Uncertainty of arrival-time picking

Introduction

This script reproduces Figures 1, 2 and 3 published in the paper:

Abakumov, I., Roeser, A., and S. A. Shapiro (2020) The arrival time picking uncertainty: theoretical estimationsand their application to microseismic data, Geophysics

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Add MLIB library

In this project we use several functions from MLIB library.

You can download the whole library at github:

https://github.com/Abakumov/MLIB

```
clear; close all; clc;
mlibfolder = '/home/ivan/Desktop/MLIB';
path(path, mlibfolder);
add_mlib_path;
```

Definition of basic values

```
Fs = 4000;                % sampling frequency
dt = 1/Fs;                % time step
fc = 200;                 % dominant frequency of the signal
T  = 1/fc;                % central period
Ta = 1;                   % length of a priori interval [0 Ta]
damp  = 400;              % attenuation factor (for signal modeling)
sigma = 0.2;              % variance of noise is equal sigma^2
tau = 0.45242;            % arrival time
t = 0:dt:1-dt;
```

Definition of signals and noise

τ is the arrival time

Symmetric signal:

$$s(t) = \cos(2\pi f_c(t-\tau))e^{-A|t-\tau|}$$

```
get_sym_signal  = @(t,tau,fc,damp)(   cos(2*pi*fc*(t-tau)).*exp(-damp*abs(t-tau))   );
get_sym_true_signal = @(t,t0,fc,damp)(   cos(2*pi*fc*(t-t0 )).*exp(-damp*abs(t-t0))
```

Non-symmetric signal:

$s(t) = h(t) \cdot \sin(2\pi f_c (t - \tau))e^{-A|t-\tau|}$

```
get_nsm_signal  = @(t,tau,fc,damp)(   myheaviside(t-tau).*sin(2*pi*fc*(t-tau)).*exp(-da
get_nsm_true_signal = @(t,t0,fc,damp)(myheaviside(t-t0).*sin(2*pi*fc*(t-t0) ).*exp(-dam
```

Noise:

```
get_noise = @(t,sigma)(   sigma*randn(size(t))   );
```

Basic definitions: spectral variance of the signal, energy of the signal, noise spectral density

1. Spectral variance

There are two definitions of spectral variance of the signal (angular bandwidth of the pulse) β

in the frequency domain (see equation 10):

$$\beta^2 = \frac{\int_{-\infty}^{\infty} \omega^2 |S(\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |S(\omega)|^2 d\omega},$$

```
signal = get_sym_signal(t,tau,fc,damp);
fftSignal = fft(signal)/numel(signal);
fftSignal = fftshift(fftSignal);
f = Fs/2*linspace(-1,1,Fs);
w = 2*pi*f;
beta = sqrt(sum(w.^2.*abs(fftSignal).^2)/sum(abs(fftSignal).^2));
disp(beta);
```

and in the time domain:

$$\beta^2 = \frac{\int |s_t'(t)|^2 dt}{\int |s(t)|^2 dt},$$

```
signal = get_sym_signal(t,tau,fc,damp);
dsignal = diff(signal,1)/dt;
beta = sqrt(sum(dsignal.^2)/sum(signal.^2));
disp(beta);
```

These definitions are identical.

```
get_beta = @(signal,dt)(   sqrt(sum((diff(signal,1)/dt).^2)/sum(signal.^2))   );
```

2. Definition of the signal energy

$$W = \int_{-\infty}^{\infty} |s(t)|^2 dt$$

3. Definition of noise:

Additive white Gaussian noise (AWGN) has a normal distribution $\mathcal{N}(\mu, \sigma^2)$, where

μ - is a mean value (we assume $\mu = 0$),

σ^2 - is a variance of noise.

```
sigma = 12;
noise = sigma*randn(1,10000);
var(noise)                  % = sigma^2
std(noise)                  % = sigma
```

$N_0/2$ **double-sided** power spectral density of noise (NB!! In our fornulas $N_0 \equiv \frac{N_0}{2}$)

Noise: **one-sided** vs **double-sided** power spectral density

B is the bandwidth

$F_s = \frac{1}{\mathrm{dt}}$ sampling frequency

$B = F_s/2$ for one-sided case (spectral density N_0)

$B = F_s$ double-sided case (spectral density $N_0/2$)

$$N_0/2 = \frac{\sigma^2}{F_s} = \sigma^2 dt$$

Note: when computing power spectral density:

- multiply the positive and negative frequencies by a factor of 2;
- zero frequency (DC) and the Nyquist frequency do not occur twice.

```
sigma = 12'
noise = sigma*randn(size(t));
N = length(noise);
xdft = fft(noise);
xdft = xdft(1:N/2+1);               % only positive frequencies
psdx = (1/(Fs*N)) * abs(xdft).^2;
psdx(2:end-1) = 2*psdx(2:end-1);
freq = 0:Fs/N:Fs/2;

mean(psdx)
N0 = 2*sigma^2*dt

figure(232)
plot(freq,10*log10(psdx),'k')
grid on
title('Periodogram Using FFT')
xlabel('Frequency (Hz)')
ylabel('Power/Frequency (dB/Hz)')
mean(10*log10(psdx))
```

4. Signal to noise ratio:

$$\mathrm{SNR} = \frac{W}{N_0}$$

$$\mathrm{SNR_{dB}} = 10\log_{10}\mathrm{SNR}$$

```
signal = get_sym_signal(t,tau,fc,damp);
noise  = get_noise(t,sigma);

beta = get_beta(signal,dt);
W  = sum(signal.^2)*dt;
N0 = var(noise)*dt;
SNR = W/N0;
SNRdb = 10*log10(SNR);
```

Lower bounds for arrival-time uncertainty

1. Cramer-Rao bound (equation 9 and 11)

$$CRB = \frac{1}{\beta^2}\frac{N_0}{W} \equiv \frac{1}{\beta^2}\frac{1}{\mathrm{SNR}}$$

```
get_CRB = @(SNR,beta)(   1./beta.^2./SNR   );
```

2. Ziv Zakai Bound (equation 17)

Original formula from:

Bell, K. L., Steinberg, Y., Ephraim, Y., & Van Trees, H. L. (1997). Extended Ziv-Zakai lower bound for vector parameter estimation. IEEE Transactions on Information Theory, 43(2), 624-637 (equation 108):

$$\mathrm{ZZB} = \frac{T_a^2}{6}\Phi\left(\sqrt{\frac{W}{2N_0}}\right) + \frac{N_0}{\beta^2 W}\Gamma_{\frac{3}{2}}\left(\frac{W}{4N_0}\right) - \left(\frac{N_0}{\beta^2 W}\right)^{\frac{3}{2}}\frac{32}{3T_a\sqrt{2\pi}}\Gamma_2\left(\frac{W}{4N_0}\right)$$

where

$$\Phi\,(z) = \int_z^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}\mathrm{dv}$$

In matlab there exists a function

$$\mathrm{ercf}\,(z) = \int_z^\infty \frac{2}{\sqrt{\pi}}e^{-t^2}\mathrm{dt}$$

$$2\Phi\,(\sqrt{2}\,x) = \mathrm{erfc}(x)$$

Hence, in our code and in the manuscript:

$$ZZB = \frac{T_a^2}{12}\mathrm{erfc}\left(\sqrt{\frac{W}{4N_0}}\right) + \frac{N_0}{\beta^2 W}\Gamma_{\frac{3}{2}}\left(\frac{W}{4N_0}\right) - \left(\frac{N_0}{\beta^2 W}\right)^{\frac{3}{2}}\frac{32}{3T_a\sqrt{2\pi}}\Gamma_2\left(\frac{\mathrm{W}}{4N_0}\right).$$

Also note that

$$\Gamma_a(x) = \text{gammainc}(x, a),$$

and

$$\text{SNR} \equiv \frac{W}{N_0}.$$

```
get_ZZB = @(SNR,beta,Ta)(    Ta^2/12*erfc(sqrt(SNR/4)) ...
    + (beta.^(-2)./SNR)*gammainc(SNR/4,3/2) ...
    - (beta.^(-2)./SNR).^1.5.*(32/(3*Ta*sqrt(2*pi))).*gammainc(SNR/4,2)   );
```

3. Seismic CR-like bound (equation 19)

$$CRB = \frac{T_d^2}{4\pi^2}\frac{1}{\text{SNR}}$$

```
get_SB  = @(SNR, Td)(   (Td/(2*pi))^2./SNR   );
```

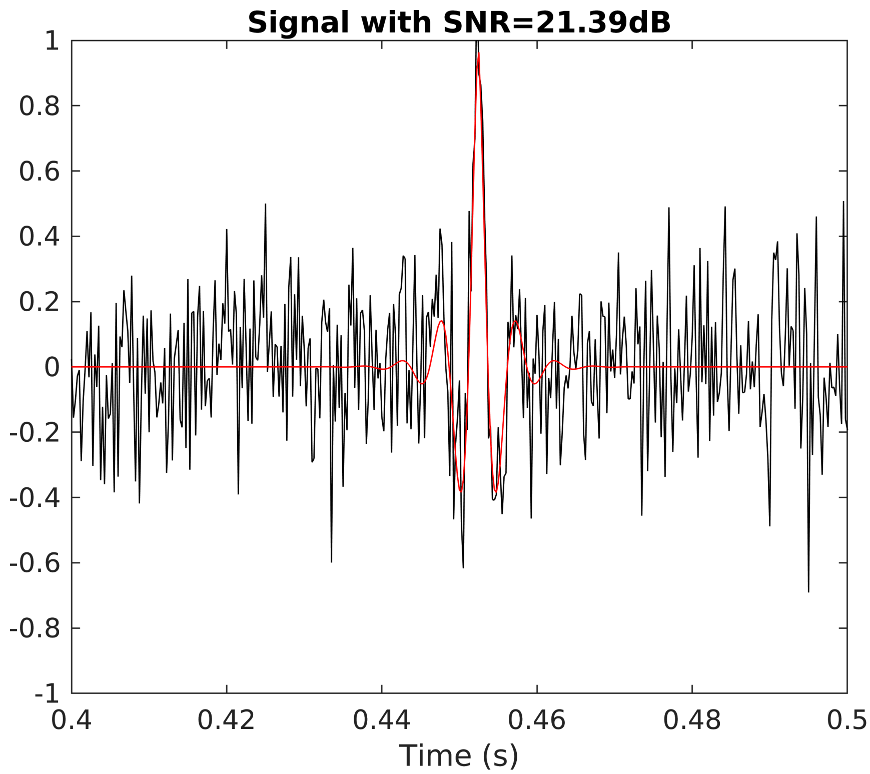
4. Seismic ZZ bound (eq. 21)

$$SZZB = \frac{T_a^2}{12}\text{erfc}\left(\sqrt{\frac{\text{SNR}}{4}}\right) + \frac{1}{\text{SNR}}\frac{T_d^2}{4\pi^2}\Gamma_{\frac{3}{2}}\left(\frac{\text{SNR}}{4}\right) - \left(\frac{1}{\text{SNR}}\frac{T_d^2}{4\pi^2}\right)^{\frac{3}{2}}\frac{32}{3T_a\sqrt{2\pi}}\Gamma_2\left(\frac{\text{SNR}}{4}\right).$$

```
get_SZZB = @(SNR,Td,Ta)(    Ta^2/12*erfc(sqrt(SNR/4)) ...
    + (Td^2/(4*pi^2)./SNR)*gammainc(SNR/4,3/2) ...
    - (Td^2/(4*pi^2)./SNR).^1.5.*(32/(3*Ta*sqrt(2*pi))).*gammainc(SNR/4,2)   );
```

Example how to use bounds

```
signal = get_sym_signal(t,tau,fc,damp);
noise  = get_noise(t,sigma);
noisySignal = signal + noise;

beta = get_beta(signal,dt);
Td = 1/fc;
W  = sum(signal.^2)*dt;
N0 = var(noise)*dt;
SNR = W/N0;
SNRdb = 10*log10(SNR);

figure(11)
fig = figure('Position', [1 1 500 400]);
plot(t, noisySignal,'k');
hold on
plot(t, signal,'r');
axis([0.4 0.5 -1 1])
xlabel('Time (s)')
title(['Signal with SNR=' num2str(SNRdb,4) 'dB'])
```




```
CRB = get_CRB(SNR,beta);
ZZB = get_ZZB(SNR,beta,Ta);
SB  = get_SB(SNR,Td);
SZZB = get_SZZB(SNR,Td,Ta);

disp(['Cramer-Rao Bound is equal: ' num2str(CRB*1e6) ' ms^2']);
```

```
Cramer-Rao Bound is equal: 0.0042827 ms^2
```

```
disp(['Ziv-Zakai Bound is equal: ' num2str(ZZB*1e6) ' ms^2']);
```

```
Ziv-Zakai Bound is equal: 0.0042815 ms^2
```

```
disp(['Seismic Cramer-Rao Bound is equal: ' num2str(SB*1e6) ' ms^2']);
```

```
Seismic Cramer-Rao Bound is equal: 0.0045978 ms^2
```

```
disp(['Seismic Ziv-Zakai Bound is equal: ' num2str(SZZB*1e6) ' ms^2']);
```

```
Seismic Ziv-Zakai Bound is equal: 0.0045965 ms^2
```

Numerical estimation of arrival-time uncertainty

```
ntest = 1000;
error = zeros(1,ntest);

t0 = 4/fc;
trueSignal = get_sym_true_signal(t,t0,fc,damp);
signal = get_sym_signal(t,tau,fc,damp);
```

```
for j=1:ntest
    noise  = get_noise(t,sigma);
    noisySignal = signal + noise;
    [time,~] = mycorr(trueSignal, noisySignal, dt);
    error(j) = time - tau + t0;
end

disp(['Mean arrival-time error after ' num2str(ntest) ' experiments is equal ' num2str(
```

```
Mean arrival-time error after 1000 experiments is equal -0.0033048ms
```

```
disp(['Variance of arrival-time error after ' num2str(ntest) ' experiments is equal ' n
```

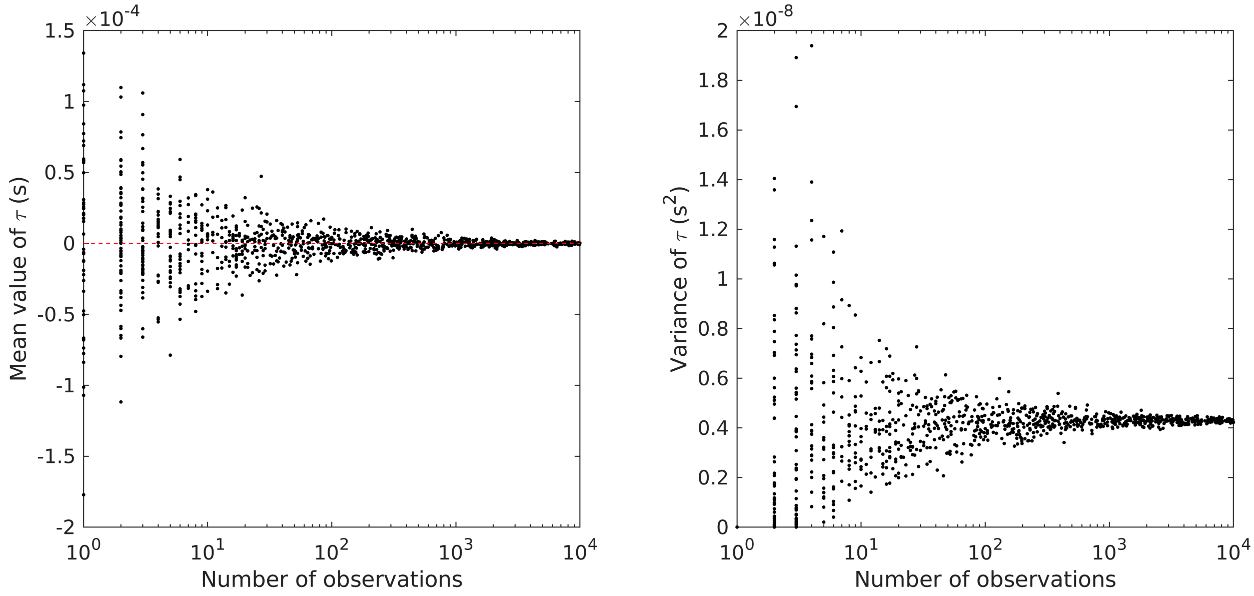
```
Variance of arrival-time error after 1000 experiments is equal 0.0041227ms^2
```

Detect the optimum number of numerical tests to achieve stable values of the variance

(Note: execution of this section is extreemly time-consuming. To run the section set the flag "repeat_computations" to True.)

```
repeat_computations = 0;
if(repeat_computations)
    NRT = 1000;
    Tmean = zeros(1,NRT);
    Tvar  = zeros(1,NRT);
    Ntest = round(10.^(4*rand(1,NRT)));

    t0 = 4/fc;
    trueSignal = get_sym_true_signal(t,t0,fc,damp);
    sigma = 0.2;

    for i=1:length(Ntest)
        ntest = Ntest(i);
        error = zeros(1,ntest);
        for j=1:ntest
            tau = 0.4+0.1*rand(1);
            signal = get_sym_signal(t,tau,fc,damp);
            noise  = get_noise(t,sigma);
            noisySignal = signal + noise;
            [time,~] = mycorr(trueSignal, noisySignal, dt);
            error(j) = time - tau + t0;
        end
        Tmean(i) = mean(error);
        Tvar(i) = var(error);
    end
    data.NTest = Ntest;
    data.Tmean = Tmean;
    data.Tvar = Tvar;
else
    data = MLD([mlibfolder '/Examples/Arrival_time_uncertainty/Data/Precomputed_data.ma
    NTest = data.NTest;
    Tmean = data.Tmean;
```

```
    Tvar  = data.Tvar;
end

fig = figure(21);
set(fig, 'Position', [100 100 1000 500])
subplot(1,2,1)
semilogx(Ntest, Tmean, '.k');
hold on
plot(linspace(1, 10000, 100), zeros(1,100), 'r--');
ylabel('Mean value of \tau (s)')
xlabel('Number of observations')
axis('square')

subplot(1,2,2)
semilogx(Ntest, Tvar, '.k');
ylabel('Variance of \tau (s^2)')
xlabel('Number of observations')
axis('square')
```



Conclusion

Precise numerical estimation of the variance requires 1000 - 10 000 experiments

The behavior of the variance for different SNR

```
SNRdb = [-5:1:17 17.2:0.2:18.8 19:35];
SNR = 10.^(SNRdb/10);
FC = [10 20 40 100 200 400];

repeat_computations = 0;
if (repeat_computations)
```

```
    Variance.numeric = zeros(length(FC),length(SNRdb));
    Variance.sigma   = zeros(length(FC),length(SNRdb));
    Variance.FC = FC;
    Variance.SNR = SNR;
    Variance.SNRdb = SNRdb;


    ntest = 10000;   % use ntest = 100000 precise estimation of MSE in transition zone
    error = zeros(1,ntest);

    for k = 1:length(FC)
        fc = FC(k);
        t0 = 4/fc;
        trueSignal = get_sym_true_signal(t,t0,fc,damp);
        tau = 0.5;
        signal = get_sym_signal(t,tau,fc,damp);
        W  = sum(signal.^2)*dt;
        for i = 1:length(SNRdb)
            sigma = sqrt(W/SNR(i)/dt);
            Variance.sigma(k,i) = sigma;
            for j=1:ntest
                tau = 0.4+0.1*rand(1);
                signal = get_sym_signal(t,tau,fc,damp);
                noise  = get_noise(t,sigma);
                noisySignal = signal + noise;
                [time,~] = mycorr(trueSignal, noisySignal, dt);
                error(j) = time - tau + t0;
            end
            Variance.numeric(k,i) = var(error);
        end
    end
else
    Variance = MLD([mlibfolder '/Examples/Arrival_time_uncertainty/Data/Precomputed_MSE
end
```

Compare numerical results with theoretical bounds

```
FC = Variance.FC;
SNR = Variance.SNR;
SNRdb = Variance.SNRdb;
Variance.CRB = zeros(length(FC),length(SNRdb));
Variance.ZZB = zeros(length(FC),length(SNRdb));
Variance.SB  = zeros(length(FC),length(SNRdb));
Variance.SZZB = zeros(length(FC),length(SNRdb));

for k = 1:length(FC)
    fc = FC(k);
    Td = 1/fc;
    tau = 0.5;
    signal = get_sym_signal(t,tau,fc,damp);
    beta = get_beta(signal,dt);
    for i = 1:length(SNRdb)
        snr = SNR(i);
```

```
        Variance.CRB(k,i) = get_CRB(snr,beta);
        Variance.ZZB(k,i) = get_ZZB(snr,beta,Ta);
        Variance.SB(k,i)  = get_SB(snr,Td);
        Variance.SZZB(k,i) = get_SZZB(snr,Td,Ta);
    end
end
```

Figure 1

```
figure(1)
fig = figure('Position', [1 1 500 500]);

k = 5;      %    fc = 200 Hz
fc = Variance.FC(k);
tau = 0.45242;
t0=0.05;

signal = get_nsm_true_signal(t,t0,fc,damp);
signal = signal/max(signal);

W  = sum(signal.^2)*dt;
snrdb = 30;
snr = 10.^(snrdb/10);
sigma = sqrt(W/snr/dt);

noise = get_noise(t,sigma);
noisySignal = signal + noise;

subplot(4,1,1);
plot(t-t0,signal,'black');
axis([-0.009 0.091 -1 1])
title('Original signal')
text(-0.019,1,'a)','FontSize',16)

subplot(4,1,2)
i = 44;
sigma = Variance.sigma(k,i);
signal = get_nsm_signal(t,tau,fc,damp);
signal = signal/max(signal);
noise  = get_noise(t,sigma);
noisySignal = signal + noise;
plot(t,noisySignal,'black');
hold on
plot(t,signal,'red');
title(['Noisy signal, SNR = ' num2str(Variance.SNRdb(i)) ' dB'])
axis([0.4 0.5 -1 1])
text(0.39,1,'b)','FontSize',16)

subplot(4,1,3)
i = 34;
sigma = Variance.sigma(k,i);
signal = get_nsm_signal(t,tau,fc,damp);
signal = signal/max(signal);
```

```
noise  = get_noise(t,sigma);
noisySignal = signal + noise;
plot(t,noisySignal,'black');
hold on
plot(t,signal,'red');
title(['Noisy signal, SNR = ' num2str(Variance.SNRdb(i)) ' dB'])
axis([0.4 0.5 -1 1])
text(0.39,1,'c)','FontSize',16)

subplot(4,1,4)
i = 16;
sigma = Variance.sigma(k,i);
signal = get_nsm_signal(t,tau,fc,damp);
signal = signal/max(signal);
noise  = get_noise(t,sigma);
noisySignal = signal + noise;
plot(t,noisySignal,'black');
hold on
plot(t,signal,'red');
title(['Noisy signal, SNR = ' num2str(Variance.SNRdb(i)) ' dB'])
axis([0.4 0.5 -1 1])
text(0.39,1,'d)','FontSize',16)
xlabel('Time (s)')
```

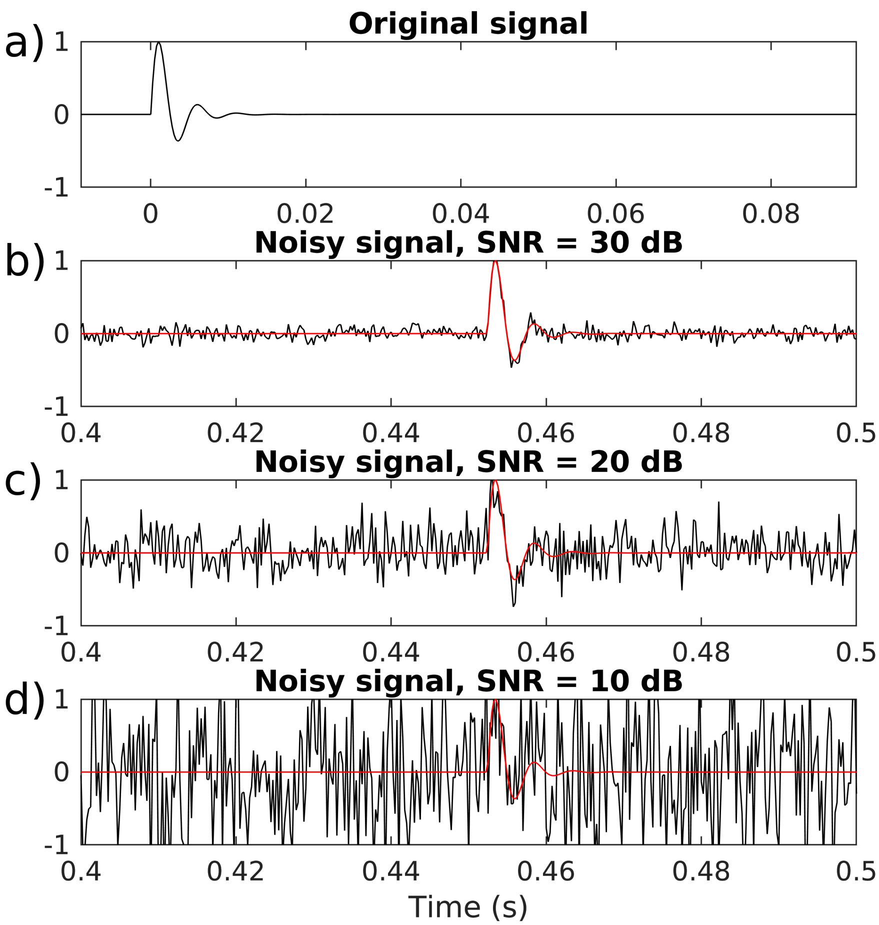


Figure 2

```
figure(353843)
fig = figure('Position', [1 1 500 400]);
semilogy(SNRdb,ones(size(SNRdb))*(1/12), '-b');
hold on
semilogy(SNRdb,Variance.numeric(1,:), 'r--')
semilogy(SNRdb,Variance.numeric(2,:), 'm--')
semilogy(SNRdb,Variance.numeric(3,:), 'c--')
semilogy(SNRdb,Variance.numeric(4,:), 'g--')
semilogy(SNRdb,Variance.numeric(5,:), 'b--')
semilogy(SNRdb,Variance.numeric(6,:), 'k--')
semilogy(SNRdb,ones(size(SNRdb))*(1/12), '--b')

line([10 10], [1e-5 1e-1],'Color','black','LineStyle','-')
text(-1,1e-3,'A priori region','FontSize',10)
text(-1,1e-4,'SNR < 10 dB','FontSize',10)
line([20 20], [1e-5 1e-9],'Color','black','LineStyle','-')
text(21,1e-6,'Asymptotic region','FontSize',10)
text(21,1e-7,'SNR > 20 dB','FontSize',10)
text(10,1e-8,'Transition','FontSize',10)
text(10,1e-9,'    region','FontSize',10)

legend('T_a^2/12', '10 Hz', '20 Hz','40 Hz','100 Hz','200 Hz','400 Hz' );
axis([-5 35 1e-11 1])
xlabel('SNR (dB)')
ylabel('MSE (s^2)')
```

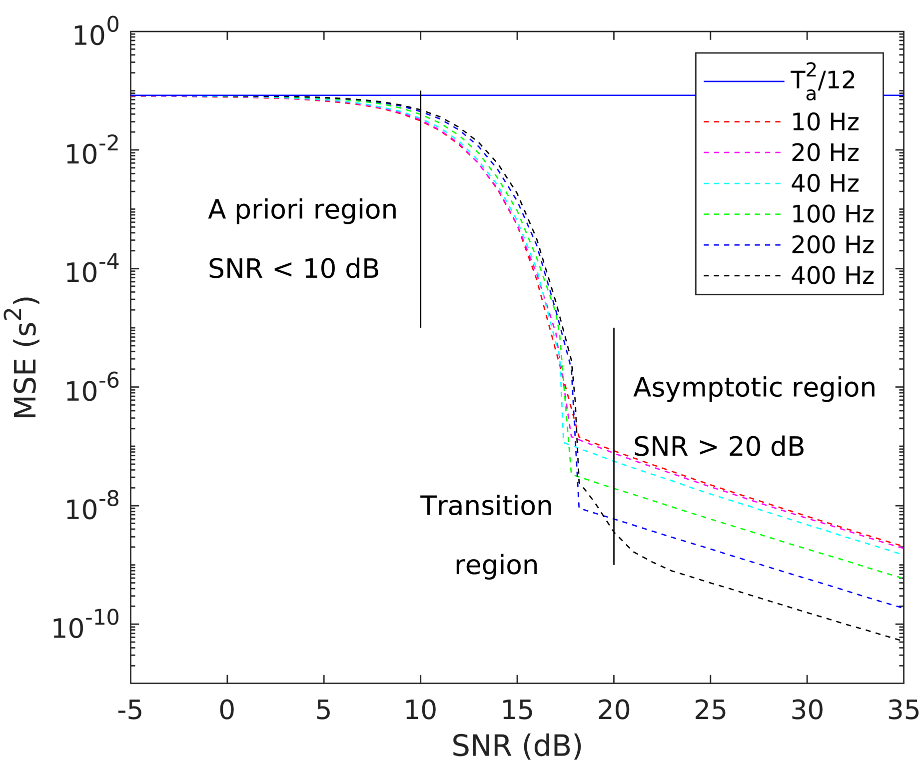


Figure 3

```
figure(353823)
fig = figure('Position', [1 1 500 400]);
semilogy(SNRdb,Variance.CRB(k,:), 'r')
hold on
semilogy(SNRdb,Variance.ZZB(k,:), 'm')
semilogy(SNRdb,Variance.SB(k,:), 'c+')
semilogy(SNRdb,Variance.SZZB(k,:), 'g+')
semilogy(SNRdb,Variance.numeric(k,:), 'k--')
semilogy(SNRdb,ones(size(SNRdb))*(1/12), '-b');
legend('CRB', 'ZZB', 'equation 19', 'equation 21', 'Numeric', 'T_a^2/12');
text(-15,1.0,'e)','FontSize',16)
axis([-5 35 1e-10 1])
axis([-5 35 1e-11 1])
xlabel('SNR (dB)')
ylabel('MSE (s^2)')
```