

# Robust fixed-time stability: application to sliding mode control

Emmanuel Moulay, Vincent Léchappé, Emmanuel Bernuau, and Franck Plestan

**Abstract**—This article deals with robust fixed-time stability and stabilization. First, new global robust fixed-time stability results are proposed for scalar systems by using constant and variable exponent coefficients. Then, they are applied to global robust fixed-time stabilization of a class of uncertain nonlinear second-order systems by using sliding mode control. All the results are illustrated in simulation.

**Index Terms**—Fixed-time stability, sliding mode control, robust control.

## I. Introduction

Sliding mode control (SMC) has been developed by Utkin in [1] and then by many authors, see [2] and the references therein for more details. The aim of SMC is to enforce a dynamical system to reach a manifold called “sliding surface” defined by a function called “sliding variable” with an appropriate controller ensuring that a constraint on the sliding variable is satisfied. After the constraint is checked, the system trajectories “slide” on the sliding surface towards the desired equilibrium. The main advantage of SMC lies in the simplicity of its feedback control strategy after choosing the sliding variable, its robustness when using discontinuous controllers and the finite-time convergence of the closed-loop system trajectories to reach the sliding surface. Moreover, it has been refined over time, for instance with the integral SMC [3].

Finite-time stability has been developed by Bhat and Bernstein in [4] and then applied for finite-time stabilization, for instance in [5], [6]. It ensures that dynamical systems reach their equilibrium in a finite-time called settling-time depending on the initial conditions. Finite-time stabilization using a SMC strategy has been developed in [7], [8] where a singularity problem is solved. The fixed-time stability was introduced by Polyakov in [9] and then developed by many authors, for instance in [10], [11], [12], [13], [14]. In addition to finite-time stability, fixed-time stability ensures that the settling-time does not depend on the initial conditions. Fixed-time stabilization provides a predefined convergence time

towards the equilibrium which is a desirable property for engineering applications. In particular, fixed-time stabilization using a SMC strategy with time-independent controllers has been proposed in [15], [16], [17] also by solving a singularity problem. The singularity problem comes from the fact that the simplest finite-time and fixed-time sliding variables are non differentiable. It results in more complex feedback controls to implement. Finally, the notion of predefined/prescribed-time SMC has been introduced in [18], [19], [20] by using time-dependent controllers.

In this article, new global robust fixed-time stability results are provided for scalar systems by using constant and state-dependent variable exponent coefficients. State-dependent variable exponent coefficients have already been used in the context of homogeneous self-triggered control in [21]. They have also been used for defining the controllers in [22] for finite-time SMC. But to the best of the authors’ knowledge, it has never been used for fixed-time stability. By employing the SMC strategy, global robust asymptotic stabilization of the global  $x$ -system of the state variable  $x$  with robust fixed-time stabilization of the  $s$ -system of the sliding variable  $s$  is obtained by using constant exponent coefficients in the sliding variable and the controllers for a class of uncertain nonlinear second-order systems. Moreover, global robust fixed-time stabilization of the global  $x$ -system is obtained by using state-dependent variable exponent coefficient in the sliding variable and the controllers. The new sliding mode controllers are time-independent, non singular, robust with respect to bounded disturbances and easy to implement. So using a variable exponent coefficient allows to obtain robust fixed-time SMC of the global  $x$ -system contrary to the constant exponent coefficient strategy. Actually, it is not easy to obtain fixed-time stabilization of the global  $x$ -system when dealing with SMC because some singularities appear when using the simplest fixed-time sliding variable, see [16] and [17]. With the use of a variable exponent coefficient in the sliding variable and the controllers, we obtain a simple solution because the controllers have no singularity.

The paper is organized as follows. After some preliminaries in Section II, the main results on robust fixed-time stability are given in Section III. The application to SMC is developed in Section IV. Finally, a conclusion is addressed in Section V.

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## II. Preliminaries

In the following, denote  $\mathbb{R}^+$  the set of positive real numbers and  $e$  the constant such that  $\ln(e) = 1$ . Recall some results on finite-time stability and fixed-time stability. Consider the following ordinary differential equation

$$\begin{aligned} \dot{x}(t) &= f(x(t)), & x(t) &\in \mathbb{R}^n \\ x(0) &= x_0 \end{aligned} \quad (1)$$

with  $f$  a continuous function such that  $f(0) = 0$ .

**Definition 1:** [6], [4] System (1) is globally finite-time stable if it is Lyapunov stable and for all  $x_0 \in \mathbb{R}^n$  there exists  $T(x_0) > 0$  dependent on the initial conditions such that  $\lim_{t \rightarrow T(x_0)} \|x(t)\| = 0$ , i.e.  $\|x(t)\| \equiv 0$  for all  $t \geq T(x_0)$ . The function  $T$  is called the settling-time.

**Definition 2:** [9] System (1) is globally fixed-time stable if:

- (1) it is globally finite-time stable;
- (2) the settling-time function  $T$  is upper bounded by a constant  $\mathbf{T} > 0$ , i.e.  $T(x_0) \leq \mathbf{T}$  and does not depend on the initial conditions.

**Lemma 1:** [9], [14] If there exists a continuously differentiable positive definite radially unbounded function  $V : \mathbb{R}^n \rightarrow \mathbb{R}^+$  such that

$$\dot{V}(x) \leq -aV(x)^\gamma - bV(x)^\alpha \quad (2)$$

where  $x \in \mathbb{R}^n$ ,  $a > 0$ ,  $b > 0$  and  $0 < \gamma < 1 < \alpha$ , then system (1) is globally fixed-time stable and the settling-time satisfies

$$T(x_0) \leq \frac{1}{a(1-\gamma)} + \frac{1}{b(\alpha-1)}. \quad (3)$$

$V$  is called a Lyapunov function for system (1).

In the following, all simulations are performed with a fixed step simulation equal to 0.1ms.

## III. Robust fixed-time stability

### A. Constant exponent coefficient

Consider the following robust fixed-time stability result.

**Theorem 1:** The system

$$\begin{aligned} \dot{x} &= -k_1 \operatorname{sgn}(x) - k_2 |x|^\alpha \operatorname{sgn}(x) - k_3 |x|^\gamma \operatorname{sgn}(x) - k_4 x + d \\ x(0) &= x_0 \end{aligned} \quad (4)$$

with  $x(t) \in \mathbb{R}$ ,  $\alpha > 1$ ,  $0 < \gamma < 1$ ,  $d(t) \in \mathbb{R}$  an external disturbance such that  $|d(t)| < \delta$  for a given  $\delta > 0$ ,  $k_1 > \delta$ ,  $k_2 > 0$ ,  $k_3 \geq 0$ ,  $k_4 \geq 0$  is globally fixed-time stable with the settling-time  $T$  satisfying

$$T(x_0) \leq \frac{1}{k_1 - \delta} + \frac{1}{k_2(\alpha - 1)}. \quad (5)$$

**Proof.** Consider the following quadratic Lyapunov function

$$V(x) = x^2 \quad (6)$$

Then it leads to

$$\begin{aligned} \dot{V}(x) &= -2k_1|x| - 2k_2|x|^{\alpha+1} - 2k_3|x|^{\gamma+1} - 2k_4x^2 + 2dx \\ &\leq -2(k_1 - \delta)|x| - 2k_2|x|^{\alpha+1} \\ &\leq -2(k_1 - \delta)V(x)^{\frac{1}{2}} - 2k_2V(x)^{\frac{\alpha+1}{2}} \end{aligned} \quad (7)$$

with  $\frac{\alpha+1}{2} > 1$ . By using Lemma 1, the result follows. ■

**Remark 1:** On the one hand, the function  $x \mapsto |x|^\gamma \operatorname{sgn}(x)$  with  $0 < \gamma < 1$  is not necessary to obtain the fixed-time stability while still used for instance in [14], [9]. On the other hand, the sign function  $x \mapsto \operatorname{sgn}(x)$  coupled with the function  $x \mapsto |x|^\alpha \operatorname{sgn}(x)$  where  $\alpha > 1$  allows the fixed-time stability and is known to reject the disturbances. This is the reason why robust fixed-time stability is obtained. Moreover, if only the first term is used, i.e. if  $k_2 = k_3 = k_4 = 0$ , one only obtains robust finite-time stability.

In the following, we compare in simulation the robustness of the state  $x(t)$  of system (4) and the term  $u(t) = -k_1 \operatorname{sgn}(x) - k_2 |x|^\alpha \operatorname{sgn}(x) - k_3 |x|^\gamma \operatorname{sgn}(x) - k_4 x$  in several cases with the disturbance  $d(t) = \sin(10t)$  leading to  $\delta = 1$ , the initial condition  $x(0) = 3$ ,  $\alpha = 1.5$  and  $\gamma = 0.5$ . Here are the three cases:

- Case 1.  $k_1 = 2$ ,  $k_2 = 2$ ,  $k_3 = 0$ ,  $k_4 = 0$ ;
- Case 2.  $k_1 = 2$ ,  $k_2 = 2$ ,  $k_3 = 2$ ,  $k_4 = 0$ ;
- Case 3.  $k_1 = 2$ ,  $k_2 = 2$ ,  $k_3 = 2$ ,  $k_4 = 2$ ;

that leads to  $T(x_0) \leq 2$ s. Figure 1 shows that system (4) is robust with respect to the disturbance for all cases and Figure 2 shows the induced chattering for the steady state  $x(t)$ . The settling-time of system (4) in case 3 is strictly lower than the settling-time in cases 1 and 2 because the time-derivative  $\dot{V}$  is rendered more negative. This explains the interest of introducing additional terms in system (4) while keeping the robust fixed-time stability.

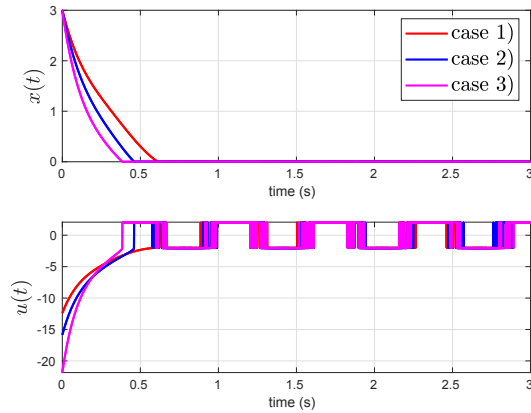


Fig. 1. Top. States  $x(t)$  versus time (sec) Bottom.  $u(t)$  versus time (sec)

### B. Variable exponent coefficient

Consider the following new robust fixed-time stability result using a state-dependent variable exponent coefficient.

**Theorem 2:** The system

$$\begin{aligned} \dot{x}(t) &= -k|x(t)|^{\frac{\lambda x(t)^2}{1+\mu x(t)^2}} \operatorname{sgn}(x(t)) + d(t) \\ x(0) &= x_0 \end{aligned} \quad (8)$$

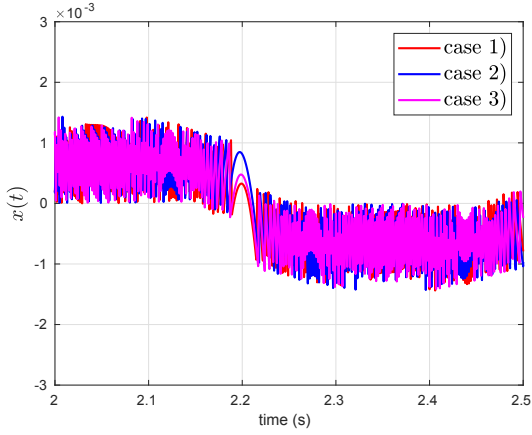


Fig. 2. Zoom on the steady state  $x(t)$  versus time (sec)

with  $x(t) \in \mathbb{R}$ ,  $\lambda > 0$  and  $\mu > 0$  such that  $\theta = \frac{\lambda}{1+\mu} > 1$ ,  $d(t) \in \mathbb{R}$  an external disturbance such that  $|d(t)| < \delta$  for a given  $\delta > 0$  and  $k > \delta e^{\frac{\lambda}{2e}}$  is globally fixed-time stable and the settling-time satisfies

$$T(x_0) \leq \frac{1}{(k-\delta)(\theta-1)} + \frac{1}{ke^{\frac{\lambda}{2e}} - \delta}. \quad (9)$$

Proof. First note that the function  $\varphi : x \mapsto |x|^{\frac{\lambda x^2}{1+\mu x^2}} = \exp\left(\frac{\lambda x^2}{1+\mu x^2} \ln(|x|)\right)$  is continuous at  $x = 0$  with  $\varphi(0) = 1$ . Therefore the right-hand side of (8) is locally bounded.

Consider the following quadratic Lyapunov function

$$V(x) = x^2. \quad (10)$$

It leads to

$$\dot{V}(x) = -2k|x|^{\frac{\lambda x^2}{1+\mu x^2}+1} + 2dx \quad (11)$$

Consider the case  $V(x) \geq 1$ . We have  $\frac{\lambda x^2}{1+\mu x^2} + 1 \geq \frac{\lambda}{1+\mu} + 1 > 2$ . As  $|x| \geq 1$  and  $\theta = \frac{\lambda}{1+\mu} > 1$  it leads to

$$\dot{V}(x) \leq -2(k-\delta)|x|^{\theta+1} \quad (12)$$

$$\leq -2(k-\delta)V(x)^{\frac{\theta+1}{2}} \quad (13)$$

As  $k-\delta > 0$  and  $\frac{\theta+1}{2} > 1$  the proof of [9, Lemma 1] ensures that all the solutions starting from  $\{V(x) \geq 1\}$  reaches the set  $\{V(x) \leq 1\}$  in a fixed time  $T_1 \leq \frac{1}{(k-\delta)(\theta-1)}$ .

Consider now the case  $V(x) \leq 1$ . We have

$$\dot{V}(x) = -2k|x||x|^{\frac{\lambda x^2}{1+\mu x^2}} + 2dx \quad (14)$$

As  $1 + \mu x^2 \geq 1$  and  $|x| \leq 1$  it leads to  $\min\left(|x|^{\frac{\lambda x^2}{1+\mu x^2}}\right) \geq \min\left(|x|^{\frac{\lambda}{2e}}\right) = e^{\frac{-\lambda}{2e}}$  and we have

$$\dot{V}(x) \leq -2\left(ke^{\frac{-\lambda}{2e}} - \delta\right)|x| \quad (15)$$

$$\leq -2\left(ke^{\frac{-\lambda}{2e}} - \delta\right)V(x)^{\frac{1}{2}} \quad (16)$$

with  $ke^{\frac{-\lambda}{2e}} - \delta > 0$ . Theorem 4.2 in [4] implies that all the solutions starting from  $\{V(x) \leq 1\}$  reach the origin in a uniform time  $T_2 \leq \frac{1}{ke^{\frac{-\lambda}{2e}} - \delta}$ .

Finally, system (8) reaches the origin in a fixed time  $T(x_0) \leq T_1 + T_2$ . ■

Figure 3 displays the time variations of the state  $x(t)$  and  $u(t) = -k|x(t)|^{\frac{\lambda x(t)^2}{1+\mu x(t)^2}} \text{sgn}(x(t))$  of system (8) with  $k = 3$ ,  $\lambda = 2$ ,  $\mu = 0.1$  and  $d(t) = \sin(10t)$ . As  $\delta = 1$ , it leads to  $T(x_0) \leq 1.5s$ . Moreover, a zoom on the steady state  $x(t)$  of system (8) is given in Figure 4.

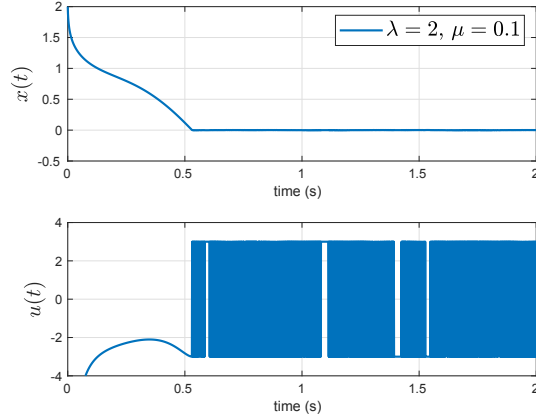


Fig. 3. Top. State  $x(t)$  versus time (sec) Bottom.  $u(t)$  versus time (sec)

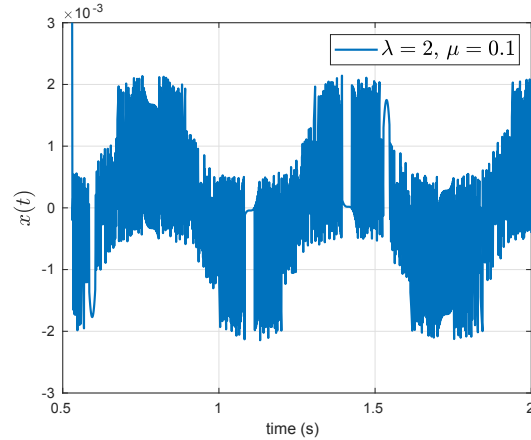


Fig. 4. Zoom on the steady state  $x(t)$  versus time (sec)

Remark 2: When  $x \rightarrow \infty$ , system (8) is equivalent to the system

$$\dot{x}(t) = -k|x(t)|^{\frac{\lambda}{\mu}} \text{sgn}(x(t)) + d(t) \quad (17)$$

with  $\frac{\lambda}{\mu} > 1$ . If  $\lambda \approx \mu$ , it is possible to obtain a linear behavior away from the origin.

When  $x = 0$ , given the continuity of  $x \mapsto |x|^{\frac{\lambda x^2}{1+\mu x^2}}$ , system (8) is equivalent to the system

$$\dot{x}(t) = -k \text{sgn}(x(t)) + d(t) \quad (18)$$

which is known to be robust with respect to the disturbances but leads to chattering. This is the reason why high frequency oscillations appears on Figure 3 for  $u(t)$ ; by a similar way chattering appear on the steady state  $x(t)$  of system (8) as shown by Figure 4.

#### IV. Application to sliding mode control

In this section, consider the following uncertain nonlinear second-order system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x) + g(x)u + d\end{aligned}\quad (19)$$

with  $x = (x_1, x_2) \in \mathbb{R}^2$  the state,  $u \in \mathbb{R}$  the control input,  $f$  and  $g$  continuous functions such that  $f(0) = 0$ ,  $g(x) \neq 0$  for all  $x \in \mathbb{R}^2$  and  $d$  the external disturbance such that  $|d(t)| < \delta$ . The second-order systems have been widely used in practice, see for instance [23]. The objective is to use the previous results on robust fixed-time stability for designing sliding mode controllers.

##### A. Constant exponent coefficient

Consider the standard sliding variable

$$s(x) = x_2 + \beta x_1 \quad (20)$$

with  $\beta > 0$  and the controller

$$\begin{aligned}u(x) &= -g^{-1}(x) \left[ f(x) + \beta x_2 + k_1 \operatorname{sgn}(s) \right. \\ &\quad \left. + k_2 |s|^\alpha \operatorname{sgn}(s) + k_3 |s|^\gamma \operatorname{sgn}(s) + k_4 s \right]\end{aligned}\quad (21)$$

with  $k_1 > \delta$ ,  $k_2 > 0$ ,  $k_3 \geq 0$ ,  $k_4 \geq 0$ ,  $\alpha > 1$  and  $0 < \gamma < 1$ .

Proposition 1: The closed-loop system (19)–(20)–(21) reaches the sliding surface  $\{s(x) = 0\}$  in a fixed-time satisfying

$$T(s_0) \leq \frac{1}{k_1 - \delta} + \frac{1}{k_2(\alpha - 1)} \quad (22)$$

is also globally asymptotically stable.

Proof.  $s$ -dynamics read as

$$\begin{aligned}\dot{s} &= f(x) + g(x)u(x) + \beta x_2 + d \\ &= -k_1 \operatorname{sgn}(s) - k_2 |s|^\alpha \operatorname{sgn}(s) - k_3 |s|^\gamma \operatorname{sgn}(s) - k_4 s + d\end{aligned}\quad (23)$$

By using Theorem 1, the first part of the proposition is deduced. When the sliding surface is reached, one has

$$\dot{x}_1 = -\beta x_1 \quad (24)$$

which ensures the asymptotic stability of the closed-loop system (19)–(20)–(21) towards the origin. ■

For the simulations, consider the functions  $f \equiv 0$ ,  $g \equiv 1$ ,  $d(t) = \sin(10t)$ , the parameters  $\beta = 1$ ,  $\alpha = 1.5$ ,  $\gamma = 0.5$ ,  $\delta = 1$  and the gains  $k_i$  where  $i = 1, \dots, 4$  given by the different cases presented in Subsection III-A. Since all the parameters are the same as in Subsection III-A, one still has  $T(s_0) \leq 2$ s. The time evolution of the sliding variable  $s(t)$  and the norms of the state variable  $\|x(t)\|$  associated to the closed-loop system (19)–(20)–(21) are shown on Figure 5.

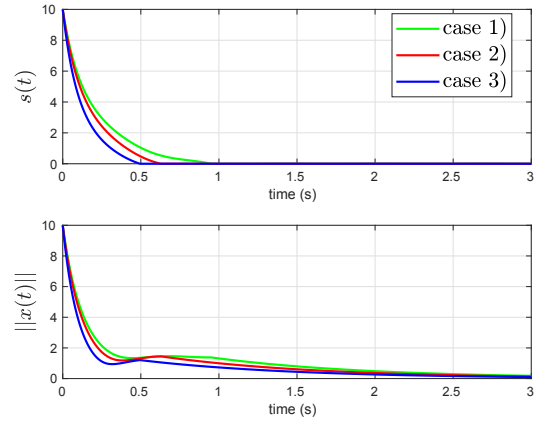


Fig. 5. Top. Sliding variables  $s(t)$  versus time (sec) Bottom. Norms of state variable  $\|x(t)\|$  versus time (sec)

Remark 3: Let us remark that if system (4) is used for building the simplest fixed-time sliding variable of the form

$$s(x) = x_2 + \beta_1 |x_1|^\alpha \operatorname{sgn}(x_1) + \beta_2 |x_1|^\gamma \operatorname{sgn}(x_1) \quad (25)$$

with  $\beta_1 > 0$ ,  $\beta_2 > 0$ ,  $\alpha > 1$  and  $0 < \gamma < 1$  it leads to a singular controller, see for instance [16], [17]. With the classical sliding surface (20), one can get the global robust fixed-time stabilization of the  $s$ -system (23), as explained in Proposition 1, but only the global robust asymptotic stabilization of the  $x$ -system (19). However, the controller (21) is easy to implement. Global robust fixed-time stabilization of the global  $x$ -system (19) is obtained in [16], [17] with complex sliding variables and controllers and in the next subsection by using a state-dependent variable power coefficient.

##### B. Variable exponent coefficient

The main objective of this subsection is to design a new simple sliding variable leading to global robust fixed-time stabilization of system (19). Consider Theorem 2 and the induced sliding variable with a state-dependent variable exponent coefficient given by

$$s(x) = x_2 + \beta |x_1|^{\frac{\lambda_1 x_1^2}{1 + \mu_1 x_1^2}} \operatorname{sgn}(x_1) \quad (26)$$

with  $\theta_1 = \frac{\lambda_1}{1 + \mu_1} > 1$ ,  $\beta > 0$  and the controller

$$\begin{aligned}u(x) &= -g(x)^{-1} \left[ f(x) + k |s|^{\frac{\lambda_2 s^2}{1 + \mu_2 s^2}} \operatorname{sgn}(s) \right. \\ &\quad \left. + \frac{\beta \lambda_1 |x_1| x_2}{1 + \mu_1 x_1^2} \left( \frac{2 \ln |x_1|}{1 + \mu_1 x_1^2} + 1 \right) |x_1|^{\frac{\lambda_1 x_1^2}{1 + \mu_1 x_1^2}} \right]\end{aligned}\quad (27)$$

with  $\theta_2 = \frac{\lambda_2}{1 + \mu_2} > 1$ ,  $k > \delta e^{\frac{\lambda_2}{2\epsilon}}$ .

Proposition 2: The closed-loop system (19)–(26)–(27) is globally fixed-time stable and the settling-time satisfies

$$T(x_0) \leq \frac{1}{(k-\delta)(\theta_2-1)} + \frac{1}{ke^{\frac{-\lambda_2}{2e}} - \delta} + \frac{1}{\beta(\theta_1-1)} + \frac{1}{\beta e^{\frac{-\lambda_1}{2e}}}. \quad (28)$$

Proof. One has

$$\begin{aligned} \dot{s} &= f(x) + g(x)u(x) \\ &+ \frac{\beta\lambda_1|x_1|x_2}{1+\mu_1x_1^2} \left( \frac{2\ln|x_1|}{1+\mu_1x_1^2} + 1 \right) |x_1|^{\frac{\lambda_1x_1^2}{1+\mu_1x_1^2}} + d \\ &= -k|s|^{\frac{\lambda_2s^2}{1+\mu_2s^2}} \operatorname{sgn}(s) + d \end{aligned} \quad (29)$$

By using Theorem 2, one deduces that system (29) starting at  $s(0) = s_0$  reaches the sliding surface  $\{s = 0\}$  in a fixed-time satisfying  $T(s_0) \leq \frac{1}{(k-\delta)(\theta_2-1)} + \frac{1}{ke^{\frac{-\lambda_2}{2e}} - \delta}$ . From (26), one has

$$\dot{x}_1 = -\beta|x_1|^{\frac{\lambda_1x_1^2}{1+\mu_1x_1^2}} \operatorname{sgn}(x_1). \quad (30)$$

By using one more time Theorem 2, it is deduced that  $x_1(t)$  starting at  $x_1(0) = x_{10}$  reaches the origin in a fixed-time satisfying  $T(x_{10}) \leq \frac{1}{\beta(\theta_1-1)} + \frac{1}{\beta e^{\frac{-\lambda_1}{2e}}}$ . Finally, the closed-loop system (19)–(26)–(27) reaches the origin in a fixed-time  $T(x_0) = T(s_0) + T(x_{10})$  that is bounded by (28). ■

Consider the closed-loop system (19)–(26)–(27) with  $f = 0$ ,  $g = 1$ ,  $\beta = 0.2$ ,  $\lambda_1 = 2$ ,  $\lambda_2 = 4$ ,  $\mu_1 = 0.1$ ,  $\mu_2 = 1$ ,  $k = 10$ ,  $x(0) = [0.1, x_2(0)]^T$  and  $d(t) = \sin(10t)$ . In the case, one gets  $T(x_0) \leq 13.7$ s. Figure 6 displays the time evolution of the sliding variable  $s(t)$  and the state variable  $x(t) = (x_1(t), x_2(t))$ .

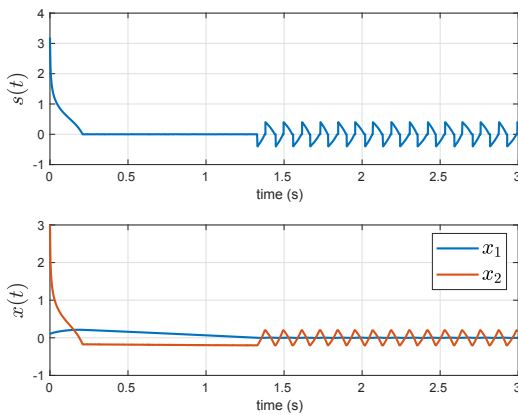


Fig. 6. Top. Sliding variable  $s(t)$  versus time (sec) Bottom. State variable  $x(t) = (x_1(t), x_2(t))$  versus time (sec)

The time evolution of the sliding variable  $s(t)$  given by system (29) is plotted on Figure 7 for different initial conditions  $x_2(0)$ .

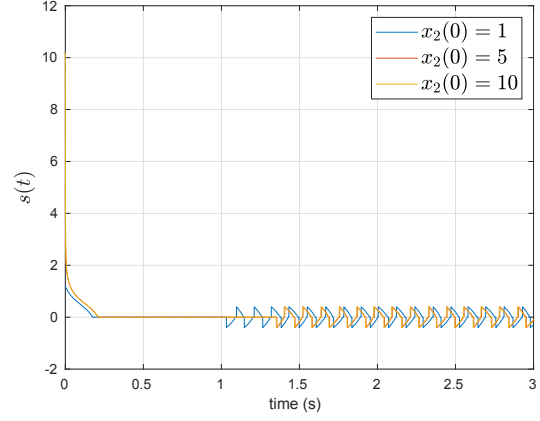


Fig. 7. Sliding variable  $s(t)$  versus time for different initial conditions  $x_2(0)$

Remark 4: First of all, the controller (27) is not singular due to the fact that  $\lim_{x_1 \rightarrow 0} |x_1| \ln(|x_1|) = 0$ . Then, if a more simple sliding variable

$$s(x) = x_2 + \beta|x_1|^{\lambda|x_1|} \operatorname{sgn}(x_1) \quad (31)$$

with  $\beta > 0$ ,  $\lambda > 0$  is chosen and if the controller reads as

$$\begin{aligned} u(x) &= -g^{-1}(x) \left[ f(x) + \beta\lambda \left( x_2 (\ln|x_1| + 1) |x_1|^{|x_1|} \right) \right. \\ &\quad \left. + k|s|^{|s|} \operatorname{sgn}(s) \right] \end{aligned} \quad (32)$$

with  $k > \delta$  then

$$\dot{s} = f(x) + g(x)u(x) \quad (33)$$

$$+ \beta\lambda \left( x_2 (\ln|x_1| + 1) |x_1|^{|x_1|} \right) + d \quad (34)$$

$$= -k|s|^{|s|} \operatorname{sgn}(s) + d \quad (35)$$

The fixed-time stabilization is obtained but the controller  $u(x)$  is singular due to the fact that  $\lim_{x_1 \rightarrow 0} |x_1|^{|x_1|} \ln|x_1| = -\infty$ .

In order to reduce the chattering induced by the use of controller (27), consider the following controller

$$\begin{aligned} u(x) &= -g^{-1}(x) \left[ f(x) + k_1 \operatorname{sgn}(s) + k_2 |s|^\alpha \operatorname{sgn}(s) \right. \\ &\quad \left. + \frac{\beta\lambda_1|x_1|x_2}{1+\mu_1x_1^2} \left( \frac{2\ln|x_1|}{1+\mu_1x_1^2} + 1 \right) |x_1|^{\frac{\lambda_1x_1^2}{1+\mu_1x_1^2}} \right] \end{aligned} \quad (36)$$

with  $k_1 > \delta$ ,  $k_2 > 0$ ,  $\alpha > 1$ .

Proposition 3: The closed-loop system (19)–(26)–(36) is globally fixed-time stable and the settling-time satisfies

$$T(x_0) \leq \frac{1}{k_1 - \delta} + \frac{1}{k_2(\alpha - 1)} \quad (37)$$

$$+ \frac{1}{\beta(\theta_1 - 1)} + \frac{1}{\beta e^{\frac{-\lambda_1}{2e}}}. \quad (38)$$

Proof. One has

$$\begin{aligned} \dot{s} &= f(x) + g(x)u(x) \\ &+ \frac{\beta\lambda_1|x_1|x_2}{1+\mu_1x_1^2} \left( \frac{2\ln|x_1|}{1+\mu_1x_1^2} + 1 \right) |x_1|^{\frac{\lambda_1x_1^2}{1+\mu_1x_1^2}} + d \\ &= -k_1 \operatorname{sgn}(s) - k_2 |s|^\alpha \operatorname{sgn}(s) + d \end{aligned} \quad (39)$$



By using Theorem 1, one deduces that system (39) starting at  $s(0) = s_0$  reaches the sliding surface  $\{s = 0\}$  in a fixed-time satisfying  $T(s_0) \leq \frac{1}{k_1 - \delta} + \frac{1}{k_2(\alpha - 1)}$ . Then it yields

$$\dot{x}_1 = -\beta |x_1|^{\frac{\lambda_1 x_1^2}{1 + \mu_1 x_1^2}} \operatorname{sgn}(x_1) \quad (40)$$

By using Theorem 2, it is deduced that  $x_1(t)$  starting at  $x_1(0) = x_{10}$  reaches the origin in a fixed-time satisfying  $T(x_{10}) \leq \frac{1}{\beta(\theta_1 - 1)} + \frac{1}{\beta e^{-\frac{\lambda_1}{2e}}}$ . Finally, the closed-loop system (19)–(26)–(36) reaches the origin in a fixed-time  $T(x_0) = T(s_0) + T(x_{10})$ . ■

The time evolution of the sliding variable  $s(t)$  and the state variable  $x(t) = (x_1(t), x_2(t))$  associated to the closed-loop system (19)–(26)–(36) is plotted on Figure 8 with the same parameters as before and  $k_1 = k_2 = 20$ ,  $\alpha = 1.5$ . So one gets  $T(x_0) \leq 13.5$ s.

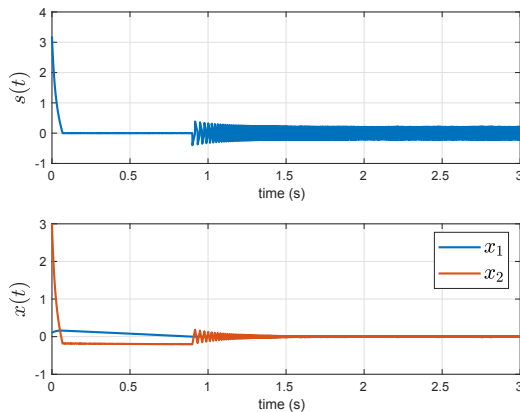


Fig. 8. Top. Sliding variable  $s(t)$  versus time (sec) Bottom. State variable  $x(t) = (x_1(t), x_2(t))$  versus time (sec)

**Remark 5:** The use of the sliding variable (26) with a state-dependent variable exponent coefficient leads to the global robust fixed-time stabilization of the global  $x$ -system (19) with the simple controllers (27) and (36) such that the closed-loop system behaves like the standard SMC around the sliding surface. So, a robust behavior of the closed-loop system is obtained similar to the standard SMC but in fixed time. When using system (1) with constant exponent coefficients for building a sliding variable for fixed-time stabilization, the associated controller is singular, see [16], [17].

**Remark 6:** Note that the proposed fixed-time SMC solution has the advantage of being simple and easy to tune with respect to the methods presented in [16], [17]. Indeed, our controllers have 6 parameters to tune whereas the controllers in [16] have 14 parameters. In [17], 6 scalar parameters need to be chosen as well as a function to define the sliding surface. The choice of this function is not obvious since it is based on properties of its time-derivative. Finally, both controllers in [16], [17] have a singularity which imposes to use a switched structure and this makes the controller more complex.

## V. Conclusion

This article deals with global robust fixed-time stability. Several robust fixed-time stability results involving constant and state-dependent variable exponent coefficients are provided and applied to the robust fixed-time stabilization of a class of uncertain nonlinear second-order systems by using sliding-mode control. For future works, a high order sliding mode strategy could be used for reducing the chattering when dealing with robust fixed-time stabilization.

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