x^2+y^2) + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + \cos x \cos hy$$

And $\quad \Phi_2 = u_y = \frac{xy}{x^2+y^2} - \tan^{-1}\left(\frac{y}{x}\right) - y \cdot \frac{1}{1+(y^2/x^2)} \cdot \frac{1}{x} + \sin x \sin hy$

$$= \frac{xy}{x^2+y^2} - \tan^{-1}\frac{y}{x} - \frac{xy}{x^2+y^2} + \sin x \sin hy$$

By Milne-Thomson method,

$$f'(z) = \Phi_1(z, 0) - i\,\Phi_2(z, 0)$$

$$\therefore\ f'(z) = \frac{1}{2}\log z^2 + \frac{z^2}{z^2} + \cos z - i\,(0)$$

$$= \log z + 1 + \cos z$$

$\therefore$ By integration,

$$f(z) = \int \log z\, dz + \int 1 dz + \int \cos z\, dz$$

$$= \log z \cdot z - \int 1 \cdot dz + z + \sin z \qquad \text{[By Integrating by parts]}$$

$$= z\log z - z + z + \sin z$$

$$= z\log z + \sin z + c.$$

Example 6 : Find the analytic function whose real part is

$$\frac{\sin 2x}{\cos h\, 2y + \cos 2x}. \qquad \textbf{(M.U. 2002, 06, 10, 11, 15)}$$

Sol. : Let $u = \dfrac{\sin 2x}{\cos h\, 2y + \cos 2x}$

$$\therefore\ \Phi_1 = u_x = \frac{(\cos h\, 2y + \cos 2x)(2\cos 2x) + \sin 2x \cdot 2\sin 2x}{(\cos h\, 2y + \cos 2x)^2}$$

$$= \frac{2\cos h\, 2y \cos 2x + 2}{(\cos h\, 2y + \cos 2x)^2}$$

$$\Phi_2 = u_y = -\frac{\sin 2x \cdot 2\sin h\, 2y}{(\cos h\, 2y + \cos 2x)^2}$$

By Milne-Thompson Method

$$\therefore\ f'(z) = \Phi_1(z, 0) - i\,\Phi_2(z, 0)$$

$$= \frac{2\cos 2z + 2}{(1 + \cos 2z)^2} - 0 = \frac{2}{1 + \cos 2z} = \sec^2 z$$

(See that $\sin hx = \dfrac{e^x - e^{-x}}{2}$ and $\sin h\, 0 = 0$, $\cos hx = \dfrac{e^x + e^{-x}}{2}$ and $\cos h\, 0 = 1$.)

$$\therefore\ f(z) = \int \sec^2 z\, dz = \tan z + c.$$

Type II : To find the analytic function whose imaginary part v is given

Procedure to find $f(z)$ when imaginary part v is given

1. From v first find v_y and v_x.
2. Then put $\Psi_1 = v_y$ and $\Psi_2 = v_y$.
3. Putting $x = z$ and $y = 0$, find $\Psi_1(z, 0)$ and $\Psi_2(z, 0)$.
4. Then $f'(z) = \Psi_1(z, 0) + i\,\Psi_2(z, 0)$.
5. $f(z) = \int f'(z)\, dz + c$

Example 1 : Find an analytic function whose imaginary part is

$$(x^4 - 6x^2y^2 + y^4) + (x^2 - y^2) + 2xy$$

Sol. : We have $v = (x^4 - 6x^2y^2 + y^4) + (x^2 - y^2) + 2xy$

$$\therefore \quad v_y = \psi_1(x, y) = -12x^2y + 4y^3 - 2y + 2x$$

$$v_x = \psi_2(x, y) = 4x^3 - 12xy^2 + 2x + 2y$$

$$\therefore \quad \psi_1(z, 0) = 2z, \quad \psi_2(z, 0) = 4z^3 + 2z$$

By Milne-Thompson method

$$f'(z) = \psi_1(z, 0) + i\,\psi_2(z, 0)$$

$$\therefore \quad f(z) = \int \psi_1(z, 0)\,dz + i\int \psi_2(z, 0)\,dz$$

$$= \int 2z\,dz + i\int (4z^3 + 2z)\,dz$$

$$= z^2 + i(z^4 + z^2) + c.$$

Example 2 : Find an analytic function whose imaginary part is $e^{-x}(y\cos y - x\sin y)$.

Sol. : We have $v = e^{-x}(y\cos y - x\sin y)$

$$\therefore \quad v_y = \psi_1(x, y) = e^{-x}(\cos y - y\sin y - x\cos y)$$

$$v_x = \psi_2(x, y) = -e^{-x}(y\cos y - x\sin y) + e^{-x}(-\sin y)$$

$$= e^{-x}(-\sin y - y\cos y + x\sin y)$$

$$\therefore \quad \psi_1(z, 0) = e^{-z}(1 - z), \quad \psi_2(z) = 0$$

By Milne-Thompson Method

$$f'(z) = \psi_1(z, 0) + i\,\psi_2(z, 0)$$

$$\therefore \quad f(z) = \int \psi_1(z, 0)\,dz + i\int \psi_2(z, 0)\,dz = \int (1 - z)e^{-z}\,dz$$

$$= (1 - z)(-e^{-z}) - \int (-e^{-z})(-1)\,dz$$

$$= -e^{-z} + ze^{-z} + e^{-z} = ze^{-z} + c.$$

Example 3 : Find an analytic function $f(z)$ whose imaginary part is $e^{-x}(y\sin y + x\cos y)$.

(M.U. 1995, 2007, 09)

Sol. : We have $v = e^{-x}(y\sin y + x\cos y)$

$$\therefore \quad v_y = \psi_1(x, y) = e^{-x}(\sin y + y\cos y - x\sin y)$$

$$v_x = \psi_2(x, y) = -e^{-x}(y\sin y + x\cos y) + e^{-x}(\cos y)$$

$$\therefore \quad v_x = \psi_2(x, y) = e^{-x}(\cos y - y\sin y - x\cos y)$$

$$\therefore \quad \psi_1(z, 0) = 0, \quad \psi_2(z, 0) = e^{-z}(1 - z)$$

By Milne-Thompson Method

$$f'(z) = \psi_1(z, 0) + i\,\psi_2(z, 0)$$

$$\therefore \quad f(z) = \int \psi_1(z, 0)\,dz + i\int \psi_2(z, 0)\,dz = i\int e^{-z}(1 - z)\,dz$$

$$= i\left[(1 - z)(-e^{-z}) - \int -e^{-z}(-1)\,dz\right] = i[(1 - z)(-e^{-z}) + e^{-z}]$$

$$= ie^{-z}z + c.$$

Example 4 : Find the analytic function whose imaginary part is $\tan^{-1}\dfrac{y}{x}$. **(M.U. 2019)**

Sol. : We have $v = \tan^{-1}\dfrac{y}{x}$.

$$\therefore \quad v_y = \psi_1(x, y) = \frac{1}{1+(y^2/x^2)} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$

$$v_x = \psi_2(x, y) = \frac{1}{1+(y^2/x^2)}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

$$\therefore \quad \psi_1(z, 0) = \frac{z}{z^2} = \frac{1}{z}; \quad \psi_2(z, 0) = 0$$

By Milne-Thompson method

$$f'(z) = \Psi_1(z,\ 0) + i\,\Psi_2(z,\ 0) = \frac{1}{z} \qquad \therefore \quad f(z) = \int \frac{1}{z}\,dz = \log z + c.$$

Example 5 : If the imaginary part of the analytic function $w = f(z)$ is $v = x^2 - y^2 + \dfrac{x}{x^2+y^2}$, show that the real part $u = -2xy + \dfrac{y}{x^2+y^2} + c$. **(M.U. 2007, 08, 13, 14)**

Sol. : We have $v = x^2 - y^2 + \dfrac{x}{x^2+y^2}$

$$\therefore \quad v_y = \psi_1(x, y) = -2y - \frac{2xy}{(x^2+y^2)^2}$$

$$v_x = \psi_2(x, y) = 2x - \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\psi_1(z, 0) = 0, \quad \psi_2(z, 0) = 2z - \frac{1}{z^2}$$

By Milne-Thompson Method

$$f'(z) = \psi_1(z,\ 0) + i\,\psi_2(z,\ 0)$$

$$\therefore \quad f'(z) = i\left(2z - \frac{1}{z^2}\right) = i\int\left(2z - \frac{1}{z^2}\right)dz = i\left(z^2 + \frac{1}{z}\right)$$

$$= i(x^2 + 2i\,xy - y^2) + i\,\frac{(x-iy)}{x^2+y^2}$$

$$= \left(-2xy + \frac{y}{x^2+y^2}\right) + i\left(x^2 - y^2 + \frac{x}{x^2+y^2}\right) + c$$

$$\therefore \quad u = -2xy + \frac{y}{x^2+y^2} + c.$$

Type III : To find the analytic function when $u + v$ or $u - v$ is given

Procedure to find $f(z)$ when $u + v$ is given

1. Let $V = u + v$ (given).
2. Find $\dfrac{\partial V}{\partial x} = \Psi_1(x, y)$ and $\dfrac{\partial V}{\partial y} = Y_2(x, y)$.

3. Find $\Psi'_2(z, 0)$ and $\Psi'_1(z, 0)$.
4. Then $(1+i)f'(z) = \Psi'_2(z, 0) + i\Psi'_1(z, 0)$.
5. By integrating both sides w.r.t. z, we get $f(z)$.

Example 1 : If $f(z) = u + iv$ is analytic and $u + v = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$, find $f(z)$.

(M.U. 1998, 2003, 04, 06, 07)

Sol. : $f(z) = u + iv \quad \therefore\ i f(z) = iu - v$

$$\therefore\ (1+i)f(z) = (u-v) + i(u+v) = U + iV \text{ say.}$$

$$\therefore\ (1+i)f'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial y} = \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x} \qquad [\because\ U_x = V_y]$$

(When $u + v$ is given we use $\dfrac{\partial V}{\partial y} + i\dfrac{\partial V}{\partial x}$)

But $\quad V = u + v = \dfrac{\sin 2x}{\{(e^{2y} + e^{-2y})/2\} - \cos 2x} = \dfrac{\sin 2x}{\cos h\, 2y - \cos 2x}$

$$\therefore\ \frac{\partial V}{\partial x} = \frac{\partial}{\partial x}(u+v) = \frac{(\cos h\, 2y - \cos 2x)(2\cos 2x) - 2\sin 2x \sin 2x}{(\cos h\, 2y - \cos 2x)^2}$$

$$= [2\cos h\, 2y \cos 2x - 2]/[\cos h\, 2y - \cos 2x]^2 = \psi_1(x, y)$$

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}(u+v) = -\frac{2\sin h\, y \sin 2x}{[\cos h\, 2y - \cos 2x]^2} = \psi_2(x, y)$$

By Milne-Thompson method

$$\therefore\ (1+i)f'(z) = \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x} = \psi_2(z, 0) + i\,\psi_1(z, 0)$$

$$\therefore\ (1+i)f(z) = \int 0 + i\frac{2(\cos 2z - 1)}{(1 - \cos 2z)^2} \cdot dz = -2i\int \frac{dz}{1 - \cos 2z} = -2i\int \frac{dz}{2\sin^2 z}$$

$$\therefore\ (1+i)f(z) = -i\int \text{cosec}^2 z\, dz = i\cot z + c'$$

$$\therefore\ f(z) = \frac{i}{1+i} \cdot \cot z + c.$$

Example 2 : Find an analytic function $f(z) = u + iv$ where $u + v = e^x(\cos y + \sin y)$.

(M.U. 2004, 05, 15)

Sol. : We have $f(z) = u + iv \ \therefore\ i f(z) = iu - v$

$$\therefore\ (1+i)f(z) = (u-v) + i(u+v) = U + iV, \text{ say}$$

$$\therefore\ \frac{\partial V}{\partial x} = \frac{\partial}{\partial x}(u+v) = e^x(\cos y + \sin y) = \psi_1(x, y)$$

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}(u+v) = e^x(-\sin y + \cos y) = \psi_2(x, y)$$

By Milne-Thompson method

$$(1+i)f'(z) = \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x} = \psi_2(z, 0) + i\,\psi_1(z, 0)$$

$$= e^z(0 + 1) + i e^z(1 + 0) = (1+i)e^z$$

$$\therefore\ f'(z) = e^z \qquad \therefore\ f(z) = \int e^z dz = e^z + c.$$

Procedure to find $f(z)$ when $u - v$ is given

1. Let $U = u - v$ (given).
2. Find $\frac{\partial U}{\partial x} = \Phi_1(x, y)$ and $\frac{\partial U}{\partial y} = \Phi_2(x, y)$.
3. Find $\Phi_1(z, 0)$ and $\Phi_2(z, 0)$.
4. Then $(1 + i) f'(z) = \Phi_1(z, 0) - i\,\Phi_2(z, 0)$.
5. Integrating both sides w.r.t. z, we get $f(z)$.

Example 3 : Find the analytic function $f(z) = u + iv$ such that

$$u - v = \frac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}} \text{ when } f\left(\frac{\pi}{2}\right) = 0.$$ **(M.U. 2003, 06, 07, 09, 13, 19)**

Sol. : We have $f(z) = u + iv \quad \therefore\ i f(z) = iu - v$ .... (A)

$\therefore (1 + i) f(z) = (u - v) + i(u + v) = U + iV$, say.

$$\therefore \frac{\partial U}{\partial x} = \frac{\partial}{\partial x}(u - v) = \frac{(2\cos x - e^{y} - e^{-y})(-\sin x + \cos x) - (\cos x + \sin x - e^{-y})(-2\sin x)}{(2\cos x - e^{y} - e^{-y})^2}$$

$$= \frac{2 + e^{y}\sin x - e^{-y}\sin x - e^{y}\cos x - e^{-y}\cos x}{(2\cos x - e^{y} - e^{-y})^2}$$

$$= \Phi_1(x, y)$$

And
$$\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}(u - v) = \frac{(2\cos x - e^{y} - e^{-y})(e^{-y}) - (\cos x + \sin x - e^{-y})(-e^{y} + e^{-y})}{(2\cos x - e^{y} - e^{-y})^2}$$

$$\frac{\partial U}{\partial y} = \frac{\cos x \cdot e^{-y} + \cos x\, e^{y} + \sin x\, e^{y} - \sin x \cdot e^{-y} - 2}{(2\cos x - e^{y} - e^{-y})^2} = \Phi_2(x, y)$$

(When $u - v$ is given we used $\frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$.)

From (A), $$(1 + i) f'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} \qquad [\because\ U_y = -V_x]$$

$$(1 + i) f'(z) = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} = \Phi_1(z, 0) - i\,\Phi_2(z, 0)$$

$$= \frac{2 + \sin z - \sin z - \cos z - \cos z}{(2\cos z - 2)^2} - i \cdot \frac{\cos z + \cos z + \sin z - \sin z - 2}{(2\cos z - 2)^2}$$

$$= \frac{2(1 - \cos z)}{4(\cos z - 1)^2} - i\frac{2(\cos z - 1)}{4(\cos z - 1)^2} = -\frac{1}{2(\cos z - 1)} - i \cdot \frac{1}{2(\cos z - 1)}$$

$$= (1 + i) \cdot \frac{1}{2(1 - \cos z)} = (1 + i) \cdot \frac{1}{4\sin^2(z/2)} = \frac{(1 + i)}{4} \cdot \operatorname{cosec}^2\frac{z}{2}.$$

$$\therefore f'(z) = \frac{1}{4}\operatorname{cosec}^2\frac{z}{2}$$

$$\therefore f(z) = \frac{1}{4}\int \operatorname{cosec}^2(z/2)\, dz = -\frac{1}{2}\cot\frac{z}{2} + c$$

But when $z = \frac{\pi}{2}$, $f(z) = 0$ $\quad \therefore \; 0 = -\frac{1}{2} + c \quad \therefore \; c = \frac{1}{2}$

$$\therefore \; f(z) = \frac{1}{2}\left(1 - \cot\frac{z}{2}\right).$$

Example 4 : Find the analytic function $f(z) = u + iv$ in terms of z if $u - v = (x-y)(x^2 + 4xy + y^2)$
(M.U. 2003, 04, 07, 08, 12, 15)

Sol. : We have $\quad f(z) = u + iv$

$$\therefore \; i\,f(z) = iu - v$$

$$\therefore \; (1+i)\,f(z) = (u-v) + i\,(u+v) = U + iV \text{ say}$$

We have

$$\begin{aligned} U &= u - v \\ &= (x-y)(x^2 + 4xy + y^2) \\ &= x^3 + 4x^2y + xy^2 - x^2y - 4xy^2 - y^3 \\ &= x^3 + 3x^2y - 3xy^2 - y^3 \end{aligned}$$

$$\therefore \; \frac{\partial U}{\partial x} = \frac{\partial(u-v)}{\partial x} = 3x^2 + 6xy - 3y^2 = \Phi_1(x,y)$$

$$\frac{\partial U}{\partial y} = \frac{\partial(u-v)}{\partial y} = 3x^2 - 6xy - 3y^2 = \Phi_2(x,y)$$

$$\therefore \; \Phi_1(z,0) = 3z^2 \text{ and } \Phi_2(z,0) = 3z^2.$$

$$\begin{aligned} \therefore \; (1+i)\,f'(z) &= \Phi_1(z,0) - i\,\Phi_2(z,0) \\ &= 3z^2 - i\cdot 3z^2 \\ &= -3\,i^2 z^2 - i\cdot 3z^2 \qquad [\because \; i^2 = -1] \end{aligned}$$

By integration,

$$\begin{aligned} (1+i)\,f(z) &= -3i^2 \int z^2\,dz - 3i\int z^2\,dz \\ &= -i^2 z^3 - i\,z^3 = -i\,z^3\,(1+i) \end{aligned}$$

$$\therefore \; f(z) = -i\,z^3 + c.$$

Example 5 : If $f(z) = u + iv$ is analytic and $u - v = e^x(\cos y - \sin y)$, find $f(z)$ in terms of z.
(M.U. 2003, 05, 14, 19)

Sol. : $\because \; f(z) = u + iv, \quad i\,f(z) = iu - v.$

$$\therefore \; (1+i)f(z) = (u-v) + i\,(u+v) = U + iV \text{ say.}$$

Now $\quad \frac{\partial U}{\partial x} = \frac{\partial}{\partial x}(u-v) = e^x(\cos y - \sin y) = \Phi_1(x,y)$

$$\frac{\partial U}{\partial y} = e^x(-\sin y - \cos y) = \Phi_2(x,y)$$

$$\therefore \; (1+i)f'(z) = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} = \Phi_1(z,0) - i\,\Phi_2(z,0)$$

$$\therefore \; (1+i)f(z) = \int [e^z + i\,e^z]\,dz = (1+i)\int e^z\,dz = (1+i)\,e^z + c'$$

$$\therefore \; f(z) = e^z + c.$$

Example 6 : Find the analytic function $f(z) = u + iv$ if $3u + 2v = y^2 - x^2 + 16xy$.

(M.U. 2002, 08, 13)

Sol. : Differentiating the given relation w.r.t. x and y

$$3\frac{\partial u}{\partial x} + 2\frac{\partial v}{\partial x} = -2x + 16y \qquad \text{.................... (i)}$$

and $$3\frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial y} = 2y + 16x \qquad \text{.................... (ii)}$$

But $u_x = v_y$ and $u_y = -v_x$. Hence, from (ii), we get,

$$-3\frac{\partial v}{\partial x} + 2\frac{\partial u}{\partial x} = 2y + 16x \qquad \text{.................... (iii)}$$

Now, multiply (i) by (3) and (iii) by (2) and add.

$$\therefore\ 13\frac{\partial u}{\partial x} = 26x + 52y \quad i.e. \quad \frac{\partial u}{\partial x} = 2x + 4y = \Phi_1(x, y)$$

Again, multiply (i) by (-2) and (iii) by (3) and add.

$$\therefore\ -13\frac{\partial v}{\partial x} = 52x - 26y \quad i.e. \quad \frac{\partial v}{\partial x} = -4x + 2y = \Phi_2(x, y)$$

But $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \Phi_1(x, y) + i\,\Phi_2(x, y) = \Phi_1(z, 0) + i\,\Phi_2(z, 0)$$

$$\therefore\ f'(z) = 2z - i \cdot 4z$$

$$\therefore\ f(z) = \int 2z\,dz - 2i\int 2z\,dz = z^2 - 2i\,z^2 + c = (1 - 2i)\,z^2 + c$$

Example 7 : State true or false with proper justification "There does not exist an analytic function whose real part is $x^3 - 3x^2 y - y^3$". **(M.U. 1995, 2004, 14)**

Sol. : We shall use the theorem (page 4-26) to check whether $u = x^3 - 3x^3 - y^3$ is a real part of some analytic function. By the result referred to above $u = x^3 - 3x^2 y - y^3$ must satisfy Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ if it is a real part of some analytic function.

Now $$\frac{\partial u}{\partial x} = 3x^2 - 6xy, \quad \frac{\partial^2 u}{\partial x^2} = 6x - 6y; \qquad \frac{\partial u}{\partial y} = -3x^2 - 3y^2, \quad \frac{\partial^2 u}{\partial y^2} = -6y$$

$$\therefore\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 12y \neq 0$$

$\therefore$ There does not exist an analytic function whose real part is $u = x^3 - 3x^2 y - y^3$.

Type IV : To find the analytic function whose real part u is given in polar form

Example 1 : State Laplace's equation in polar form and verify it for $u = r^2 \cos 2\theta$ and also find v and $f(z)$.

Sol. : Laplace's equation in polar form is $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

$$\because\ u = r^2 \cos 2\theta$$

$$\therefore\ \frac{\partial u}{\partial r} = 2r\cos 2\theta, \quad \frac{\partial^2 u}{\partial r^2} = 2\cos 2\theta; \qquad \frac{\partial u}{\partial \theta} = -2r^2 \sin 2\theta, \quad \frac{\partial^2 u}{\partial \theta^2} = -4r^2\cos 2\theta$$

$$\therefore \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 2\cos 2\theta + \frac{1}{r}(2r\cos 2\theta) + \frac{1}{r^2}(-4r^2\cos 2\theta)$$

$$= 4\cos 2\theta - 4\cos 2\theta = 0$$

$\therefore$ Laplace's equation is satisfied.

By Cauchy-Riemann equations in polar form $u_r = \frac{1}{r}v_\theta \quad \therefore \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$.

$$\therefore \quad \frac{\partial v}{\partial \theta} = r(2r\cos 2\theta) = 2r^2\cos 2\theta$$

Integrating w.r.t. θ, $\quad v = r^2 \sin 2\theta + c$

Hence, $f(z) = u + iv$

$$\therefore \quad f(z) = r^2\cos 2\theta + i\,r^2 \sin 2\theta + c = r^2(\cos 2\theta + i\sin 2\theta) + c$$

$$= r^2 e^{i2\theta} = (r e^{i\theta})^2 + c = z^2 + c$$

Example 2 : Verify Laplace's equation for $u = \left(r + \frac{a^2}{r}\right)\cos\theta$. Also find v and $f(z)$.

(M.U. 2004, 14)

Sol. : $\because \quad u = \left(r + \frac{a^2}{r}\right)\cos\theta$

$$\therefore \quad \frac{\partial u}{\partial r} = \left(1 - \frac{a^2}{r^2}\right)\cos\theta, \quad \frac{\partial^2 u}{\partial r^2} = \frac{2a^2}{r^3}\cos\theta$$

$$\frac{\partial u}{\partial \theta} = -\left(r + \frac{a^2}{r}\right)\sin\theta, \quad \frac{\partial^2 u}{\partial \theta^2} = -\left(r + \frac{a^2}{r}\right)\cos\theta$$

$$\therefore \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{2a^2}{r^3}\cos\theta + \frac{1}{r}\cdot\left(1 - \frac{a^2}{r^2}\right)\cos\theta - \frac{1}{r^2}\left(r + \frac{a^2}{r}\right)\cos\theta$$

$$= 0.$$

$\therefore$ Laplace's equation is satisfied.

By Cuachy - Riemann equations in polar form $u_r = \frac{1}{r}v_\theta \quad \therefore \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$.

$$\therefore \quad \left(1 - \frac{a^2}{r^2}\right)\cos\theta = \frac{1}{r}\cdot\frac{\partial v}{\partial \theta} \qquad \therefore \quad \frac{\partial v}{\partial \theta} = \left(r - \frac{a^2}{r}\right)\cos\theta$$

Integrating w.r.t. θ, $\quad v = \left(r - \frac{a^2}{r}\right)\sin\theta + c$.

Hence, $f(z) = u + iv = \left(r + \frac{a^2}{r}\right)\cos\theta + i\left(r - \frac{a^2}{r}\right)\sin\theta + c$

$$= r(\cos\theta + i\sin\theta) + \frac{a^2}{r}(\cos\theta - i\sin\theta) + c = z + \frac{a^2}{z} + c.$$

Alternatively we can express u in terms of x and y and use cartesian form of Laplace's equation. However, it may be noted that this method is rather tedius.

Example 3 : If $u = k(1 + \cos\theta)$, find v so that $u + iv$ is analytical. **(M.U. 2007, 17, 18)**

Sol. : Since, $u = k + k\cos\theta$, $\dfrac{\partial u}{\partial r} = 0$ and $\dfrac{\partial u}{\partial \theta} = -k\sin\theta$.

But by C-R equations in polar coordinates.

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \qquad \therefore \frac{\partial v}{\partial \theta} = 0, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}(-k\sin\theta)$$

Integrating the first equation partially w.r.t θ,

$v = f(r)$ where $f(r)$ is an arbitrary function.

(If $v = f(r)$ then $\dfrac{\partial v}{\partial \theta} = 0$)

$$\therefore \frac{\partial v}{\partial r} = f'(r) = \frac{k\sin\theta}{r} \qquad \therefore v = k\sin\theta\log r + c$$

Hence, the analytic function is

$$f(z) = u + iv = k(1 + \cos\theta) + ik\sin\theta\log r + c$$

Example 4 : If $u = -r^3\sin 3\theta$, find the analytic function $f(z)$ whose real part is u. **(M.U. 2004, 05, 16)**

Sol. : We have $\dfrac{\partial u}{\partial r} = -3r^2\sin\theta$ and $\dfrac{\partial u}{\partial \theta} = -3r^3\cos 3\theta$

By Cauchy-Riemann equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \qquad \therefore \frac{\partial v}{\partial \theta} = -3r^3\sin 3\theta.$$

Integrating w.r.t. θ, $\quad v = r^3\cos 3\theta$.

$$\therefore f(z) = u + iv = -r^3\sin 3\theta + i\,r^3\cos 3\theta$$
$$= i\,r^3(\cos 3\theta + i\sin 3\theta) = i\,r^3 e^{i3\theta} = i\,z^3 + c.$$

Example 5 : Show that $u = r - \dfrac{a^2}{r}\sin\theta$ cannot be the real part of an analytic function $f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$. **(M.U. 2003, 04)**

Sol. : Left to you. Show that the Cauchy equation in polar form, page 4-23 is not satisfied by u.

EXERCISE - III

1. Prove that $u(x, y) = x^2 - y^2$ and $v(x, y) = -y/(x^2 + y^2)$ are both harmonic functions but $u + iv$ is not analytic. **(M.U. 2003)**
2. Show that there does not exist an analytic function whose real part is
   (i) $3x^2 + \sin x + y^2 + 5y + 4$ **(M.U. 2002)** **(ii)** $3x^2 - 2x^2y + y^2$ **(M.U. 2003)**
3. Show that the following functions are harmonic
   (i) $e^x\cos y + x^3 - 3xy^2$ **(M.U. 2003)** **(ii)** $e^{2x}(x\cos 2y - y\sin 2y)$ **(M.U. 2003)**
   (iii) $\log\sqrt{x^2 + y^2}$ **(M.U. 2003)**

4. Find the analytic function whose real part is

1. $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ **(M.U. 2011, 13)**
2. $u = (x-1)^3 - 3xy^2 + 3y^2$
3. $u = x^2 - y^2 - 5x + y + 2$
4. $u = \log\sqrt{x^2 + y^2}$
5. $u = \sin x \cos h\, y$
6. $u = e^{2x}(x\cos 2y - y\sin 2y)$
7. $u = e^x(x\cos y - y\sin y)$ **(M.U. 2014)**
8. $u = e^{-x}(x\sin y - y\cos y)$
9. $u = \dfrac{\sin 2x}{\cos h\, 2y - \cos 2x}$ **(M.U. 1995)**
10. $u = \dfrac{x}{2}\log(x^2 + y^2) - y\tan^{-1}\left(\dfrac{y}{x}\right) + \sin x \cos hy$ **(M.U. 2003, 08)**

[**Ans. :** **(1)** $z^3 + 3z^2 + c$, **(2)** $(z-1)^3 + c$, **(3)** $z^2 - 5z - iz + c$, **(4)** $\log z + c$, **(5)** $\sin z + c$, **(6)** $z e^{2z} + c$, **(7)** $z e^z + c$, **(8)** $ize^{-z} + c$, **(9)** $\cot z + c$, **(10)** $z\log z + \sin z + c$.]

5. Find the analytic function whose imaginary part is

1. $v = \log(x^2 + y^2) + x - 2y$
2. $v = \dfrac{x - y}{x^2 + y^2}$
3. $v = \cos x \cos h\, y$
4. $v = \sin h\, x \cos y$
5. $v = e^x(x\sin y + y\cos y)$ **(M.U. 2005)**
6. $v = e^{-x}(x\sin y - y\cos y)$
7. $v = \dfrac{\sin h\, 2y}{\cos 2x + \cos h\, 2y}$
8. $v = \sin h\, x \sin y$
9. $v = e^{-x}\left[2xy\cos y + (y^2 - x^2)\sin y\right]$ **(M.U. 2003)**
10. $\dfrac{x}{x^2 + y^2} + \cos hx \cos y$ **(M.U. 2002, 09, 14)**
11. $v = \dfrac{y}{x^2 + y^2}$

[**Ans. :** **(1)** $(i-2)\,z + i\log z + c$, **(2)** $(1+i)\dfrac{1}{z}$, **(3)** $i\cos z + 1$, **(4)** $i\sin hz$, **(5)** $z e^z + c$, **(6)** $-z e^{-z} + c$, **(7)** $\tan z + c$, **(8)** $\sin hz + c$, **(9)** $e^{-z} \cdot z^2 + c$, **(10)** $i\left(\dfrac{1}{z} + \cos hz\right) + c$, **(11)** $-\dfrac{1}{z} + c$.]

6. Find the analytic function $f(z) = u + iv$ in terms of z if

(i) $u - v = (x - y)(x^2 + 4xy + y^2)$ **(M.U. 2003, 04, 07, 08, 12)**

(ii) $u + v = \dfrac{x}{x^2 + y^2}$ **(M.U. 2003, 15)**

(iii) $u - v = \dfrac{\sin x + \sin h\, y}{\cos h\, y - \cos x}$

(iv) $u + v = \dfrac{\sin 2x}{\cos h\, 2y - \cos 2x}$

(v) $u - v = \dfrac{e^y - \cos x + \sin x}{\cos h\, y - \cos x}$ and $f(\pi/2) = 0$ **(M.U. 1996)**

(vi) $u - v = x^3 + x^2 - 3xy^2 - y^2 - 3x^2y + y^3 - 2xy$

(vii) $u + v = e^x(\cos y + \sin y) + \dfrac{x - y}{x^2 + y^2}$ **(M.U. 2005)**

[Ans. : (i) $f(z) = -iz^3 + c$, (ii) $\frac{1}{1+i}\left(\frac{i}{z}\right) + c$, (iii) $\cot\frac{z}{2} + c$,
(iv) $\frac{(1+i)}{2}\cot z + c$, (v) $\cot\left(\frac{z}{2}\right) - 1$, (vi) $z^3 + z^2 + c$, (vii) $e^z + \frac{1}{z} + c$.]

7. Find the analytic function $f(z)$ whose real part is
(i) $r^2\cos 2\theta - r\sin\theta + 2$ **(M.U. 2014)** (ii) $r^n \cos n\theta$ **[Ans. : (i)** $z^2 + z + c$, **(ii)** $z^n + c$.]

13. To Find an Analytic Function when Harmonic Function is Given

Example 1 : Show that $u = y^3 - 3x^2y$ is a harmonic function. Find its harmonic conjugate and the corresponding analytic function. **(M.U. 2003)**

Sol. : Method 1 : Since, $u = y^3 - 3x^2y$

$$u_x = -6xy,\ u_{xx} = -6y \ ; \quad u_y = 3y^2 - 3x^2,\ u_{yy} = 6y$$

$$\therefore \quad \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6y + 6y = 0$$

$\therefore$ $u = x^3 - 3x^2y$ is a harmonic function.

Now by the note 1 after § 12, page 4-29.

$$\int -\frac{\partial u}{\partial y}\,dx = \int -(3y^2 - 3x^2)\,dx = -3xy^2 + x^3$$

and $\int(\text{terms in } \frac{\partial u}{\partial x} \text{ free from } x)\,dy = \int 0\,dy = 0$

$\therefore$ $v = x^3 - 3xy^2 + c$.

$\therefore$ $f(z) = u + iv = (y^3 - 3x^2y) + i(x^3 - 3xy^2) + c$.

Putting $x = z,\ y = 0$ $\quad\therefore\quad f(z) = z^3 i + c$

Method 2 : Since, $u = y^3 - 3x^2y$ by Milne-Thompson method given on page 4-29.

$$u_x = \Phi_1 = -6xy, \quad u_y = \Phi_2 = 3y^2 - 3x^2$$

$$\therefore \quad f'(z) = \Phi_1(z, 0) - i\,\Phi_2(z, 0) = 0 + 3iz^2$$

$\therefore$ $f(z) = \int 3iz^2\,dz = iz^3 + c$ as above is the required analytic function.

Now, $f(z) = i(x + iy)^3 = i(x^3 + 3ix^2y - 3xy^2 - iy^3)$

$\therefore$ $u + iv = -3x^2y + y^3 + i(x^3 - 3xy^2)$

$\therefore$ $v = x^3 - 3xy^2$ is the harmonic conjugate.

Note ....

This method is more convenient.

Method 3 : $u = y^3 - 3x^2y$

$$\therefore \quad u\left(\frac{z}{2}, \frac{z}{2i}\right) = \frac{z^3}{8i^3} - 3\left(\frac{z}{2}\right)^2\left(\frac{z}{2i}\right) = -\frac{z^3}{8i} - \frac{3z^3}{8i} = -\frac{z^3}{2i} = \frac{z^3}{2}i$$

$$u(0, 0) = 0$$

$$\therefore \quad f(x) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + ci = z^3 i + ci\,.$$

Example 2 : Show that the function $u = \sin x \cos hy + 2 \cos x \sin hy + x^2 - y^2 + 4xy$ satisfies Laplace's equation and find it corresponding analytic function $f(z) = u + iv$.

(M.U. 1998, 2005, 09, 13)

Sol. : We have $\dfrac{\partial u}{\partial x} = \cos x \cos hy - 2 \sin x \sin hy + 2x + 4y$

$$\frac{\partial^2 u}{\partial x^2} = -\sin x \cos hy - 2 \cos x \sin hy + 2$$

$$\frac{\partial u}{\partial y} = \sin x \sin hy + 2 \cos x \cos hy - 2y + 4x$$

$$\frac{\partial^2 u}{\partial y^2} = \sin x \cos hy + 2 \cos x \sin hy - 2$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence, u satisfies Laplace's equation.

Now $u_x = \Phi_1(x, y) = \cos x \cos hy - 2 \sin x \sin hy + 2x + 4y$

$$\Phi_1(z, 0) = \cos z + 2z$$

$$u_y = \Phi_2(x, y) = \sin x \sin hy + 2 \cos x \cos hy - 2y + 4x$$

$$\Phi_2(z, 0) = 2 \cos z + 4z$$

Now we use Milne-Thompson Method given on page 4-29.

(Read the procedure given on page 4-31)

$$\therefore \quad f'(z) = \Phi_1(z, 0) - i\,\Phi_2(z, 0) = (\cos z + 2z) - i\,(2 \cos z + 4z)$$

$$\therefore \quad f(z) = \int [(\cos z + 2z) - i\,(2 \cos z + 4z)]\, dz = \sin z + z^2 - i\,(2 \sin z + 2z^2) + c$$

which is the required analytic function.

Example 3 : If $v = e^x \sin y$, prove that v is a harmonic function. Also find the corresponding harmonic conjugate function and analytic function.

(M.U. 2011, 18)

Sol. : We have $\dfrac{\partial v}{\partial x} = e^x \sin y, \quad \dfrac{\partial^2 v}{\partial x^2} = e^x \sin y\,; \quad \dfrac{\partial v}{\partial y} = e^x \cos y, \quad \dfrac{\partial^2 v}{\partial y^2} = -e^x \sin y.$

$$\therefore \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0\,. \qquad \therefore \; v \text{ satisfies Laplace's equation.}$$

Now, we use, Milne-Thompson Method given on page 4-29.

(Read the procedure given on page 4-34)

$v_x = e^x \sin y \quad \therefore \; \psi_2(z,0) = 0; \qquad v_y = e^x \cos y \quad \therefore \; \psi_1(z,0) = e^z$

$\therefore \; f'(z) = \psi_1(z,0) + i\,\psi_2(z,0) = e^z + 0$

$\therefore \; f(z) = e^z + c$

Now, $f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$

$\therefore \; u = e^x \cos y$

Example 4 : Show that the function $v = e^x\,(x \sin y + y \cos y)$ satisfies Laplace equation and find its corresponding analytic function and its harmonic conjugate. **(M.U. 2013)**

Sol. : We have

$$\frac{\partial v}{\partial x} = e^x\,(x \sin y + y \cos y) + e^x \sin y$$

$$\therefore \; \frac{\partial^2 v}{\partial x^2} = e^x\,(x \sin y + y \cos y) + e^x\,(\sin y) + e^x \sin y$$

$$\frac{\partial v}{\partial y} = e^x\,(x \cos y + \cos y - y \sin y)$$

$$\therefore \; \frac{\partial^2 v}{\partial y^2} = e^x\,(-x \sin y - \sin y - \sin y - y \cos y)$$

$$\therefore \; \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \qquad \therefore \; v \text{ satisfies Laplace equation.}$$

Now, $v_x = e^x\,(x \sin y + y \cos y + \sin y)$

$\therefore \; \Psi_2\,(z, 0) = 0$

$v_y = e^x\,(x \cos y + \cos y - y \sin y)$

$\therefore \; \Psi_1\,(z, 0) = e^z\,(z + 1)$

$\therefore \; f'(z) = \Psi_1(z, 0) + i\,\Psi_2(z, 0) = e^z\,(z + 1) + 0$

$\therefore \; f(z) = \int e^z\,(z + 1)\,dz = z\,e^z$ which is the required analytic function.

Now, $f(z) = (x + iy)\,(e^{x + iy}) = (x + iy)\,(e^x \cdot e^{iy})$

$= (x + iy)\,e^x\,(\cos y + i \sin y)$

$\therefore \; u = e^x\,(x \cos y - y \sin y)$ which is the required harmonic conjugate.

Example 5 : If $v = 3x^2y + 6xy - y^3$, show that v is harmonic and find the corresponding analytic function. **(M.U. 2003, 07, 13, 14, 19)**

Sol. : We have

$$\frac{\partial v}{\partial x} = 6xy + 6y, \quad \frac{\partial^2 v}{\partial x^2} = 6y; \qquad \frac{\partial v}{\partial y} = 3x^2 + 6x - 3y^2, \quad \frac{\partial^2 v}{\partial y^2} = -6y$$

$$\therefore \; \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 6y - 6y = 0. \qquad \therefore \; v \text{ is satisfies Laplace's equation.}$$

Now we use Milne-Thompson Method given on page 4-29.

$\therefore \; v_x = 6xy + 6y, \; \psi_2(z,0) = 0$

$v_y = 3x^2 + 6x - 3y^2, \; \psi_1(z,0) = 3z^2 + 6z$

$$\therefore \quad f'(z) = \psi_1(z,0) + i\,\psi_2(z,0) = (3z^2 + 6z) + 0$$

$$\therefore \quad f(z) = \int (3z^2 + 6z)\,dz = (z^3 + 3z^2) + c$$

Example 6 : Show that $u = \cos x \cos hy$ is a harmonic function. Find its harmonic conjugat and corresponding analytic function. **(M.U. 2005, 19**

Sol. : Since, $u = \cos x \cos hy$

$$\frac{\partial u}{\partial x} = -\sin x \cos hy, \quad \frac{\partial^2 u}{\partial x^2} = -\cos x \cos hy$$

$$\frac{\partial u}{\partial y} = \cos x \sin hy, \quad \frac{\partial^2 u}{\partial y^2} = \cos x \cos hy$$

$$\therefore \quad \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \qquad \therefore \quad u = \cos x \cos hy \text{ is a harmonic functions.}$$

Now, we use Milne-Thompson Method.

Now, $u_x = \Phi_1(x, y) = -\sin x \cos hy \qquad \therefore \quad \Phi_1(z, 0) = -\sin z$

$u_y = \Phi_2(x, y) = \cos x \sin hy \qquad \therefore \quad \Phi_2(z, 0) = 0$

$$\therefore \quad f'(z) = \Phi_1(z, 0) - i\,\Phi_2(z, 0) = -\sin z$$

$\therefore \quad f(z) = \int -\sin z\,dz = \cos z + c$ is the required analytic function.

Now, $f(z) = \cos(x + iy) = \cos x \cos iy - \sin x \sin iy$

$$\therefore \quad u + iv = \cos x \cos hy - i \sin x \sin hy$$

$\therefore \quad v = -\sin x \sin hy$ is the required harmonic conjugate.

Example 7 : Prove that $u = x^2 - y^2,\ v = -\dfrac{y}{x^2 + y^2}$ both u and v satisfy Laplace's equation, but that $u + iv$ is not an analytic function of z. **(M.U. 1996, 2003, 09)**

Sol. : $u_x = 2x,\ u_{xx} = 2\ ;\ u_y = -2y,\ u_{yy} = -2$

$$v_x = \frac{2xy}{(x^2 + y^2)^2}$$

$$\therefore \quad v_{xx} = 2y\left[\frac{(x^2 + y^2)^2 \cdot 1 - x \cdot 2(x^2 + y^2) \cdot 2x}{(x^2 + y^2)^4}\right]$$

$$= \frac{2y(x^2 + y^2)[x^2 + y^2 - 4x^2]}{(x^2 + y^2)^4} = 2y\frac{(y^2 - 3x^2)}{(x^2 + y^2)^3}$$

$$v_y = -\left[\frac{(x^2 + y^2) \cdot 1 - y \cdot 2y}{(x^2 + y^2)^2}\right] = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$v_{yy} = \frac{(x^2 + y^2)^2 \cdot 2y - (y^2 - x^2) \cdot 2(x^2 + y^2) \cdot 2y}{(x^2 + y^2)^4}$$

$$= 2y(x^2 + y^2)\frac{[x^2 + y^2 - 2y^2 + 2x^2]}{(x^2 + y^2)^4} = 2y\frac{(3x^2 - y^2)}{(x^2 + y^2)^3} = -2y\frac{(y^2 - 3x^2)}{(x^2 + y^2)^3}$$

$$\therefore \quad u_{xx} + u_{yy} = 0 \text{ and } v_{xx} + v_{yy} = 0$$

Hence, u, v satisfy Laplace's Equations.

But Cauchy-Riemann equations are not satisfied as $u_x \neq v_y$ and $u_y \neq -v_x$.

Hence, $u + iv$ is not analytic.

Example 8 : Show that the following function satisfies Laplace's equation and find its corresponding analytic function and the harmonic conjugate, $u = \frac{1}{2}\log(x^2 + y^2)$.

(M.U. 2002, 05, 06, 08, 10)

Sol. : We have $\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}$ $\quad \therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{(x^2 + y^2) \cdot 1 - x \cdot 2x}{(x^2 + y^2)^2} = \frac{-x^2 + y^2}{(x^2 + y^2)^2}$

Similarly, $\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$ $\quad \therefore \quad \frac{\partial^2 u}{\partial y^2} = \frac{(x^2 + y^2) \cdot 1 - y \cdot 2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{-x^2 + y^2 + x^2 - y^2}{(x^2 + y^2)^2} = 0$$

$\therefore$ u satisfies Laplace's equation.

Now $u_x = \Phi_1(x, y) = \frac{x}{x^2 + y^2}$ $\quad \therefore \quad \Phi_1(z, 0) = \frac{z}{z^2 + 0} = \frac{1}{z}$

$u_y = \Phi_2(x, y) = \frac{y}{x^2 + y^2}$ $\quad \therefore \quad \Phi_2(z, 0) = 0$

By Milne-Thompson method,

$$\therefore \quad f'(z) = \Phi_1(z, 0) - i\,\Phi_2(z, 0) = \frac{1}{z} - i\,0 = \frac{1}{z}$$

$$\therefore \quad f(z) = \int \frac{1}{z}\,dz = \log z + c = \log(x + iy) + c$$

$$\therefore \quad u + iv = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\frac{y}{x} + c$$ **[Note this]**

(See Applied Mathematics I by the same author.)

$\therefore \quad v = \tan^{-1}\frac{y}{x} + c$ is the corresponding harmonic conjugate.

Example 9 : Prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}$.

Sol. : We have $z = x + iy$, $\bar{z} = x - iy$

$$\therefore \quad x = \frac{1}{2}(z + \bar{z}), \quad y = \frac{1}{2i}(z - \bar{z})$$

Treating z and $\bar{z}$ as independent variables

$$\therefore \quad \frac{\partial x}{\partial z} = \frac{1}{2}, \quad \frac{\partial x}{\partial \bar{z}} = \frac{1}{2}, \quad \frac{\partial y}{\partial z} = \frac{1}{2i}, \quad \frac{\partial y}{\partial \bar{z}} = -\frac{1}{2i}.$$

Now, $\frac{\partial}{\partial z} = \frac{\partial}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial}{\partial y} \cdot \frac{\partial y}{\partial z} = \frac{\partial}{\partial x} \cdot \frac{1}{2} + \frac{\partial}{\partial y} \cdot \frac{1}{2i} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$

And $\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}} = \frac{\partial}{\partial x} \cdot \frac{1}{2} + \frac{\partial}{\partial y} \cdot \left(-\frac{1}{2i}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

$$\therefore \quad \frac{\partial^2}{\partial z\, \partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \cdot \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

Hence, the result.

Example 10 : If $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x^2 - y^2$, find u.

Sol. : Let $z = x + iy \quad \therefore \bar{z} = x - iy \qquad \therefore \quad x = \frac{z + \bar{z}}{2}, \; y = \frac{z - \bar{z}}{2i}$

$$\therefore \quad x^2 - y^2 = \frac{1}{4}[(z + \bar{z})^2 + (z - \bar{z})^2] = \frac{1}{2}(z^2 + \bar{z}^2) \qquad \text{.......... (1)}$$

By the above result $\quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4\frac{\partial^2 u}{\partial z\, \partial \bar{z}} \qquad$ .... (2)

$$\therefore \quad 4\frac{\partial^2 u}{\partial z\, \partial \bar{z}} = x^2 - y^2 \qquad \text{[By data]}$$

$$\therefore \quad 4\frac{\partial^2 u}{\partial z\, \partial \bar{z}} = \frac{1}{2}(z^2 + \bar{z}^2) \qquad \text{[By (1)]}$$

$$\therefore \quad \frac{\partial^2 u}{\partial z\, \partial \bar{z}} = \frac{1}{8}(z^2 + \bar{z}^2)$$

$\therefore$ By integration w.r.t. $\bar{z} \qquad \frac{\partial u}{\partial z} = \frac{1}{8} \cdot z^2 \bar{z} + \frac{\bar{z}^3}{24} + f_1(\bar{z})$

where $f_1(\bar{z})$ if a constant of integration and f_1 is an arbitrary function.

Integrating again, w.r.t. z,

$u = \frac{z^3}{24} \cdot \bar{z} + \frac{\bar{z}^3 \cdot z}{24} + f_2(z) + f_1(\bar{z})$ where $f_2(z)$ is an arbitrary function of z.

$$\therefore \quad u = \frac{z\bar{z}}{24}(z^2 + \bar{z}^2) + f_1(\bar{z}) + f_2(z)$$

$$= \frac{(x + iy)(x - iy)}{24}[(x + iy)^2 + (x - iy)^2] + f_1(x - iy) + f_2(x + iy)$$

$$= \frac{(x^2 - y^2)}{12} \cdot (x^2 + y^2) + f_1(x - iy) + f_2(x + iy)$$

$$= \frac{x^4 - y^4}{12} + f_1(x - iy) + f_2(x + iy).$$

Example 11 : If $f(z)$ is analytic function, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$.

(M.U. 1993, 96, 98, 2003, 06, 07)

Sol. : We first note that if $f(z) = u + iv, \quad f(\bar{z}) = u - iv$

$$\therefore \quad |f(z)|^2 = u^2 + v^2 = (u + iv)(u - iv) = f(z) \cdot f(\bar{z}) \qquad \text{.......... (1)}$$

Also, $\quad f'(z) = u_x + iv_x, \quad f'(\bar{z}) = u_x - iv_x$

$$\therefore \quad |f'(z)|^2 = u_x^2 + v_x^2 = (u_x + iv_x)(u_x - iv_x) = f'(z) \cdot f'(\bar{z}) \qquad \text{.......... (2)}$$

Now, by the result obtained in Ex. 9 above.

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}|f(z)|^2 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}} f(z)\cdot f(\bar{z})$$

$$= 4\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}} f(z)\cdot f(\bar{z})\right] = 4\frac{\partial}{\partial z}\left[f(z)\cdot\frac{\partial}{\partial \bar{z}} f(\bar{z})\right]$$

$$= 4\frac{\partial}{\partial z}\left[f(z)\cdot f'(\bar{z})\right] = 4\cdot f'(\bar{z})\frac{\partial}{\partial z} f(z) = 4\cdot f'(\bar{z})\cdot f'(z)$$

$$= 4|f'(z)|^2 \qquad \text{[By (2)]}$$

Restatement

Since $|f(z)|^2 = u^2+v^2$ and $|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$, the above example can be restated in the following way.

"If $f(z)$ is an anylatic function, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^2+v^2) = 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]."$$

Example 12 : If $f(z)$ is an analytic function, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^n = n^2|f'(z)|^{n-2}\cdot|f'(z)|^2.$$ **(M.U. 1994, 2002)**

Sol. : We first note that

$$|f(z)| = \left\{|f(z)|^2\right\}^{1/2} = \left[f(z)\cdot f(\bar{z})\right]^{1/2} \quad \therefore \quad |f(z)|^n = \left[f(z)\cdot f(\bar{z})\right]^{n/2} \qquad \text{.................... (1)}$$

Now $$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^n = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}|f(z)|^n = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left[f(z)\cdot f(\bar{z})\right]^{n/2}$$

$$= 4\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}}[f(z)]^{n/2}\cdot[f(\bar{z})]^{n/2}\right] = 4\left[\frac{\partial}{\partial z}\left\{[f(z)]^{n/2}\cdot\frac{\partial}{\partial \bar{z}}[f(\bar{z})]^{n/2}\right\}\right]$$

$$= 4\left[\frac{\partial}{\partial z}[f(z)]^{n/2}\cdot\frac{n}{2}f(\bar{z})^{(n/2)-1}\cdot f'(\bar{z})\right]$$

$$= 4\cdot\frac{n}{2}f(\bar{z})^{(n/2)-1}\cdot f'(\bar{z})\cdot\frac{\partial}{\partial z}\cdot[f(z)]^{n/2}$$

$$= 4\cdot\frac{n}{2}f(\bar{z})^{(n/2)-1}\cdot f'(\bar{z})\cdot\frac{n}{2}\cdot[f(z)]^{(n/2)-1}\cdot f'(z)$$

$$= n^2\left[f(z)\cdot f(\bar{z})\right]^{(n/2)-1}\cdot\left[f'(z)\cdot f'(\bar{z})\right]$$

But $f(z)\cdot f(\bar{z}) = |f(z)|^2$ and $f'(z)\cdot f'(\bar{z}) = |f'(z)|^2$

$$\therefore \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^n = n^2\left\{|f(z)|^2\right\}^{(n/2)-1}\cdot|f'(z)|^2$$

$$= n^2|f(z)|^{n-2}\cdot|f'(z)|^2$$

Example 13 : If $f(z) = u + iv$ is an analytic function, prove that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|Rf(z)|^2 = 2|f'($

(M.U. 2

Sol. : We first note that if $f(z) = u + iv,\ f(\bar{z}) = u - iv$

$$\therefore\ |R\,f(z)|^2 = \left|\frac{1}{2}\left[f(z) + f(\bar{z})\right]\right|^2 = \frac{1}{4}|f(z) + f(\bar{z})|^2 \quad \ldots\ldots$$

And $|f(z)|^2 = u^2 + v^2 = (u + iv)(u - iv) = f(z)\cdot f(\bar{z})$ ....

$$\therefore\ |R\,f(z)|^2 = \frac{1}{4}\left[f(z) + f(\bar{z})\right]\left[f(\bar{z}) + f(\bar{z})\right] = \frac{1}{4}\left[f(z) + f(\bar{z})\right]\cdot\left[f(\bar{z}) + f(z)\right]$$

$$= \frac{1}{4}\left[f(z) + f(\bar{z})\right]^2$$

$$\therefore\ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|R\,f(z)|^2 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}|R\,f(z)|^2 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left\{\frac{1}{4}\left[f(z) + f(\bar{z})\right]^2\right\}$$

$$= \frac{\partial^2}{\partial z\,\partial \bar{z}}\left[f(z) + f(\bar{z})\right]^2 = \frac{\partial}{\partial z}2\left[f(z) + f(\bar{z})\right]\cdot f'(\bar{z})$$

$$\therefore\ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|R\,f(z)|^2 = 2f'(\bar{z})\cdot\frac{\partial}{\partial z}\left[f(z) + f(\bar{z})\right] = 2f'(\bar{z})\cdot f'(z)$$

$$= 2|f'(z)|^2 \qquad [\text{ By (2) }]$$

Example 14 : If $f(z)$ is an analytic function, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|R\,f(z)|^n = n(n-1)|R\,f(z)|^{n-2}\cdot|f'(z)|^2.$$

Sol. : We know that

$$|R\,f(z)|^2 = \left|\frac{1}{2}\left[f(z) + f(\bar{z})\right]\right|^2 = \frac{1}{4}|f(z) + f(\bar{z})|^2 \quad \ldots\ldots (1)$$

But $|f(z)|^2 = f(z)\cdot f(\bar{z})$ .... (2)

$$\therefore\ |R\,f(z)|^2 = \frac{1}{4}|f(z) + f(\bar{z})|^2 = \frac{1}{4}\left[f(z) + f(\bar{z})\right]\cdot\left[f(\bar{z}) + f(\bar{z})\right]$$

$$= \frac{1}{4}\left[f(z) + f(\bar{z})\right]\cdot\left[f(\bar{z}) + f(z)\right] = \frac{1}{4}\left[f(z) + f(\bar{z})\right]^2 \quad \ldots\ldots (3)$$

$$\therefore\ |R\,f(z)|^n = \left[|R\,f(z)|^2\right]^{n/2} = \left\{\frac{1}{4}\left[f(z) + f(\bar{z})\right]\right\}^{n/2} = \frac{1}{2^n}\left[f(z) + f(\bar{z})\right]^n \quad \ldots\ldots (4)$$

Now, $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|R\,f(z)|^n = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}|R\,f(z)|^n = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left\{\frac{1}{2^n}\left[f(z) + f(\bar{z})\right]^n\right\}$$

$$= 4\cdot\frac{1}{2^n}\frac{\partial}{\partial z}\left\{n\left[f(z) + f(\bar{z})\right]^{n-1}\cdot f'(\bar{z})\right\} = \frac{n}{2^{n-2}}f'(\bar{z})\cdot\frac{\partial}{\partial z}\left[f(z) + f(\bar{z})\right]^{n-1}$$

$$= \frac{n}{2^{n-2}}f'(z)\cdot(n-1)\left[f(z) + f(\bar{z})\right]^{n-2}\cdot f'(z)$$

$$\therefore \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|R\,f(z)|^n = \frac{n\,(n-1)}{2^{n-2}}\left[f(z)+f(\bar{z})\right]^{n-2}\cdot f'(z)\cdot f'(\bar{z})$$

By (2), $$f'(z)\cdot f'(\bar{z}) = |f'(z)|^2$$

By (4), $$\frac{1}{2^{n-2}}\left[f(z)+f(\bar{z})\right]^{n-2} = |R\,f(z)|^{n-2}$$

$$\therefore \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|R\,f(z)|^n = n\,(n-1)\,|R\,f(z)|^{n-2}\cdot|f'(z)|^2$$

Example 15 : If $f(z)$ is an analytic function, prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0.$

(M.U. 2000, 11)

Sol. : We first note that $\log|f'(z)| = \frac{1}{2}\log|f'(z)|^2$

But $|f'(z)|^2 = f'(z)\cdot f'(\bar{z})$

$$\therefore \log|f'(z)| = \frac{1}{2}\log\left[f'(z)\cdot f'(\bar{z})\right] = \frac{1}{2}\log f'(z) + \frac{1}{2}\log f'(\bar{z})$$

Now, $$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 4\,\frac{\partial^2}{\partial z\,\partial \bar{z}}\log|f'(z)|$$

$$= 4\,\frac{\partial^2}{\partial z\,\partial \bar{z}}\left[\frac{1}{2}\log f'(z) + \frac{1}{2}\log f'(\bar{z})\right] = 2\,\frac{\partial^2}{\partial z\,\partial \bar{z}}\log f'(z) + 2\,\frac{\partial^2}{\partial z\,\partial \bar{z}}\log f'(\bar{z})$$

$$= 2\,\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}}\log f'(z)\right] + 2\,\frac{\partial}{\partial \bar{z}}\left[\frac{\partial}{\partial z}\log f'(\bar{z})\right] = 2\,\frac{\partial}{\partial z}(0) + 2\,\frac{\partial}{\partial z}(0) = 0.$$

Example 16 : If Φ and Ψ are functions satisfying Laplace equation, then show that $s + it$ is holomorphic (analytic) where $s = \frac{\partial \Phi}{\partial y} - \frac{\partial \Psi}{\partial x}$ and $t = \frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial y}$. **(M.U. 1999, 2003, 05)**

Sol.: Since Φ and Ψ satisfy Laplace equation

$$\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 \Psi}{\partial x^2}+\frac{\partial^2 \Psi}{\partial y^2} = 0 \qquad \text{.................... (1)}$$

Now, $$\frac{\partial s}{\partial x} = \frac{\partial^2 \Phi}{\partial y\,\partial x} - \frac{\partial^2 \Psi}{\partial x^2} = \frac{\partial^2 \Phi}{\partial y\,\partial x} + \frac{\partial^2 \Psi}{\partial y^2} \qquad [\text{ By } (1)] \qquad \text{.................... (2)}$$

$$\frac{\partial s}{\partial y} = \frac{\partial^2 \Phi}{\partial y^2} - \frac{\partial^2 \Psi}{\partial x\,\partial y} = -\frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial^2 \Psi}{\partial x\,\partial y} \qquad [\text{ By } (1)] \qquad \text{.................... (3)}$$

Also, $$\frac{\partial t}{\partial x} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y\,\partial x} \qquad \text{.................... (4)}$$

$$\frac{\partial t}{\partial y} = \frac{\partial^2 \Phi}{\partial x\,\partial y} + \frac{\partial^2 \Psi}{\partial y^2} \qquad \text{.................... (5)}$$

From (2) and (5), we have $\frac{\partial s}{\partial x} = \frac{\partial t}{\partial y}$

From (3) and (4), we have $\frac{\partial s}{\partial y} = -\frac{\partial t}{\partial x}$

Since, $s + it$ satisfies Cauchy-Riemann equations it is analytic.

EXERCISE - IV

1. Prove that $u = e^x \cos y + x^3 - 3xy^2$ is harmonic. **(M.U. 2003)**

2. Check whether $u = x + e^{xy} + y + e^{-xy}$ is harmonic. **(M.U. 2004)** [**Ans.** : No]

3. Prove that $u = e^x \cos y$ is harmonic. Determine its harmonic conjugate v and the analytic function $f(z)$. [**Ans.** : $f(z) = e^z + c,\ v = e^x \sin y$]

4. Prove that $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. Find the corresponding analytic function $f(z)$. Also find the conjugate function.
[**Ans.** : $f(z) = (1+i)z^2 - (z+3i)z,\ v = x^2 + 2xy - y^2 - 3x - 2y$]

5. Show that $f(z) = e^x(\cos y + i \sin y)$ is analytic, it also satisfies Laplace's equation. Find its derivative. [**Ans.** : $f(z) = e^z + c$]

6. Show that the function $u = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic. Find the harmonic conjugate function v and express $u + iv$ as an analytic function of z. **(M.U. 2006, 19)**
[**Ans.** : $v = 4xy - x^3 + 3xy^2$; $f(z) = -iz^3 + 2z^2 + c$]

7. Show that $u = e^x(x \cos y - y \sin y)$ is harmonic. Find the harmonic conjugate v and the analytic function $f(z)$. **(M.U. 2009)** [**Ans.** : $v = e^x(x \sin y + y \cos y)$; $f(z) = ze^z + c$]

8. Show that the following functions are harmonic and find the corres-ponding analytic function $f(z) = u + iv$

   (i) $v = e^{-x}(x \cos y + y \sin y)$, **(ii)** $v = e^{2x}(y \cos 2y + x \sin 2y)$.

   (iii) $u = e^x \cos y - x^2 + y^2$ **(iv)** $u = x^4 - 6x^2y^2 + y^4$ **(M.U. 2013)**

   [**Ans.** : **(i)** $ize^{-z} + c$, **(ii)** $ze^{2z} + c$, **(iii)** $e^z - z^2 + c$, **(iv)** $z^4 + c$]

9. Show that the following functions are harmonic and find their corres-ponding analytic functions $f(z) = u + iv$

   (i) $u = x^2 - y^2$, (ii) $u = 2x(1 - y)$ **(iii)** $u = 3x^2y - y^3$,

   (iv) $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ **(v)** $u = (x-1)^3 - 3xy^2 + 3y^2$ **(M.U. 2004)**

   [**Ans.** : (i) $z^2 + c$, **(ii)** $2z + iz^2 = c$, **(iiii)** $z^3 + c$, **(iv)** $z^3 + 3z^2 + c$, **(v)** $(z-1)^3 + c$]

10. Find the analytic function $f(z) = u + iv$ where $u = 2a\,xy + b(y^2 - x^2)$. Also verify that u is harmonic.
(M.U. 2004) [**Ans.** : $f(z) = (a - bi)z^2$]

11. Show that $u = e^{-2xy} \sin(x^2 - y^2)$ is harmonic. Find the corresponding analytic function $f(z)$ and also find the conjugate function v.

[**Ans.** : $f(z) = -ie^{iz^2} + c$; $v = -e^{-2xy} \cos(x^2 - y^2) + c'$]

12. Show that $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$ is harmonic. Hence, find its harmonic conjugate v and corresponding analytic function $f(z) = u + iv$.

(M.U. 1997, 2006) [**Ans.** : $f(z) = \tan z + c$]

13. If u and v are harmonic functions of x and y and $s = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$ and $t = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$, prove that $s + it$ is an analytic function of $z = x + iy$.

14. If $u(x, y)$ and $v(x, y)$ are harmonic functions in a region R, prove that the function $\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ is an analytic function of $z = x + iy$. **(M.U. 2005)**

Remark ... The solved Example No. 16 above and this example are the same with the difference in wording them.

14. Orthogonal Curves - Orthogonal Trajectories

(a) If $f(z) = u(x, y) + iv(x, y)$ is an analytic function then the curves $u = c_1$ and $v = c_2$ intersect orthogonally. **(M.U. 2003, 06)**

Proof : Let $u = f(x, y) = c_1$ and $v = \Phi(x, y) = c_2$

Then $\left(\frac{dy}{dx}\right)_{u=c_1} = -\frac{\partial f / \partial x}{\partial f / \partial y} = -\frac{\partial u / \partial x}{\partial u / \partial y}$ and $\left(\frac{dy}{dx}\right)_{v=c_2} = -\frac{\partial \Phi / \partial x}{\partial \Phi / \partial y} = -\frac{\partial v / \partial x}{\partial v / \partial y}$

Since, $f(z)$ is analytic C-R equations give $u_x = v_y$ and $u_y = -v_x$

$$\therefore \left(\frac{\partial y}{\partial x}\right)_{u=c_1} \times \left(\frac{dy}{dx}\right)_{v=c_2} = \frac{\partial u / \partial x}{\partial u / \partial y} \times \frac{\partial v / \partial x}{\partial v / \partial y} = \frac{v_y}{-v_x} \cdot \frac{v_x}{v_y} = -1.$$

Hence, $u = c_1$ and $v = c_2$ intersect orthogonally.

(b) Orthogonal Trajectories : By an **orthogonal trajectory** of a family of curves we mean a curve which cuts every member of the given family at right angles. For example, consider a family of straight lines passing through the origin given by $y = mx$, where m is an arbitrary constant.

It is easy to see that these straight lines are cut by a circle with centre at the origin at right angles at every point of intersection. Its equation is of the form $x^2 + y^2 = a^2$ where a is a parameter.

Thus, the family of circles $x^2 + y^2 = a^2$ represents the family of orthogonal trajectories to the family of straight lines given by $y = mx$.

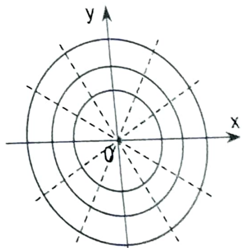


Fig. 4.4

(c) Orthogonal trajectories of the family of curves given by $u = c$.

We have seen that if $f(z) = u + iv$ is an analytic function then the curves $u = c_1$ and $v = c_2$ intersect orthogonally *i.e.* $v = c_2$ is the family of orthogonal trajectories of the family of curves given by $u = c_1$.

Hence, to find the orthogonal trajectory of $u = c_1$ (or $v = c_2$) we find the harmonic conjugate v (or $u = c_1$) of u (or v).

Example 1 : Find the orthogonal trajectory of the family of curves $x^3y - xy^3 = c$.
(M.U. 2003, 04, 05, 06, 07, 11, 14, 19)

Sol. : As seen above we have to find the harmonic conjugate of u.

$\because \quad u = x^3y - xy^3 \qquad \therefore \quad u_x = 3x^2y - y^3$ and $u_y = x^3 - 3xy^2$

$\therefore \quad f'(z) = u_x + iv_x = u_x - iu_y$ [By C-R equations]

$= (3x^2y - y^3) - i(x^3 - xy^2)$

By Milne-Thompson's method, we put $x = z,\ y = 0$

$\therefore \quad f'(z) = -iz^3$

$$\therefore \quad f(z) = -\int iz^3 dz = -i\frac{z^4}{4} + c = -\frac{i}{4}(x+iy)^4 + c$$

$$= -\frac{i}{4}(x^4 + 4x^3iy - 6x^2y^2 - 4xiy^3 + y^4) + c$$

$$\therefore \text{ Imaginary part } v = -\frac{1}{4}(x^4 - 6x^2y^2 + y^4)$$

Hence, the required orthogonal trajectories are $x^4 - 6x^2y^2 + y^4 = c'$.

Example 2 : Find the orthogonal trajectories of the family of curves $e^{-x}\cos y + xy = \alpha$, where α is the real constant in the xy-plane. **(M.U. 2004, 05, 13, 16, 19)**

Sol. : We have seen above that the orthogonal trajectories of $u = c_1$ are given by $v = c_2$ where v is the harmonic conjugate of u.

$\therefore \quad u = e^{-x}\cos y + xy \qquad \therefore \quad u_x = -e^{-x}\cos y + y$ and $u_y = -e^{x}\sin y + x$

Also $f'(z) = u_x + iv_x = u_x - iu_y$ [By C-R equations]

$= (-e^{-x}\cos y + y) - i(-e^{-x}\sin y + x)$

By Milne-Thompson's method, we replace x by z and y by zero.

$\therefore \quad f'(z) = -e^{-z} - iz$

$$\text{By integration } f(z) = e^{-z} - i\frac{z^2}{2} + c$$

$$\therefore \quad f(z) = e^{-(x+iy)} - i\frac{(x+iy)^2}{2} + c$$

$$= e^{-x}(\cos y - i\sin y) - \frac{i}{2}(x^2 + 2i\,xy - y^2) + c$$

$$\therefore \text{ Imaginary part, } \quad v = -e^{-x}\sin y - \frac{1}{2}(x^2 - y^2)$$

Hence, the required orthogonal trajectories are $e^{-x}\sin y + \frac{1}{2}(x^2 - y^2) = c_2$.

Example 3 : Find the orthogonal trajectory of the family of curves given by $2x - x^3 + 3xy^2 = a$. **(M.U. 2002, 04, 16)**

Sol. : Let $u = 2x - x^3 + 3xy^2$

$\therefore \quad u_x = 2 - 3x^2 + 3y^2,\ u_y = 6xy$

$\therefore \quad f'(z) = u_x + i\,v_x = u_x - i\,u_y$ [By C-R equations]

$= 2 - 3x^2 + 3y^2 - i\cdot 6xy$

By Milne-Thomson method, we put $x = z$, $y = 0$.

$\therefore \quad f'(z) = 2 - 3z^2$

Integrating w.r.t. z, we get

$$f(z) = 2z - z^3 + c = 2(x + iy) - (x + iy)^3 + c$$
$$= 2x + 2iy - x^3 - 3ix^2y + 3xy^2 + iy^3 + c$$
$$= 2x + 2iy - x^3 - 3ix^2y + 3xy^2 + iy^3 + c$$

$\therefore$ Imaginary part is $v = 2y - 3x^2y + y^3$.

$\therefore$ The required orthogonal trajectory is $2y - 3x^2y + y^3 = c$.

Example 4 : For the function $f(z) = z^3$, verify that the families of curves $u = c_1$ and $v = c_2$ cut orthogonally where c_1 and c_2 are constants and $f(z) = u + iv$. **(M.U. 2004)**

Sol. : $f(z) = (x + iy)^3 = x^3 + 3ix^2y - 3xy^2 - iy^3$

$\therefore$ $u = x^3 - 3xy^2,\ v = 3x^2y - y^3;\quad u_x = 3x^2 - 3y^2,\ u_y = -6xy$

$v_x = 6xy,\ v_y = 3x^2 - 3y^2$

$$\therefore\ m_1 = \left(\frac{dy}{dx}\right)_{u=c_1} = -\frac{u_x}{u_y} = -\frac{3(x^2 - y^2)}{-6xy}$$

$$m_2 = \left(\frac{dy}{dx}\right)_{u=c_2} = -\frac{v_x}{v_y} = -\frac{6xy}{3(x^2 - y^2)}$$

$$\therefore\ m_1 \times m_2 = \frac{3(x^2 - y^2)}{6xy} \cdot \left(-\frac{6xy}{3(x^2 - y^2)}\right) = -1$$

Hence, the families cut orthogonally.

EXERCISE - V

Find the orthogonal trajectories of the family of curves

1. $3x^2y - y^3 = c$ **(M.U. 2002, 04, 07, 15)**
2. $x^2 - y^2 + x = c$ **(M.U. 2019)**
3. $e^{-x}(x \sin y - y \cos y) = c$
4. $x^2 - y^2 - 2xy + 2x - 3y = c$ **(M.U. 2004, 10)**
5. $e^x \cos y - xy = c$ **(M.U. 2007, 08)**
6. $3x^2y + 2x^2 - y^3 - 2y^2 = c$ **(M.U. 2003, 12, 14)**

[**Ans. :** **(1)** $3xy^2 - x^3 = c$, **(2)** $2xy + y = c$, **(3)** $e^{-x}(x \cos y + y \sin y) = c$
(4) $2xy + 2y + x^2 - y^2 + 3y = c$ **(5)** $e^x \sin y + (x^2 - y^2)/2 = c$,
(6) $4y - 3x^2 + 3xy^2 = c$.]

(d) Angle of Intersection of Two Curves in Polar Coordinates : If the equation of a curve is given in polar coordinates by $r = f(\theta)$ and if Φ is the angle between the radius vector at P and the tangent at P, then $\tan \Phi = r\dfrac{d\theta}{dr}$. (We accept this result without proof.)

If two curves given by polar equations $r = f(\theta)$ and $r = F(\theta)$ intersect at P and if Φ_1 and Φ_2 are the angles between the common radius vector and the tangents to the two curves then it is clear from the figure that the angle between the curves *i.e.* the angle between the tangents is equal to $\Phi_1 \sim \Phi_2$ *i.e.* the difference between Φ_1 and Φ_2.

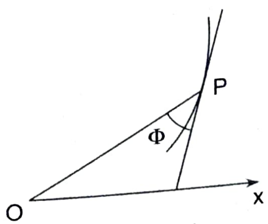


Fig. 4.5 (a)

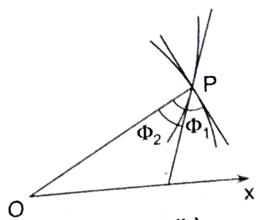


Fig. 4.5 (b)

The two curves intersect orthogonally if the angle between the tangents to the two curves at the common point is $\pi/2$ i.e. if $\Phi_1 - \Phi_2 = \pi/2$.

Example 1 : Prove that the curves $r^n = a \sec n\theta$ and $r^n = b \operatorname{cosec} n\theta$ intersect orthogonally. **(M.U. 2005)**

Sol. : Let $P(r, \theta)$ be the point of intersection.

For the first curve $\quad n \log r = \log a + \log \sec n\theta$

$$\therefore \frac{n}{r} \cdot \frac{dr}{d\theta} = \frac{1}{\sec n\theta} \cdot \sec n\theta \cdot \tan n\theta \cdot n \qquad \therefore \frac{dr}{d\theta} = r \tan n\theta$$

$$\therefore \tan \Phi_1 = r\frac{d\theta}{dr} = \frac{r}{r \tan n\theta} = \cot n\theta \qquad \therefore \Phi_1 = \frac{\pi}{2} - n\theta.$$

For the second curve, $\quad n \log r = \log b + \log \operatorname{cosec} n\theta$

$$\therefore \frac{n}{r}\frac{dr}{d\theta} = -\frac{1}{\operatorname{cosec} n\theta} \cdot \operatorname{cosec} n\theta \cdot \cot n\theta \cdot n \qquad \therefore \frac{dr}{d\theta} = -r \cot n\theta$$

$$\therefore \tan \Phi_2 = r\frac{d\theta}{dr} = -\frac{r}{r \cot n\theta} = -\tan n\theta \qquad \therefore \Phi_2 = -n\theta$$

$$\therefore \Phi_1 - \Phi_2 = \frac{\pi}{2} - n\theta - (-n\theta) = \frac{\pi}{2}.$$

$\therefore$ The curves intersect orthogonally.

Example 2 : Find the angle of intersection of the curves $r = \sin\theta + \cos\theta$ and $r = 2\sin\theta$.

Sol. : If the two curves intersect at $P(r, \theta)$, then by eliminating r, we have $\sin\theta + \cos\theta = 2\sin\theta$.

$$\therefore \sin\theta = \cos\theta \quad \therefore \tan\theta = 1 \quad \therefore \theta = \frac{\pi}{4} \text{ at the point of intersection.}$$

Now, for the curve $r = \sin\theta + \cos\theta$

$$\frac{dr}{d\theta} = \cos\theta - \sin\theta \qquad \therefore \tan\Phi = r\frac{d\theta}{dr} \qquad \therefore \tan\Phi = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}$$

$$\therefore \text{At } P, \text{ i.e. at } \theta = \frac{\pi}{4}, \quad \tan\Phi = \frac{(1/\sqrt{2}) + (1/\sqrt{2})}{(1/\sqrt{2}) - (1/\sqrt{2})} = \infty.$$

$$\therefore \text{At } P, \ \Phi_1 = \frac{\pi}{2}$$

Also for the curve $r = 2\sin\theta$.

$$\frac{dr}{d\theta} = 2\cos\theta \qquad \therefore \tan\Phi = r\frac{d\theta}{dr} = \frac{2\sin\theta}{2\cos\theta}$$

$$\therefore \tan\Phi = \tan\theta \qquad \therefore \Phi = \theta$$

$$\therefore \text{At } P, \text{ i.e., at } \theta = \frac{\pi}{4}, \ \Phi_2 = \frac{\pi}{4}.$$

$$\therefore \text{Angle of intersection} = \Phi_1 - \Phi_2 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Example 3 : Prove that the curves $r^2 = a^2 \cos 2\theta$ and $r = a(1 + \cos\theta)$ intersect at an angle $3\sin^{-1}(3/4)^{1/4}$.

Sol. : Let us first find θ at the point of intersection of the two curves.

Eliminating r, $a^2 \cos 2\theta = a^2(1+\cos\theta)^2$ $\therefore$ $2\cos^2\theta - 1 = 1 + 2\cos\theta + \cos^2\theta$

$\therefore$ $\cos^2\theta - 2\cos\theta - 2 = 0$ $\therefore$ $\cos\theta = \dfrac{2 \pm \sqrt{4-4(1)(-2)}}{2} = 1 \pm \sqrt{3}.$

Since, $\cos\theta$ lies between -1 and $+1$.

$$\cos\theta = 1-\sqrt{3} \quad \therefore \quad 1 - 2\sin^2\frac{\theta}{2} = 1-\sqrt{3}$$

$$\therefore \sin^2\left(\frac{\theta}{2}\right) = \frac{\sqrt{3}}{2} = \left[\left(\frac{\sqrt{3}}{2}\right)^2\right]^{1/2} = \left(\frac{3}{4}\right)^{1/2} \quad \therefore \quad \sin\left(\frac{\theta}{2}\right) = \left(\frac{3}{4}\right)^{1/4} \quad \therefore \quad \frac{\theta}{2} = \sin^{-1}\left(\frac{3}{4}\right)^{1/4}$$

Now, for the curve $r^2 = a^2 \cos 2\theta$.

$$2r\frac{dr}{d\theta} = a^2 \cdot (-2)\sin 2\theta \quad \therefore \quad \frac{dr}{d\theta} = -\frac{a^2 \sin 2\theta}{r}$$

$$\therefore \tan\Phi = r\frac{d\theta}{dr} = r\left(\frac{-r}{a^2\sin 2\theta}\right) = -\frac{r^2}{a^2 \sin 2\theta}$$

$$= -\frac{a^2\cos 2\theta}{a^2 \sin 2\theta} = -\cot 2\theta = \tan\left(\frac{\pi}{2} + 2\theta\right)$$

$$\therefore \Phi_1 = \frac{\pi}{2} + 2\theta$$

Further for the second curve, $r = a(1+\cos\theta)$

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$\therefore \tan\Phi = r\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta} = -\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = -\cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$$\therefore \Phi_2 = \frac{\pi}{2} + \frac{\theta}{2}$$

$\therefore$ The angle of intersection $= \Phi_1 - \Phi_2 = \dfrac{\pi}{2} + 2\theta - \dfrac{\pi}{2} - \dfrac{\theta}{2} = 3\left(\dfrac{\theta}{2}\right) = 3\sin^{-1}\left(\dfrac{3}{4}\right)^{1/4}.$

EXERCISE - VI

Prove that the following curves in each case intersect orthogonally.

1. $r = 2\sin\theta$, $r = 2\cos\theta$.
2. $r = a(1+\cos\theta)$, $r = b(1-\cos\theta)$
3. $r^2 \sin 2\theta = a^2$, $r^2 \cos 2\theta = b^2$

15. Applications of Analytic Functions

Analytic functions are useful in the study of 'field problems'. For example, in the study of flow of a fluid or of heat or of electricity in a given region analytic functions can be used.

In physical applications of analytic functions, the complex function w is denoted by

$$w = u(x, y) + iv(x, y) = \Phi(x, y) + i\,\psi(x, y)$$

(a) In the flow of fluids $\Phi(x, y)$ is called velocity potential and $\psi(x, y)$ is called stream functions. The function $w = \Phi(x, y) + i\,\psi(x, y)$ is called potential function $\Phi(x, y) = c_1$ gives family of equipotential curves and $\psi(x, y) = c_2$ gives family of stream lines. $\Phi(x, y) = c_1$ are orthogonal to $\psi(x, y) = c_2$. Stream lines are the paths of fluid particles.

(b) In electrostatic fields $\Phi(x, y)$ is called potential function and $\psi(x, y)$ is known as stream function $\Phi(x, y) = c_1$ gives family of equipotential curves and $\psi(x, y) = c_2$ gives family of lines of force of electrostatic field. As before $\Phi(x, y) = c_1$ and $\psi(x, y) = c_2$ are orthogonal.

(c) In the study of heat flow, $\Phi(x, y)$ is known as temperature function and $\psi(x, y)$ is known as stream function $\Phi(x, y) = c_1$ gives a family of isothermals and $\psi(x, y) = c_2$ gives a family of heat flow lines.

Example 1 : If $w = \Phi + i\psi$ represents complex potential for an electric field and $\psi = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$, determine Φ. Also show that ψ is harmonic. **(M.U. 2003, 06)**

Sol. : See Ex. 5 on page 4-36.

Example 2 : The potential function of an electrostatic field is given by $\Phi = 3x^2y - y^3$ find the corresponding stream function.

Sol. : See Ex. 1, page 4-44.

EXERCISE - VII

1. In aerodynamics and fluid mechanics the functions Φ and ψ in $f(z) = \Phi + i\psi$ where $f(z)$ is analytic are called potential and stream functions. If potential function $\Phi = x^2 + 4x - y^2 + 2y$, find the stream function.

   [**Ans. :** $\psi = 2xy + 4y - 2x$]

2. If the potential function is $\log(x^2 + y^2)$, find the flux function and the complex potential function.

   [**Ans. :** $2\tan^{-1}\dfrac{y}{x}$, $2\log z + c$]

3. In a two dimensional fluid flow, the stream function is $\psi = \tan^{-1}\left(\dfrac{y}{x}\right)$ find the velocity potential.

   [**Ans. :** $\dfrac{1}{2}\log(x^2 + y^2)$]

4. An electrostatic field in the x-y plane is given by the potential function $\Phi = x^2 - y^2$ find the stream function.

   [**Ans. :** $\psi = 2xy + c$]

EXERCISE - VIII

Theory

1. Define analytic function and harmonic function. **(M.U. 2007)**
2. State and prove the conditions for a function $w = f(z)$ to be analytic. **(M.U. 1997, 2003, 04)**
3. State and prove Cauchy - Riemann equations. **(M.U. 1999, 2002, 03)**
4. State and prove Cauchy - Riemann equations in polar form. **(M.U. 2002, 04, 08, 11)**
5. If $f(z)$ and $\overline{f(z)}$ are both analytic then show that $f(z)$ is constant. **(M.U. 1993)**

6. If $f(z)$ is analytic and $|f(z)|$ is constant, prove that $f(z)$ is constant. **(M.U. 1994, 1997, 2003)**
7. Prove that real and imaginary parts of an analytic function satisfy Laplace Equation. **(M.U. 1993)**
8. If $f(z)$ is analytic, show that $\frac{\partial f}{\partial \bar{z}} = 0$.
9. If $f(z)$ is analytic, show that $f(z)$ is independent of $\bar{z}$.
10. If $f(z)$ is analytic, show that x and y can occur in $f(z)$ in the combination of $x + iy$ only.
11. Show that real and imaginary parts of an analytic function are harmonic. **(M.U. 1996, 2005, 06, 07)**
12. State true or false with proper justification.
    (i) There does not exist an analytic function whose real part is $x^3 - 3x^2 y - y^3$. **(M.U. 2013)**
    (ii) If $f(z)$ and $\overline{f(z)}$ are both analytic then $f(z)$ is a constant function. **(M.U. 1995)**
13. Show that an analytic function with constant modulus is constant. **(M.U. 1994, 99, 2003)**
14. If $f(z) = u + iv$ is analytic in a region R then show that
    (i) u and v are harmonic functions and
    (ii) u = constant and v = constant intersect orthogonally. **(M.U. 1994, 97, 98, 2003)**
15. If $f(z) = u + iv$ is analytic in a region R show that
    (i) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$
    (ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y} = 0, \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ **(M.U. 1995)**
16. If $f(z) = u + iv$ is analytic and $z = re^{i\theta}$, show that
    $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$ **(M.U. 1996, 2002, 03)**
17. If $f(z) = u + iv$ is an analytic function, prove that
    (i) u = constant and v constant are orthogonal trajectories.
    (ii) u and v are harmonic functions.
    (iii) $f(z)$ is constnat if $f'(z) = 0$.
    (iv) $u - iv$ is also analytic. **(M.U. 2004)**
18. If u and v are conjugate harmonic functions, prove that uv is also harmonic. **(M.U. 2003)**