EL9343

Data Structure and Algorithm

Lecture 8: Graph Basics

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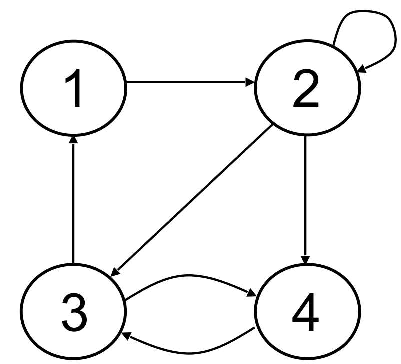
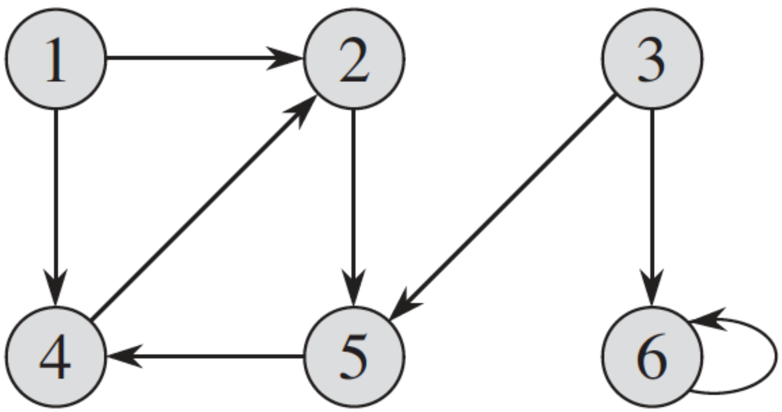
Graphs

- **Definition** = a set of vertices (nodes) with edges (links) between them.
  - $G = (V, E)$ - graph
  - V = set of vertices
  - E = set of edges = subset of $V \times V$
  - Thus $|E| = O(|V|^2)$

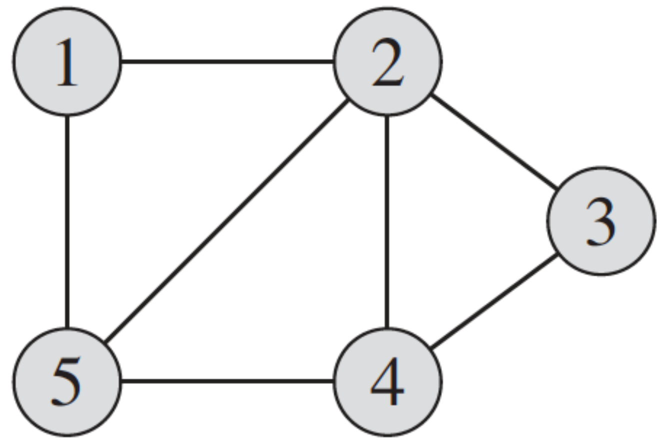


Directed Graphs (Digraph) VS. Undirected Graph

Directed Graphs (digraphs)
(ordered pairs of vertices)



Undirected Graphs
(unordered pairs of vertices)



in-degree of v: # of edges entering v
out-degree of v: # of edges leaving v

degree of v: # of edges incident on v

v is **adjacent** to u if there is an edge (u,v)

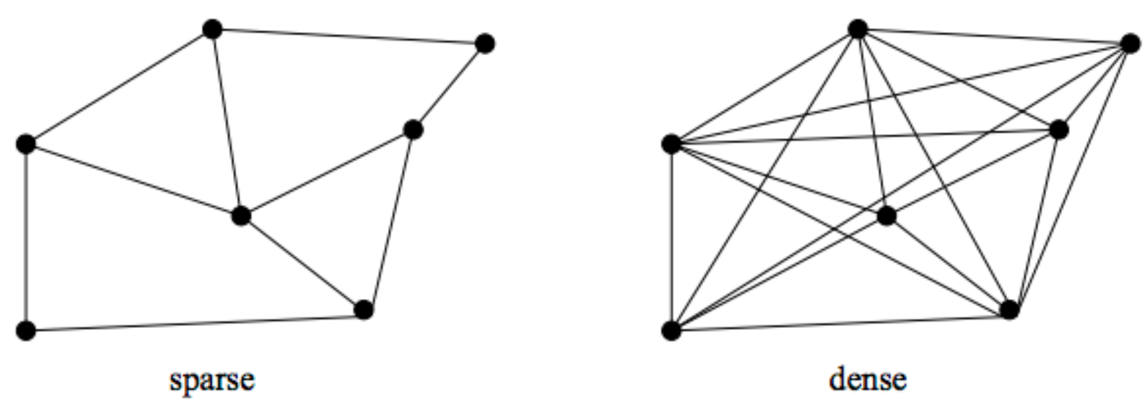
v is **adjacent** to u and u is adjacent to v is if there is an edge between v and u

More Graph variations

- A *weighted graph* associates weights with either the edges or the vertices
  - e.g., a road map: edges might be weighted w/ distance
- A *multigraph* allows multiple edges between the same pair of vertices
  - e.g., the call graph in a program (a function can get called from multiple points in another function)

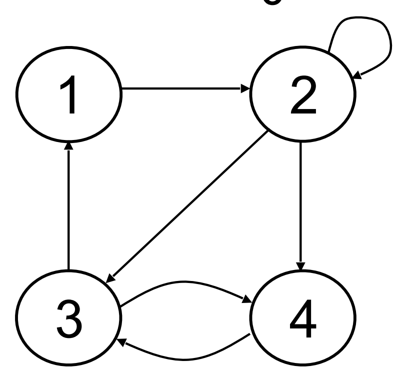
Sparse VS. Dense Graphs

- We will typically express running times in terms of |E| and |V|
  - If $|E| \approx |V|^2$ the graph is *dense*
    - have a quadratic number of edges
  - If $|E| \approx |V|$ the graph is *sparse*
    - linear in size, only a small fraction of the possible number of vertex pairs actually have edges defined between them



Terminology

- Complete graph
  - A graph with an edge between each pair of vertices
- Subgraph
  - A graph (V', E') such that V'⊆V and E'⊆E
- Path from v to w
  - A sequence of vertices $<v_0, v_1, \ldots, v_k>$ such that $v_0=v$ and $v_k=w$
- Length of a path
  - Number of edges along the path

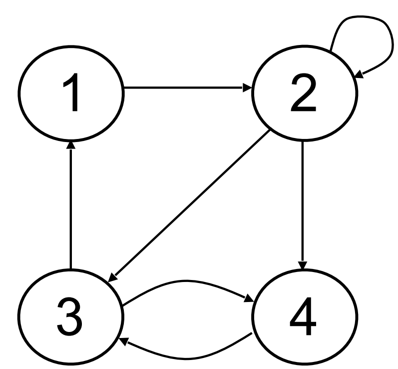


path from v_1 to v_4

$<v_1, v_2, v_4>$

Terminology (cont'd)

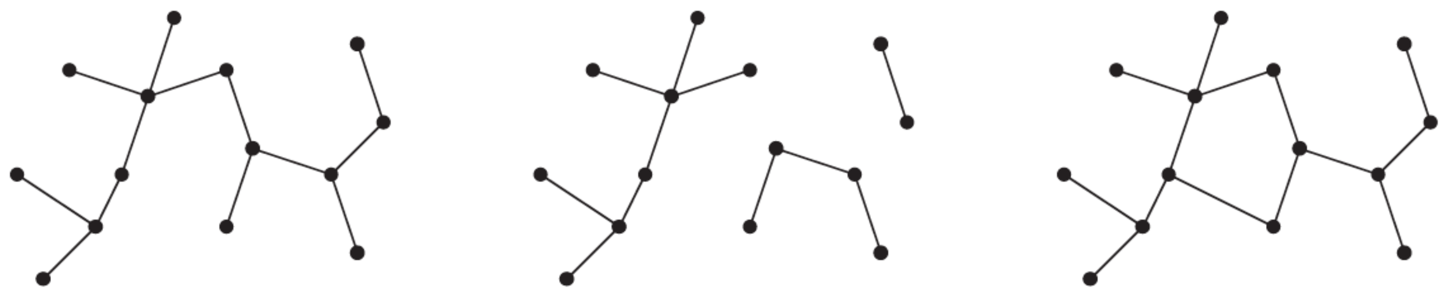
- w is **reachable** from v
  - If there is a path from v to w
- **Simple** path
  - All the vertices in the path are distinct
- Cycles
  - A path $<v_0, v_1, \ldots, v_k>$ forms a cycle if $v_0=v_k$ and $k\geq2$
- Acyclic graph
  - A graph without any cycles



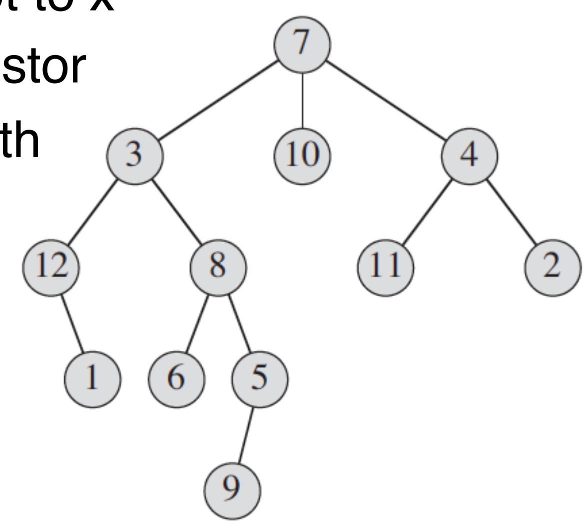
cycle from v_1 to v_1
$<v_1, v_2, v_3, v_1>$

Special case: Tree

- **(Free) Tree**: connected, acyclic, undirected graph
- **Forest:** acyclic, undirected graph, possibly disconnected



- **Rooted Tree**: a free tree with special **root** node
  - **Ancestor** of node x: any node on the path from root to x
  - **Descendant** of node x: any node with x as its ancestor
  - **Parent** of node x: node immediately before x on path from root
  - **Child** of node x: any node with x as its parent
  - **Siblings** of node x: nodes sharing parent with x
  - **Leaf/external node**: without child
  - **Internal node**: with at least one child

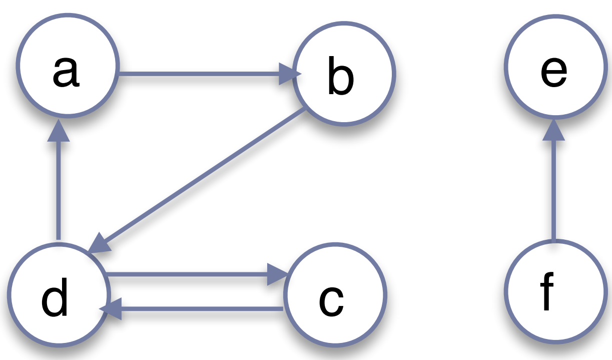


Strongly connected VS. Connected

Directed Graphs

Strongly connected: every two vertices are reachable from each other

Strongly connected components: all possible strongly connected subgraphs

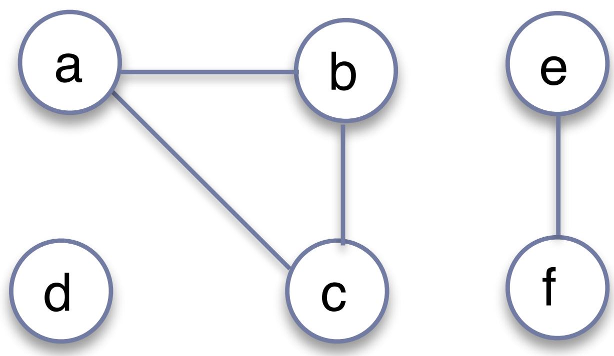


strongly connected components:
{a,b,c,d},{e},{f}

Undirected Graphs

connected: every pair of vertices are connected by a path

connected components: all possible connected subgraphs



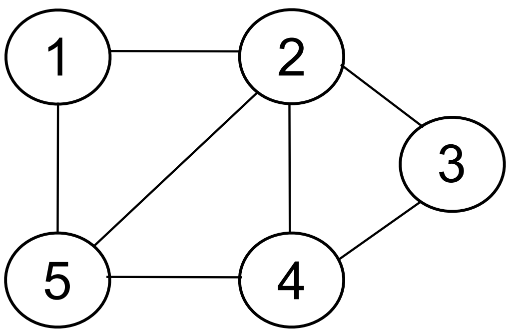
connected components:
{a,b,c},{d},{e,f}

Representing Graphs

- **Adjacency matrix representation** of G = (V, E)
  - Assume vertices are numbered 1, 2, … $|V|$
  - The representation consists of a matrix $A_{|V| \times |V|}$:
  - $a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$

Graphs: Adjacency Matrix

- Example



Undirected graph

	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

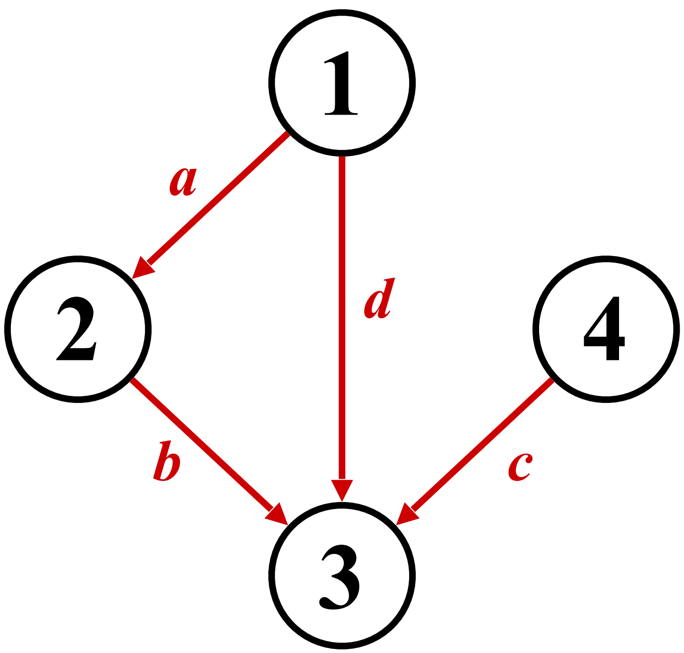
For undirected graphs, matrix A is symmetric:

$a_{ij} = a_{ji}$

$A = A^T$

Graphs: Adjacency Matrix

- Another Example



directed graph

	1	2	3	4
1	0	1	1	0
2	0	0	1	0
3	0	0	0	0
4	0	0	1	0

Properties of Adjacency Matrix Representation

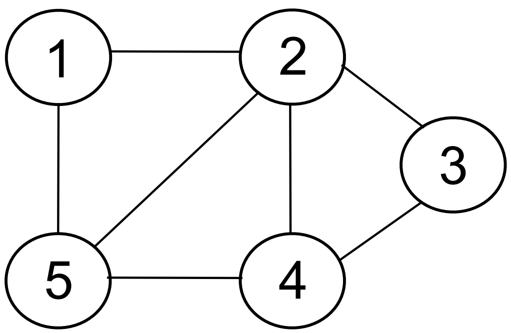
- Memory required
  - $\Theta(V^2)$, independent on the number of edges in G
- Preferred when
  - The graph is **dense:** $|E|$ is close to $|V|^2$
  - We need to quickly determine if there is an edge between two vertices
- Time to determine if $(u, v) \in E$:
  - $\Theta(1)$
- Disadvantage
  - No quick way to determine the vertices adjacent to a vertex
- Time to list all vertices adjacent to u:
  - $\Theta(V)$

Graph Adjacency List

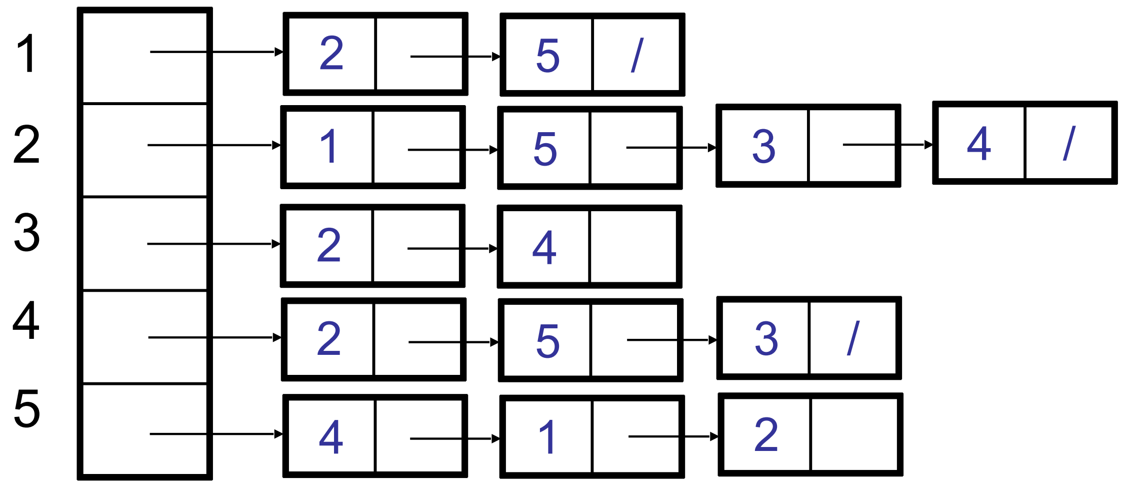
- **Adjacency list representation** of G = (V, E)
  - An array of $|V|$ lists, one for each vertex in V
  - Each list Adj[u] contains all the vertices v that are adjacent to u (i.e., there is an edge from u to v)
  - Can be used for both directed and undirected graphs

Graph Adjacency List

- Example

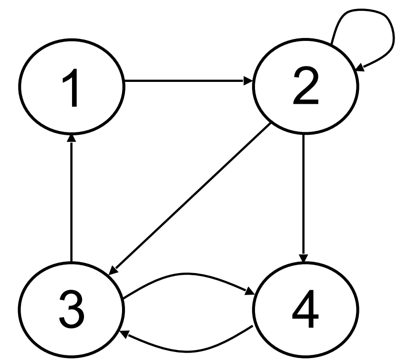


Undirected graph

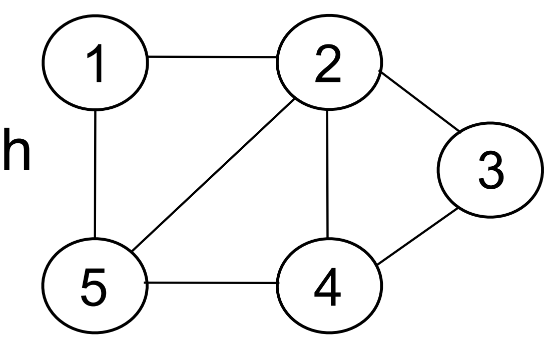


Properties of Adjacency-List Representation

- Sum of “lengths” of all adjacency lists
  - Directed graph: $|E|$
    - edge (u, v) appears only once (i.e., in the list of **u**)
  - Undirected graph: $2|E|$
    - edge (u, v) appears twice (i.e., in the lists of both **u** and **v**)



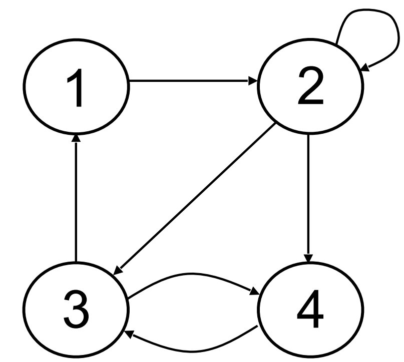
Directed graph



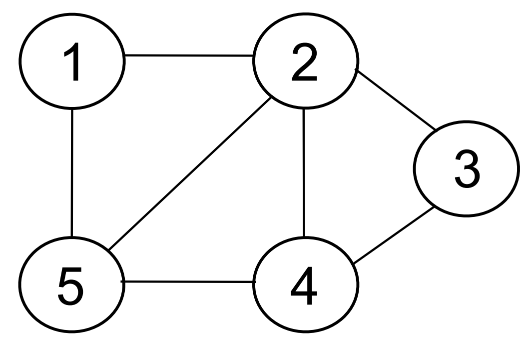
Undirected graph

Properties of Adjacency-List Representation

- Memory required
  - $\Theta(V + E)$
- Preferred when
  - The graph is **sparse**: $|E| \ll |V|^2$
  - We need to quickly determine the nodes adjacent to a given node.
- Disadvantage
  - No quick way to determine whether there is an edge between node u and v
- Time to determine if $(u, v) \in E$:
  - $O(\text{degree}(u))$
- Time to list all vertices adjacent to u:
  - $\Theta(\text{degree}(u))$



Directed graph



Undirected graph

Graph Search

- Given: a graph G = (V, E), directed or undirected
  - In general, given a vertex s, we want to locate some vertex t.
    - Find a path in G
  - We want to visit all vertices in a "local" organized manner

Breadth-First Search (BFS)

- "Explore" a graph, turning it into a tree
  - start with a *source vertex,* explore all other vertices *reachable* from the source, one vertex at a time
  - expand frontier of explored vertices across the *breadth* of the frontier
  - compute the distance (smallest number of edges) from source to each reachable vertex
- Builds a breadth-first tree over the graph
  - source is the root, cover all reachable vertices
  - find ("discover") its children, then their children, etc.
    - discover vertices at distance k from source before discovering vertices at distance k+1
  - the path from source to a vertex in breadth-first tree is the shortest path in the original graph

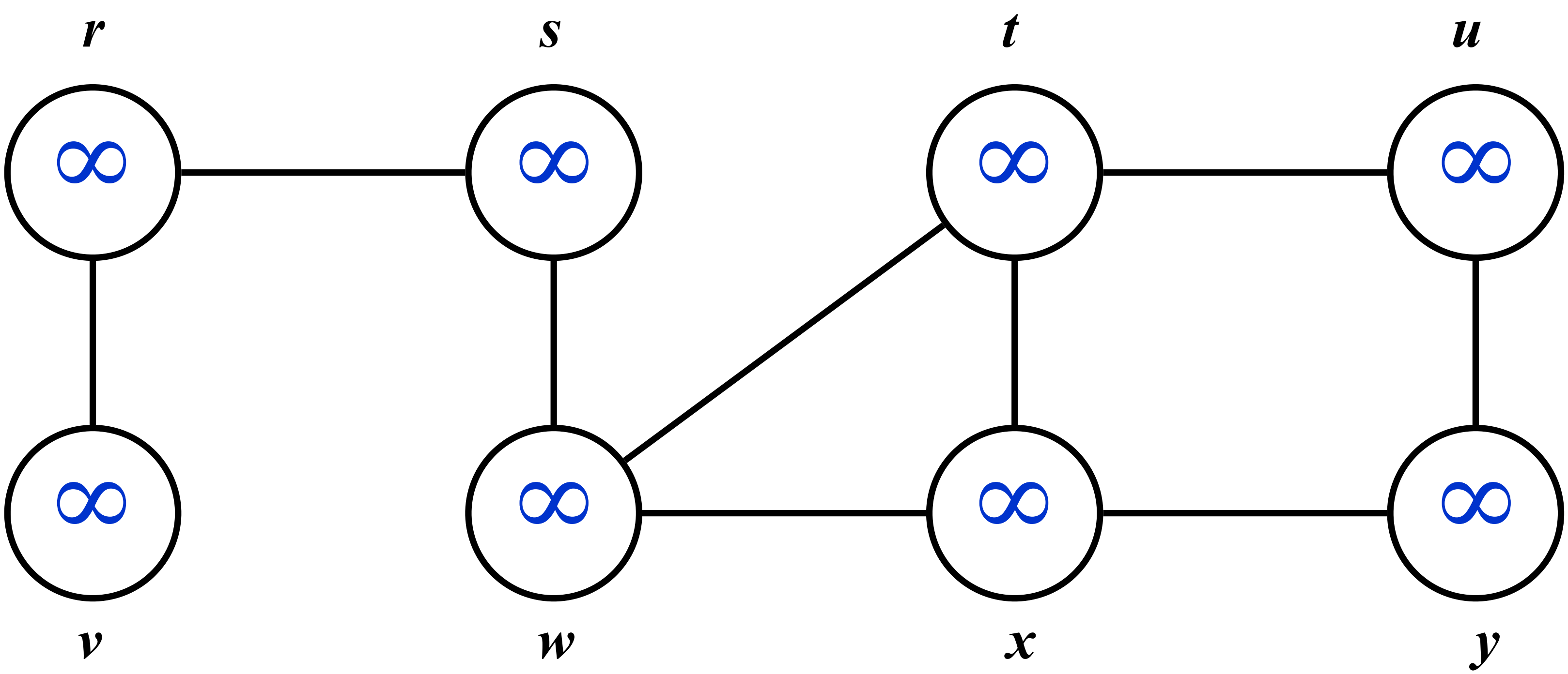
Breadth-First Search (BFS)

- Associate vertex “colors” to guide the algorithm
  - White vertices have not been discovered
    - All vertices start out white
  - Gray vertices are discovered but not fully explored
    - They may have some adjacent white vertices
  - Black vertices are discovered and fully explored
    - adjacent vertices of a black vertices are either black or gray
- Explore vertices by scanning adjacency list of gray vertices

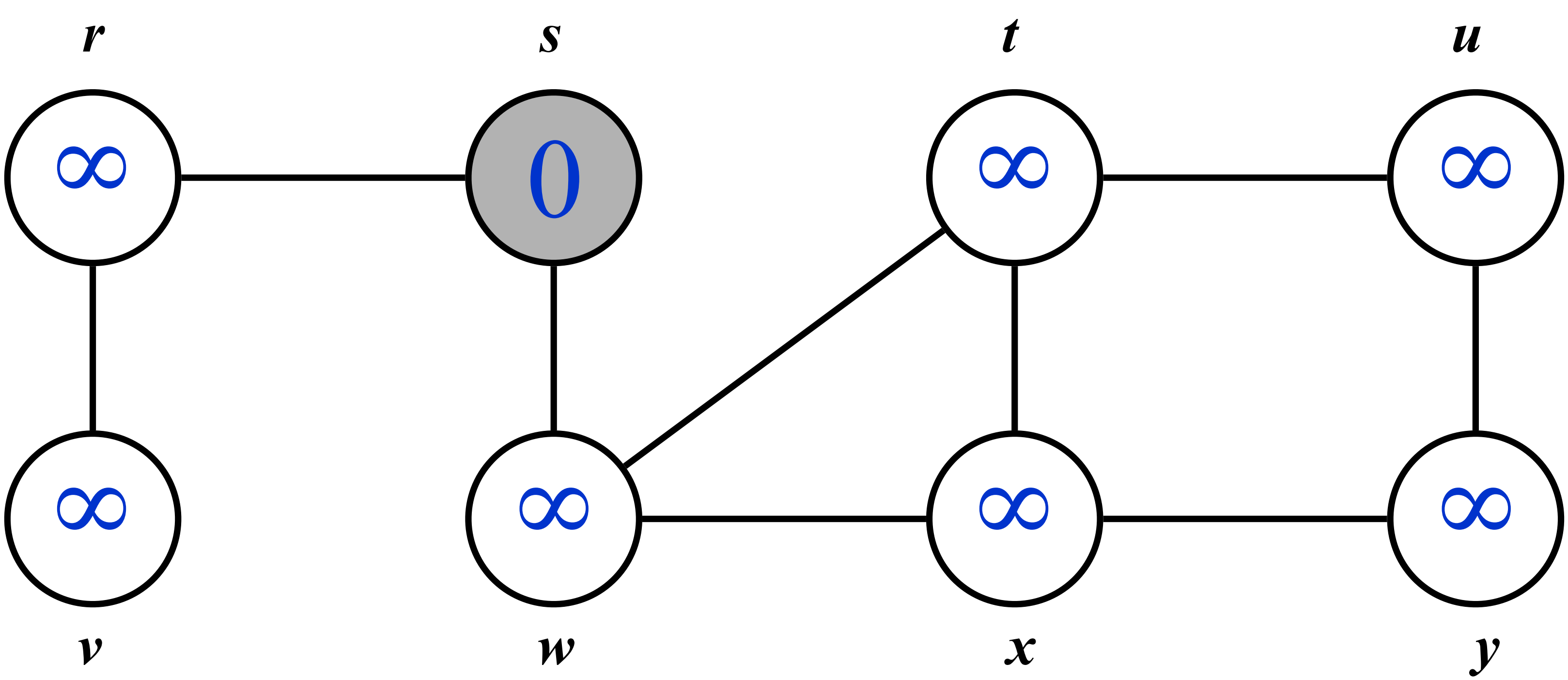
Breadth-First Search (BFS)

```
BFS(G,s) {
    for each u in V {                              // initialization
        color[u] = white
        d[u]     = infinity
        pred[u]  = null
    }
    color[s] = gray                                // initialize source s
    d[s] = 0
    Q = {s}                                        // put s in the queue
    while (Q is nonempty) {
        u = Q.Dequeue()                            // u is the next to visit
        for each v in Adj[u] {
            if (color[v] == white) {               // if neighbor v undiscovered
                color[v] = gray                    // ...mark it discovered
                d[v]     = d[u]+1                  // ...set its distance
                pred[v]  = u                       // ...and its predecessor
                Q.Enqueue(v)                       // ...put it in the queue
            }
        }
        color[u] = black                           // we are done with u
    }
}
```

Breadth-First Search (BFS)-Example

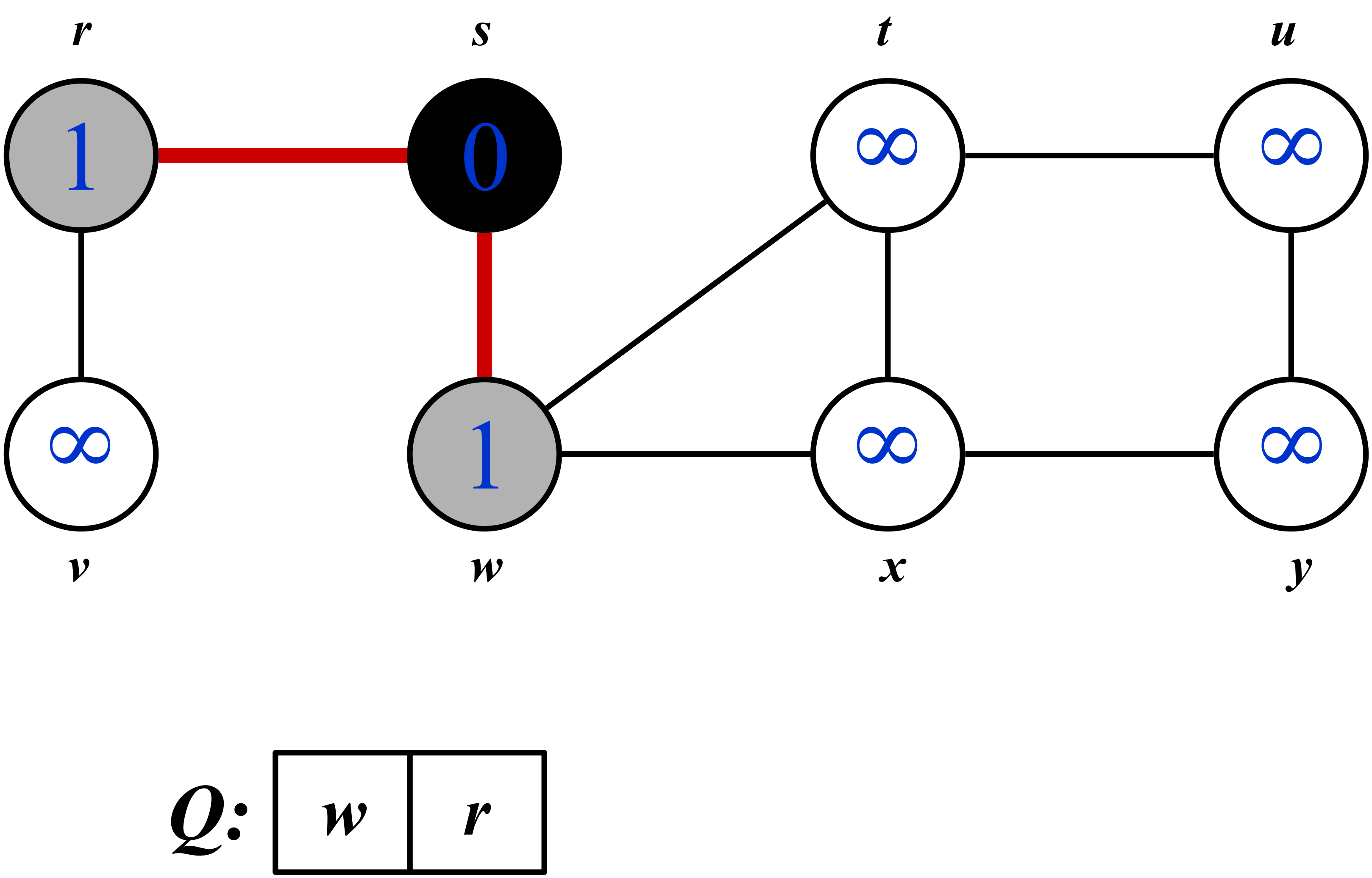


Breadth-First Search (BFS)-Example

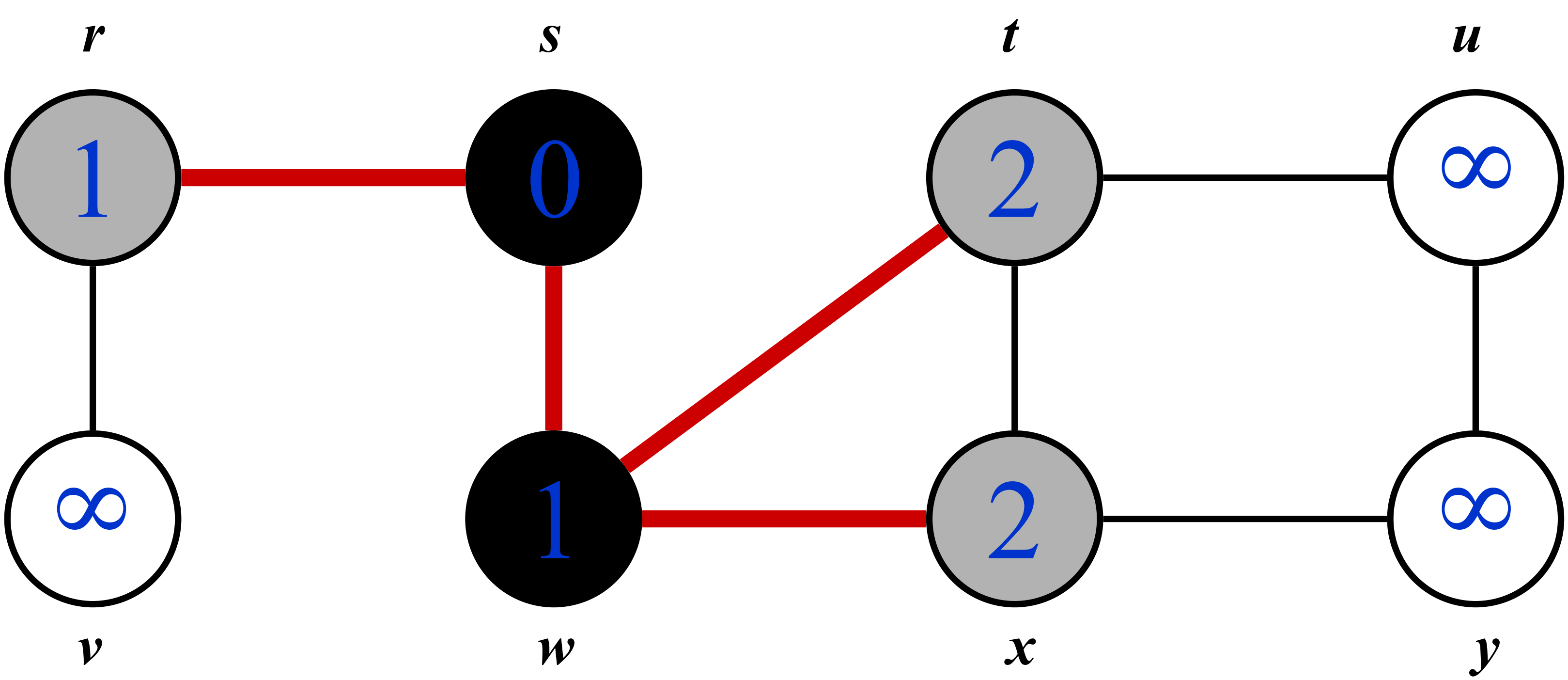


Q: $\boxed{s}$

Breadth-First Search (BFS)-Example

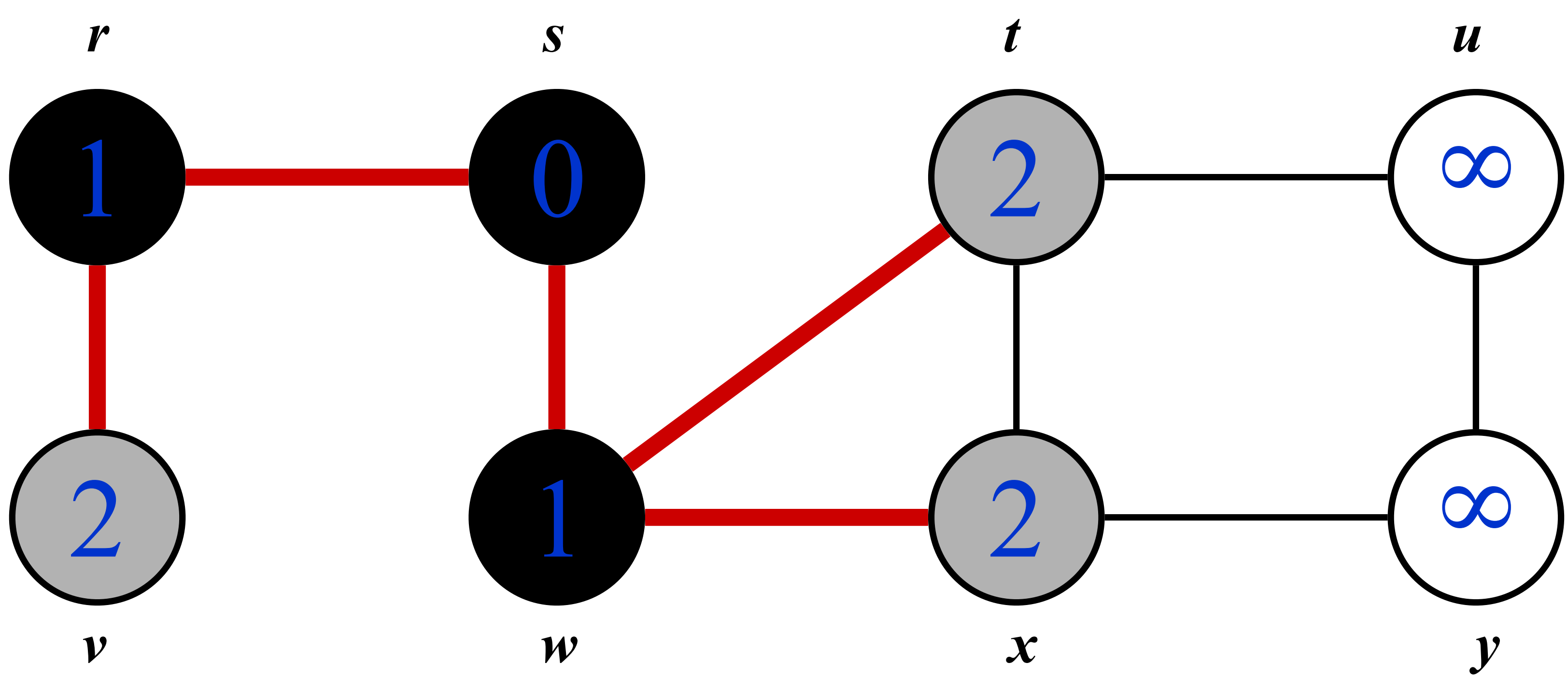


Breadth-First Search (BFS)-Example



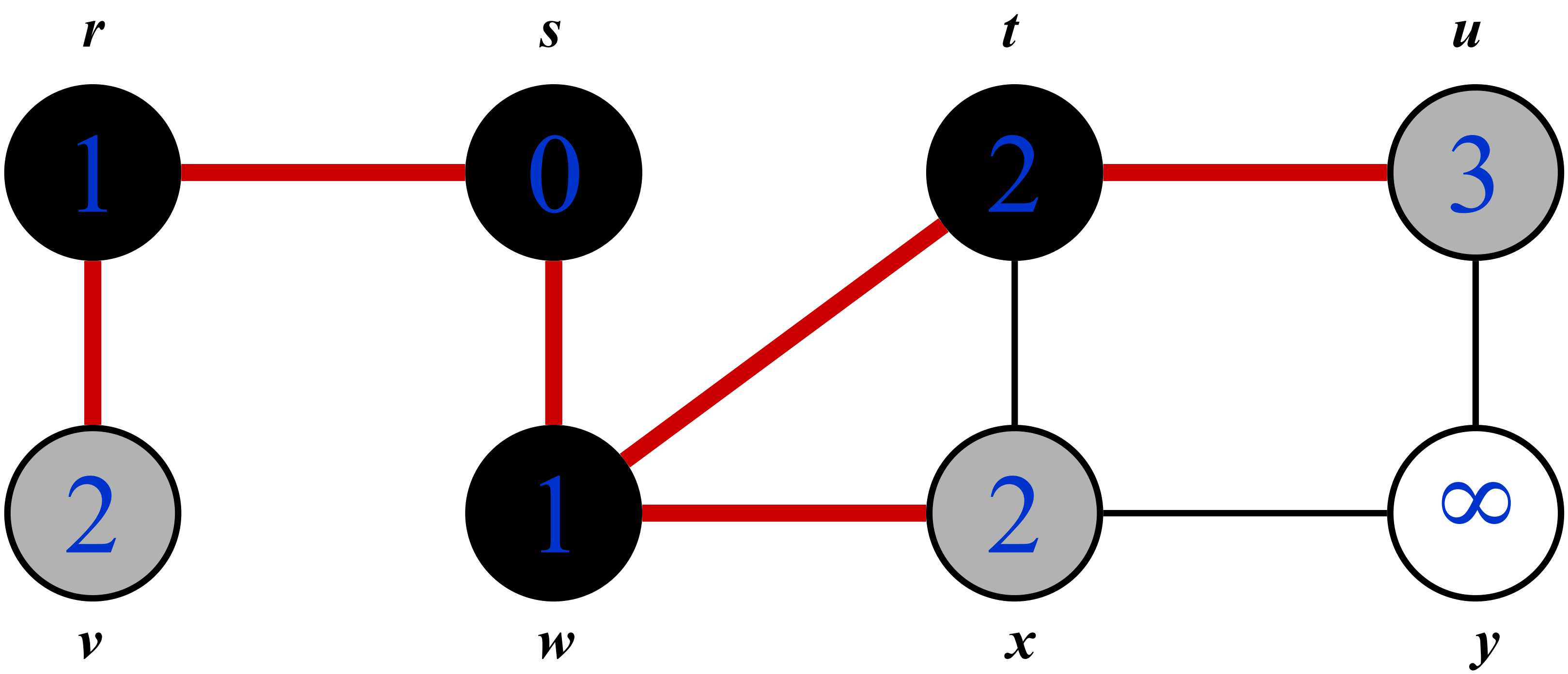
Q:		
	r	t	x

Breadth-First Search (BFS)-Example



Q: | t | x | v |

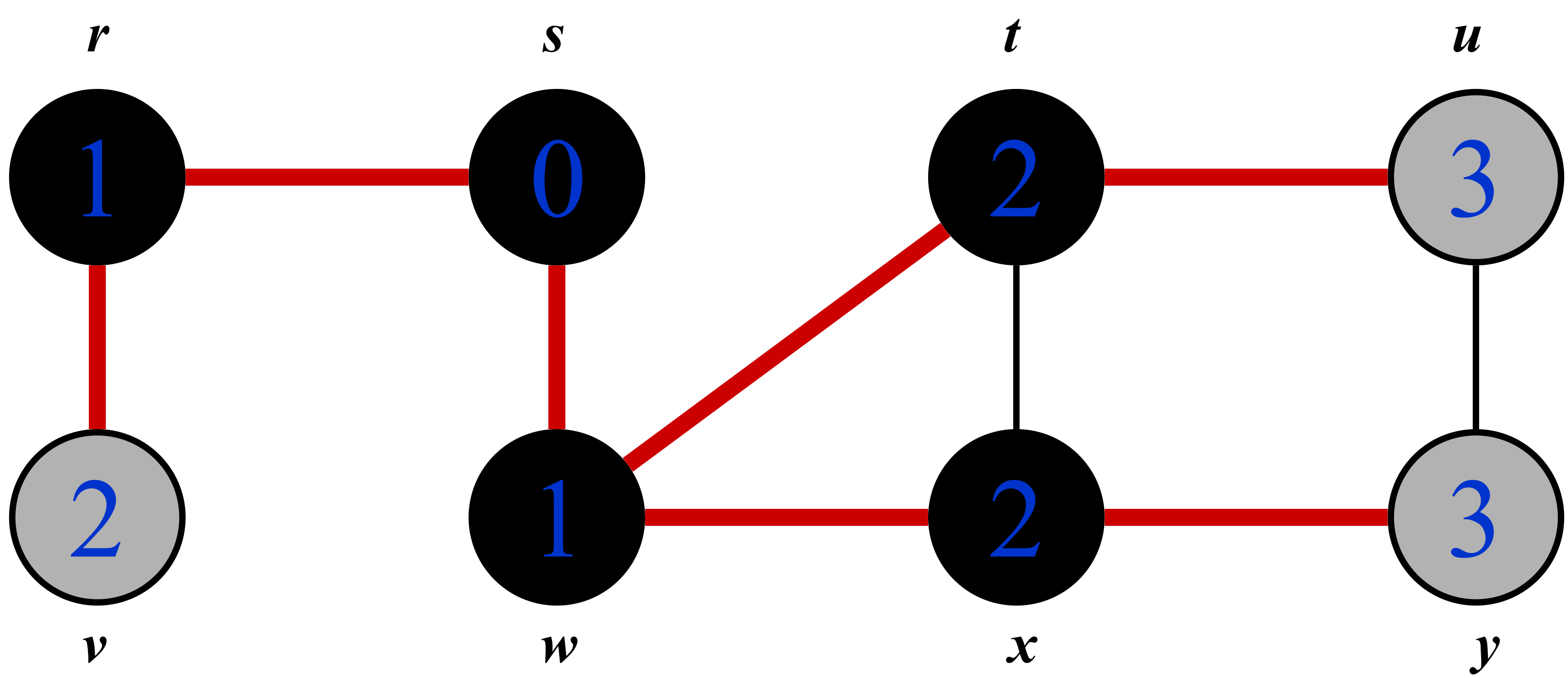
Breadth-First Search (BFS)-Example

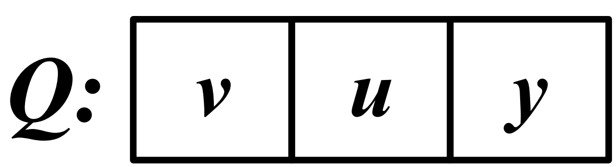


Q:

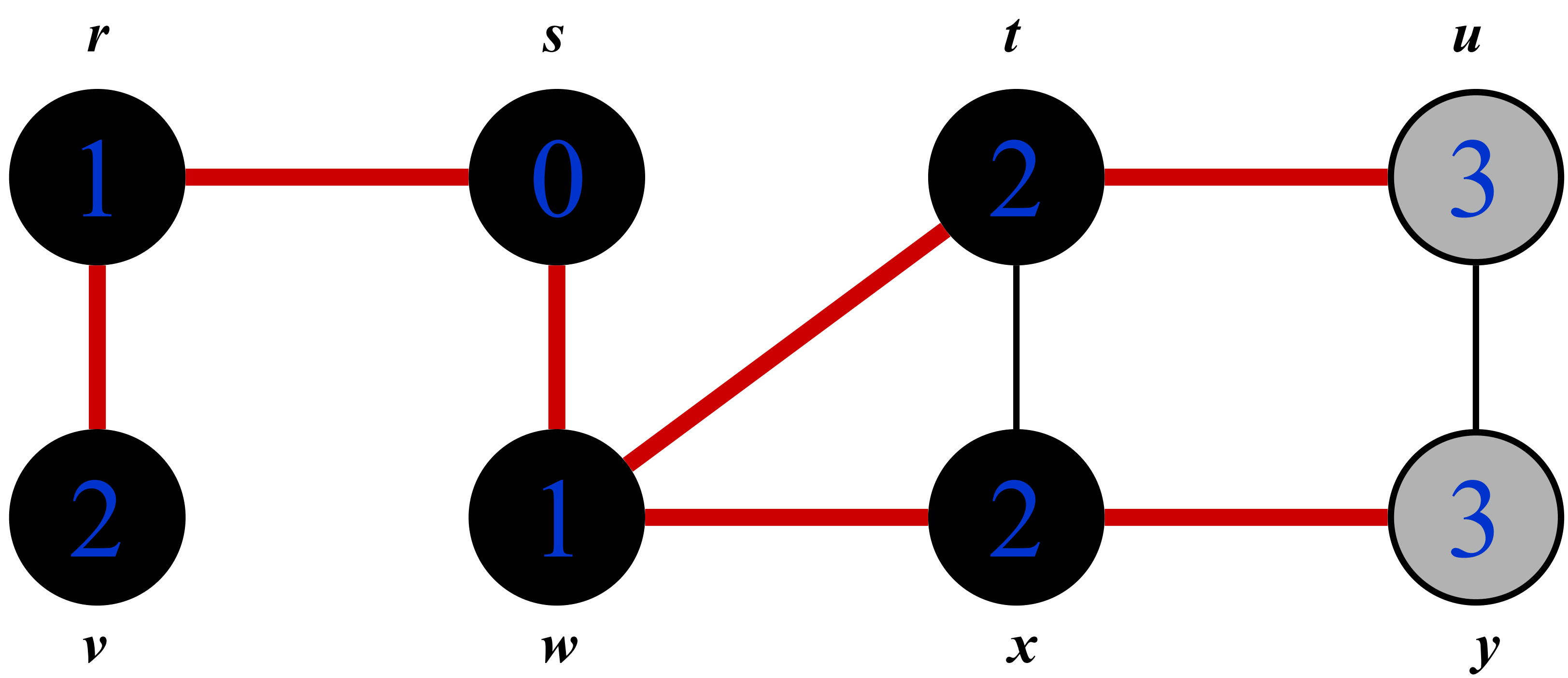
x	*v*	*u*

Breadth-First Search (BFS)-Example





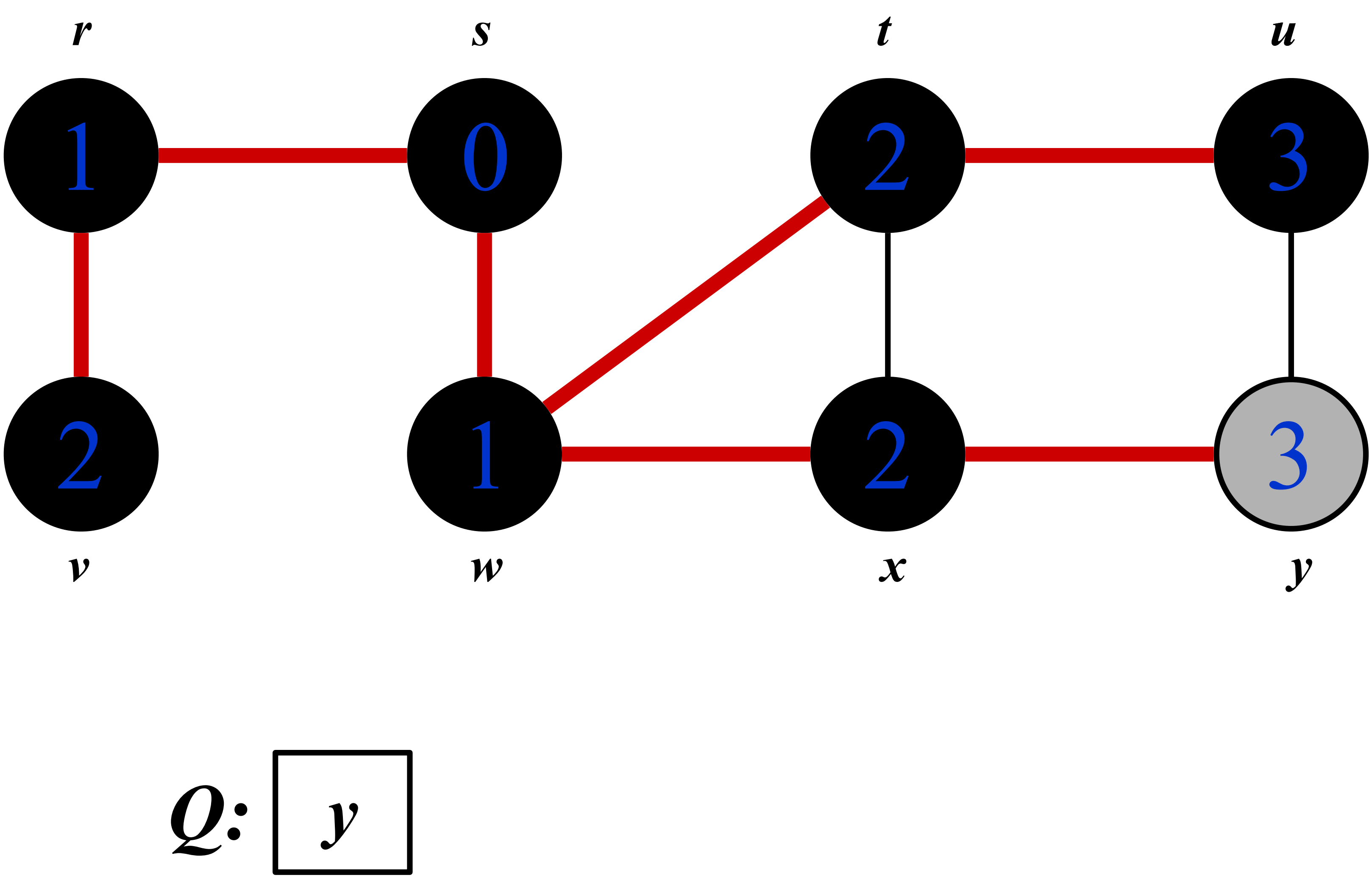
Breadth-First Search (BFS)-Example



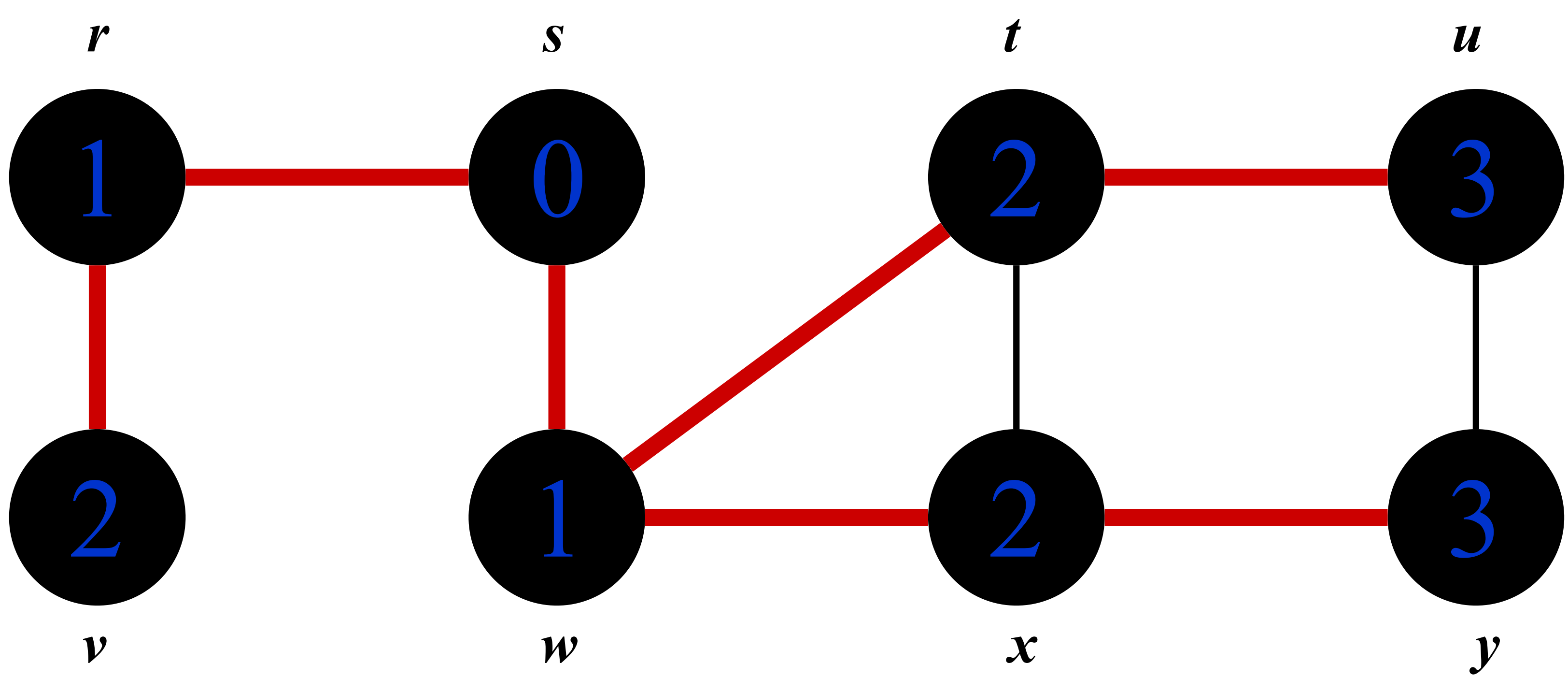
Q:

u	*y*

Breadth-First Search (BFS)-Example



Breadth-First Search (BFS)-Example



***Q*: Ø**

BFS Properties

- BFS calculates a *shortest-path distance* from the source node to all other nodes
  - Shortest-path distance $\delta(s,v)$ = minimum number of edges from s to v, or ∞ if v not reachable from s
  - $d(v) = \delta(s,v)$, see proof in the book
- BFS builds a *breadth-first tree*
  - s is the root, pred(v) is the predecessor/parent of v in breadth-first tree (relative to s)
  - path from s to v in tree is a shortest path from s to v in G
  - Thus can use BFS to calculate shortest path from one vertex to another in $O(V+E)$ time

Depth-First Search

- *Depth-first search* is another strategy for exploring a graph
  - Explore “deeper” in the graph whenever possible
  - Edges are explored out of the most recently discovered vertex *v* that still has unexplored edges
  - When all of *v*’s edges have been explored, backtrack to the vertex from which *v* was discovered

Depth-First Search

Again will associate vertex “colors” to guide the algorithm

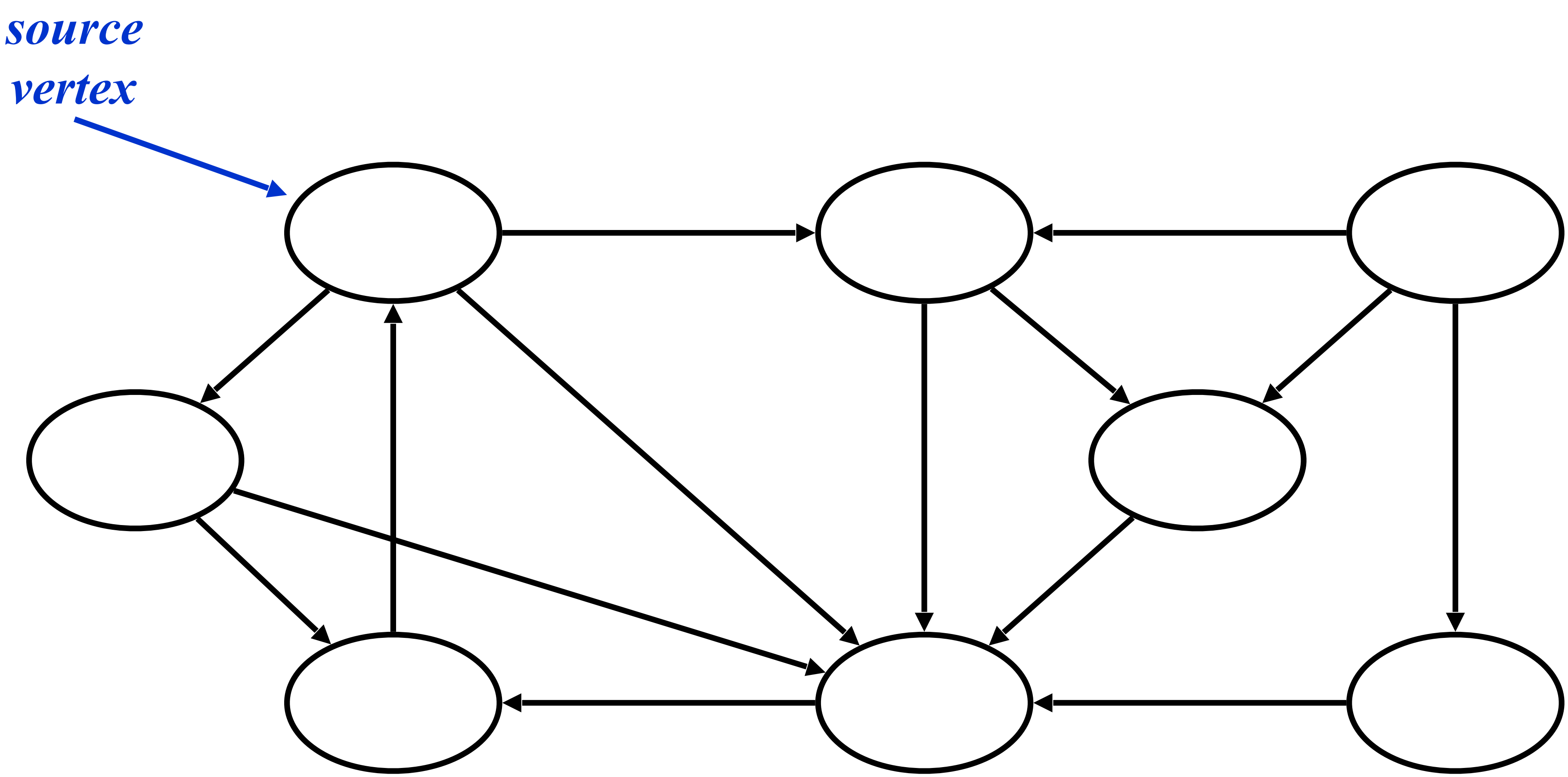
- Vertices initially colored white
- Then colored gray when discovered, not finished
- Then black when finished

Depth-First Search (DFS)

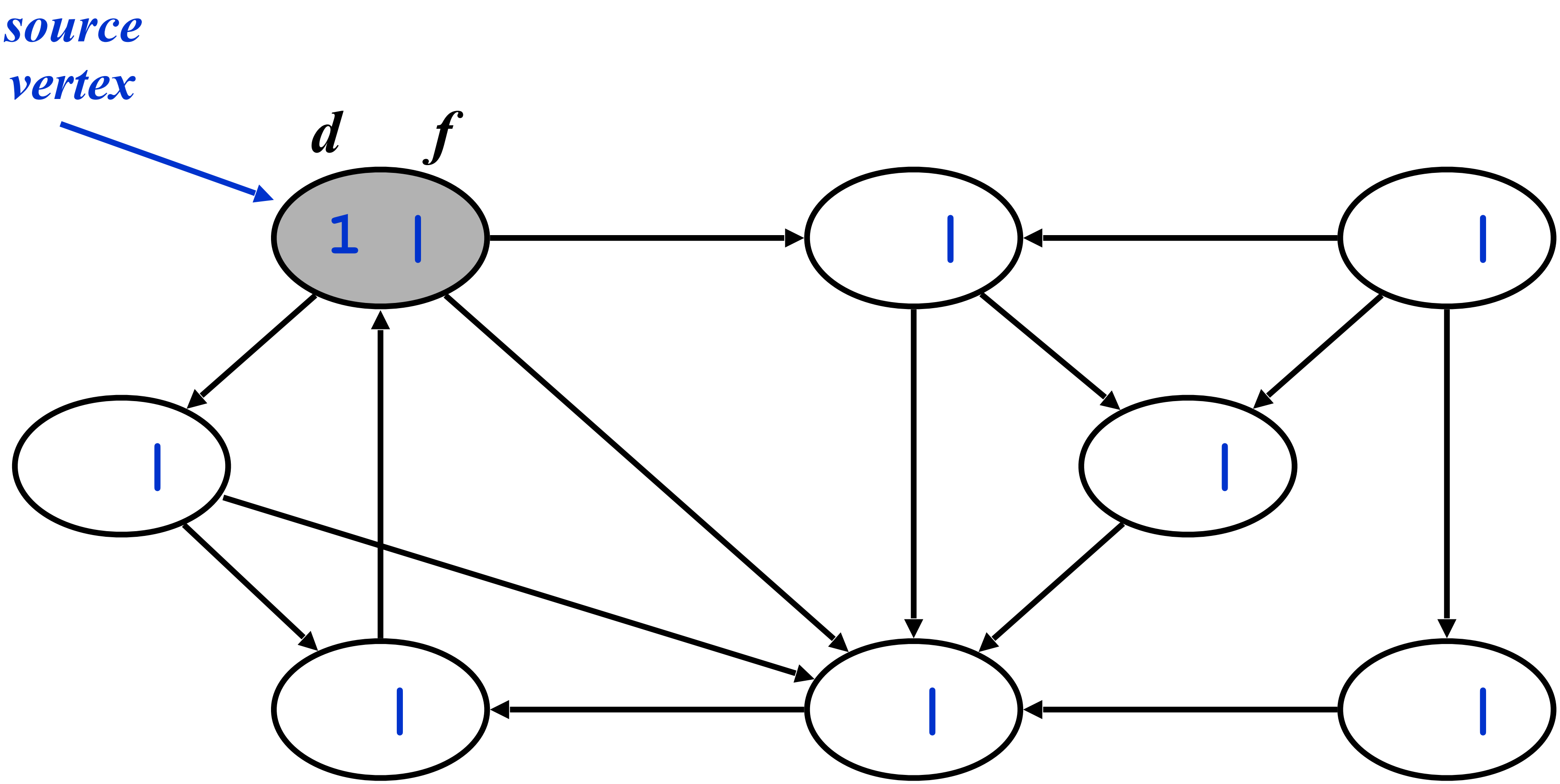
```
DFS(G) {                                    // main program
    for each u in V {                       // initialization
        color[u] = white;
        pred[u]  = null;
    }
    time = 0;
    for each u in V
        if (color[u] == white)              // found an undiscovered vertex
            DFSVisit(u);                    // start a new search here
}

DFSVisit(u) {                               // start a search at u
    color[u] = gray;                        // mark u visited
    d[u] = ++time;
    for each v in Adj(u) do
        if (color[v] == white) {            // if neighbor v undiscovered
            pred[v] = u;                    // ...set predecessor pointer
            DFSVisit(v);                    // ...visit v
        }
    color[u] = black;                       // we're done with u
    f[u] = ++time;
}
```

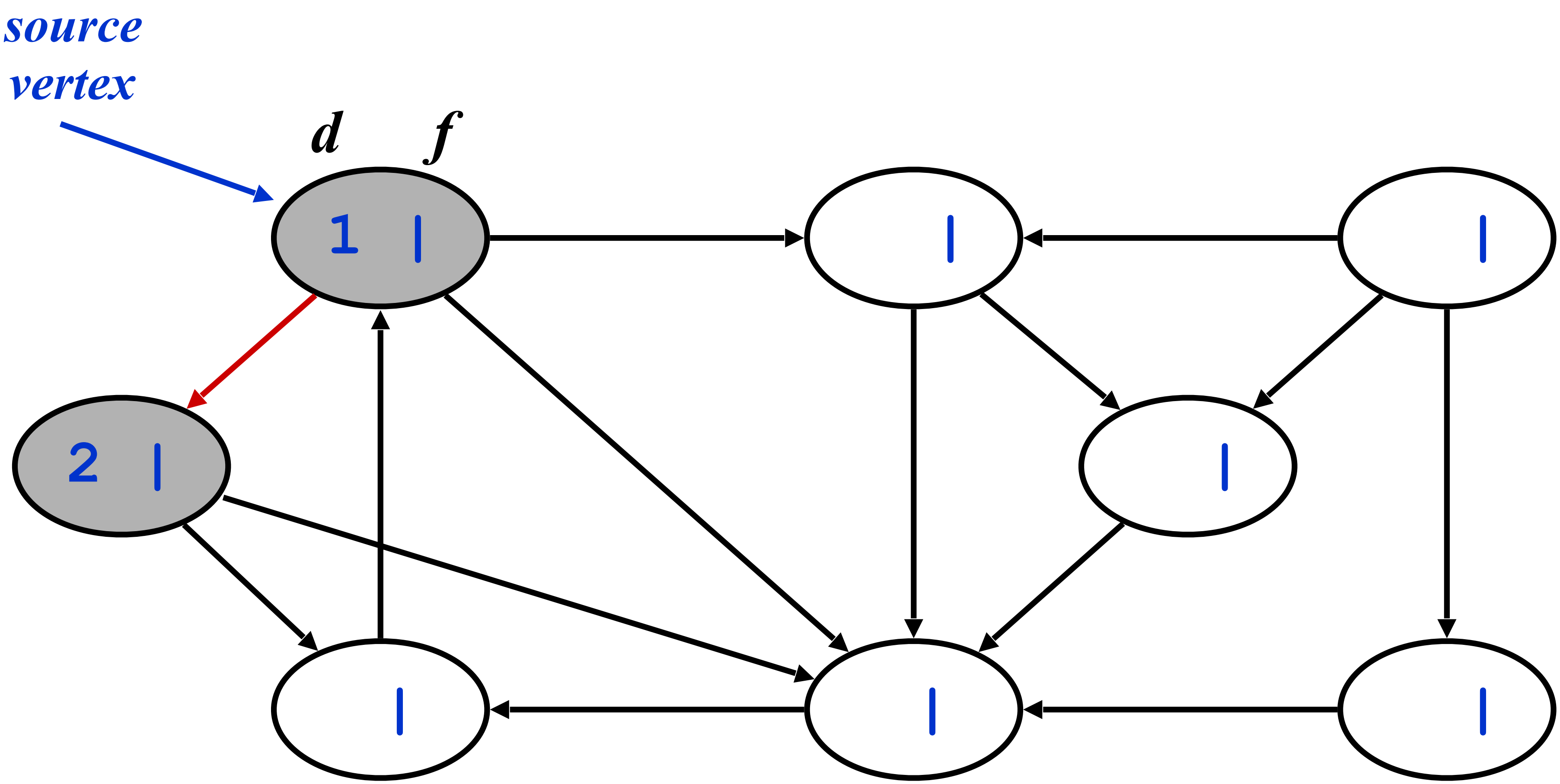
DFS Example



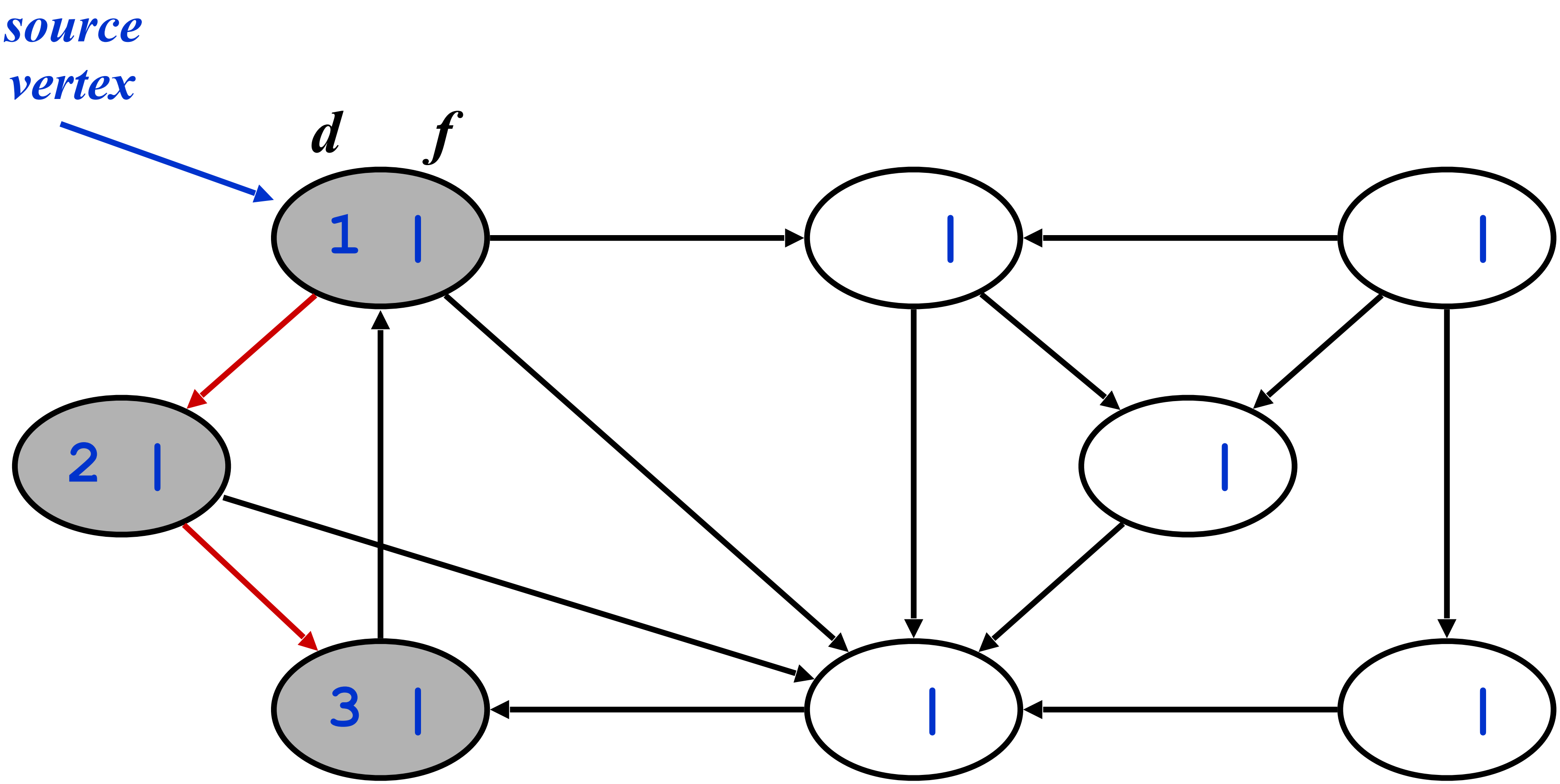
DFS Example



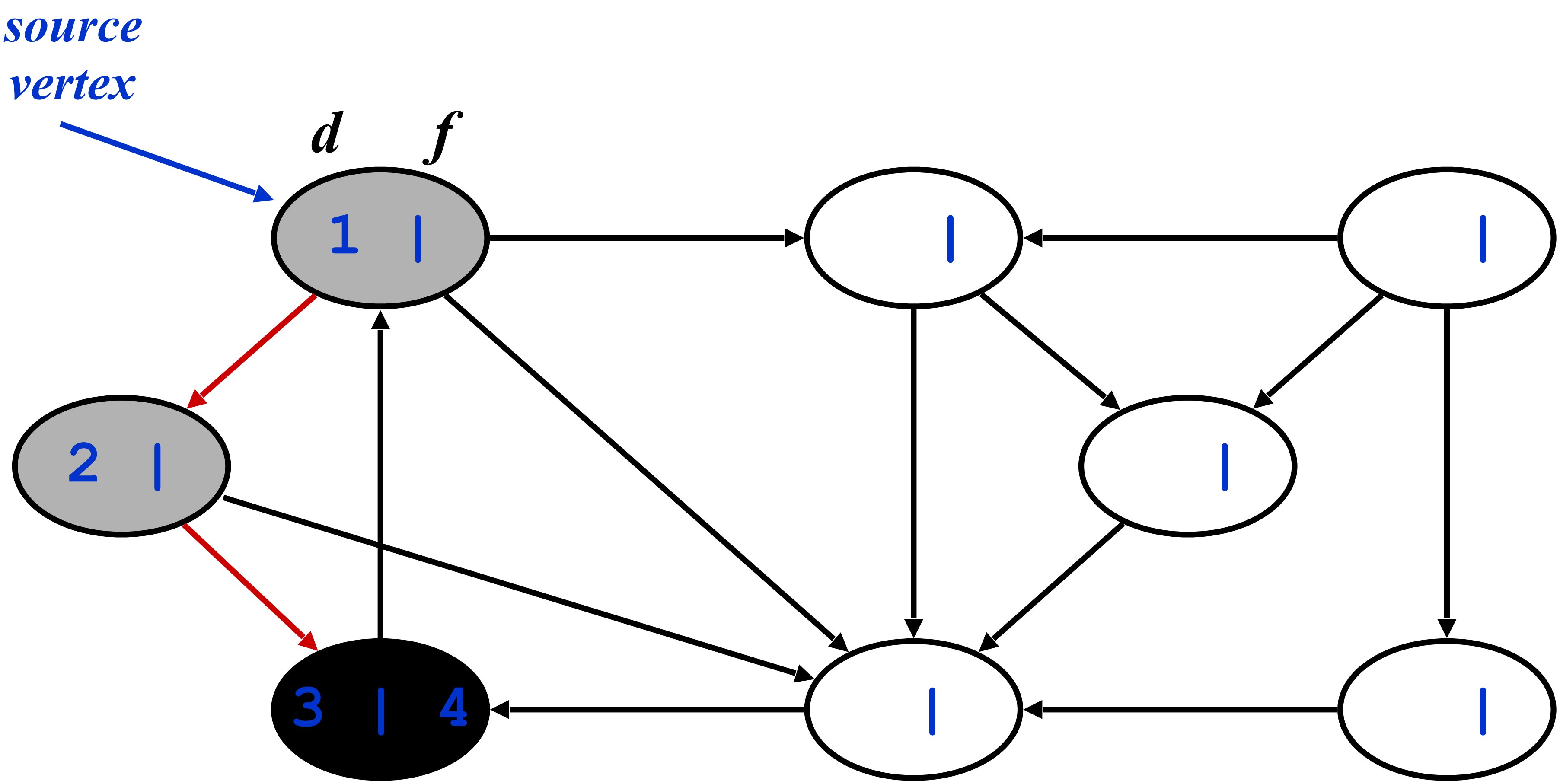
DFS Example



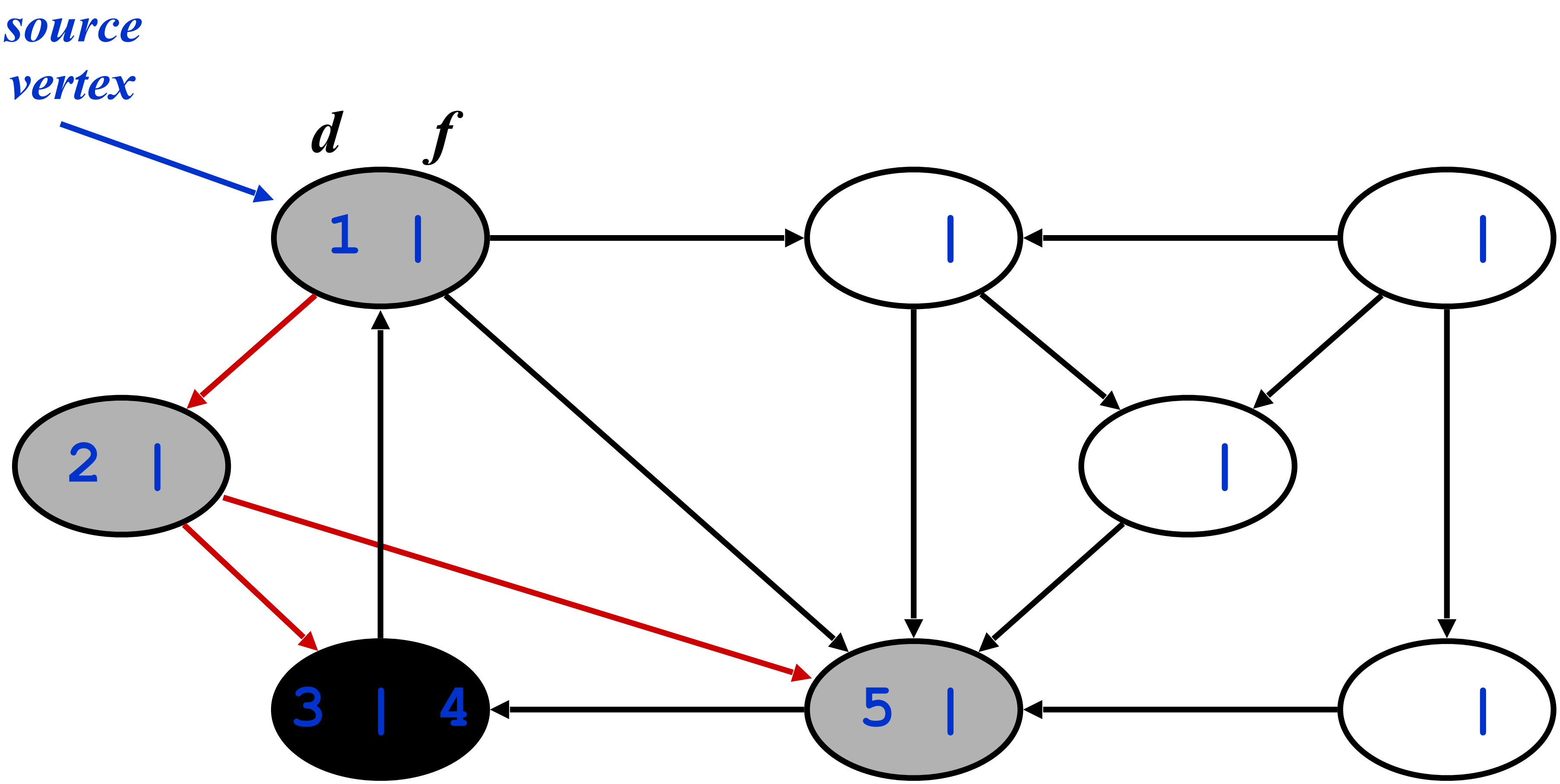
DFS Example



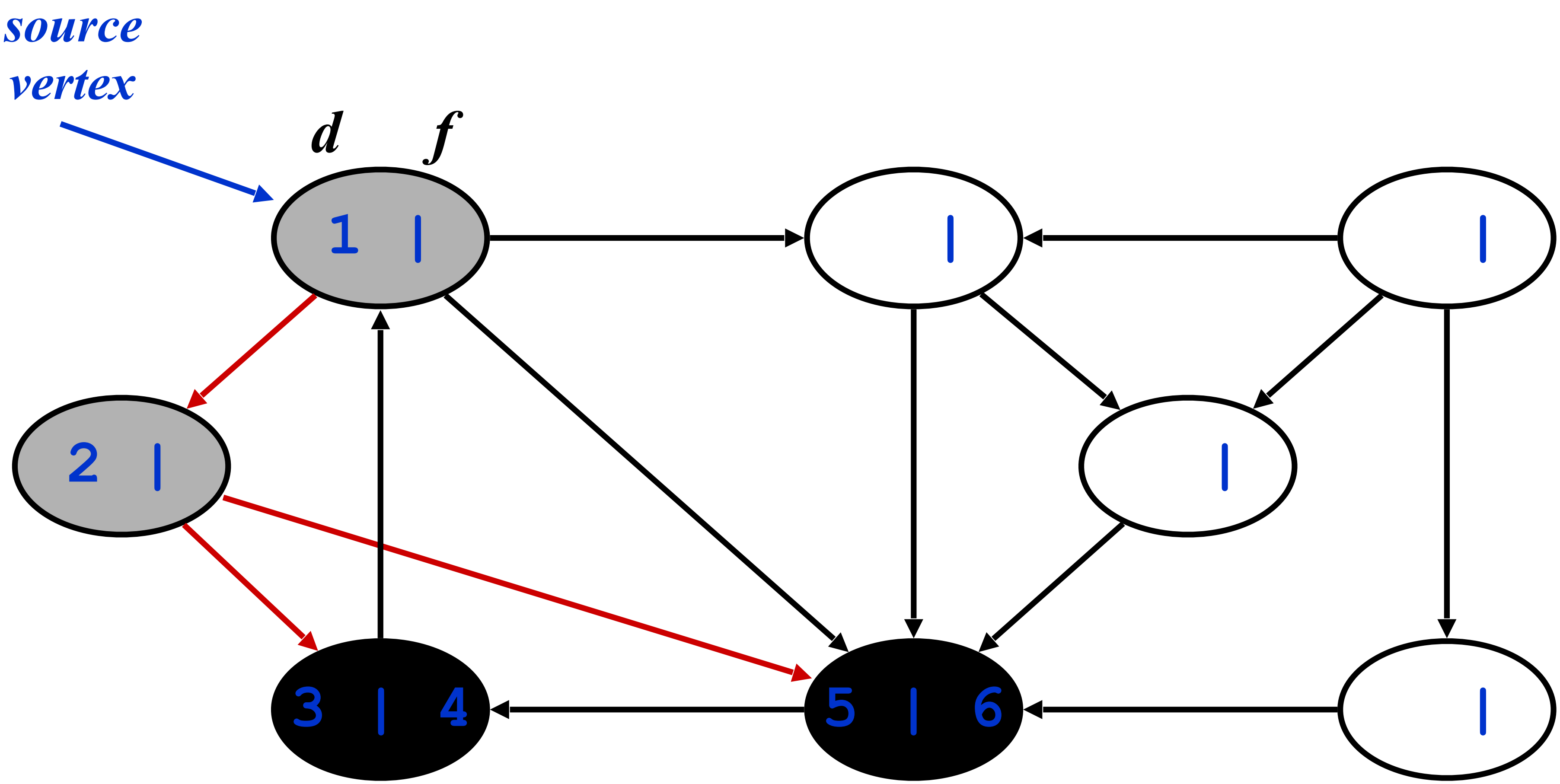
DFS Example



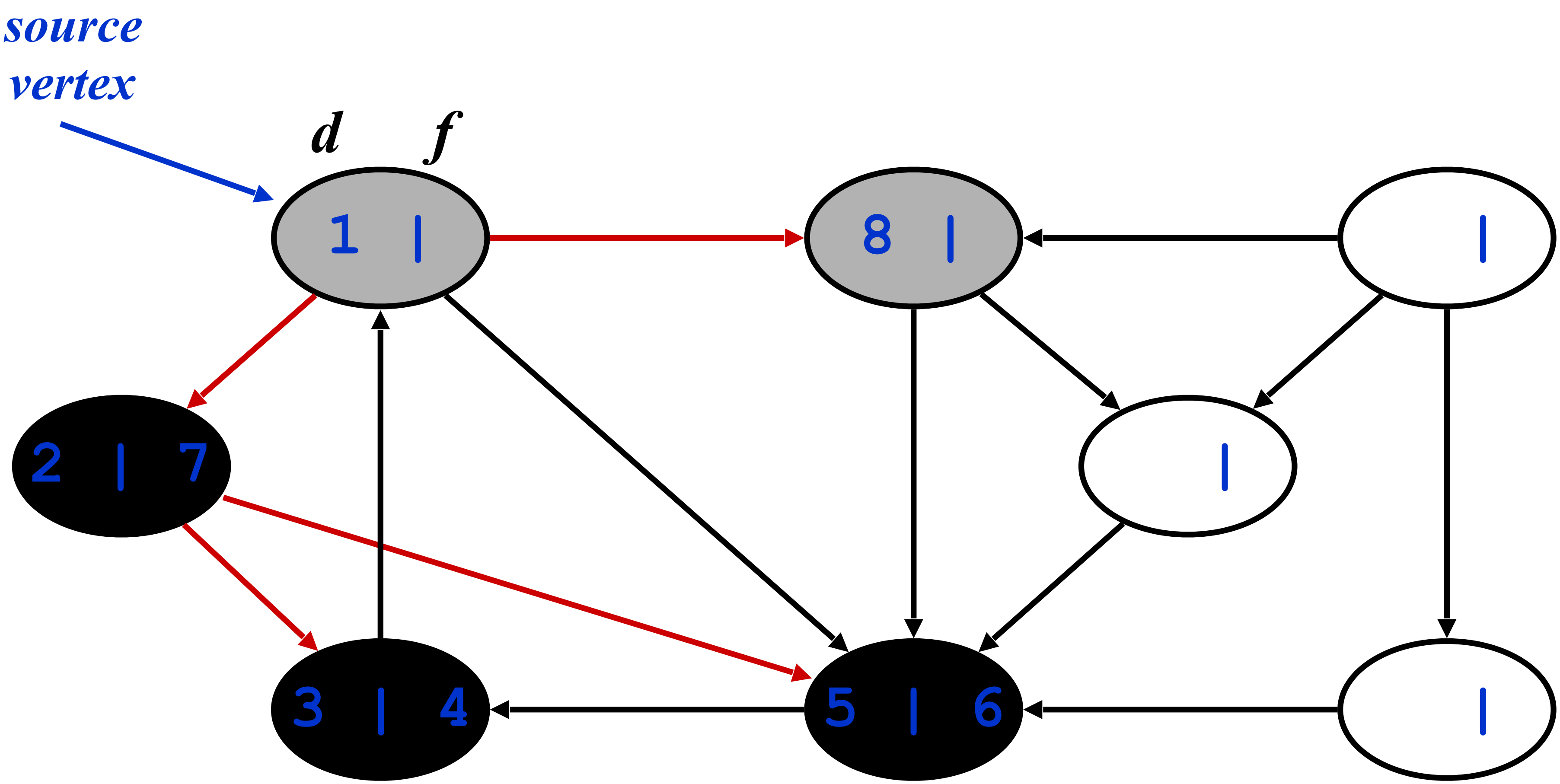
DFS Example



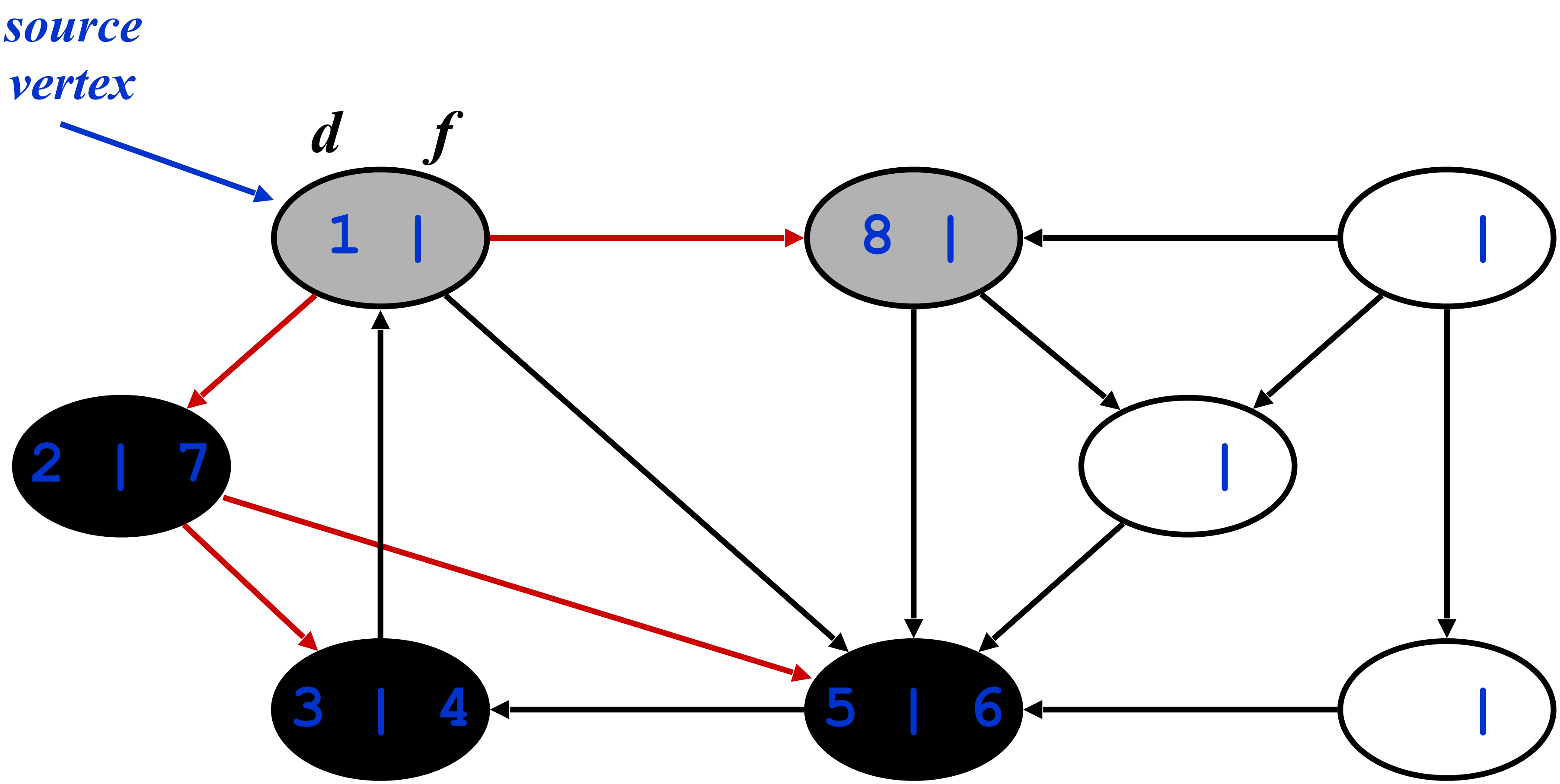
DFS Example



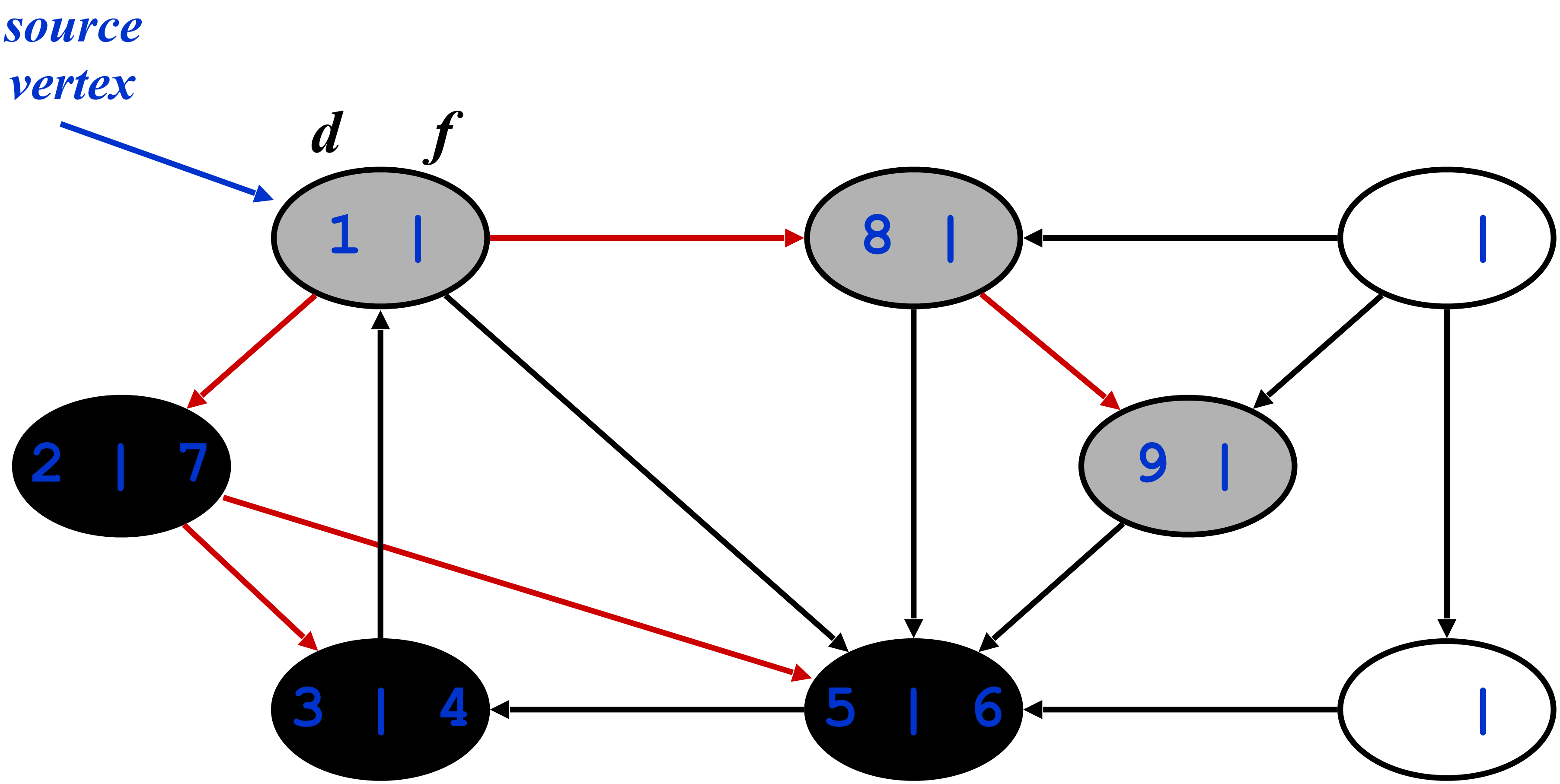
DFS Example



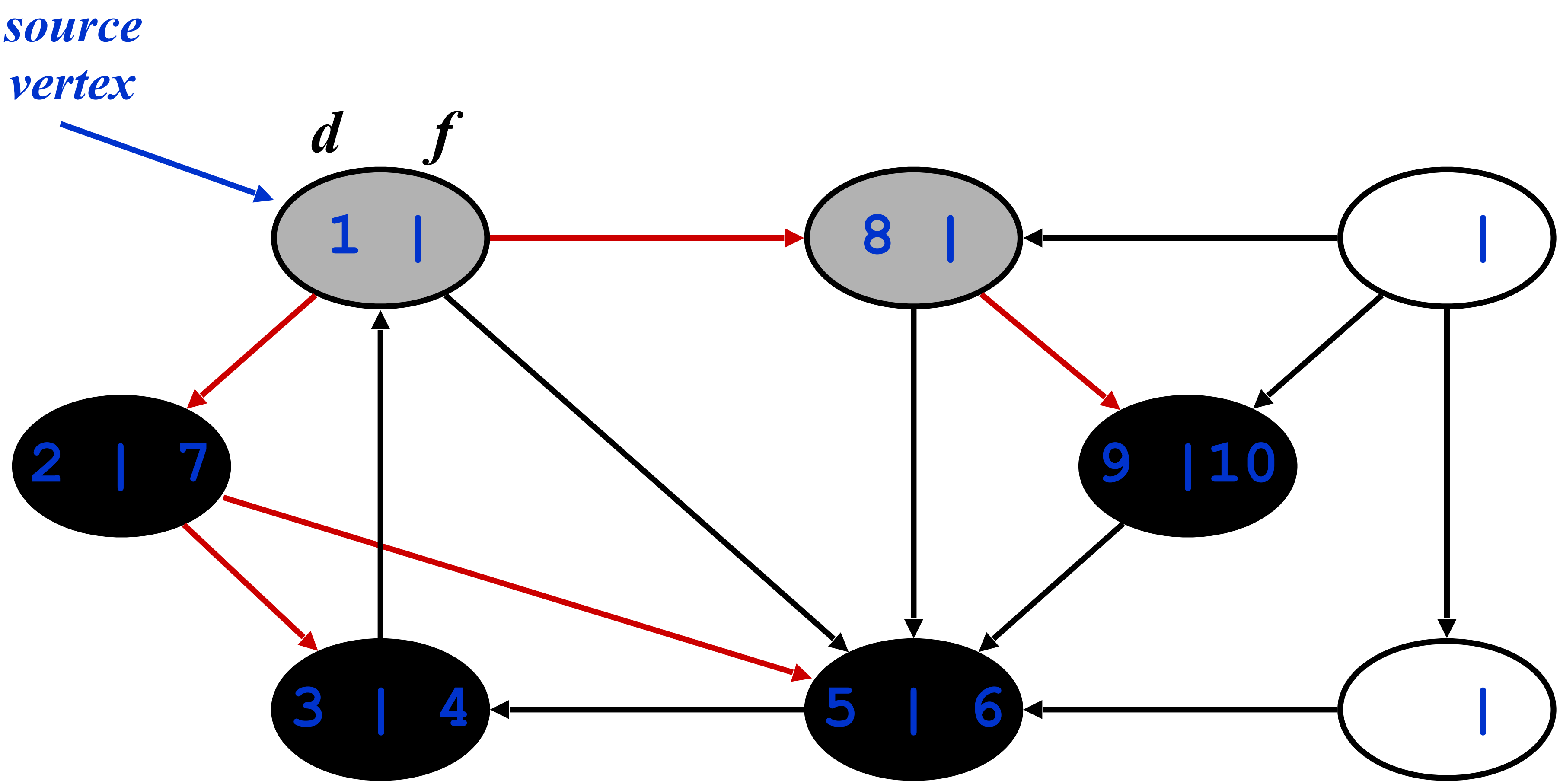
DFS Example



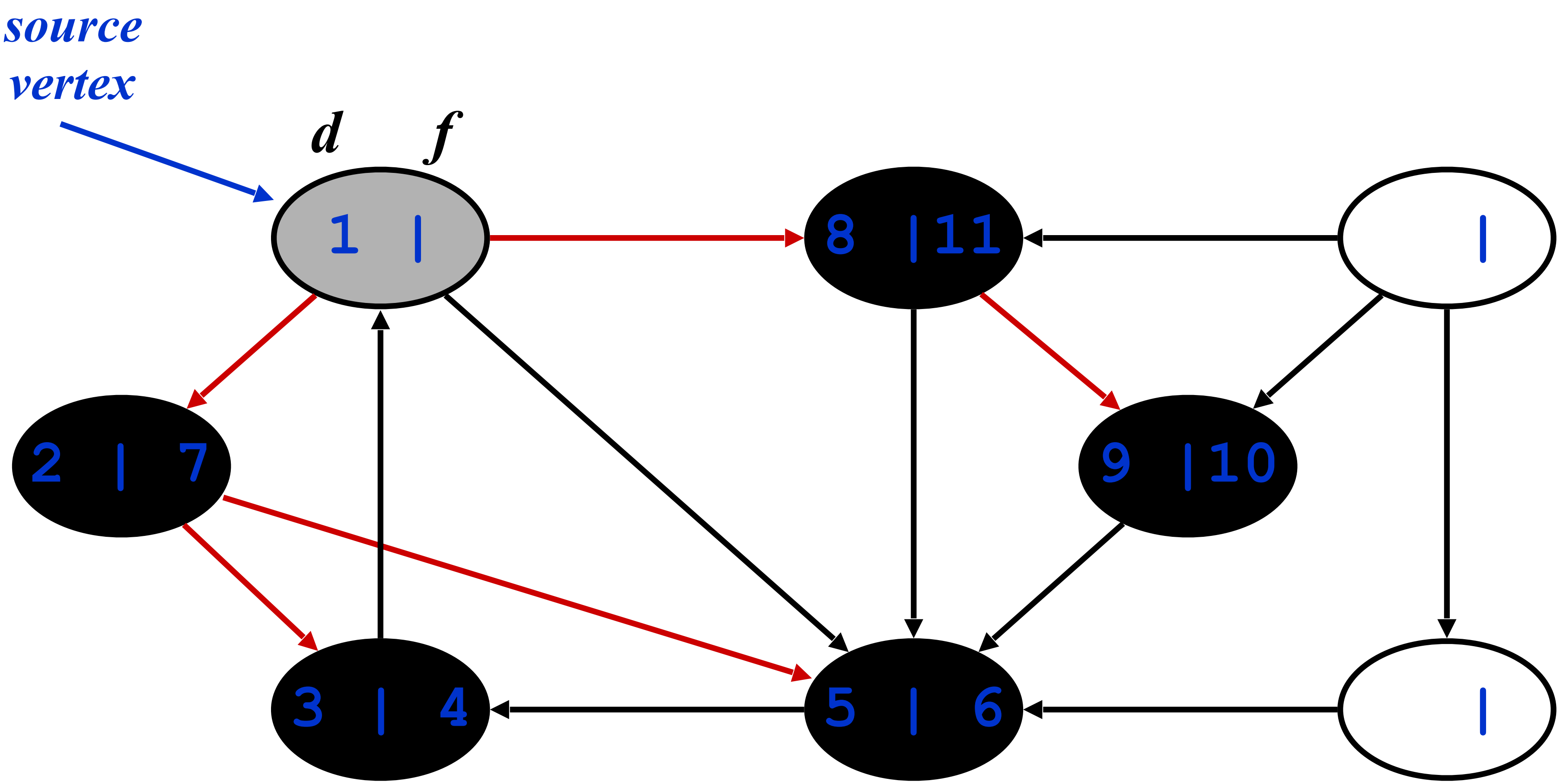
DFS Example



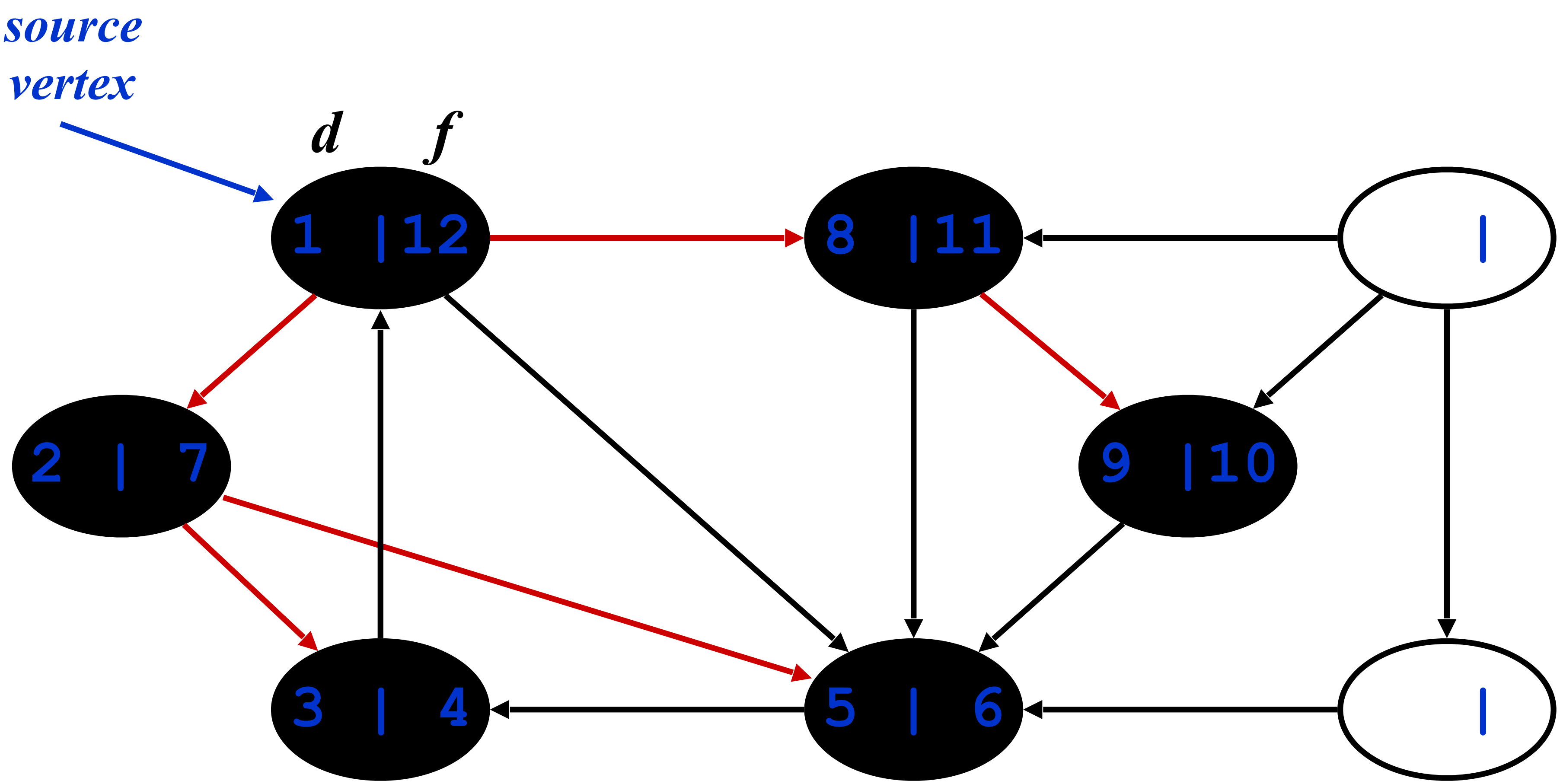
DFS Example



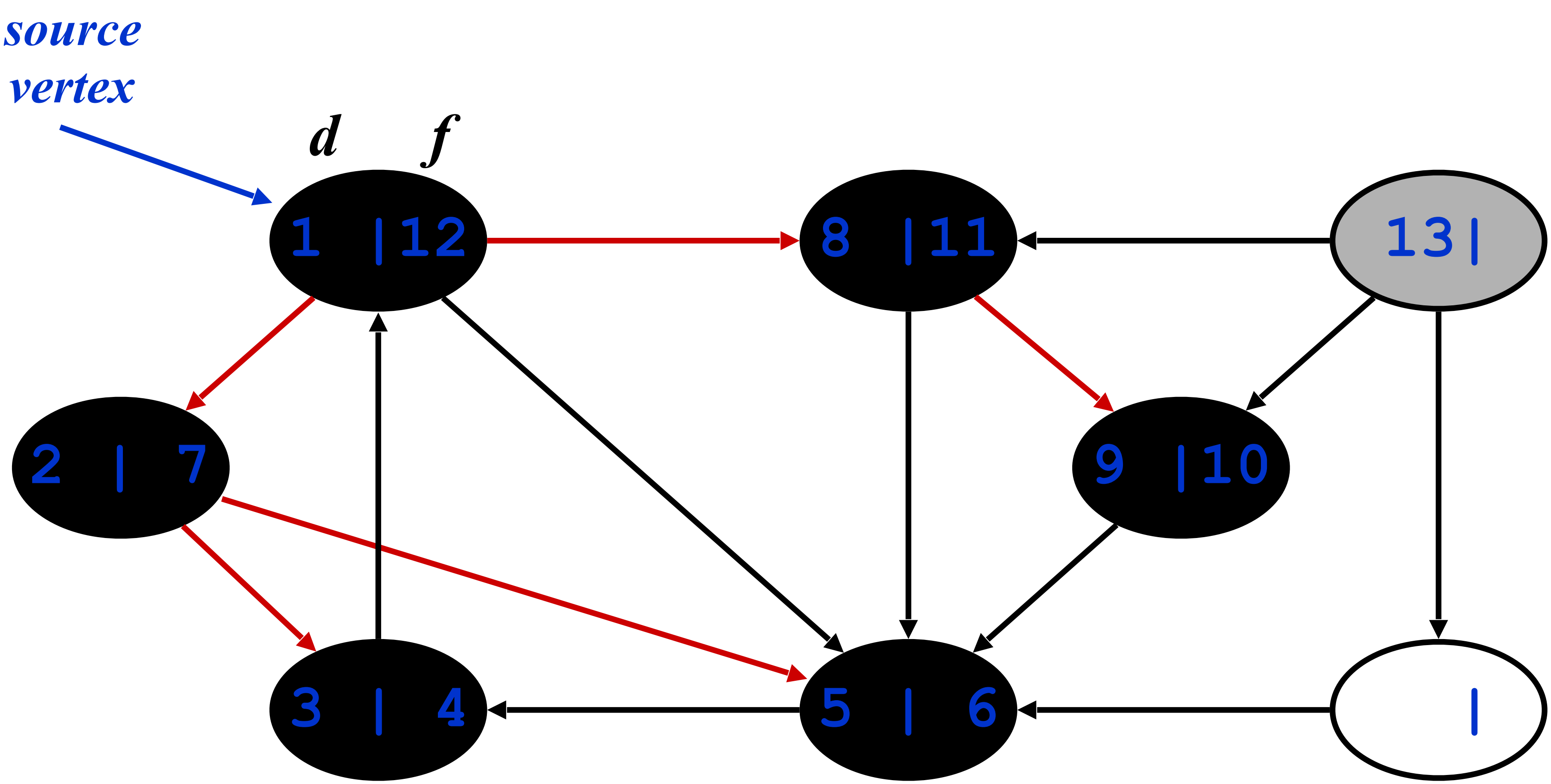
DFS Example



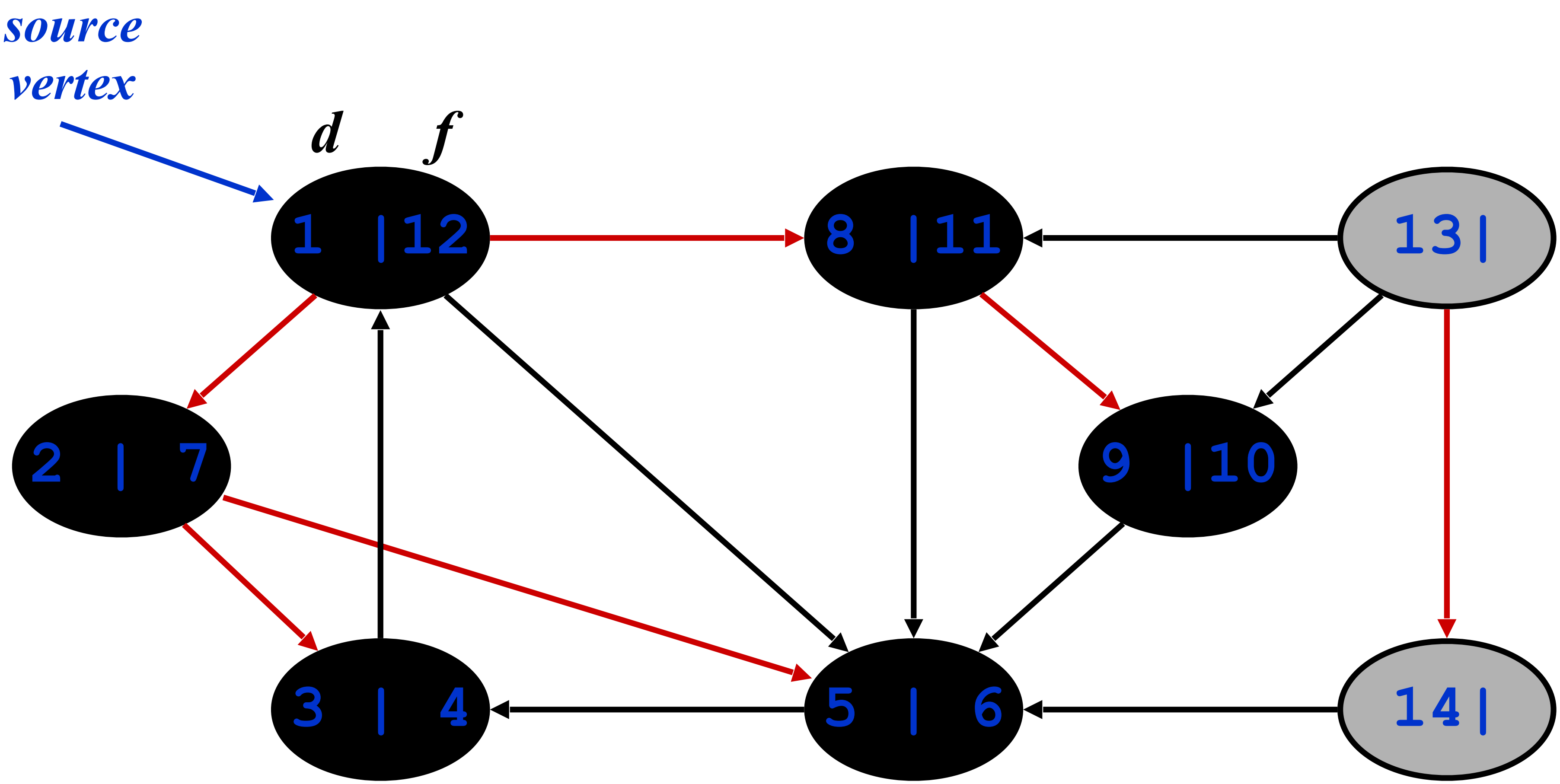
DFS Example



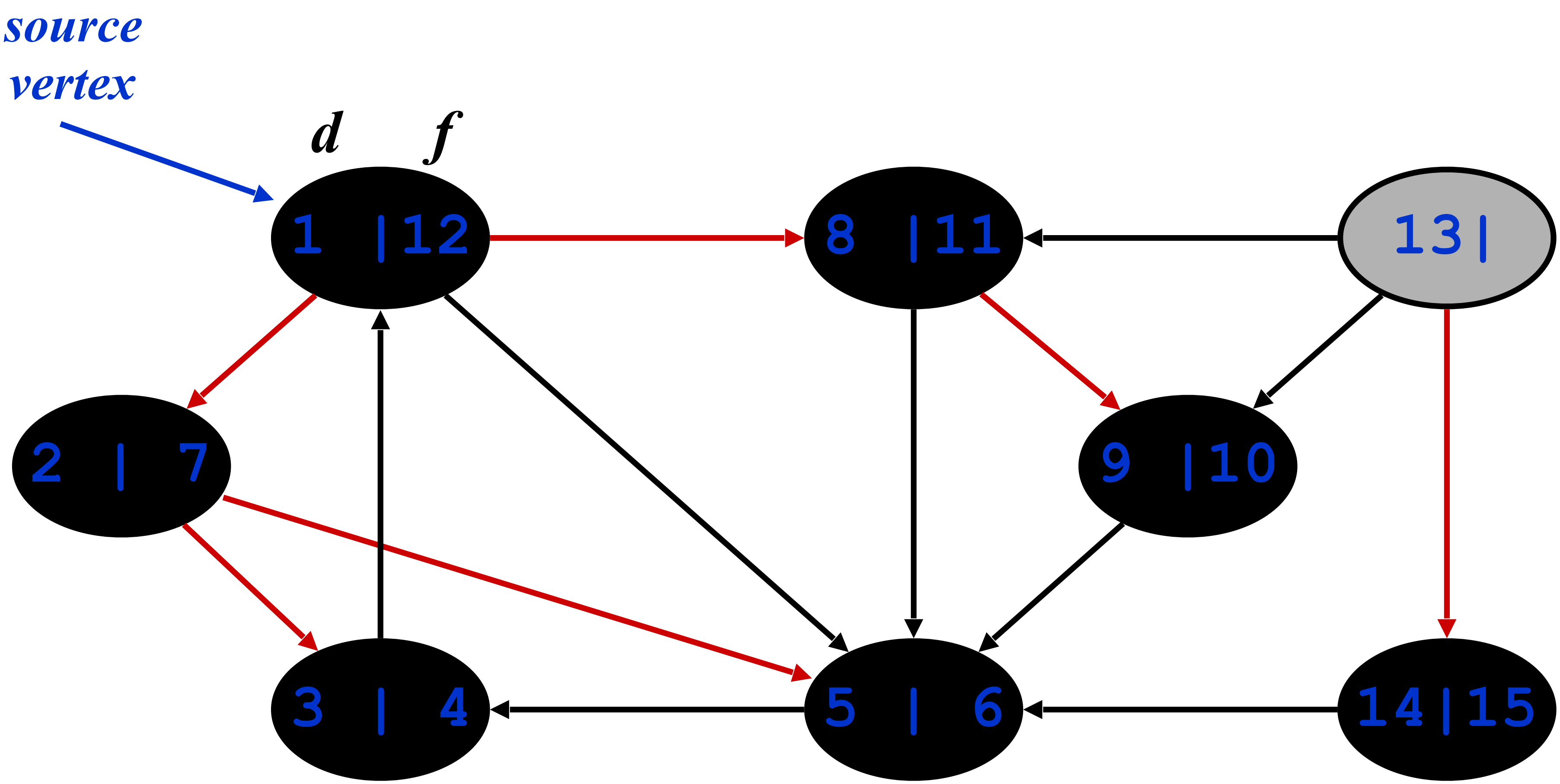
DFS Example



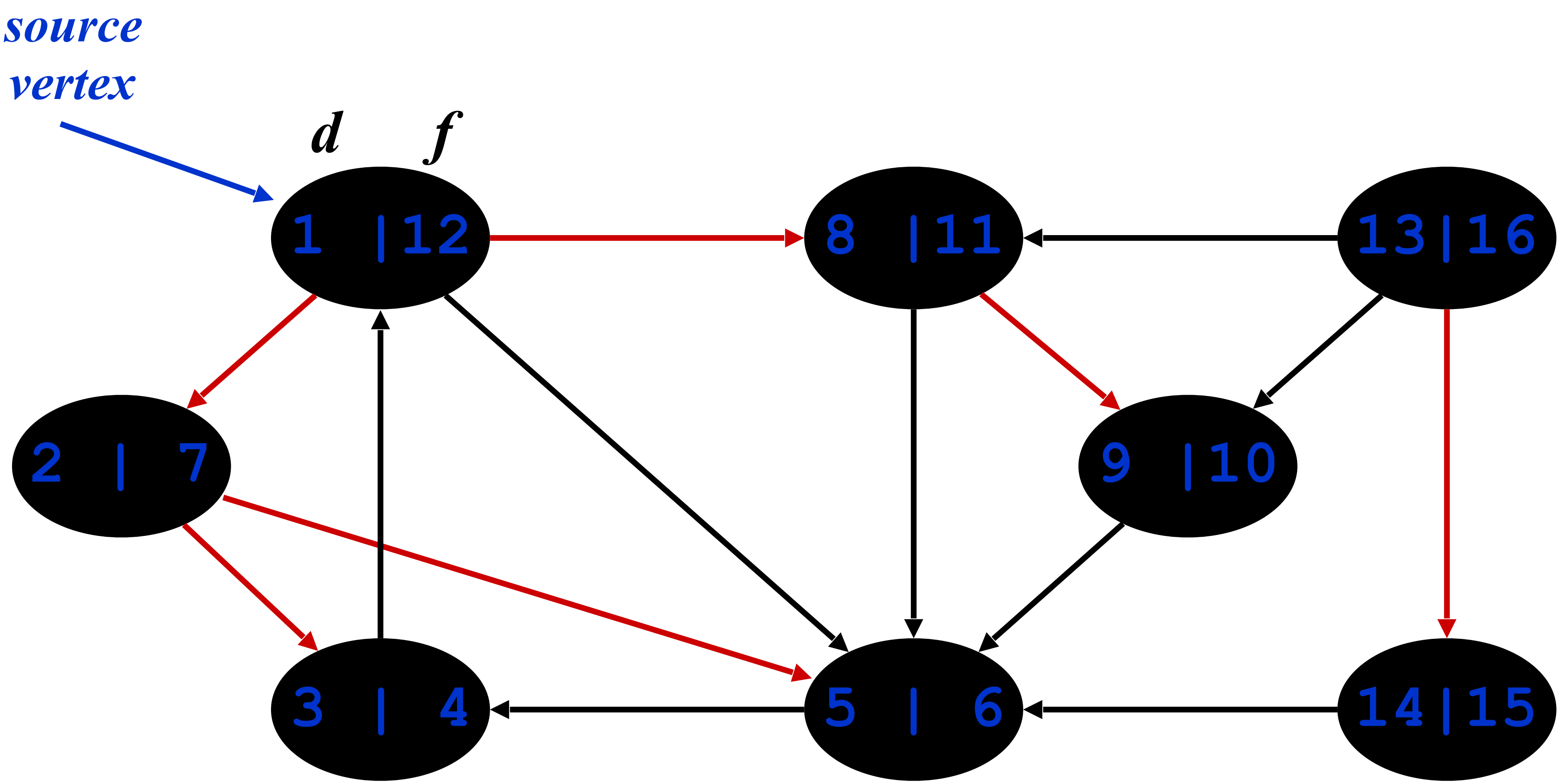
DFS Example



DFS Example



DFS Example

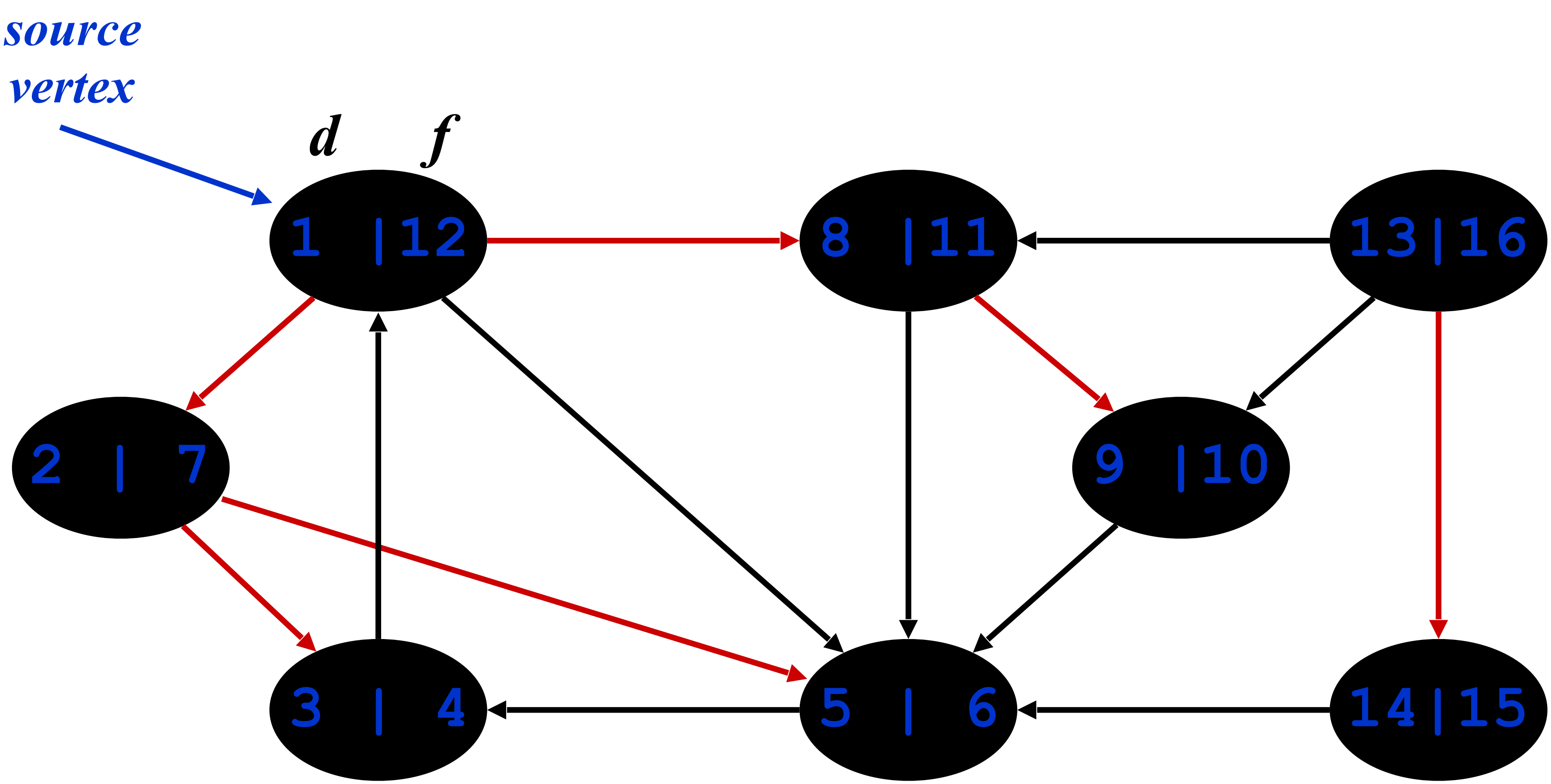


DFS: Kinds of Edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
    - The tree edges form a spanning forest, called depth-first forest consisting of depth-first trees
      - pred(v) is the parent of v in its depth-first tree

```
DFSVisit(u) {
    color[u] = gray;
    d[u] = ++time;
    for each v in Adj(u) do
        if (color[v] == white) {
            pred[v] = u;
            DFSVisit(v);
        }
    color[u] = black;
    f[u] = ++time;
}
```

DFS Example

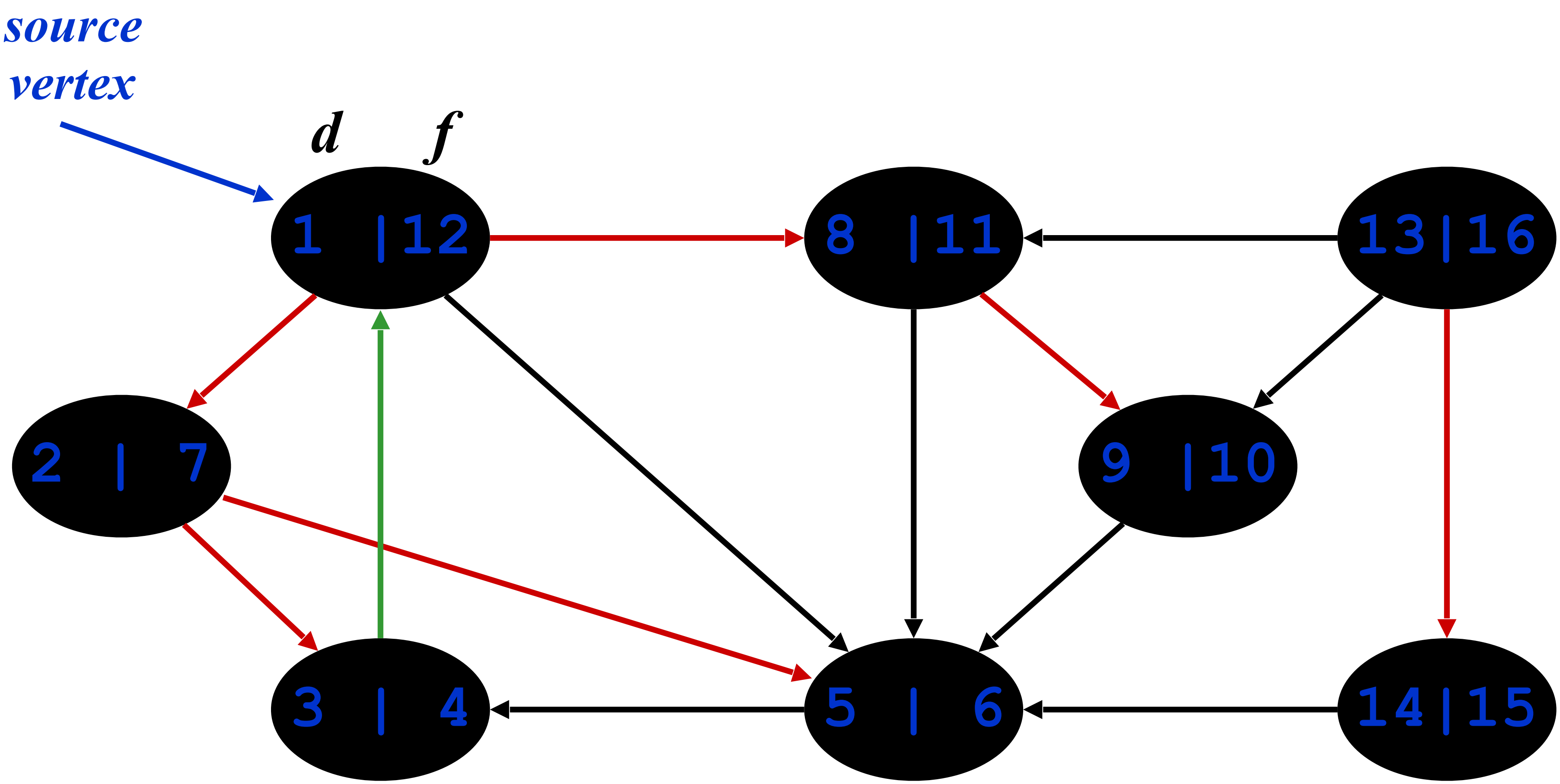


Tree edges

DFS: Kinds of Edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor (w.r. depth-first tree)
    - Encounter a grey vertex (grey to grey)

DFS Example

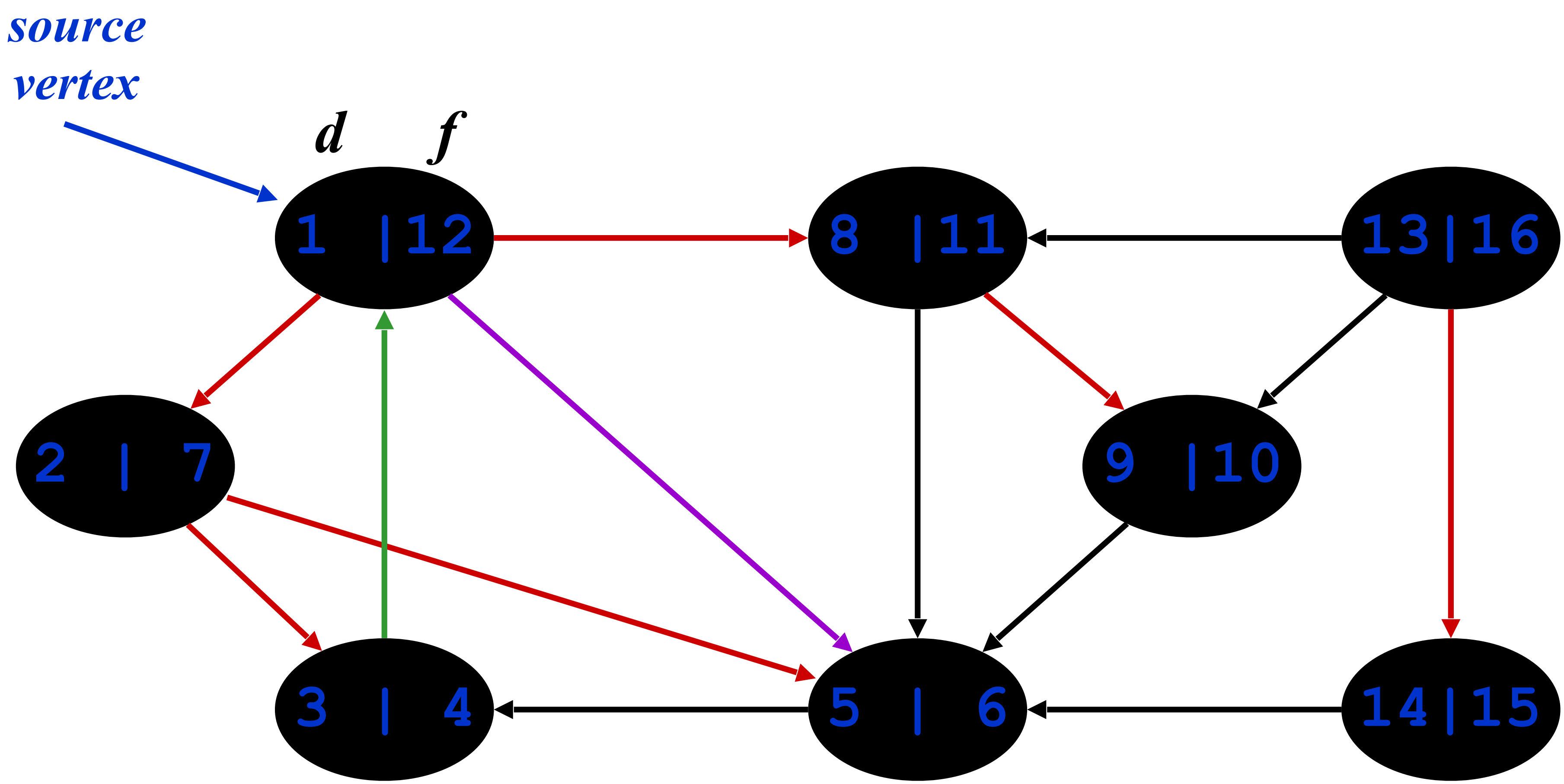


Tree edges *Back edges*

DFS: Kinds of Edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor (w.r. depth-first tree)
  - *Forward edge*: from ancestor to descendent (w.r. depth-first tree)
    - not a tree edge, though
    - from grey node to black node

DFS Example

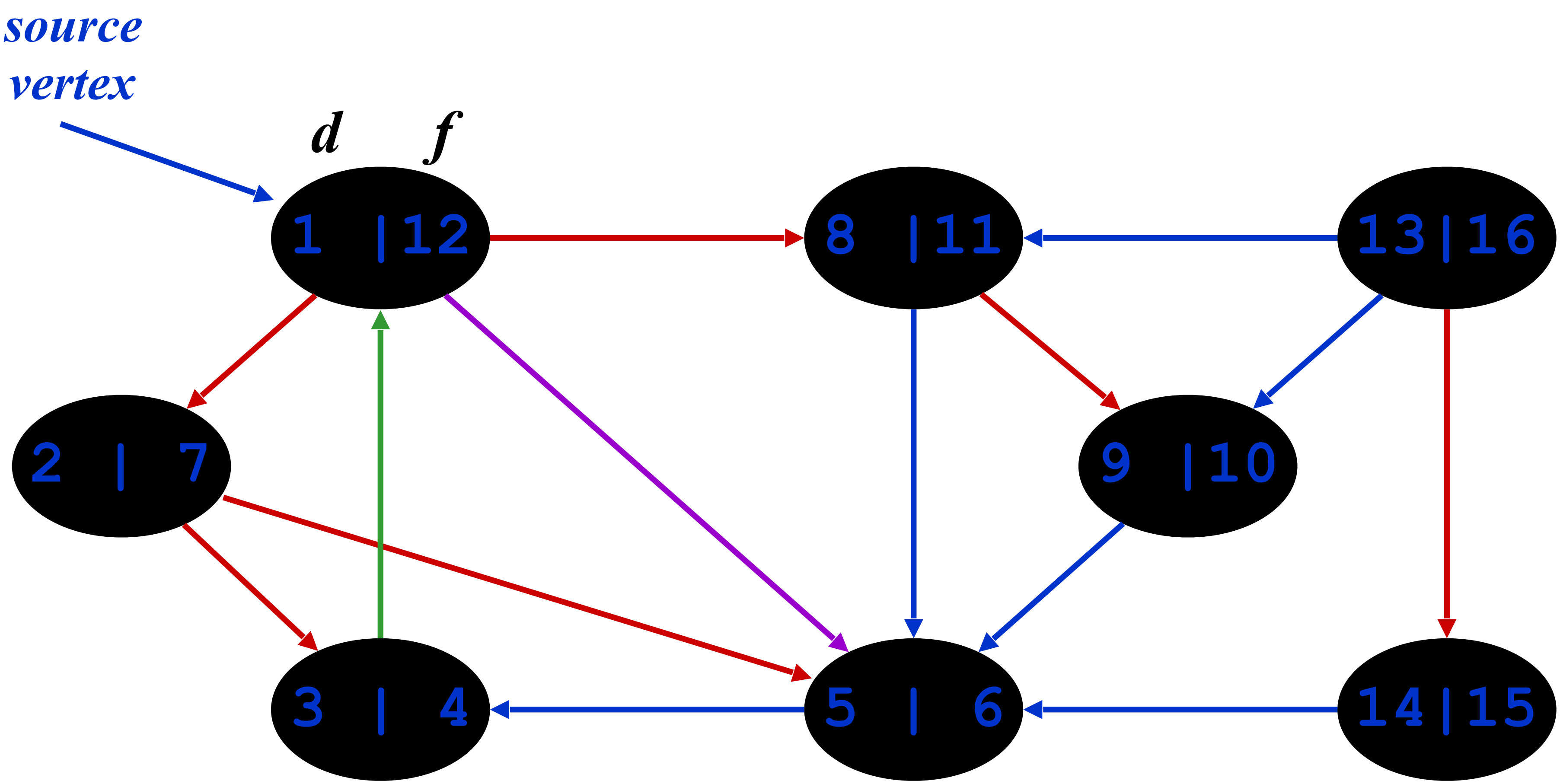


Tree edges *Back edges* *Forward edges*

DFS: Kinds of Edges

- DFS introduces an important distinction among edge (u,v) in the original graph:
  - *Tree edge*: encounter new (white) vertex, edge from parent to child in depth-first tree
    - v is white color when (u,v) is first explored
  - *Back edge*: from descendant to ancestor in depth-first tree
    - v is gray color when (u,v) is first explored
  - *Forward edge*: from ancestor to descendant in depth-first tree
    - v is black color when (u,v) is first explored
  - *Cross edge*: between two nodes w/o ancestor-descendant relation in a depth-first tree or two nodes in two different depth-first trees
    - v is black color when (u,v) is first explored
- Note: tree & back edges are important; most algorithms don't distinguish forward & cross

DFS Example



Tree edges *Back edges* *Forward edges* *Cross edges*

Parenthesis Theorem

Theorem: In any DFS of directed or undirected graph, for any two vertices u and v, one of the following three conditions holds:

1. intervals $[d(u), f(u)]$ and $[d(v), f(v)]$ are disjoint;
2. $[d(v), f(v)]$ entirely inside $[d(u), f(u)]$, i.e. $d(u) < d(v) < f(v) < f(u)$;
3. $[d(u), f(u)]$ entirely inside $[d(v), f(v)]$, i.e. $d(v) < d(u) < f(u) < f(v)$.

Proof: Without loss of generality (w.l.o.g), assume u is discovered first, $d(u)<d(v)$. There are two subclasses:

a. $d(v)<f(u)$: v was discovered while u was still gray => v is a descendant of u in DFS tree, search will return to u after all outgoing edge of v are explored, so that $f(v)<f(u)$, this is case 2 in the theorem.
b. $d(v)>f(u)$: v was discovered after u was fully explored, case 1 in theorem.

Corollary: Nesting of Descendant's Intervals
v is a proper descendant of u in DFS forest if and only if $d(u) < d(v) < f(v) < f(u)$

White-path Theorem

Theorem: Vertex v is a descendant of u in a DFS tree **if and only if** at time d(u) that u was discovered, vertex v can be reached from u along a path consisting entirely of white vertices.

Proof: **=> (only if),** let w be any vertex on the path between u and v in DFS tree, w is a descendant of u, then according to Parenthesis theorem, $d[w]>d[u]$, i.e., w was white at time d[u]. The path between u and v in DFS tree corresponds to a white path from u to v in the original graph.

<= (if) let p be the white path at time d(u), w.l.o.g., let w1 be the node which is the closest to u on p but not a descendant of u in DFS, let w2 be the predecessor of w1 on p, then w2 is a descendant of u in DFS, according to Nesting Corollary, $d(u)<d(w2)<f(w2)<f(u)$, since (w2,w1) is an edge in G, w1 will be discovered before w2 is finished, so $d(u)<d(w1)<f(w2)<f(u)$, according to Parenthesis theorem, $d(u)<d(w1)<f(w1)<f(u)$, w1 is a descendant of u in DFS according to Nesting Corollary.

DFS in Undirected Graph

- Theorem: In a depth-first search of an undirected graph G, every edge of G is either a tree edge or a back edge.
- Proof: for any (u,v), w.l.o.g, assume $d(u)<d(v)$, then $d(v)<f(v)<f(u)$. (white path theorem)
  1) if the first time (u,v) is processed, it is from u's adjacency list, then v must not have been discovered (v is white), then (u,v) is a tree edge.
  2) if the first time (u,v) is processed, it is from v's adjacency list, then u is still gray, then (u,v) is a back edge.

DFS & Graph Cycle: undirected

- Theorem: An undirected graph is *acyclic* iff a DFS yields no back edges
  - If acyclic, no back edges (because a back edge implies a cycle)
  - If no back edges, acyclic
    - No back edges implies only tree edges
    - Only tree edges implies we have a tree or a forest
    - Which by definition is acyclic

DFS & Graph Cycle: undirected

- *What will be the running time?*
- A: O(V+E)
- We can actually determine if cycles exist in O(V) time:
  - In an undirected acyclic forest, $|E| \leq |V| - 1$
  - So count the edges: if ever see $|V|$ distinct edges, must have seen a back edge along the way

DFS & Graph Cycle: directed

- Theorem: A directed graph is *acyclic* iff a DFS yields no back edges
- Proof: (sketch, details in book)
  => DFS produces a back edge (u,v), v is an ancestor of u in depth-first tree, then in G there is a path from v to u, then back edge (u,v) completes a cycle
  <= suppose G has a cycle c, let v be the first vertex to be discovered by DFS, u is the predecessor of v in cycle c, at time d(v), all vertices of c are white, form a white path from v to u, then u becomes a descendant of v in depth-first tree, (white path theorem), (u,v) is a back edge.