Mathematical Formulae

A Book of High School and Engineering Common Core Mathematical Formulae

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December, 2020

Contents

Part I

Algebra

Chapter 1

Logarithm

1.1 Basic Formulae

For $a^x = b$:

$$\log_a x, \text{ for all } x \leq 0 \text{ is undefined} \tag{1.1}$$

$$\log_a b = x, \ bax \neq 1, \ a \neq 1 \tag{1.2}$$

$$\log_b a^m = m \log_b a, \text{ for } a^m = b \tag{1.3}$$

$$a^{\log_a x} = x \tag{1.4}$$

$$a^{\log_b c} = c^{\log_b a} \tag{1.5}$$

$$\frac{1}{\log_a b} = \log_b a \tag{1.6}$$

$$\log_c(ab) = \log_c a + \log_c b \tag{1.7}$$

$$\log_c(\frac{a}{b}) = \log_c a - \log_c b \tag{1.8}$$

$$|\log_a x| = \begin{cases} -\log_a x, \ \text{if } \ 0 < x < 1 \\ \log_a x, \ \text{if } \ 1 \leq x < \infty \end{cases} \tag{1.9}$$

1.2 Series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\infty = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{x^i}{i!} \tag{1.10}$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\infty = \lim_{n\to\infty} \sum_{i=1}^{n} (-1)^{(i-1)} \frac{x^i}{i} \tag{1.11}$$

Chapter 2

Complex Number

2.1 Basic Formulae

For $z = x + iy$,

$$|z| = \sqrt{x^2 + y^2} \tag{2.1}$$

$$\tan\theta = \frac{y}{x} \tag{2.2}$$

$$\bar{z} = x - iy \tag{2.3}$$

2.2 Arithmetic Operation of Complex Number

For two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \tag{2.4}$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \tag{2.5}$$

$$|z_1 z_2| = |z_1||z_2| \tag{2.6}$$

$$\frac{z_1}{z_2} = \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{{a_2}^2 + {b_2}^2} \tag{2.7}$$

$$|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|} \tag{2.8}$$

2.3 Euler's Formula

$$z = re^{i\theta}, \text{ where } r = |z|,\ e^{i\theta} = \cos\theta + i\sin\theta, \text{ and } \theta = \tan^{-1}\frac{y}{x} \tag{2.9}$$

2.4 Trigonometric Ratios in Complex Form

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta \tag{2.10}$$

$$\Rightarrow \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{2.11}$$

$$e^{i\theta} - e^{-i\theta} = 2\sin\theta \tag{2.12}$$

$$\Rightarrow \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} \tag{2.13}$$

2.5 De Moivre's Formula

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \tag{2.14}$$

2.6 Application of Euler's and De Moivre's Formula

For $z_1 = |r_1|e^{i\theta_1}$ and $z_2 = |r_2|e^{i\theta_2}$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} \tag{2.15}$$

2.7 Roots of Unity

$$\sqrt[n]{1} = e^{i\frac{2k\pi}{n}}, \text{ where } k \in [0, n-1] \tag{2.16}$$

2.8 Important Relations of Complex Numbers

$$|z_1 + z_2| \leq |z_1| + |z_2| \tag{2.17}$$

$$|z_1 - z_2| \leq |z_1| + |z_2| \tag{2.18}$$

$$|z_1 - z_2| \geq |z_1| - |z_2| \tag{2.19}$$

$$|z_1 + z_2| \geq ||z_1| - |z_2|| \tag{2.20}$$

$$|z_1 + z_2|^2 = 2(|z_1|^2 + |z_2|^2) \tag{2.21}$$

Chapter 3

Progression

3.1 Arithmetic Progression (A.P.)

An arithmetic sequence is $a, a+n, a+2n, ...\infty$ or $t_n = a+(n-1)d$, where a is the first term, d is the common difference, and n is the n^{th}-term.

An arithmetic series is $a + (a+d) + (a+2d) + ...\infty$.

3.1.1 Sum of A.P. Series

$$\begin{aligned} S_n &= a + (a+d) + ... + (a + \overline{n-2}d) + (a + \overline{n-1}d) \\ S_n &= (a + \overline{n-1}d) + (a + \overline{n-2}d + ... + (a+d) + a \\ &\Rightarrow 2S_n = n(2a + \overline{n-1}d) \\ &\Rightarrow S_n = \frac{n}{2}(2a + \overline{n-1}d) \end{aligned} \tag{3.1}$$

3.1.2 Important Relation

If the three terms a, b, c are in A.P., then

$$2b = a + c \tag{3.2}$$

3.2 Geometric Progression (G.P.)

An geometric sequence is $a, ar, ar^2, ...\infty$ or $t_n = ar^{n-1}$, where a is the first term, r is the common ratio, and n is the n^{th}-term.

An geometric series is $a + ar + ar^2 + ...\infty$.

3.2.1 The Value of 'r'

$$r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = ... = \frac{t_n}{t_{n-1}} \tag{3.3}$$

3.2.2 Sum of a G.P. Series

For a definite G.P. series, where there are n terms in the series, the sum of the series is:

$$S_n = \frac{a|r^n - 1|}{|r - 1|} \tag{3.4}$$

For an infite G.P. series the sum of the series is defined for $r < 1$. Sum of such a series is:

$$S_\infty = \frac{a}{1 - r} \tag{3.5}$$

3.2.3 Important relations

If the three terms a, b, c are in G.P., then:

$$b^2 = ac \tag{3.6}$$

3.3 Harmonic Progression (H.P.)

If a, b, c are terms of an H.P. then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

$$\therefore \frac{2}{b} = \frac{1}{a} + \frac{1}{c} \tag{3.7}$$

$$\Rightarrow b = \frac{2ac}{a + c} \tag{3.8}$$

3.4 Arithmetico-Geometric Progression (A.G.P.)

Sequence $a, (a + d)r, (a + 2d)r^2, ..., (a + \overline{n - 1}d)r^{n-1}$, where $a \rightarrow$first term of A.G.P., $d \rightarrow$common difference, and $r \rightarrow$common ratio.

3.4.1 Sum of A.G.P.:

For an infinite A.G.P. series, the sum is defined for $r < 1$:

$$S_\infty = \frac{a}{1 - r} + \frac{dr}{(1 - r)^2} \tag{3.9}$$

3.5 Special Series

For $n \in \mathbb{N}$

$$1 + 2 + 3 + ... + (n-1) + n = \frac{n(n-1)}{2} \tag{3.10}$$

$$1^2 + 2^2 + 3^2 + ... + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{3.11}$$

$$1^3 + 2^3 + 3^3 + ... + (n-1)^3 + n^3 = [\frac{n(n-1)}{2}]^2 \tag{3.12}$$

3.5.1 Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{3.13}$$

3.5.2 Riemann's Infinite Series as an Integration

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=r_1}^{r_2} f(\frac{i}{n}) = \int_{\frac{r_1}{n}}^{\frac{r_2}{n}} f(x)dx \tag{3.14}$$

Chapter 4

Test of Convergence of Infinite Series

If $a_1, a_2, a_3, ..., a_n$ is a sequence by a_n and their sum of series is S_n, then the following apply.

4.1 Definition

If

$$\lim_{n \to \infty} S_n = l$$

where l is a finite value, the series S_n is said to converge. A non-convergent series is called a divergent series.

4.2 Tests of Convergence

4.2.1 Comparison Test

If u_n and v_n are two positive series, then:

1. (a) v_n converges
   (b) $u_n \leq v_n \forall n$ Then u_n converges.
2. (a) v_n diverges
   (b) $u_n \geq v_n \forall n$ Then u_n diverges.

4.2.2 Limit Form

If

$$\lim_{x \to \infty} \frac{u_n}{v_n} = l$$

where l is a finite quantity $\neq 0$, then u_n and v_n converge and diverge together.

4.2.3 Integral Test or Maclaurin-Cauchy Test

For a series

$$\sum_{i=N}^{\infty} f(x), \text{ where } N \in \mathbb{Z} \tag{4.1}$$

will only converge if the improper integral

$$\int_{N}^{\infty} f(x)dx \tag{4.2}$$

is finite.

If the improper integral is finite, the upper and lower limit of the infinite series is given by:

$$\int_{N}^{\infty} f(x)dx \leq \sum_{i=N}^{\infty} f(x) \leq f(N) + \int_{N}^{\infty} f(x)dx \tag{4.3}$$

4.2.4 Ratio Test

If, for two series $\sum u_n$ and $\sum v_n$:

1. (a) $\sum v_n$ converges
   (b) from or after a particular term $\dfrac{u_n}{u_{n+1}} > \dfrac{v_n}{v_{n+1}}$, then u_n converges.
2. (a) $\sum v_n$ diverges
   (b) from or after a particular term $\dfrac{u_n}{u_{n+1}} < \dfrac{v_n}{v_{n+1}}$, then u_n diverges.

4.2.5 D'Alembert's Ratio Test

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lambda \tag{4.4}$$

- series converges if $\lambda < 1$
- series diverges if $\lambda > 1$
- fails if $\lambda = 1$

4.2.6 Rabbe's Test

$$\lim_{n\to\infty} n[\frac{u_n}{u_{n+1}} - 1] = \kappa \tag{4.5}$$

- series converges if $\kappa < 1$
- series diverges if $\kappa > 1$
- fails if $\kappa = 1$

4.2.7 Cauchy's Root Test

$$\lim_{n\to\infty} |u_n| = \lambda \tag{4.6}$$

- series converges for $\lambda < 1$
- series diverges for $\lambda > 1$
- test fails for $\lambda = 1$

4.2.8 Logarithmic Test

$$\lim_{n\to\infty} n \log(\frac{u_n}{u_{n+1}}) = \kappa \tag{4.7}$$

- series converges for $\kappa < 1$
- series diverges for $\kappa > 1$
- test fails for $\kappa = 1$

Chapter 5

Determinants

5.1 Definition

For a determinant:

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \tag{5.1}$$

5.1.1 Minor and Cofactor

For a third order determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, the minor of a_{11} is $M_{11} = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, i.e., all the terms of the determinant expect those in the same row and columns as the one of which the minor is being calculated.

Cofactor $C_{ij} = (-1)^{i+j} M_{ij}$

5.2 Important Properties

1. Transposing a determinant does not alter its value.
2. If rows and columns are interchanges m times, the value of the new determinant is
$$\Delta' = (-1)^m \Delta \tag{5.2}$$
3. If two parallel lines are equal, then $\Delta = 0$

4. For $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then $\Delta_1 = k\Delta$

5. For $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{vmatrix}$, then $\Delta_1 = k\Delta$

6. For $C_n \to k_1C_l + k_2C_m + k_3C_n$ or $R_n \to k_1R_l + k_2R_m + k_3R_n$, $\Delta' = \Delta$

5.3 Cramer's Rule

For a system of equations:

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

the following determinants are defined:

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

The solution of the system of equations is:

$$x = \frac{D_x}{D} \tag{5.3}$$

$$y = \frac{D_y}{D} \tag{5.4}$$

$$z = \frac{D_z}{D} \tag{5.5}$$

5.3.1 Consistency Test

1. If $D \neq 0$, the system is consistent and has unique solutions.
2. If $D = D_x = D_y = D_z = 0$, the system may or may not be consisten and it will have infinite solutions and the system will be dependent.
3. If $D = 0$ and at least one of D_x, D_y, D_z is non zero, the system is inconsistent

Chapter 6

Matrices

For a matrix,

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

and where I_p is an identity matrix of the p^{th} order, the following relations are applicable.

6.1 Sum of Two Matrices

$$A_{m\times n} + B_{m\times n} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{21} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix} \tag{6.1}$$

6.2 Multiplication of Two Matrices

If

$$C_{m\times p} = A_{m\times n} \cdot B_{n\times p}$$

then,

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk} \tag{6.2}$$

6.2.1 Multiplicative Properties

1. Multiplication of matrices is associative, hence $(AB)C = A(BC)$.

2. $AI = A$
3. $A \cdot A^{-1} = I$
4. $A \cdot (adjA) = (adjA) \cdot A = |A|I$
5. $A^{-1} = \frac{1}{|A|}(adjA)^t$
6. $(AB)^t = B^t A^t$

6.3 Adjoint of a Matrix

$$adjA = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m1} & M_{m2} & \cdots & M_{mn} \end{bmatrix}^t, \text{ where } M_{ij} \text{ is the minor of } a_{ij} \tag{6.3}$$

6.4 Martin's Rule

For a system of equation,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n \end{aligned}$$

The system can be written as:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \tag{6.4}$$

$$\Rightarrow AX = B \tag{6.5}$$

$$\Rightarrow X = A^{-1}B \tag{6.6}$$

Chapter 7

Binomial Theorem

For a binomial expansion $(a+b)^n$, there are $(n+1)$ terms and $(a+b+c)^n$ has $\dfrac{(n+1)(n+2)}{2}$ terms.

7.1 Expansion of a binomial expression

$$\begin{aligned}(a+b)^n =& {}^nC_0 a^n b^0 + {}^n C_1 a^{n-1} b^1 + {}^n C_2 a^{n-2} b^2 + \\ & \cdots + {}^n C_n a^0 b^n \ \forall n \in \mathbb{N} \\ =& \sum_{i=0}^{n} {}^nC_i a^{n-i} b^i \ \forall n \in \mathbb{N}\end{aligned} \tag{7.1}$$

$$\begin{aligned}(a+b)^n =& a^n b^0 + n a^{n-1} b + \frac{n(n-1)}{2!} a^{n-2} b^2 \\ & + \cdots + \frac{n(n-1)\cdots 3 \cdot 2 \cdot 1}{n!} a^0 b^n + \cdots \infty \ \forall n \in \mathbb{R}\end{aligned} \tag{7.2}$$

7.2 Trinomial Expansion

For $(a+b+c)^n$:

$$\begin{aligned}(a+b+c+)^n = \sum \frac{n!}{i!j!k!} a^i b^j c^k \\ \forall \ (i+j+k) = n;\ i,j,k,n \in \mathbb{N}\end{aligned} \tag{7.3}$$

7.3 Properties of Coefficients

$$\text{Sum of Co-efficients: } C_0 + C_1 + C_2 + \cdots + C_{n-1} + C_n = 2^n \tag{7.4}$$

$$\text{Sum of Odd Co-efficients: } C_0 + C_2 + C_4 + \cdots + C_{2n-3} + C_{2n-1} = 2^{n-1} \tag{7.5}$$

$$C_0 - C_1 + C_2 - \cdots + C_{2n-1} - C_{2n} = 0 \tag{7.6}$$

7.4 Pascal's Rule

For $1 \leq k \leq n$ and $k, n \in \mathbb{N}$:

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k} \tag{7.7}$$

Chapter 8

Boolean Algebra

Let B be a set of a, b, c with operations sum $(+)$ and product $(\cdot)$.
Then B is said to belong to the Boolean Structure if the following conditions are satisfied:

Property	Name of Property
$a + b \in B$ $a \cdot b \in B$	Closure Property
$a + b = b + a$ $a \cdot b = b \cdot a$	Associative Law
$a(b + c) = ab + ac$ $a + bc = (a + b)(a + c)$	Commutative Law
$\{0, 1\} \in B$ $a + 0 = a$ $a + 1 = 1$ $a \cdot 0 = 0$ $a \cdot 1 = a$	Laws of 1 and 0
$a + ab = a$ $a(a + b) = a$	Absorption Law
$(a + b)' = (a'b')$	De'Morgan's Law

Table 8.1: Properties of Boolean Algebraic Structure

Chapter 9

Remainder Theorems

9.1 Remainder Theorem

If a function $f(x)$ is divided by a binomial $x - a$, then the remainder is provided by $f(a)$.

$$\frac{f(x)}{x-a} \equiv f(a) \mod (x-a) \tag{9.1}$$

Worked Example

Find the remainder when $f(x) = x^3 - 4x^2 - 7x + 10$ is divided by $(x - 2)$.

The remainder:

$$R = \left(x^3 - 4x^2 - 7x + 10\right) \mod (x-2)$$

is given by:

$$\begin{aligned} R = f(2) &= (2)^3 - 4(2)^2 - 7(2) + 10 \\ &= 8 - 16 - 14 + 10 = -12 \end{aligned}$$

9.2 Euler's Remainder Theorem

According to Euler's Remainder Theorem, if x and n are two co-prime numbers:

$$x^{\varphi(n)} \equiv 1 \mod n, x, n \in \mathbb{Z}^+ \tag{9.2}$$

where, $\varphi(n)$ is Euler's totient function.

9.2.1 Euler's Totient Function

For a number defined as:

$$n = \prod_{i=1}^{r} {a_r}^{b_r} \tag{9.3}$$

then Euler's totient function is defined as:

$$\begin{aligned}\varphi(n) &= n \cdot \left[\left(1 - \frac{1}{a_1}\right) \cdot \left(1 - \frac{1}{a_2}\right) \cdot \left(1 - \frac{1}{a_3}\right) \cdots\right] \\ &= n \prod_{i=1}^{r} \left(1 - \frac{1}{a_r}\right) \end{aligned} \tag{9.4}$$

Worked Example

Find the remainder if 3^{76} is divided by 35.

Since:

$$35 = 5^1 \times 7^1$$

Hence the totient quotient of 35 is:

$$\begin{aligned}\varphi(35) &= 35 \cdot \left(1 - \frac{1}{5}\right) \cdot \left(1 - \frac{1}{7}\right) \\ &= 35 \times \frac{4}{5} \times \frac{6}{7} \\ &= 24\end{aligned}$$

Hence Euler's Theorem yields:

$$\begin{aligned}3^{24} &\equiv 1 \mod 35 \\ 3^{76} &\equiv 3^{24 \times 3 + 4} \\ &\equiv \left(3^{24}\right)^3 \times 3^4 \mod 35 \\ &\equiv (1)^3 \times 3^4 \mod 35 \\ &\equiv 81 \mod 35 \\ &\equiv 11 \mod 35\end{aligned}$$

The remainder when 3^{76} is divided by 35 is 11.

9.3 Wilson Theorem

According to Wilson Theorem:

$$(n-1)! \equiv -1 \mod n \tag{9.5}$$

Worked Example

Find the remainder when 28! is divided by 31.

By Wilson's Theorem:

$$\begin{aligned}
30! &\equiv -1 \mod 31 \\
\Rightarrow 30 \cdot 29 \cdot 28! &\equiv -1 \mod 31 \\
\text{Let } 28! \mod 31 &= x \\
\Rightarrow (-1) \cdot (-2) \cdot x &\equiv 30 \mod 31 \\
\Rightarrow 2x &= 30 \\
\Rightarrow x &= 15
\end{aligned}$$

The remainder when 28! is divided by 31 is 15.

Part II

Co-Ordinate Geometry

Chapter 10

2-D Co-ordinate Geometry

For the ordered pairs, $A(x_1, y_1)$ and $B(x_2, y_2)$:

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \tag{10.1}$$

$$\text{Mid point of AB} = (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}) \tag{10.2}$$

$$\text{Point C, which divides AB in the ratio } m : n = (\frac{nx_1 + mx_2}{m + n}, \frac{ny_1 + my_2}{m + n}) \tag{10.3}$$

Chapter 11

Triangles

For a triangle defined with three vertices $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ and corresponding sides of length a, b, c, then:

$$\text{Centroid of } \triangle ABC = (\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}) \tag{11.1}$$

$$\text{Area of } \triangle ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \tag{11.2}$$

For a triangle, the semiperimeter, s, is defined as:

$$s = \frac{a + b + c}{2}$$

Then the radius, r, and centre of incircle, o, is:

$$o = \left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right) \tag{11.3}$$

$$r = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}} \tag{11.4}$$

The radius, R, and centre, O, of circumcircle is defined as:

$$O = \left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right) \tag{11.5}$$

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \tag{11.6}$$

Chapter 12

Straight Line

A straight line can be defined as:

$$y = mx + c \tag{12.1}$$

$$\frac{x}{a} + \frac{y}{b} = 1, \text{ where } a \text{ and } b \text{ are the intercepts at x and y axes respectively} \tag{12.2}$$

$$x \cos\alpha + y \sin\alpha = p \text{ (Normal Form)} \tag{12.3}$$

$$Ax + By + C = 0 \text{ (General Form)} \tag{12.4}$$

Equation of Straight Line Passing Through (x_0, y_0) and Slope m

$$(y - y_0) = m(x - x_0) \tag{12.5}$$

Distance Between Two Points on a Line

$$\frac{y_1 - y_2}{\sin\theta} = \frac{x_1 - x_2}{\cos\theta} = \gamma \tag{12.6}$$

$$\theta = \tan^{-1} m \tag{12.7}$$

Angle Between Two Lines

For two lines with slopes m_1, m_2, the angle between them, θ:

$$\theta = \arctan\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right) \tag{12.8}$$

Distance of a Point from a Line
Line: $ax + by + c = 0$ Point: (g, h)

$$S = \frac{ag + bh + c}{\sqrt{a^2 + b^2}} \tag{12.9}$$

Angle Bisector of a Line For the two lines: $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, the angle bisector is:

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \tag{12.10}$$

If the sign of c_1 and c_2 is the same, then the equation obtained is the internal bisector.

Equation of a Straight Line Passing through the Intersection of Two Lines

$$(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0 \ \forall k \in \mathbb{R} \tag{12.11}$$

Relative Position of Points w.r.t. a Line For the points (x_1, y_1) and (x_2, y_2):

$$k_1 = ax_1 + by_1 + c$$
$$k_2 = ax_2 + by_2 + c$$

If k_1 and k_2 have the same sign, they are on the same side of a line, otherwise on opposite sides.

Ratio of Division of Line Segment For any line, $f(x, y) = 0$, the ratio in which it divides (x_1, y_1) and (x_2, y_2) is given by:

$$r = -\frac{f(x_1, y_1)}{f(x_2, y_2)} \tag{12.12}$$

If $\begin{cases} r > 0, \text{ then division is internal} \\ r < 0, \text{ then division is external} \end{cases}$.

Chapter 13

General Theory of Second Degree Equation

For any general equation of the form:

$$ax^2 + by^2 + 2gx + 2fy + 2hxy + c = 0 \tag{13.1}$$

Δ is defined as:

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \tag{13.2}$$

If $\Delta = 0$ then the equation is a pair of straight lines. If $a + b = 0$ then the lines are $\perp$.
If the $\Delta \neq 0$:

1. $a = b, h = 0$ $\rightarrow$circle

2. $h^2 = ab$ $\rightarrow$parabola

3. $h^2 < ab$ $\rightarrow$ellipse

4. $h^2 > ab$ $\rightarrow$hyperbola

Chapter 14

Conics

The four conic sections are: circle, parabola, ellipse, and hyperbola. Circle has been done separately in the next chapter.

14.1 Parabola

Property	$y^2 = 4ax$	$x^2 = 4ay$
Axis	$y = 0$	$x = 0$
Eccentricity	1	1
Directrix	$x + a = 0$	$y + a = 0$
Focus	$(a, 0)$	$(0, a)$
Vertex	$(0, 0)$	$(0, 0)$
Length of latus rectum	$\lvert 4a \rvert$	$\lvert 4a \rvert$
Equation of latus rectum	$x - a = 0$	$y - a = 0$

Table 14.1: Properties of a Parabola

14.2 Ellipse and Hyperbola

For $a > b$:

Property	$\frac{x^2}{a} + \frac{y^2}{b} = 1$ Ellipse	$\frac{x^2}{a} - \frac{y^2}{b} = 1$ Hyperbola
Length of Major Axis	$2a$	$2a$
Length of Minor Axis	$2b$	$2b$
Equation of Major Axis	$x = 0$	$x = 0$
Equation of Minor Axis	$y = 0$	$y = 0$
Eccentricity e	$\sqrt{1 - \frac{b^2}{a^2}}$	$\sqrt{1 + \frac{b^2}{a^2}}$
Vertices	$(\pm a, 0)$	$(\pm a, 0)$
Foci	$(\pm ae, 0)$	$(\pm ae, 0)$
Equation of Directrix	$x \pm \frac{a}{e} = 0$	$x = \pm\frac{a}{e}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Equation of latus rectum	$x \pm ae = 0$	
Centre	$(0, 0)$	$(0, 0)$

Table 14.2: Properties of Ellipse and Hyperbola

14.3 Parametric Form of Conics

14.3.1 Hyperbola

$$x = a\sec\theta \tag{14.1}$$

$$y = b = \tan\theta \tag{14.2}$$

14.3.2 Ellipse

$$x = a\cos\phi \tag{14.3}$$

$$y = b\sin\phi \tag{14.4}$$

14.3.3 Parabola

$$x = at^2 \tag{14.5}$$

$$y = 2at \tag{14.6}$$

Chapter 15

Circles

15.1 Locus Form

$$(x-g)^2 + (y-h)^2 = r^2 \tag{15.1}$$

where the centre is (g, h) and the radius is r.

15.2 Diameter Form

$$(x-a)(x-c) + (y-b)(y-d) = 0 \tag{15.2}$$

where (a, b) and (c, d) are the two ends of the diamter.

15.3 General Form

If the equation of a circle is in the form:

$$x^2 + y^2 + 2gx + 2fy + c = 0 \tag{15.3}$$

Then the following is true about the circle:

1. centre of the circle is $(-g, -f)$
2. radius of circle is $\sqrt{g^2 + f^2 - c}$

15.4 Important Relations

1. If the circle passes through the origin, $g = 0, f = 0$.
2. If the circle touches the x-axis $c = g^2$.
3. If the circle touches the y-axis $c = f^2$.

Common for Two Circles

1. The common chord passing between two circles S_1 and S_2 are:

$$S_1 - S_2 = 0 \tag{15.4}$$

2. Circles passing through the intersection of two circles is:

$$S_2 + k(S_1 - S_2) = 0 \ \forall k \in \mathbb{R} \tag{15.5}$$

Chapter 16

Vectors

Let two vectors be $\vec{a} = a\hat{i} + b\hat{j} + c\hat{k}$ and $\vec{b} = x\hat{i} + y\hat{j} + z\hat{k}$:

16.1 Modulus of a Vector

For a vector $\vec{a}$, the modulus of the vector is:

$$|\vec{a}| = \sqrt{a^2 + b^2 + c^2} \tag{16.1}$$

16.2 Sum of Vectors

The sum of two vectors is:

$$|\vec{a} + \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{a}|\cos\theta} \tag{16.2}$$

$$\vec{a} + \vec{b} = (a + x)\hat{i} + (b + y)\hat{j} + (c + z)\hat{k} \tag{16.3}$$

The direction of the resultant vector is:

$$\tan\alpha = \frac{b\sin\theta}{a + b\cos\theta} \tag{16.4}$$

where, θ is the angle between the two vectors.

16.3 Product of Vectors

16.3.1 Dot Product

$$\vec{a} \cdot \vec{b} = |a||b|\cos\theta \tag{16.5}$$

$$\vec{a} \cdot \vec{b} = ax + by + cz \tag{16.6}$$

16.3.2 Cross Product

$$\vec{a} \times \vec{b} = |a||b| \sin \theta \hat{n} \tag{16.7}$$

where $\hat{n}$ is a vector $\perp$ $\vec{a}, \vec{b}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ x & y & z \end{vmatrix} \tag{16.8}$$

16.4 Test of Co-planarity

Three vectors are called co-planar if:

$$\lambda \vec{a} + \mu \vec{b} = \vec{c} \tag{16.9}$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = 0 \tag{16.10}$$

Chapter 17

3-D Geometry

17.1 Distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$:

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \tag{17.1}$$

17.2 Section Formula of a Line Segment Divided in the ratio $m : n$

$$P = \left(\frac{nx_1 + mx_2}{m + n}, \frac{ny_1 + my_2}{m + n}, \frac{nz_1 + mz_2}{m + n} \right) \tag{17.2}$$

17.3 Centroid of a Triangle

$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right) \tag{17.3}$$

Chapter 18

Line in 3-D Space

For a line which is defined as $\vec{a} = a\hat{i} + b\hat{j} + c\hat{k}$:

1. Line numbers of the line is

$$< a, b, c > \tag{18.1}$$

2. The line cosines are:

$$< \frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}} > \tag{18.2}$$

$$=< l, m, n > \tag{18.3}$$

18.1 Angle between Two Lines

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + b_1b_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} \tag{18.4}$$

$$\Rightarrow \cos\theta = l_1l_2 + m_1m_2 + n_1n_2 \tag{18.5}$$

When two lines are $\perp, l_1l_2 + m_1m_2 + n_1n_2 = 0$.
When two lines are $\parallel \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = 1$.

18.2 Skew and Co-planar Lines

Let there be two lines $\vec{r_1}$ and $\vec{r_2}$,

$$\vec{r_1} = \vec{a_1} + \mu\vec{b_1}\vec{r_2} = \vec{a_2} + \lambda\vec{b_2} \tag{18.6}$$

18.3 Distances

18.3.1 The shortest distance between r_1 and r_2

$$S = \left| \frac{(\vec{a_1} - \vec{a_2}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right| \tag{18.7}$$

If $S = 0$, the lines intersect.

18.3.2 Cartesian Form

For two lines defined as $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$:

$$S = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \tag{18.8}$$

18.3.3 Distance Between Parallel Lines

$$S = \left| \frac{\vec{b} \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b}|} \right| \tag{18.9}$$

18.3.4 Distance of a Point to a Line

For a point, (x_1, y_1, z_1) the distance to a line $\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}$:

$$S = \left(\begin{vmatrix} x_1 - \alpha & y_1 - \beta \\ l & m \end{vmatrix} + \begin{vmatrix} y_1 - \beta & z_1 - \gamma \\ m & n \end{vmatrix} + \begin{vmatrix} z_1 - \gamma & x_1 - \alpha \\ n & l \end{vmatrix} \right)^{\frac{1}{2}} \tag{18.10}$$

Chapter 19

3-D Plane

A plane in 3-D space can be defined as:

1. Cartesian Form:
$$ax + by + cz + d = 0 \tag{19.1}$$
2. Vectorial Form:
$$\vec{r} \cdot \vec{n} = p \tag{19.2}$$
, where $\vec{r}$ is a line on the plane, $\vec{n}$ is a normal to the plane, and p is perpendicular distance to the plane from the origin.

19.1 Angle Between Two Planes

For two planes, $\vec{r_1} \cdot \vec{n_1} = p_1$ and $\vec{r_2} \cdot \vec{n_2} = p_2$, the angle between the planes, θ is:
$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|} \tag{19.3}$$
In the Cartesian Form:
$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{{a_1}^2 + {b_1}^2 + {c_1}^2}\sqrt{{a_2}^2 + {b_2}^2 + {c_2}^2}} \tag{19.4}$$

19.2 Distance of a Point from a Plane

19.2.1 Catesian Form

For the point (p, q, r) and the plane, $ax + by + cz + d = 0$:
$$S = \frac{ap + bq + cr + d}{\sqrt{a^2 + b^2 + c^2}} \tag{19.5}$$

19.2.2 Vector Form

For the point $\vec{g} = p\hat{i} + q\hat{j} + r\hat{k}$ and the plane $\vec{r} \cdot (a\hat{i} + b\hat{j} + c\hat{k}) + d = 0$:

$$S = \frac{(a\hat{i} + b\hat{j} + c\hat{k}) \cdot (p\hat{i} + q\hat{j} + r\hat{k})}{\sqrt{a^2 + b^2 + c^2}} \tag{19.6}$$

$$\Rightarrow S = \frac{(a\hat{i} + b\hat{j} + c\hat{k}) \cdot \vec{g}}{|a\hat{i} + b\hat{j} + c\hat{k}|} \tag{19.7}$$

Part III

Statistics

Chapter 20

Descriptive Statistics

20.1 Measure of Location

20.1.1 Mean

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{20.1}$$

20.1.2 Median

For odd number of elements in a dataset:

$$\tilde{x} = x_{\frac{n+1}{2}} \tag{20.2}$$

For even number of elements in a dataset:

$$\tilde{x} = \frac{x_{\frac{n}{2}} + x_{\left(\frac{n}{2}+1\right)}}{2} \tag{20.3}$$

20.1.3 Mode

$$Mo(x) = \max(f(x_i)) \tag{20.4}$$

20.1.4 Quartile

Measure of percentage of elements less than or equal to a term

20.2 Measure of Spread

20.2.1 Variance

Variance measured on the whole population

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \tag{20.5}$$

20.2.2 Sample Variance

Variance measured on a sample population

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 \tag{20.6}$$

20.2.3 Standard Deviation and Sample Standard

$$\sigma = \sqrt{\sigma^2} \tag{20.7}$$

$$s = \sqrt{s^2} \tag{20.8}$$

20.2.4 Co-efficient of Variance

$$v = \frac{s}{\bar{x}} \tag{20.9}$$

20.3 Skewness

20.3.1 Types of Skewness

Name	Other Name	Characteristic
Right Skew	Positive Skew	Data concentrated on the lower side
Symmetric Distribution	Normal Distribution	Data distributed evenly
Left Skew	Negative Skew	Data concentrated on the higher side

20.3.2 Measure of Skewness

Skewness is measured by the Moment Co-efficient of Skewness.

$$g_m = \frac{m_3}{s^3}, \text{ where} \tag{20.10}$$

$$m_3 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^3 \tag{20.11}$$

Type of Skewness

The type of skewness from the value is g_m is:

Value of g_m	Type
$g_m = 0$	Symmetric
$g_m > 0$	Positive Skew
$g_m < 0$	Negative Skew

Degree of Skewness

The degree of skewness from the value is g_m is:

Value of g_m	Degree
$\lvert g_m \rvert > 1$	High Skewness
$0.5 < \lvert g_m \rvert \geq 1$	Moderate Skewness
$\lvert g_m \rvert \leq 0.5$	Low Skewness

20.4 Kurtosis

Kurtosis is the measure of peakedness of data. Fisher's kurtosis measure is defined as:

$$\gamma = \frac{m_4}{s^4}, \text{ where} \tag{20.12}$$

$$m_4 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^4 \tag{20.13}$$

20.4.1 Type of Kurtosis

The types of kurtosis from the value of γ are:

Value of γ	Type
$\gamma = 0$	Normal Distribution or Mesokurtic
$\gamma < 0$	Flattened or Platykurtic
$\gamma > 0$	Peaked or Lepokurtic

Chapter 21

Hypothesis Testing

21.1 T-Test

$$T = \frac{\bar{X} - \mu}{\frac{s}{\sqrt{n}}} \tag{21.1}$$

where:

$$\begin{aligned}
\bar{X} &= \text{Sample Mean} \\
\mu &= \text{Assumed Mean} \\
s &= \text{Number of Samples} \\
n &= \text{Number of observations}
\end{aligned}$$

If $T < t_c$ the H_0 is not rejected. t_c is a functions of level of significance (α) and degrees of freedom ($v = n - 1$).

21.2 χ^2 Test

$$\chi^2 = \sum_i \sum_j \frac{(h_{ij}^o - h_{ij}^e)^2}{h_{ij}^e} \tag{21.2}$$

where:

$$\begin{aligned}
h_e &= \text{Expected Value} \\
h_o &= \text{Actual Value}
\end{aligned}$$

If $\chi^2 < \chi_c^2$ then H_0 is not rejected. χ_c is a functions of level of significance (α) and degrees of freedom ($v = (i - 1)(j - 1)$).

Chapter 22

Research and Survey Design

22.1 Population Covariance

$$\mathrm{Cov}(x, y) = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu_x)(y_i - \mu_y) \tag{22.1}$$

22.2 Sample Covariance

$$\mathrm{Cov}(x, y) = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) \tag{22.2}$$

22.3 Bravais-Pearson Correlation Co-efficient

$$r = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n} (x_i - \bar{x})^2} \cdot \sqrt{\sum_{i=1}^{n} (y_i - \bar{y})^2}} \tag{22.3}$$

$$= \frac{\mathrm{Cov}(x, y)}{\sqrt{\mathrm{Var}(x) \cdot \mathrm{Var}(y)}} \tag{22.4}$$

$$= \frac{\mathrm{Cov}(x, y)}{\sigma_x \cdot \sigma_y} \tag{22.5}$$

22.4 Spearman's Rank Correlation Co-efficient

$$r_s = 1 - \frac{6 \sum d_i^2}{n\,(n^2 - 1)} \tag{22.6}$$

$$d_i = R(X_i) - R(Y_i) \tag{22.7}$$

Chapter 23

Estimation of Regression Function

For the regression functions:

$$Y_i = \beta_0 + \beta_1 X_1 \tag{23.1}$$

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_1 \tag{23.2}$$

$$\tag{23.3}$$

where Y_i is the observed dependent variable (DV), $\hat{Y}_i$ is the estimated DV, and X_i is the independent variable (IV).

$$u_i = Y_i - \hat{Y}_i \tag{23.4}$$

$$\Rightarrow Y_i = \hat{Y}_i + u_i \tag{23.5}$$

$$\Rightarrow Y_i = \hat{\beta}_0 + \hat{\beta}_1 X_1 + u_i \tag{23.6}$$

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i \tag{23.7}$$

The objective function is:

$$\min_{u_i} \sum u_i = \min \sum_i \left[Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i\right]^2$$

Since the regression function passes through: $\left(\bar{X}, \bar{Y}\right)$

$$\beta_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$$

$$\begin{aligned}
\min_{u_i} \sum u_i &= \min \sum_i \left[Y_i - \bar{Y} + \hat{\beta}_1 \bar{X} - \hat{\beta}_1 X_i\right]^2 \\
&= \min \sum_i \left[\left(Y_i - \bar{Y}\right) - \hat{\beta}_1 \left(X_i - \bar{X}\right)\right]^2 \\
&= \min \sum_i \left[\left(Y_i - \bar{Y}\right)^2 - 2 \cdot \left(Y_i - \bar{Y}\right) \cdot \hat{\beta}_1 \left(X_i - \bar{X}\right) + \hat{\beta}_1^{\ 2} \left(X_i - \bar{X}\right)^2\right] \\
&= \min \left[\sum_i \left(Y_i - \bar{Y}\right)^2 - 2 \cdot \hat{\beta}_1 \sum_i \left(Y_i - \bar{Y}\right) \cdot \left(X_i - \bar{X}\right) + \hat{\beta}_1^{\ 2} \sum_i \left(X_i - \bar{X}\right)^2\right] \\
\Rightarrow u_i^{\beta_1} &= -2 \sum_i \left(Y_i - \bar{Y}\right) + 2\hat{\beta}_1 \left(X_i - \bar{X}\right)^2 = 0
\end{aligned}$$

(For optima Conditions)

$$\Rightarrow \hat{\beta}_1 = \boxed{\frac{\sum_i (Y_i - \bar{Y})(X_i - \bar{X})}{\sum_i (X_i - \bar{X})^2}}$$

$$\Rightarrow \hat{\beta}_0 = \boxed{\bar{Y} - \hat{\beta}_1 \bar{X}}$$

23.1 Sum of Squares Error

$$TSS = \sum_i (Y_i - \bar{Y})^2 \tag{23.8}$$

$$= \underbrace{\sum_i (\hat{Y}_i - \bar{Y})}_{\text{Explained Sum of Square Error (ESS)}} + \underbrace{\sum_i u_i^2}_{\text{Residual Sum of Squares Error (RSS)}} \tag{23.9}$$

23.1.1 R^2: Coefficient of Determination

$$R^2 = \frac{\text{ESS}}{\text{TSS}} \tag{23.10}$$
$$= 1 - \frac{\text{RSS}}{\text{TSS}} \tag{23.11}$$
$$= 1 - \frac{\sum_i u_i^2}{\sum_i (Y_i - \bar{Y})^2} \tag{23.12}$$

For a regression analysis with single IV:

$$\sqrt{R^2} = v$$

23.1.2 $\bar{R}^2$: Coefficient of Determination

$$\bar{R}^2 = 1 - \frac{\dfrac{\sum_i u_i^2}{(N-K-1)}}{\dfrac{\sum_i (Y_i - \bar{Y})^2}{(N-1)}} \tag{23.13}$$

where, N is the number of observations and K is the number of independent variables.

23.2 T-Test

Test for statistical significance of a single IV.

$$T = \frac{\hat{\beta}_1 - 0}{S_e(\hat{\beta}_1)} \tag{23.14}$$

23.3 F-Test

Test for statistical significance of all IVs together.

$$F = \frac{\dfrac{\text{ESS}}{(K-1)}}{\dfrac{\text{RSS}}{(N-K)}} \qquad (F \geq F_c, H_0 \text{ is rejected})$$

23.4 Test for Heteroskedasticity

23.4.1 Definition

$\sigma_{\epsilon_i} \forall \epsilon_i \in [X_a, X_b] = \sigma_{\epsilon_i} \forall \epsilon_i \in [X_{b+1}, X_c]$

23.4.2 Durbin-Watson Test

$$d_e = \frac{\sum_{t=2}^{n} (\hat{u}_t - \hat{u_{t-1}})^2}{\sum_{t=2}^{n} \hat{u_t}^2} \tag{23.15}$$

For the H_0: No autocorrelation:

d	H_0
$0 \leq d_e \leq d_L$ & $(4 - d_L) \leq d_e \leq 4$	Rejected
$d_L < d_e \leq d_U$ & $(4 - d_U) < d_e \leq (4 - d_L)$	Decision Free Zone
$d_L < d_e < D_U$	Not rejected

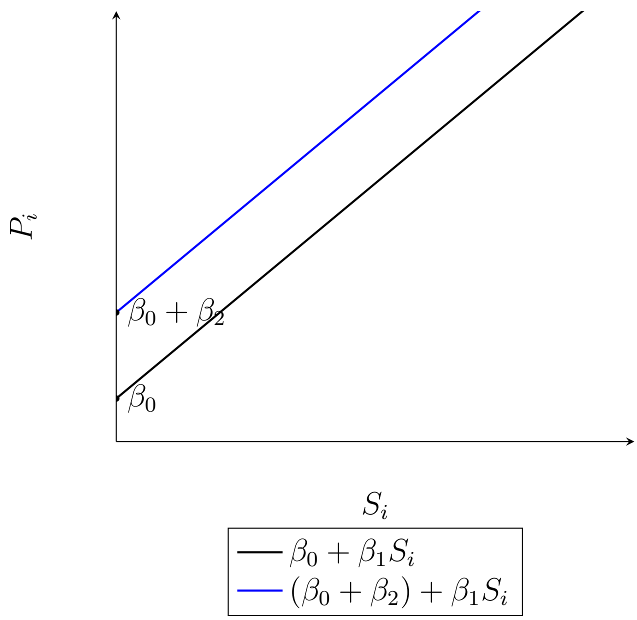
Chapter 24

Dummy Variables

24.1 Dummy Variable

$$P_i = \beta_0 + \beta_1 S_1 + \beta_2 D_i + \epsilon_i \tag{24.1}$$

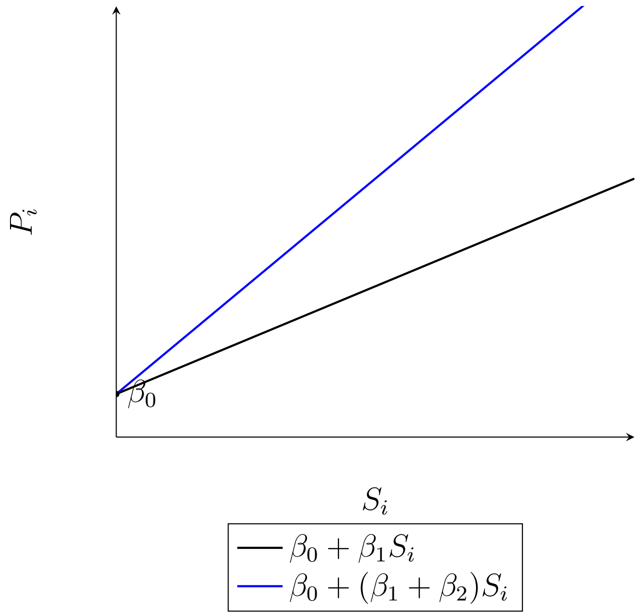
$$E(P_i) = \begin{cases} (\hat{\beta}_0 + \hat{\beta}_2) + \hat{\beta}_1 S_i, & D_i = 1 \\ \hat{\beta}_0 + \hat{\beta}_1 S_i, & D_i = 0 \end{cases} \tag{24.2}$$



24.2 Slope Dummy Variable

$$P_i = \beta_0 + \beta_1 S_1 + \beta_2 (S_i \cdot D_i) + \epsilon_i \tag{24.3}$$

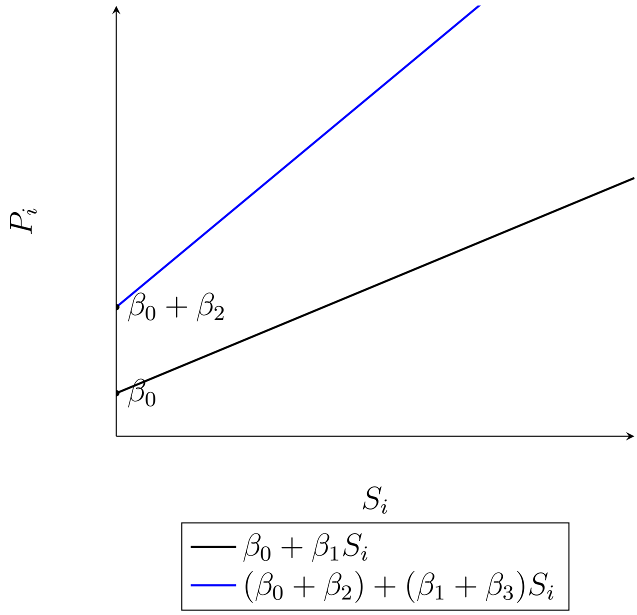
$$E(P_i) = \begin{cases} \hat{\beta}_0 + \left(\hat{\beta}_1 + \hat{\beta}_2\right) S_i, & D_i = 1 \\ \hat{\beta}_0 + \hat{\beta}_1 S_i, & D_i = 0 \end{cases} \tag{24.4}$$



24.3 Slope & Dummy Variable

$$P_i = \beta_0 + \beta_1 S_1 + \beta_2 D_i + \beta_3 S_i D_i + \epsilon_i \tag{24.5}$$

$$E(P_i) = \begin{cases} (\hat{\beta}_0 + \hat{\beta}_2) + \left(\hat{\beta}_1 + \hat{\beta}_3\right) S_i, & D_i = 1 \\ \hat{\beta}_0 + \hat{\beta}_1 S_i, & D_i = 0 \end{cases} \tag{24.6}$$



24.4 Multi-Categories Dummy Variable

$$P_0 = b_0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \underbrace{b_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + b_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{Leads to Perfect Multicollinearlity}} \tag{24.7}$$

Alternatives

- B_n captures the mean of each category, but F-Test is impossible

$$y = \beta_1 D_{1i} + \beta_2 D_{2i} + \beta_3 D_{3i} \tag{24.8}$$

- Computer drops automatically drops a variable

$$y = \beta_0 + \beta_1 D_{1i} + \beta_2 D_{2i} + \beta_3 D_{3i} \tag{24.9}$$

- Manually dropping a variable

$$y = \beta_0 + \beta_1 D_{1i} + \beta_2 D_{2i} \tag{24.10}$$

Chapter 25

Logistic Regression

For $Y_i \in \{0, 1\}$:

$$z_k = \beta_0 + \sum_{j=1}^{n} \beta_{jk} x_j + \epsilon_k, \beta_j \rightarrow \text{Logit Coefficient} \tag{25.1}$$

$$p = \frac{\exp^k}{1 + \exp^k} = \frac{1}{1 + \exp^{-k}} \tag{25.2}$$

where p is the probability of $y = 1$.

Part IV

Trigonometry

Chapter 26

Circular Trigonometric Functions

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$	0	1	0
$15°$	$\frac{1}{4}$	$\frac{1}{4(2-\sqrt{3})}$	$2-\sqrt{3}$
$18°$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$
$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$36°$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}$
$45°$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$60°$	$\frac{1}{\sqrt{3}}$	$\frac{1}{2}$	$\sqrt{3}$
$72°$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}$
$90°$	1	0	∞

Table 26.1: Trigonometric Ratios of Standard Angles

For any given triangle:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \tag{26.1}$$

, where $2R$ is the radius of circumcircle.

26.1 Negative Angle Formula

$$\sin(-\theta) = -\sin\theta \tag{26.2}$$
$$\cos(-\theta) = \cos\theta \tag{26.3}$$
$$\tan(-\theta) = -\tan\theta \tag{26.4}$$
$$\csc(-\theta) = -\csc\theta \tag{26.5}$$
$$\sec(-\theta) = \sec\theta \tag{26.6}$$
$$\cot(-\theta) = -\cot\theta \tag{26.7}$$

26.2 Sum of Angles Formula

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \tag{26.8}$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B \tag{26.9}$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B} \tag{26.10}$$

26.3 Difference of Angles Formula

$$\sin(A-B) = \sin A\cos B - \cos A\sin B \tag{26.11}$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B \tag{26.12}$$
$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B} \tag{26.13}$$

26.4 Multiples and Sub-multiples of π and $\frac{\pi}{2}$

$$\forall k \in \mathbb{Z}$$
$$\sin\left((4k+1)\frac{\pi}{2}\right) = 1 \tag{26.14}$$
$$\sin\left((4k-1)\frac{\pi}{2}\right) = -1 \tag{26.15}$$
$$\sin k\pi = 0 \tag{26.16}$$
$$\sin\left((2k+1)\frac{\pi}{2}\right) = 0 \tag{26.17}$$
$$\sin\left((2k-1)\frac{\pi}{2}\right) = 0 \tag{26.18}$$
$$\sin k\pi = (-1)^k \tag{26.19}$$

26.5 $\left(\frac{\pi}{2} \pm \theta\right)$ Formula

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \tag{26.20}$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta \tag{26.21}$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \tag{26.22}$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta \tag{26.23}$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \tag{26.24}$$

$$\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta \tag{26.25}$$

$$\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta \tag{26.26}$$

$$\cot\left(\frac{\pi}{2} + \theta\right) = -\tan\theta \tag{26.27}$$

$$\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta \tag{26.28}$$

$$\csc\left(\frac{\pi}{2} + \theta\right) = \sec\theta \tag{26.29}$$

$$\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta \tag{26.30}$$

$$\sec\left(\frac{\pi}{2} + \theta\right) = -\csc\theta \tag{26.31}$$

26.6 $\left(\frac{\pi}{4} \pm \theta\right)$ Formula

$$\tan\left(\frac{\pi}{4} + \theta\right) = \frac{1 + \tan\theta}{1 - \tan\theta} \tag{26.32}$$

$$\tan\left(\frac{\pi}{4} - \theta\right) = \frac{1 - \tan\theta}{1 + \tan\theta} \tag{26.33}$$

26.7 Trigonometric Identities

$$\sin^2\theta + \cos^2\theta = 1 \tag{26.34}$$

$$\tan^2\theta + 1 = \sec^2\theta \tag{26.35}$$

$$\cot^2\theta + 1 = \csc^2\theta \tag{26.36}$$

26.8 Double Angle Formula

$$\begin{aligned} \sin 2\theta &= 2\sin\theta\cos\theta \\ &= \frac{2\tan\theta}{1+\tan^2\theta} \end{aligned} \tag{26.37}$$

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \\ &= \frac{1-\tan^2\theta}{1+\tan^2\theta} \end{aligned} \tag{26.38}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \tag{26.39}$$

26.9 Triple Angle Formula

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta \tag{26.40}$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \tag{26.41}$$

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^3\theta} \tag{26.42}$$

26.10 Sum and Product of Two Ratios

For $A > B$:

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \tag{26.43}$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \tag{26.44}$$

$$2\sin A\cos B = \sin(A+B) + \sin(A-B) \tag{26.45}$$

$$2\cos A\sin B = \sin(A+B) - \sin(A-B) \tag{26.46}$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \tag{26.47}$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \tag{26.48}$$

$$2\cos A\cos B = \cos(A+B) + \cos(A-B) \tag{26.49}$$

$$2\cos A\sin B = \cos(A+B) - \cos(A-B) \tag{26.50}$$

$$\sin(A-B)\sin(A+B) = \sin^2 A - \sin^2 B \tag{26.51}$$

$$\cos(A-B)\cos(A+B) = \cos^2 A - \sin^2 B \tag{26.52}$$

$$\tan(A-B)\tan(A+B) = \frac{\tan^2 A - \tan^2 B}{1 - \tan^2 A\tan^2 B} \tag{26.53}$$

26.11 General Solutions

If $\sin\theta = \sin\alpha$:

$$\theta = n\pi + (-1)^n\alpha \tag{26.54}$$

$n \in \mathbb{Z}$
If $\cos\theta = \cos\alpha$:

$$\theta = 2n\pi \pm \alpha \tag{26.55}$$

$n \in \mathbb{Z}$
If $\tan\theta = \tan\alpha$:

$$\theta = n\pi \pm \alpha \tag{26.56}$$

$n \in \mathbb{Z}$

26.12 Taylor Series Expansion of Trigonometric Ratios

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \infty = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^{2i-1}}{(2i-1)!} \tag{26.57}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \infty = \sum_{i=0}^{\infty} (-1)^{i} \frac{x^{2i}}{(2i)!} \tag{26.58}$$

Chapter 27

Inverse Circular Trigonometric Function

27.1 Definition of Inverse Circular Trigonometric Function

27.1.1 For $\sin x$

$y = \arcsin x$ iff $x = \sin y$, then:

1. $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
2. domain of $x \in [-1, 1]$
3. $\sin(\arcsin x) = x, \forall x \in [-1, 1]$
4. $\arcsin(\sin y) = y, \forall y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
5. $\sin x$ is a strictly increasing in the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and one-one.

27.1.2 For $\cos x$

$y = \arccos x$ iff $x = \cos y$, then:

1. $y \in [0, \pi]$
2. domain of $x \in [-1, 1]$
3. $\cos(\arccos x) = x, \forall x \in [-1, 1]$
4. $\arccos(\cos y) = y, \forall y \in [0, \pi]$
5. $\cos x$ is a strictly decreasing in the domain $[0, \pi]$ and one-one.

27.1.3 For $\tan x$

$y = \arctan x$ iff $x = \tan y$, then:

1. $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
2. domain of $x \in \mathbb{R}$
3. $\tan(\arctan x) = x, \forall x \in \mathbb{R}$
4. $\arctan(\tan y) = y, \forall y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
5. $\tan x$ is a strictly increasing in the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and one-one.

27.1.4 For $\cot x$

$y = \cot^{-1} x$ iff $x = \cot y$, then:

1. $y \in (0, \pi)$
2. domain of $x \in \mathbb{R}$
3. $\cot(\cot^{-1} x) = x, \forall x \in \mathbb{R}$
4. $\cot^{-1}(\cot y) = y, \forall y \in (0, \pi)$
5. $\cot x$ is a strictly decreasing in the domain $(0, \pi)$ and one-one.

For $\sec x$

$y = \sec^{-1} x$ iff $x = \sec y$

1. $y \in \left\{[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]\right\}$
2. $x \in \{(-\infty, -1] \cup [1, \infty)\}$
3. $\sec(\sec^{-1} x) = x, \forall |x| \geq 1$
4. $\sec^{-1}(\sec y) = y, \forall y \in \left\{[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]\right\}$

27.1.5 For $\csc x$

$y = \csc^{-1} x$ iff $x = \csc y$

1. $y \in \left\{[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]\right\}$
2. $x \in \{(-\infty, -1] \cup [1, \infty)\}$
3. $\csc(\csc^{-1} x) = x, \forall |x| \geq 1$
4. $\csc^{-1}(\csc y) = y, \forall y \in \left\{[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]\right\}$

27.2 Negative Arguments

$$\arcsin(-x) = -\arcsin x \tag{27.1}$$
$$\arctan(-x) = -\arctan x \tag{27.2}$$
$$\csc^{-1}(-x) = -\csc^{-1} x \tag{27.3}$$
$$\arccos(-x) = \pi - \arccos x \tag{27.4}$$
$$\cot^{-1}(-x) = \pi - \cot^{-1} x \tag{27.5}$$
$$\sec^{-1}(-x) = \pi - \sec^{-1} x \tag{27.6}$$

27.3 Reciprocal Relations

$$\csc^{-1} x = \arcsin \frac{1}{x} \tag{27.7}$$
$$\sec^{-1} x = \arccos \frac{1}{x} \tag{27.8}$$
$$\sec^{-1} x = \begin{cases} \arctan \frac{1}{x}, x > 0 \\ \pi + \arctan \frac{1}{x}, x < 0 \end{cases} \tag{27.9}$$

27.4 I.T.F. Identities

$$\arcsin x + \arccos x = \frac{\pi}{2}, |x| \leq 1 \tag{27.10}$$
$$\arctan x + \cot^{-1} x = \frac{\pi}{2}, x \in \mathbb{R} \tag{27.11}$$
$$\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}, |x| \geq 1 \tag{27.12}$$

27.5 Sum of Two Angles

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) \tag{27.13}$$
$$\arcsin x + \arcsin y = \arcsin(y\sqrt{1-x^2} + x\sqrt{1-y^2}) \tag{27.14}$$
$$\arccos x + \arccos y = \arccos(xy - \sqrt{1-x^2}\sqrt{1-y^2}) \tag{27.15}$$

27.6 Difference of Two Angles

$$\arctan x - \arctan y = \arctan\left(\frac{x-y}{1+xy}\right) \tag{27.16}$$

$$\arcsin x - \arcsin y = \arcsin(x\sqrt{1-y^2} - y\sqrt{1-x^2}) \tag{27.17}$$

$$\arccos x - \arccos y = \arccos(xy + \sqrt{1-x^2}\sqrt{1-y^2}) \tag{27.18}$$

27.7 Interconversion of Ratios

$$\begin{aligned} \arcsin x = &\quad \arccos\sqrt{1-x^2} \\ = &\quad \arctan\left(\frac{x}{\sqrt{1-x^2}}\right) \end{aligned} \tag{27.19}$$

$$\begin{aligned} \arccos x = &\quad \arcsin\sqrt{1-x^2} \\ = &\quad \arctan\left(\frac{\sqrt{1-x^2}}{x}\right) \end{aligned} \tag{27.20}$$

$$\begin{aligned} 2\arctan x = &\quad \arcsin\left(\frac{2x}{1+x^2}\right) \\ = &\quad \arccos\left(\frac{1-x^2}{1+x^2}\right) \\ = &\quad \arctan\left(\frac{2x}{1-x^2}\right) \end{aligned} \tag{27.21}$$

27.8 Miscellaneous Relations

$$\cos(\arcsin x) = \sin(\arccos x) = \sqrt{1-x^2} \tag{27.22}$$

$$\arctan x = \frac{\pi}{2} - \arctan\left(\frac{1}{x}\right), x > 1 \tag{27.23}$$

Chapter 28

Hyperbolic Trigonometric Function

28.1 Definition

Hyperbolic trigonometric functions are defined such that $(\cosh t, \sinh t)$ form the right half of an equilateral hyperbola. The functions are defined as follows:

$$\sinh x = \frac{\exp(x) - \exp(-x)}{2} \tag{28.1}$$

$$\cosh x = \frac{\exp(x) + \exp(-x)}{2} \tag{28.2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\exp(x) - \exp(-x)}{\exp(x) + \exp(-x)} \tag{28.3}$$

$$\coth x = \frac{1}{\tanh x} = \frac{\exp(x) + \exp(-x)}{\exp(x) - \exp(-x)} \tag{28.4}$$

$$cschx = \frac{1}{\sinh x} = \frac{2}{\exp(x) - \exp(-x)} \tag{28.5}$$

$$sechx = \frac{1}{\cosh x} = \frac{2}{\exp(x) + \exp(-x)} \tag{28.6}$$

28.2 Identities

$$\coth^2 x - \sinh^2 x = 1 \tag{28.7}$$

$$\tanh^2 x + sech^2 x = 1 \tag{28.8}$$

$$\coth^2 x - csch^2 x = 1 \tag{28.9}$$

28.3 Inverse Hyperbolic Function

$$\sinh^{-1} z = \ln(z + \sqrt{z^2 + 1}) \tag{28.10}$$

$$\cosh^{-1} z = \ln(z \pm \sqrt{z^2 - 1}) \tag{28.11}$$

$$\tanh^{-1} z = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right) \tag{28.12}$$

$$\coth^{-1} z = \frac{1}{2} \ln\left(\frac{z+1}{z-1}\right) \tag{28.13}$$

$$csch^{-1} z = \ln\left(\frac{1 \pm \sqrt{z^2+1}}{z}\right) \tag{28.14}$$

$$sech^{-1} z = \ln\left(\frac{1 \pm \sqrt{1-z^2}}{2}\right) \tag{28.15}$$

28.4 Relation to Circular Trigonometric Functions

$$\sinh(z) = -i \sin(iz) \tag{28.16}$$

$$\coth(z) = \cos(iz) \tag{28.17}$$

$$\tanh(z) = -i \tan(iz) \tag{28.18}$$

$$csch(z) = i \csc(iz) \tag{28.19}$$

$$sech(z) = \sec(iz) \tag{28.20}$$

$$\coth(z) = i \cot(iz) \tag{28.21}$$

Part V

Calculus

Chapter 29

Limits

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \tag{29.1}$$

$$\lim_{x\to 0} \frac{\tan x}{x} = 1 \tag{29.2}$$

$$\lim_{x\to 0} \cos x = 1 \tag{29.3}$$

$$\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1} \tag{29.4}$$

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x) \tag{29.5}$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \lim_{x\to a} g(x) \neq 0 \tag{29.6}$$

$$\lim_{x\to 0} \exp(x) = 1 \tag{29.7}$$

$$\lim_{x\to a} \exp(x) = \exp(c) \tag{29.8}$$

$$\lim_{x\to 0} \frac{\exp(x) - 1}{x} = 1 \tag{29.9}$$

$$\lim_{x\to a} c^x = c^a \tag{29.10}$$

$$\lim_{x \to a} \ln x = \ln a \tag{29.11}$$

$$\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e \tag{29.12}$$

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1 \tag{29.13}$$

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a \tag{29.14}$$

$$\lim_{n \to \infty} \frac{x^n}{n!} = 0, \forall x \in \mathbb{R} \tag{29.15}$$

29.1 L'Hospital Rule

If:

$$L = \lim_{x \to a} \frac{f(x)}{g(x)}$$

is such that $f(a) = 0$ and $g(a) = 0$, or $f(a) = \infty$ and $g(a) = \infty$, then:

$$L = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Chapter 30

Differentiation

30.1 Differentiation by First Principle

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{30.1}$$

30.2 Standard Differentiation Formulae

$$\frac{dk}{dx} = 0 \tag{30.2}$$

$$\frac{dx^n}{dx} = nx^{n-1} \tag{30.3}$$

$$\frac{da^x}{dx} = \ln a \cdot a^x \tag{30.4}$$

$$\frac{d\exp(x)}{dx} = \exp(x) \tag{30.5}$$

$$\frac{d\ln x}{dx} = \frac{1}{x} \tag{30.6}$$

$$\frac{d\sqrt{x}}{dx} = \frac{1}{2\sqrt{2}} \tag{30.7}$$

$$\tag{30.8}$$

30.2.1 Circular Trigonometric Functions

$$\frac{d\sin x}{dx} = \cos x \tag{30.9}$$
$$\frac{d\cos x}{dx} = -\sin x \tag{30.10}$$
$$\frac{d\tan x}{dx} = \sec^2 x \tag{30.11}$$
$$\frac{d\sec x}{dx} = \sec x \tan x \tag{30.12}$$
$$\frac{d\csc x}{dx} = -\csc x \cot x \tag{30.13}$$
$$\frac{d\cot x}{dx} = -\csc^2 x \tag{30.14}$$

30.2.2 Inverse Circular Trigonometric Functions

$$\frac{d\arcsin x}{dx} = \frac{1}{\sqrt{1-x^2}}, |x| \leq 1 \tag{30.15}$$
$$\frac{d\arccos x}{dx} = -\frac{1}{\sqrt{1-x^2}}, |x| \leq 1 \tag{30.16}$$
$$\frac{d\arctan x}{dx} = \frac{1}{1+x^2} \tag{30.17}$$
$$\frac{d\cot^{-1} x}{dx} = -\frac{1}{1+x^2} \tag{30.18}$$
$$\frac{d\sec^{-1} x}{dx} = \frac{1}{x\sqrt{x^2-1}}, |x| \geq 1 \tag{30.19}$$
$$\frac{d\csc^{-1} x}{dx} = -\frac{1}{x\sqrt{x^2-1}}, |x| \geq 1 \tag{30.20}$$

30.2.3 Hyperbolic Trigonometric Function

$$\frac{d\sinh x}{dx} = \cosh x \tag{30.21}$$

$$\frac{d\cosh x}{dx} = \sinh x \tag{30.22}$$

$$\frac{d\tanh x}{dx} = 1 - \tanh^2 x = sech^2(x) \tag{30.23}$$

$$\frac{d\coth x}{dx} = 1 - \coth^2 x = -csch^2(x) \tag{30.24}$$

$$\frac{d[sech(x)]}{dx} = -\tanh x \ \mathrm{sech} x \tag{30.25}$$

$$\frac{d[csch(x)]}{dx} = -\coth x \ \mathrm{csch} x \tag{30.26}$$

30.2.4 Inverse Hyperbolic Trigonometric Function

$$\frac{d\sinh x}{dx} = \frac{1}{\sqrt{x^2+1}} \tag{30.27}$$

$$\frac{d\cosh x}{dx} = \frac{1}{\sqrt{x^2-1}} \tag{30.28}$$

$$\frac{d\tanh x}{dx} = \frac{1}{1-x^2} \tag{30.29}$$

$$\frac{d\coth x}{dx} = \frac{1}{1-x^2} \tag{30.30}$$

$$\frac{d[sech(x)]}{dx} = \frac{1}{x\sqrt{1-x^2}} \tag{30.31}$$

$$\frac{d[csch(x)]}{dx} = \frac{1}{|x|\sqrt{1+x^2}} \tag{30.32}$$

30.3 Rules of Differentiation

$$\frac{d[cf(x)]}{dx} = c\frac{df(x)}{dx} \tag{30.33}$$

$$\frac{d[f_1(x)+f_2(x)]}{dx} = \frac{d[f_1(x)]}{dx} + \frac{d[f_2(x)]}{dx} \tag{30.34}$$

$$\frac{d[f_1 f_2]}{dx} = f_1 f_2' + f_2 f_1' \tag{30.35}$$

$$\frac{d\left(\frac{f_1}{f_2}\right)}{dx} = \frac{f_2 f_1' - f_1 f_2'}{f_2^2} \tag{30.36}$$

30.4 Chain Rule

If two functions are defined as $z = f(y)$ and $y = g(x)$:

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} \tag{30.37}$$

If two functions are defined as $x = f(\theta)$ and $y = g(\theta)$:

$$\frac{d^2y}{dx^2} = \left[\frac{d}{d\theta}\left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right)\right]\frac{d\theta}{dx} \tag{30.38}$$

Chapter 31

Successive Differentiation

$$D^n(ax+b)^m = m(m-1)\cdots(m-n+1)a^n(ax+b)^{m-n} \tag{31.1}$$

$$D^n\left(\frac{1}{ax+b}\right) = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}} \tag{31.2}$$

$$D^n \ln(ax+b) = \frac{(-1)^{n-1}(n-1)! a^n}{(ax+b)^n}, n \geq 2 \tag{31.3}$$

$$D^n(a^{mx}) = m^n(\ln a)^n a^{mx} \tag{31.4}$$

$$D^n(e^{mx}) = m^n e^{mx} \tag{31.5}$$

$$D^n \sin(ax+b) = a^n \sin(ax+b+n\frac{\pi}{2}) \tag{31.6}$$

$$D^n \cos(ax+b) = a^n \cos(ax+b+n\frac{\pi}{2}) \tag{31.7}$$

$$D^n[e^{ax}\sin(bx+c)] = (a^2+b^2)^{\frac{n}{2}} e^{ax} \sin(bx+c+n\arctan\frac{b}{a}) \tag{31.8}$$

$$D^n[e^{ax}\cos(bx+c)] = (a^2+b^2)^{\frac{n}{2}} e^{ax} \cos(bx+c+n\arctan\frac{b}{a}) \tag{31.9}$$

31.1 Leibnitz's Theorem

For two functions u and v of x, the successive differentiation of their product is defined as:

$$\begin{aligned}(uv)_n &= {}^nC_0 u_n v +^n C_1 u_{n-1} v_1 + \cdots +^n C_0 u v_n \\ &= \sum_{i=0}^{n} {}^nC_i u_{n-i} v_i\end{aligned} \tag{31.10}$$

Chapter 32

Partial Derivative

If $f(x, y)$ is a function of (x, y), then $\dfrac{\delta f(x,y)}{\delta x}$ is the differentiation of $f(x, y)$ w.r.t. x, keeping all other parameters constant.

32.1 Chain Rule

If f is a function of u and v, which are functions of x and y, then:

$$\frac{\delta f}{\delta x} = \frac{\delta f}{\delta u}\frac{\delta u}{\delta x} + \frac{\delta f}{\delta v}\frac{\delta v}{\delta x} \tag{32.1}$$

$$\frac{\delta f}{\delta y} = \frac{\delta f}{\delta u}\frac{\delta u}{\delta y} + \frac{\delta f}{\delta v}\frac{\delta v}{\delta y} \tag{32.2}$$

If f is a function of x and y, which are functions of t, then:

$$\frac{df}{dt} = \frac{\delta f}{\delta x}\frac{dx}{dt} + \frac{\delta f}{\delta y}\frac{dy}{dt} \tag{32.3}$$

32.2 Euler's Theorem

For a homogeneous function [1], $f(x_i)$ of degree n:

$$\sum x_i \frac{\delta f}{\delta x_i} = n f(x_i) \tag{32.4}$$

[1] Homogeneous functions are defined as $f(ax, ay) = a^{\kappa} f(x, y)$, where κ is the degree of homogeneity. E.g. $f(x, y) = x^2 + y^2$, then $f(tx, ty) = t^2(x^2 + y^2)$, and the degree of homogeneity is 2.

Chapter 33

Application of Differentiation

33.1 Rolle's Theorem

For a function $f(x)$:

1. is continuous in $[a, b]$
2. is differentiable in (a, b)
3. $f(a) = f(b)$,

then there exists a point $x = c$ such that $f'(c) = 0$, $c \in (a, b)$

33.2 Mean Value Theorem or LaGrange's Theorem

For a function $f(x)$:

1. is continuous in $[a, b]$
2. is differentiable in (a, b),

then there exists a point $x = c$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$, $c \in (a, b)$, i.e., the tangent is parallel to the line joining the the points $(a, f(a))$ and $(b, f(b))$.

33.3 Cauchy's Mean Value Theorem

For a function $f(x)$ and $g(x)$:

1. are continuous in $[a, b]$
2. are differentiable in (a, b)
3. $g'(x) \neq 0$ in (a, b),

then there exists a point $c \in (a, b)$, such that $\dfrac{f(x)}{g(x)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$.

33.4 Maxima and Minima

33.4.1 Maxima

For the local maxima of a function $f(x)$:

1. $f'(c) = 0$ and

$$\lim_{\epsilon \to c^-} f'(\epsilon) > 0$$
$$\lim_{\epsilon \to c^+} f'(\epsilon) < 0$$

OR

2. $f'(c) = 0$ and $f''(x) < 0$,

then $f(c)$ is the local maxima point of the function $f(x)$.

33.4.2 Minima

For the local minima of a function $f(x)$:

1. $f'(c) = 0$ and

$$\lim_{\epsilon \to c^-} f'(\epsilon) < 0$$
$$\lim_{\epsilon \to c^+} f'(\epsilon) > 0$$

OR

2. $f'(c) = 0$ and $f''(x) > 0$,

then $f(c)$ is the local minima point of the function $f(x)$.

33.5 Taylor's Theorem

For a function which is differentiable n times:

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{x!}R_n \quad (33.1)$$

where R_n is the remainder term.

33.5.1 Remainder Term

LeGrange's Form

$$R_n = f^n(a + \theta h), \theta \in (0, 1) \quad (33.2)$$

Cauchy's Form

$$R_n = n(1-\theta)^{n-1} f^n(a+\theta h), \theta \in (0,1) \tag{33.3}$$

33.5.2 Conditions for Validity of Expansion

For validity of Taylor Expansion,the condition

$$\lim_{n\to\infty} R_n = 0 \tag{33.4}$$

needs to be satisfied either where R_n is the remainder term in either LeGrange's Form or Cauchy's Form. If the condition is satisfied in a certain domain, then the expansion is valid within that domain only.

33.5.3 Taylor's Theorem for Two Variables

$$\begin{aligned} f(a+x,b+y) = & \quad f(x,y) + \left(a\frac{\delta}{\delta x} + b\frac{\delta}{\delta y}\right) f(x,y) + \\ & \frac{1}{2!}\left(a^2\frac{\delta^2}{\delta x^2} + b^2\frac{\delta^2}{\delta y^2}\right) f(x,y) + \cdots + \\ & \frac{1}{n!}\left(a^n\frac{\delta^n}{\delta x^n} + b^n\frac{\delta^n}{\delta y^n}\right) f(x+\theta a, y+\theta b), \theta \in (0,1) \end{aligned} \tag{33.5}$$

33.6 Maclaurin's Series

$$\begin{aligned} f(x) = & \quad f(0) + xf'(0) + \frac{1}{2!}x^2 f''(0) + \frac{1}{3!}x^3 f'''(0) + \cdots \infty \\ = & \quad \sum_{i=0}^{\infty} \frac{1}{i!} x^i f^i(0) \end{aligned} \tag{33.6}$$

33.6.1 Maclaurin's Series with Two Variables

$$\begin{aligned} f(a,b) = & \quad f(0,0) + \left(a\frac{\delta}{\delta x} + b\frac{\delta}{\delta x}\right) f(0,0) + \\ & \frac{1}{2!}\left(a^2\frac{\delta^2}{\delta x^2} + b^2\frac{\delta^2}{\delta x^2}\right) f(0,0) + \cdots \infty \\ = & \quad \sum_{i=0}^{\infty} \frac{1}{n!}\left(a^i\frac{\delta^i}{\delta x^i} + b^i\frac{\delta^i}{\delta x^i}\right) f(0,0) \end{aligned} \tag{33.7}$$

33.7 Curvature

Curvature is the rate of change of direction w.r.t. arc. Mathematically:

$$\text{Curvature} = \frac{d(\text{direction})}{d(\text{arc})}$$

$$\lim_{\Delta s \to 0} \frac{\Delta \psi}{\Delta s} = \frac{d\psi}{ds} \tag{33.8}$$

33.7.1 Radius of Curvature

Cartesian Form

For a curve $y = f(x)$:

$$\rho = \frac{(1 + y'^2)^{\frac{3}{2}}}{y''} \tag{33.9}$$

However, this formula fails for $y' \to \infty$.

Parametric Form

For a curve defined as $x = \phi(t)$ and $y = \psi(t)$:

$$\rho = \frac{(\ddot{x}^2 + \ddot{y}^2)^{\frac{3}{2}}}{x\ddot{y} - y\ddot{x}} \tag{33.10}$$

33.7.2 Newton's Formula

1. If the curve passes through origin, and the tangent at origin is the x-axis:

$$\rho = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2}{2y} \tag{33.11}$$

2. If the curve passes through origin, and the tangent at origin is the y-axis:

$$\rho = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{y^2}{2x} \tag{33.12}$$

3. If the curve passes through origin and $ax + by + c = 0$ is the tangent at origin:

$$\rho(0,0) = \frac{1}{2}\sqrt{a^2 + b^2} \lim_{\substack{x \to 0 \\ y \to 0}} \frac{a^2 + y^2}{ax + by} \tag{33.13}$$

33.7.3 Tangent at Origin

For a curve

$$\sum c_i x^j y^k = 0, i \in \mathbb{N} \text{ and } j, k \in \mathbb{Z} - \{0\} \tag{33.14}$$

The curve passes through origin $\because c = 0$. Then the lowest degree term equated to x gives the tangent at origin.

33.8 Asymptotes

If the distance between a line P and a curve $f(x)$, s is such that $s \to 0$, as $x \to \infty$, then P is the asymptote of $f(x)$. For asymptotes not parallel to x-axis:
Let $y = mx + c$ be the asymptote of the function $y = f(x)$, then:

$$m = \lim_{x \to \infty} \frac{y}{x} \tag{33.15}$$

$$c = \lim_{x \to \infty} (y - mx) \tag{33.16}$$

33.8.1 Asymptote of Algebraic Curves

For an algebraic curve, passing through origin, defined as:

$$\begin{aligned} (a_0 x^n + a_1 x^{n-1} y^1 + a_2 x^{n-2} y^2 + \cdots + a_{n-1} x y^{n-1} + a_n y^n) + \\ (b_0 x^{n-1} + b_1 x^{n-2} y^1 + b_2 x^{n-3} y^2 + \cdots + b_{n-1} x y^{n-2} + a_n y^{n-1}) + \\ \cdots = 0 \end{aligned}$$

$$\Rightarrow x^n \phi_n \left(\frac{y}{x}\right) + x^{n-1} \phi_{n-1} \left(\frac{y}{x}\right) + \cdots + x \phi_1 \left(\frac{y}{x}\right) = 0$$

The asymptote(s) defined as $y = mx + c$,

1. m is the solution for the equation

$$\phi_n(m) = 0 \tag{33.17}$$

2.

$$c = -\frac{\phi_{n-1}(m)}{\phi_n(m)} \tag{33.18}$$

   where c is a finite value.

Chapter 34

Integration

34.1 General Formulae

[1]

$$\int nx^{n-1}dx = x^n + A \tag{34.1}$$

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + A \tag{34.2}$$

$$\int e^x dx = e^x + A \tag{34.3}$$

$$\int \frac{1}{x}dx = \ln x + A \tag{34.4}$$

$$\int \ln x dx = x(\ln x - 1) + A \tag{34.5}$$

[1] A is the constant of integration in all cases

34.2 Circular Trigonometric Functions

$$\int \sin x dx = -\cos x + A \tag{34.6}$$

$$\int \cos x dx = \sin x + A \tag{34.7}$$

$$\int \sec^2 x dx = \tan x + A \tag{34.8}$$

$$\int \csc^2 x dx = -\cot x + A \tag{34.9}$$

$$\int \sec x \tan x dx = \sec x + A \tag{34.10}$$

$$\int \csc x \cot x dx = -\csc x + A \tag{34.11}$$

$$\int \sec x dx = \ln(\sec x + \tan x) + A \tag{34.12}$$

$$\int \csc x dx = -\ln(\csc x + \cot x) + A \tag{34.13}$$

$$\begin{aligned} \int \tan x dx &= -\ln(\cos x) + A \\ &= \ln(\sec x) + A \end{aligned} \tag{34.14}$$

$$\int \cot x dx = \ln(\sin x) + A \tag{34.15}$$

34.3 Inverse Circular Trigonometric Function

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + A \tag{34.16}$$

$$\int \frac{-1}{\sqrt{1-x^2}} dx = \cos^{-1} x + A \tag{34.17}$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + A \tag{34.18}$$

$$\int \frac{-1}{1+x^2} dx = \cot^{-1} x + A = -\tan^{-1} x + A \tag{34.19}$$

$$\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + A = -\csc^{-1} x + A \tag{34.20}$$

$$\int \frac{-1}{x\sqrt{x^2-1}} dx = \csc^{-1} x + A = -\sec^{-1} x + A \tag{34.21}$$

34.4 Standard Integrals

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + A \tag{34.22}$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2}) + A \tag{34.23}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + A \tag{34.24}$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln(x + \sqrt{x^2 - a^2}) + A \tag{34.25}$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} + A \tag{34.26}$$

$$\int \sqrt{a^2 - x^2} dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + A \tag{34.27}$$

$$\int \sqrt{a^2 + x^2} dx = \frac{x\sqrt{a^2 + x^2}}{2} + \frac{a^2}{2} \ln(x + \sqrt{x^2 + a^2}) + A \tag{34.28}$$

$$\int \sqrt{x^2 - a^2} dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} \ln(x + \sqrt{a^2 - x^2}) + A \tag{34.29}$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + A \tag{34.30}$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a + x}{a - x} \right| + A \tag{34.31}$$

[2]

34.5 Special Forms

For a function $f(x)$:

$$\int [f(x)]^n f'(x) dx = \begin{cases} \dfrac{[f(x)]^{n+1}}{n+1} + A, n \neq 1 \\ \ln|f(x)| + A, n = 1 \end{cases} \tag{34.32}$$

[2] a is a constant $\in \mathbb{R}$

34.5.1 Integration by Part

For two functions $u(x)$ and $v(x)$:

$$\int u(x)v(x)dx = u(x)\left[\int v(x)dx\right] - \int\left[\frac{du(x)}{dx}\left(\int v(x)dx\right)dx\right] \quad (34.33)$$

Chapter 35

Definite Integral

35.1 Definition

For a function $f(x)$ for which $\int f(x)dx = F(x) + A$,

$$\int_a^b f(x)dx = F(b) - F(a) \tag{35.1}$$

35.2 Properties of Definite Integration

$$\int_a^b f(x)dx = \int_a^b f(t)dt \tag{35.2}$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx \tag{35.3}$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \tag{35.4}$$

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx \tag{35.5}$$

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx \tag{35.6}$$

$$\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, \ f(2a-x) = f(x) \\ 0, \ f(2a-x) = f-(x) \end{cases} \tag{35.7}$$

$$\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx, \ f(x) \text{ is even} \\ 0, \ f(x) \text{ is odd} \end{cases} \tag{35.8}$$

35.3 Approximation

$$f(a)(b-a) \leq \int_a^b f(x)dx \leq f(b)(b-a) \tag{35.9}$$

35.4 Sum of Infinite Series as a Definite Integral

Refer to 3.5.2.

Chapter 36

Reduction Formulae

$$\int \sin^n x dx = -\frac{1}{n}\sin^{n-1} x \cos x dx + \frac{n-1}{n}\int \sin^{n-2} x dx \tag{36.1}$$

$$\int \cos^n x dx = -\frac{1}{n}\cos^{n-1} x \sin x dx + \frac{n-1}{n}\int \cos^{n-2} x dx \tag{36.2}$$

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx = \begin{cases} \frac{(n-1)\cdot(n-3)\cdots 3\cdot 1}{n\cdot(n-2)\cdots 4\cdot 2}\left(\frac{\pi}{2}\right), n \to \text{even} \\ \frac{(n-1)\cdot(n-3)\cdots 4\cdot 2}{n\cdot(n-2)\cdots 3\cdot 1}, n \to \text{odd} \end{cases} \tag{36.3}$$

$$\int \sin^m x \cos^n x dx = \frac{-\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x dx \tag{36.4}$$

For $I(m,n) = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx$:
When m and n are both even:

$$I(m,n) = \frac{[(m-1).(m-3)\cdots 3.1][(n-1).(n-3)\cdots 3.1]}{(m+n).(m+n-1)\cdots(4).(2)} \cdot \frac{\pi}{2} \tag{36.5}$$

Otherwise:

$$I(m,n) = \frac{[(m-1).(m-3)\cdots(2 \text{ or } 1)][(n-1).(n-3)\cdots()(2 \text{ or } 1)]}{(m+n).(m+n-1)\cdots(2 \text{ or } 1))} \tag{36.6}$$

$$I_n = \int \tan^n x dx$$

$$\Rightarrow I_n = \frac{\tan^{n-2} x}{n-1} - I_{n-2} \tag{36.7}$$

$$I_n = \int \cot^n x dx$$

$$\Rightarrow I_n = -\frac{\cot^{n-2} x}{n-1} - I_{n-2} \tag{36.8}$$

$$I_n = \int \sec^n x dx$$

$$\Rightarrow I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2} \tag{36.9}$$

$$I_n = \int \csc^n x dx$$

$$\Rightarrow I_n = -\frac{\csc^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2} \tag{36.10}$$

$$I_n = \int x^n e^{ax} dx \tag{36.11}$$

$$\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-2} \tag{36.12}$$

$$I(m,n) = \int x^m (\ln x)^n dx \tag{36.13}$$

$$\int x^m (\ln x)^n dx = \frac{x^{m+1}}{m+1} (\ln x)^n - \frac{n}{m+1} I_{m,n-1} \tag{36.14}$$

Chapter 37

Multiple Integrals

37.1 Two Variables

For

$$I = \iint_R f(x, y) dxdy \tag{37.1}$$

The following substitution are made:

$$x = g(r, \theta) \tag{37.2}$$

$$y = h(r, \theta) \tag{37.3}$$

$$\therefore dxdy = |J| dr d\theta \tag{37.4}$$

Where J is the Jacobian, defined as:

$$J = \begin{vmatrix} \frac{\delta x}{\delta r} & \frac{\delta y}{\delta r} \\ \frac{\delta x}{\delta \theta} & \frac{\delta y}{\delta \theta} \end{vmatrix} \tag{37.5}$$

The equivalent integral is:

$$I = \iint_{R_1} f(g(r, \theta), h(r, \theta)) |J| dr d\theta \tag{37.6}$$

37.2 Three Variables

For

$$I = \iiint_R f(x, y, z) dxdydz \tag{37.7}$$

The following substitution are made:

$$x = g(r, \theta, \phi) \tag{37.8}$$
$$y = h(r, \theta, \phi) \tag{37.9}$$
$$z = k(r, \theta, \phi) \tag{37.10}$$

$$\therefore dxdydz = |J| dr d\theta d\phi \tag{37.11}$$

Where J is the Jacobian, defined as:

$$J = \begin{vmatrix} \frac{\delta x}{\delta r} & \frac{\delta y}{\delta r} & \frac{\delta z}{\delta r} \\ \frac{\delta x}{\delta \theta} & \frac{\delta y}{\delta \theta} & \frac{\delta z}{\delta \theta} \\ \frac{\delta x}{\delta \phi} & \frac{\delta y}{\delta \phi} & \frac{\delta z}{\delta \phi} \end{vmatrix} \tag{37.12}$$

The equivalent integral is:

$$I = \iiint_{R_1} f(g(r, \theta, \phi), h(r, \theta, \phi), k(r, \theta, \phi)) |J| dr d\theta d\phi \tag{37.13}$$

Chapter 38

Differential Equation

38.1 1^{st} Order, 1^{st} Degree Differential Equation

For the equation:

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{38.1}$$

Then an Integral Function (I.F.) is defined as:

$$I.F. = e^{\int P(x)dx} \tag{38.2}$$

Then the solution of the equation 38.1 is given by:

$$y(I.F.) = \int Q(I.F.)dx \tag{38.3}$$

38.2 2^{nd} Order, 1^{st} Degree Differential Equation

For the equation:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0 \tag{38.4}$$

OR

$$y'' + ay' + by = 0 \tag{38.5}$$

By substituting $y = e^{\lambda x}$, the equation obtained is:

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} + be^{\lambda x} = 0$$

$$\because e^{\lambda x} \neq 0$$

$$\Rightarrow \lambda^2 + a\lambda + b = 0 \tag{38.6}$$

If α and β are the solutions of the equation 38.6, then the solution of 38.4 can be:

1. If $\alpha = \beta$ and $\alpha, \beta \in \mathbb{R}$:
$$y = (c_1 + c_2 x)e^{\alpha x} \tag{38.7}$$
2. If $\alpha \neq \beta$ and $\alpha, \beta \in \mathbb{R}$:
$$y = c_1 e^{\alpha x} + c_2 e^{\beta x} \tag{38.8}$$
3. If $\lambda = \alpha + i\beta$:
$$y = e^{\alpha x}\left[A\cos(\beta x) + B\sin(\beta x)\right] \tag{38.9}$$

38.3 Special Cases of Differential Equation

38.3.1 Definition of Inverse Operator

The operator D is equivalent to $\frac{d}{dx}$. If $Df(x) = X$, then $f(x) = \frac{1}{D}X = \int X dx$.

38.3.2 Special Cases

1. $$f(x) = \frac{1}{D-a}X = e^{ax}\int Xe^{-ax}dx \tag{38.10}$$
2. $$\frac{1}{f(D)}e^{ax} = \begin{cases} \frac{e^{ax}}{f(a)}, f(a) \neq 0 \\ x\frac{e^{ax}}{f'(a)}, f(x) = 0 \text{ and } f'(a) \neq 0 \\ x^2\frac{e^{ax}}{f''(a)}, f(x) = 0 \text{ and } f'(a) = 0 \end{cases} \tag{38.11}$$
3. $$\frac{1}{f(D)}x^m = [f(D)]^{-1}x^m \tag{38.12}$$
   $[f(D)]^{-1}$ is expanded and arranged in terms of ascending powers of D and operated on x^m.
4. (a)
$$\begin{aligned} \frac{1}{f(D)}\sin(ax) &= \frac{1}{\phi(D^2)}\sin(ax) \\ &= \frac{1}{\phi(-a^2)}\sin(ax) \end{aligned} \tag{38.13}$$

(b)

$$\begin{aligned} \frac{1}{f(D)}\cos(ax) &= \frac{1}{\phi(D^2)}\cos(ax) \\ &= \frac{1}{\phi(-a^2)}\cos(ax) \end{aligned} \tag{38.14}$$

5. (a)

$$\begin{aligned} \frac{1}{f(D)}\sin(ax) &= \frac{1}{\phi(D^2, D)}\sin(ax) \\ &= \frac{1}{\phi(-a^2, D)}\sin(ax) \end{aligned} \tag{38.15}$$

(b)

$$\begin{aligned} \frac{1}{f(D)}\cos(ax) &= \frac{1}{\phi(D^2, D)}\cos(ax) \\ &= \frac{1}{\phi(-a^2, D)}\cos(ax) \end{aligned} \tag{38.16}$$

6. (a)

$$\begin{aligned} \frac{1}{f(D)}\sin(ax) &= \frac{\psi(D)}{\phi(D^2)}\sin(ax) \\ &= \frac{\psi(D)}{\phi(-a^2)}\sin(ax) \end{aligned} \tag{38.17}$$

(b)

$$\begin{aligned} \frac{1}{f(D)}\cos(ax) &= \frac{\psi(D)}{\phi(D^2)}\cos(ax) \\ &= \frac{\psi(D)}{\phi(-a^2)}\cos(ax) \end{aligned} \tag{38.18}$$

7. (a)

$$\frac{1}{f(D)}\sin(ax) = x\frac{1}{f'(D)}\sin(ax) \tag{38.19}$$

(b)

$$\frac{1}{f(D)}\cos(ax) = x\frac{1}{f'(D)}\cos(ax) \tag{38.20}$$

38.4 Method of Variation of Parameters

If the equation is of the form:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f \tag{38.21}$$

where a, b, f are functions of x. The solution for 38.21 is obtained by solving for:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0 \tag{38.22}$$

If y_1 and y_2 are the two independent solution of equation 38.22.
Then the general solution of the equation is:

$$y = c_1y_1 + c_2y_2 \tag{38.23}$$

where c_1 and c_2 are the constants.
The particular solution of equation 38.22 will be:

$$y = y_1\left(\int \frac{y_2(-f)}{W}dx\right) + y_2\left(\int \frac{y_1 f}{W}dx\right) \tag{38.24}$$

W is the Wronskian, which is defined by:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{38.25}$$

38.5 Singular and Ordinary Point

For a differential equation:

$$P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_{n-1}\frac{dy}{dx} + P_ny = R(x) \tag{38.26}$$

where $P_0 \cdots P_n$ are functions of x.
If at a point $x = x_0$:

1. $P_0(x_0) \neq 0$, x_0 is an ordinary point.
2. $P_0(x_0) = 0$, x_0 is an singular point:
   (a)
   $$\lim_{x\to x_0}(x - x_0)P_1(x) = c_1 \tag{38.27}$$
   $$\lim_{x\to x_0}(x - x_0)^2P_2(x) = c_2 \tag{38.28}$$
   $$\tag{38.29}$$
   where c_1 and c_2 are both finite quantities x_0 is a regular singular point.

   (b) otherwise it is an irregular singular point.

Chapter 39

Beta and Gamma Functions

For $m, n > 0$:

$$\begin{aligned}\beta(m,n) = & \int_0^1 x^{m-1}(1-x)^{n-1}dx \\ = & \ 2\int_0^{\frac{\pi}{2}} \sin^{2m-1} x \cos^{2n-1} x dx\end{aligned} \tag{39.1}$$

$$\Gamma(n) = \int_0^\infty e^{-1}x^{n-1}dx \tag{39.2}$$

39.1 Important Relations between $\beta(m, n)$ and $\Gamma(n)$ Functions

$$\Gamma(n) = \frac{\Gamma(n+1)}{n} \tag{39.3}$$

$$\Gamma(1) = 1 \tag{39.4}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \approx 1.772 \tag{39.5}$$

$$\Gamma(n+1) = n!, n \in \mathbb{N} \tag{39.6}$$

$$\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \sqrt{\pi}\Gamma(2m) \tag{39.7}$$

$$\Gamma(m)\Gamma(m-1) = \pi\csc(m\pi) \tag{39.8}$$

$$\beta(m,n) = \beta(n,m) \tag{39.9}$$

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \tag{39.10}$$

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \pi \tag{39.11}$$

$$\int_0^{\frac{\pi}{2}} \sin^p x \cos^q x = \frac{1}{2}\frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{\Gamma(\frac{p+2}{2})} \tag{39.12}$$

$$\tag{39.13}$$

Chapter 40

Laplace Transformations

The Laplace Transformation of a function $f(t)$ is defined as:

$$F(s) = \mathcal{L}\{f(t)\} = \lim_{x \to \infty} \int_0^x e^{-st} f(t) dt \tag{40.1}$$

40.1 Basic Transformations

$f(t)$	F(s)
$af(t) + bg(t)$	$aF(s) + bG(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$

Table 40.1: Table of Laplace Transformations

40.2 Important Relations

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a) \tag{40.2}$$
$$\mathcal{L}\{tf(t)\} = -F'(s) \tag{40.3}$$
$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^n(s) \tag{40.4}$$
$$\mathcal{L}\{\frac{f(t)}{t}\} = \lim_{x\to\infty}\int_s^x F(u)du \tag{40.5}$$
$$\mathcal{L}\{\frac{f(t)}{t^n}\} = \lim_{x\to\infty}\int_1\int_2\cdots\int_s^x{}_n F(u)du\cdots du \tag{40.6}$$

40.3 Convolution

For two functions $f(t)$ and $g(t)$ be given such that their Laplace transforms are $F(s)$ and $G(s)$, then:

$$\mathcal{L}\{f(t)\star g(t)\} = F(s)G(s) \tag{40.7}$$

where $f(t)\star g(t)$ is defined as:

$$\int_0^t f(u)g(t-u)du \tag{40.8}$$

40.4 Laplace Transforms of Differentials

If the Laplace Transform of $f(t)$ is $F(s)$[1]:

$$\mathcal{L}\{f'(t)\} = sF(s) - y(0) \tag{40.9}$$
$$\mathcal{L}\{f''(t)\} = s^2F(s) - [sy(0) + y'(0)] \tag{40.10}$$
$$\vdots$$
$$\mathcal{L}\{f^n(t)\} = s^nF(s) - [\sum_{i=0}^{n-1} s^{n-i}y^i(0)] \tag{40.11}$$

[1] Used in initial value problems

Part VI

Operations Research

Chapter 41

Linear Programming Problems

41.1 Basic Feasible Solution

The standard LPP problem has an objective function and conditions.

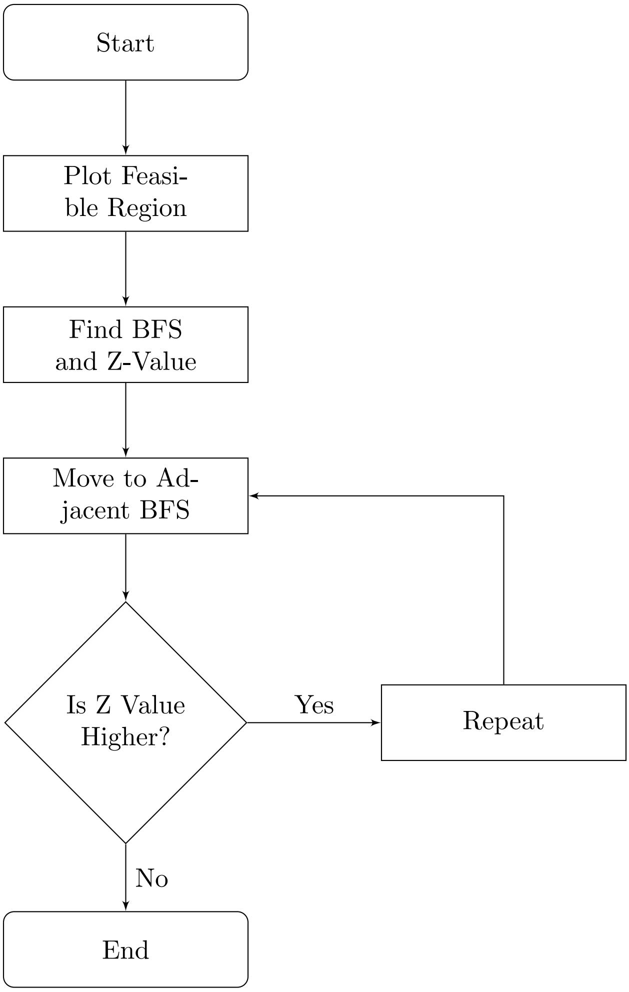
$$\begin{aligned} Z = a_1x_1 + a_2x_2 + \cdots + a_nx_n \\ \text{Subject to:} \\ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \end{aligned}$$

For a system with n variables and m conditions, the number of basic solutions are: $\binom{n}{k}$. For any n - m system there are $n - m$ non-basic variables (NBV) and m basic variables (BV).
For the above system, the basic solutions are obtained by:

NBV	BV	BFS
$x_1, x_2, \cdots, x_{n-m} = 0$ $\vdots$	$x_{n-m+1} = c_1, \cdots, x_n = c_n$ $\vdots$	If $x_{n-m+1,\cdots,x_n,} \geq 0$ then it is a basic feasible solution.

41.1.1 Adjacent Basic Feasible Solutions

If two adjacent BFS share $m-1$ BV then they are called adjacent varibales. The optimal solution is always a extreme point. Thus, graphically:



41.2 Simplex Method