EL-GY 9343 Homework 2

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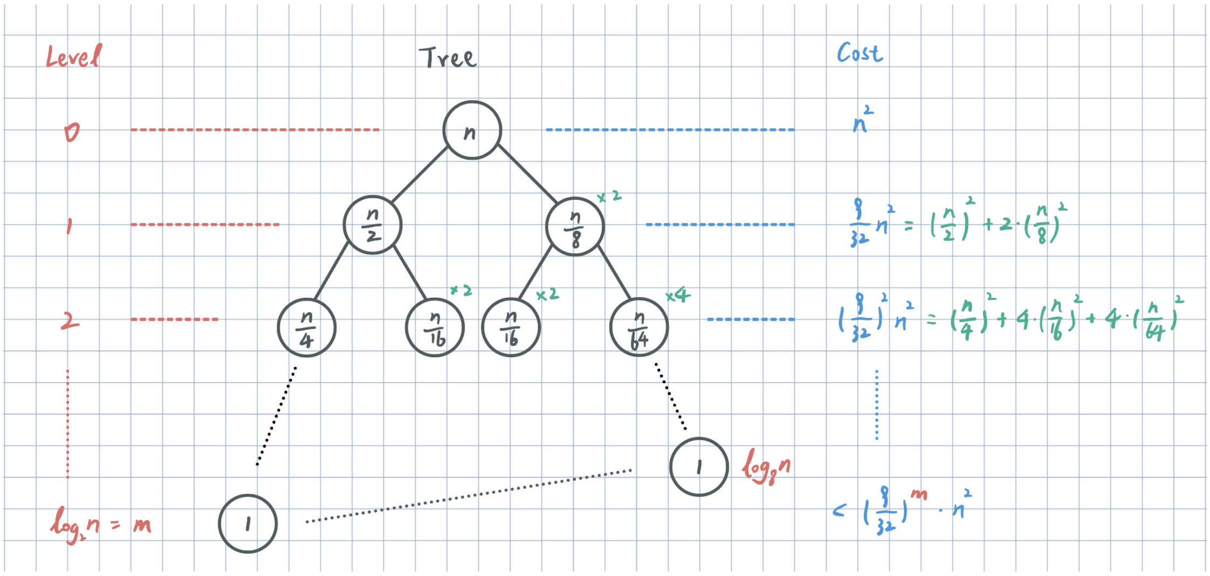
1. First use the iteration method to solve the recurrence, draw the recursion tree to analyze.

$$T(n) = T(\frac{n}{2}) + 2T(\frac{n}{8}) + n^2$$

Then use the substitution method to verify your solution.

Analysis with recursion tree:



$$\begin{aligned}
T(n) &< [1 + \frac{9}{32} + (\frac{9}{32})^2 + \cdots + (\frac{9}{32})^{\log_2 n}] \cdot n^2 \\
&< [1 + \frac{9}{32} + (\frac{9}{32})^2 + \cdots + (\frac{9}{32})^{+\infty}] \cdot n^2 \\
&= \frac{1}{1 - 9/32} \cdot n^2 \implies \underline{T(n) = O(n^2)}
\end{aligned}$$

Verify with substitution method:

Assume $T(k) \le c \cdot k^2$, $\forall k < n$, we have

$$\begin{aligned}
T(n) &= T(\frac{n}{2}) + 2T(\frac{n}{8}) + n^2 \\
&\le c \cdot (\frac{n}{2})^2 + 2c \cdot (\frac{n}{8})^2 + n^2 \\
&= (1 + \frac{9}{32}c) \cdot n^2 = \underline{O(n^2)}
\end{aligned}$$

2. Use the substitution method to prove that,

$$T(n) = 2T(\frac{n}{2}) + cn\log n$$

is $O(n(\log n)^2)$, where $c > 0$ is a constant. ($\log \equiv \log 2$, in this and the following questions)

Assume $T(k) \leq c_1 \cdot k(\log k)^2,\ \forall k < n$, we have

$$\begin{aligned}
T(n) &= 2T(\frac{n}{2}) + cn\ \log n \\
&\leq 2c_1 \cdot \frac{n}{2}(\log \frac{n}{2})^2 + c \cdot n\ \log n \\
&= c_1 \cdot n(\log n - 1)^2 + c \cdot n\ \log n \\
&= c_1 \cdot n(\log n)^2 + (c - 2c_1) \cdot n\ \log n + c_1 \cdot n = \underline{O(n(\log n)^2)}
\end{aligned}$$

3. Solve the recurrence:

$$T(n) = 2T(\sqrt{n}) + (\log\log n)^2$$

(Hint: How to make change of variables so that you can apply Master's method?)

If we set $m = \log n$, so $n = 2^m$, we have

$$T(2^m) = 2T(\sqrt{2^m}) + \log m = 2T(2^{\frac{m}{2}}) + (\log m)^2$$

Then we let $S(m) = T(2^m)$, we have

$$S(m) = 2S(\frac{m}{2}) + (\log m)^2$$

which is in the form of

$$S(m) = aS(\frac{m}{b}) + f(m),$$

where $a = 2$, $b = 2$ and $f(m) = (\log m)^2$. Since we know that

$$\log m = O(\sqrt{m}),$$

we can also show that

$$f(m) = (\log m)^2 = O(m) = m^{\log_b a}$$

Therefore, we can readily apply Master's method and get the solution:

$$S(m) = \Theta(m) \implies \underline{T(n) = \Theta(\log n)}$$

4. You want to solve the following three recurrence formulas:

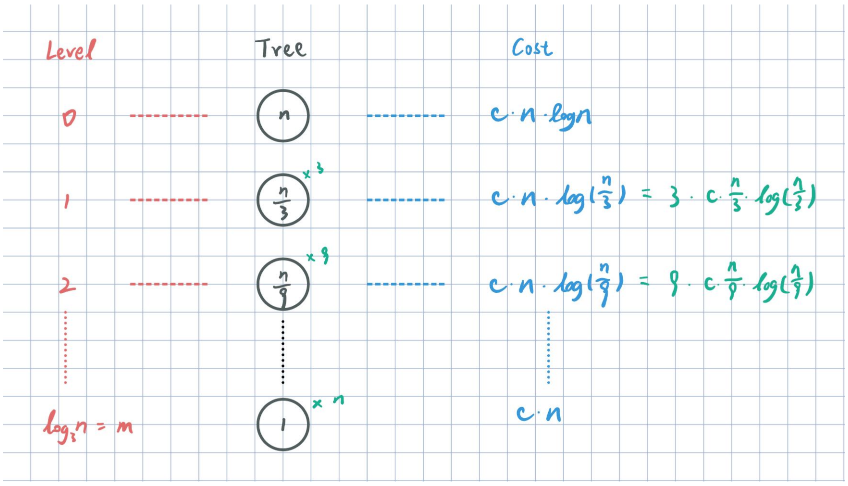
$$A: T(n) = 5T(\frac{n}{2}) + an$$
$$B: T(n) = T(\frac{n}{3}) + bn^2$$
$$C: T(n) = 3T(\frac{n}{3}) + cn\log n$$

Can you use Master's method for each of these? If yes, write down how you check the conditions and the answer. If not, briefly explain why and solve using other methods.
(Hint: You may need the harmonic series, i.e. when n is very large, $\log n \approx \sum_{k=1}^{n} \frac{1}{k}$)

A: Using Master's Method, $T(n) = aT(\frac{n}{b}) + f(n)$, where $a = 5$, $b = 2$ and $f(n) = an$. Therefore, $f(n) = O(n^{\log_b a - \epsilon}) = n^{\log_2 5 - \epsilon}$, where $\epsilon = \log_2 5 - 1 > 0$. So $\underline{T(n) = \Theta(n^{\log_2 5})}$.

B: Using Master's Method, $T(n) = aT(\frac{n}{b}) + f(n)$, where $a = 1$, $b = 3$ and $f(n) = bn^2$. Therefore, $f(n) = \Omega(n^{\log_b a + \epsilon}) = n^{\log_3 1 + \epsilon}$, where $\epsilon = 2 > 0$. Moreover, $\forall c \in [1/9, 1)$, we have $af(n/b) \leq cf(n)$. So $\underline{T(n) = \Theta(n^2)}$.

C: This recursion can not be solved using Master's method because $f(n) = cn\log n$ is not of exponential form of n, thus can not be compared accordingly. Moreover, $\forall \epsilon > 0, \log n = O(n^\epsilon)$. Therefore, we use recursion tree to analyse:



$$T(n) \simeq c \cdot n[\log n + \log(\frac{n}{3}) + \log(\frac{n}{3^2}) + \cdots + \log(\frac{n}{3^{\log_3 n}})]$$
$$= c \cdot n \sum_{i=0}^{\log_3 n} (\log n - i \cdot \log 3)$$
$$= c \cdot n \log n \log_3 n - c \cdot n \log 3 \cdot \sum_{i=0}^{\log_3 n} i$$
$$= c \cdot n \log n \log_3 n - c \cdot n \log 3 \, \frac{\log_3 n(\log_3 n + 1)}{2} = \underline{\Theta(n(\log n)^2)}$$