A Book of High School and Engineering Common Core Mathematical Formulae

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Contents

Part I
Algebra

Chapter 1
Logarithm

1.1 Basic Formulae

For $a^x = b$:

$$\log_a x, \forall x \leq 0 \text{ is undefined} \tag{1.1}$$
$$\log_a b = x, bax \neq 1, a \neq 1 \tag{1.2}$$
$$\log_b a^m = m \log_b a, \text{ for } a^m = b \tag{1.3}$$
$$a^{\log_a x} = x \tag{1.4}$$
$$a^{\log_b c} = c^{\log_b a} \tag{1.5}$$
$$\frac{1}{\log_a b} = \log_b a \tag{1.6}$$
$$\log_c(ab) = \log_c a + \log_c b \tag{1.7}$$
$$\log_c \left(\frac{a}{b}\right) = \log_c a - \log_c b \tag{1.8}$$
$$|\log_a x| = \begin{cases} -\log_a x, \text{ if } 0 < x < 1 \\ \log_a x, \text{ if } 1 \leq x < \infty \end{cases} \tag{1.9}$$

1.2 Series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \infty = \lim_{n \to \infty} \sum_{i=0}^{n} \frac{x^i}{i!} \tag{1.10}$$
$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \infty = \lim_{n \to \infty} \sum_{i=1}^{n} (-1)^{(i-1)} \frac{x^i}{i} \tag{1.11}$$

Chapter 2
Complex Numbers

2.1 Basic Formulae

For $z = x + iy$,

$$|z| = \sqrt{x^2 + y^2} \tag{2.1}$$
$$\tan\theta = \frac{y}{x} \tag{2.2}$$
$$\bar{z} = x - iy \tag{2.3}$$

2.2 Arithmetic Operation of Complex Number

For two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \tag{2.4}$$
$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \tag{2.5}$$
$$|z_1 z_2| = |z_1| \cdot |z_2| \tag{2.6}$$
$$\frac{z_1}{z_2} = \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{{a_2}^2 + {b_2}^2} \tag{2.7}$$
$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \tag{2.8}$$

2.3 Euler's Formula

$$z = re^{i\theta}, \text{ where} \tag{2.9}$$
$$r = |z| \tag{2.10}$$
$$e^{i\theta} = \cos\theta + i\sin\theta \tag{2.11}$$
$$\theta = \arctan\left(\frac{y}{x}\right) \tag{2.12}$$

2.4 Trigonometric Ratios in Complex Form

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta \tag{2.13}$$

$$\Rightarrow \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{2.14}$$

$$e^{i\theta} - e^{-i\theta} = 2\sin\theta \tag{2.15}$$

$$\Rightarrow \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} \tag{2.16}$$

2.5 De Moivre's Formula

According to DeMoivre's Formula:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \tag{2.17}$$

Proof

$$\begin{aligned}
\cos\theta + i\sin\theta &= e^{i\theta} \\
\Rightarrow (\cos\theta + i\sin\theta)^n &= e^{n(i\theta)} \\
&= \cos(n\theta) + i\sin(n\theta) \\
&\text{Q.E.D.}
\end{aligned}$$

2.6 Application of Euler's and De Moivre's Formula

For $z_1 = |r_1|\, e^{i\theta_1}$ and $z_2 = |r_2|\, e^{i\theta_2}$

$$z_1 z_2 = (r_1 r_2)\, e^{i(\theta_1+\theta_2)} \tag{2.18}$$

$$\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1-\theta_2)} \tag{2.19}$$

2.7 Roots of Unity

$$\sqrt[n]{1} = e^{i\dfrac{2k\pi}{n}}, \text{ where } k \in [0, n-1] \tag{2.20}$$

2.8 Important Relations of Complex Numbers

$$|z_1 + z_2| \le |z_1| + |z_2| \tag{2.21}$$

$$|z_1 - z_2| \le |z_1| + |z_2| \tag{2.22}$$

$$|z_1 - z_2| \ge |z_1| - |z_2| \tag{2.23}$$

$$|z_1 + z_2| \ge ||z_1| - |z_2|| \tag{2.24}$$

$$|z_1 + z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right) \tag{2.25}$$

Chapter 3
Progression

3.1 Arithmetic Progression (A.P.)

An arithmetic sequence is $a, a+n, a+2n, ...\infty$ or $t_n = a + (n-1)d$, where a is the first term, d is the common difference, and n is the n^{th}-term.
An arithmetic series is $a + (a+d) + (a+2d) + ...\infty$.

3.1.1 Sum of A.P. Series

$$\begin{aligned} S_n &= a + (a+d) + ... + (a + \overline{n-2}d) + (a + \overline{n-1}d) \\ S_n &= (a + \overline{n-1}d) + (a + \overline{n-2}d + ... + (a+d) + a \\ \Rightarrow 2S_n &= n(2a + \overline{n-1}d) \\ \Rightarrow S_n &= \frac{n}{2}(2a + \overline{n-1}d) \end{aligned} \tag{3.1}$$

3.1.2 Important Relation

If the three terms a, b, c are in A.P., then

$$2b = a + c \tag{3.2}$$

3.2 Geometric Progression (G.P.)

An geometric sequence is $a, ar, ar^2, ...\infty$ or $t_n = ar^{n-1}$, where a is the first term, r is the common ratio, and n is the n^{th}-term. An geometric series is $a + ar + ar^2 + ...\infty$.

3.2.1 The Value of 'r'

$$r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = ... = \frac{t_n}{t_{n-1}} \tag{3.3}$$

3.2.2 Sum of a G.P. Series

For a definite G.P. series, where there are n terms in the series, the sum of the series is:

$$S_n = \frac{a|r^n - 1|}{|r-1|} \tag{3.4}$$

For an infite G.P. series the sum of the series is defined for $r < 1$. Sum of such a series is:

$$S_\infty = \frac{a}{1-r} \tag{3.5}$$

3.2.3 Important relations

If the three terms a, b, c are in G.P., then:

$$b^2 = ac \tag{3.6}$$

3.3 Harmonic Progression (H.P.)

If a, b, c are terms of an H.P. then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

$$\frac{2}{b} = \frac{1}{a} + \frac{1}{c} \tag{3.7}$$

$$\Rightarrow b = \frac{2ac}{a+c} \tag{3.8}$$

3.4 Arithmetico-Geometric Progression (A.G.P.)

Sequence $a, (a+d)r, (a+2d)r^2, ..., (a+\overline{n-1}d)r^{n-1}$, where a →first term of A.G.P., d →common difference, and r →common ratio.

3.4.1 Sum of A.G.P.:

For an infinite A.G.P. series, the sum is defined for $r < 1$:

$$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2} \tag{3.9}$$

3.5 Special Series

For $n \in \mathbb{N}$

$$1 + 2 + 3 + ... + (n-1) + n = \frac{n(n-1)}{2} \tag{3.10}$$

$$1^2 + 2^2 + 3^2 + ... + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{3.11}$$

$$1^3 + 2^3 + 3^3 + ... + (n-1)^3 + n^3 = [\frac{n(n-1)}{2}]^2 \tag{3.12}$$

3.5.1 Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{3.13}$$

3.5.2 Riemann's Infinite Series as an Integration

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=r_1}^{r_2} f(\frac{i}{n}) = \int_{\frac{r_1}{n}}^{\frac{r_2}{n}} f(x)dx \tag{3.14}$$

Chapter 4

Test of Convergence of Infinite Series

If $a_1, a_2, a_3, ..., a_n$ is a sequence by a_n and their sum of series is S_n, then the following apply.

4.1 Definition

If

$$\lim_{n \to \infty} S_n = l$$

where l is a finite value, the series S_n is said to converge. A non-convergent series is called a divergent series.

4.2 Tests of Convergence

4.2.1 Comparison Test

If u_n and v_n are two positive series, then:

1. (a) v_n converges
   (b) $u_n \leq v_n \forall n$ Then u_n converges.
2. (a) v_n diverges
   (b) $u_n \geq v_n \forall n$ Then u_n diverges.

4.2.2 Limit Form

If

$$\lim_{x \to \infty} \frac{u_n}{v_n} = l$$

where l is a finite quantity $\neq 0$, then u_n and v_n converge and diverge together.

4.2.3 Integral Test or Maclaurin-Cauchy Test

For a series

$$\sum_{i=N}^{\infty} f(x), \text{ where } N \in \mathbb{Z} \tag{4.1}$$

will only converge if the improper integral

$$\int_N^\infty f(x)dx \tag{4.2}$$

is finite.

If the improper integral is finite, the upper and lower limit of the infinite series is given by:

$$\int_N^\infty f(x)dx \leq \sum_{i=N}^\infty f(x) \leq f(N) + \int_N^\infty f(x)dx \tag{4.3}$$

4.2.4 Ratio Test

If, for two series $\sum u_n$ and $\sum v_n$:

1. (a) $\sum v_n$ converges
   (b) from or after a particular term $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$, then u_n converges.
2. (a) $\sum v_n$ diverges
   (b) from or after a particular term $\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$, then u_n diverges.

4.2.5 D'Alembert's Ratio Test

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lambda \tag{4.4}$$

- series converges if $\lambda < 1$
- series diverges if $\lambda > 1$
- fails if $\lambda = 1$

4.2.6 Rabbe's Test

$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = \kappa \tag{4.5}$$

- series converges if $\kappa < 1$
- series diverges if $\kappa > 1$
- fails if $\kappa = 1$

4.2.7 Cauchy's Root Test

$$\lim_{n\to\infty} |u_n| = \lambda \tag{4.6}$$

- series converges for $\lambda < 1$
- series diverges for $\lambda > 1$
- test fails for $\lambda = 1$

4.2.8 Logarithmic Test

$$\lim_{n\to\infty} n \log\left(\frac{u_n}{u_{n+1}}\right) = \kappa \tag{4.7}$$

- series converges for $\kappa < 1$
- series diverges for $\kappa > 1$
- test fails for $\kappa = 1$

Chapter 5

Determinants

5.1 Definition

For a determinant:

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \tag{5.1}$$

5.1.1 Minor and Cofactor

For a third order determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Minor

$$M\left(a_{11}\right) = M_{11} = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \tag{5.2}$$

i.e., all the terms of determinant expect those in the same row and columns as the one of which the minor is being calculated.

Cofactor

$$C_{ij} = (-1)^{i+j} M_{ij} \tag{5.3}$$

5.2 Properties of Determinants

1. Transposing a determinant does not alter its value.

$$\Delta = \Delta^T \tag{5.4}$$

2. If rows and columns are interchanges m times, the value of the new determinant is

$$\Delta' = (-1)^m \Delta \tag{5.5}$$

3. If two parallel lines are equal, then $\Delta = 0$

4. For $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then $\Delta_1 = k\Delta$

5. For $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{vmatrix}$, then $\Delta_1 = k\Delta$

6. For $C_n \to k_1 C_l + k_2 C_m + k_3 C_n$ or $R_n \to k_1 R_l + k_2 R_m + k_3 R_n$, $\Delta' = \Delta$

5.3 Cramer's Rule

For a system of equations:

$$\begin{aligned} a_1 x + b_1 y + c_1 z &= d_1 \\ a_2 x + b_2 y + c_2 z &= d_2 \\ a_3 x + b_3 y + c_3 z &= d_3 \end{aligned}$$

the following determinants are defined:

$$\begin{aligned} D &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ D_x &= \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \\ D_y &= \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \\ D_z &= \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \end{aligned}$$

The solution for the system of equations is:

$$x = \frac{D_x}{D} \tag{5.6}$$

$$y = \frac{D_y}{D} \tag{5.7}$$

$$z = \frac{D_z}{D} \tag{5.8}$$

5.3.1 Consistency Test

1. If $D \neq 0$, the system is consistent and has unique solutions.
2. If $D = D_x = D_y = D_z = 0$, the system may or may not be consisten and it will have infinite solutions and the system will be dependent.
3. If $D = 0$ and at least one of D_x, D_y, D_z is non zero, the system is inconsistent

Chapter 6
Matrices

For a matrix,

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

and where I_p is an identity matrix of the p^{th} order, the following relations are applicable.

6.1 Sum of Two Matrices

$$A_{m\times n} + B_{m\times n} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{21} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix} \tag{6.1}$$

6.2 Multiplication of Two Matrices

If

$$C_{m\times p} = A_{m\times n} \cdot B_{n\times p} \tag{6.2}$$

then,

$$c_{ik} = \sum_{j=1}^{n} a_{ij} \cdot b_{jk} \tag{6.3}$$

6.2.1 Multiplicative Properties

1. Multiplication of matrices is associative, hence $(AB)C = A(BC)$.
2. $AI = A$

3. $A \cdot A^{-1} = I$
4. $A \cdot (adj A) = (adj A) \cdot A = |A| I$
5. $A^{-1} = \frac{1}{|A|}(adj A)^t$
6. $(AB)^t = B^t A^t$

6.3 Adjoint of a Matrix

$$adj A = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m1} & M_{m2} & \cdots & M_{mn} \end{bmatrix}^t, \text{ where } M_{ij} \text{ is the minor of } a_{ij} \tag{6.4}$$

6.4 Martin's Rule

For a system of equation,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n \end{aligned}$$

The system can be written as:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \tag{6.5}$$

$$\Rightarrow AX = B \tag{6.6}$$

$$\Rightarrow X = A^{-1}B \tag{6.7}$$

Chapter 7
Binomial Theorem

For a binomial expansion $(a+b)^n$, there are $(n+1)$ terms and $(a+b+c)^n$ has $\dfrac{(n+1)(n+2)}{2}$ terms.

7.1 Expansion of a binomial expression

$$\begin{aligned}(a+b)^n &=^n C_0 a^n b^0 +^n C_1 a^{n-1} b^1 +^n C_2 a^{n-2} b^2 + \cdots +^n C_n a^0 b^n \\ &= \sum_{i=0}^{n} {}^nC_i a^{n-i} b^i \\ &\forall n \in \mathbb{N} \end{aligned} \tag{7.1}$$

$$\begin{aligned}(a+b)^n &= a^n b^0 + n a^{n-1} b + \frac{n(n-1)}{2!} a^{n-2} b^2 + \cdots + \frac{n(n-1)\cdots 3\cdot 2\cdot 1}{n!} a^0 b^n + \cdots \infty \\ &\forall n \in \mathbb{R}\end{aligned} \tag{7.2}$$

7.2 Trinomial Expansion

For $(a+b+c)^n$:

$$\begin{aligned}(a+b+c+)^n &= \sum \frac{n!}{i!j!k!} a^i b^j c^k \\ &\forall (i+j+k) = n;\ i,j,k,n \in \mathbb{N}\end{aligned} \tag{7.3}$$

7.3 Properties of Coefficients

$$\text{Sum of Co-efficients: } C_0 + C_1 + C_2 + \cdots + C_{n-1} + C_n = 2^n \tag{7.4}$$

$$\text{Sum of Odd Co-efficients: } C_0 + C_2 + C_4 + \cdots + C_{2n-3} + C_{2n-1} = 2^{n-1} \tag{7.5}$$

$$C_0 - C_1 + C_2 - \cdots + C_{2n-1} - C_{2n} = 0 \tag{7.6}$$

7.4 Pascal's Rule

For $1 \leq k \leq n$ and $k, n \in \mathbb{N}$:

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k} \tag{7.7}$$

Chapter 8
Boolean Algebra

Let B be a set of a, b, c with operations sum $(+)$ and product $(\cdot)$.
Then B is said to belong to the Boolean Structure if the following conditions are satisfied:

Table 8.1: Properties of Boolean Algebraic Structure

Property	Name of Property
$a + b \in B$	
$a \cdot b \in B$	Closure Property
$a + b = b + a$	
$a \cdot b = b \cdot a$	Associative Law
$a(b + c) = ab + ac$	
$a + bc = (a + b)(a + c)$	Commutative Law
$\{0, 1\} \in B$	
$a + 0 = a$	
$a + 1 = 1$	
$a \cdot 0 = 0$	
$a \cdot 1 = a$	Laws of 1 and 0
$a + ab = a$	
$a(a + b) = a$	Absorption Law
$(a + b)' = (a'b')$	De'Morgan's Law

Chapter 9

Remainder Theorems

9.1 Remainder Theorem

If a function $f(x)$ is divided by a binomial $x - a$, then the remainder is provided by $f(a)$.

$$\frac{f(x)}{x-a} \equiv f(a) \mod (x-a) \tag{9.1}$$

Worked Example

Find the remainder when $f(x) = x^3 - 4x^2 - 7x + 10$ is divided by $(x - 2)$.
The remainder:

$$R = \left(x^3 - 4x^2 - 7x + 10\right) \mod (x-2)$$

is given by:

$$\begin{aligned} R = f(2) &= (2)^3 - 4(2)^2 - 7(2) + 10 \\ &= 8 - 16 - 14 + 10 = -12 \end{aligned}$$

9.2 Euler's Remainder Theorem

According to Euler's Remainder Theorem, if x and n are two co-prime numbers:

$$x^{\varphi(n)} \equiv 1 \mod n, x, n \in \mathbb{Z}^+ \tag{9.2}$$

where, $\varphi(n)$ is Euler's totient function.

9.2.1 Euler's Totient Function

For a number defined as:

$$n = \prod_{i=1}^{r} {a_r}^{b_r} \tag{9.3}$$

then Euler's totient function is defined as:

$$\begin{aligned} \varphi(n) &= n \cdot \left[\left(1 - \frac{1}{a_1}\right) \cdot \left(1 - \frac{1}{a_2}\right) \cdot \left(1 - \frac{1}{a_3}\right) \cdots \right] \\ &= n \prod_{i=1}^{r} \left(1 - \frac{1}{a_r}\right) \end{aligned} \tag{9.4}$$

Worked Example

Find the remainder if 3^{76} is divided by 35.

Since:

$$35 = 5^1 \times 7^1$$

Hence the totient quotient of 35 is:

$$\begin{aligned} \varphi(35) &= 35 \cdot \left(1 - \frac{1}{5}\right) \cdot \left(1 - \frac{1}{7}\right) \\ &= 35 \times \frac{4}{5} \times \frac{6}{7} \\ &= 24 \end{aligned}$$

Hence Euler's Theorem yields:

$$\begin{aligned} 3^{24} &\equiv 1 \mod 35 \\ 3^{76} &\equiv 3^{24 \times 3 + 4} \\ &\equiv \left(3^{24}\right)^3 \times 3^4 \mod 35 \\ &\equiv (1)^3 \times 3^4 \mod 35 \\ &\equiv 81 \mod 35 \\ &\equiv 11 \mod 35 \end{aligned}$$

The remainder when 3^{76} is divided by 35 is 11.

9.3 Wilson Theorem

According to Wilson Theorem:

$$(n-1)! \equiv -1 \mod n \tag{9.5}$$

Worked Example

Find the remainder when 28! is divided by 31.
By Wilson's Theorem:

$$\begin{aligned}
30! &\equiv -1 \mod 31 \\
\Rightarrow 30 \cdot 29 \cdot 28! &\equiv -1 \mod 31 \\
\text{Let } 28! \mod 31 &= x \\
\Rightarrow (-1) \cdot (-2) \cdot x &\equiv 30 \mod 31 \\
\Rightarrow 2x &= 30 \\
\Rightarrow x &= 15
\end{aligned}$$

The remainder when 28! is divided by 31 is 15.

Part II
Co-ordinate Geometry

Chapter 10
2-D Co-ordinate Geometry

For the ordered pairs, $A(x_1, y_1)$ and $B(x_2, y_2)$:

10.1 Distance between Two Points

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \tag{10.1}$$
$$\tag{10.2}$$

10.2 Section Formula

If point C divides AB in the ratio $m : n$:

$$C = \left(\frac{nx_1 + mx_2}{m+n}, \frac{ny_1 + my_2}{m+n}\right) \tag{10.3}$$

10.2.1 Corollary: Mid - Point Formula

If C is the mid-point of AB, and $m : n = 1 : 1$:

$$C = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \tag{10.4}$$
$$\tag{10.5}$$

Chapter 11
Triangles

For a triangle defined with three vertices $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ and corresponding sides of length a, b, c, then:

11.1 Centroid of a Traiangle

$$\text{Centroid of } \triangle ABC = (\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}) \tag{11.1}$$

$$\tag{11.2}$$

11.2 Area of Triangle

11.2.1 Determinant Method

$$\text{Area of } \triangle ABC = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \tag{11.3}$$

11.2.2 Heron's Formula

For a triangle, the semi-perimeter, s, is defined as:

$$s = \frac{a + b + c}{2}$$

The area of the triangle can be defined as:

$$\text{Area of } \triangle ABC = \sqrt{s \cdot (s - a) \cdot (s - b) \cdot (s - c)} \tag{11.4}$$

11.3 Incircle of a Triangle

The radius, r, and centre of incircle, o, is:

$$o = \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c} \right) \tag{11.5}$$

$$r = \sqrt{\frac{(s-a)\cdot(s-b)\cdot(s-c)}{s}} \tag{11.6}$$

$$\tag{11.7}$$

11.4 Circumcircle of a Triangle

The radius, R, and centre, O, of circumcircle is defined as:

$$O = \left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C} \right) \tag{11.8}$$

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \tag{11.9}$$

Chapter 12
Straight Line

A straight line can be defined as:

$$y = mx + c \tag{12.1}$$

$$\frac{x}{a} + \frac{y}{b} = 1, \text{ where } a \text{ and } b \text{ are the intercepts at x and y axes respectively} \tag{12.2}$$

$$x \cos \alpha + y \sin \alpha = p \text{ (Normal Form)} \tag{12.3}$$

$$Ax + By + C = 0 \text{ (General Form)} \tag{12.4}$$

12.1 Equation of Straight Line Passing Through (x_0, y_0) and Slope m

$$(y - y_0) = m(x - x_0) \tag{12.5}$$

12.2 Distance Between Two Points on a Line

$$\frac{y_1 - y_2}{\sin \theta} = \frac{x_1 - x_2}{\cos \theta} = \gamma \tag{12.6}$$

$$\theta = \tan^{-1} m \tag{12.7}$$

12.3 Angle Between Two Lines

For two lines with slopes m_1, m_2, the angle between them, θ:

$$\theta = \arctan\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right) \tag{12.8}$$

12.4 Distance of a Point from a Line

Line: $ax + by + c = 0$ Point: (g, h)

$$S = \frac{ag + bh + c}{\sqrt{a^2 + b^2}} \tag{12.9}$$

12.5 Angle Bisector of a Line

For the two lines: $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, the angle bisector is:

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \tag{12.10}$$

If the sign of c_1 and c_2 is the same, then the equation obtained is the internal bisector.

12.6 Equation of a Straight Line Passing through the Intersection of Two Lines

$$(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0 \ \forall k \in \mathbb{R} \tag{12.11}$$

12.7 Relative Position of Points w.r.t. a Line

For the points (x_1, y_1) and (x_2, y_2):

$$k_1 = ax_1 + by_1 + c$$
$$k_2 = ax_2 + by_2 + c$$

If k_1 and k_2 have the same sign, they are on the same side of a line, otherwise on opposite sides.

12.8 Ratio of Division of Line Segment

For any line, $f(x, y) = 0$, the ratio in which it divides (x_1, y_1) and (x_2, y_2) is given by:

$$r = -\frac{f(x_1, y_1)}{f(x_2, y_2)} \tag{12.12}$$

If $\begin{cases} r > 0, \text{ then division is internal} \\ r < 0, \text{ then division is external} \end{cases}$.

Chapter 13
General Theory of Second Degree Equation

For any general equation of the form:

$$ax^2 + by^2 + 2gx + 2fy + 2hxy + c = 0 \tag{13.1}$$

Δ is defined as:

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \tag{13.2}$$

If $\Delta = 0$ then the equation is a pair of straight lines. If $a + b = 0$ then the lines are $\perp$.
If the $\Delta \neq 0$:

1. $a = b, h = 0$ →circle
2. $h^2 = ab$ →parabola
3. $h^2 < ab$ →ellipse
4. $h^2 > ab$ →hyperbola

Chapter 14
Conics

The four conic sections are: circle, parabola, ellipse, and hyperbola. Circle has been done separately in the next chapter.

14.1 Parametric Form of Conics

14.1.1 Hyperbola

$$x = a \sec \theta \tag{14.1}$$
$$y = b = \tan \theta \tag{14.2}$$

14.1.2 Ellipse

$$x = a \cos \phi \tag{14.3}$$
$$y = b \sin \phi \tag{14.4}$$

14.1.3 Parabola

$$x = at^2 \tag{14.5}$$
$$y = 2at \tag{14.6}$$

14.2 Equation form of Conics

14.2.1 Parabola

Table 14.1: Properties of a Parabola

Property	$y^2 = 4ax$	$x^2 = 4ay$
Axis	$y = 0$	$x = 0$
Eccentricity	1	1
Directrix	$x + a = 0$	$y + a = 0$
Focus	$(a, 0)$	$(0, a)$
Vertex	$(0, 0)$	$(0, 0)$
Length of latus rectum	$\|4a\|$	$\|4a\|$
Equation of latus rectum	$x - a = 0$	$y - a = 0$

14.2.2 Ellipse and Hyperbola

For $a > b$:

Table 14.2: Properties of Ellipse and Hyperbola

Property	$\frac{x^2}{a} + \frac{y^2}{b} = 1$ Ellipse	$\frac{x^2}{a} - \frac{y^2}{b} = 1$ Hyperbola
Length of Major Axis	$2a$	$2a$
Length of Minor Axis	$2b$	$2b$
Equation of Major Axis	$x = 0$	$x = 0$
Equation of Minor Axis	$y = 0$	$y = 0$
Eccentricity e	$\sqrt{1 - \frac{b^2}{a^2}}$	$\sqrt{1 + \frac{b^2}{a^2}}$
Vertices	$(\pm a, 0)$	$(\pm a, 0)$
Foci	$(\pm ae, 0)$	$(\pm ae, 0)$
Equation of Directrix	$x \pm \frac{a}{e} = 0$	$x = \pm\frac{a}{e}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Equation of latus rectum	$x \pm ae = 0$	
Centre	$(0, 0)$	$(0, 0)$

Chapter 15
Circles

15.1 Locus Form

$$(x-g)^2 + (y-h)^2 = r^2 \tag{15.1}$$

where the centre is (g, h) and the radius is r.

15.2 Diameter Form

$$(x-a)(x-c) + (y-b)(y-d) = 0 \tag{15.2}$$

where (a, b) and (c, d) are the two ends of the diamter.

15.3 General Form

If the equation of a circle is in the form:

$$x^2 + y^2 + 2gx + 2fy + c = 0 \tag{15.3}$$

Then the following is true about the circle:

1. centre of the circle is $(-g, -f)$
2. radius of circle is $\sqrt{g^2 + f^2 - c}$

15.4 Important Relations

1. If the circle passes through the origin, $g = 0, f = 0$.
2. If the circle touches the x-axis $c = g^2$.
3. If the circle touches the y-axis $c = f^2$.

15.5 Common for Two Circles

1. The common chord passing between two circles S_1 and S_2 are:

$$S_1 - S_2 = 0 \tag{15.4}$$

2. Circles passing through the intersection of two circles is:

$$S_2 + k(S_1 - S_2) = 0 \ \forall k \in \mathbb{R} \tag{15.5}$$

Chapter 16
Vectors

Let two vectors be $\vec{a} = a\hat{i} + b\hat{j} + c\hat{k}$ and $\vec{b} = x\hat{i} + y\hat{j} + z\hat{k}$:

16.1 Modulus of a Vector

For a vector $\vec{a}$, the modulus of the vector is:

$$|\vec{a}| = \sqrt{a^2 + b^2 + c^2} \tag{16.1}$$

16.2 Sum of Vectors

The sum of two vectors is:

$$|\vec{a} + \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{a}|\cos\theta} \tag{16.2}$$

$$\vec{a} + \vec{b} = (a + x)\hat{i} + (b + y)\hat{j} + (c + z)\hat{k} \tag{16.3}$$

The direction of the resultant vector is:

$$\tan\alpha = \frac{b\sin\theta}{a + b\cos\theta} \tag{16.4}$$

where, θ is the angle between the two vectors.

16.3 Product of Vectors

16.3.1 Dot Product

$$\vec{a} \cdot \vec{b} = |a||b|\cos\theta \tag{16.5}$$

$$\vec{a} \cdot \vec{b} = ax + by + cz \tag{16.6}$$

16.3.2 Cross Product

$$\vec{a} \times \vec{b} = |a||b|\sin\theta\hat{n} \tag{16.7}$$

$$\tag{16.8}$$

where $\hat{n}$ is a vector $\perp \vec{a}, \vec{b}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ x & y & z \end{vmatrix} \tag{16.9}$$

16.4 Test of Co-planarity

Three vectors are called co-planar if:

$$\lambda\vec{a} + \mu\vec{b} = \vec{c} \tag{16.10}$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = 0 \tag{16.11}$$

Chapter 17
3D - Space

17.1 Line segments in 3D - Space

For points defined in a 3D space as $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$:

17.1.1 Distance between Two Points

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \tag{17.1}$$

17.1.2 Section Formula of a Line Segment Divided in the ratio $m : n$

$$P = \left(\frac{nx_1 + mx_2}{m+n}, \frac{ny_1 + my_2}{m+n}, \frac{nz_1 + mz_2}{m+n} \right) \tag{17.2}$$

17.2 Line in 3D - Space

For a line which is defined as $\vec{a} = a\hat{i} + b\hat{j} + c\hat{k}$:

1. Line numbers of the line is

$$< a, b, c > \tag{17.3}$$

2. The line cosines are:

$$< \frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}} > \tag{17.4}$$

$$=< l, m, n > \tag{17.5}$$

17.2.1 Angle between Two Lines

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + b_1b_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} \tag{17.6}$$

$$\Rightarrow \cos\theta = l_1l_2 + m_1m_2 + n_1n_2 \tag{17.7}$$

When two lines are $\perp, l_1l_2 + m_1m_2 + n_1n_2 = 0$.
When two lines are $\parallel \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = 1$.

17.2.2 Skew and Co-planar Lines

Let there be two lines $\vec{r_1}$ and $\vec{r_2}$,

$$\vec{r_1} = \vec{a_1} + \mu\vec{b_1}\vec{r_2} = \vec{a_2} + \lambda\vec{b_2} \tag{17.8}$$

17.2.3 Distance between Lines

The shortest distance between r_1 and r_2

$$S = \left| \frac{(\vec{a_1} - \vec{a_2}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right| \tag{17.9}$$

If $S = 0$, the lines intersect.

Cartesian Form

For two lines defined as $\dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1}$ and $\dfrac{x - x_2}{a_2} = \dfrac{y - y_2}{b_2} = \dfrac{z - z_2}{c_2}$:

$$S = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \tag{17.10}$$

Distance Between Parallel Lines

$$S = \left| \frac{\vec{b} \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b}|} \right| \tag{17.11}$$

Distance of a Point to a Line

For a point, (x_1, y_1, z_1) the distance to a line $\dfrac{x - \alpha}{l} = \dfrac{y - \beta}{m} = \dfrac{z - \gamma}{n}$:

$$S = \left(\begin{vmatrix} x_1 - \alpha & y_1 - \beta \\ l & m \end{vmatrix} + \begin{vmatrix} y_1 - \beta & z_1 - \gamma \\ m & n \end{vmatrix} + \begin{vmatrix} z_1 - \gamma & x_1 - \alpha \\ n & l \end{vmatrix} \right)^{\frac{1}{2}} \tag{17.12}$$

17.3 Triangular Plane

17.3.1 Centroid of a Triangle

$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right) \tag{17.13}$$

Chapter 18
3D - Plane

A plane in 3-D space can be defined as:

1. Cartesian Form:

$$ax + by + cz + d = 0 \tag{18.1}$$

2. Vectorial Form:

$$\vec{r} \cdot \vec{n} = p \tag{18.2}$$

, where $\vec{r}$ is a line on the plane, $\vec{n}$ is a normal to the plane, and p is perpendicular distance to the plane from the origin.

18.1 Angle Between Two Planes

For two planes, $\vec{r_1} \cdot \vec{n_1} = p_1$ and $\vec{r_2} \cdot \vec{n_2} = p_2$, the angle between the planes, θ is:

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|} \tag{18.3}$$

In the Cartesian Form:

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{{a_1}^2 + {b_1}^2 + {c_1}^2}\sqrt{{a_2}^2 + {b_2}^2 + {c_2}^2}} \tag{18.4}$$

18.2 Distance of a Point from a Plane

18.2.1 Catesian Form

For the point (p, q, r) and the plane, $ax + by + cz + d = 0$:

$$S = \frac{ap + bq + cr + d}{\sqrt{a^2 + b^2 + c^2}} \tag{18.5}$$

18.2.2 Vector Form

For the point $\vec{g} = p\hat{i} + q\hat{j} + r\hat{k}$ and the plane $\vec{r} \cdot (a\hat{i} + b\hat{j} + c\hat{k}) + d = 0$:

$$S = \frac{\left(a\hat{i} + b\hat{j} + c\hat{k}\right) \cdot \left(p\hat{i} + q\hat{j} + r\hat{k}\right)}{\sqrt{a^2 + b^2 + c^2}} \tag{18.6}$$

$$\Rightarrow S = \frac{\left(a\hat{i} + b\hat{j} + c\hat{k}\right) \cdot \vec{g}}{|a\hat{i} + b\hat{j} + c\hat{k}|} \tag{18.7}$$

Part III
Statistics

Chapter 19

Descriptive Statistics

19.1 Measure of Location

19.1.1 Mean

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{19.1}$$

19.1.2 Median

For odd number of elements in a dataset:

$$\tilde{x} = x_{\frac{n+1}{2}} \tag{19.2}$$

For even number of elements in a dataset:

$$\tilde{x} = \frac{x_{\frac{n}{2}} + x_{\left(\frac{n}{2}+1\right)}}{2} \tag{19.3}$$

19.1.3 Mode

$$Mo(x) = \max(f(x_i)) \tag{19.4}$$

19.1.4 Quartile

Measure of percentage of elements less than or equal to a term

19.2 Measure of Spread

19.2.1 Variance

Variance measured on the whole population

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^2 \tag{19.5}$$

19.2.2 Sample Variance

Variance measured on a sample population

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 \tag{19.6}$$

19.2.3 Standard Deviation and Sample Standard

$$\sigma = \sqrt{\sigma^2} \tag{19.7}$$

$$s = \sqrt{s^2} \tag{19.8}$$

19.2.4 Co-efficient of Variance

$$v = \frac{s}{\bar{x}} \tag{19.9}$$

19.3 Skewness

19.3.1 Types of Skewness

Name	Other Name	Characteristic
Right Skew	Positive Skew	Data concentrated on the lower side
Symmetric Distribution	Normal Distribution	Data distributed evenly
Left Skew	Negative Skew	Data concentrated on the higher side

19.3.2 Measure of Skewness

Skewness is measured by the Moment Co-efficient of Skewness.

$$g_m = \frac{m_3}{s^3}, \text{ where} \tag{19.10}$$

$$m_3 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^3 \tag{19.11}$$

Type of Skewness

The type of skewness from the value is g_m is:

Value of g_m	Type
$g_m = 0$	Symmetric
$g_m > 0$	Positive Skew
$g_m < 0$	Negative Skew

Value of g_m	Degree
$\lvert g_m \rvert > 1$	High Skewness
$0.5 < \lvert g_m \rvert \geq 1$	Moderate Skewness
$\lvert g_m \rvert \leq 0.5$	Low Skewness

Degree of Skewness

The degree of skewness from the value is g_m is:

19.4 Kurtosis

Kurtosis is the measure of peakedness of data. Fisher's kurtosis measure is defined as:

$$\gamma = \frac{m_4}{s^4}, \text{ where} \tag{19.12}$$

$$m_4 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^4 \tag{19.13}$$

19.4.1 Type of Kurtosis

The types of kurtosis from the value of γ are:

Value of γ	Type
$\gamma = 0$	Normal Distribution or Mesokurtic
$\gamma < 0$	Flattened or Platykurtic
$\gamma > 0$	Peaked or Lepokurtic

Chapter 20
Hypothesis Testing

20.1 T-Test

$$T = \frac{\bar{X} - \mu}{\frac{s}{\sqrt{n}}} \tag{20.1}$$

where:

$$\begin{aligned}
\bar{X} &= \text{Sample Mean} \\
\mu &= \text{Assumed Mean} \\
s &= \text{Number of Samples} \\
n &= \text{Number of observations}
\end{aligned}$$

If $T < t_c$ the H_0 is not rejected. t_c is a functions of level of significance (α) and degrees of freedom ($v = n - 1$).

20.2 χ^2 Test

$$\chi^2 = \sum_i \sum_j \frac{(h_{ij}^o - h_{ij}^e)^2}{h_{ij}^e} \tag{20.2}$$

where:

$$\begin{aligned}
h_e &= \text{Expected Value} \\
h_o &= \text{Actual Value}
\end{aligned}$$

If $\chi^2 < \chi_c^2$ then H_0 is not rejected. χ_c is a functions of level of significance (α) and degrees of freedom ($v = (i-1)(j-1)$).

Chapter 21
Research and Survey Design

21.1 Population Covariance

$$\mathrm{Cov}(x,y) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_x)(y_i - \mu_y) \tag{21.1}$$

21.2 Sample Covariance

$$\mathrm{Cov}(x,y) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) \tag{21.2}$$

21.3 Bravais-Pearson Correlation Co-efficient

$$\begin{align}
r &= \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2} \cdot \sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}} \tag{21.3}\\
&= \frac{\mathrm{Cov}(x,y)}{\sqrt{\mathrm{Var}(x)\cdot \mathrm{Var}(y)}} \tag{21.4}\\
&= \frac{\mathrm{Cov}(x,y)}{\sigma_x \cdot \sigma_y} \tag{21.5}
\end{align}$$

21.4 Spearman's Rank Correlation Co-efficient

$$\begin{align}
r_s &= 1 - \frac{6\sum d_i^2}{n\,(n^2-1)} \tag{21.6}\\
d_i &= R(X_i) - R(Y_i) \tag{21.7}
\end{align}$$

Chapter 22

Estimation of Regression Function

For the regression functions:

$$Y_i = \beta_0 + \beta_1 X_1 \tag{22.1}$$

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_1 \tag{22.2}$$

$$\tag{22.3}$$

where Y_i is the observed dependent variable (DV), $\hat{Y}_i$ is the estimated DV, and X_i is the independent variable (IV).

$$u_i = Y_i - \hat{Y}_i \tag{22.4}$$

$$\Rightarrow Y_i = \hat{Y}_i + u_i \tag{22.5}$$

$$\Rightarrow Y_i = \hat{\beta}_0 + \hat{\beta}_1 X_1 + u_i \tag{22.6}$$

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i \tag{22.7}$$

The objective function is:

$$\min_{u_i} \sum u_i = \min \sum_i \left[Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i\right]^2$$

Since the regression function passes through: $\left(\bar{X}, \bar{Y}\right)$

$$\beta_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$$

$$\begin{aligned}
\min_{u_i} \sum u_i &= \min \sum_i \left[Y_i - \bar{Y} + \hat{\beta}_1 \bar{X} - \hat{\beta}_1 X_i\right]^2 \\
&= \min \sum_i \left[\left(Y_i - \bar{Y}\right) - \hat{\beta}_1 \left(X_i - \bar{X}\right)\right]^2 \\
&= \min \sum_i \left[\left(Y_i - \bar{Y}\right)^2 - 2 \cdot \left(Y_i - \bar{Y}\right) \cdot \hat{\beta}_1 \left(X_i - \bar{X}\right) + \hat{\beta}_1{}^2 \left(X_i - \bar{X}\right)^2\right] \\
&= \min \left[\sum_i \left(Y_i - \bar{Y}\right)^2 - 2 \cdot \hat{\beta}_1 \sum_i \left(Y_i - \bar{Y}\right) \cdot \left(X_i - \bar{X}\right) + \hat{\beta}_1{}^2 \sum_i \left(X_i - \bar{X}\right)^2\right] \\
\Rightarrow u_i^{\beta_1} &= -2 \sum_i \left(Y_i - \bar{Y}\right) + 2\hat{\beta}_1 \left(X_i - \bar{X}\right)^2 = 0 \qquad \text{(For optima Conditions)} \\
\Rightarrow \hat{\beta}_1 &= \boxed{\frac{\sum_i (Y_i - \bar{Y})(X_i - \bar{X})}{\sum_i (X_i - \bar{X})^2}} \\
\Rightarrow \hat{\beta}_0 &= \boxed{\bar{Y} - \hat{\beta}_1 \bar{X}}
\end{aligned}$$

22.1 Sum of Squares Error

$$TSS = \sum_i (Y_i - \bar{Y})^2 \tag{22.8}$$

$$= \underbrace{\sum_i (\hat{Y}_i - \bar{Y})}_{\text{Explained Sum of Square Error (ESS)}} + \underbrace{\sum_i u_i^2}_{\text{Residual Sum of Squares Error (RSS)}} \tag{22.9}$$

22.1.1 R^2: Coefficient of Determination

$$R^2 = \frac{\text{ESS}}{\text{TSS}} \tag{22.10}$$

$$= 1 - \frac{\text{RSS}}{\text{TSS}} \tag{22.11}$$

$$= 1 - \frac{\sum_i u_i^2}{\sum_i (Y_i - \bar{Y})^2} \tag{22.12}$$

For a regression analysis with single IV:

$$\sqrt{R^2} = v$$

22.1.2 $\bar{R^2}$: Coefficient of Determination

$$\bar{R^2} = 1 - \frac{\dfrac{\sum_i u_i^2}{(N-K-1)}}{\dfrac{\sum_i (Y_i - \bar{Y})^2}{(N-1)}} \tag{22.13}$$

where, N is the number of observations and K is the number of independent variables.

22.2 T-Test

Test for statistical significance of a single IV.

$$T = \frac{\hat{\beta}_1 - 0}{S_e(\hat{\beta}_1)} \tag{22.14}$$

22.3 F-Test

Test for statistical significance of all IVs together.

$$F = \frac{\dfrac{\text{ESS}}{(K-1)}}{\dfrac{\text{RSS}}{(N-K)}} \qquad (F \geq F_c, H_0 \text{ is rejected})$$

22.4 Test for Heteroskedasticity

22.4.1 Definition

$\sigma_{\epsilon_i} \forall \epsilon_i \in [X_a, X_b] = \sigma_{\epsilon_i} \forall \epsilon_i \in [X_{b+1}, X_c]$

22.4.2 Durbin-Watson Test

$$d_e = \frac{\sum_{t=2}^{n} (\hat{u_t} - \hat{u_{t-1}})^2}{\sum_{t=2}^{n} \hat{u_t}^2} \tag{22.15}$$

For the H_0: No autocorrelation:

d	H_0
$0 \leq d_e \leq d_L$ & $(4 - d_L) \leq d_e \leq 4$	Rejected
$d_L < d_e \leq d_U$ & $(4 - d_U) < d_e \leq (4 - d_L)$	Decision Free Zone
$d_L < d_e < D_U$	Not rejected

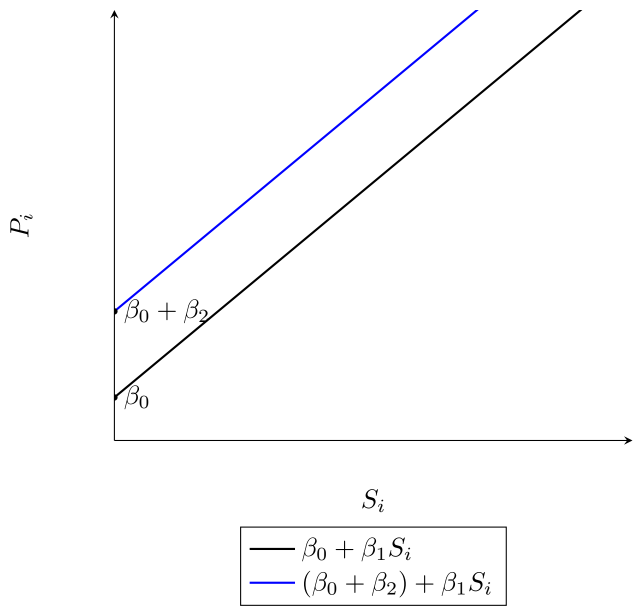
Chapter 23

Dummy Variables

23.1 Dummy Variable

$$P_i = \beta_0 + \beta_1 S_1 + \beta_2 D_i + \epsilon_i \tag{23.1}$$

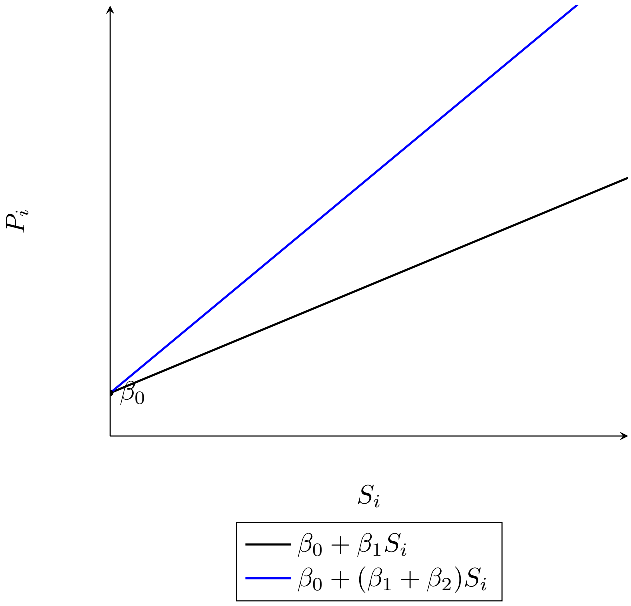
$$E(P_i) = \begin{cases} (\hat{\beta}_0 + \hat{\beta}_2) + \hat{\beta}_1 S_i, & D_i = 1 \\ \hat{\beta}_0 + \hat{\beta}_1 S_i, & D_i = 0 \end{cases} \tag{23.2}$$



23.2 Slope Dummy Variable

$$P_i = \beta_0 + \beta_1 S_1 + \beta_2 (S_i \cdot D_i) + \epsilon_i \tag{23.3}$$

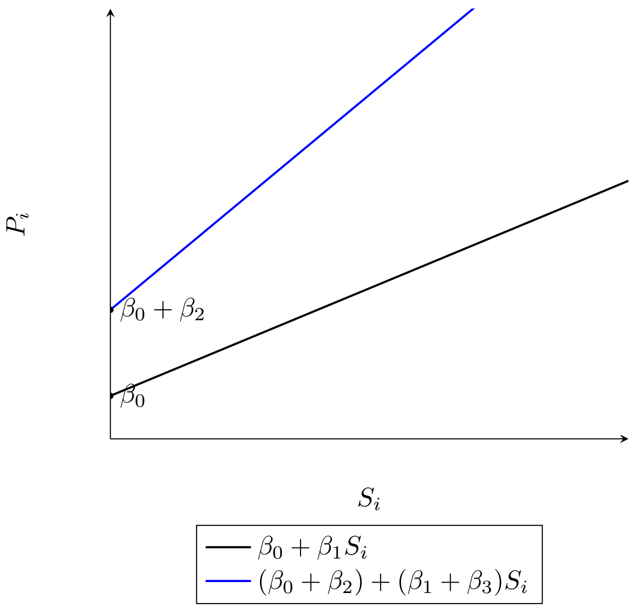
$$E(P_i) = \begin{cases} \hat{\beta}_0 + \left(\hat{\beta}_1 + \hat{\beta}_2\right) S_i, & D_i = 1 \\ \hat{\beta}_0 + \hat{\beta}_1 S_i, & D_i = 0 \end{cases} \tag{23.4}$$



23.3 Slope & Dummy Variable

$$P_i = \beta_0 + \beta_1 S_1 + \beta_2 D_i + \beta_3 S_i D_i + \epsilon_i \tag{23.5}$$

$$E(P_i) = \begin{cases} (\hat{\beta}_0 + \hat{\beta}_2) + \left(\hat{\beta}_1 + \hat{\beta}_3\right) S_i, & D_i = 1 \\ \hat{\beta}_0 + \hat{\beta}_1 S_i, & D_i = 0 \end{cases} \tag{23.6}$$



23.4 Multi-Categories Dummy Variable

$$P_0 = b_0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \underbrace{b_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + b_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{Leads to Perfect Multicollinearlity}} \tag{23.7}$$

Alternatives

- B_n captures the mean of each category, but F-Test is impossible

$$y = \beta_1 D_{1i} + \beta_2 D_{2i} + \beta_3 D_{3i} \tag{23.8}$$

- Computer drops automatically drops a variable

$$y = \beta_0 + \beta_1 D_{1i} + \beta_2 D_{2i} + \beta_3 D_{3i} \tag{23.9}$$

- Manually dropping a variable

$$y = \beta_0 + \beta_1 D_{1i} + \beta_2 D_{2i} \tag{23.10}$$

Chapter 24

Logistic Regression

For $Y_i \in \{0, 1\}$:

$$z_k = \beta_0 + \sum_{j=1}^{n} \beta_{jk} x_j + \epsilon_k, \beta_j \rightarrow \text{Logit Coefficient} \tag{24.1}$$

$$p = \frac{\exp^k}{1 + \exp^k} = \frac{1}{1 + \exp^{-k}} \tag{24.2}$$

where p is the probability of $y = 1$.