Adversarial Training and Robustness for Multiple Perturbations

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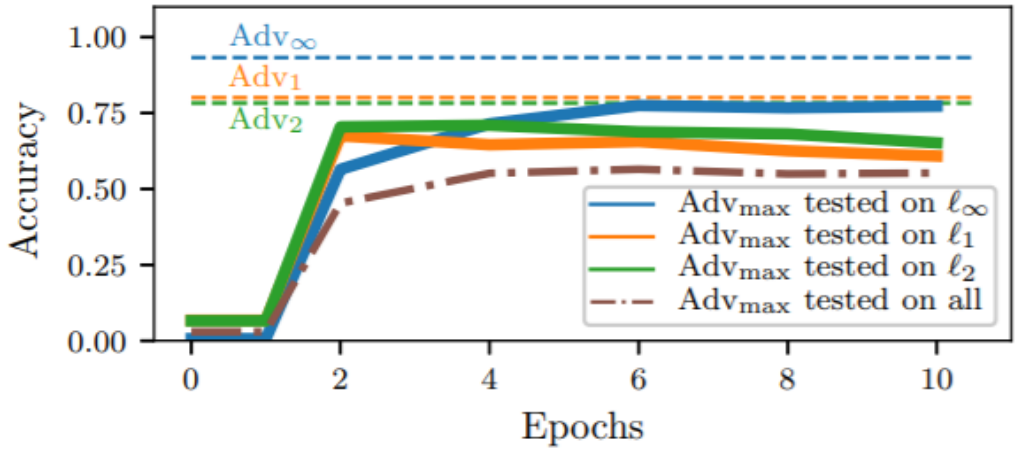
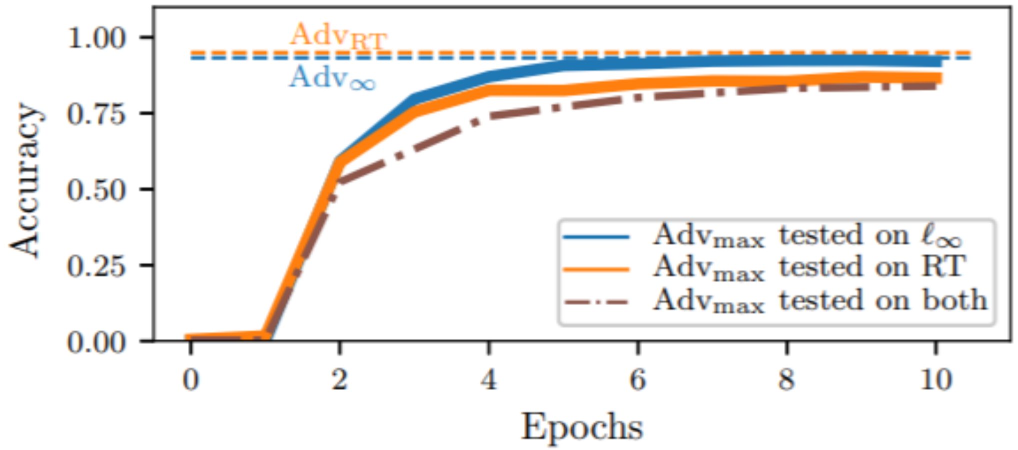
Can we achieve adversarial robustness to different types of perturbations simultaneously?

For now, the answer is NO

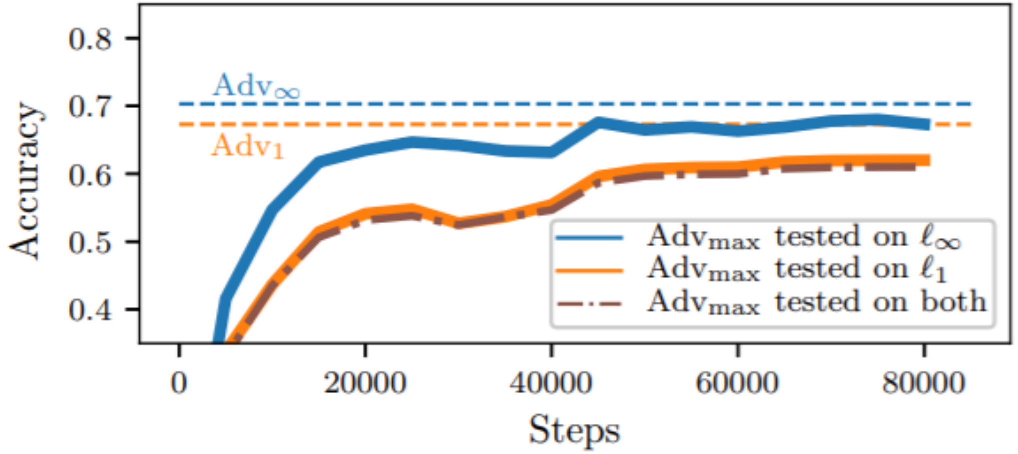


(a) MNIST models trained on ℓ_1, ℓ_2 & ℓ_∞ attacks.

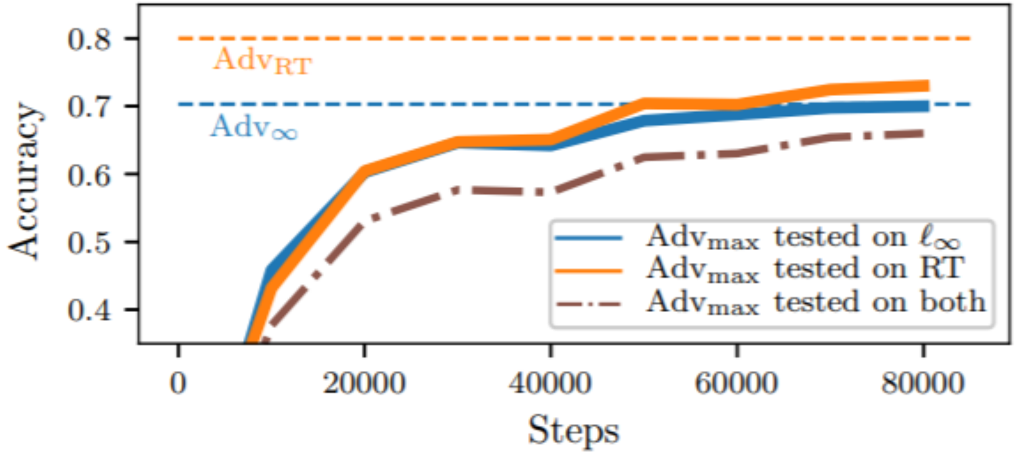


(b) MNIST models trained on ℓ_∞ and RT attacks.

RT: Rotation and translation



(c) CIFAR10 models trained on ℓ_1 and ℓ_∞ attacks.



(d) CIFAR10 models trained on ℓ_∞ and RT attacks.

Theoretical Proof

Notations:

- Data distribution $\mathcal{D}, (\boldsymbol{x}, y) \sim \mathcal{D}, \boldsymbol{x} \in \mathbb{R}^d, y \in [C]$
- Classifier $f: \mathbb{R}^d \longrightarrow [C]$
- Zero-one loss: $l(f(x), y) = \mathbb{1}_{f(\boldsymbol{x}) \neq y}$
- A type of perturbation: a set S, e.g., S can be an l_p-ball for a l_p perturbation

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- A type of perturbation: a set S, e.g., S can be an l_p-ball for a l_p perturbation
- Adversarial risk: $\mathcal{R}_{adv}(f; S) := E_{(\boldsymbol{x},y)\sim\mathcal{D}}[\max_{\boldsymbol{r} \in S} l(f(\boldsymbol{x} + \boldsymbol{r}), y)]$
- For multiple perturbation sets $S_1, \ldots, S_n$:

$$\mathcal{R}_{\text{adv}}^{\max}(f; S_1, \ldots, S_n) := \mathcal{R}_{\text{adv}}(f; \cup_i S_i) , \quad \mathcal{R}_{\text{adv}}^{\text{avg}}(f; S_1, \ldots, S_n) := \frac{1}{n} \sum_i \mathcal{R}_{\text{adv}}(f; S_i) .$$

- Define S_1, S_2 are *Mutually Exclusive Perturbations* (MEPs), if $\mathcal{R}_{\text{adv}}^{\text{avg}}(f; S_1, S_2) \geq 1/|C|$

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l_∞ and l_1 perturbations are *Mutually Exclusive*

- Construct a specific distribution $\mathcal{D}$
- Prove that for $\forall\, f, \mathcal{R}_{\mathrm{adv}}^{\mathrm{avg}}(f; S_\infty, S_1) \geq 1/2$

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$$y \overset{u.a.r}{\sim} \{-1, +1\}, \quad x_0 = \begin{cases} +y, & \text{w.p. } p_0, \\ -y, & \text{w.p. } 1 - p_0 \end{cases}, \quad x_1, \ldots, x_d \overset{i.i.d}{\sim} \mathcal{N}(y\eta, 1), \tag{2}$$

where $p_0 \geq 0.5$, $\mathcal{N}(\mu, \sigma^2)$ is the normal distribution and $\eta = \alpha/\sqrt{d}$ for some positive constant α.

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Theorem 1. *Let f be a classifier for $\mathcal{D}$. Let S_∞ be the set of ℓ_∞-bounded perturbations with $\epsilon = 2\eta$, and S_1 the set of ℓ_1-bounded perturbations with $\epsilon = 2$. Then, $\mathcal{R}_{adv}^{avg}(f; S_\infty, S_1) \geq 1/2$.*

Theoretical Proof

l_∞ and Spatial perturbations are (nearly) *Mutually Exclusive*

Theorem 2. *Let f be a classifier for $\mathcal{D}$ (with $x_0 \sim \mathcal{N}(y, \alpha^{-2})$). Let S_∞ be the set of ℓ_∞-bounded perturbations with $\epsilon = 2\eta$, and S_{RT} be the set of perturbations for an RT adversary with budget N. Then,* $\boxed{\mathcal{R}_{adv}^{avg}(f; S_\infty, S_{RT}) \geq 1/2 - O(1/\sqrt{N})}$.

As the distribution $\mathcal{D}$ is constructed, these two theorem might not hold in a real dataset.

Multi-perturbation ADV training

For a normal adversarial training: $L(f(\mathcal{A}(\boldsymbol{x})), y)$

where $\mathcal{A}(\cdot)$ is an attack procedure.

1. **"Max" strategy:** For each input $\boldsymbol{x}$, we train on the strongest adversarial example from all attacks, i.e., the max in $\hat{\mathcal{R}}_{\text{adv}}$ is replaced by $L(f(\mathcal{A}_{k^*}(\boldsymbol{x})), y)$, for $k^* = \arg\max_k L(f(\mathcal{A}_k(\boldsymbol{x})), y)$.

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2. **"Avg" strategy:** This strategy simultaneously trains on adversarial examples from all attacks. That is, the max in $\hat{\mathcal{R}}_{\text{adv}}$ is replaced by $\underline{\frac{1}{n}\sum_{i=1}^{n} L(f(\mathcal{A}_i(\boldsymbol{x}), y))}$.

Affine Attacks

- $\beta \in [0,1]$
- $\boldsymbol{r} = \boldsymbol{r_1} + \boldsymbol{r_2}$
- $\boldsymbol{r_1} \in \beta \cdot S_1$
- $\boldsymbol{r_2} \in (1 - \beta) \cdot S_2$

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Claim 3. *For a linear classifier $f(\boldsymbol{x}) = sign(\boldsymbol{w}^T\boldsymbol{x}+b)$, we have $\mathcal{R}_{adv}^{max}(f; S_p, S_q) = \mathcal{R}_{adv}(f; S_{affine})$.*

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Theorem 4. *Let $f(\boldsymbol{x}) = sign(\boldsymbol{w}^T\boldsymbol{x} + b)$ be a linear classifier for $\mathcal{D}$ (with $x_0 \sim \mathcal{N}(y, \alpha^{-2})$). Let S_∞ be some ℓ_∞-ball and S_{RT} be rotation-translations with budget $N > 2$. Define S_{affine} as above. Assume $w_0 > w_i > 0, \forall i \in [1, d]$. Then $\mathcal{R}_{adv}(f; S_{affine}) > \mathcal{R}_{adv}^{max}(f; S_\infty, S_{RT})$.*

Experiments

Experiments on MNIST

Model	Acc.	ℓ_∞	ℓ_1	ℓ_2	$1-\mathcal{R}_{\text{adv}}^{\max}$	$1-\mathcal{R}_{\text{adv}}^{\text{avg}}$
Nat	**99.4**	0.0	12.4	8.5	0.0	7.0
Adv_∞	**99.1**	**91.1**	12.1	11.3	6.8	38.2
Adv_1	98.9	0.0	**78.5**	50.6	0.0	43.0
Adv_2	98.5	0.4	68.0	**71.8**	0.4	46.7
Adv_{avg}	97.3	76.7	53.9	58.3	**49.9**	**63.0**
$\text{Adv}_{\max}$	97.2	71.7	62.6	56.0	**52.4**	**63.4**

Model	Acc.	ℓ_∞	RT	$1-\mathcal{R}_{\text{adv}}^{\max}$	$1-\mathcal{R}_{\text{adv}}^{\text{avg}}$
Nat	**99.4**	0.0	0.0	0.0	0.0
Adv_∞	**99.1**	**91.4**	0.2	0.2	45.8
Adv_{RT}	**99.3**	0.0	**94.6**	0.0	47.3
Adv_{avg}	**99.2**	88.2	86.4	**82.9**	**87.3**
$\text{Adv}_{\max}$	98.9	89.6	85.6	**83.8**	**87.6**

Experiments

Experiments on CIFAR10

Model	Acc.	ℓ_∞	ℓ_1	$1-\mathcal{R}_{\text{adv}}^{\text{max}}$	$1-\mathcal{R}_{\text{adv}}^{\text{avg}}$
Nat	**95.7**	0.0	0.0	0.0	0.0
Adv_∞	92.0	**71.0**	16.4	16.4	44.9
Adv_1	90.8	53.4	**66.2**	53.1	60.0
Adv_{avg}	91.1	64.1	60.8	**59.4**	**62.5**
Adv_{max}	91.2	65.7	62.5	**61.1**	**64.1**

Model	Acc.	ℓ_∞	RT	$1-\mathcal{R}_{\text{adv}}^{\text{max}}$	$1-\mathcal{R}_{\text{adv}}^{\text{avg}}$
Nat	**95.7**	0.0	5.9	0.0	3.0
Adv_∞	92.0	**71.0**	8.9	8.7	40.0
Adv_{RT}	**94.9**	0.0	**82.5**	0.0	41.3
Adv_{avg}	93.6	67.8	78.2	**65.2**	**73.0**
Adv_{max}	93.1	**69.6**	75.2	**65.7**	**72.4**

Experiments

Experiments on affine attacks

Dataset	Attacks	acc. on S_{U}	acc. on S_{affine}
MNIST	ℓ_∞ & RT	83.8	62.6
CIFAR10	ℓ_∞ & RT	65.7	56.0
CIFAR10	ℓ_∞ & ℓ_1	61.1	58.0

Thanks！