# The Complexity Classes QMA

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# The class NP

**Definition**

Recall the **class NP** consists of languages $L$ such that there is a polynomial time verifier(Turing Machine) $V$ with

1. If $x \in L$, then there is a proof $y$ such that $V(x, y) = 1$.
2. If $x \notin L$, then there is a disproof $y$ of polynomial length such that $V(x, y) = 0$.

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1. If $x \in L$, then there is a proof $y$ such that $Pr(V(x, y) = 1) \geq 3/4$.
2. If $x \notin L$, then there is a disprove $y$ of polynomial length such that $Pr(V(x, y) = 1) \leq 1/4$.

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**Definition**

A **Polynomial time quantum algorithm** is a polynomial time turing machine that on $n$ inputs outputs quantum circuit $Q_n$.

# The class QMA

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The **class QMA** consists of languages $L$ such that there is a Quantum verifier $V$ with

1. If $x \in L$, then there is a proof state $|\phi\rangle$ such that $Pr(V(x, |\phi\rangle) = 1) \geq 3/4$.
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# Local Hamiltonian problem

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# 3SAT reduces to 3-LH

**Proof.**

- Let $\phi(x_1 \dots x_n) = \bigwedge_i R_i$ be a formula for 3Sat.
- $R_i = x_j \vee \dots$ : Define $H_i$ to do $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ to $j$th qubit.

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- all the $|x\rangle$ are eigenvalues of $H_i$, with eigenvalue $0 \iff x$ satisfies $R_i$.
- Hence $H = \sum H_i$ has smallest eigenvalue $0 \iff \phi$ is satisfiable.

□

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$$\frac{1}{n} \sum_{i=1}^{m} \langle\psi| H_i |\psi\rangle = \frac{1}{n} \langle\psi| H |\psi\rangle .$$

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- If $|\psi\rangle^{\otimes N}$ for large $N$ was the proof, we could figure out $\langle\psi| H |\psi\rangle$ to high precision.

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- Note 3-LH is **not** in NP. This is because a proof state would have exponential embedding.
- Locality assures input can be embedded independent of number of qubits $n$.
- In 3Sat, the number of variables of the formula doesnt matter, the number of clauses does.

# $O(\log(n))$-LH is QMA complete

## Proof: Kataev-Shen-Viyali.

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- The quantum circuit for a given QMA problem is $U = U_T \dots U_1$.
- Has $N$ inputs, with $m$ ancilla qubits set to 0.
- Find $H$ on $(\mathbb{C}^2)^{\otimes n} \otimes \mathbb{C}^{T+1}$ which will be of the form

$$H = H_{in} + H_{out} + H_{prop}.$$

# The Local Hamiltonians

**Proof.**

$$H_{in} = \left( \sum_{i=m+1}^{N} |1\rangle_i \langle 1|_i \right) \otimes |0\rangle\langle 0|$$

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H_{prop} &= \sum_{j=0}^{T} H_j \\
2H_j &= I \otimes (|j\rangle\langle j| + |j-1\rangle\langle j-1|) - U_j \otimes |j\rangle\langle j-1| - U_j^* \otimes |j-1\rangle\langle j|
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- Clearly $\langle\psi| H_{in} |\psi\rangle = \langle\psi| H_{prop} |\psi\rangle = 0$.
- Note
$$\langle\psi| H_{out} |\psi\rangle = \frac{1}{L+1} |\langle 0, 0| U |\xi, 0\rangle|^2 = \frac{P(V(x, |\xi\rangle) = 0)}{L+1}.$$

# References

Kataev, Shen, Vyali, *Classical and Quantum Computation*, 1999 O' Donnell, *Lecture 24: QMA: Quantum Merlin-Arthur*, 2015. Kempe, Regev, *3-Local Hamiltonian is QMA-complete*, 2003