Quaternions

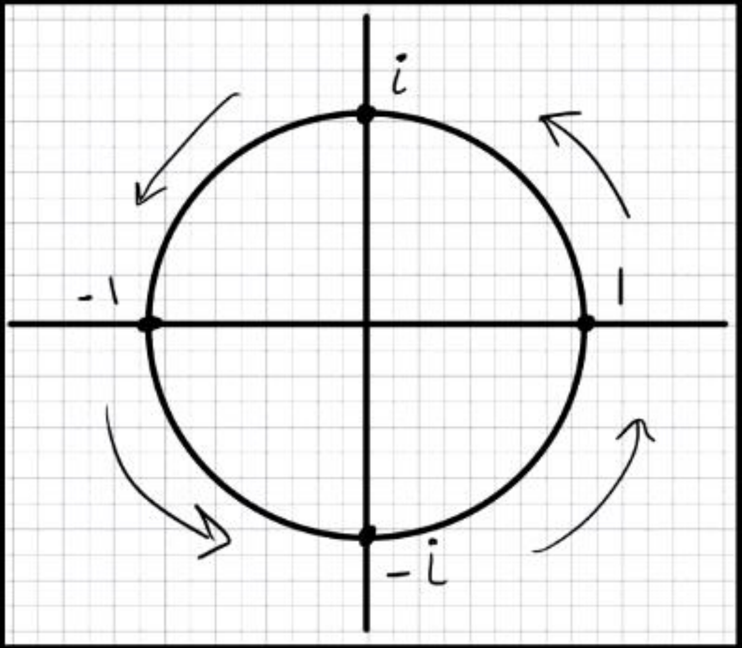
$$i^2 = -1$$

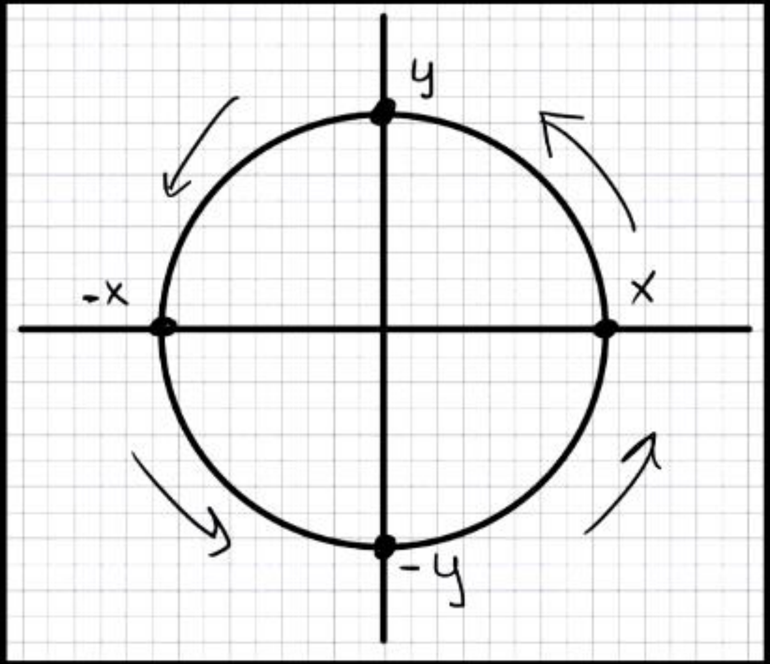
$$i = \sqrt{-1}$$

$1, i, -1, -i, 1, i, -1, -i \ldots$

Like we rotate a complex number we can also do the same with a real number on x-y plane and this series results.

$x, y, -x, -y, x, y, -x, -y \ldots$





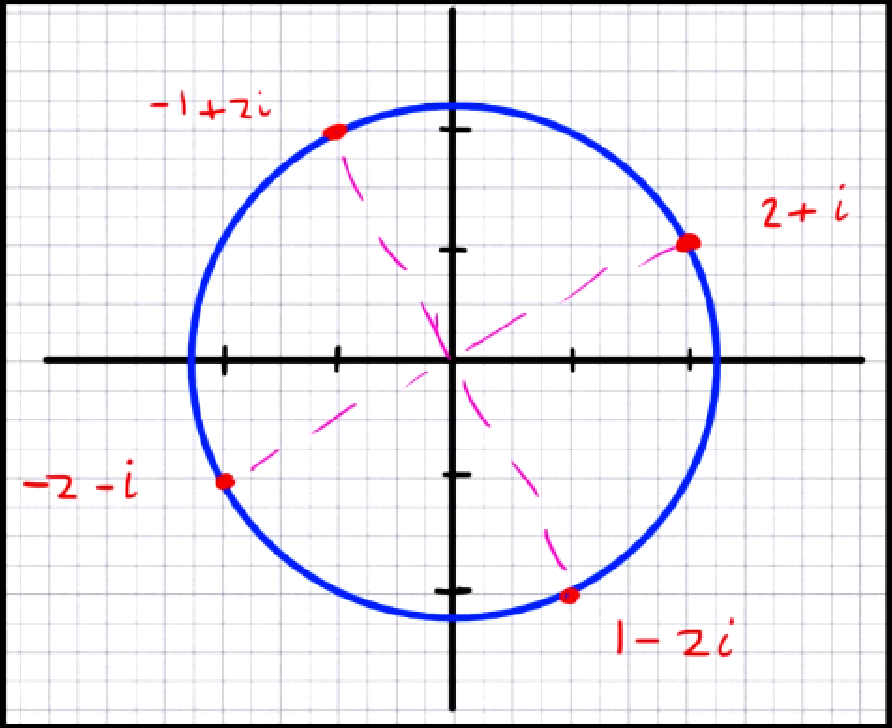
- We Know what are simple complex numbers and what are there properties.
- We can add them multiply them.

$a = 2 + i$

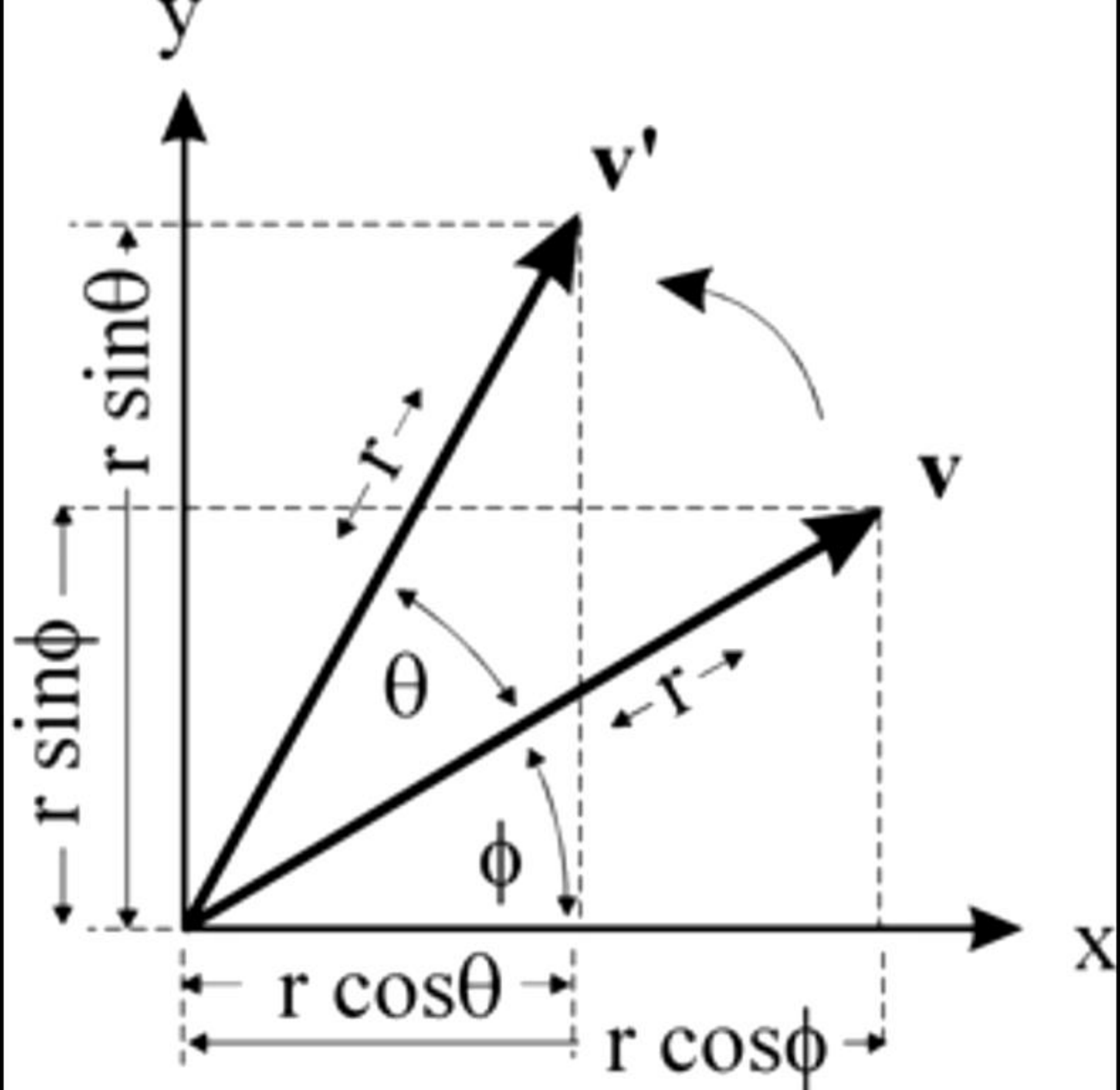
$b = a * i = 2i - 1 = -1 + 2i$

$c = b * i = -i - 2 = -2 -$

A complex number when multiplied by ‘I’ we Get a rotated complex number by an angle of 90 degrees on argand plane.



For rotating a Complex number by theta we Use multiplication with [e^I*theta]



$$r = \cos\theta + \sin\theta\, i$$

$$p = a + bi$$

$$p * r = (a + bi) * (\cos\theta + \sin\theta\, i)$$

$$p' = a\cos\theta - b\sin\theta + (a\sin\theta + b\cos\theta)\, i$$

Quaternions:
They problem they solved

- What if we need two complex numbers to work with.
- We can assume a j complex axis .
- We can add sub but we cannot multiply the numbers.

- WHY?

- Because we don’t know the multiplication of i*j OR i*j*k.

$$i^2 = -1 \qquad j^2 = -1$$

$$(1 + i + j) * (-1 - i - j)$$

$$= -1 - i - j - i - i^2 - ij - i^2 - ij - j^2$$

$$= -1 - 2i - 2j - 2ij - i^2 - j^2$$

$$= -3 - 2i - 2j - 2ij$$

- Later this problem was solved and these formulas were developed.

- But now due to addition of I,j,k along with real axis we not have 4 dimensions to deal with.

$$i^2 = j^2 = k^2 = ijk = -1$$

Some additional formulas

$$i^2 = -1 \quad j^2 = -1 \quad k^2 = -1$$

$$ij = k \quad jk = i \quad ki = j$$

$$ji = -k \quad kj = -i \quad ik = -j$$

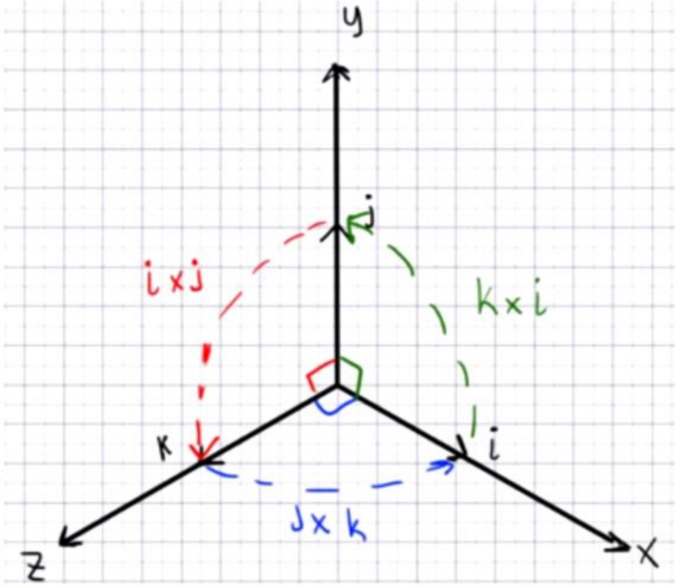
Here we take axis== i,j,k.

$$i = x, j = y, k = z$$

The math we dealt now is equivalent to vector maths.

$$i \times j = k \qquad j \times k = i \qquad k \times i = j$$

$$j \times i = -k \quad k \times j = -i \quad i \times k = -j$$



Our quaternion looks like this ! COOL

$$a + bi + cj + dk$$

We can also write the real and imaginary part as a ordered pair.

$$a + bi + cj + dk$$

$$[a, bi + cj + dk]$$

- It can also be represented like this manner also with real and vectors parts.

$$a + bi + cj + dk$$

$$\vec{v} = < b, c, d >$$

$$[a, \vec{v}]$$

Some math with quaternions

Addition is simple

- You need to add real and imaginary parts separately.

$$q_1 = [a_1, \vec{v}_1] \qquad q_2 = [a_2, \vec{v}_2]$$

$$q_1 + q_2 = [a_1 + a_2, \vec{v}_1 + \vec{v}_2]$$

$$q_1 - q_2 = [a_1 - a_2, \vec{v}_1 - \vec{v}_2]$$

$$q_1 = a_1 + b_1 i + c_1 j + d_1 k \qquad q_2 = a_2 + b_2 i + c_2 j + d_2 k$$

$$q_1 * q_2 = (a_1 + b_1 i + c_1 j + d_1 k) * (a_2 + b_2 i + c_2 j + d_2 k)$$

$$= a_1 a_2 + a_1 b_2 i + a_1 c_2 j + a_1 d_2 k + b_1 i a_2 + b_1 i b_2 i + b_1 i c_2 j + b_1 i d_2 k + c_1 j a_2 + c_1 j b_2 i + c_1 j c_2 j + c_1 j d_2 k + d_1 k a_2 + d_1 k b_2 i + d_1 k c_2 j + d_1 k d_2 k$$

$$= a_1 a_2 + a_1 b_2 i + b_1 a_2 i + a_1 c_2 j + c_1 a_2 j + a_1 d_2 k + d_1 a_2 k + b_1 b_2 i^2 + c_1 c_2 j^2 + d_1 d_2 k^2 + b_1 c_2 ij + b_1 d_2 ik + c_1 b_2 ji + c_1 d_2 jk + d_1 b_2 ki + d_1 c_2 kj$$

Now comes the multiplication part

On solving with the rules defined we get this

$$= a_1a_2 + a_1b_2i + b_1a_2i + a_1c_2j + c_1a_2j + a_1d_2k + d_1a_2k - b_1b_2 - c_1c_2 - d_1d_2 + b_1c_2k - b_1d_2j - c_1b_2k + c_1d_2i + d_1b_2j - d_1c_2i$$

On simplification we get this

$$a_1a_2 - (b_1b_2 + c_1c_2 + d_1d_2) + (a_1b_2i + a_1c_2j + a_1d_2k) + (b_1a_2i + c_1a_2j + d_1a_2k) + (c_1d_2i - d_1c_2i + d_1b_2j - b_1d_2j + b_1c_2k - c_1b_2k)$$

$$a_1a_2 - (b_1b_2 + c_1c_2 + d_1d_2) + a_1(b_2i + c_2j + d_2k) + a_2(b_1i + c_1j + d_1k) + (c_1d_2i - d_1c_2i + d_1b_2j - b_1d_2j + b_1c_2k - c_1b_2k)$$

It is also possible to write the same form with this cute formula

$$q_1 = [a_1, \vec{v}_1] \qquad q_2 = [a_2, \vec{v}_2]$$

$$q_1 * q_2 = [a_1 a_2 - \vec{v}_1 \cdot \vec{v}_2, a_1 \vec{v}_2 + a_2 \vec{v}_1 + \vec{v}_1 \times \vec{v}_2]$$

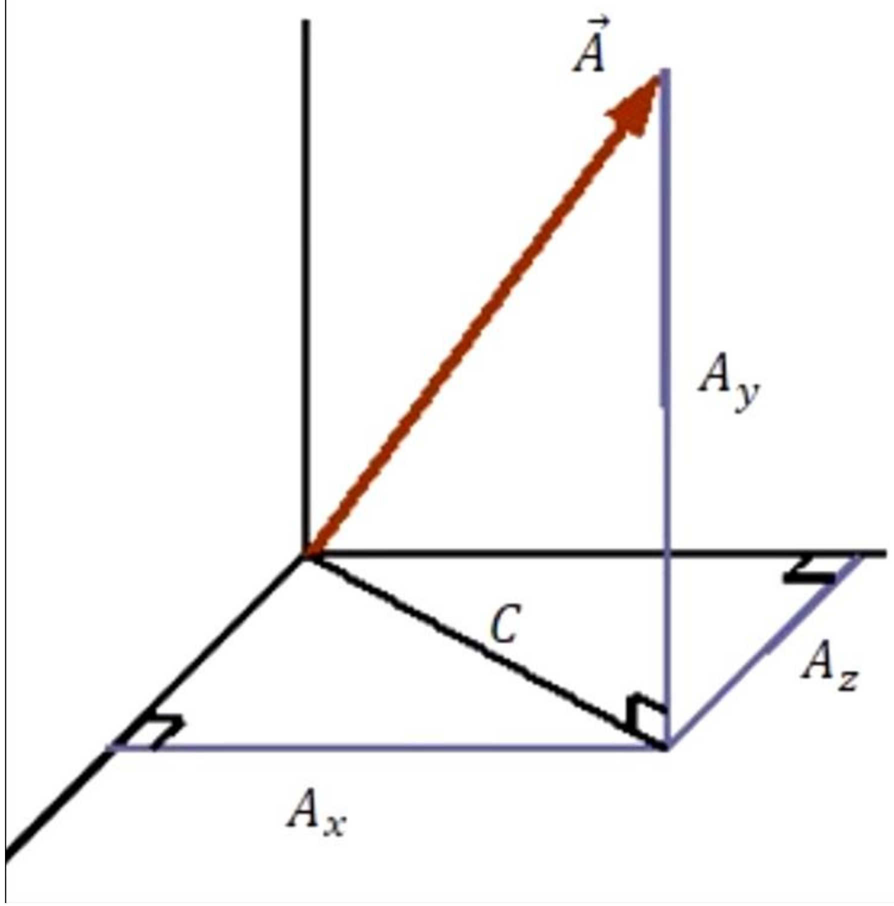
Conjugate of a quaternion q* looks like this and it follows the same properties of modulus and other math which a simple complex number follows

$$q = [a, \vec{v}] \qquad\qquad q^* = [a, -\vec{v}]$$

$$q * q^* = [aa + \vec{v} \cdot \vec{v}, -a\vec{v} + a\vec{v} + \vec{v} \times -\vec{v}]$$

$$q * q^* = \left[a^2 + \left|\left|\vec{v}\right|\right|^2, < 0, 0, 0 >\right]$$

The modulus of a quaternion follows the pythagores theorem but in four diimensions



Pythagorean theorem:

$$C^2 = A_x^2 + A_z^2$$

$$\|\vec{A}\|^2 = C^2 + A_y^2$$

$$\|\vec{A}\|^2 = A_x^2 + A_y^2 + A_z^2$$

$$\|\vec{A}\| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$(\text{Modulus})^2 = qq^*$ and also see inverse property

$$q = [a, \vec{v}] \qquad q^* = [a, -\vec{v}] \qquad ||q||^2 = a^2 + ||\vec{v}||^2 \qquad q^{-1} = \frac{q^*}{||q||^2}$$

$$q^{-1} = \left[\frac{a}{a^2+||\vec{v}||^2}, \frac{-\vec{v}}{a^2+||\vec{v}||^2}\right]$$

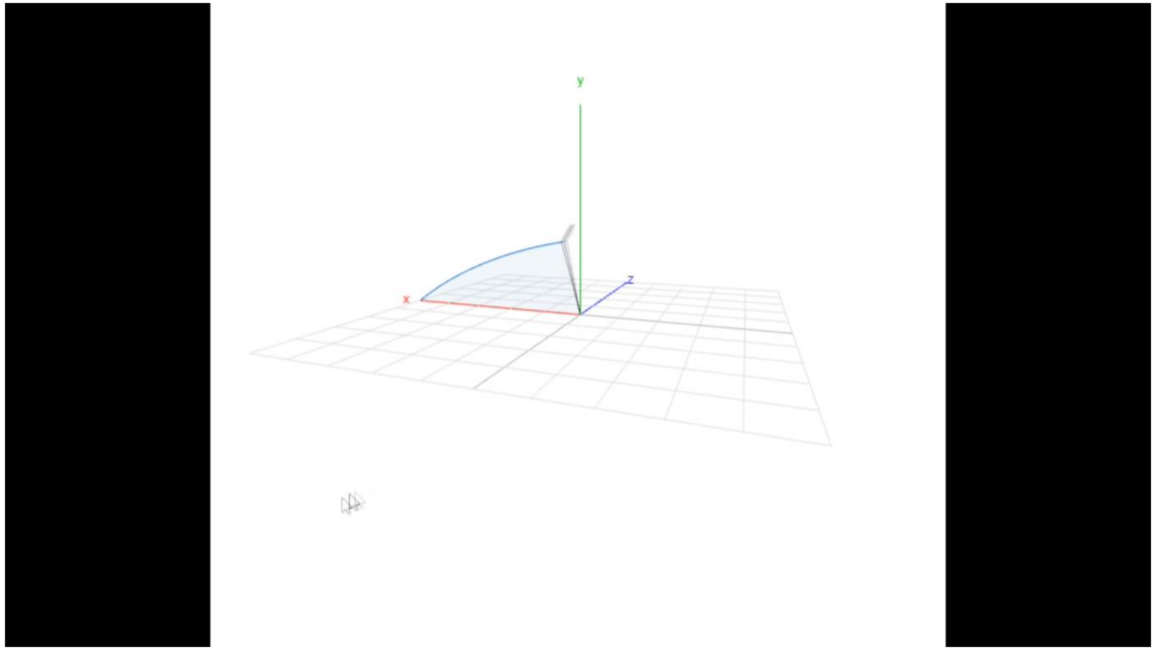
$$q * q^{-1} = [1, < 0,0,0 >]$$

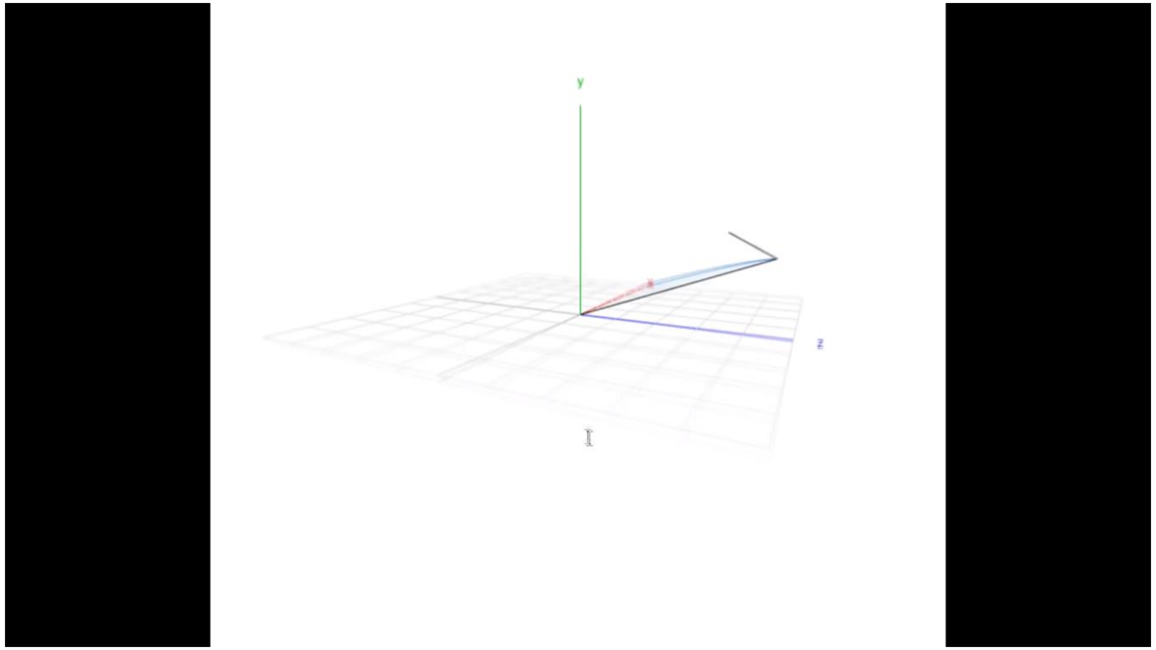
$\vec{v} =< 0.5, 0{,}0 >$ $\qquad ||\vec{v}|| = 0.5$

$$q = \left[\frac{\sqrt{3}}{2}, \vec{v}\right] \qquad ||q|| = 1$$

$$q = [0, \vec{v}]$$

Unit quaternions and pure quaternions





We use quaternions because they help us in rotations I form of simple multiplication

Advantage over other type of maths

- Using this means we can avoid gimble lock like situations as was in euler rotations .

Rodrigues’ rotation

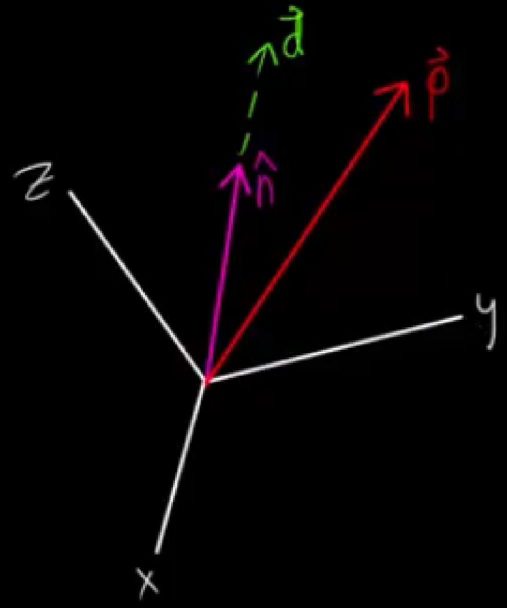
First let us understand this then later onrotation using quaternions

- It let you to rotate a vector (which is point basically in space) about any unit vector n^ which you have defined.

- Watch https://www.youtube.com/watch?v=OKr0YCj0BW4 for greater detail.

Let's rotate p vector around n^ to other point

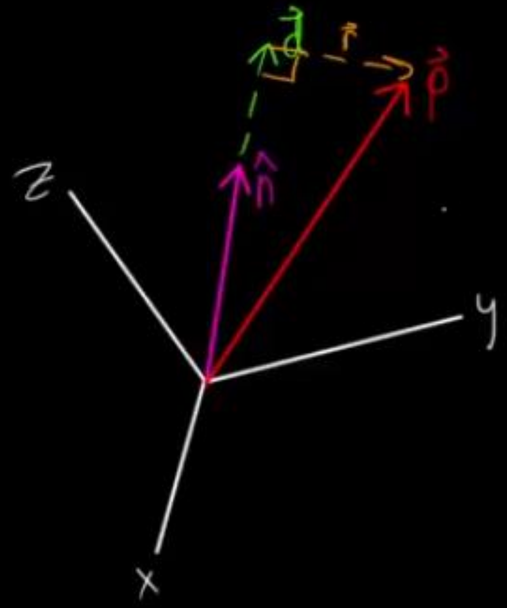
- Position
- Angle (counter clockwise) θ



$$\vec{d} = (\hat{n} \cdot \vec{p}) * \hat{n}$$

Now r vector is perpendicular to the n^ unit vector

- Position
- Angle (counter clockwise) θ

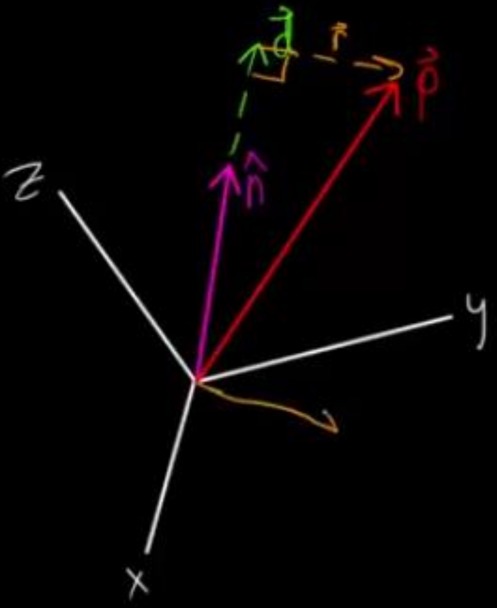


$$\vec{d} = (\hat{n} \cdot \vec{p}) * \hat{n}$$

$$\vec{r} = \vec{p} - \vec{d}$$

Projection of r vector to the origin.

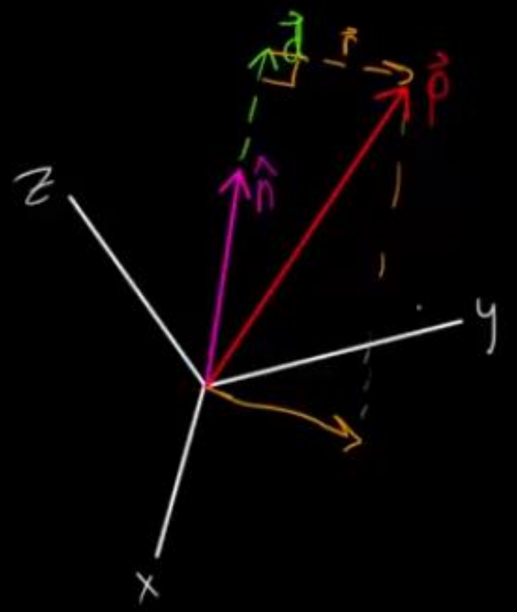
- Position
- Angle (counter clockwise) θ



$\vec{d} = (\hat{n} \cdot \vec{p}) * \hat{n}$

$\vec{r} = \vec{p} - \vec{d}$

- Axis
- Position
- Angle (counter clockwise) θ

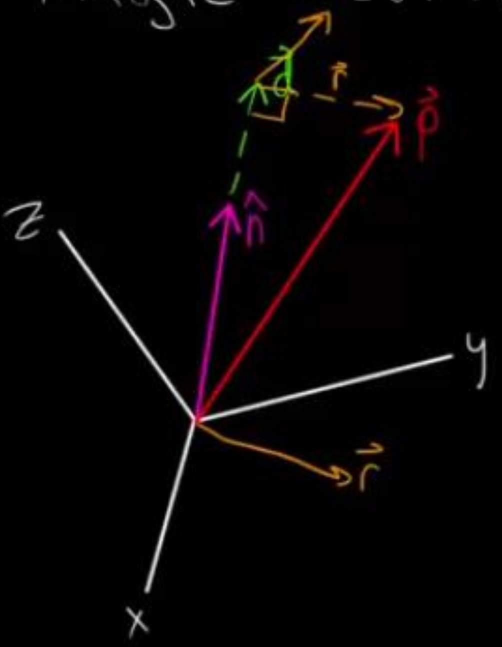


$$\vec{d} = (\hat{n} \cdot \vec{p}) * \hat{n}$$

$$\vec{r} = \vec{p} - \vec{d}$$

Now taking a perpendicular vector to the plane containing the n,p and r vector.

- Axis
- Position
- Angle (counter clockwise) θ

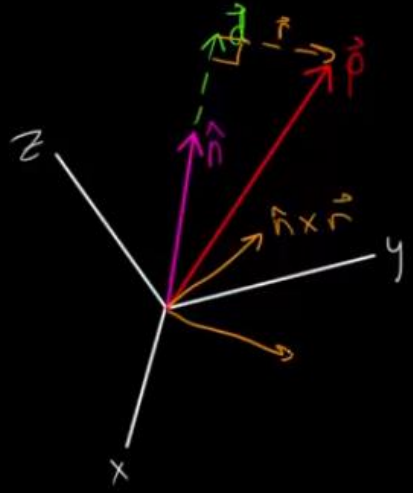


$$\vec{d} = (\hat{n} \cdot \vec{p}) * \hat{n}$$

$$\vec{r} = \vec{p} - \vec{d}$$

$$\hat{n} \times \vec{r}$$

- Axis
- Position
- Angle (counter clockwise) θ

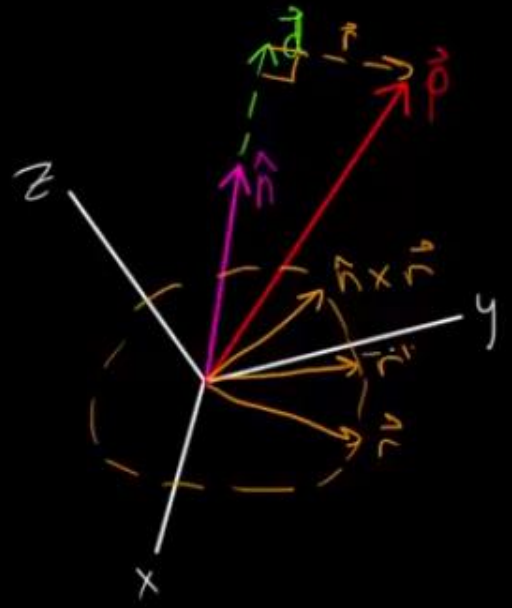


$$\vec{d} = (\hat{n} \cdot \vec{p}) * \hat{n}$$

$$\vec{r} = \vec{p} - \vec{d}$$

With the help of n*r vector and r vector we can point to any vector in the circular plane drawn

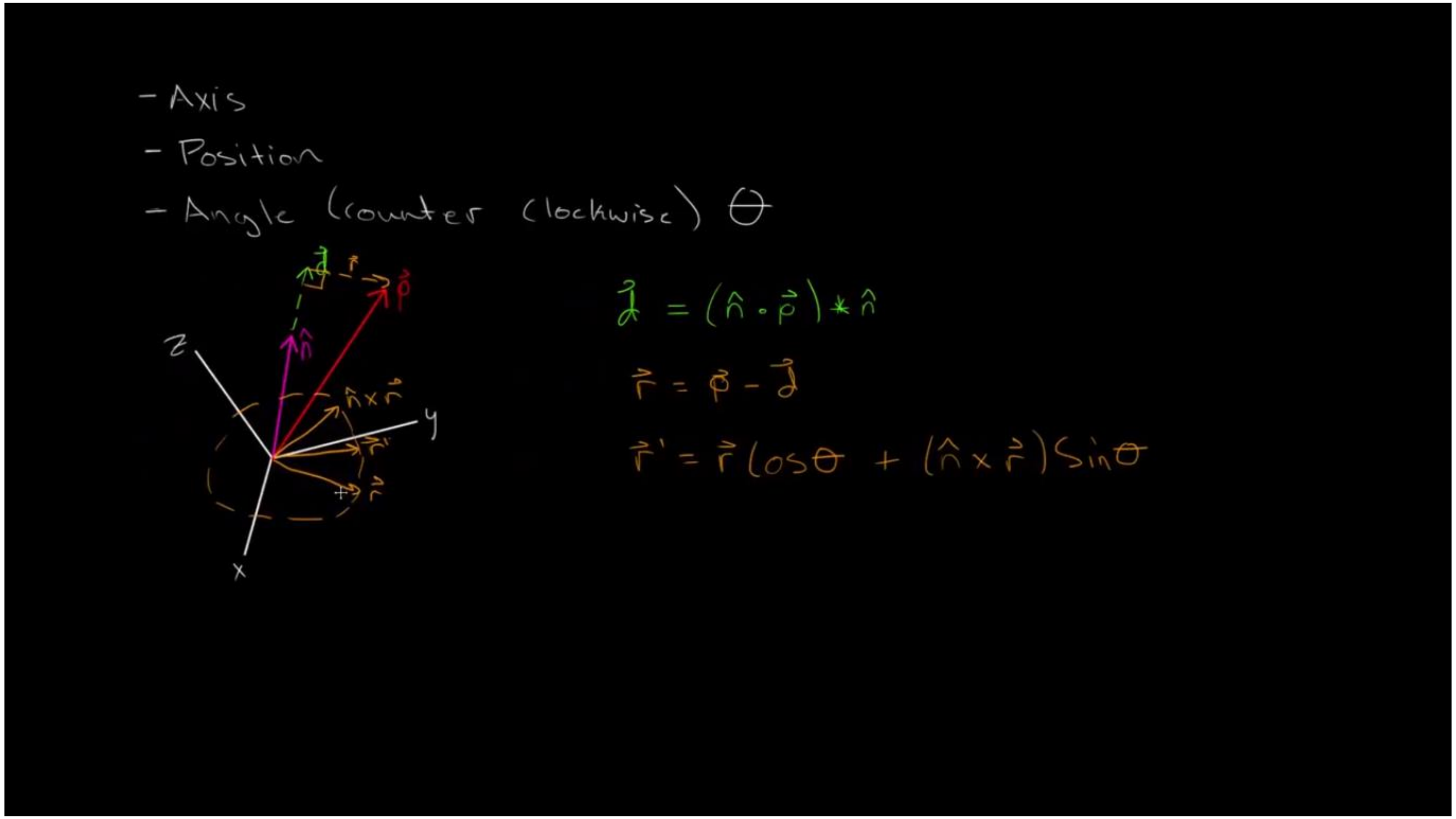
- Axis
- Position
- Angle (counter clockwise) θ



$$\vec{d} = (\hat{n} \cdot \vec{p}) * \hat{n}$$

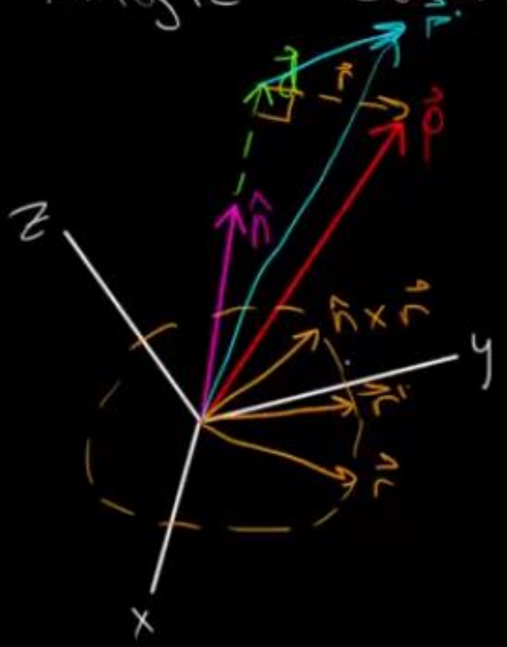
$$\vec{r} = \vec{p} - \vec{d}$$

Let a vector be r’



Now let's take the r' to the plane from where we actually borrowed the r vector

- Axis
- Position
- Angle (counter clockwise) θ



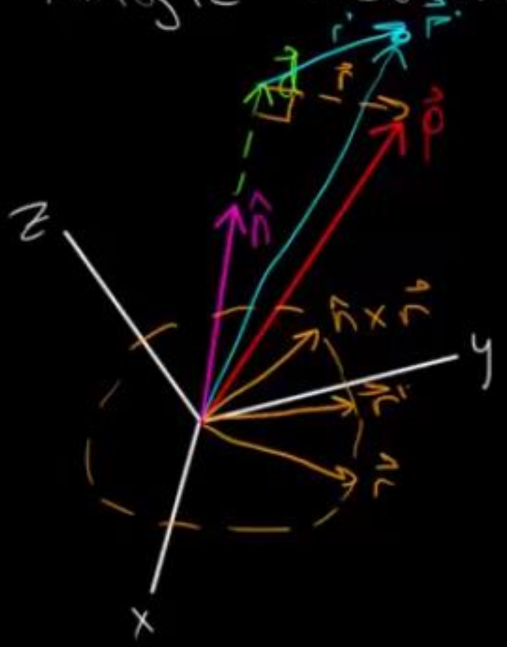
$$\vec{d} = (\hat{n}\cdot\vec{p}) * \hat{n}$$

$$\vec{r} = \vec{p} - \vec{d}$$

$$\vec{r}' = \vec{r}\cos\theta + (\hat{n}\times\vec{r})\sin\theta$$

Now the point r’ in space is same like rotating the p around n vector . This is p’.

- Angle (counter clockwise) θ



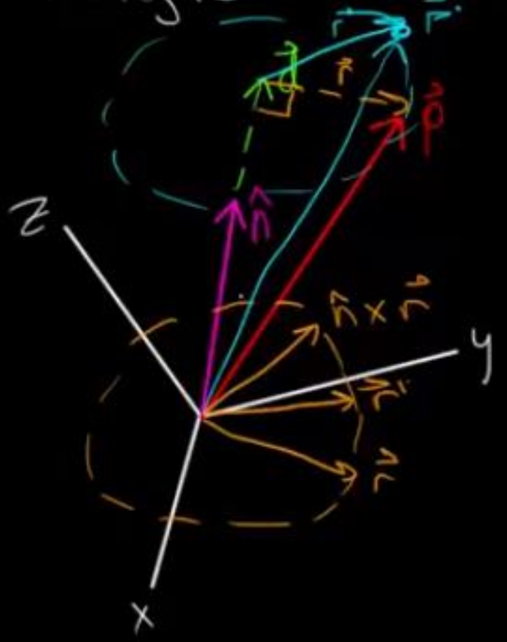
$$\vec{d} = (\hat{n}\cdot\vec{p}) * \hat{n}$$

$$\vec{r} = \vec{p} - \vec{d}$$

$$\vec{r}\,' = \vec{r}\cos\theta + (\hat{n}\times\vec{r})\sin\theta$$

$$\vec{p}\,' = \vec{d} + \vec{r}\,'$$

- Axis
- Position
- Angle (counter clockwise) θ



$\vec{d} = (\hat{n} \cdot \vec{p}) * \hat{n}$

$\vec{r} = \vec{p} - \vec{d}$

$\vec{r}' = \vec{r}\cos\theta + (\hat{n} \times \vec{r})\sin\theta$

$\vec{p}' = \vec{d} + \vec{r}'$

Now lets handle some real quaternions

- Quaernions when rotated magnitude does not changes but the angels and other stuff do change.

- Note q2 the vector to be rotated is pure Quaternions.
- Like here we want to rotate q2 about q1 but the rotated vector we got a vector which is not pure quaternion . But it should be.

- So we due to addition of fourth dimension we are getting troubled.

$$q_1 = [Cos\theta, \hat{n}Sin\theta] \qquad q_2 = [0, \vec{v}]$$

$$q_1 * q_2 = [-Sin\theta * \hat{n} \cdot \vec{v}, Cos\theta * \vec{v} + Sin\theta * \hat{n} \times \vec{v}]$$

$$-Sin\theta * \hat{n} \cdot \vec{v} = -1 \qquad Cos\theta * \vec{v} + Sin\theta * \hat{n} \times \vec{v} = < 1.41421354, 1, 0 >$$

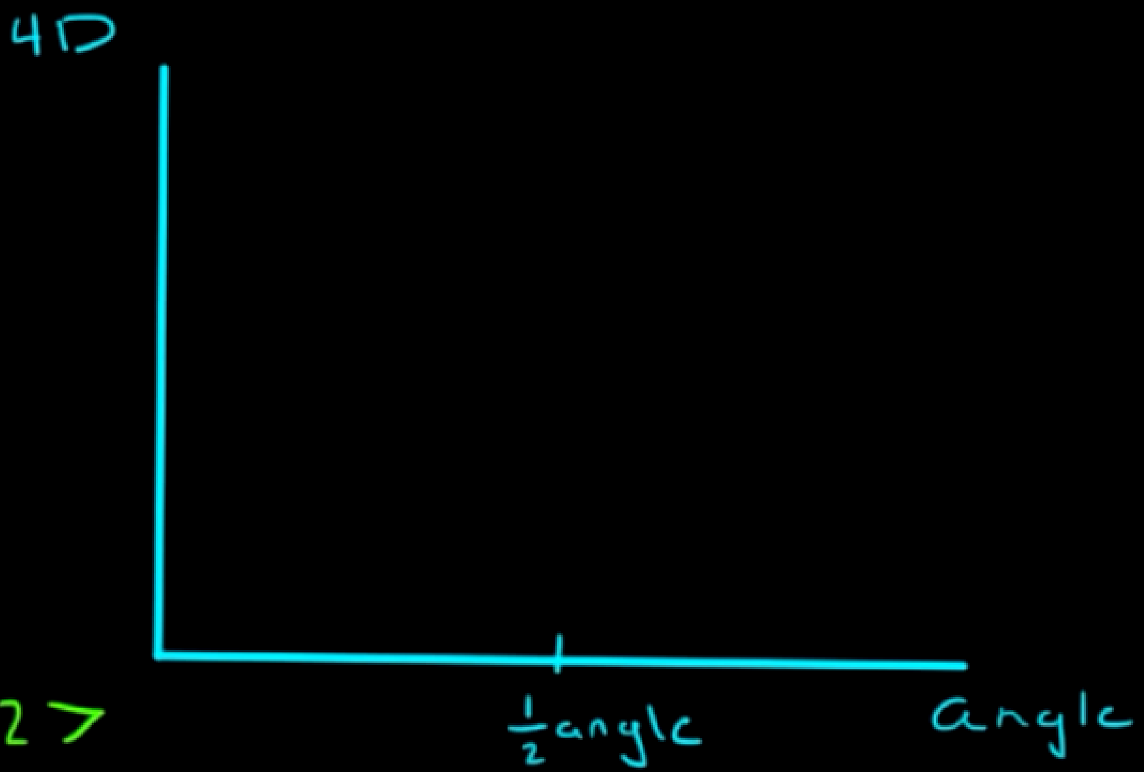
$$\theta = \frac{45^\circ}{2}$$

$$q_1 = [\cos\theta, \hat{n}\sin\theta]$$

$$q_1^{-1} = [\cos\theta, -\hat{n}\sin\theta]$$

$$q_2 = [0, \vec{v}]$$

$$q_1 * q_2 * q_1^{-1} = [0, \langle 1.7, 1, 0.2 \rangle$$



So to make corrections we multiply by q1*q2*q1^{-1} .

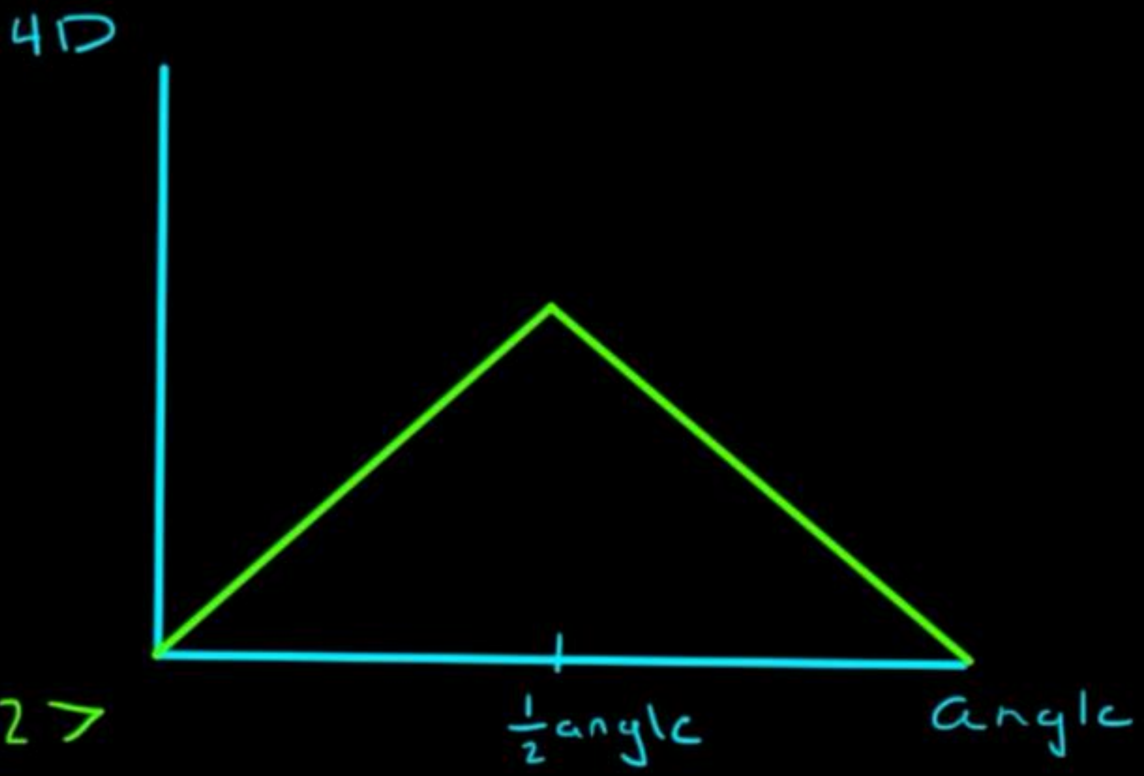
$$\theta = \frac{45^\circ}{2}$$

$$q_1 = [\cos\theta, \hat{n}\sin\theta]$$

$$q_1^{-1} = [\cos\theta, -\hat{n}\sin\theta]$$

$$q_2 = [0, \vec{v}]$$

$$q_1 * q_2 * q_1^{-1} = [0, \langle 1.7, 1, 0.2 \rangle$$



So what it actually does that q1 does half of rotation and $q1^{-1}$ does the other half.

Cancelling all side effects.

$\theta = 45° /2 \qquad q_1 = [Cos\theta, Sin\theta\hat{n}] \qquad q_1^{-1} = [Cos\theta, -Sin\theta\hat{n}] \qquad q_2 = [0, \vec{v}]$

$$q_1 * q_2 * q_1^{-1} = [0, < 1.70710683, 1, 0.292893231 >]$$