K_2 and algebraic cycles

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0. Introduction

The purpose of this paper is to apply techniques from algebraic K-theory to study codimension two algebraic cycles on a scheme X. The principal tool is the Zariski cohomology group $H^2(X, K_2(\mathcal{O}_X))$ where $K_2(\mathcal{O}_X)$ is a sheaf of abelian groups obtained by sheafifying the K_2 functor of Milnor [17].

Let $\tilde{K}_0(X)$ be the reduced Grothendieck group of vector bundles on X modulo short exact sequences, and let $K_0^{(2)}(X)$ be the kernel

$$0 \longrightarrow K_0^{(2)}(X) \longrightarrow \tilde{K}_0(X) \xrightarrow{\det} \operatorname{Pic}(X) \longrightarrow 0 .$$

We construct a sort of "universal second Chern class" map

$$CK_2\colon K_0^{(2)}(X) \longrightarrow H^2(X, K_2(\mathcal{O}_X)) . \tag{0.1}$$

If X is such that coherent $\mathcal{O}_X$-modules admit global, finite, locally-free resolutions, we get a map

$$CK_2\colon \{\text{codim } 2 \text{ subschemes } Z \subset X\} \longrightarrow H^2(X, K_2(\mathcal{O}_X)) . \tag{0.2}$$

Now assume X is a regular, algebraic k-scheme (k a field), and let $CH^2(X)$ be the Chow group of codimension two algebraic cycles mod rational equivalence [8]. There is a natural map $\bar{j}\colon H^2(X, K_2(\mathcal{O}_X)) \to CH^2(X)$ such that the triangle

$$\begin{tikzcd} \{\text{codim } 2 \text{ subschemes}\} \arrow[rr, "CK_2"] \arrow[dr, "\text{cycle map}"'] & & H^2(X, K_2) \arrow[dl, "\bar{j}"] \\ & CH^2(X) & \end{tikzcd} \tag{0.3}$$

commutes.

Writing Ω^2_{cl} for the sheaf of closed Kähler two-forms on X, there is a natural map $d\log \wedge d\log\colon K_2(\mathcal{O}_X) \to \Omega^2_{cl}$ and the composition

$$\{\text{codim } 2 \text{ subschemes}\} \longrightarrow H^2(X, K_2) \xrightarrow{d\log \wedge d\log} H^2(X, \Omega^2_{cl}) \tag{0.4}$$

is the cycle class map in the sense of Hodge–de Rham [10]. Moreover, the image of $d\log \wedge d\log$ in (0.4) is precisely the subgroup of algebraic cycle classes in $H^2(X, \Omega^2_{cl})$.

The last two sections of the paper discuss variants and applications of these ideas.

I would like to stress the analogy between the above and the more familiar constructions in the theory of divisors, involving the Picard group $H^1(X, \mathcal{O}_X^*)$. In fact, sheafifying the K_1 functor of Bass [7] yields $\mathcal{O}_X^*$.

I am endebted to B. Messing for noticing that I was doing K-theory in disguise, as well as to S. Gersten for many helpful communications.

1. The sheaves $K_2(\mathcal{O}_X)$, Σ_2

A standard reference for this section is [7].

Let A be a ring, and let $E(A) \subset GL(A) = \varinjlim_n GL_n(A)$ be the subgroup of *elementary matrices*. $E(A)$ is generated by matrices e_{ij}^a $(i \neq j)$ with 1's on the diagonal, $a \in A$ as $(i, j)^{\text{th}}$ entry, and zeros elsewhere. Define the *Steinberg Group* $St(A)$ to be the group with generators X_{ij}^a, $a \in A$, $1 \leqq i \neq j < \infty$; and relations:

$$X_{ij}^a X_{ij}^b = X_{ij}^{a+b} \tag{1.1}$$

$$[X_{ij}^a, X_{jl}^b] = X_{ij}^a X_{jl}^b X_{ij}^{-a} X_{jl}^{-b} = X_{il}^{ab} \qquad i \neq l \tag{1.2}$$

$$[X_{ij}^a, X_{kl}^b] = 1\,, \qquad j \neq k,\ i \neq l\,. \tag{1.3}$$

There is a natural surjection $St(A) \xrightarrow{\phi} E(A)$, $X_{ij}^a \mapsto e_{ij}^a$, and K_2 is defined to be the kernel

$$0 \longrightarrow K_2(A) \longrightarrow St(A) \xrightarrow{\phi} E(A) \longrightarrow 1\,. \tag{1.4}$$

One shows $K_2(A) = \text{Center}\,\big(St(A)\big)$, so it is an abelian group.

Given matrices $M, N \in E(A)$ which commute, define $M*N = [\tilde{M}, \tilde{N}] \in K_2(A)$, where $\tilde{M}$, $\hat{N} \in St(A)$ are liftings of M, N. Since the extension (1.4) is central, $M*N$ is independent of the choice of $\tilde{M}$, $\tilde{N}$.

Suppose now that A is commutative with 1, write A^* for the multiplicative group of units, and let a, $b \in A^*$. Let $d_{ij}(a)$ be the diagonal matrix with a in the $(i, i)^{\text{th}}$ place, a^{-1} in the $(j, j)^{\text{th}}$ place, and 1's elsewhere. We compute

$$d_{ij}(a) = e_{ij}^a e_{ji}^{-a^{-1}} e_{ij}^a e_{ij}^1 e_{ji}^{-1} e_{ij}^1 \tag{1.5}$$

so $d_{ij}(a) \in E(A)$. The *symbol* $\{a, b\}$ is defined by

$$\{a, b\} = d_{ij}(a) * d_{ik}(b)\,. \tag{1.6}$$

The symbol $\{a, b\}$ is independent of the choice of i, j, $k \neq j$, and it satisfies the following identities:

$$\{a, b\} = \{b, a\}^{-1} \tag{1.7}$$

$$\{a, b\}\{a', b\} = \{aa', b\} \tag{1.8}$$

$$\{a, -a\} = 1 \tag{1.9}$$

$$\text{If } a, 1 - a \in A^*,\ \{a, 1 - a\} = 1\,. \tag{1.10}$$

Assume further that A is a local ring. In this case, it is known that $E(A) = SL(A)$, the special linear group, and $K_2(A)$ is generated by symbols. One has

$$\begin{tikzcd} & A^* \otimes_{\mathbf{Z}} A^* \arrow[d] & & & \\ 0 \arrow[r] & K_2(A) \arrow[r] \arrow[d] & St(A) \arrow[r] & SL(A) \arrow[r] & 1 . \\ & 0 & & & \end{tikzcd} \tag{1.11}$$

LEMMA 1.12. *The functors K_2, St, E, and SL all commute with filtering direct limits.*

Proof. This is clear for St, E, and SL. It follows for $K_2 = \operatorname{Ker}(\phi)$ since taking filtering direct limits is exact.

Let X be a scheme with structure sheaf $\mathcal{O}_X$. Define sheaves $St(\mathcal{O}_X)$ and $K_2(\mathcal{O}_X)$ by sheafifying the presheaves

$$U \longmapsto St\big(\Gamma(U, \mathcal{O}_X)\big)$$
$$U \longmapsto K_2\big(\Gamma(U, \mathcal{O}_X)\big) .$$

PROPOSITION 1.13. *There is an exact sequence*

$$0 \longrightarrow K_2(\mathcal{O}_X) \longrightarrow St(\mathcal{O}_X) \longrightarrow SL(\mathcal{O}_X) \longrightarrow 1 .$$

Moreover, $K_2(\mathcal{O}_X)$ is a quotient of $\mathcal{O}_X^* \otimes_{\mathbf{Z}} \mathcal{O}_X^*$.

Proof. It suffices to check on the stalks. For $x \in X$, we have by (1.12)

$$K_2(\mathcal{O}_X)_x \cong K_2(\mathcal{O}_{X,x})$$
$$St(\mathcal{O}_X)_x \cong St(\mathcal{O}_{X,x}) .$$

The proposition now follows from (1.11).

THEOREM 1.14. (*Matsumoto*). *For F a field, $K_2(F)$ is the group generated by symbols $\{a, b\}$, a, $b \in F^*$ modulo relations*

$$\{a, b\} = \{b, a\}^{-1}; \ \{a, b\}\{a', b\} = \{aa', b\}; \ \{a, 1 - a\} = 1 .$$

(*The relation $\{a, -a\} = 1$ is a consequence of these three.*)

A map $\varphi\colon F^* \times F^* \to G$ with target an abelian group G and satisfying (1.7)–(1.10) induces by (1.14) a unique $\bar{\varphi}\colon K_2(F) \to G$. Such a φ is called a *symbol* on F with values in G.

For example, let ν be a discrete valuation on F with residue field k_ν. The *tame symbol* $T_\nu\colon F^* \times F^* \to k_\nu^*$ is defined by

$$T_\nu(x, y) = (-1)^{\nu(x)\nu(y)} x^{\nu(y)} y^{-\nu(x)} . \tag{1.15}$$

THEOREM 1.16. (*Bass, Dennis, Stein*). *Let A be a discrete valuation ring with quotient field F and associated valuation ν. Then the sequence*

$$0 \longrightarrow K_2(A) \longrightarrow K_2(F) \xrightarrow{T_\nu} k_\nu^* \longrightarrow 0$$

is exact [14].

To apply these ideas to geometry, let X be a normal scheme with function field F_X and consider

$$T = \bigoplus T_D : K_2(F_X) \longrightarrow \bigoplus_D F_D^* \; . \tag{1.17}$$

Here D runs through all the irreducible divisors on X, F_D is the function field of D, and T_D is the tame symbol of the valuation associated to D. Note that T maps to the direct sum because any function has only a finite number of zeros and poles.

View both sides of (1.17) as sheaves on X for the Zariski topology (with $K_2(F_X)$ constant and F_D^* supported on D) and define *sheaves* Σ_2 and $\mathcal{C}_2$ so the sequence

$$0 \longrightarrow \Sigma_2 \longrightarrow K_2(F_X) \xrightarrow{T} \bigoplus_D F_D^* \longrightarrow \mathcal{C}_2 \longrightarrow 0 \tag{1.18}$$

is exact. It follows from (1.16) that

$$\Gamma(U, \Sigma_2) = \bigcap_{D \cap U \neq \varnothing} K_2(\mathcal{O}_{X,D}) \tag{1.19}$$

where $\mathcal{O}_{X,D}$ is the discrete valuation ring of functions with no poles on D and the intersection is taken in $K_2(F_X)$.

As a consequence of the last assertion in (1.13) and (1.19), there is a natural map

$$\alpha : K_2(\mathcal{O}_X) \longrightarrow \Sigma_2 \tag{1.20}$$

which I conjecture is an isomorphism. In algebraic terms, this is equivalent to:

Conjecture 1.21. Let A be a local noetherian integral domain with quotient field F. Then

(i) $K_2(A) \hookrightarrow K_2(F)$.

(ii) Assume in addition that A is normal (take A to be regular if you like). Then

$$K_2(A) = \bigcap_{p \,\mathrm{ht}\, 1} K_2(A_p)$$

where p runs through all height 1 primes in A.

2. The second Chern class

Throughout this section, X will be a separated quasi-compact regular scheme.

LEMMA 2.1. *Let*

$$0 \longrightarrow A \longrightarrow H \xrightarrow{\alpha} G \longrightarrow 1$$

be a central extension (i.e., $A \subset$ Center H) of sheaves of groups for the Zariski topology on X. Assume for local ring R on X and any $f \in R$ we have $G(\operatorname{Spec} R_f)/\alpha(H(\operatorname{Spec} R_f)) = (e)$, where

$$G(\operatorname{Spec} R_f) = \varinjlim G(V_f) ,$$

the direct limit being taken over affine $V \subset X$ containing the point corresponding to R. Then there is a long exact sequence of pointed sets

$$0 \longrightarrow \Gamma(X, A) \longrightarrow \Gamma(X, H) \longrightarrow \cdots \longrightarrow \check{H}^1(X, H) \longrightarrow \check{H}^1(X, G) \xrightarrow{\partial} \check{H}^2(X, A) .$$

($\check{H}^i$ = Čech cohomology.)

Proof. All this is well known except perhaps the existence of ∂ and exactness at $\check{H}^1(X, G)$. Let g_{ij} be a Čech 1-cocycle with values in G with respect to an open affine cover $\{U_i\}$ of X, and assume g_{ij} lifts to a cochain h_{ij} with values in H.

Clearly $a_{ijk} = h_{ik}h_{jk}^{-1}h_{ij}^{-1}$ is a two-cochain with values in A. We will show that a_{ijk} is a two-cocycle which cobounds if and only if the class of g_{ij} lies in the image of $H^1(X, H) \to H^1(X, G)$. We compute

$$\begin{aligned} a_{jkl}^{-1}a_{ijl}^{-1}a_{ikl}a_{ijk} &= (h_{jk}h_{kl}h_{jl}^{-1})(h_{ij}h_{jl}h_{il}^{-1})(h_{il}h_{kl}^{-1}h_{ik}^{-1})\cdot(h_{ik}h_{jk}^{-1}h_{ij}^{-1}) \\ &= (h_{jk}h_{kl}h_{jl}^{-1})(h_{ij}h_{jl}h_{kl}^{-1}h_{jk}^{-1}h_{ij}^{-1}) \\ &= h_{ij}(h_{jk}h_{kl}h_{jl}^{-1})h_{jl}h_{kl}^{-1}h_{jk}^{-1}h_{ij}^{-1} = 1 \end{aligned}$$

where terms in parentheses lie in A and hence commute with terms in H. It follows that a_{ijk} is a two-cocycle. If $a_{ijk} = a'_{ik}a'^{-1}_{jk}a'^{-1}_{ij}$ then $h'_{ij} = h_{ij}\cdot a'^{-1}_{ij}$ is a two-cocycle lifting g_{ij}. The reader can check that the cohomology class of a_{ijk} is independent of the various choices made.

It remains to show that by refining the cover, we can lift g_{ij} to an h_{ij}. Using quasi-compactness, this reduces to the following assertion:

Let $U, U' \in \{U_i\}$, and let g be the value of the cocycle on $U \cap U'$. Let $x \in U' - U$ be a point. Then there exists an affine V, with $x \in V \subset U'$ such that $g \mid U \cap V$ lifts to a section of H.

Let $U' = \operatorname{Spec} A$; by separation $U \cap U' = \operatorname{Spec} B$ is affine, so $U' - U = U' - U \cap U'$ is a divisor D on U'. By regularity D is principal near x so restricting U' we may assume $D: f = 0$, $U \cap U' = \operatorname{Spec} A_f$. If $\mathcal{P} \subset A$ is the ideal of x, we may further restrict by inverting functions in $A - \mathcal{P}$, so it suffices to know $G(\operatorname{Spec}(A_f[A - \mathcal{P}]^{-1})) = \alpha(H(\operatorname{Spec}(A_f[A - \mathcal{P}]^{-1})))$. Since

$A_f[A - \mathcal{P}]^{-1} = R_f$, $R = A_{\mathcal{P}}$, this holds by hypothesis. Q.E.D.

Example. The above criterion works also in the étale topology, and can be applied, for example to the sequence

$$0 \longrightarrow \mathrm{G}_m \longrightarrow GL_n \longrightarrow PGL_{n-1} \longrightarrow 0 .$$

Indeed, $PGL_{n-1}(R_f)/\mathrm{Im}\,(GL_n(R_f)) \hookrightarrow H^1(\mathrm{Spec}\, R_f, \mathrm{G}_m) = (0)$ for R regular local. I am indebted to R. Hoobler for pointing out the need for caution in trying to define ∂.

Of interest to us is the sequence in (1.13),

$$0 \longrightarrow K_2(\mathcal{O}_X) \longrightarrow St(\mathcal{O}_X) \longrightarrow SL(\mathcal{O}_X) \longrightarrow 1 .$$

To apply (2.1), it suffices to show $SL(R_f)/E(R_f) \underset{\text{def.}}{=\!=} SK_1(R_f) = (0)$.

PROPOSITION. *Let R be a local noetherian ring, $f \in R$ such that R_f is regular. Then $SK_1(R_f) = (0)$.*

Proof. By induction on the dimension $d = \dim R$. If $d = 0$ we are done by known results in K-theory. Similarly it is known that $SK_1(Q) = (0)$ where Q is the quotient field of R_f, so it suffices to show $K_1(R_f) \hookrightarrow K_1(Q)$.

Given any $g \in R_f$, we can find elements $f_i \in R_f$ and a diagram of rings

$$\begin{array}{ccc} & R_{fg} & \\ \nearrow & & \searrow \\ R_f \subset R_1 \subset R_2 \subset & \cdots & \subset R_n \end{array}$$

such that $R_i = (R_{i-1})_{f_i}$ and such that $R_{i-1}/f_i R_{i-1}$ is regular. Thus it suffices to show $K_1(R_f) \hookrightarrow K_1(R_{fg})$ when $R_f/gR_f = (R/gR)_f$ is regular.

In this case, the exact sequence for localization gives

$$\begin{array}{ccccccc} K_2(R_{fg}) & \xrightarrow{\partial} & K_1((R/gR)_f) & \longrightarrow & K_1(R_f) & \longrightarrow & K_1(R_{fg}) . \\ & & \Big\| \text{by induction} & & & & \\ & & (R/gR)_f^* & & & & \end{array}$$

Since ∂ is surjective (proof: $R_f^* \to (R/gR)_f^*$ is surjective because R is local. Given $\bar{u} \in (R/gR)_f^*$ let u be a lifting. Then $\bar{u} = \partial\{g, u\}$.), we are done. Q.E.D.

Applying (2.1) to the sequence (1.13) yields

$$(2.2) \qquad \check{H}^1(X, St(\mathcal{O}_X)) \longrightarrow \check{H}^1(X, SL(\mathcal{O}_X)) \xrightarrow{CK_2} H^2(X, K_2(\mathcal{O}_X)) .$$

Geometrically, $H^1(X, SL(\mathcal{O}_X))$ is the set of stable isomorphism classes of SL-bundles. Given such a class ξ, define the *K-theoretic second Chern class* of ξ to be $CH_2(\xi) \in H^2(X, K_2(\mathcal{O}_X))$.

Remark 2.3. The sequence

$$1 \longrightarrow SL \longrightarrow GL \xrightarrow{\det} \mathbf{G}_m \longrightarrow 0$$

is split, so $H^1(X, SL) \hookrightarrow H^1(X, GL)$. We may therefore interpret $H^1(X, SL)$ as the set of stable isomorphism classes of vector bundles whose determinant bundle is trivial.

PROPOSITION 2.4. *Let $0 \to E' \to E \to E'' \to 0$ be an exact sequence of vector bundles with trivial determinant bundles on X. Then $CK_2(E) = CK_2(E') \cdot CK_2(E'')$.*

Proof. The reader can easily check that there exists an open covering $\{U_i\}$ of X and transition matrices α'_{ij}, α_{ij}, α''_{ij} for E', E, E'' with respect to $\{U_i\}$ such that $\det \alpha'_{ij} = \det \alpha_{ij} = \det \alpha''_{jj} = 1$ and moreover

$$\alpha_{ij} = \begin{pmatrix} \alpha'_{ij} & * \\ 0 & \alpha''_{ij} \end{pmatrix}.$$

Shrinking the covering if necessary, we may assume that α'_{ij} and α''_{ij} lie in the image of

$$St(\Gamma(U_{ij}, \mathcal{O}_X)) \longrightarrow SL(\Gamma(U_{ij}, \mathcal{O}_X)) .$$

It follows that

$$\alpha_{ij} = \begin{pmatrix} \alpha'_{ij} & q_{ij} \\ 0 & \alpha''_{ij} \end{pmatrix} = \begin{pmatrix} I_m & q_{ij}\alpha''^{-1}_{ij} \\ 0 & I_{n-m} \end{pmatrix} \begin{pmatrix} \alpha'_{ij} & 0 \\ 0 & \alpha''_{ij} \end{pmatrix} \tag{2.5}$$

lies in the image as well (see the discussion of T below).

Choose liftings a'_{ij}, $a''_{ij} \in St$ of the matrices

$$\begin{pmatrix} \alpha'_{ij} & 0 \\ 0 & I_{n-1} \end{pmatrix},$$
$$\begin{pmatrix} I_m & 0 \\ 0 & \alpha''_{ij} \end{pmatrix}$$

respectively. Note that a'_{ij} and a''_{ij} commute by (1.3). Let $T \subset St$ be the subgroup (subgroup functor, really) generated by all X^{λ}_{pq} for $p < q$. An element in T can be written as a word in which the X_{pq} appear in lexicographic order (compare [7, p. 78]), from which it follows that T maps isomorphically onto the group of upper triangular matrices (with 1's on the diagonal) in SL. Thus, writing

$$\beta_{ij} = \begin{pmatrix} I_m & q_{ij}\alpha''^{-1}_{ij} \\ 0 & I_{n-m} \end{pmatrix},$$

there is a unique $t_{ij} \in T$ mapping to β_{ij}.

LEMMA 2.6. $t_{ik} = t_{ij}a'_{ij}a''_{ij}t_{jk}a''^{-1}_{ij}a'^{-1}_{ij}$.

Proof. Note

$$I_n = \alpha_{ij}\alpha_{jk}\alpha_{ik}^{-1} = \beta_{ij}\begin{pmatrix}\alpha'_{ij} & 0\\ 0 & \alpha''_{ij}\end{pmatrix}\beta_{jk}\begin{pmatrix}\alpha'_{jk} & 0\\ 0 & \alpha''_{jk}\end{pmatrix}\begin{pmatrix}\alpha'^{-1}_{ik} & 0\\ 0 & \alpha''^{-1}_{ik}\end{pmatrix}\beta_{ik}^{-1}$$

so

$$\beta_{ik} = \beta_{ij}\begin{pmatrix}\alpha'_{ij} & 0\\ 0 & \alpha''_{ij}\end{pmatrix}\beta_{jk}\begin{pmatrix}\alpha'^{-1}_{ij} & 0\\ 0 & \alpha''^{-1}_{ij}\end{pmatrix};$$

i.e., the identity (2.6) holds in SL. To prove the lemma, it suffices to show $a'_{ij}a''_{ij}t_{jk}a''^{-1}_{ij}a'^{-1}_{ij} \in T$. Dropping the subscripts, we have

$$\begin{aligned} t &= \text{word on } X^{\lambda}_{pq}\,, && p<q,\ 1 \leqq p \leqq m,\ m<q\leqq n\\ a' &= \text{word on } X^{\lambda}_{pq}\,, && 1\leqq p\leqq m,\ 1\leqq q\leqq m\\ a'' &= \text{word on } X^{\lambda}_{pq}\,, && m+1\leqq p\leqq n,\ m+1\leqq q\leqq n\,. \end{aligned}$$

We must show whenever $p \leqq m < q$ and either $r,\ s > m$ or $r,\ s \leqq m$,

$$X^{\mu}_{rs}X^{\lambda}_{pq}X^{-\mu}_{rs} = \text{word on } X_{p',q'}\,, \qquad p' \leqq m < q'\,.$$

Suppose, for example, $r,\ s \leqq m$. Then either $s = p$ and $X^{\mu}_{rp}X^{\lambda}_{pq}X^{-\mu}_{rp} = X^{\mu\lambda}_{rq}X^{\lambda}_{pq}$, or $s \neq p$ in which case $X^{\mu}_{rs}X^{\lambda}_{pq}X^{-\mu}_{rs} = X^{\lambda}_{pq}$. The case $r,\ s > m$ is equally easy, completing the proof of (2.6).

To prove (2.4), take the liftings a'_{ij}, a''_{ij}, and $a_{ij} = t_{ij}a'_{ij}a''_{ij} \in St(\Gamma(U_{ij}, \mathcal{O}_X))$ for the transition matrices α'_{ij}, α''_{ij}, α_{ij}. The cocycle representing $CK_2(E)$ is given by

$$\begin{aligned} a_{ik}a^{-1}_{jk}a^{-1}_{ij} &= t_{ik}a'_{ik}a''_{ik}a''^{-1}_{jk}a'^{-1}_{jk}t^{-1}_{jk}a''^{-1}_{ij}a'^{-1}_{ij}t^{-1}_{ij}\\ &= t_{ij}a'_{ij}a''_{ij}t_{jk}a''^{-1}_{ij}a'^{-1}_{ij}a'_{ik}a''_{ik}a''^{-1}_{jk}a'^{-1}_{jk}t^{-1}_{jk}a''^{-1}_{ij}a'^{-1}_{ij}t^{-1}_{ij}\,. \end{aligned} \tag{2.6}$$

Since this expression lies in K_2, it is invariant under conjugation. Conjugating by the last four letters successively gives

$$\begin{aligned} a_{ik}a^{-1}_{jk}a^{-1}_{ij} &= a''^{-1}_{ij}a'^{-1}_{ij}a'_{ik}a''_{ik}a''^{-1}_{jk}a'^{-1}_{jk}\\ &= a'_{ik}a''_{ik}a''^{-1}_{jk}a'^{-1}_{jk}a'^{-1}_{ij}a''^{-1}_{ij}\,. \end{aligned}$$

Finally, since the a''s commute with the a'''s, one gets

$$\begin{aligned} CK_2(E) &= (a'_{ik}a'^{-1}_{jk}a'^{-1}_{ij})(a''_{ik}a''^{-1}_{jk}a''^{-1}_{ij})\\ &= CK_2(E')CK_2(E'')\,, \end{aligned}$$

proving (2.4).

Let $\widetilde{K}_0(X)$ denote the reduced Grothendieck group of vector bundles on X and define $K_0^{(2)}(X)$ by the exact sequence

$$0 \longrightarrow K_0^{(2)}(X) \longrightarrow \widetilde{K}_0(X) \xrightarrow{\det} \mathrm{Pic}\,(X) \longrightarrow 0\,.$$

COROLLARY 2.7. *CK_2 induces a homomorphism* $K_0^{(2)}(X) \xrightarrow{CK_2} H^2(X, K_2(\mathcal{O}_X))$.

Proof. Let A be the free abelian group generated by elements (E, F), where E and F are isomorphism classes of vector bundles on X with rank $E =$ rank F and $\det E = \det F$. Define $f: A \to H^2(X, K_2(\mathcal{O}_X))$ by

$$f(E, F) = CK_2(E \oplus \det E^*) - CK_2(F \oplus \det E^*) .$$

Let $A_0 \subset A$ be the subgroup generated by elements

$$(1) \qquad (E \oplus E', F \oplus F') - (E, F) - (E', F') ,$$

$$(2) \qquad (E, E' \oplus E'') \text{ for all exact sequences}$$

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0 .$$

LEMMA. $f(A_0) = 0$.

Proof. We have

$$\begin{aligned} f(E \oplus E', F \oplus F') = {} & CK_2(E \oplus E' \oplus (\det E^* \otimes \det E'^*)) \\ & - CK_2(F \oplus F' \oplus (\det E^* \otimes \det E'^*)) . \end{aligned}$$

On the other hand,

$$\begin{aligned} f(E, F) + f(E', F') = {} & CK_2(E \oplus E' \oplus \det E^* \oplus \det E'^*) \\ & - CK_2(F \oplus F' \oplus \det E^* \oplus \det E'^*) . \end{aligned}$$

Subtracting these two equations and using the additivity of CK_2 for vector bundles with trivial determinant, one gets $f(E \oplus E', F \oplus F') - f(E, F) - f(E', F') = 0$.

Now let $0 \to E' \to E \to E'' \to 0$ be an exact sequence of vector bundles. I must show $f(E, E' \oplus E'') = 0$. This is clear if $E' = (0)$, so by the above it suffices to show $f(E \oplus G, E' \oplus E'' \oplus G) = 0$ for some G. Take $G = \det E'^* \oplus \det E''^*$. The sequence

$$0 \longrightarrow E' \oplus \det E'^* \longrightarrow E \oplus G \longrightarrow E'' \oplus \det E''^* \longrightarrow 0$$

is a sequence of bundles with trivial determinant, so $f(E \oplus G, E' \oplus E'' \oplus G) = 0$ by (2.4). Q.E.D.

LEMMA. $A/A_0 \xrightarrow{\sim} K_0^{(2)}(X)$.

Proof. There is an obvious surjective map φ, $\varphi(E, F) = [E] - [F]$. An easy consequence of relation (1) above is that any element in A/A_0 can be written in the form (E, F). The assertion $\varphi(E, F) = 0$ amounts to saying there exist exact sequences $0 \to G' \to G \to G'' \to 0$, $0 \to H' \to H \to H'' \to 0$ such that $E \oplus G \oplus H' \oplus H'' \cong F \oplus H \oplus G'' \oplus G'$. It follows that $0 = (E \oplus G \oplus H' \oplus H'', F \oplus H \oplus G'' \oplus G') = (E, F)$ as claimed.

The existence of $CK_2: K_0^{(2)}(X) \to H^2(X, K_2(\mathcal{O}_X))$ follows from these lemmas.

PROPOSITION 2.8. *Let E be a vector bundle with trivial determinant bundle on X. Assume E has a filtration whose successive quotients are line bundles $L^{(1)}, \cdots, L^{(n)}$, and let $u_{ij}^{(p)}$ be the transition functions for $L^{(p)}$ with respect to some fixed open cover independent of p. Suppose the $u_{ij}^{(p)}$ normalized so that $\prod_{p=1}^{n} u_{ij}^{(p)} = 1$ for all i, j. Then a 2-cocycle representing $CK_2(E)$ is*

$$\prod_{p\leq q\leq n-1} \{u_{ij}^{(q)}, u_{jk}^{(p)}\}^{-1} .$$

Proof. Suppose $E = \bigoplus L^{(p)}$. Let $d_{lm}(u)$ denote the diagonal matrix with u in the l^{th} column and u^{-1} in the m^{th}. There is a "standard" lifting $h_{lm}(u) \in \text{St}$ of $d_{lm}(u)$, and

$$\{u, v\} = [h_{lm}(u), h_{ln}(v)] = h_{ln}(uv)h_{ln}^{-1}(u)h_{ln}^{-1}(v) , \qquad m \neq n , \tag{2.9}$$

([7, § 8, 9]).

The transition matrices for E have the form

$$\alpha_{ij} = d_{1n}(u_{ij}^{(1)})d_{2n}(u_{ij}^{(2)}) \cdots d_{n-1,n}(u_{ij}^{(n-1)}) ,$$

so we must show

$$\begin{aligned} &h_{1n}(u_{ik}^{(1)})\cdot \ \ldots\ \cdot h_{n-1,n}(u_{ik}^{(n-1)})h_{n-1,n}^{-1}(u_{jk}^{(n-1)})\cdot \ \ldots \\ &\cdot h_{1,n}^{-1}(u_{jk}^{(1)})\cdot h_{n-1,n}^{-1}(u_{ij}^{(n-1)})\cdot \ \ldots\ \cdot h_{1n}^{-1}(u_{ij}^{(1)}) \\ &= \prod_{p\leq q\leq n-1} \{u_{ij}^{(q)}, u_{ik}^{(p)}\}^{-1} . \end{aligned} \tag{2.10}$$

When $n = 2$, (2.10) reads

$$h_{1n}(u_{jk}^{(1)})h_{1n}^{-1}(u_{jk}^{(1)})h_{1n}^{-1}(u_{ij}^{(1)}) = \{u_{ij}^{(1)}, u_{jk}^{(1)}\}^{-1} , \tag{2.11}$$

an equation which follows from (2.9).

The proof is now by induction on n. Write $\theta(1, \cdots, n)$ for the left hand side of (2.10). Since $\theta(1, \cdots, n) \in K_2$, $\theta = h_{1n}^{-1}(u_{ik}^{(1)})\theta h_{1n}(u_{ik}^{(1)})$. Use the identities

$$h_{1n}^{-1}(u_{jk}^{(1)})h_{l,n}^{-1}(u_{ij}^{(l)}) = h_{ln}^{-1}(u_{ij}^{(l)})h_{1n}^{-1}(u_{jk}^{(1)})[h_{1n}(u_{jk}^{(1)}), h_{ln}(u_{ij}^{(l)})] \tag{2.12}$$

to bring $h_{1,n}^{-1}(u_{jk}^{(1)})$ to the right. Since $h_{l,n} = h_{n,l}^{-1}$ ([7, p. 76]), the commutation in (2.12) can be rewritten

$$[h_{1n}, h_{ln}] = h_{n1}^{-1}h_{nl}^{-1}h_{n1}h_{nl} = h_{1n}h_{ln}h_{1n}^{-1}h_{ln}^{-1} = \{u_{ij}^{(l)}, u_{jk}^{(1)}\}^{-1} .$$

These symbols can be shifted to the right as well, so one gets finally

$$\begin{aligned} \theta(1, \cdots, n) &= \theta(2, \cdots, n)h_{1n}^{-1}(u_{jk}^{(1)})h_{1n}^{-1}(u_{ij}^{(1)})h_{1n}(u_{ik}^{(1)})\cdot\prod_{l=2}^{n-1} \{u_{ij}^{(l)}, u_{jk}^{(1)}\}^{-1} \\ &= \theta(2, \cdots, n) \prod_{l=1}^{n-1} \{u_{ij}^{(l)}, u_{jk}^{(1)}\}^{-1} . \end{aligned}$$

The desired formula follows by induction.

Now assume $E \neq \bigoplus L^{(p)}$, and take $E' = \bigoplus L^{(p)}$. Write $F = E \bigoplus E'^*$. By (2.4), $CK_2(F) = CK_2(E)\cdot CK_2(E'^*)$. On the other hand, F has a filtration with successive quotients $F^{(p)} = L^{(p)} \oplus L^{(p)*}$ so again by (2.4), $CK_2(F) = \prod_{p\leq n} CK_2(F^{(p)})$. From what has been proven, $CK_2(E')$ and $CK_2(F^{(p)})$ are represented by cocycles

$$\prod_{p\leq q\leq n-1} \{u_{ij}^{(p)}, u_{jk}^{(q)}\}^{-1}$$

and

$$\{u_{ij}^{(p)}, u_{jk}^{(p)}\}^{-1}$$

respectively. Using these formulae and the relation $u_{ij}^{(n)} = [\prod_{1\leq p\leq n-1} u_{ij}^{(p)}]^{-1}$, one easily verifies the desired formula for $CK_2(E)$. Q.E.D.

Now assume that X is a regular scheme, and that any coherent sheaf on X can be realized as a quotient of a locally-free sheaf (e.g. X quasi-projective). Given a subscheme $Z \subset X$ of codimension 2, there exists a resolution $E^{\bullet} \to \mathcal{O}_Z$ of $\mathcal{O}_Z$ by a finite complex of locally-free sheaves E^i, $0 \leqq i \leqq n = \dim X$. Define

$$CK_2(Z) = CK_2(E^{\bullet}) \ . \tag{2.13}$$

Note that $[E^{\bullet}] \in K_0(X)$ is independent of the resolution $E^{\bullet}$ [12], and $\det [E^{\bullet}] = 0$ so (2.13) is defined. Moreover, a computation of Grothendieck [3] shows that for $Y \subset X$ of codimension p and $F^{\bullet}$ a resolution of $\mathcal{O}_Y$, then at least if X is projective [15]

$$CH_2(F^{\bullet}) = (p-1)!\, CH_2(Y) \ ,$$

where $CH_2(F^{\bullet})$ and $CH_2(Y)$ are respectively the second Chern class and the cycle class taken in the Chow group of codimension two cycles modulo rational equivalence. In our case $p = 2$, so it is plausible to view $CK_2(Z)$ as defining a cycle class in $H^2(X, K_2(\mathcal{O}_X))$. Unfortunately, except in the case $\dim X = 2$ (see § 5 bis below) we have been unable to show that CK_2 is a cycle map in the classical sense, i.e., that $CK_2(Z)$ depends only on the support of $\mathcal{O}_Z$ and the length of the artinian local rings $\mathcal{O}_{Z,z}$ for $z \in Z$ generic. This would follow from a standard dévissage argument if one knew that the *local cohomology sheaves* $\underline{H}_Y^i(X, K_2(\mathcal{O}_X))$ were zero for $Y \subset X$ a closed set of codimension $\geqq 3$ and $i = 0, 1, 2$.

Remark 2.14. For any scheme X, there is a natural pairing

$$\begin{array}{ccccc} H^1(X, \mathcal{O}_X^*) \times H^1(X, \mathcal{O}_X^*) & \xrightarrow{\text{cup product}} & H^2(X, \mathcal{O}_X^* \otimes_{\mathbf{Z}} \mathcal{O}_X^*) & \xrightarrow{\{\ \}} & H^2(X, K_2(\mathcal{O}_X)) \\ \{u_{ij}\} \times \{v_{jk}\} & \longmapsto & & & \{u_{ij}, v_{jk}\} \end{array}$$

which can be viewed as defining a cycle class for subschemes $Z \subset X$ given as global complete intersections. This construction is easily seen to agree with CK_2, e.g. on bundles of the form $L \oplus L^{-1}$ (L a line bundle) both define the same second Chern class.

3. Cycles in $H^2(X, \Sigma_2)$

Throughout this section, X is assumed to be a regular algebraic k-scheme, where k is a field. Recall the sheaves Σ_2, $\mathcal{C}_2$ on X are defined by the sequence (1.18)

$$0 \longrightarrow \Sigma_2 \longrightarrow K_2(F_X) \xrightarrow{T} \bigoplus_D F_D^* \longrightarrow \mathcal{C}_2 \longrightarrow 0 .$$

Note that the higher Zariski cohomology of the sheaves $K_2(F_X)$ and $\bigoplus F_D^*$ is trivial, so one gets an exact sequence

$$\bigoplus_D F_D^* \longrightarrow \Gamma(X, \mathcal{C}_2) \xrightarrow{\partial} H^2(X, \Sigma_2) \longrightarrow 0 . \tag{3.1}$$

Definition 3.2. $\mathcal{C}_2$ will be called the sheaf of *codimension two Cartier cycles* on X.

Let $\tilde{\mathfrak{Z}}^2$ (resp. $\tilde{\mathfrak{Z}}^2_{\text{l.c.i}}$) be the sheaf on X whose sections over an open U are the free abelian group generated by integral subschemes $Z \subset U$ (resp. free abelian group generated by local complete intersections $Z \subset U$) of codimension two. There is a natural map

$$\begin{aligned} \tilde{\mathfrak{Z}}^2_{\text{l.c.i}} &\longrightarrow \tilde{\mathfrak{Z}}^2 \\ Z &\longmapsto \sum n_i Z_{i,\text{red}} \end{aligned}$$

where Z_i runs through the irreducible components of Z, n_i is the multiplicity of Z_i on X, and $Z_{i,\text{red}}$ indicates reduced structure.

THEOREM 3.3. *There exists a commutative triangle*

$$\begin{tikzcd} & \mathcal{C}^2 \arrow[dr, "j"] & \\ \tilde{\mathfrak{Z}}^2_{\text{l.c.i}} \arrow[ur, "i"] \arrow[rr] & & \tilde{\mathfrak{Z}}^2 . \end{tikzcd} \tag{3.4}$$

Proof. Definition of j. It suffices to define a map $j' \colon \bigoplus_D F_D^* \to \tilde{\mathfrak{Z}}^2$ such that the composition

$$K_2(F_X) \xrightarrow{T} \bigoplus_D F_D^* \xrightarrow{j'} \tilde{\mathfrak{Z}}^2$$

is zero. Working locally, let $f \in F_D^*$ where $D\colon g = 0$. Assume first that f is a regular function on D, and lift to $\tilde{f} \in \mathcal{O}_X$. Let γ_i be the irreducible components of $\{\tilde{f} = g = 0\}$, μ_i the multiplicity of the intersection $\{\tilde{f} = 0\} \cap \{g = 0\}$ along γ_i, and take

$$j'(f|_D) = \sum \mu_i \gamma_i .$$

LEMMA 3.5. *$j'(f|_D)$ is independent of the choice of $\tilde{f}$. Moreover, $j'(ff'|_D) = j'(f|_D) + j'(f'|_D)$.*

Proof of Lemma. Clearly the γ_i are independent of the lifting $\tilde{f}$. Let A_i

be the local ring of the generic point of γ_i on X. The integer μ_i can be computed as the alternating sum of the lengths of the homology groups of the complex

$$\underset{\deg -1}{A_i/\tilde{f}A_i} \xrightarrow{g} \underset{\deg 0}{A_i/\tilde{f}A_i} .$$

Since A_i is Cohen-Macauley (it is regular) and the ideal $(\tilde{f}, g)$ has height 2, $\tilde{f}$, g is a regular sequence and the above complex has cohomology only in degree 0. But $H_0 = A_i/(\tilde{f}, g)A_i$ is clearly independent of the lifting $\tilde{f}$.

Writing $\tilde{F} = \{\tilde{f} = 0\}$, we have (in the notation of Weil [13]) $\mu_i = i(\tilde{F}, D{:}\,\gamma_i)$. The assertion about $j'(ff'|_D)$ thus follows from standard results in intersection theory.

As a consequence of the lemma, we can define $j'(f|_D)$ for any $f \in F_D^*$ by writing (locally) $f = f_1/f_2$ with f_1, $f_2 \in \mathcal{O}_D$ and taking

$$j'(f|_D) = j'(f_1|_D) - j'(f_2|_D) .$$

To show $j' \circ T = 0$, it suffices by (1.14) to show $j' \circ T\{x, y\} = 0$, x, $y \in F_X^*$. Writing (locally) $x = x_1/x_2$, $y = y_1/y_2$ with x_i, y_i regular functions, and using the multiplicative properties of $\{\ \}$, we reduce to the case x and y regular and $F{:}\ x = 0$, $G{:}\ y = 0$ distinct irreducible divisors with multiplicity 1. The assertion now becomes

$$j'(x|_G) = j'(y|_F)$$

which is clear.

Definition of i in (3.4). Let $\gamma \subset X$ be a local complete intersection of codimension two in X. Given $x \in \gamma$ let $\gamma{:}\, f = g = 0$ in some neighborhood of x and define the map on stalks in i_x by

$$i_x(\gamma) \equiv f|_{\{g=0\}} \pmod{\text{Image } T} .$$

We must show i independent of the choice of f and g. Suppose $\gamma{:}\ f' = g' = 0$ near x. Clearly there exists a g'' such that $\gamma{:}\ f = g'' = 0$ and $\gamma{:}\ f' = g'' = 0$. It therefore suffices to show

$$\text{(3.6)} \qquad f|_{\{g=0\}} \equiv f|_{\{g''=0\}} \underset{(1)}{\equiv} f'|_{\{g''=0\}} \equiv f'|_{\{g'=0\}} \pmod{\text{Im } T} .$$

Note that

$$f|_{\{g=0\}} = g|_{\{f=0\}} \cdot T\{f, g\} .$$

Using identities of this sort, the verification of (3.6) reduces to that of congruence (1).

The ideals (f, g'') and (f', g'') coincide at x so we have

$$f = af' + bg'' ,$$
$$f' = cf + dg''$$

with a, b, c, d regular at x. Thus

(3.7)
$$f|_{\{g''=0\}} = af'|_{\{g''=0\}} ,$$
$$f'|_{\{g''=0\}} = cf|_{\{g''=0\}}$$

so $f/f'|_{\{g''=0\}} = u|_{\{g''=0\}}$ where u is a unit in some neighborhood of x on X. Hence

$$f|_{\{g''=0\}} = f'|_{\{g''=0\}} \cdot T\{u, g''\} ,$$

proving (1).

This completes the definition of i. The fact that the triangle in (3.3) commutes is clear, so the theorem is proved.

Remarks 3.8. (i) Combining (3.1) and (3.3), one gets

(3.9)
$$\begin{tikzcd} \bigoplus_D F_D^* \arrow[r] & L(X, \mathcal{C}_2) \arrow[r, "\partial"] \arrow[d, "j"] & H^2(X, \Sigma_2) \arrow[r] & 0 \\ \mathcal{Z}^2_{\mathrm{l.c.i}} \arrow[ur, "i"] \arrow[r] & \mathcal{Z}^2 & & \end{tikzcd}$$

where $\mathcal{Z}^2 = \Gamma(\tilde{\mathcal{Z}}^2)$.

(ii) Given a local complete intersection $Z \subset X$ of codimension two, the cycle class $C\Sigma_2(Z)$ is defined by

(3.10)
$$C\Sigma_2(Z) = \partial \circ i(Z) .$$

(iii) The fact that ∂ in (3.9) is surjective says that $H^2(X, \Sigma_2)$ satisfies *inversion*, i.e., to any $\xi \in H^2(X, \Sigma_2)$ one can associate a non-empty class of cycles $j(\partial^{-1}(\xi)) \subset \mathcal{Z}^2$. Since there is a natural map $\alpha: H^2(X, K_2(\mathcal{O}_X)) \to H^2(X, \Sigma_2)$ (1.20), a similar remark holds for $H^2(X, K_2)$.

Notice that the above proof of inversion does not work in topologies where constant sheaves have non-trivial higher cohomology. One of the problems in carrying what has been done over to the analytic category (a necessity, it seems, if one wants to attack the Hodge conjecture) is to find some alternative argument for inversion.

Definition 3.11. Two cycles $z_0, z_\infty \in \mathcal{Z}^2$ are said to be *rationally equivalent* if there exists a (not necessarily effective) $D \in \mathcal{Z}^2_{\mathbf{P}^1 \times X}$ such that $z_i = D \cap (\{i\} \times X)$, $i = 0, \infty$ where components of the intersection are counted with appropriate multiplicity. To insure that the intersections are defined, we assume the components of D are flat over $\mathbf{P}^1$.

Definition 3.12. Let Y be an integral noetherian scheme. Two (Weil) divisors γ_1, γ_2 on Y are said to be linearly equivalent if $\pi^*\gamma_1 - \pi^*\gamma_2$ is the divisor of a function, where $\pi\colon \tilde{Y} \to Y$ is the normalization.

Given z_0, $z_\infty \in \mathcal{Z}^2$, write $z_0 \underset{\text{pre}}{\sim} z_\infty$ (z_0 is pre-equivalent to z_∞) if there exists an irreducible divisor $D \subset X$ such that $\operatorname{supp}(z_i) \subset D$, $i = 0, \infty$, and z_0 is linearly equivalent to z_∞ on D.

PROPOSITION 3.13. *Rational equivalence is the equivalence relation generated by pre-equivalence.*

Proof. Suppose $z_0 \underset{\text{pre}}{\sim} z_\infty$ and let D be the corresponding divisor. There is an $f \in F_D^*$ defining the linear equivalence, and f gives a rational map $D \dashrightarrow \mathbf{P}^1$. Let $\Gamma \subset \mathbf{P}^1 \times X$ be the closure of the graph of f. Γ defines a rational equivalence between z_0 and z_∞.

Conversely, let $Z = \sum a_i Z_i \in \mathcal{Z}^2_{\mathbf{P}^1 \times X}$ with the Z_i flat over $\mathbf{P}^1$. Let $z_0 = Z \cap (\{0\} \times X)$, $z_\infty = Z \cap (\{\infty\} \times X)$, and let D_i be the image of Z_i in X. We may assume the D_i are divisors, else z_0 and z_∞ will have a common component which can be eliminated.

Note that $z_{0i} - z_{\infty i} = Z_i \cdot (\{0\} \times X - \{\infty\} \times X)$ is the divisor of a function f_i on Z_i. Since Z_i is generically finite over D_i, we can define g_i to be the norm of f_i in the function field of D_i. Then g_i defines a pre-equivalence between z_{0i} and $z_{\infty i}$, proving the proposition.

THEOREM 3.14. *Given $\xi \in H^2(X, \Sigma_2)$, the set $j(\partial^{-1}(\xi)) \subset \mathcal{Z}^2$ consists of rationally equivalent cycles. Hence there is an induced map $H^2(X,\Sigma_2) \xrightarrow{\bar{j}} CH^2(X)$.*

Proof. Let $\mu\colon \bigoplus F_D^* \to \mathcal{C}_2$ be the natural map. If c_1, $c_2 \in \Gamma(X, \mathcal{C}_2)$ with $\partial(c_1) = \partial(c_2)$, necessarily $c_1 - c_2 = \mu(\sum_i f_i|_{D_i})$. Let $\pi_i\colon \hat{D}_i \to X$ be the normalization of D_i, so

$$j(c_1) - j(c_2) = j\mu\left(\sum f_i|_{D_i}\right) = \sum \pi_{i*}((f_i)) \underset{\text{rational}}{\sim} 0 .$$

This proves (3.14).

4. Compatibility

Keeping the notations of the previous sections, there are maps

$$(4.1)\qquad \begin{array}{ccccc} H^2(X, K_2(\mathcal{O}_X)) & \xrightarrow{\alpha} & H^2(X, \Sigma_2) & \xrightarrow{\bar{j}} & CH^2(X) . \\ \uparrow{\scriptstyle CK_2} & & & \nearrow{\scriptstyle CH^2} & \\ \left\{\begin{matrix}\text{Codim 2 subschemes} \\ Z \subset X\end{matrix}\right\} & & & & \end{array}$$

THEOREM 4.2. *The diagram* (4.1) *is commutative.*

Proof. Let $\pi\colon P \to X$ be a flag scheme over X (e.g. $P = \mathbf{P}(E)$, E a vector bundle on X). Since the pull-back $\pi^*\colon CH^2(X) \to CH^2(P)$ is injective [3], it is permissible to first pull (4.1) back to P.

(*Remark*. We are unable to show Σ_2 is functorial, for example with respect to a closed immersion $\pi\colon P \hookrightarrow X$. However, when π is surjective functoriality is straightforward.)

Let $E^\bullet \to \mathcal{O}_Z \to 0$ be a finite global resolution for a given codimension two subscheme $Z \subset X$. Let $\pi\colon P \to X$ be such that $\pi^* E^i$ admits a filtration with successive quotients line bundles for all i. Replacing X by P in (4.1) and untangling, it suffices to prove

THEOREM 4.2′. *Let $E = \bigoplus L_i$ be a direct sum of line bundles on X such that* $\det(E)$ *is trivial. Then $\bar{j} \circ \alpha \circ CK_2(E) = CH^2(E)$, where $CH^2(E) \in CH^2(X)$ is the second Chern class in the sense of* [3].

Proof of 4.2′. We write $L \times L' \mapsto \{L, L'\}$ for the map defined in (2.14). Writing $E = (\bigoplus_{i=1}^{n-1} L_i) \oplus (L_1^{-1} L_2^{-1} \cdots L_{n-1}^{-1})$, it follows from (2.8) that $CK_2(E) = \prod_{p \leq q \leq n-1} \{L_p, L_q\}^{-1}$.

On the other hand, let $l_i = CH^1(L_i)$ in the rational equivalence group of divisors. One knows that $CH^2(E)$ is the coefficient of the term of degree $n-2$ in the polynomial

$$[\textstyle\prod_{i=1}^{n-1} (T - l_i)](T + l_1 + \cdots + l_{n-1}) ,$$

that is, $CH^2(E) = -\sum_{p \leq q \leq n-1} l_p \cdot l_q$.

Comparing $CK_2(E)$ and $CH^2(E)$, we must only show $\bar{j} \circ \alpha\{L_p, L_q\} = l_p \cdot l_q$. This is immediate from the following lemma.

LEMMA 4.3. *Let D, D' be divisors on X. Let $D\colon f_i = 0$, $D'\colon f_i' = 0$ be local defining equations with respect to a fixed open cover $\{U_i\}$ of X. Let $u_{ij} = f_i/f_j$, $u_{ij}' = f_i'/f_j'$ on $U_i \cap U_j$. Then $\sigma_i = f_i|_{D'}$ patches to give a global section $\sigma \in \Gamma(X, \mathcal{C}_2)$ such that $j(\sigma) = D \cdot D'$ and $\partial(\sigma) = \{u_{ij}, u_{jk}'\}$ where j and ∂ are as in* (3.1), (3.4) *and $D \cdot D'$ is cycle-theoretic intersection.*

The proof of the lemma is a straightforward cocycle computation and is omitted.

This completes the proof of (4.2).

Remark 4.4. One expects the cycle classes $CK_2(Z)$ and $C\Sigma_2(Z)$ to be compatible when both are defined, but we are not able to prove this. One problem is we cannot show the map $H^2(X, \Sigma_2) \xrightarrow{\pi^*} H^2(P, \Sigma_2)$ is injective when $\pi\colon P \to X$ is a flag scheme (splitting principle).

5. Local cohomology

We continue to assume X is a regular algebraic k-scheme. Given an abelian sheaf F on X and a subset $Z \subset X$, recall there are defined local cohomology groups [4] $H_Z^*(X, F)$ which fit into an exact sequence

$$(5.1) \quad \cdots \longrightarrow H_Z^i(X, F) \longrightarrow H^i(X, F) \longrightarrow H^i(X - Z, F) \longrightarrow H_Z^{i+1}(X, F) \longrightarrow \cdots .$$

One also has local cohomology sheaves $\underline{H}_Z^*(X, F)$ related to $H_Z^*(X, F)$ by a spectral sequence

$$(5.2) \qquad E_2^{p,q} = H^p\big(X, \underline{H}_Z^q(X, F)\big) \Longrightarrow H_Z^{p+q}(X, F) .$$

PROPOSITION 5.3 *and Definition. Let $Z \subset X$ be a local complete intersection of codimension 2. Then there is a canonical cycle class*

$$\langle Z \rangle \in \Gamma\big(X, \underline{H}_Z^2(X, K_2(\mathcal{O}_X))\big) .$$

Proof. Let $m: X - Z \to X$ be the inclusion. We have ([4, p. 9])

$$(5.4) \qquad \underline{H}_Z^2\big(X, K_2(\mathcal{O}_X)\big) \cong R^1 m_*\big(K_2(\mathcal{O}_X) \mid X - Z\big) .$$

Let $z \in Z$ and let U be a neighborhood of z on X such that $Z \cap U: f = g = 0$. Write D_f, D_g for the open sets where f and g are invertible. Then $U - Z = D_f \cup D_g$ and the symbol

$$\{f, g\} \in \Gamma\big(D_{fg}, K_2(\mathcal{O}_X)\big)$$

defines a Čech 1-cocycle and hence a section of $R^1 m_*(K_2 \mid X - Z)$.

Let

$$[\{f, g\}] \in \Gamma\big(U, \underline{H}_Z^2(X, K_2)\big)$$

denote the corresponding section. We must show $[\{f, g\}]$ is independent of f and g, at least locally near z. Suppose $Z: f_1 = g_1 = 0$ near z. Arguing as in (3.6), we may assume $(f, g) = (f, g_1) = (g, g_1)$, and it suffices to show $[\{f, g\}] = [\{f, g_1\}]$. Write $g_1 = af + bg$ with a, b units at z. We compute

$$(5.5) \qquad \begin{aligned} \{f, g_1\}\{f, g\}^{-1} &= \left\{f, b + \frac{af}{g}\right\} = \{f, b\}\left\{f, 1 + \frac{af}{bg}\right\} \\ &= \{f, b\}\left\{\frac{-bg}{a}, 1 + \frac{af}{bg}\right\} \end{aligned}$$

(the last step uses the Steinberg relation $\{-af/bg, 1 + af/bg\} = 1$).

Restricting U so that a and b are units over U, the right-hand terms of (5.5) are sections of K_2 over D_f, D_{gg_1} respectively. Since $U - Z = D_f \cup D_{gg_1}$, (5.5) represents $[\{f, g_1\}][\{f, g\}]^{-1}$ as a coboundary near z. This completes the proof of (5.3).

Remark 5.6. If one knew that

$$\underline{H}^0_Z(X, K_2(\mathcal{O}_X)) = \underline{H}^1_Z(X, K_2(\mathcal{O}_X)) = (0) \ , \tag{5.7}$$

$\langle Z\rangle$ would give a class in $H^2_Z(X, K_2)$ by (5.2), and hence, by (5.1), a class $[Z]$ in $H^2(X, K_2)$. Presumably, one could then show $[Z] = CK_2(Z)$. If, in addition,

$$\underline{H}^i_W(X, K_2(\mathcal{O}_X)) = (0) \ , \qquad i = 0,\ 1,\ 2 \tag{5.8}$$

for all $W \subset X$ of codimension $\geqq 3$, one could identify $CH^2(X)$ as a direct summand of $H^2(X, K_2)$.

LEMMA 5.9. *Let $Z \subset X$ be a subscheme of codimension 2. Then $\underline{H}^i_Z(X, \Sigma_2) =$ (0), $i = 0, 1$.*

Proof. There is an exact sequence ([4, p. 9])

$$0 \longrightarrow \underline{H}^0_Z(X, \Sigma_2) \longrightarrow \Sigma_2 \xrightarrow{\ \phi\ } m_*m^*\Sigma_2 \longrightarrow \underline{H}^1_Z(X, \Sigma_2) \longrightarrow 0 \ .$$

ϕ is injective because Σ_2 is a subsheaf of the constant sheaf $K_2(F_X)$. On the other hand,

$$\Gamma(U, \Sigma_2) = \textstyle\bigcap_{A \in S(U)} K_2(A)$$

where $S(U)$ is the set of all discrete valuation rings $A \subset F_X$ centered on U. Since $S(U) = S(U - Z)$, it follows that ϕ is surjective, proving (5.9).

THEOREM 5.10. (i) $H^2_Z(X, \Sigma_2) \cong \Gamma(X, \underline{H}^2_Z(X, \Sigma_2))$.

(ii) *The composition*

$$\Gamma(X, \underline{H}^2_Z(X, K_2)) \longrightarrow \Gamma(X, \underline{H}^2_Z(X, \Sigma_2)) \cong H^2_Z(X, \Sigma_2) \longrightarrow H^2(X, \Sigma_2)$$

carries a class $\langle Z\rangle$ to $C\Sigma_2(Z)$.

Proof. (i) follows from (5.2) and (5.9). (ii) can be verified by noting $i(Z) \in \Gamma_Z(X, \mathcal{C}_2) \subset \Gamma(X, \mathcal{C}_2)$ and computing the lower boundary map in the diagram

$$\begin{tikzcd}
\Gamma(X, \mathcal{C}_2) \arrow[r, "\partial"] & H^2(X, \Sigma_2) \\
\Gamma_Z(X, \mathcal{C}_2) \arrow[u, hook] \arrow[r, "\partial"] & H^2_Z(X, \Sigma_2) \arrow[u]
\end{tikzcd} \ .$$

Details are omitted.

The local cohomology sequence is also of interest when taken with supports on a divisor:

THEOREM 5.11. *Let $D \subset X$ be a regular divisor. Then*

(i) Pic $(D) = H^1(D, \mathcal{O}^*_D)$ *is a direct summand of $H^2_D(X, \Sigma_{2,X})$; i.e., there exist natural maps* Pic $(D) \xrightarrow{a} H^2_D(X, \Sigma_2) \xrightarrow{b}$ Pic (D) *with $b \circ a =$ identity.*

(ii) *The diagram*

$$\begin{array}{ccc} H^2_D(X, \Sigma_2) & \longrightarrow & H^2(X, \Sigma_2) \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle \bar{j}} \\ \operatorname{Pic}(D) & \longrightarrow & CH^2(X) \end{array}$$

commutes.

Proof. Let $\mathcal{C}_{1,D}$ be the sheaf of cartier divisors on D, defined by the exact sequence

$$0 \longrightarrow \mathcal{O}_D^* \longrightarrow F_D^* \longrightarrow \mathcal{C}_{1,D} \longrightarrow 0 .$$

There is a natural map $\mathcal{C}_{1,D} \to \underline{H}^0_D(X, \mathcal{C}_2)$ obtained by mapping $f(\operatorname{mod} \mathcal{O}_D^*) \mapsto f|_D$ $(\operatorname{mod} \operatorname{Im} T)$. This gives a diagram with exact rows

$$(5.12)\qquad \begin{array}{ccccccc} F_D^* & \longrightarrow & H^0_D(X, \mathcal{C}_2) & \longrightarrow & H^2_D(X, \Sigma_2) & \longrightarrow & 0 \\ \| & & \uparrow & & \uparrow{\scriptstyle a} & & \\ F_D^* & \longrightarrow & \Gamma(\mathcal{C}_{1,D}) & \longrightarrow & \operatorname{Pic}(D) & \longrightarrow & 0 \end{array}$$

and so a map a. The map b arises from

$$H^0_D(X, \mathcal{C}_2) \xrightarrow{j} H^0_D(X, \tilde{\mathcal{Z}}^2) = \Gamma(D, \mathcal{C}_{1,D}) \longrightarrow \operatorname{Pic}(D) .$$

Commutativity in (ii) is left for the reader.

Remark 5.13. Combining (5.11) with (5.1), one gets an exact sequence

$$H^1(X - D, \Sigma_2) \longrightarrow \operatorname{Pic}(D) \longrightarrow CH^2(X) \longrightarrow CH^2(X - D) \longrightarrow 0 .$$

(Surjectivity on the right is clear, since any cycle on $X - D$ can be completed to a cycle on X.) If moreover X is a surface, $U = X - D$, it follows from (5.14) below that $H^1(U, K_2)$ maps onto $H^1(U, \Sigma_2)$, so the left-hand term in the above sequence can be replaced by $H^1(U, K_2(\mathcal{O}_U))$.

5 bis. The case of a surface

PROPOSITION 5.14. *Let X be a normal noetherian scheme, and define sheaves R_0, R_1 on X by the exact sequence*

$$0 \longrightarrow R_0 \longrightarrow K_2(\mathcal{O}_X) \longrightarrow \Sigma_2 \longrightarrow R_1 \longrightarrow 0 .$$

Then R_0 and R_1 are supported on sets of codimension $\geqq 2$ in X. In particular, when $\dim X = 2$, *and $Z \subset X$ is any subset (including $Z = X$)*

$$H^i_Z(X, K_2(\mathcal{O}_X)) \xrightarrow{\sim} H^i_Z(X, \Sigma_2) , \qquad i \geqq 2 ,$$

$$\underline{H}^i_Z(X, K_2) \xrightarrow{\sim} \underline{H}^i_Z(X, \Sigma_2) .$$

Proof. The first assertion follows because

$$(\Sigma_2)_x = K_2(\mathcal{O}_{X,x}) = K_2(\mathcal{O}_X)_x$$

for any $x \in X$ of codimension $\leqq 1$. The assertion about cohomology is an immediate consequence.

THEOREM 5.15. *Let X be a regular algebraic k-scheme of dimension two. Then*

(i) $H^2(X, K_2(\mathcal{O}_X)) \overset{\alpha}{\cong} H^2(X, \Sigma_2) \overset{\bar{j}}{\cong} CH^2(X)$.

(ii) *The diagram*

$$\begin{tikzcd} H^2(X, K_2) \arrow[r, phantom, "\cong"] & H^2(X, \Sigma_2) \\ \left\{\begin{matrix}\textit{codim two local}\\ \textit{complete intersections}\end{matrix}\right\} \arrow[u, "CK_2"] \arrow[ur, "\partial\circ i"'] & \end{tikzcd}$$

commutes.

Proof. (ii) follows from (i) and (4.2). For (i), we already know α is an isomorphism and $\bar{j}$ is surjective. To show injectivity, it suffices to define a map $C\Sigma_2 \colon CH^2(X) \to H^2(X, \Sigma_2)$ such that $C\Sigma_2 \cdot \bar{j} =$ identity. The main step is the following:

THEOREM 5.16. *Let X be a regular algebraic k-scheme of dimension two. Then*

$$j \colon \mathcal{C}_{2,X} \longrightarrow \mathcal{Z}_X^2 \cong \bigoplus_{x \in X \text{ closed}} \mathbf{Z}_{\{x\}}$$

is an isomorphism.

Proof. Fix $x \in X$, let z_1, z_2 be regular parameters at x, and let $(f, g) \subset \mathcal{O}_{X,x}$ be an ideal primary to the maximal ideal. Let $\mu \colon \bigoplus F_D^* \to \mathcal{C}_{2,X}$ and let $n =$ length $\mathcal{O}_{X,x}/(f, g)$. We will show by induction on the multiplicity n that $\mu(f|_{g=0}) = n \cdot \mu(z_1|_{z_2=0})$.

Actually, $\mu(f|_{g=0})$ depends only on the ideal (f, g) (which follows from the construction of i in (3.3)), so one can write $\mu(f, g)$. Note:

(i) When $n = 1$, $(f, g) = (z_1, z_2)$ so the assertion is clear.

(ii) When $g = z_2$, $f \equiv \text{unit} \cdot z_1^n (\operatorname{mod} z_2)$ and the assertion is again clear.

(iii) If (f, g_1) and (f, g_2) are two such ideals, so is $(f, g_1 \cdot g_2)$, and $\mu(f, g_1 g_2) = \mu(f, g_1) + \mu(f, g_2)$. A similar additive property holds for the multiplicities.

(iv) Using (ii), (iii), and the induction hypothesis, the assertion holds whenever g (or, by symmetry, f) lies in the ideal (z_2).

Now assume $f, g \notin (z_2)$ and let $I_p = (z_1^p, z_2) \subset \mathcal{O}_{X,x}$. Since $\bigcap_p I_p = (z_2)$, we have $f \in I_p - I_{p+1}$, $g \in I_q - I_{q+1}$ for some p and q and we may assume $p \leqq q$. In other words, $f = az_1^p + bz_2$, $g = cz_1^q + dz_2$ with $a, c \notin m_x$ (if say $a \in m_x$, $a = uz_1 + vz_2$, it would follow that $f \in I_{p+1}$). One computes

$$g = \alpha f + \beta z_2 ,$$

$$\alpha = \frac{c}{a} z_1^{q-p} ,$$

$$\beta = d - \frac{cb}{a} z_1^{q-p} ,$$

so $(f, g) = (f, \beta z_2)$. The proof is now completed using (iv).

Construction of $C\Sigma_2$: We have

$$\begin{array}{ccc} & & \bigoplus F_D^* \\ & & \downarrow \mu \\ \mathcal{Z}^2 & \xrightarrow{j^{-1}} & \Gamma(X, \mathcal{C}_2) \\ \downarrow & & \downarrow \partial \\ CH^2(X) & \xrightarrow{C\Sigma_2} & H^2(X, \Sigma_2) \\ \downarrow & & \downarrow \\ 0 & & 0 . \end{array}$$

To construct the dotted arrow, it suffices by (3.13) to show $j^{-1}(z) \in \operatorname{Im} \mu$ whenever $z \underset{\text{pre}}{\sim} 0$. If $f \in F_D^*$ defines the pre-equivalence, $z = j\mu(f|_D)$ so $j^{-1}(z) = \mu(f|_D)$. This proves (5.15).

6. The $d\log$ symbol

Let $k \subset F$ be an extension of fields. Write $\Omega^1_{F/k}$ for the F-vector space of Kähler differentials of F over k, and let $\Omega^2_{F/k} = \Lambda^2_F \Omega^1_{F/k}$. The map

$$d\log \wedge d\log \colon F^* \times F^* \longrightarrow \Omega^2_{F/k}$$

$$d\log \wedge d\log \{f, g\} = \frac{df}{f} \wedge \frac{dg}{g}$$

satisfies (1.7)–(1.10). For example

$$d\log \wedge d\log \{f, 1-f\} = \frac{df}{f} \wedge \frac{-df}{1-f} = 0 .$$

By (1.14) there is an induced map

$$d\log \wedge d\log \colon K_2(F) \longrightarrow \Omega^2_{F/k} .$$

Now suppose that $F = F_X$ where X is a smooth, algebraic k-scheme.

LEMMA 6.1. *View $K_2(F_X)$ and $\Omega^2_{F_X/k}$ as constant sheaves on X, let $\Sigma_2 = \Sigma_{2,X} \subset K_2(F_X)$ be as above, and let $\Omega^2_{X/k} \subset \Omega^2_{F_X/k}$ be the sheaf of regular two-forms. Then there are maps*

(6.2) $$d\log \wedge d\log \colon \Sigma_2 \longrightarrow \Omega^2_{X/k} ,$$

(6.3) $$d\log \wedge d\log\colon K_2(\mathcal{O}_X) \longrightarrow \Omega^2_{X/k}\,.$$

Proof. (6.3) is constructed from (6.2) by composing with $\alpha\colon K_2(\mathcal{O}_X) \to \Sigma_2$. To get (6.2), notice that if f is a unit at a point x, df/f is a regular differential form at x. It follows that $d\log \wedge d\log(\Sigma_2) \subset \Omega^2_{F_X/k}$ consists of forms which are regular at every point of codimension one on X. Such a form is necessarily regular everywhere, so (6.2) is defined.

Now recall there is a Hodge cycle map

$$Hg^2\colon CH^2(X) \longrightarrow H^2(X, \Omega^2_{X/k})\,.$$

THEOREM 6.4. *The triangles*

$$\begin{tikzcd} H^2(X, K_2(\mathcal{O}_X)) \arrow[r, "\alpha"] \arrow[dr, "d\log\wedge d\log"'] & H^2(X, \Sigma_2) \arrow[r, "\bar{j}"] \arrow[d, "d\log\wedge d\log"] & CH^2(X) \arrow[dl, "Hg^2"] \\ & H^2(X, \Omega^2_{X/k}) & \end{tikzcd}$$

are commutative.

Proof. The left hand triangle commutes by definition. For the right hand one, I recall some facts about local cohomology and the cycle class:

(6.5) For $Z \subset X$ a closed subset of codimension r, $\underline{H}^i_Z(\Omega^2_{X/k}) = (0)$, $0 \leqq i < r$ [4], so by (5.2)

$$H^r_Z(X, \Omega^2_{X/k}) \cong \Gamma(X, \underline{H}^r_Z(X, \Omega^2_{X/k}))\,.$$

(6.6) Let $Z \subset X$ be a local complete intersection of codimension two. Let $\{U_i\}$ be an open affine cover of X such that $Z \cap U_i\colon f_i = g_i = 0$, and let $m\colon X - Z \to X$. Then $df_i/f_i \wedge dg_i/g_i$ gives rise to a class in $\Gamma(U_i, Rm^1_*(\Omega^2_{U_i-Z}))$. These sections patch ([4], [10], and compare (5.3) above) to give $\sigma \in \Gamma(X, \underline{H}^2_Z(X, \Omega^2_{X/k}))$. The image of σ in $H^2(X, \Omega^2_{X/k})$ (using (6.5)) is $Hg^2(Z)$.

Now let $g|_{f=0}$ give a local section of $\mathcal{C}_2$, and suppose g regular near a given point x. Let $Z\colon f = g = 0$ near x. As in the proof of (5.3), the symbol $\{f, g\}$ gives a local section of $Rm^1_*(K_2 | X - Z)$. Applying $d\log \wedge d\log$, one gets a section $df/f \wedge dg/g$ of $Rm^1_*(\Omega^2_{X-Z})$ which corresponds, by (6.6), to the class $Hg^2(Z)$. Theorem 6.4 follows easily.

Remark 6.7. Actually, Σ_2 maps via $d\log \wedge d\log$ into the subsheaf $\Omega^2_{\text{closed}} \subset \Omega^2_{X/k}$ consisting of closed two-forms. Let $F^p\Omega^\bullet_{X/k} \subset \Omega^\bullet_{X/k}$ be the subcomplex of the de Rham complex defined by

$$(F^p\Omega^\bullet)^q = \begin{cases} 0 & q < p \\ \Omega^q & q \geqq p\,. \end{cases}$$

There is a natural map of complexes

$$\Omega^2_{\text{closed}}[-2] \longrightarrow F^2\Omega^{\bullet}_{X/k}$$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega^2_{\text{closed}} & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \big\downarrow \cap & & \big\downarrow & & \\ 0 & \longrightarrow & \Omega^2 & \xrightarrow{\quad d \quad} & \Omega^3 & \longrightarrow & \cdots . \end{array}$$

Composing with the $d\log$ map, one gets

$$\Sigma_2[-2] \longrightarrow F^2\Omega^{\bullet}_{X/k} ,$$

$$H^2(X, \Sigma_2) \longrightarrow \mathbf{H}^4(X, F^2\Omega^{\bullet})$$

($\mathbf{H}^4$ = *hypercohomology*).

When $k = \mathbf{C}$ and X is projective, it is known [2] that

$$\mathbf{H}^4(X, F^2\Omega^{\bullet}) \cong F^2H^4(X, \mathbf{C})$$

where F^2H^4 refers to the second level of the *Hodge filtration*. In any case, using a more elaborate version of (6.6) involving local hypercohomology, one can show the triangle

$$\begin{array}{ccc} H^2(X, \Sigma_2) & \xrightarrow{\ \bar{j}\ } & CH^2(X) \\ & {\scriptstyle d\log \wedge d\log} \searrow \quad \swarrow {\scriptstyle dR^2} & \\ & H^4(X, F^2\Omega^{\bullet}) & \end{array}$$

commutes, where dR^2 is the de Rham cycle class.

THEOREM 6.8. *Let* $A^2(X) \subset H^2(X, \Omega^2_{X/k})$ (resp. $A^2(X) \subset H^4(X, F^2\Omega^{\bullet})$) *be the subgroup* Im (Hg^2) (resp. Im (dR^2)) *generated by algebraic cycle classes. Then*

$$d\log \wedge d\log\big(H^2(X, K_2(\mathcal{O}_X))\big) = d\log \wedge d\log\big(H^2(X, \Sigma_2)\big) = A^2(X) .$$

Proof. Straightforward, using diagrams like

$$\begin{array}{ccccc} \{\text{Codim 2 subschemes}\} & \xrightarrow{CK_2} & H^2(X, K_2) & \xrightarrow{d\log \wedge d\log} & H^2(X, \Omega^2) . \\ & {\scriptstyle CH^2} \searrow & \big\downarrow {\scriptstyle \bar{j}\circ\alpha} & \nearrow {\scriptstyle Hg^2} & \\ & & CH^2(X) & & \end{array}$$

7. Homotoping bundles

In this section I sketch briefly a geometric interpretation of the K-theoretic second Chern class (§ 2), using the K-theory of Karoubi–Villamayor [5]. Their K_2 is called K^{-2} (they have K^n for all $n \in \mathbf{Z}$) and it is a quotient of the Milnor K_2, $K_2 \twoheadrightarrow K^{-2}$.

Let me first give a few of the constructions in the Karoubi–Villamayor theory. Given a ring R (possibly without 1) let $GL(R)$ be the group of invertible matrices of the form $I + M$, where M has coefficients in R. Write

$$PR = XR[X] , \tag{7.1}$$
$$\Omega R = (1 - X)\cdot X\cdot R[X]$$

and view these as rings without 1. There is an exact sequence

$$1 \longrightarrow GL(\Omega R) \longrightarrow GL(PR) \xrightarrow[X \mapsto 1]{} GL(R) . \tag{7.2}$$

Let $GL^\circ(R)$ be the image of the right hand map in (7.2) and define

$$K^{-1}(R) = GL(R)/GL^\circ(R) . \tag{7.3}$$

One shows that K^{-1} is an abelian group.

Thinking of $GL(PR)$ as paths and $GL(\Omega R)$ as loops, one is motivated to define

$$K^{-n}(R) = K^{-1}(\Omega^{n-1}R) \qquad (\Omega^{n-1} = \underbrace{\Omega\circ\cdots\circ\Omega}_{n-1 \text{ times}}) . \tag{7.4}$$

Let S be a scheme. Localizing the above construction gives sheaves of rings $P\mathcal{O}_S$, $\Omega\mathcal{O}_S$, $P\Omega\mathcal{O}_S$, etc. on S, and exact sequence

$$1 \longrightarrow GL(\Omega\mathcal{O}_S) \longrightarrow GL(P\mathcal{O}_S) \longrightarrow GL^\circ(\mathcal{O}_S) \longrightarrow 1 . \tag{7.5}$$

In general $SL(\mathcal{O}_S) \subset GL^\circ(\mathcal{O}_S)$, with equality when S is reduced. Define

$$\begin{aligned} GL^\circ(\Omega\mathcal{O}_S) &= \mathrm{Image}\big(GL(P\Omega\mathcal{O}_S) \longrightarrow GL(\Omega\mathcal{O}_S)\big) , \\ G^\circ(\mathcal{O}_S) &= GL(P\mathcal{O}_S)/GL^\circ(\Omega\mathcal{O}_S) , \\ K^{-2}(\mathcal{O}_S) &= GL(\Omega\mathcal{O}_S)/GL^\circ(\Omega\mathcal{O}_S) . \end{aligned} \tag{7.6}$$

One checks that $GL^\circ(\Omega\mathcal{O}_S)$ is normal in $GL(P\mathcal{O}_S)$ so G° and K^{-2} are sheaves of groups. In fact $K^{-2}(\mathcal{O}_S)$ is abelian and is the sheaf obtained from the presheaf $R \rightarrow K^{-2}(R)$.

Taking a quotient of (7.5) gives

$$0 \longrightarrow K^{-2}(\mathcal{O}_S) \longrightarrow G^\circ(\mathcal{O}_S) \longrightarrow GL^\circ(\mathcal{O}_S) \longrightarrow 1 . \tag{7.7}$$

PROPOSITION 7.8. ([5, p. 304]). *There exists a map of exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_2(\mathcal{O}_S) & \longrightarrow & St(\mathcal{O}_S) & \longrightarrow & SL(\mathcal{O}_S) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \cap\!\downarrow & & \\ 0 & \longrightarrow & K^{-2}(\mathcal{O}_S) & \longrightarrow & G^\circ(\mathcal{O}_S) & \longrightarrow & GL^\circ(\mathcal{O}_S) & \longrightarrow & 1 . \end{array}$$

When S is regular, G° is a quotient of St, the sequence (7.7) *is central, and there is a Chern class map*

$$CK^{-2}\colon \{GL^\circ(\mathcal{O}_S)\text{-bundles}\} \longrightarrow H^2(S, K^{-2}(\mathcal{O}_S)) .$$

Proof. The only problem is to define a map

$$St(\mathcal{O}_S) \longrightarrow G^\circ(\mathcal{O}_S) = GL(P\mathcal{O}_S)/GL^\circ(\Omega\mathcal{O}_S)\ .$$

Recall $St(\mathcal{O}_S)$ is generated (locally) by sections $X^a_{i,j}$, $a \in \mathcal{O}_S$, and $GL(P\mathcal{O}_S)$ contains matrices e^{aX}_{ij} (§ 1). Map

$$X^a_{ij} \longmapsto e^{aX}_{ij} \left(\mathrm{mod}\ GL^\circ(\Omega\mathcal{O}_S)\right)\ .$$

To show the identities (1.1)–(1.3) hold in G°, note

$$\begin{aligned}[e^{aX}_{ij}, e^{bX}_{jl}] &= e^{abX^2}_{il} = e^{abX}_{il}\cdot e^{ab(X^2-X)}_{il}\\ &\equiv e^{abX}_{il} \left(\mathrm{mod}\ GL^\circ(\Omega\mathcal{O}_S)\right)\ .\end{aligned}$$

For surjectivity, see op. sit. p. 305.

In what follows, I use the notation $E \approx E'$ to indicate *stable equivalence* of vector bundles on S, i.e., $E \approx E'$ if and only if there exist integers n, m such that $E \oplus \mathcal{O}_S^{\oplus n} \cong E' \oplus \mathcal{O}_S^{\oplus m}$.

PROPOSITION 7.10. *Let E be a $GL^\circ(\mathcal{O}_S)$ bundle, $\xi \in H^1(S, GL^\circ(\mathcal{O}_S))$ the corresponding cohomology classes. Then the following are equivalent:*

(i) *ξ lifts to a class $\xi' \in H^1(S, GL(P\mathcal{O}_S))$.*

(ii) *There exists a bundle F on the affine line over S, $\mathbf{A}^1_S$, such that*

(a) $F|_{S\times\{0\}} \approx \bigoplus \mathcal{O}_S$

(b) $F|_{S\times\{1\}} \approx E$

(c) *For all $s \in S$, $T = \mathrm{Spec}\,(\mathcal{O}_{S,s})$, we have $F|_{\mathbf{A}^1_T} \cong \bigoplus \mathcal{O}_{\mathbf{A}^1_T}$.*

Proof. $GL(P\mathcal{O}_S) = \pi_* GL(I\mathcal{O}_{\mathbf{A}^1_S})$, where $\pi\colon \mathbf{A}^1_S \to S$ is the projection and $I \subset \mathcal{O}_{\mathbf{A}^1_S}$ is the ideal of $S \times \{0\}$. As a consequence, there is a natural map $H^1(S, GL(P\mathcal{O}_S)) \xrightarrow{\phi} H^1(\mathbf{A}^1_S, GL(I\mathcal{O}_{\mathbf{A}^1}))$ and Image (ϕ) consists of those $GL(I\mathcal{O}_{\mathbf{A}^1})$ bundles which admit a trivialization of the form $\{\mathbf{A}^1_{U_i}\}$ with $\{U_i\}$ an open cover of S. The implication (i) $\Rightarrow$ (ii) follows.

Conversely, using the maps $S \underset{\pi}{\overset{0\text{-sect}}{\rightleftarrows}} \mathbf{A}^1_S$ it is not hard to show that $GL(I\mathcal{O}_{\mathbf{A}^1})$-bundles correspond to vector bundles F on $\mathbf{A}^1_S$ which are trivial on $S \times \{0\}$. Finally,

$$\begin{gathered}F|_{\mathbf{A}^1_T} \cong \bigoplus \mathcal{O}_{\mathbf{A}^1_T}\ ,\\ T = \mathrm{Spec}\,(\mathcal{O}_{S,s})\end{gathered}$$

implies a similar isomorphism with T replaced by some open neighborhood U of s in S, so (c) implies a trivialization $\{\mathbf{A}^1_{U_i}\}$ for F. This proves (7.10).

Remark 7.11. Combining (7.8) and (7.10) and noting that $G^\circ(\mathcal{O}_S)$ is a quotient of $GL(P\mathcal{O}_S)$, we get a realization of the Karoubi–Villamayor second

Chern class of a $GL^\circ(\mathcal{O}_S)$-bundle E as an obstruction to "rationally homotoping" E to the trivial bundle.

8. Analytic K_2

The complicated nature of the Steinberg relations in K_2 can in some circumstances be simplified by completing with respect to a suitable topology. I apply this idea to construct a sheaf $K_2^{an}(\mathcal{O}_X)$ for the complex topology on a complex manifold X. The principal result is a structure theorem relating K_2^{an} to the sheaf Ω^2_{cl} of closed analytic two-forms.

Let X be a complex manifold, with structure sheaf $\mathcal{O}_X$. Let $K_2(\mathcal{O}_X)$ be the sheaf associated to the presheaf

$$U \longmapsto K_2(\Gamma(U, \mathcal{O}_X)) . \tag{8.1}$$

Since the stalks $\mathcal{O}_{X,x}$ are local rings, one has by Stein's Theorem (1.11) an exact sequence

$$0 \longrightarrow R \longrightarrow \mathcal{O}_X^* \otimes_{\mathbf{Z}} \mathcal{O}_X^* \longrightarrow K_2(\mathcal{O}_X) \longrightarrow 0 \tag{8.2}$$

where $\mathcal{O}_X^*$ is the sheaf of units in $\mathcal{O}_X$.

Let $K \subset X$ be a compact, simply-connected neighborhood on X, and let $A = \Gamma(K, \mathcal{O}_X^*)$ (i.e., elements of A are units defined in some open neighborhood of K on X). Define a function

$$|\ \ |_K\colon A \longrightarrow \mathbf{R}_+$$

by

$$|u|_K = \min|\log u|_K$$

where $|\log u|_K$ is the sup norm and min denotes the minimum over all determinations of $\log u$ (note that each determination is single-valued since K is simply connected). Define

$$||\ \ ||_K\colon A \otimes_{\mathbf{Z}} A \longrightarrow \mathbf{R}_+$$

by

$$||v||_K = \inf_{v=\Pi(v_i'\otimes v_i'')} \left(\textstyle\sum_i |v_i'|_K \cdot |v_i''|_K\right) . \tag{8.3}$$

$||\ \ ||_K$ induces a structure of topological group on $A \otimes_{\mathbf{Z}} A$.

LEMMA 8.4. *Let v_i, v, w_i, $w \in A$ and suppose*

$$\lim_{i\to\infty} \sup_K |v - v_i| = \lim_{i\to\infty} \sup_K |w - w_i| = 0 .$$

Then $v_i \otimes w_i \to v \otimes w$ in the topological group $A \otimes_{\mathbf{Z}} A$.

Proof. $(v \otimes w)\cdot(v_i \otimes w_i)^{-1} = (v \otimes ww_i^{-1})(vv_i^{-1} \otimes w_i)$, so replacing w_i by ww_i^{-1}, it suffices to show $w_i \to 1 \Rightarrow v \otimes w_i \to 1$. This is clear from (8.3).

Sheafifying the above, one gets a topology on $\mathcal{O}_X^* \otimes_{\mathbf{Z}} \mathcal{O}_X^*$. Let $\mathcal{O}_X^* \widehat{\otimes}_{\mathbf{Z}} \mathcal{O}_X^*$

be the separated completion with respect to this topology, and define $K_2^{an}(\mathcal{O}_X)$ by the exact sequence

$$(8.5) \qquad 0 \longrightarrow \hat{R} \longrightarrow \mathcal{O}_X^* \widehat{\otimes_{\mathbf{Z}}} \mathcal{O}_X^* \longrightarrow K_2^{an}(\mathcal{O}_X) \longrightarrow 0 ,$$

where $\hat{R}$ is the closure of the image of the sheaf R in (8.2).

Arguing as in § 6, there is a map

$$(8.6) \qquad d\log \wedge d\log\colon K_2(\mathcal{O}_X) \longrightarrow \Omega^2_{cl}$$

where Ω^2_{cl} is the sheaf of closed, analytic two-forms on X. Topologize Ω^2_{cl} as follows: first give $\mathcal{O}_X$ the topology of uniform convergence on compact subsets. The sheaves Ω^j of analytic j-forms on X are coherent, so they inherit a topology, and the differentials $d^j\colon \Omega^j \to \Omega^{j+1}$ are continuous. Give $\Omega^2_{cl} = \operatorname{Ker}(d^2)$ the induced topology as a closed subsheaf of Ω^2.

With this topology, the map

$$d\log \wedge d\log\colon \mathcal{O}_X^* \otimes_{\mathbf{Z}} \mathcal{O}_X^* \longrightarrow \Omega^2_{cl}$$

is easily seen to be continuous, hence, using (8.5) and (8.6), there is a natural map

$$(8.7) \qquad d\log \wedge d\log\colon K_2^{an}(\mathcal{O}_X) \longrightarrow \Omega^2_{cl} .$$

As usual, I write $\{a, b\}$ for the image of $a \otimes b \in \mathcal{O}^* \otimes \mathcal{O}^*$ in K_2^{an}. Let $Q \subset K_2^{an}$ be the closure of the image of the natural map

$$(8.8) \qquad \begin{aligned} \mathcal{O}_X^* \otimes_{\mathbf{Z}} \mathbf{C}^* &\longrightarrow K_2^{an}(\mathcal{O}_X) \\ a \otimes b &\longmapsto \{a, b\} . \end{aligned}$$

THEOREM 8.9. *Let X be a complex manifold. The sequence*

$$(8.10) \qquad 0 \longrightarrow Q \longrightarrow K_2^{an}(\mathcal{O}_X) \xrightarrow{d\log \wedge d\log} \Omega^2_{cl} \longrightarrow 0$$

is exact.

Proof. Step 1: $d\log \wedge d\log$ is surjective. This follows from the commutative diagram

$$\begin{tikzcd} \mathcal{O} \otimes_{\mathbf{Z}} \mathcal{O}^* \arrow[r, "\exp\otimes 1"] \arrow[d, "1\otimes d\log"'] & \mathcal{O}^* \otimes_{\mathbf{Z}} \mathcal{O}^* \arrow[d, "\frac{1}{2\pi i} d\log \wedge d\log"] & \\ \Omega^1 \arrow[r, "d"] \arrow[d] & \Omega^2_{cl} \arrow[r] & 0 \\ 0 & & \end{tikzcd}$$

where $\exp(z) = e^{2\pi i z}$.

Step 2: $d\log \wedge d\log(Q) = (0)$. In fact, Q is generated topologically by symbols with one argument constant.

Now write $L = K_2^{an}/Q$. The idea in the rest of the proof is to define an isomorphism α

$$\begin{tikzcd} \Omega^1/d(\mathcal{O}_X) \arrow[r, "\alpha", "\sim"'] \arrow[d, equal] & L \arrow[dl, "d\log\wedge d\log"] \\ \Omega^2_{cl} & \end{tikzcd}$$

inverse to $d\log \wedge d\log$. Let $|a, b|$ denote the image of $\{a, b\}$ in L.

Step 3: Let u be a function and let $f(T)$, $g(T)$ be power series in one variable such that $f(u)$, $g(u)$ converge in the neighborhood of some point $x_\circ \in X$ and such that $f(u(x_\circ)) \neq 0 \neq g(u(x_\circ))$. Then $|f(u), g(u)| = 1$.

Indeed, there exist polynomials $f^{(N)}(T)$, $g^{(N)}(T)$ of degree $\leqq N$ which approximate f and g to order N. Clearly $f^{(N)}(u) \to f(u)$, $g^{(N)}(u) \to g(u)$ uniformly in some compact neighborhood K of $x_\circ$. For $N \gg 0$, $f^{(N)}(u)$ and $g^{(N)}(u)$ will be units on K and by (8.4)

$$\{f^{(N)}(u), g^{(N)}(u)\} \longrightarrow \{f(u), g(u)\}\ .$$

It therefore suffices to show $|f(u), g(u)| = 1$ when f and g are polynomials. Factoring $f(u) = \prod(u - \alpha_i)$, $g(u) = \prod(u - \beta_j)$, we are reduced to showing $|u - \alpha, u - \beta| = 1$ where $u - \alpha$, $u - \beta$ are units, $\alpha, \beta \in \mathbf{C}$. Finally, replacing $u - \alpha$ by u, β by $\beta - \alpha$, we must prove $|u, u - \beta| = 1$. If $\beta = 0$, a standard relation in K_2 gives $|u, u| = |u, -1| = 1$ (symbols with one argument constant die in L). If $\beta \neq 0$,

$$\begin{aligned}\{u, u - \beta\} &= \{u, -\beta\}\Big\{u, 1 - \frac{u}{\beta}\Big\} = \{u, -\beta\}\Big\{\beta, 1 - \frac{u}{\beta}\Big\}\Big\{\frac{u}{\beta}, 1 - \frac{u}{\beta}\Big\} \\ &= \{u, -\beta\}\Big\{\beta, 1 - \frac{u}{\beta}\Big\}\end{aligned}$$

so again $|u, u - \beta| = 1$. This proves Step 3.

Remark. Step 3 is the essential reason for introducing a topology on K_2.

Step 4: Define a map of sheaves of sets

$$\alpha' \colon \mathcal{O} \times \mathcal{O}^* \longrightarrow L$$
$$\alpha'(b, c) = |\exp(bc), c|\ .$$

Then for $u, v \in \mathbf{C}$, $\alpha'(b + u, c + v) = \alpha'(b, c)$. In particular, α' extends to a map $\alpha' \colon \mathcal{O} \times \mathcal{O} \to L$. Indeed,

$$(8.11) \qquad \begin{aligned}\alpha'(b, c) &= |\exp(bc), c| = |\exp((b + u)c), c|\ |\exp(-uc), c| \\ &\underset{(\text{Step } 3)}{=} |\exp((b + u)c), c| = \alpha'(b + u, c)\ .\end{aligned}$$

Now assume $b' = b + u$ is a unit. We compute

$$\begin{aligned} \alpha'(b, c) &= \alpha'(b', c) = |\exp(b'c), c| = |\exp(b'c), b'c|\ |\exp(b'c), b'|^{-1} \\ &= |\exp(b'c), b'|^{-1} = |\exp(b'(c+v)), b'|^{-1} \\ \text{(Step 3)} \quad &= |\exp(b'(c+v)), b'(c+v)|^{-1}|\exp(b'(c+v)), c+v| \\ &= |\exp(b'(c+v)), c+v| = \alpha'(b', c+v) \underset{(8.11)}{=\!=} \alpha'(b, c+v) . \end{aligned}$$

This proves Step 4.

Step 5: $\alpha'\colon \mathcal{O} \times \mathcal{O} \to L$ is $\mathbf{Z}$-linear in both variables. Moreover for $u \in \mathbf{C}$,

$$\alpha'(ub, c) = \alpha'(b, uc) , \tag{8.12}$$

$$\alpha'(\mathbf{C} \times \mathcal{O}) = \alpha'(\mathcal{O} \times \mathbf{C}) = 1 . \tag{8.13}$$

Indeed

$$\alpha'(b + b', c) = |\exp((b + b')c), c| = |\exp(bc), c|\ |\exp(b'c), c| = \alpha'(b, c)\alpha'(b', c) .$$

To show linearity in the second variable, we may assume by Step 4 that b is a unit and write

$$\begin{aligned} \alpha'(b, c)\alpha'(b, c') &= |\exp(bc), c|\ |\exp(bc'), c'| \\ &= |\exp(bc), b^{-1}|\ |\exp(bc'), b^{-1}| = |\exp(b(c+c')), b|^{-1} \\ \text{(Step 3)} \quad &= |\exp(b(c+c')), c+c'| = \alpha'(b, c+c') . \end{aligned}$$

When $u = 0$, (8.12) follows from bilinearity. For $u \in \mathbf{C}^*$

$$\begin{aligned} \alpha'(ub, c) &= |\exp(ubc), c| = |\exp(ubc), ub|^{-1} = |\exp(ubc), b|^{-1} \\ &= |\exp(ubc), uc| = \alpha'(b, uc) . \end{aligned}$$

Finally, (8.13) follows from (8.12) and Step 4.

Step 6: Let Ω' denote the sheaf of non-separated Kähler differentials on X. That is, Ω' is the free $\mathcal{O}_X$-module generated by elements da, $a \in \mathcal{O}_X$, modulo relations

$$\begin{aligned} d(a + b) &= d(a) + d(b) \\ d(ab) &= ad(b) + bd(a) \\ d(u) &= 0, \quad u \in \mathbf{C} . \end{aligned}$$

Then α' induces a map $\alpha\colon \Omega' \to L$, $\alpha(bdc) = \alpha'(b, c)$. Moreover $\alpha(d(\mathcal{O}_X)) = 1$.

Indeed, the only thing that must be checked is that the relation $c(d(ab) - bda - adb) = 0$ carries over to K_2, i.e., that

$$\alpha'(c, ab)\alpha'(ca, b)^{-1}\alpha'(cb, a)^{-1} = 1 .$$

But the left hand side is

$$|\exp(abc), ab|\ |\exp(abc), b|^{-1}|\exp(abc), a|^{-1} = 1 .$$

Finally $\alpha\big(d(\mathcal{O}_X)\big) = 1$ because $|\exp(a), a| = 1$.

Step 7: Let Ω^1 denote the coherent analytic sheaf of 1-forms on X. Then α induces a map $\alpha\colon \Omega^1 \to L$.

Indeed, we must show α factors through the quotient map $\Omega' \to \Omega^1 \to 0$. The question is local, so fix local parameters $t_1, \cdots, t_r$ at a point $x_0 \in X$. For f and g functions at z_0, we must show

$$\alpha\Big(g\Big(df - \sum \frac{\partial f}{\partial t_i} dt_i\Big)\Big) = 1 \, .$$

This is clear when f is a polynomial in the t_i (the expression in parentheses is trivial in Ω' in this case). For the general case, write f as a uniformly convergent limit of polynomials, $f = \lim_N f^{(N)}$ and note

$$1 = \alpha\Big(g\Big(df^{(N)} - \sum \frac{\partial f^{(N)}}{\partial t_i} dt_i\Big)\Big) \longrightarrow \alpha\Big(g\Big(df - \sum \frac{\partial f}{\partial t_i} dt_i\Big)\Big)$$

where convergence is in the natural quotient topology on $L = K_2^{an}/Q$. This proves Step 7.

Step 8: α induces an isomorphism

$$\Omega^1/d(\mathcal{O}) \xrightarrow{\sim} L \, .$$

Indeed, the diagram

$$\begin{tikzcd} & L \arrow[dr, "\frac{1}{2\pi i} d\log \wedge d\log"] & \\ \Omega^1/d(\mathcal{O}_X) \arrow[ur, "\alpha"] \arrow[rr, "d", "\cong"'] & & \Omega^2_{cl} \end{tikzcd}$$

commutes, so α is injective. To show surjectivity, note that the image of α is dense (it contains all the symbols). on the other hand, Im (α) is the kernel of the continuous operator

$$1 - \alpha \circ d^{-1} \circ \frac{1}{2\pi i}(d\log \wedge d\log) \, ,$$

hence Im (α) is closed. This completes the proof of (8.9).

Remark 8.14. One would like to understand the image of the map on cohomology

$$d\log \wedge d\log\colon H^2\big(X, K_2^{an}(\mathcal{O}_X)\big) \longrightarrow H^2(X, \Omega^2_{cl}) \, .$$

When X is projective $H^2(X, \Omega^2_{cl}) \cong F^2H^4(X, \mathbf{C})$ $\big(F^2H^4$ denotes the Hodge filtration [2]$\big)$. I conjecture in this case

(8.15) $$\text{Image}\,(d\log \wedge d\log) \cong F^2H^4(X, \mathbf{C}) \cap H^4(X, \mathbf{R}) \, .$$

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