Mathematical Formulae
A Book of High School and Engineering Common Core Mathematical Formulae

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Contents

Part I
Algebra

Chapter 1

Logarithm

1.1 Basic Formulae

For $a^x = b$:

$$\log_a x, \forall x \leq 0 \text{ is undefined} \tag{1.1}$$
$$\log_a b = x, bax \neq 1, a \neq 1 \tag{1.2}$$
$$\log_b a^m = m \log_b a, \text{ for } a^m = b \tag{1.3}$$
$$a^{\log_a x} = x \tag{1.4}$$
$$a^{\log_b c} = c^{\log_b a} \tag{1.5}$$
$$\frac{1}{\log_a b} = \log_b a \tag{1.6}$$
$$\log_c(ab) = \log_c a + \log_c b \tag{1.7}$$
$$\log_c \left(\frac{a}{b}\right) = \log_c a - \log_c b \tag{1.8}$$
$$|\log_a x| = \begin{cases} -\log_a x, & \text{if } 0 < x < 1 \\ \log_a x, & \text{if } 1 \leq x < \infty \end{cases} \tag{1.9}$$

1.2 Series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \infty = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{x^i}{i!} \tag{1.10}$$
$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \infty = \lim_{n\to\infty} \sum_{i=1}^{n} (-1)^{(i-1)} \frac{x^i}{i} \tag{1.11}$$

Chapter 2
Complex Numbers

2.1 Basic Formulae

For $z = x + iy$,

$$|z| = \sqrt{x^2 + y^2} \tag{2.1}$$

$$\tan\theta = \frac{y}{x} \tag{2.2}$$

$$\bar{z} = x - iy \tag{2.3}$$

2.2 Arithmetic Operation of Complex Number

For two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \tag{2.4}$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \tag{2.5}$$

$$|z_1 z_2| = |z_1| \cdot |z_2| \tag{2.6}$$

$$\frac{z_1}{z_2} = \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{{a_2}^2 + {b_2}^2} \tag{2.7}$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \tag{2.8}$$

2.3 Euler's Formula

$$z = re^{i\theta}, \text{ where} \tag{2.9}$$

$$r = |z| \tag{2.10}$$

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{2.11}$$

$$\theta = \arctan\left(\frac{y}{x}\right) \tag{2.12}$$

2.4 Trigonometric Ratios in Complex Form

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta \tag{2.13}$$

$$\Rightarrow \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{2.14}$$

$$e^{i\theta} - e^{-i\theta} = 2\sin\theta \tag{2.15}$$

$$\Rightarrow \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} \tag{2.16}$$

2.5 De Moivre's Formula

According to DeMoivre's Formula:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \tag{2.17}$$

Proof

$$\begin{aligned} \cos\theta + i\sin\theta &= e^{i\theta} \\ \Rightarrow (\cos\theta + i\sin\theta)^n &= e^{n(i\theta)} \\ &= \cos(n\theta) + i\sin(n\theta) \\ &\text{Q.E.D.} \end{aligned}$$

2.6 Application of Euler's and De Moivre's Formula

For $z_1 = |r_1|\, e^{i\theta_1}$ and $z_2 = |r_2|\, e^{i\theta_2}$

$$z_1 z_2 = (r_1 r_2)\, e^{i(\theta_1+\theta_2)} \tag{2.18}$$

$$\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1-\theta_2)} \tag{2.19}$$

2.7 Roots of Unity

$$\sqrt[n]{1} = e^{i\dfrac{2k\pi}{n}}, \text{ where } k \in [0, n-1] \tag{2.20}$$

2.8 Important Relations of Complex Numbers

$$|z_1 + z_2| \leq |z_1| + |z_2| \tag{2.21}$$

$$|z_1 - z_2| \leq |z_1| + |z_2| \tag{2.22}$$

$$|z_1 - z_2| \geq |z_1| - |z_2| \tag{2.23}$$

$$|z_1 + z_2| \geq ||z_1| - |z_2|| \tag{2.24}$$

$$|z_1 + z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right) \tag{2.25}$$

Chapter 3

Progression

3.1 Arithmetic Progression (A.P.)

An arithmetic sequence is $a, a+n, a+2n, ...\infty$ or $t_n = a+(n-1)d$, where a is the first term, d is the common difference, and n is the n^{th}-term.
An arithmetic series is $a+(a+d)+(a+2d)+...\infty$.

3.1.1 Sum of A.P. Series

$$\begin{aligned} S_n &= a+(a+d)+...+(a+\overline{n-2}d)+(a+\overline{n-1}d) \\ S_n &= (a+\overline{n-1}d)+(a+\overline{n-2}d+...+(a+d)+a \\ \Rightarrow 2S_n &= n(2a+\overline{n-1}d) \\ \Rightarrow S_n &= \frac{n}{2}(2a+\overline{n-1}d) \end{aligned} \tag{3.1}$$

3.1.2 Important Relation

If the three terms a, b, c are in A.P., then

$$2b = a+c \tag{3.2}$$

3.2 Geometric Progression (G.P.)

An geometric sequence is $a, ar, ar^2, ...\infty$ or $t_n = ar^{n-1}$, where a is the first term, r is the common ratio, and n is the n^{th}-term. An geometric series is $a+ar+ar^2+...\infty$.

3.2.1 The Value of 'r'

$$r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = ... = \frac{t_n}{t_{n-1}} \tag{3.3}$$

3.2.2 Sum of a G.P. Series

For a definite G.P. series, where there are n terms in the series, the sum of the series is:

$$S_n = \frac{a|r^n - 1|}{|r - 1|} \tag{3.4}$$

For an infite G.P. series the sum of the series is defined for $r < 1$. Sum of such a series is:

$$S_\infty = \frac{a}{1 - r} \tag{3.5}$$

3.2.3 Important relations

If the three terms a, b, c are in G.P., then:

$$b^2 = ac \tag{3.6}$$

3.3 Harmonic Progression (H.P.)

If a, b, c are terms of an H.P. then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

$$\frac{2}{b} = \frac{1}{a} + \frac{1}{c} \tag{3.7}$$

$$\Rightarrow b = \frac{2ac}{a + c} \tag{3.8}$$

3.4 Arithmetico-Geometric Progression (A.G.P.)

Sequence $a, (a+d)r, (a+2d)r^2, ..., (a+\overline{n-1}d)r^{n-1}$, where a →first term of A.G.P., d →common difference, and r →common ratio.

3.4.1 Sum of A.G.P.:

For an infinite A.G.P. series, the sum is defined for $r < 1$:

$$S_\infty = \frac{a}{1 - r} + \frac{dr}{(1 - r)^2} \tag{3.9}$$

3.5 Special Series

For $n \in \mathbb{N}$

$$1 + 2 + 3 + ... + (n - 1) + n = \frac{n(n - 1)}{2} \tag{3.10}$$

$$1^2 + 2^2 + 3^2 + ... + (n - 1)^2 + n^2 = \frac{n(n + 1)(2n + 1)}{6} \tag{3.11}$$

$$1^3 + 2^3 + 3^3 + ... + (n - 1)^3 + n^3 = [\frac{n(n - 1)}{2}]^2 \tag{3.12}$$

3.5.1 Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{3.13}$$

3.5.2 Riemann's Infinite Series as an Integration

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=r_1}^{r_2} f(\frac{i}{n}) = \int_{\frac{r_1}{n}}^{\frac{r_2}{n}} f(x)dx \tag{3.14}$$

Chapter 4

Test of Convergence of Infinite Series

If $a_1, a_2, a_3, ..., a_n$ is a sequence by a_n and their sum of series is S_n, then the following apply.

4.1 Definition

If

$$\lim_{n \to \infty} S_n = l$$

where l is a finite value, the series S_n is said to converge. A non-convergent series is called a divergent series.

4.2 Tests of Convergence

4.2.1 Comparison Test

If u_n and v_n are two positive series, then:

1. (a) v_n converges
   (b) $u_n \leq v_n \forall n$ Then u_n converges.
2. (a) v_n diverges
   (b) $u_n \geq v_n \forall n$ Then u_n diverges.

4.2.2 Limit Form

If

$$\lim_{x \to \infty} \frac{u_n}{v_n} = l$$

where l is a finite quantity $\neq 0$, then u_n and v_n converge and diverge together.

4.2.3 Integral Test or Maclaurin-Cauchy Test

For a series

$$\sum_{i=N}^{\infty} f(x), \text{ where } N \in \mathbb{Z} \tag{4.1}$$

will only converge if the improper integral

$$\int_{N}^{\infty} f(x)dx \tag{4.2}$$

is finite.

If the improper integral is finite, the upper and lower limit of the infinite series is given by:

$$\int_{N}^{\infty} f(x)dx \leq \sum_{i=N}^{\infty} f(x) \leq f(N) + \int_{N}^{\infty} f(x)dx \tag{4.3}$$

4.2.4 Ratio Test

If, for two series $\sum u_n$ and $\sum v_n$:

1. (a) $\sum v_n$ converges
   (b) from or after a particular term $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$, then u_n converges.
2. (a) $\sum v_n$ diverges
   (b) from or after a particular term $\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$, then u_n diverges.

4.2.5 D'Alembert's Ratio Test

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lambda \tag{4.4}$$

- series converges if $\lambda < 1$
- series diverges if $\lambda > 1$
- fails if $\lambda = 1$

4.2.6 Rabbe's Test

$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = \kappa \tag{4.5}$$

- series converges if $\kappa < 1$
- series diverges if $\kappa > 1$
- fails if $\kappa = 1$

4.2.7 Cauchy's Root Test

$$\lim_{n\to\infty} |u_n| = \lambda \tag{4.6}$$

- series converges for $\lambda < 1$
- series diverges for $\lambda > 1$
- test fails for $\lambda = 1$

4.2.8 Logarithmic Test

$$\lim_{n\to\infty} n \log\left(\frac{u_n}{u_{n+1}}\right) = \kappa \tag{4.7}$$

- series converges for $\kappa < 1$
- series diverges for $\kappa > 1$
- test fails for $\kappa = 1$

Chapter 5
Determinants

5.1 Definition

For a determinant:

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \tag{5.1}$$

5.1.1 Minor and Cofactor

For a third order determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Minor

$$M\left(a_{11}\right) = M_{11} = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \tag{5.2}$$

i.e., all the terms of determinant expect those in the same row and columns as the one of which the minor is being calculated.

Cofactor

$$C_{ij} = (-1)^{i+j} M_{ij} \tag{5.3}$$

5.2 Properties of Determinants

1. Transposing a determinant does not alter its value.

$$\Delta = \Delta^T \tag{5.4}$$

2. If rows and columns are interchanges m times, the value of the new determinant is

$$\Delta' = (-1)^m \Delta \tag{5.5}$$

3. If two parallel lines are equal, then $\Delta = 0$

4. For $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then $\Delta_1 = k\Delta$

5. For $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{vmatrix}$, then $\Delta_1 = k\Delta$

6. For $C_n \to k_1C_l + k_2C_m + k_3C_n$ or $R_n \to k_1R_l + k_2R_m + k_3R_n$, $\Delta' = \Delta$

5.3 Cramer's Rule

For a system of equations:

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

the following determinants are defined:

$$\begin{aligned} D &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ D_x &= \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \\ D_y &= \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \\ D_z &= \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \end{aligned}$$

The solution for the system of equations is:

$$x = \frac{D_x}{D} \tag{5.6}$$

$$y = \frac{D_y}{D} \tag{5.7}$$

$$z = \frac{D_z}{D} \tag{5.8}$$

5.3.1 Consistency Test

1. If $D \neq 0$, the system is consistent and has unique solutions.
2. If $D = D_x = D_y = D_z = 0$, the system may or may not be consisten and it will have infinite solutions and the system will be dependent.
3. If $D = 0$ and at least one of D_x, D_y, D_z is non zero, the system is inconsistent

Chapter 6
Matrices

For a matrix,

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

and where I_p is an identity matrix of the p^{th} order, the following relations are applicable.

6.1 Sum of Two Matrices

$$A_{m\times n} + B_{m\times n} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{21} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix} \tag{6.1}$$

6.2 Multiplication of Two Matrices

If

$$C_{m\times p} = A_{m\times n} \cdot B_{n\times p} \tag{6.2}$$

then,

$$c_{ik} = \sum_{j=1}^{n} a_{ij} \cdot b_{jk} \tag{6.3}$$

6.2.1 Multiplicative Properties

1. Multiplication of matrices is associative, hence $(AB)C = A(BC)$.
2. $AI = A$

3. $A \cdot A^{-1} = I$

4. $A \cdot (adjA) = (adjA) \cdot A = |A|I$

5. $A^{-1} = \dfrac{1}{|A|}(adjA)^t$

6. $(AB)^t = B^t A^t$

6.3 Adjoint of a Matrix

$$adjA = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m1} & M_{m2} & \cdots & M_{mn} \end{bmatrix}^t \text{, where } M_{ij} \text{ is the minor of } a_{ij} \tag{6.4}$$

6.4 Martin's Rule

For a system of equation,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n \end{aligned}$$

The system can be written as:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \tag{6.5}$$

$$\Rightarrow AX = B \tag{6.6}$$

$$\Rightarrow X = A^{-1}B \tag{6.7}$$

Chapter 7
Binomial Theorem

For a binomial expansion $(a+b)^n$, there are $(n+1)$ terms and $(a+b+c)^n$ has $\dfrac{(n+1)(n+2)}{2}$ terms.

7.1 Expansion of a binomial expression

$$\begin{aligned}(a+b)^n &= ^n C_0 a^n b^0 + ^n C_1 a^{n-1} b^1 + ^n C_2 a^{n-2} b^2 + \cdots + ^n C_n a^0 b^n \\ &= \sum_{i=0}^{n} {}^nC_i a^{n-i} b^i \\ &\forall n \in \mathbb{N} \end{aligned} \tag{7.1}$$

$$\begin{aligned}(a+b)^n &= a^n b^0 + n a^{n-1} b + \frac{n(n-1)}{2!} a^{n-2} b^2 + \cdots + \frac{n(n-1)\cdots 3\cdot 2\cdot 1}{n!} a^0 b^n + \cdots \infty \\ &\forall n \in \mathbb{R}\end{aligned} \tag{7.2}$$

7.2 Trinomial Expansion

For $(a+b+c)^n$:

$$\begin{aligned}(a+b+c+)^n = \sum \frac{n!}{i!j!k!} a^i b^j c^k \\ \forall (i+j+k) = n;\ i,j,k,n \in \mathbb{N}\end{aligned} \tag{7.3}$$

7.3 Properties of Coefficients

$$\text{Sum of Co-efficients: } C_0 + C_1 + C_2 + \cdots + C_{n-1} + C_n = 2^n \tag{7.4}$$

$$\text{Sum of Odd Co-efficients: } C_0 + C_2 + C_4 + \cdots + C_{2n-3} + C_{2n-1} = 2^{n-1} \tag{7.5}$$

$$C_0 - C_1 + C_2 - \cdots + C_{2n-1} - C_{2n} = 0 \tag{7.6}$$

7.4 Pascal's Rule

For $1 \leq k \leq n$ and $k, n \in \mathbb{N}$:

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k} \tag{7.7}$$

Chapter 8

Boolean Algebra

Let B be a set of a, b, c with operations sum $(+)$ and product $(\cdot)$.
Then B is said to belong to the Boolean Structure if the following conditions are satisfied:

Table 8.1: Properties of Boolean Algebraic Structure

Property	Name of Property
$a + b \in B$	
$a \cdot b \in B$	Closure Property
$a + b = b + a$	
$a \cdot b = b \cdot a$	Associative Law
$a(b + c) = ab + ac$	
$a + bc = (a + b)(a + c)$	Commutative Law
$\{0, 1\} \in B$	
$a + 0 = a$	
$a + 1 = 1$	
$a \cdot 0 = 0$	
$a \cdot 1 = a$	Laws of 1 and 0
$a + ab = a$	
$a(a + b) = a$	Absorption Law
$(a + b)' = (a'b')$	De'Morgan's Law

Chapter 9

Remainder Theorems

9.1 Remainder Theorem

If a function $f(x)$ is divided by a binomial $x - a$, then the remainder is provided by $f(a)$.

$$\frac{f(x)}{x-a} \equiv f(a) \mod (x-a) \tag{9.1}$$

Worked Example

Find the remainder when $f(x) = x^3 - 4x^2 - 7x + 10$ is divided by $(x - 2)$.
The remainder:

$$R = \left(x^3 - 4x^2 - 7x + 10\right) \mod (x-2)$$

is given by:

$$\begin{aligned} R = f(2) &= (2)^3 - 4(2)^2 - 7(2) + 10 \\ &= 8 - 16 - 14 + 10 = -12 \end{aligned}$$

9.2 Euler's Remainder Theorem

According to Euler's Remainder Theorem, if x and n are two co-prime numbers:

$$x^{\varphi(n)} \equiv 1 \mod n, x, n \in \mathbb{Z}^+ \tag{9.2}$$

where, $\varphi(n)$ is Euler's totient function.

9.2.1 Euler's Totient Function

For a number defined as:

$$n = \prod_{i=1}^{r} {a_r}^{b_r} \tag{9.3}$$

then Euler's totient function is defined as:

$$\begin{aligned}\varphi(n) &= n \cdot \left[\left(1 - \frac{1}{a_1}\right) \cdot \left(1 - \frac{1}{a_2}\right) \cdot \left(1 - \frac{1}{a_3}\right) \cdots \right] \\ &= n \prod_{i=1}^{r} \left(1 - \frac{1}{a_r}\right) \end{aligned} \tag{9.4}$$

Worked Example

Find the remainder if 3^{76} is divided by 35.

Since:

$$35 = 5^1 \times 7^1$$

Hence the totient quotient of 35 is:

$$\begin{aligned}\varphi(35) &= 35 \cdot \left(1 - \frac{1}{5}\right) \cdot \left(1 - \frac{1}{7}\right) \\ &= 35 \times \frac{4}{5} \times \frac{6}{7} \\ &= 24\end{aligned}$$

Hence Euler's Theorem yields:

$$\begin{aligned}3^{24} &\equiv 1 \mod 35 \\ 3^{76} &\equiv 3^{24\times 3+4} \\ &\equiv \left(3^{24}\right)^3 \times 3^4 \mod 35 \\ &\equiv (1)^3 \times 3^4 \mod 35 \\ &\equiv 81 \mod 35 \\ &\equiv 11 \mod 35\end{aligned}$$

The remainder when 3^{76} is divided by 35 is 11.

9.3 Wilson Theorem

According to Wilson Theorem:

$$(n-1)! \equiv -1 \mod n \tag{9.5}$$

Worked Example

Find the remainder when 28! is divided by 31.
By Wilson's Theorem:

$$\begin{aligned}
30! &\equiv -1 \mod 31 \\
\Rightarrow 30 \cdot 29 \cdot 28! &\equiv -1 \mod 31 \\
\text{Let } 28! \mod 31 &= x \\
\Rightarrow (-1) \cdot (-2) \cdot x &\equiv 30 \mod 31 \\
\Rightarrow 2x &= 30 \\
\Rightarrow x &= 15
\end{aligned}$$

The remainder when 28! is divided by 31 is 15.

Part II
Co-ordinate Geometry

Chapter 10

2-D Co-ordinate Geometry

For the ordered pairs, $A(x_1, y_1)$ and $B(x_2, y_2)$:

10.1 Distance between Two Points

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \tag{10.1}$$

$$\tag{10.2}$$

10.2 Section Formula

If point C divides AB in the ratio $m : n$:

$$C = \left(\frac{nx_1 + mx_2}{m+n}, \frac{ny_1 + my_2}{m+n} \right) \tag{10.3}$$

10.2.1 Corollary: Mid - Point Formula

If C is the mid-point of AB, and $m : n = 1 : 1$:

$$C = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \tag{10.4}$$

$$\tag{10.5}$$

Chapter 11
Triangles

For a triangle defined with three vertices $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ and corresponding sides of length a, b, c, then:

11.1 Centroid of a Traiangle

$$\text{Centroid of } \triangle ABC = (\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}) \tag{11.1}$$

$$\tag{11.2}$$

11.2 Area of Triangle

11.2.1 Determinant Method

$$\text{Area of } \triangle ABC = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \tag{11.3}$$

11.2.2 Heron's Formula

For a triangle, the semi-perimeter, s, is defined as:

$$s = \frac{a + b + c}{2}$$

The area of the triangle can be defined as:

$$\text{Area of } \triangle ABC = \sqrt{s \cdot (s - a) \cdot (s - b) \cdot (s - c)} \tag{11.4}$$

11.3 Incircle of a Triangle

The radius, r, and centre of incircle, o, is:

$$o = \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c} \right) \tag{11.5}$$

$$r = \sqrt{\frac{(s-a)\cdot(s-b)\cdot(s-c)}{s}} \tag{11.6}$$

$$\tag{11.7}$$

11.4 Circumcircle of a Triangle

The radius, R, and centre, O, of circumcircle is defined as:

$$O = \left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C} \right) \tag{11.8}$$

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \tag{11.9}$$

Chapter 12
Straight Line

A straight line can be defined as:

$$y = mx + c \tag{12.1}$$

$$\frac{x}{a} + \frac{y}{b} = 1, \text{ where } a \text{ and } b \text{ are the intercepts at x and y axes respectively} \tag{12.2}$$

$$x \cos\alpha + y \sin\alpha = p \text{ (Normal Form)} \tag{12.3}$$

$$Ax + By + C = 0 \text{ (General Form)} \tag{12.4}$$

12.1 Equation of Straight Line Passing Through (x_0, y_0) and Slope m

$$(y - y_0) = m(x - x_0) \tag{12.5}$$

12.2 Distance Between Two Points on a Line

$$\frac{y_1 - y_2}{\sin\theta} = \frac{x_1 - x_2}{\cos\theta} = \gamma \tag{12.6}$$

$$\theta = \tan^{-1} m \tag{12.7}$$

12.3 Angle Between Two Lines

For two lines with slopes m_1, m_2, the angle between them, θ:

$$\theta = \arctan\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right) \tag{12.8}$$

12.4 Distance of a Point from a Line

Line: $ax + by + c = 0$ Point: (g, h)

$$S = \frac{ag + bh + c}{\sqrt{a^2 + b^2}} \tag{12.9}$$

12.5 Angle Bisector of a Line

For the two lines: $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, the angle bisector is:

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \tag{12.10}$$

If the sign of c_1 and c_2 is the same, then the equation obtained is the internal bisector.

12.6 Equation of a Straight Line Passing through the Intersection of Two Lines

$$(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0 \ \forall k \in \mathbb{R} \tag{12.11}$$

12.7 Relative Position of Points w.r.t. a Line

For the points (x_1, y_1) and (x_2, y_2):

$$k_1 = ax_1 + by_1 + c$$
$$k_2 = ax_2 + by_2 + c$$

If k_1 and k_2 have the same sign, they are on the same side of a line, otherwise on opposite sides.

12.8 Ratio of Division of Line Segment

For any line, $f(x, y) = 0$, the ratio in which it divides (x_1, y_1) and (x_2, y_2) is given by:

$$r = -\frac{f(x_1, y_1)}{f(x_2, y_2)} \tag{12.12}$$

If $\begin{cases} r > 0, \text{ then division is internal} \\ r < 0, \text{ then division is external} \end{cases}$.

Chapter 13

General Theory of Second Degree Equation

For any general equation of the form:

$$ax^2 + by^2 + 2gx + 2fy + 2hxy + c = 0 \tag{13.1}$$

Δ is defined as:

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \tag{13.2}$$

If $\Delta = 0$ then the equation is a pair of straight lines. If $a + b = 0$ then the lines are $\perp$.
If the $\Delta \neq 0$:

1. $a = b, h = 0$ →circle
2. $h^2 = ab$ →parabola
3. $h^2 < ab$ →ellipse
4. $h^2 > ab$ →hyperbola

Chapter 14
Conics

The four conic sections are: circle, parabola, ellipse, and hyperbola. Circle has been done separately in the next chapter.

14.1 Parametric Form of Conics

14.1.1 Hyperbola

$$x = a \sec \theta \tag{14.1}$$
$$y = b = \tan \theta \tag{14.2}$$

14.1.2 Ellipse

$$x = a \cos \phi \tag{14.3}$$
$$y = b \sin \phi \tag{14.4}$$

14.1.3 Parabola

$$x = at^2 \tag{14.5}$$
$$y = 2at \tag{14.6}$$

14.2 Equation form of Conics

14.2.1 Parabola

Table 14.1: Properties of a Parabola

Property	$y^2 = 4ax$	$x^2 = 4ay$
Axis	$y = 0$	$x = 0$
Eccentricity	1	1
Directrix	$x + a = 0$	$y + a = 0$
Focus	$(a, 0)$	$(0, a)$
Vertex	$(0, 0)$	$(0, 0)$
Length of latus rectum	$\lvert 4a \rvert$	$\lvert 4a \rvert$
Equation of latus rectum	$x - a = 0$	$y - a = 0$

14.2.2 Ellipse and Hyperbola

For $a > b$:

Table 14.2: Properties of Ellipse and Hyperbola

Property	$\frac{x^2}{a} + \frac{y^2}{b} = 1$ Ellipse	$\frac{x^2}{a} - \frac{y^2}{b} = 1$ Hyperbola
Length of Major Axis	$2a$	$2a$
Length of Minor Axis	$2b$	$2b$
Equation of Major Axis	$x = 0$	$x = 0$
Equation of Minor Axis	$y = 0$	$y = 0$
Eccentricity e	$\sqrt{1 - \frac{b^2}{a^2}}$	$\sqrt{1 + \frac{b^2}{a^2}}$
Vertices	$(\pm a, 0)$	$(\pm a, 0)$
Foci	$(\pm ae, 0)$	$(\pm ae, 0)$
Equation of Directrix	$x \pm \frac{a}{e} = 0$	$x = \pm\frac{a}{e}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Equation of latus rectum	$x \pm ae = 0$	
Centre	$(0, 0)$	$(0, 0)$

Chapter 15
Circles

15.1 Locus Form

$$(x - g)^2 + (y - h)^2 = r^2 \tag{15.1}$$

where the centre is (g, h) and the radius is r.

15.2 Diameter Form

$$(x - a)(x - c) + (y - b)(y - d) = 0 \tag{15.2}$$

where (a, b) and (c, d) are the two ends of the diamter.

15.3 General Form

If the equation of a circle is in the form:

$$x^2 + y^2 + 2gx + 2fy + c = 0 \tag{15.3}$$

Then the following is true about the circle:

1. centre of the circle is $(-g, -f)$
2. radius of circle is $\sqrt{g^2 + f^2 - c}$

15.4 Important Relations

1. If the circle passes through the origin, $g = 0, f = 0$.
2. If the circle touches the x-axis $c = g^2$.
3. If the circle touches the y-axis $c = f^2$.

15.5 Common for Two Circles

1. The common chord passing between two circles S_1 and S_2 are:

$$S_1 - S_2 = 0 \tag{15.4}$$

2. Circles passing through the intersection of two circles is:

$$S_2 + k(S_1 - S_2) = 0 \ \forall k \in \mathbb{R} \tag{15.5}$$

Chapter 16
Vectors

Let two vectors be $\vec{a} = a\hat{i} + b\hat{j} + c\hat{k}$ and $\vec{b} = x\hat{i} + y\hat{j} + z\hat{k}$:

16.1 Modulus of a Vector

For a vector $\vec{a}$, the modulus of the vector is:

$$|\vec{a}| = \sqrt{a^2 + b^2 + c^2} \tag{16.1}$$

16.2 Sum of Vectors

The sum of two vectors is:

$$|\vec{a} + \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{a}|\cos\theta} \tag{16.2}$$

$$\vec{a} + \vec{b} = (a + x)\hat{i} + (b + y)\hat{j} + (c + z)\hat{k} \tag{16.3}$$

The direction of the resultant vector is:

$$\tan\alpha = \frac{b\sin\theta}{a + b\cos\theta} \tag{16.4}$$

where, θ is the angle between the two vectors.

16.3 Product of Vectors

16.3.1 Dot Product

$$\vec{a} \cdot \vec{b} = |a||b|\cos\theta \tag{16.5}$$

$$\vec{a} \cdot \vec{b} = ax + by + cz \tag{16.6}$$

16.3.2 Cross Product

$$\vec{a} \times \vec{b} = |a||b|\sin\theta\hat{n} \tag{16.7}$$

$$\tag{16.8}$$

where $\hat{n}$ is a vector $\perp \vec{a}, \vec{b}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ x & y & z \end{vmatrix} \tag{16.9}$$

16.4 Test of Co-planarity

Three vectors are called co-planar if:

$$\lambda\vec{a} + \mu\vec{b} = \vec{c} \tag{16.10}$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = 0 \tag{16.11}$$

Chapter 17

3D - Space

17.1 Line segments in 3D - Space

For points defined in a 3D space as $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$:

17.1.1 Distance between Two Points

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \tag{17.1}$$

17.1.2 Section Formula of a Line Segment Divided in the ratio $m : n$

$$P = \left(\frac{nx_1 + mx_2}{m+n}, \frac{ny_1 + my_2}{m+n}, \frac{nz_1 + mz_2}{m+n} \right) \tag{17.2}$$

17.2 Line in 3D - Space

For a line which is defined as $\vec{a} = a\hat{i} + b\hat{j} + c\hat{k}$:

1. Line numbers of the line is

$$< a, b, c > \tag{17.3}$$

2. The line cosines are:

$$< \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}} > \tag{17.4}$$

$$=< l, m, n > \tag{17.5}$$

17.2.1 Angle between Two Lines

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + b_1 b_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} \tag{17.6}$$

$$\Rightarrow \cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2 \tag{17.7}$$

When two lines are $\perp, l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$.
When two lines are $\parallel \dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2} = 1$.

17.2.2 Skew and Co-planar Lines

Let there be two lines $\vec{r_1}$ and $\vec{r_2}$,

$$\vec{r_1} = \vec{a_1} + \mu\vec{b_1}\vec{r_2} = \vec{a_2} + \lambda\vec{b_2} \tag{17.8}$$

17.2.3 Distance between Lines

The shortest distance between r_1 and r_2

$$S = \left| \frac{(\vec{a_1} - \vec{a_2}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right| \tag{17.9}$$

If $S = 0$, the lines intersect.

Cartesian Form

For two lines defined as $\dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1}$ and $\dfrac{x - x_2}{a_2} = \dfrac{y - y_2}{b_2} = \dfrac{z - z_2}{c_2}$:

$$S = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \tag{17.10}$$

Distance Between Parallel Lines

$$S = \left| \frac{\vec{b} \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b}|} \right| \tag{17.11}$$

Distance of a Point to a Line

For a point, (x_1, y_1, z_1) the distance to a line $\dfrac{x - \alpha}{l} = \dfrac{y - \beta}{m} = \dfrac{z - \gamma}{n}$:

$$S = \left(\begin{vmatrix} x_1 - \alpha & y_1 - \beta \\ l & m \end{vmatrix} + \begin{vmatrix} y_1 - \beta & z_1 - \gamma \\ m & n \end{vmatrix} + \begin{vmatrix} z_1 - \gamma & x_1 - \alpha \\ n & l \end{vmatrix} \right)^{\frac{1}{2}} \tag{17.12}$$

17.3 Triangular Plane

17.3.1 Centroid of a Triangle

$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right) \tag{17.13}$$

Chapter 18
3D - Plane

A plane in 3-D space can be defined as:

1. Cartesian Form:

$$ax + by + cz + d = 0 \tag{18.1}$$

2. Vectorial Form:

$$\vec{r} \cdot \vec{n} = p \tag{18.2}$$

   , where $\vec{r}$ is a line on the plane, $\vec{n}$ is a normal to the plane, and p is perpendicular distance to the plane from the origin.

18.1 Angle Between Two Planes

For two planes, $\vec{r_1} \cdot \vec{n_1} = p_1$ and $\vec{r_2} \cdot \vec{n_2} = p_2$, the angle between the planes, θ is:

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|} \tag{18.3}$$

In the Cartesian Form:

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{{a_1}^2 + {b_1}^2 + {c_1}^2}\sqrt{{a_2}^2 + {b_2}^2 + {c_2}^2}} \tag{18.4}$$

18.2 Distance of a Point from a Plane

18.2.1 Catesian Form

For the point (p, q, r) and the plane, $ax + by + cz + d = 0$:

$$S = \frac{ap + bq + cr + d}{\sqrt{a^2 + b^2 + c^2}} \tag{18.5}$$

18.2.2 Vector Form

For the point $\vec{g} = p\hat{i} + q\hat{j} + r\hat{k}$ and the plane $\vec{r} \cdot (a\hat{i} + b\hat{j} + c\hat{k}) + d = 0$:

$$S = \frac{\left(a\hat{i} + b\hat{j} + c\hat{k}\right) \cdot \left(p\hat{i} + q\hat{j} + r\hat{k}\right)}{\sqrt{a^2 + b^2 + c^2}} \tag{18.6}$$

$$\Rightarrow S = \frac{\left(a\hat{i} + b\hat{j} + c\hat{k}\right) \cdot \vec{g}}{|a\hat{i} + b\hat{j} + c\hat{k}|} \tag{18.7}$$