Deep Convolutional Framelet Denoising for Low-Dose CT via Wavelet Residual Network

Authors: Eunhee Kang, *et al.*

Presented by Qingsong

Frame-Based Denoising

- Frame:

$$\alpha \| f \|^2 \le \|\mathcal{W} f \|^2 \le \beta \| f \|^2, \quad \forall f \in H$$

- Tight frame:

$$\alpha = \beta \qquad \tilde{\mathcal{W}} = \mathcal{W} \text{ or } \mathcal{W}^\top \mathcal{W} = I$$

- Tight frame based denoising:

$$\min_{f,\alpha} \frac{\mu}{2} \|g - f\|^2 + \frac{1-\mu}{2} \left\{ \|\mathcal{W} f - \alpha\|^2 + \lambda \|\alpha\|_1 \right\}$$

- Proximal update equation

$$f_{n+1} = \mu g + (1-\mu) \mathcal{W}^\top T_\lambda (\mathcal{W} f_n)$$

Convolutional Framelet

- Circular convolution can be represented by Hankel matrix operation

single-input single-output (SISO)

$$y = f \circledast \overline{\psi} = \mathbb{H}_d(f)\psi$$

$$\mathbb{H}_d(f) = \begin{bmatrix} f[1] & f[2] & \cdots & f[d] \\ f[2] & f[3] & \cdots & f[d+1] \\ \vdots & \vdots & \ddots & \vdots \\ f[n] & f[1] & \cdots & f[d-1] \end{bmatrix}$$

multi-input multi-output (MIMO)

$$Y = F \circledast \overline{\Psi} = \mathbb{H}_{d|p}(F)\Psi$$

$$F := [f_1 \cdots f_p]$$

$$\overline{\Psi} := \begin{bmatrix} \overline{\psi}_1^1 & \cdots & \overline{\psi}_q^1 \\ \vdots & \ddots & \vdots \\ \overline{\psi}_1^p & \cdots & \overline{\psi}_q^p \end{bmatrix} \in \mathbb{R}^{dp \times q}$$

$$\mathbb{H}_{d|p}(F) := \left[\mathbb{H}_d(f_1)\ \mathbb{H}_d(f_2) \cdots \mathbb{H}_d(f_p)\right]$$

Convolutional Framelet

- Define filter bases

$$\Phi = [\phi_1, \cdots, \phi_n] \text{ and } \tilde{\Phi} = [\tilde{\phi}_1, \cdots, \tilde{\phi}_n] \in \mathbb{R}^{n\times n}$$

$$\Psi = [\psi_1, \cdots, \psi_q] \text{ and } \tilde{\Psi} = [\tilde{\psi}_1, \cdots, \tilde{\psi}_q] \in \mathbb{R}^{d\times q}$$

- Encoder-decoder pair

$$f = \left(\tilde{\Phi} C\right) \circledast \nu(\tilde{\Psi}), \quad \tilde{\mathcal{W}}^\top C := (\Phi C) \circledast \nu(\tilde{\Psi}).$$

$$C := \Phi^\top \left(f \circledast \overline{\Psi}\right) \qquad \mathcal{W} f := C = \Phi^\top (f \circledast \overline{\Psi})$$

Deep Convolutional Framelet Denoising

- Low-rank Hankel structured matrix constraint for image denoising

$$\min_{f \in \mathbb{R}^n} \|f^* - f\|^2$$

$$\text{subject to} \quad \textsc{rank}\,\mathbb{H}_d(f) \leq r < d.$$

2.2. Low-rank property of Hankel matrices. One of the most intriguing features of the Hankel matrix is that it often has a low-rank structure, and its low-rankness is related to the sparsity in the Fourier domain (for the case of Fourier samples, it is related to the sparsity in the spatial domain) [71, 36].

- Define a space

$$\mathcal{H}_r = \left\{ f \in \mathbb{R}^n \mid f = \left(\tilde{\Phi} C\right) \circledast \nu(\tilde{\Psi}), C = \Phi^\top \left(f \circledast \overline{\Psi}\right) \right\}$$

- Low-rank Hankel regress problem

$$\min_{f \in \mathcal{H}_r} \|f^* - f\|^2$$

Neural Network Training

In deep convolutional framelets, Φ and $\tilde{\Phi}$ correspond to the generalized pooling and unpooling which are chosen based on the application-specific knowledge

the main goal of the neural network training is to learn $(\Psi, \tilde{\Psi})$ from training data $\{(f_{(i)}, f^*_{(i)})\}_{i=1}^N$ assuming that $\{f^*_{(i)}\}$ are associated with rank-r Hankel matrices.

Neural Network Training

$$\min_{(\Psi,\tilde{\Psi})} \sum_{i=1}^{N} \left\| f^*_{(i)} - \mathcal{Q}(f_{(i)}; \Psi, \tilde{\Psi}) \right\|^2$$

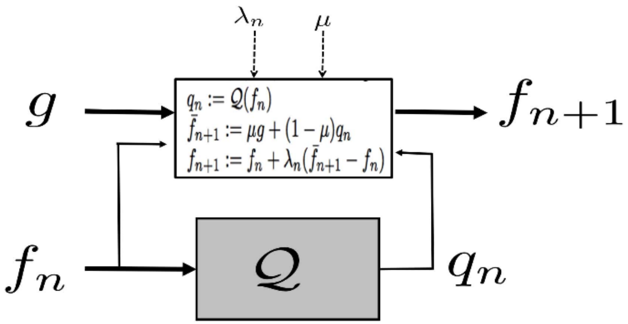
where

$$\mathcal{Q}(f_{(i)}; \Psi, \tilde{\Psi}) = \left(\tilde{\Phi} C[f_{(i)}]\right) \circledast \nu(\tilde{\Psi})$$
$$C[f_{(i)}] = \Phi^\top \left(f_{(i)} \circledast \overline{\Psi}\right).$$

Convolutional Framelet Denoising Algorithm

Algorithm 1 Pseudocode Implementation

Train a deep network $\mathcal{Q}$ using training data set.
Set $0 \leq \mu \leq 1$ and $0 < \lambda_n < 1, \forall n$.
Set initial guess of f_0 and f_1.
for $n = 1, 2, \ldots,$ until convergence **do**
  $q_n := \mathcal{Q}(f_n)$
  $\bar{f}_{n+1} := \mu g + (1 - \mu) q_n$
  $f_{n+1} := f_n + \lambda_n(\bar{f}_{n+1} - f_n)$
end for



Denoising Structure

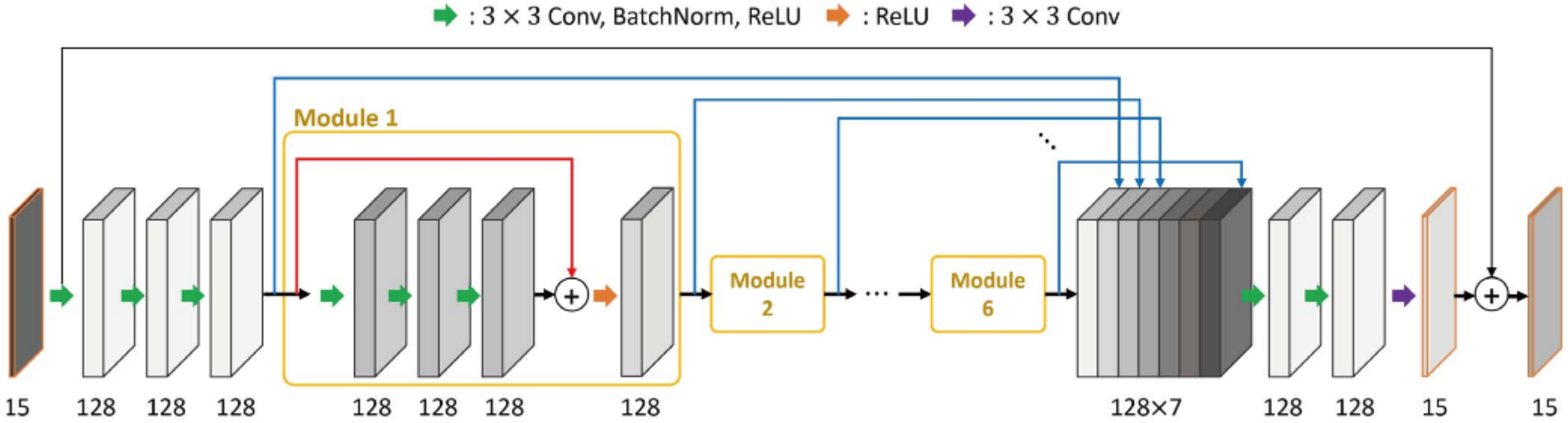


Fig. 2. The proposed WavResNet backbone (i.e. $\mathcal{Q}(f)$ in Algorithm 1 and Fig. 1) for low-dose X-ray CT restoration.

Network Training

- Dataset: Mayo dataset
- Stage 1: train the network using original database which consists of a pair of quarter-dose and full dose CT images
- Stage 2: add databases gradually which consists of quarter-dose input, inference results from $Q_k(f_i)$.
- Stage 3: add a database whose both input and target images are the full dose CT images.
- The learned bases should be robust enough not only for the strongly aliased input but also for the near artifact-free images.

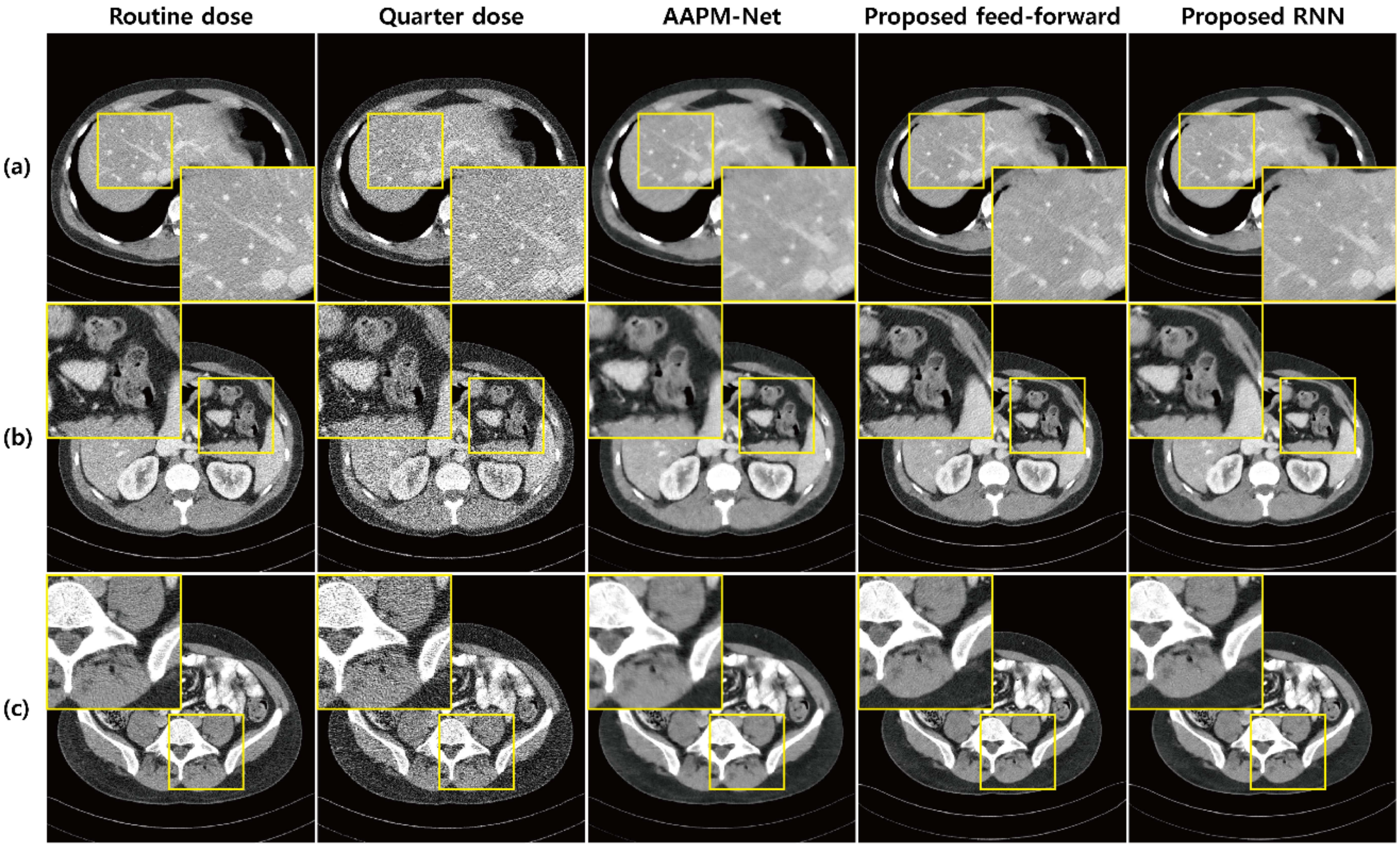


Fig. 5. Transverse view restoration results with routine-dose and quarter-dose images. AAPM-Net is the algorithm which we applied to the "2016 Low-Dose CT Grand Challenge". Intensity range is (−160,240) [HU] (Hounsfield Unit). (a) Example of liver, (b) example of intestine and (c) example of pelvic bone.

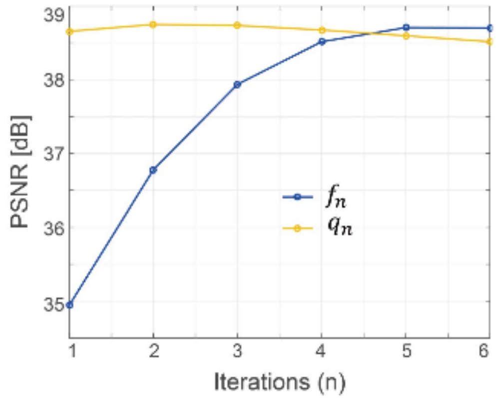


Fig. 6. PSNR values of restoration results according to iterations are plotted.

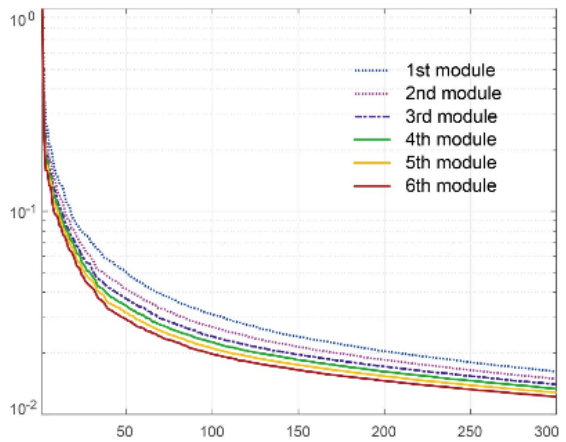


Fig. 10. The singular value spectrum of the extended Hankel matrix along layers.

TABLE I
ANALYSIS OF NETWORK STRUCTURE

	RMSE	**PSNR** [dB]	**SSIM index**
Exclude external bypass connection	48.09	33.63	0.828
Exclude concatenation layer (symmetric)	28.43	38.20	0.893
Proposed feed-forward (128 channels)	27.32	38.54	0.899
Proposed RNN (128 channels)	26.90	38.70	0.893

Summary

- RNN network for recursively denoising

- Deep convolutional framelet

SIAM J. IMAGING SCIENCES
Vol. 11, No. 2, pp. 991–1048

Deep Convolutional Framelets: A General Deep Learning Framework for Inverse Problems*

Jong Chul Ye†, Yoseob Han‡, and Eunju Cha‡