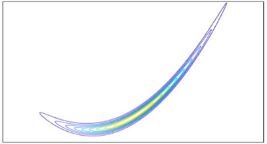
Bayesian Model Updating: Going On-line with Engineering Problems

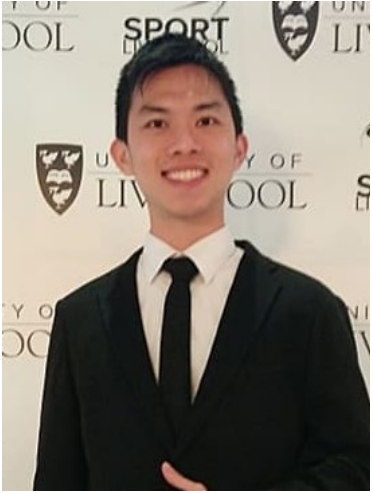
Speaker: Adolphus Lye

Institute for Risk and Uncertainty

12-February-2021



Introduction: Who Am I?



- 3^{rd}-Year PhD Student;
- Supervisors:
  - (a) Professor E. Patelli [University of Strathclyde]
  - (b) Professor A. Cicirello [T. U. Delft]
- Project: On-line Bayesian Model Updating for Real-time Damage Assessment in Dynamical Structures;
- Funding: Singapore Nuclear Research and Safety Initiatives (SNRSI);
- Research Interests:
  - (i) Probabilistic Safety Assessment for Light Water Reactors;
  - (ii) Bayesian Model Updating for Structural Health Monitoring;
  - (iii) Model Uncertainty in Bayesian Inference Problems
  - (iv) Sampling Techniques for Bayesian Inference

Introduction: Presentation Outline

Today's Special

1. Overview of Bayesian Model Updating;
2. Tools for Bayesian Model Updating + Simple Application Problems;
3. Introduction of On-line Bayesian Model Updating;
4. Tool for On-line Bayesian Model Updating + Application Problem;
5. Overall Evaluation of Techniques;
6. Concluding Remarks

1. Overview of Bayesian Model Updating

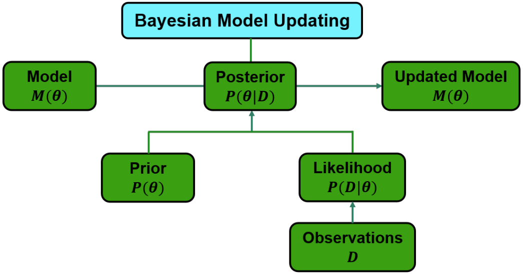
Overview of Bayesian Model Updating

- A Probabilistic Model Updating technique based on Bayes' Inference:

$$P(\boldsymbol{\theta}|\boldsymbol{D}) = \frac{P(\boldsymbol{\theta}) \cdot P(\boldsymbol{D}|\boldsymbol{\theta})}{P(\boldsymbol{D})} \qquad (1)$$

whereby:

Variable:	**Description:**
$\boldsymbol{\theta}$	Vector of Epistemic parameter(s)
$\boldsymbol{D}$	Vector of Data / Observations
$P(\boldsymbol{\theta})$	Prior distribution
$P(\boldsymbol{D}\|\boldsymbol{\theta})$	Likelihood function
$P(\boldsymbol{D})$	Evidence / Normalization constant
$P(\boldsymbol{\theta}\|\boldsymbol{D})$	Posterior distribution



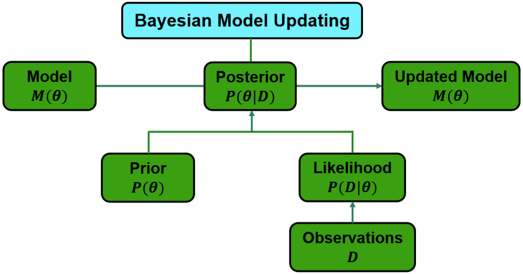
- Yields a distribution of $\boldsymbol{\theta}$ rather than just a point estimate.

Overview of Bayesian Model Updating

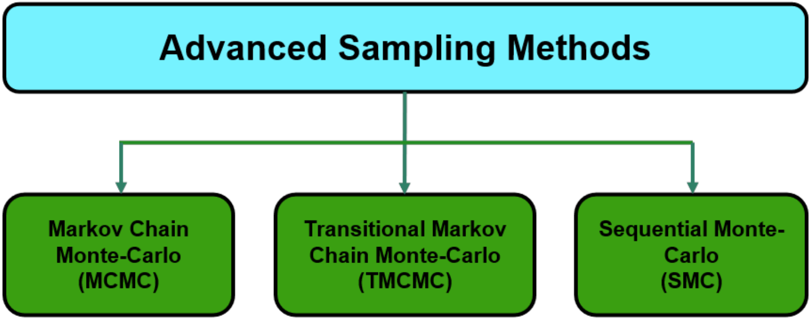
- The Evidence $P(\boldsymbol{D})$ is a constant and independent of $\boldsymbol{\theta}$;
- The Posterior $P(\boldsymbol{\theta}|\boldsymbol{D})$ can therefore be expressed in its un-normalized form as:

$$P(\boldsymbol{\theta}|\boldsymbol{D}) \propto P(\boldsymbol{\theta}) \cdot P(\boldsymbol{D}|\boldsymbol{\theta}) \tag{2}$$

- Thus, the main ingredients required are: $P(\boldsymbol{\theta})$ and $P(\boldsymbol{D}|\boldsymbol{\theta})$;
- But due to $P(\boldsymbol{D}|\boldsymbol{\theta})$ being un-normalized, standard Monte-Carlo approach becomes inapplicable;
- Sophisticated tools are needed in this case.



Overview of Bayesian Model Updating

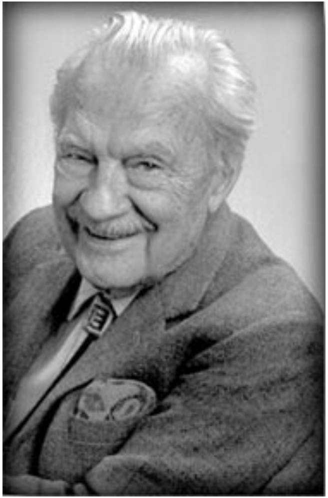


2. Tools for Bayesian Model Updating

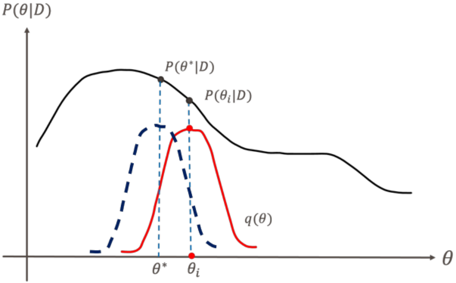
[Markov Chain Monte-Carlo (MCMC)]

Markov Chain Monte-Carlo (MCMC)

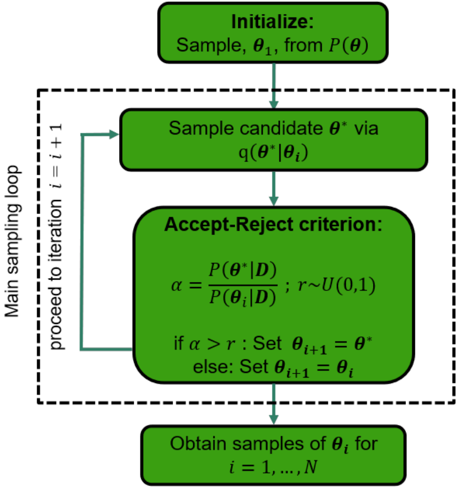
- Conceptualised by Nicholas Metropolis;
(1915 – 1999)
- Adopts the use of Markov Chains to generate samples;
- New samples are generated based on current sample via a Proposal distribution $q(\boldsymbol{\theta}^*|\boldsymbol{\theta}) \sim N(\boldsymbol{\theta}, \sigma_p)$;
- The chain will run until it approaches "Stationary distribution";
- This "Stationary distribution" is assumed to correspond to $P(\boldsymbol{\theta}|\boldsymbol{D})$;
- Accept-Reject algorithm: Metropolis-Hastings.



Markov Chain Monte-Carlo (MCMC)



Note: Should θ^* be accepted, the Proposal distribution will shift from its current location (in red), to the new one represented by the blue dotted curve. Otherwise, the Proposal distribution remains in its current location.



Application Problem A

Application Problem A: $1DOF$ Spring-Mass System

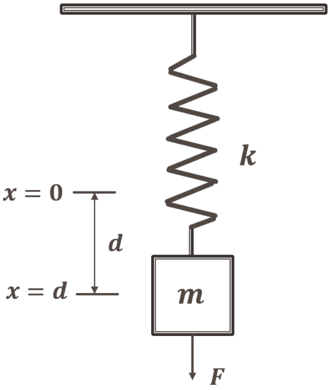
- Consider a $1DOF$ Spring-Mass system whose spring stiffness k obeys Hooke's Law:

$$F = -k \cdot d$$

  where F is the measured restoring force, d is the displacement length of the spring from its rest length;
- True value of $k = 263.0\ N/m$;
- 10 sets of synthetic measurements of F are obtained from 10 values of d are generated according to:

$$\tilde{F}_n = F_n + \epsilon_n$$

  where $n = 1, \ldots, 10$; $\epsilon_n \sim N(0, \sigma_n)$ for which $\sigma_n = 1.0\ N$;
- Least-square estimate of $k = 245.73\ N/m$.



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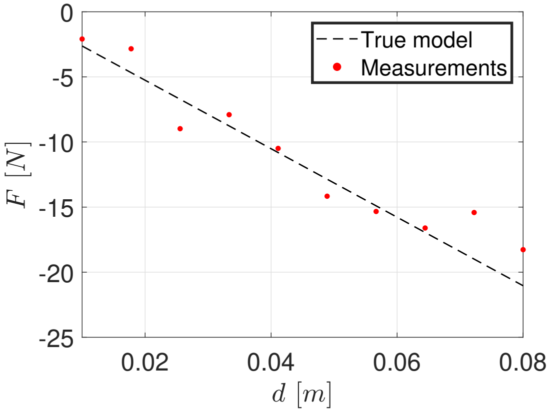


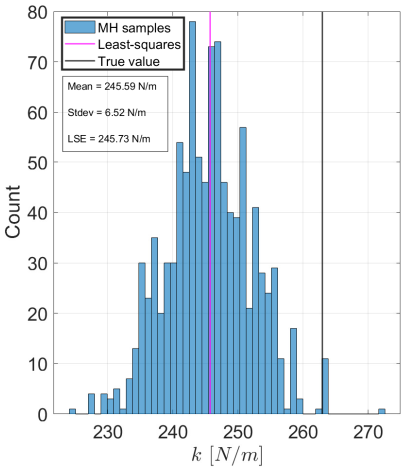
Application Problem A: $1DOF$ Spring-Mass System

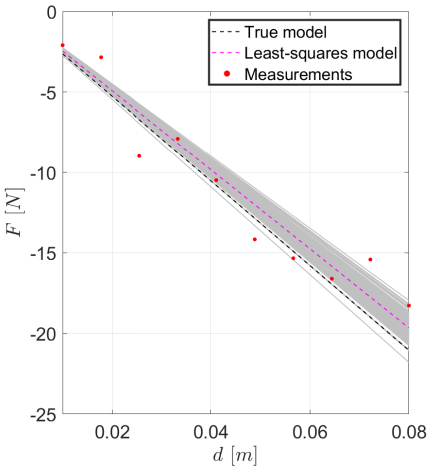
- The Bayesian model updating set-up to infer $\boldsymbol{\theta} = k$ follows:
  1. $P(k)$ is set as a Uniform prior with range: $k_1 \in [1.0, 1.0 \times 10^3]$ N/m;
  2. $P(\boldsymbol{D}|\boldsymbol{\theta})$ is set as a Normal distribution:

$$P(\boldsymbol{D}|\boldsymbol{\theta}) = \prod_{n=1}^{10} \frac{1}{\sigma_n \cdot \sqrt{2\pi}} \cdot exp\left[-\frac{(\tilde{F}_n - F_n(\boldsymbol{\theta}))^2}{2{\sigma_n}^2}\right]$$

- $\sigma_p = 30.0$ N/m to ensure acceptance rates of the MCMC sampler is as close to 0.234;
- $N_{samples} = 1000$ obtained from posterior via MCMC (i.e. MH).

Application Problem A: $1DOF$ Spring-Mass System





Note: MH took 0.31 s with acceptance-rate 0.254.

2. Tools for Bayesian Model Updating

[Transitional Markov Chain Monte-Carlo (TMCMC)]

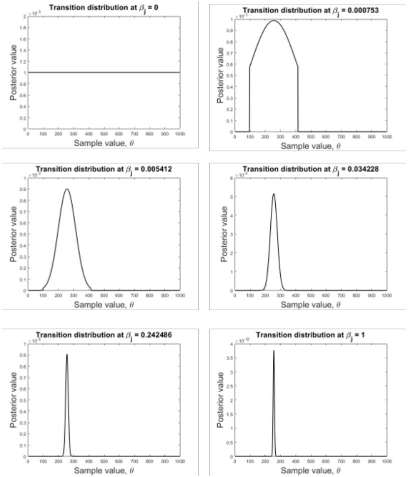
Transitional Markov Chain Monte-Carlo (TMCMC)

- Based on Adaptive Metropolis-Hastings (AMH) algorithm;
- Adopts the use "transitional" distributions, P^j:

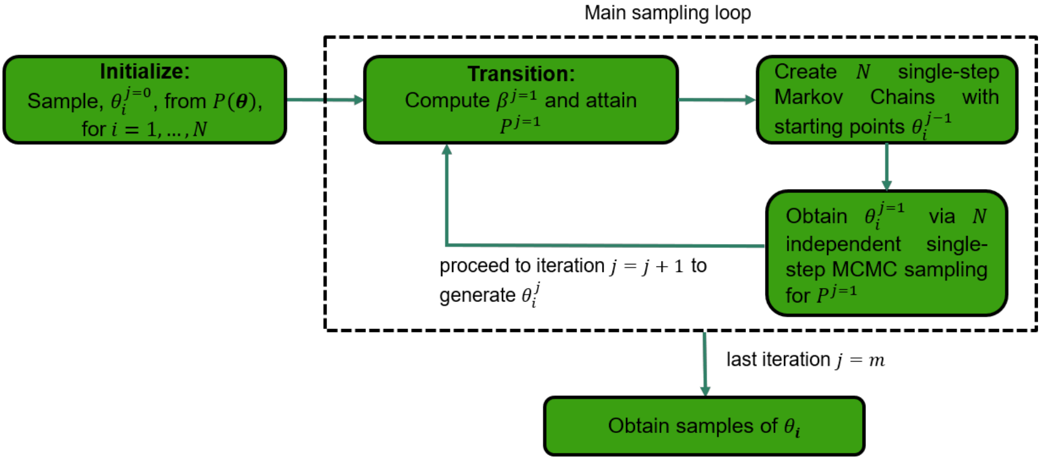
$$P^j = P(\boldsymbol{D}|\boldsymbol{\theta})^{\beta_j} \cdot P(\boldsymbol{\theta})$$

  where $j = 1, ..., m$ denotes the iteration number, and β_j is such that $\beta_0 = 0 < \beta_1 <, ..., < \beta_{m-1} < \beta_m = 1$

- Change in β_j has to be small to ensure smooth, gradual transition;
- Performs parallel sampling: N samples obtained per iteration;
- Generates the solution for $P(\boldsymbol{D})$ as by-product.



Transitional Markov Chain Monte-Carlo (TMCMC)



Application Problem B

Application Problem B: $2DOF$ Shear Building

- Consider a $2DOF$ 2-storey Shear building structure whose dynamics is described by:

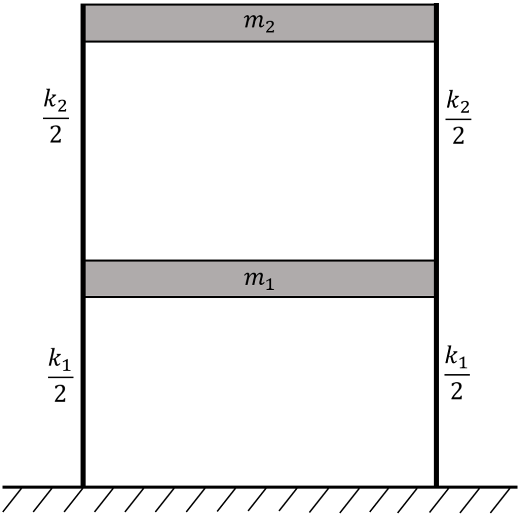
$$\mathbb{M}\ddot{\boldsymbol{x}} + \mathbb{K}\boldsymbol{x} = F(t)$$

where $F(t)$ is the external force function, $\mathbb{M}$ is the Mass matrix, and $\mathbb{K}$ is the Stiffness matrix defined respectively as:

$$\mathbb{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$\mathbb{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

- Solving the second-order differential equation for $F(t) = 0$ yields 2 distinct eigenfrequencies: $\{f_1, f_2\} = \{3.271, 0.229\}\ Hz$.



Application Problem B: $2DOF$ Shear Building

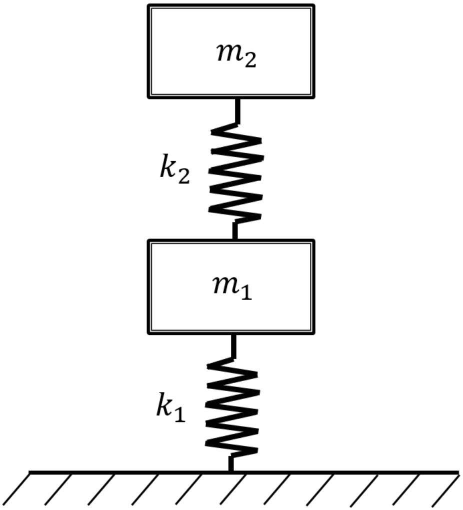
- The set-up can be re-represented as a coupled oscillator as seen on the left;
- The true values of the defined parameters are:

Parameter:	**Value:**
m_1	$1.0 \times 10^4 \; kg$
m_2	$1.0 \times 10^4 \; kg$
k_1	$5.0 \times 10^3 \; N/m$
k_2	$1.5 \times 10^4 \; N/m$

- 10 sets of synthetic measurements of f_1 and f_2 are generated according to:

$$\tilde{f}_l = f_l + \epsilon_l$$

where $l = 1, 2$; $\epsilon_l \sim N(0, \sigma_l)$ for which $\{\sigma_1, \sigma_2\} = \{1.0, 0.1\} \; Hz$.



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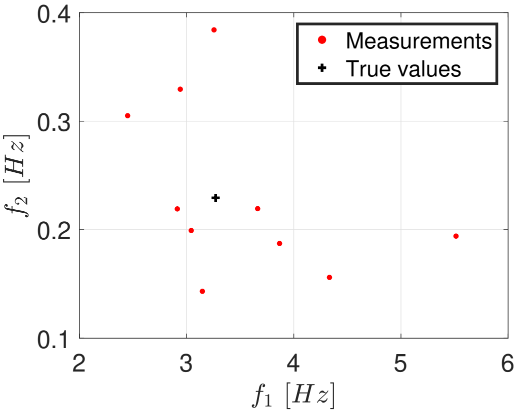


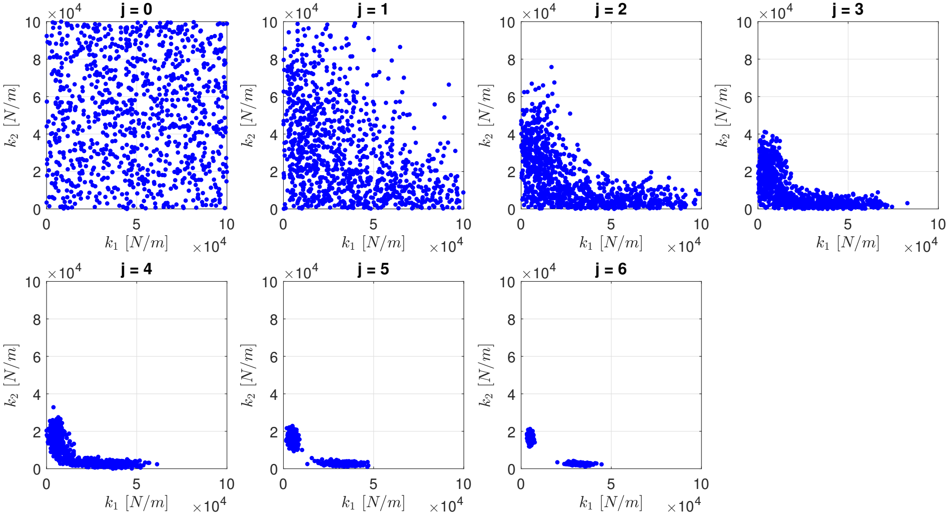
Application Problem B: $2DOF$ Shear Building

- The Bayesian model updating set-up to infer $\boldsymbol{\theta} = \{k_1, k_2\}$ follows:
  1. $P(k_1)$ is set as a Uniform prior with range: $k_1 \in [1.0, 1.0 \times 10^5]\ N/m$;
  2. $P(k_2)$ is set as a Uniform prior with range: $k_2 \in [1.0, 1.0 \times 10^5]\ N/m$;
  3. $P(\boldsymbol{D}|\boldsymbol{\theta})$ is set as a Bi-Normal distribution:

$$P(\boldsymbol{D}|\boldsymbol{\theta}) = \prod_{n=1}^{10} \frac{1}{\sigma_1 \cdot \sigma_2 \cdot 2\pi} \cdot exp\left[-\frac{(\tilde{f}_{1,n} - f_1(\boldsymbol{\theta}))^2}{2{\sigma_1}^2} - \frac{(\tilde{f}_{2,n} - f_2(\boldsymbol{\theta}))^2}{2{\sigma_2}^2}\right]$$

- $N_{samples} = 1000$ obtained from posterior via TMCMC;
- Results will be compared against MCMC (i.e. MH) with $\Sigma_p = 6.0 \times 10^6 \cdot I$ to achieve acceptance rates as close to 0.234.

Application Problem B: 2DOF Shear Building



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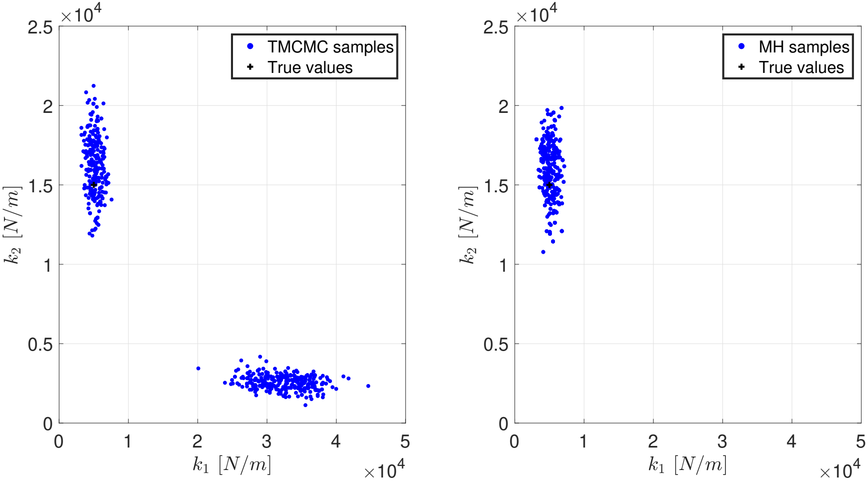
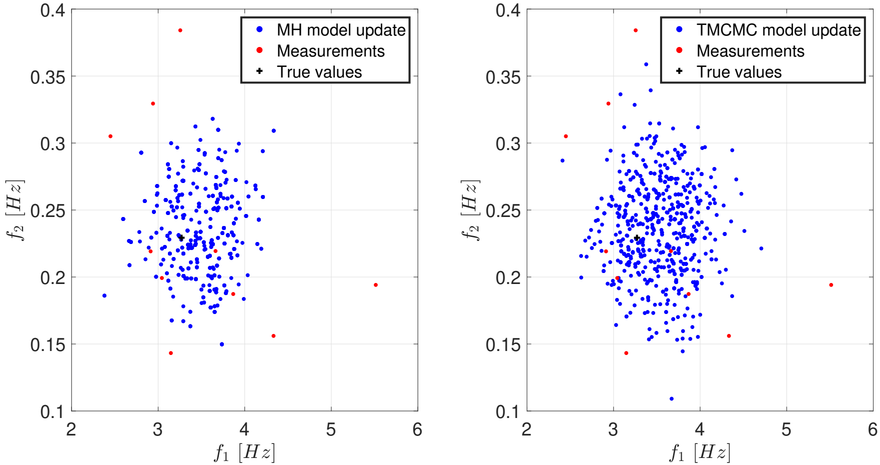


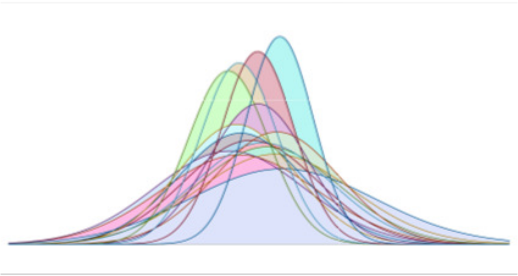
Note: TMCMC took 22.00 s while MH took 0.27 s (with acceptance rate 0.239).

Application Problem B: $2DOF$ Shear Building



Some Remarks

- So far, we only consider the case where a complete data-set is made "available";
- As such, samples are generated from static $P(\boldsymbol{\theta}|\boldsymbol{D})$;
- In reality, such assumptions are never true at all times as data can be obtained On-line;
- This is especially true for real-time monitoring of infrastructures such as Bridges, Building structures, and Dynamical systems;
- Such problems would give-rise to dynamical $P(\boldsymbol{\theta}|\boldsymbol{D})$;
- This can be addressed using On-line Bayesian Model Updating.

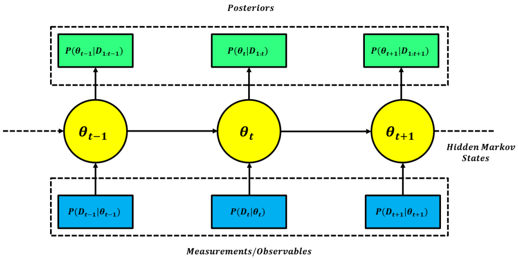


On-line Bayesian Model Updating

- Consider a stream of data: $\boldsymbol{D}_{1:t} = \{\boldsymbol{D}_1, \ldots, \boldsymbol{D}_t\}$;
- A recursive approach is adopted to update $P(\boldsymbol{\theta}|\boldsymbol{D}_{1:t})$ where:

$$P(\boldsymbol{\theta}|\boldsymbol{D}_{1:t}) \propto P(\boldsymbol{\theta}) \cdot \prod_{j=1}^{t} P(\boldsymbol{D}_j|\boldsymbol{\theta}) \qquad (3)$$

  assuming independence between data-set $\boldsymbol{D}_t$ obtained at time-step t, and $\boldsymbol{\theta}$ is static;
- At each t, the previous posterior $P(\boldsymbol{\theta}|\boldsymbol{D}_{1:t-1})$ is set as the new Prior to be updated with $P(\boldsymbol{D}_t|\boldsymbol{\theta})$;
- Sampling is done sequentially: Sequential Monte Carlo (SMC) sampler

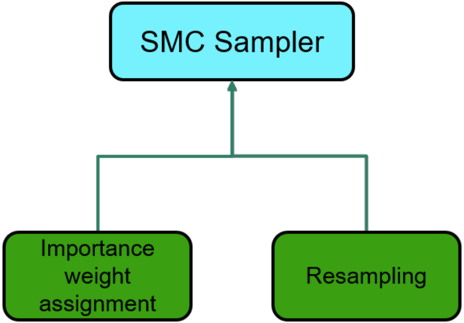


4. Tool for On-line Bayesian Model Updating

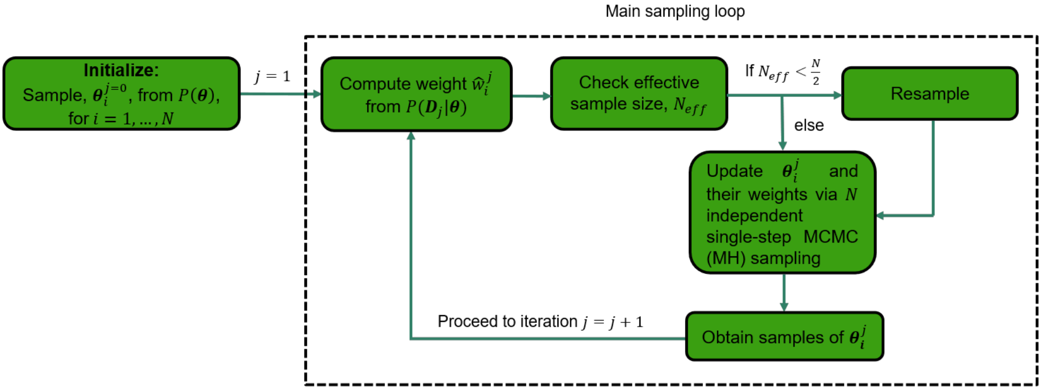
[Sequential Monte-Carlo (SMC)]

Sequential Monte-Carlo (SMC)

- Based on the Sequential Importance Resampling (SIR) Particle-Filter algorithm;
- Designed to sample from dynamic posteriors in a sequential manner;
- Recursive algorithm;
- Utilises weights to approximate the distribution of $\boldsymbol{\theta}$;
- Generates the solution for $P(\boldsymbol{D}_{1:t})$ at each iteration.



Sequential Monte-Carlo (SMC)



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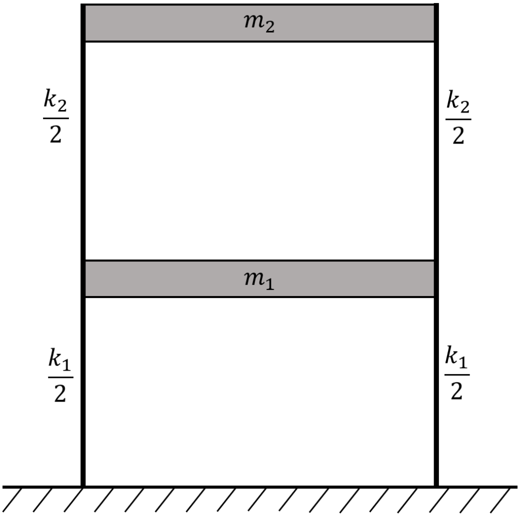

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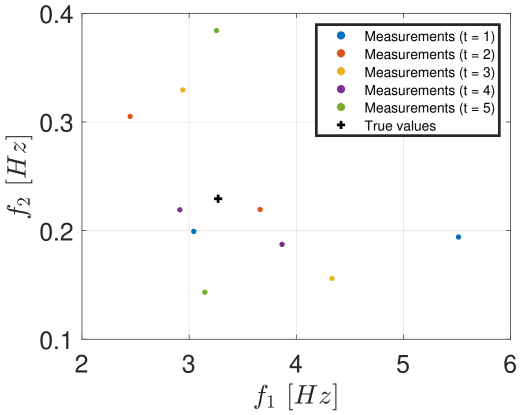
where $l = 1, 2$; $\epsilon_l \sim N(0, \sigma_l)$ for which $\{\sigma_1, \sigma_2\} = \{1.0, 0.1\}\ Hz$;

- Measurements are obtained in batches of 2 at each time-step t, across 5 time-steps.



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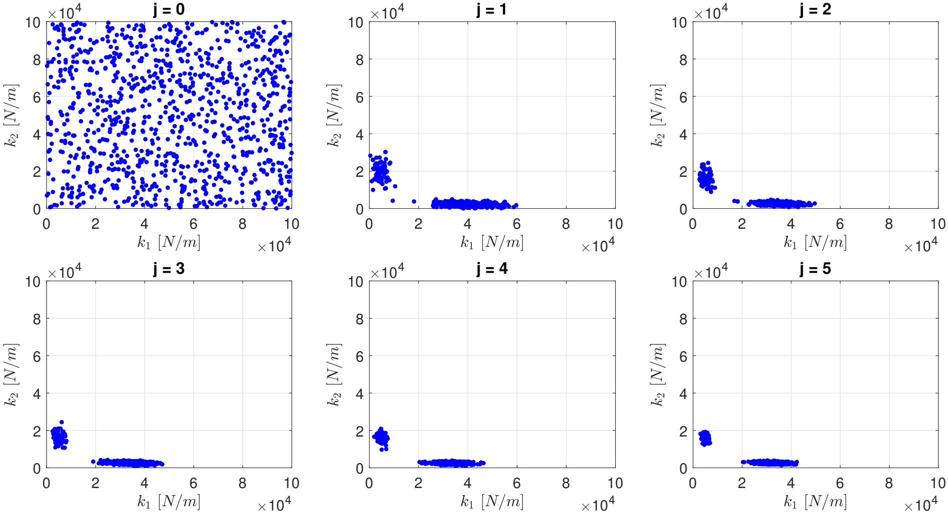
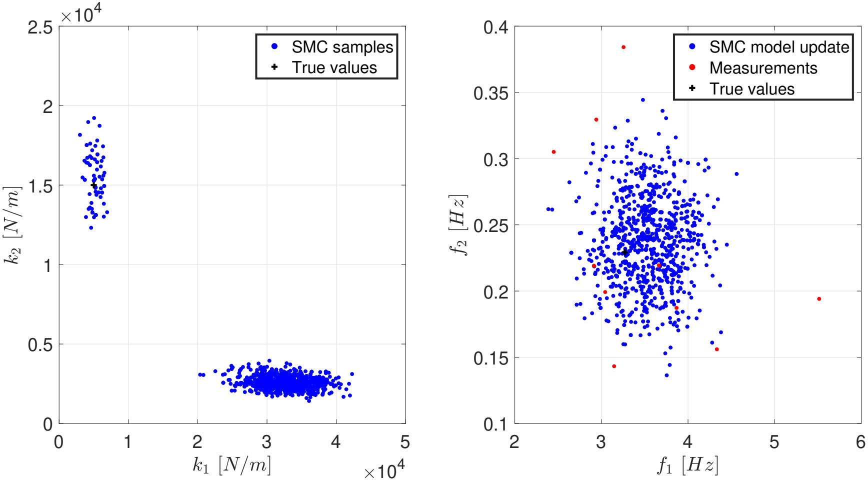
$$P(\boldsymbol{D}_t|\boldsymbol{\theta}) = \frac{1}{\sigma_1 \cdot \sigma_2 \cdot 2\pi} \cdot exp\left[\sum_{n=2t-1}^{2t} -\frac{(\tilde{f}_{1,n} - f_1(\boldsymbol{\theta}))^2}{2{\sigma_1}^2} - \frac{(\tilde{f}_{2,n} - f_2(\boldsymbol{\theta}))^2}{2{\sigma_2}^2}\right]$$

- $N_{samples} = 1000$ obtained from posterior via TMCMC;
- Results will be compared against MCMC (i.e. MH).

Application Problem B: 2DOF Shear Building



Application Problem B: $2DOF$ Shear Building



Note: SMC took 16.93 s (Ref: TMCMC took 22.00 s; MH took 0.27 s).

5. Overall Evaluation of Techniques

Overall Evaluation of Techniques

	Advantage(s):	**Disadvantage(s):**
MCMC	- Short computation time - Easy to implement	- Not tune-free - Suffers under high-dimensionality - Fails when $P(\boldsymbol{\theta}\|\boldsymbol{D})$ is multi-modal - Doesn't compute $P(\boldsymbol{D})$ - Only works for static $P(\boldsymbol{\theta}\|\boldsymbol{D})$ - Cannot be parallelized at chain-level
TMCMC	- Robust under high-dimensions - Applicable for multi-modal $P(\boldsymbol{\theta}\|\boldsymbol{D})$ - Computes $P(\boldsymbol{D})$ - Tune-free - Can be parallelized	- Long computational time - Only works for static $P(\boldsymbol{\theta}\|\boldsymbol{D})$
SMC	- Applicable for multi-modal $P(\boldsymbol{\theta}\|\boldsymbol{D}_{1:t})$ - Works on static and dynamical $P(\boldsymbol{\theta}\|\boldsymbol{D}_{1:t})$ - Computes $P(\boldsymbol{D}_{1:t})$ sequentially - Can also infer time-varying $\boldsymbol{\theta}_t$ - Can be parallelized	- Long computational time (if data is large) - Suffers under high-dimensionality - Not tune-free

6. Concluding Remarks

Concluding Remarks

Technique:	Reference(s):
MCMC	W. K. Hastings (1970). Monte Carlo Sampling Methods using Markov Chains and their Applications, Biometrika **57**, 97-109. doi: `10.1093/biomet/57.1.97`
TMCMC	J. Y. Ching, and Y. C. Chen (2007). Transitional Markov Chain Monte Carlo Method for Bayesian Model Updating, Model Class Selection, and Model Averaging, Journal of Engineering Mechanics **133**. doi: `10.1061/(ASCE)0733-9399(2007)133:7(816)`
SMC	P. D. Moral, A. Doucet, and A. Jasra (2006). Sequential Monte Carlo Samplers, Journal of the Royal Statistical Society. Series B (Statistical Methodology) **68**, 411-436. N. Chopin (2002). A Sequential Particle Filter Method for Static Models, Biometrika **89**, 539-552. doi: `10.1093/biomet/89.3.539`

Concluding Remarks

- Contact details:
  1. Email: `adolphus.lye@liverpool.ac.uk`
  2. LinkedIn/ResearchGate: Adolphus Lye
  3. GitHub: `Adolphus8`
- MATLAB codes of presented examples are available on GitHub:
- Further details to Bayesian Model Updating is available on YouTube:
  1. Introduction to Bayesian Model Updating I: `https://youtu.be/A-cjvg741is`
  2. Introduction to Bayesian Model Updating II: `https://youtu.be/87b2-Fb4uas`

The End

Thank you so much for your Undivided attention!