Mathematical Formula Sheet

A Book of High School and Engineering Common Course Mathematical Formulae

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Contents

Part I

Algebra

Chapter 1

Logarithm

1.1 Basic Formulae

For $a^x = b$:

$$\log_a x, \text{ for all } x \leq 0 \text{ is undefined} \tag{1.1}$$
$$\log_a b = x,\ bax \neq 1,\ a \neq 1 \tag{1.2}$$
$$\log_b a^m = m \log_b a, \text{ for } a^m = b \tag{1.3}$$
$$a^{\log_a x} = x \tag{1.4}$$
$$a^{\log_b c} = c^{\log_b a} \tag{1.5}$$
$$\frac{1}{\log_a b} = \log_b a \tag{1.6}$$
$$\log_c(ab) = \log_c a + \log_c b \tag{1.7}$$
$$\log_c(\frac{a}{b}) = \log_c a - \log_c b \tag{1.8}$$
$$|\log_a x| = \begin{cases} -\log_a x, & \text{if } 0 < x < 1 \\ \log_a x, & \text{if } 1 \leq x < \infty \end{cases} \tag{1.9}$$

1.2 Series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\infty = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{x^i}{i!} \tag{1.10}$$
$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\infty = \lim_{n\to\infty} \sum_{i=1}^{n} (-1)^{(i-1)} \frac{x^i}{i} \tag{1.11}$$

Chapter 2

Complex Number

2.1 Basic Formulae

For $z = x + iy$,

$$|z| = \sqrt{x^2 + y^2} \tag{2.1}$$

$$\tan\theta = \frac{y}{x} \tag{2.2}$$

$$\bar{z} = x - iy \tag{2.3}$$

2.2 Arithmetic Operation of Complex Number

For two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \tag{2.4}$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \tag{2.5}$$

$$|z_1 z_2| = |z_1||z_2| \tag{2.6}$$

$$\frac{z_1}{z_2} = \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{{a_2}^2 + {b_2}^2} \tag{2.7}$$

$$|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|} \tag{2.8}$$

2.3 Euler's Formula

$$z = re^{i\theta}\text{, where } r = |z|,\ e^{i\theta} = \cos\theta + i\sin\theta\text{, and } \theta = \tan^{-1}\frac{y}{x} \tag{2.9}$$

2.4 Trigonometric Ratios in Complex Form

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta \tag{2.10}$$

$$\Rightarrow \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{2.11}$$

$$e^{i\theta} - e^{-i\theta} = 2\sin\theta \tag{2.12}$$

$$\Rightarrow \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} \tag{2.13}$$

2.5 De Moivre's Formula

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \tag{2.14}$$

2.6 Application of Euler's and De Moivre's Formula

For $z_1 = |r_1|e^{i\theta_1}$ and $z_2 = |r_2|e^{i\theta_2}$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} \tag{2.15}$$

2.7 Roots of Unity

$$\sqrt[n]{1} = e^{i\frac{2k\pi}{n}}, \text{ where } k \in [0, n-1] \tag{2.16}$$

2.8 Important Relations of Complex Numbers

$$|z_1 + z_2| \leq |z_1| + |z_2| \tag{2.17}$$

$$|z_1 - z_2| \leq |z_1| + |z_2| \tag{2.18}$$

$$|z_1 - z_2| \geq |z_1| - |z_2| \tag{2.19}$$

$$|z_1 + z_2| \geq ||z_1| - |z_2|| \tag{2.20}$$

$$|z_1 + z_2|^2 = 2(|z_1|^2 + |z_2|^2) \tag{2.21}$$

Chapter 3

Progression

3.1 Arithmetic Progression (A.P.)

An arithmetic sequence is $a, a+n, a+2n, ...\infty$ or $t_n = a+(n-1)d$, where a is the first term, d is the common difference, and n is the n^{th}-term.

An arithmetic series is $a + (a + d) + (a + 2d) + ...\infty$.

3.1.1 Sum of A.P. Series

$$
\begin{aligned}
S_n = a + (a + d) + ... + (a + \overline{n-2}d) + (a + \overline{n-1}d) \\
S_n = (a + \overline{n-1}d) + (a + \overline{n-2}d + ... + (a + d) + a \\
\Rightarrow 2S_n = n(2a + \overline{n-1}d) \\
\Rightarrow S_n = \frac{n}{2}(2a + \overline{n-1}d)
\end{aligned}
\tag{3.1}
$$

3.1.2 Important Relation

If the three terms a, b, c are in A.P., then

$$2b = a + c \tag{3.2}$$

3.2 Geometric Progression (G.P.)

An geometric sequence is $a, ar, ar^2, ...\infty$ or $t_n = ar^{n-1}$, where a is the first term, r is the common ratio, and n is the n^{th}-term.

An geometric series is $a + ar + ar^2 + ...\infty$.

3.2.1 The Value of 'r'

$$r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = ... = \frac{t_n}{t_{n-1}} \tag{3.3}$$

3.2.2 Sum of a G.P. Series

For a definite G.P. series, where there are n terms in the series, the sum of the series is:

$$S_n = \frac{a|r^n - 1|}{|r - 1|} \tag{3.4}$$

For an infite G.P. series the sum of the series is defined for $r < 1$. Sum of such a series is:

$$S_\infty = \frac{a}{1 - r} \tag{3.5}$$

3.2.3 Important relations

If the three terms a, b, c are in G.P., then:

$$b^2 = ac \tag{3.6}$$

3.3 Harmonic Progression (H.P.)

If a, b, c are terms of an H.P. then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

$$\therefore \frac{2}{b} = \frac{1}{a} + \frac{1}{c} \tag{3.7}$$

$$\Rightarrow b = \frac{2ac}{a + c} \tag{3.8}$$

3.4 Arithmetico-Geometric Progression (A.G.P.)

Sequence $a, (a + d)r, (a + 2d)r^2, ..., (a + \overline{n - 1}d)r^{n-1}$, where $a \rightarrow$first term of A.G.P., $d \rightarrow$common difference, and $r \rightarrow$common ratio.

3.4.1 Sum of A.G.P.:

For an infinite A.G.P. series, the sum is defined for $r < 1$:

$$S_\infty = \frac{a}{1 - r} + \frac{dr}{(1 - r)^2} \tag{3.9}$$

3.5 Special Series

For $n \in \mathbb{N}$

$$1 + 2 + 3 + ... + (n-1) + n = \frac{n(n-1)}{2} \tag{3.10}$$

$$1^2 + 2^2 + 3^2 + ... + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{3.11}$$

$$1^3 + 2^3 + 3^3 + ... + (n-1)^3 + n^3 = [\frac{n(n-1)}{2}]^2 \tag{3.12}$$

3.5.1 Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{3.13}$$

3.5.2 Riemann's Infinite Series as an Integration

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=r_1}^{r_2} f(\frac{i}{n}) = \int_{\frac{r_1}{n}}^{\frac{r_2}{n}} f(x)dx \tag{3.14}$$

Chapter 4

Test of Convergence of Infinite Series

If $a_1, a_2, a_3, ..., a_n$ is a sequence by a_n and their sum of series is S_n, then the following apply.

4.1 Definition

If

$$\lim_{n \to \infty} S_n = l$$

where l is a finite value, the series S_n is said to converge. A non-convergent series is called a divergent series.

4.2 Tests of Convergence

4.2.1 Comparison Test

If u_n and v_n are two positive series, then:

1. (a) v_n converges
   (b) $u_n \leq v_n \forall n$ Then u_n converges.
2. (a) v_n diverges
   (b) $u_n \geq v_n \forall n$ Then u_n diverges.

4.2.2 Limit Form

If

$$\lim_{x \to \infty} \frac{u_n}{v_n} = l$$

where l is a finite quantity $\neq 0$, then u_n and v_n converge and diverge together.

4.2.3 Integral Test or Maclaurin-Cauchy Test

For a series

$$\sum_{i=N}^{\infty} f(x), \text{ where } N \in \mathbb{Z} \tag{4.1}$$

will only converge if the improper integral

$$\int_{N}^{\infty} f(x)dx \tag{4.2}$$

is finite.

If the improper integral is finite, the upper and lower limit of the infinite series is given by:

$$\int_{N}^{\infty} f(x)dx \leq \sum_{i=N}^{\infty} f(x) \leq f(N) + \int_{N}^{\infty} f(x)dx \tag{4.3}$$

4.2.4 Ratio Test

If, for two series $\sum u_n$ and $\sum v_n$:

1. (a) $\sum v_n$ converges
   (b) from or after a particular term $\dfrac{u_n}{u_{n+1}} > \dfrac{v_n}{v_{n+1}}$, then u_n converges.
2. (a) $\sum v_n$ diverges
   (b) from or after a particular term $\dfrac{u_n}{u_{n+1}} < \dfrac{v_n}{v_{n+1}}$, then u_n diverges.

4.2.5 D'Alembert's Ratio Test

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lambda \tag{4.4}$$

- series converges if $\lambda < 1$
- series diverges if $\lambda > 1$
- fails if $\lambda = 1$

4.2.6 Rabbe's Test

$$\lim_{n\to\infty} n[\frac{u_n}{u_{n+1}} - 1] = \kappa \tag{4.5}$$

- series converges if $\kappa < 1$
- series diverges if $\kappa > 1$
- fails if $\kappa = 1$

4.2.7 Cauchy's Root Test

$$\lim_{n\to\infty} |u_n| = \lambda \tag{4.6}$$

- series converges for $\lambda < 1$
- series diverges for $\lambda > 1$
- test fails for $\lambda = 1$

4.2.8 Logarithmic Test

$$\lim_{n\to\infty} n \log(\frac{u_n}{u_{n+1}}) = \kappa \tag{4.7}$$

- series converges for $\kappa < 1$
- series diverges for $\kappa > 1$
- test fails for $\kappa = 1$

Chapter 5

Determinants

5.1 Definition

For a determinant:

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \tag{5.1}$$

5.1.1 Minor and Cofactor

For a third order determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, the minor of a_{11} is $M_{11} = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, i.e., all the terms of the determinant expect those in the same row and columns as the one of which the minor is being calculated.

Cofactor $C_{ij} = (-1)^{i+j} M_{ij}$

5.2 Important Properties

1. Transposing a determinant does not alter its value.
2. If rows and columns are interchanges m times, the value of the new determinant is
$$\Delta' = (-1)^m \Delta \tag{5.2}$$
3. If two parallel lines are equal, then $\Delta = 0$

4. For $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then $\Delta_1 = k\Delta$

5. For $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{vmatrix}$, then $\Delta_1 = k\Delta$

6. For $C_n \to k_1C_l + k_2C_m + k_3C_n$ or $R_n \to k_1R_l + k_2R_m + k_3R_n$, $\Delta' = \Delta$

5.3 Cramer's Rule

For a system of equations:

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

the following determinants are defined:

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

The solution of the system of equations is:

$$x = \frac{D_x}{D} \tag{5.3}$$

$$y = \frac{D_y}{D} \tag{5.4}$$

$$z = \frac{D_z}{D} \tag{5.5}$$

5.3.1 Consistency Test

1. If $D \neq 0$, the system is consistent and has unique solutions.
2. If $D = D_x = D_y = D_z = 0$, the system may or may not be consisten and it will have infinite solutions and the system will be dependent.
3. If $D = 0$ and at least one of D_x, D_y, D_z is non zero, the system is inconsistent

Chapter 6

Matrices

For a matrix,

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

and where I_p is an identity matrix of the p^{th} order, the following relations are applicable.

6.1 Sum of Two Matrices

$$A_{m\times n} + B_{m\times n} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{21} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix} \tag{6.1}$$

6.2 Multiplication of Two Matrices

If

$$C_{m\times p} = A_{m\times n} \cdot B_{n\times p}$$

then,

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk} \tag{6.2}$$

6.2.1 Multiplicative Properties

1. Multiplication of matrices is associative, hence $(AB)C = A(BC)$.

2. $AI = A$
3. $A \cdot A^{-1} = I$
4. $A \cdot (adj A) = (adj A) \cdot A = |A| I$
5. $A^{-1} = \frac{1}{|A|}(adj A)^t$
6. $(AB)^t = B^t A^t$

6.3 Adjoint of a Matrix

$$adj A = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m1} & M_{m2} & \cdots & M_{mn} \end{bmatrix}^t, \text{ where } M_{ij} \text{ is the minor of } a_{ij} \tag{6.3}$$

6.4 Martin's Rule

For a system of equation,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n \end{aligned}$$

The system can be written as:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \tag{6.4}$$

$$\Rightarrow AX = B \tag{6.5}$$

$$\Rightarrow X = A^{-1}B \tag{6.6}$$

Chapter 7

Binomial Theorem

For a binomial expansion $(a+b)^n$, there are $(n+1)$ terms and $(a+b+c)^n$ has $\dfrac{(n+1)(n+2)}{2}$ terms.

7.1 Expansion of a binomial expression

$$\begin{aligned}(a+b)^n =& {}^nC_0 a^n b^0 +{}^n C_1 a^{n-1} b^1 +{}^n C_2 a^{n-2} b^2 + \\ & \cdots +{}^n C_n a^0 b^n \ \forall n \in \mathbb{N} \\ =& \sum_{i=0}^{n} {}^nC_i a^{n-i} b^i \ \forall n \in \mathbb{N}\end{aligned} \tag{7.1}$$

$$\begin{aligned}(a+b)^n =& a^n b^0 + n a^{n-1} b + \frac{n(n-1)}{2!} a^{n-2} b^2 \\ & + \cdots + \frac{n(n-1)\cdots 3 \cdot 2 \cdot 1}{n!} a^0 b^n + \cdots \infty \ \forall n \in \mathbb{R}\end{aligned} \tag{7.2}$$

7.2 Trinomial Expansion

For $(a+b+c)^n$:

$$\begin{aligned}(a+b+c+)^n = \sum \frac{n!}{i!j!k!} a^i b^j c^k \\ \forall\ (i+j+k) = n;\ i,j,k,n \in \mathbb{N}\end{aligned} \tag{7.3}$$

7.3 Properties of Coefficients

$$\text{Sum of Co-efficients: } C_0 + C_1 + C_2 + \cdots + C_{n-1} + C_n = 2^n \tag{7.4}$$

$$\text{Sum of Odd Co-efficients: } C_0 + C_2 + C_4 + \cdots + C_{2n-3} + C_{2n-1} = 2^{n-1} \tag{7.5}$$

$$C_0 - C_1 + C_2 - \cdots + C_{2n-1} - C_{2n} = 0 \tag{7.6}$$

7.4 Pascal's Rule

For $1 \leq k \leq n$ and $k, n \in \mathbb{N}$:

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k} \tag{7.7}$$

Chapter 8

Boolean Algebra

Let B be a set of a, b, c with operations sum $(+)$ and product $(\cdot)$.
Then B is said to belong to the Boolean Structure if the following conditions are satisfied:

Property	Name of Property
$a + b \in B$	
$a \cdot b \in B$	Closure Property
$a + b = b + a$	
$a \cdot b = b \cdot a$	Associative Law
$a(b + c) = ab + ac$	
$a + bc = (a + b)(a + c)$	Commutative Law
$\{0, 1\} \in B$	
$a + 0 = a$	
$a + 1 = 1$	
$a \cdot 0 = 0$	
$a \cdot 1 = a$	Laws of 1 and 0
$a + ab = a$	
$a(a + b) = a$	Absorption Law
$(a + b)' = (a'b')$	De'Morgan's Law

Table 8.1: Properties of Boolean Algebraic Structure

Chapter 9

Remainder Theorems

9.1 Remainder Theorem

If a function $f(x)$ is divided by a binomial $x - a$, then the remainder is provided by $f(a)$.

$$\frac{f(x)}{x-a} \equiv f(a) \mod (x-a) \tag{9.1}$$

Worked Example

Find the remainder when $f(x) = x^3 - 4x^2 - 7x + 10$ is divided by $(x - 2)$.

The remainder:

$$R = \left(x^3 - 4x^2 - 7x + 10\right) \mod (x-2)$$

is given by:

$$\begin{aligned} R = f(2) &= (2)^3 - 4(2)^2 - 7(2) + 10 \\ &= 8 - 16 - 14 + 10 = -12 \end{aligned}$$

9.2 Euler's Remainder Theorem

According to Euler's Remainder Theorem, if x and n are two co-prime numbers:

$$x^{\varphi(n)} \equiv 1 \mod n, x, n \in \mathbb{Z}^+ \tag{9.2}$$

where, $\varphi(n)$ is Euler's totient function.

9.2.1 Euler's Totient Function

For a number defined as:

$$n = \prod_{i=1}^{r} {a_r}^{b_r} \tag{9.3}$$

then Euler's totient function is defined as:

$$\begin{aligned}\varphi(n) &= n \cdot \left[\left(1 - \frac{1}{a_1}\right) \cdot \left(1 - \frac{1}{a_2}\right) \cdot \left(1 - \frac{1}{a_3}\right) \cdots\right] \\ &= n \prod_{i=1}^{r} \left(1 - \frac{1}{a_r}\right) \end{aligned} \tag{9.4}$$

Worked Example

Find the remainder if 3^{76} is divided by 35.

Since:

$$35 = 5^1 \times 7^1$$

Hence the totient quotient of 35 is:

$$\begin{aligned}\varphi(35) &= 35 \cdot \left(1 - \frac{1}{5}\right) \cdot \left(1 - \frac{1}{7}\right) \\ &= 35 \times \frac{4}{5} \times \frac{6}{7} \\ &= 24\end{aligned}$$

Hence Euler's Theorem yields:

$$\begin{aligned}3^{24} &\equiv 1 \mod 35 \\ 3^{76} &\equiv 3^{24\times 3+4} \\ &\equiv \left(3^{24}\right)^3 \times 3^4 \mod 35 \\ &\equiv (1)^3 \times 3^4 \mod 35 \\ &\equiv 81 \mod 35 \\ &\equiv 11 \mod 35\end{aligned}$$

The remainder when 3^{76} is divided by 35 is 11.

9.3 Wilson Theorem

According to Wilson Theorem:

$$(n-1)! \equiv -1 \mod n \tag{9.5}$$

Worked Example

Find the remainder when 28! is divided by 31.

By Wilson's Theorem:

$$\begin{aligned}
30! &\equiv -1 \mod 31\\
\Rightarrow 30 \cdot 29 \cdot 28! &\equiv -1 \mod 31\\
\text{Let } 28! \mod 31 &= x\\
\Rightarrow (-1) \cdot (-2) \cdot x &\equiv 30 \mod 31\\
\Rightarrow 2x &= 30\\
\Rightarrow x &= 15
\end{aligned}$$

The remainder when 28! is divided by 31 is 15.

Part II

Co-Ordinate Geometry

Chapter 10

2-D Co-ordinate Geometry

For the ordered pairs, $A(x_1, y_1)$ and $B(x_2, y_2)$:

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \tag{10.1}$$

$$\text{Mid point of AB} = (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}) \tag{10.2}$$

$$\text{Point C, which divides AB in the ratio } m : n = (\frac{nx_1 + mx_2}{m+n}, \frac{ny_1 + my_2}{m+n}) \tag{10.3}$$

Chapter 11

Triangles

For a triangle defined with three vertices $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ and corresponding sides of length a, b, c, then:

$$\text{Centroid of } \triangle ABC = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right) \tag{11.1}$$

$$\text{Area of } \triangle ABC = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \tag{11.2}$$

For a triangle, the semiperimeter, s, is defined as:

$$s = \frac{a + b + c}{2}$$

Then the radius, r, and centre of incircle, o, is:

$$o = \left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right) \tag{11.3}$$

$$r = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}} \tag{11.4}$$

The radius, R, and centre, O, of circumcircle is defined as:

$$O = \left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right) \tag{11.5}$$

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \tag{11.6}$$

Chapter 12

Straight Line

A straight line can be defined as:

$$y = mx + c \tag{12.1}$$

$$\frac{x}{a} + \frac{y}{b} = 1, \text{ where } a \text{ and } b \text{ are the intercepts at x and y axes respectively} \tag{12.2}$$

$$x\cos\alpha + y\sin\alpha = p \text{ (Normal Form)} \tag{12.3}$$

$$Ax + By + C = 0 \text{ (General Form)} \tag{12.4}$$

Equation of Straight Line Passing Through (x_0, y_0) and Slope m

$$(y - y_0) = m(x - x_0) \tag{12.5}$$

Distance Between Two Points on a Line

$$\frac{y_1 - y_2}{\sin\theta} = \frac{x_1 - x_2}{\cos\theta} = \gamma \tag{12.6}$$

$$\theta = \tan^{-1} m \tag{12.7}$$

Angle Between Two Lines

For two lines with slopes m_1, m_2, the angle between them, θ:

$$\theta = \arctan\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right) \tag{12.8}$$

Distance of a Point from a Line
Line: $ax + by + c = 0$ Point: (g, h)

$$S = \frac{ag + bh + c}{\sqrt{a^2 + b^2}} \tag{12.9}$$

Angle Bisector of a Line For the two lines: $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, the angle bisector is:

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \tag{12.10}$$

If the sign of c_1 and c_2 is the same, then the equation obtained is the internal bisector.

Equation of a Straight Line Passing through the Intersection of Two Lines

$$(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0 \ \forall k \in \mathbb{R} \tag{12.11}$$

Relative Position of Points w.r.t. a Line For the points (x_1, y_1) and (x_2, y_2):

$$k_1 = ax_1 + by_1 + c$$
$$k_2 = ax_2 + by_2 + c$$

If k_1 and k_2 have the same sign, they are on the same side of a line, otherwise on opposite sides.

Ratio of Division of Line Segment For any line, $f(x, y) = 0$, the ratio in which it divides (x_1, y_1) and (x_2, y_2) is given by:

$$r = -\frac{f(x_1, y_1)}{f(x_2, y_2)} \tag{12.12}$$

If $\begin{cases} r > 0, \text{ then division is internal} \\ r < 0, \text{ then division is external} \end{cases}$.

Chapter 13

General Theory of Second Degree Equation

For any general equation of the form:

$$ax^2 + by^2 + 2gx + 2fy + 2hxy + c = 0 \tag{13.1}$$

Δ is defined as:

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \tag{13.2}$$

If $\Delta = 0$ then the equation is a pair of straight lines. If $a + b = 0$ then the lines are $\perp$.
If the $\Delta \neq 0$:

1. $a = b, h = 0$ →circle
2. $h^2 = ab$ →parabola
3. $h^2 < ab$ →ellipse
4. $h^2 > ab$ →hyperbola

Chapter 14

Conics

The four conic sections are: circle, parabola, ellipse, and hyperbola. Circle has been done separately in the next chapter.

14.1 Parabola

Property	$y^2 = 4ax$	$x^2 = 4ay$
Axis	$y = 0$	$x = 0$
Eccentricity	1	1
Directrix	$x + a = 0$	$y + a = 0$
Focus	$(a, 0)$	$(0, a)$
Vertex	$(0, 0)$	$(0, 0)$
Length of latus rectum	$\|4a\|$	$\|4a\|$
Equation of latus rectum	$x - a = 0$	$y - a = 0$

Table 14.1: Properties of a Parabola

14.2 Ellipse and Hyperbola

For $a > b$:

Property	$\frac{x^2}{a}+\frac{y^2}{b}=1$ Ellipse	$\frac{x^2}{a}-\frac{y^2}{b}=1$ Hyperbola
Length of Major Axis	$2a$	$2a$
Length of Minor Axis	$2b$	$2b$
Equation of Major Axis	$x=0$	$x=0$
Equation of Minor Axis	$y=0$	$y=0$
Eccentricity e	$\sqrt{1-\frac{b^2}{a^2}}$	$\sqrt{1+\frac{b^2}{a^2}}$
Vertices	$(\pm a,0)$	$(\pm a,0)$
Foci	$(\pm ae,0)$	$(\pm ae,0)$
Equation of Directrix	$x\pm\frac{a}{e}=0$	$x=\pm\frac{a}{e}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Equation of latus rectum	$x\pm ae=0$	
Centre	$(0,0)$	$(0,0)$

Table 14.2: Properties of Ellipse and Hyperbola

14.3 Parametric Form of Conics

14.3.1 Hyperbola

$$x = a\sec\theta \tag{14.1}$$
$$y = b = \tan\theta \tag{14.2}$$

14.3.2 Ellipse

$$x = a\cos\phi \tag{14.3}$$
$$y = b\sin\phi \tag{14.4}$$

14.3.3 Parabola

$$x = at^2 \tag{14.5}$$
$$y = 2at \tag{14.6}$$

Chapter 15

Circles

15.1 Locus Form

$$(x-g)^2 + (y-h)^2 = r^2 \tag{15.1}$$

where the centre is (g, h) and the radius is r.

15.2 Diameter Form

$$(x-a)(x-c) + (y-b)(y-d) = 0 \tag{15.2}$$

where (a, b) and (c, d) are the two ends of the diamter.

15.3 General Form

If the equation of a circle is in the form:

$$x^2 + y^2 + 2gx + 2fy + c = 0 \tag{15.3}$$

Then the following is true about the circle:

1. centre of the circle is $(-g, -f)$
2. radius of circle is $\sqrt{g^2 + f^2 - c}$

15.4 Important Relations

1. If the circle passes through the origin, $g = 0, f = 0$.
2. If the circle touches the x-axis $c = g^2$.
3. If the circle touches the y-axis $c = f^2$.

Common for Two Circles

1. The common chord passing between two circles S_1 and S_2 are:

$$S_1 - S_2 = 0 \tag{15.4}$$

2. Circles passing through the intersection of two circles is:

$$S_2 + k(S_1 - S_2) = 0 \ \forall k \in \mathbb{R} \tag{15.5}$$

Chapter 16

Vectors

Let two vectors be $\vec{a} = a\hat{i} + b\hat{j} + c\hat{k}$ and $\vec{b} = x\hat{i} + y\hat{j} + z\hat{k}$:

16.1 Modulus of a Vector

For a vector $\vec{a}$, the modulus of the vector is:

$$|\vec{a}| = \sqrt{a^2 + b^2 + c^2} \tag{16.1}$$

16.2 Sum of Vectors

The sum of two vectors is:

$$|\vec{a} + \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{a}|\cos\theta} \tag{16.2}$$

$$\vec{a} + \vec{b} = (a + x)\hat{i} + (b + y)\hat{j} + (c + z)\hat{k} \tag{16.3}$$

The direction of the resultant vector is:

$$\tan\alpha = \frac{b\sin\theta}{a + b\cos\theta} \tag{16.4}$$

where, θ is the angle between the two vectors.

16.3 Product of Vectors

16.3.1 Dot Product

$$\vec{a} \cdot \vec{b} = |a||b|\cos\theta \tag{16.5}$$

$$\vec{a} \cdot \vec{b} = ax + by + cz \tag{16.6}$$

16.3.2 Cross Product

$$\vec{a} \times \vec{b} = |a||b| \sin\theta \hat{n} \tag{16.7}$$

where $\hat{n}$ is a vector $\perp \vec{a}, \vec{b}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ x & y & z \end{vmatrix} \tag{16.8}$$

16.4 Test of Co-planarity

Three vectors are called co-planar if:

$$\lambda\vec{a} + \mu\vec{b} = \vec{c} \tag{16.9}$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = 0 \tag{16.10}$$

Chapter 17

3-D Geometry

17.1 Distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$:

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \tag{17.1}$$

17.2 Section Formula of a Line Segment Divided in the ratio $m : n$

$$P = \left(\frac{nx_1 + mx_2}{m+n}, \frac{ny_1 + my_2}{m+n}, \frac{nz_1 + mz_2}{m+n} \right) \tag{17.2}$$

17.3 Centroid of a Triangle

$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right) \tag{17.3}$$

Chapter 18

Line in 3-D Space

For a line which is defined as $\vec{a} = a\hat{i} + b\hat{j} + c\hat{k}$:

1. Line numbers of the line is

$$< a, b, c > \tag{18.1}$$

2. The line cosines are:

$$< \frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}} > \tag{18.2}$$

$$=< l, m, n > \tag{18.3}$$

18.1 Angle between Two Lines

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + b_1b_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} \tag{18.4}$$

$$\Rightarrow \cos\theta = l_1l_2 + m_1m_2 + n_1n_2 \tag{18.5}$$

When two lines are $\perp, l_1l_2 + m_1m_2 + n_1n_2 = 0$.
When two lines are $\parallel \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = 1$.

18.2 Skew and Co-planar Lines

Let there be two lines $\vec{r_1}$ and $\vec{r_2}$,

$$\vec{r_1} = \vec{a_1} + \mu\vec{b_1}\vec{r_2} = \vec{a_2} + \lambda\vec{b_2} \tag{18.6}$$

18.3 Distances

18.3.1 The shortest distance between r_1 and r_2

$$S = \left| \frac{(\vec{a_1} - \vec{a_2}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right| \tag{18.7}$$

If $S = 0$, the lines intersect.

18.3.2 Cartesian Form

For two lines defined as $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$:

$$S = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \tag{18.8}$$

18.3.3 Distance Between Parallel Lines

$$S = \left| \frac{\vec{b} \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b}|} \right| \tag{18.9}$$

18.3.4 Distance of a Point to a Line

For a point, (x_1, y_1, z_1) the distance to a line $\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}$:

$$S = \left(\begin{vmatrix} x_1 - \alpha & y_1 - \beta \\ l & m \end{vmatrix} + \begin{vmatrix} y_1 - \beta & z_1 - \gamma \\ m & n \end{vmatrix} + \begin{vmatrix} z_1 - \gamma & x_1 - \alpha \\ n & l \end{vmatrix} \right)^{\frac{1}{2}} \tag{18.10}$$

Chapter 19

3-D Plane

A plane in 3-D space can be defined as:

1. Cartesian Form:
$$ax + by + cz + d = 0 \tag{19.1}$$
2. Vectorial Form:
$$\vec{r} \cdot \vec{n} = p \tag{19.2}$$
, where $\vec{r}$ is a line on the plane, $\vec{n}$ is a normal to the plane, and p is perpendicular distance to the plane from the origin.

19.1 Angle Between Two Planes

For two planes, $\vec{r_1} \cdot \vec{n_1} = p_1$ and $\vec{r_2} \cdot \vec{n_2} = p_2$, the angle between the planes, θ is:

$$\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|} \tag{19.3}$$

In the Cartesian Form:

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{{a_1}^2 + {b_1}^2 + {c_1}^2}\sqrt{{a_2}^2 + {b_2}^2 + {c_2}^2}} \tag{19.4}$$

19.2 Distance of a Point from a Plane

19.2.1 Catesian Form

For the point (p, q, r) and the plane, $ax + by + cz + d = 0$:

$$S = \frac{ap + bq + cr + d}{\sqrt{a^2 + b^2 + c^2}} \tag{19.5}$$

19.2.2 Vector Form

For the point $\vec{g} = p\hat{i} + q\hat{j} + r\hat{k}$ and the plane $\vec{r} \cdot (a\hat{i} + b\hat{j} + c\hat{k}) + d = 0$:

$$S = \frac{(a\hat{i} + b\hat{j} + c\hat{k}) \cdot (p\hat{i} + q\hat{j} + r\hat{k})}{\sqrt{a^2 + b^2 + c^2}} \tag{19.6}$$

$$\Rightarrow S = \frac{(a\hat{i} + b\hat{j} + c\hat{k}) \cdot \vec{g}}{|a\hat{i} + b\hat{j} + c\hat{k}|} \tag{19.7}$$

Part III

Statistics

Chapter 20

Statistics

For a set a data $(x_1, y_1), (x_2, y_2), \cdots, (n_n, y_n), \cdots$:

Mean of x:

$$barx = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{20.1}$$

Variance of x:

$$\sigma^2 = \frac{(x_1 - \bar{x})^2 + ((x_2 - \bar{x})^2 + \cdots + ((x_n - \bar{x})^2}{n} = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n} = \frac{\sum {x_i}^2}{n} - \bar{x}^2 \tag{20.2}$$

Standard Deviation of x:

$$\begin{aligned} \sigma &= \sqrt{\frac{(x_1 - \bar{x})^2 + ((x_2 - \bar{x})^2 + \cdots + ((x_n - \bar{x})^2}{n}} \\ &= \sqrt{\sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n}} \\ &= \sqrt{\frac{\sum {x_i}^2}{n} - \bar{x}^2} \end{aligned} \tag{20.3}$$

Covariance of (x, y):

$$Cov(x, y) = \frac{\sum_{i=1}^{N}(x_i - x)(y_i - y)}{N} = \sum xy - \frac{1}{N}\sum x \sum y \tag{20.4}$$

Correlation Co-efficient, $\gamma(x, y)$:

$$\gamma(x, y) = \frac{Cov(x, y)}{\sigma_x \sigma_y} \tag{20.5}$$

Chapter 21

Lines of Regression

An assumption is made for the line of regression. It is assumed to be:

$$y = ax + b$$

For a given set of data (x_i, y_i), the solutions of a and b are obtained by solving the following equations simultaneously:

$$\sum y_i = a \sum x_i + nb \tag{21.1}$$

$$\sum x_i y_i = a \sum {x_i}^2 + b \sum x_i \tag{21.2}$$

If the regressive function is defined as:

$$y = cx^a \tag{21.3}$$

, where c is a constant, then the following conversions are performed:

$$y = cx^a \tag{21.4}$$

$$\Rightarrow \log y = \log c + a \log x \tag{21.5}$$

Making the substitutions $\log y = Y$, $\log x = X$, and $\log c = C$, the required equation becomes:

$$Y = aX + C \tag{21.6}$$

This transformed equation can be solved using the method describes in equations **??** and 21.2.

21.1 Karl Pearson's Co-efficient of Correlation (20.5)

$$r = \rho(x, y) = \frac{Cov(x, y)}{\sigma_x \sigma_y} \tag{21.7}$$

21.1.1 Degree of Correlation

Value	Relation
$0 \leq \lvert r \rvert < \frac{1}{4}$	Low
$\frac{1}{4} \leq \lvert r \rvert < \frac{3}{4}$	Moderate
$\frac{3}{4} \leq \lvert r \rvert \leq 1$	High

Table 21.1: Degree of Correlation

Part IV

Trigonometry

Chapter 22

Circular Trigonometric Functions

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	1	0
15°	$\frac{1}{4}$	$\frac{1}{4(2-\sqrt{3})}$	$2-\sqrt{3}$
18°	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
36°	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{1}{\sqrt{3}}$	$\frac{1}{2}$	$\sqrt{3}$
72°	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}$
90°	1	0	∞

Table 22.1: Trigonometric Ratios of Standard Angles

For any given triangle:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \tag{22.1}$$

, where $2R$ is the radius of circumcircle.

22.1 Negative Angle Formula

$$\sin(-\theta) = -\sin\theta \tag{22.2}$$
$$\cos(-\theta) = \cos\theta \tag{22.3}$$
$$\tan(-\theta) = -\tan\theta \tag{22.4}$$
$$\csc(-\theta) = -\csc\theta \tag{22.5}$$
$$\sec(-\theta) = \sec\theta \tag{22.6}$$
$$\cot(-\theta) = -\cot\theta \tag{22.7}$$

22.2 Sum of Angles Formula

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \tag{22.8}$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B \tag{22.9}$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B} \tag{22.10}$$

22.3 Difference of Angles Formula

$$\sin(A-B) = \sin A\cos B - \cos A\sin B \tag{22.11}$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B \tag{22.12}$$
$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B} \tag{22.13}$$

22.4 Multiples and Sub-multiples of π and $\frac{\pi}{2}$

$$\forall k \in \mathbb{Z}$$
$$\sin\left((4k+1)\frac{\pi}{2}\right) = 1 \tag{22.14}$$
$$\sin\left((4k-1)\frac{\pi}{2}\right) = -1 \tag{22.15}$$
$$\sin k\pi = 0 \tag{22.16}$$
$$\sin\left((2k+1)\frac{\pi}{2}\right) = 0 \tag{22.17}$$
$$\sin\left((2k-1)\frac{\pi}{2}\right) = 0 \tag{22.18}$$
$$\sin k\pi = (-1)^k \tag{22.19}$$

22.5 $\left(\frac{\pi}{2} \pm \theta\right)$ Formula

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \tag{22.20}$$
$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta \tag{22.21}$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \tag{22.22}$$
$$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta \tag{22.23}$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \tag{22.24}$$
$$\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta \tag{22.25}$$

$$\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta \tag{22.26}$$
$$\cot\left(\frac{\pi}{2} + \theta\right) = -\tan\theta \tag{22.27}$$

$$\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta \tag{22.28}$$
$$\csc\left(\frac{\pi}{2} + \theta\right) = \sec\theta \tag{22.29}$$

$$\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta \tag{22.30}$$
$$\sec\left(\frac{\pi}{2} + \theta\right) = -\csc\theta \tag{22.31}$$

22.6 $\left(\frac{\pi}{4} \pm \theta\right)$ Formula

$$\tan\left(\frac{\pi}{4} + \theta\right) = \frac{1 + \tan\theta}{1 - \tan\theta} \tag{22.32}$$
$$\tan\left(\frac{\pi}{4} - \theta\right) = \frac{1 - \tan\theta}{1 + \tan\theta} \tag{22.33}$$

22.7 Trigonometric Identities

$$\sin^2\theta + \cos^2\theta = 1 \tag{22.34}$$
$$\tan^2\theta + 1 = \sec^2\theta \tag{22.35}$$
$$\cot^2\theta + 1 = \csc^2\theta \tag{22.36}$$

22.8 Double Angle Formula

$$\begin{aligned} \sin 2\theta &= 2\sin\theta\cos\theta \\ &= \frac{2\tan\theta}{1+\tan^2\theta} \end{aligned} \tag{22.37}$$

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \\ &= \frac{1-\tan^2\theta}{1+\tan^2\theta} \end{aligned} \tag{22.38}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \tag{22.39}$$

22.9 Triple Angle Formula

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta \tag{22.40}$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \tag{22.41}$$

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^3\theta} \tag{22.42}$$

22.10 Sum and Product of Two Ratios

For $A > B$:

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \tag{22.43}$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \tag{22.44}$$

$$2\sin A\cos B = \sin(A+B) + \sin(A-B) \tag{22.45}$$

$$2\cos A\sin B = \sin(A+B) - \sin(A-B) \tag{22.46}$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \tag{22.47}$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \tag{22.48}$$

$$2\cos A\cos B = \cos(A+B) + \cos(A-B) \tag{22.49}$$

$$2\cos A\sin B = \cos(A+B) - \cos(A-B) \tag{22.50}$$

$$\sin(A-B)\sin(A+B) = \sin^2 A - \sin^2 B \tag{22.51}$$

$$\cos(A-B)\cos(A+B) = \cos^2 A - \sin^2 B \tag{22.52}$$

$$\tan(A-B)\tan(A+B) = \frac{\tan^2 A - \tan^2 B}{1 - \tan^2 A\tan^2 B} \tag{22.53}$$

22.11 General Solutions

If $\sin\theta = \sin\alpha$:

$$\theta = n\pi + (-1)^n\alpha \tag{22.54}$$

$n \in \mathbb{Z}$
If $\cos\theta = \cos\alpha$:

$$\theta = 2n\pi \pm \alpha \tag{22.55}$$

$n \in \mathbb{Z}$
If $\tan\theta = \tan\alpha$:

$$\theta = n\pi \pm \alpha \tag{22.56}$$

$n \in \mathbb{Z}$

22.12 Taylor Series Expansion of Trigonometric Ratios

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \infty = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^{2i-1}}{(2i-1)!} \tag{22.57}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \infty = \sum_{i=0}^{\infty} (-1)^{i} \frac{x^{2i}}{(2i)!} \tag{22.58}$$

Chapter 23

Inverse Circular Trigonometric Function

23.1 Definition of Inverse Circular Trigonometric Function

23.1.1 For $\sin x$

$y = \arcsin x$ iff $x = \sin y$, then:

1. $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
2. domain of $x \in [-1, 1]$
3. $\sin(\arcsin x) = x, \forall x \in [-1, 1]$
4. $\arcsin(\sin y) = y, \forall y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
5. $\sin x$ is a strictly increasing in the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and one-one.

23.1.2 For $\cos x$

$y = \arccos x$ iff $x = \cos y$, then:

1. $y \in [0, \pi]$
2. domain of $x \in [-1, 1]$
3. $\cos(\arccos x) = x, \forall x \in [-1, 1]$
4. $\arccos(\cos y) = y, \forall y \in [0, \pi]$
5. $\cos x$ is a strictly decreasing in the domain $[0, \pi]$ and one-one.

23.1.3 For $\tan x$

$y = \arctan x$ iff $x = \tan y$, then:

1. $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
2. domain of $x \in \mathbb{R}$
3. $\tan(\arctan x) = x, \forall x \in \mathbb{R}$
4. $\arctan(\tan y) = y, \forall y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
5. $\tan x$ is a strictly increasing in the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and one-one.

23.1.4 For $\cot x$

$y = \cot^{-1} x$ iff $x = \cot y$, then:

1. $y \in (0, \pi)$
2. domain of $x \in \mathbb{R}$
3. $\cot(\cot^{-1} x) = x, \forall x \in \mathbb{R}$
4. $\cot^{-1}(\cot y) = y, \forall y \in (0, \pi)$
5. $\cot x$ is a strictly decreasing in the domain $(0, \pi)$ and one-one.

For $\sec x$

$y = \sec^{-1} x$ iff $x = \sec y$

1. $y \in \left\{[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]\right\}$
2. $x \in \{(-\infty, -1] \cup [1, \infty)\}$
3. $\sec(\sec^{-1} x) = x, \forall |x| \geq 1$
4. $\sec^{-1}(\sec y) = y, \forall y \in \left\{[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]\right\}$

23.1.5 For $\csc x$

$y = \csc^{-1} x$ iff $x = \csc y$

1. $y \in \left\{[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]\right\}$
2. $x \in \{(-\infty, -1] \cup [1, \infty)\}$
3. $\csc(\csc^{-1} x) = x, \forall |x| \geq 1$
4. $\csc^{-1}(\csc y) = y, \forall y \in \left\{[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]\right\}$

23.2 Negative Arguments

$$\arcsin(-x) = -\arcsin x \tag{23.1}$$
$$\arctan(-x) = -\arctan x \tag{23.2}$$
$$\csc^{-1}(-x) = -\csc^{-1} x \tag{23.3}$$
$$\arccos(-x) = \pi - \arccos x \tag{23.4}$$
$$\cot^{-1}(-x) = \pi - \cot^{-1} x \tag{23.5}$$
$$\sec^{-1}(-x) = \pi - \sec^{-1} x \tag{23.6}$$

23.3 Reciprocal Relations

$$\csc^{-1} x = \arcsin \frac{1}{x} \tag{23.7}$$
$$\sec^{-1} x = \arccos \frac{1}{x} \tag{23.8}$$
$$\sec^{-1} x = \begin{cases} \arctan \frac{1}{x}, x > 0 \\ \pi + \arctan \frac{1}{x}, x < 0 \end{cases} \tag{23.9}$$

23.4 I.T.F. Identities

$$\arcsin x + \arccos x = \frac{\pi}{2}, |x| \leq 1 \tag{23.10}$$
$$\arctan x + \cot^{-1} x = \frac{\pi}{2}, x \in \mathbb{R} \tag{23.11}$$
$$\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}, |x| \geq 1 \tag{23.12}$$

23.5 Sum of Two Angles

$$\arctan x + \arctan y = \arctan \left(\frac{x+y}{1-xy} \right) \tag{23.13}$$
$$\arcsin x + \arcsin y = \arcsin(y\sqrt{1-x^2} + x\sqrt{1-y^2}) \tag{23.14}$$
$$\arccos x + \arccos y = \arccos(xy - \sqrt{1-x^2}\sqrt{1-y^2}) \tag{23.15}$$

23.6 Difference of Two Angles

$$\arctan x - \arctan y = \arctan\left(\frac{x-y}{1+xy}\right) \tag{23.16}$$

$$\arcsin x - \arcsin y = \arcsin(x\sqrt{1-y^2} - y\sqrt{1-x^2}) \tag{23.17}$$

$$\arccos x - \arccos y = \arccos(xy + \sqrt{1-x^2}\sqrt{1-y^2}) \tag{23.18}$$

23.7 Interconversion of Ratios

$$\begin{aligned} \arcsin x = & \quad \arccos\sqrt{1-x^2} \\ = & \quad \arctan\left(\frac{x}{\sqrt{1-x^2}}\right) \end{aligned} \tag{23.19}$$

$$\begin{aligned} \arccos x = & \quad \arcsin\sqrt{1-x^2} \\ = & \quad \arctan\left(\frac{\sqrt{1-x^2}}{x}\right) \end{aligned} \tag{23.20}$$

$$\begin{aligned} 2\arctan x = & \quad \arcsin\left(\frac{2x}{1+x^2}\right) \\ = & \quad \arccos\left(\frac{1-x^2}{1+x^2}\right) \\ = & \quad \arctan\left(\frac{2x}{1-x^2}\right) \end{aligned} \tag{23.21}$$

23.8 Miscellaneous Relations

$$\cos(\arcsin x) = \sin(\arccos x) = \sqrt{1-x^2} \tag{23.22}$$

$$\arctan x = \frac{\pi}{2} - \arctan\left(\frac{1}{x}\right), x > 1 \tag{23.23}$$

Chapter 24

Hyperbolic Trigonometric Function

24.1 Definition

Hyperbolic trigonometric functions are defined such that $(\cosh t, \sinh t)$ form the right half of an equilateral hyperbola. The functions are defined as follows:

$$\sinh x = \frac{\exp(x) - \exp(-x)}{2} \tag{24.1}$$

$$\cosh x = \frac{\exp(x) + \exp(-x)}{2} \tag{24.2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\exp(x) - \exp(-x)}{\exp(x) + \exp(-x)} \tag{24.3}$$

$$\coth x = \frac{1}{\tanh x} = \frac{\exp(x) + \exp(-x)}{\exp(x) - \exp(-x)} \tag{24.4}$$

$$cschx = \frac{1}{\sinh x} = \frac{2}{\exp(x) - \exp(-x)} \tag{24.5}$$

$$sechx = \frac{1}{\cosh x} = \frac{2}{\exp(x) + \exp(-x)} \tag{24.6}$$

24.2 Identities

$$\coth^2 x - \sinh^2 x = 1 \tag{24.7}$$

$$\tanh^2 x + sech^2 x = 1 \tag{24.8}$$

$$\coth^2 x - csch^2 x = 1 \tag{24.9}$$

24.3 Inverse Hyperbolic Function

$$\sinh^{-1} z = \ln(z + \sqrt{z^2+1}) \tag{24.10}$$

$$\cosh^{-1} z = \ln(z \pm \sqrt{z^2-1}) \tag{24.11}$$

$$\tanh^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) \tag{24.12}$$

$$\coth^{-1} z = \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right) \tag{24.13}$$

$$csch^{-1} z = \ln\left(\frac{1 \pm \sqrt{z^2+1}}{z}\right) \tag{24.14}$$

$$sech^{-1} z = \ln\left(\frac{1 \pm \sqrt{1-z^2}}{2}\right) \tag{24.15}$$

24.4 Relation to Circular Trigonometric Functions

$$\sinh(z) = -i\sin(iz) \tag{24.16}$$

$$\coth(z) = \cos(iz) \tag{24.17}$$

$$\tanh(z) = -i\tan(iz) \tag{24.18}$$

$$csch(z) = i\csc(iz) \tag{24.19}$$

$$sech(z) = \sec(iz) \tag{24.20}$$

$$\coth(z) = i\cot(iz) \tag{24.21}$$

Part V

Calculus

Chapter 25

Limits

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \tag{25.1}$$

$$\lim_{x\to 0} \frac{\tan x}{x} = 1 \tag{25.2}$$

$$\lim_{x\to 0} \cos x = 1 \tag{25.3}$$

$$\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1} \tag{25.4}$$

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x) \tag{25.5}$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \lim_{x\to a} g(x) \neq 0 \tag{25.6}$$

$$\lim_{x\to 0} \exp(x) = 1 \tag{25.7}$$

$$\lim_{x\to a} \exp(x) = \exp(c) \tag{25.8}$$

$$\lim_{x\to 0} \frac{\exp(x) - 1}{x} = 1 \tag{25.9}$$

$$\lim_{x\to a} c^x = c^a \tag{25.10}$$

$$\lim_{x \to a} \ln x = \ln a \tag{25.11}$$

$$\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e \tag{25.12}$$

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1 \tag{25.13}$$

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a \tag{25.14}$$

$$\lim_{n \to \infty} \frac{x^n}{n!} = 0, \forall x \in \mathbb{R} \tag{25.15}$$

25.1 L'Hospital Rule

If:

$$L = \lim_{x \to a} \frac{f(x)}{g(x)}$$

is such that $f(a) = 0$ and $g(a) = 0$, or $f(a) = \infty$ and $g(a) = \infty$, then:

$$L = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Chapter 26

Differentiation

26.1 Differentiation by First Principle

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{26.1}$$

26.2 Standard Differentiation Formulae

$$\frac{dk}{dx} = 0 \tag{26.2}$$

$$\frac{dx^n}{dx} = nx^{n-1} \tag{26.3}$$

$$\frac{da^x}{dx} = \ln a \cdot a^x \tag{26.4}$$

$$\frac{d\exp(x)}{dx} = \exp(x) \tag{26.5}$$

$$\frac{d\ln x}{dx} = \frac{1}{x} \tag{26.6}$$

$$\frac{d\sqrt{x}}{dx} = \frac{1}{2\sqrt{2}} \tag{26.7}$$

$$\tag{26.8}$$

26.2.1 Circular Trigonometric Functions

$$\frac{d\sin x}{dx} = \cos x \tag{26.9}$$
$$\frac{d\cos x}{dx} = -\sin x \tag{26.10}$$
$$\frac{d\tan x}{dx} = \sec^2 x \tag{26.11}$$
$$\frac{d\sec x}{dx} = \sec x \tan x \tag{26.12}$$
$$\frac{d\csc x}{dx} = -\csc x \cot x \tag{26.13}$$
$$\frac{d\cot x}{dx} = -\csc^2 x \tag{26.14}$$

26.2.2 Inverse Circular Trigonometric Functions

$$\frac{d\arcsin x}{dx} = \frac{1}{\sqrt{1-x^2}}, |x| \leq 1 \tag{26.15}$$
$$\frac{d\arccos x}{dx} = -\frac{1}{\sqrt{1-x^2}}, |x| \leq 1 \tag{26.16}$$
$$\frac{d\arctan x}{dx} = \frac{1}{1+x^2} \tag{26.17}$$
$$\frac{d\cot^{-1} x}{dx} = -\frac{1}{1+x^2} \tag{26.18}$$
$$\frac{d\sec^{-1} x}{dx} = \frac{1}{x\sqrt{x^2-1}}, |x| \geq 1 \tag{26.19}$$
$$\frac{d\csc^{-1} x}{dx} = -\frac{1}{x\sqrt{x^2-1}}, |x| \geq 1 \tag{26.20}$$

26.2.3 Hyperbolic Trigonometric Function

$$\frac{d\sinh x}{dx} = \cosh x \tag{26.21}$$

$$\frac{d\cosh x}{dx} = \sinh x \tag{26.22}$$

$$\frac{d\tanh x}{dx} = 1 - \tanh^2 x = sech^2(x) \tag{26.23}$$

$$\frac{d\coth x}{dx} = 1 - \coth^2 x = -csch^2(x) \tag{26.24}$$

$$\frac{d[sech(x)]}{dx} = -\tanh x \ \mathrm{sech} x \tag{26.25}$$

$$\frac{d[csch(x)]}{dx} = -\coth x \ \mathrm{csch} x \tag{26.26}$$

26.2.4 Inverse Hyperbolic Trigonometric Function

$$\frac{d\sinh x}{dx} = \frac{1}{\sqrt{x^2+1}} \tag{26.27}$$

$$\frac{d\cosh x}{dx} = \frac{1}{\sqrt{x^2-1}} \tag{26.28}$$

$$\frac{d\tanh x}{dx} = \frac{1}{1-x^2} \tag{26.29}$$

$$\frac{d\coth x}{dx} = \frac{1}{1-x^2} \tag{26.30}$$

$$\frac{d[sech(x)]}{dx} = \frac{1}{x\sqrt{1-x^2}} \tag{26.31}$$

$$\frac{d[csch(x)]}{dx} = \frac{1}{|x|\sqrt{1+x^2}} \tag{26.32}$$

26.3 Rules of Differentiation

$$\frac{d[cf(x)]}{dx} = c\frac{df(x)}{dx} \tag{26.33}$$

$$\frac{d[f_1(x) + f_2(x)]}{dx} = \frac{d[f_1(x)]}{dx} + \frac{d[f_2(x)]}{dx} \tag{26.34}$$

$$\frac{d[f_1 f_2]}{dx} = f_1 f_2' + f_2 f_1' \tag{26.35}$$

$$\frac{d\left(\frac{f_1}{f_2}\right)}{dx} = \frac{f_2 f_1' - f_1 f_2'}{f_2^2} \tag{26.36}$$

26.4 Chain Rule

If two functions are defined as $z = f(y)$ and $y = g(x)$:

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} \tag{26.37}$$

If two functions are defined as $x = f(\theta)$ and $y = g(\theta)$:

$$\frac{d^2y}{dx^2} = \left[\frac{d}{d\theta}\left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right)\right]\frac{d\theta}{dx} \tag{26.38}$$

Chapter 27

Successive Differentiation

$$D^n(ax+b)^m = m(m-1)\cdots(m-n+1)a^n(ax+b)^{m-n} \tag{27.1}$$

$$D^n\left(\frac{1}{ax+b}\right) = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}} \tag{27.2}$$

$$D^n \ln(ax+b) = \frac{(-1)^{n-1}(n-1)! a^n}{(ax+b)^n}, n \geq 2 \tag{27.3}$$

$$D^n(a^{mx}) = m^n(\ln a)^n a^{mx} \tag{27.4}$$

$$D^n(e^{mx}) = m^n e^{mx} \tag{27.5}$$

$$D^n \sin(ax+b) = a^n \sin(ax+b+n\frac{\pi}{2}) \tag{27.6}$$

$$D^n \cos(ax+b) = a^n \cos(ax+b+n\frac{\pi}{2}) \tag{27.7}$$

$$D^n[e^{ax}\sin(bx+c)] = (a^2+b^2)^{\frac{n}{2}} e^{ax} \sin(bx+c+n\arctan\frac{b}{a}) \tag{27.8}$$

$$D^n[e^{ax}\cos(bx+c)] = (a^2+b^2)^{\frac{n}{2}} e^{ax} \cos(bx+c+n\arctan\frac{b}{a}) \tag{27.9}$$

27.1 Leibnitz's Theorem

For two functions u and v of x, the successive differentiation of their product is defined as:

$$\begin{aligned}(uv)_n &= {}^nC_0 u_n v +^n C_1 u_{n-1} v_1 + \cdots +^n C_0 u v_n \\ &= \sum_{i=0}^{n} {}^nC_i u_{n-i} v_i\end{aligned} \tag{27.10}$$

Chapter 28

Partial Derivative

If $f(x, y)$ is a function of (x, y), then $\dfrac{\delta f(x,y)}{\delta x}$ is the differentiation of $f(x, y)$ w.r.t. x, keeping all other parameters constant.

28.1 Chain Rule

If f is a function of u and v, which are functions of x and y, then:

$$\frac{\delta f}{\delta x} = \frac{\delta f}{\delta u}\frac{\delta u}{\delta x} + \frac{\delta f}{\delta v}\frac{\delta v}{\delta x} \tag{28.1}$$

$$\frac{\delta f}{\delta y} = \frac{\delta f}{\delta u}\frac{\delta u}{\delta y} + \frac{\delta f}{\delta v}\frac{\delta v}{\delta y} \tag{28.2}$$

If f is a function of x and y, which are functions of t, then:

$$\frac{df}{dt} = \frac{\delta f}{\delta x}\frac{dx}{dt} + \frac{\delta f}{\delta y}\frac{dy}{dt} \tag{28.3}$$

28.2 Euler's Theorem

For a homogeneous function [1], $f(x_i)$ of degree n:

$$\sum x_i \frac{\delta f}{\delta x_i} = nf(x_i) \tag{28.4}$$

[1] Homogeneous functions are defined as $f(ax, ay) = a^{\kappa} f(x, y)$, where κ is the degree of homogeneity. E.g. $f(x, y) = x^2 + y^2$, then $f(tx, ty) = t^2(x^2 + y^2)$, and the degree of homogeneity is 2.

Chapter 29

Application of Differentiation

29.1 Rolle's Theorem

For a function $f(x)$:

1. is continuous in $[a, b]$
2. is differentiable in (a, b)
3. $f(a) = f(b)$,

then there exists a point $x = c$ such that $f'(c) = 0$, $c \in (a, b)$

29.2 Mean Value Theorem or LaGrange's Theorem

For a function $f(x)$:

1. is continuous in $[a, b]$
2. is differentiable in (a, b),

then there exists a point $x = c$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$, $c \in (a, b)$, i.e., the tangent is parallel to the line joining the the points $(a, f(a))$ and $(b, f(b))$.

29.3 Cauchy's Mean Value Theorem

For a function $f(x)$ and $g(x)$:

1. are continuous in $[a, b]$
2. are differentiable in (a, b)
3. $g'(x) \neq 0$ in (a, b),

then there exists a point $c \in (a, b)$, such that $\dfrac{f(x)}{g(x)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$.

29.4 Maxima and Minima

29.4.1 Maxima

For the local maxima of a function $f(x)$:

1. $f'(c) = 0$ and

$$\lim_{\epsilon \to c^-} f'(\epsilon) > 0$$
$$\lim_{\epsilon \to c^+} f'(\epsilon) < 0$$

OR

2. $f'(c) = 0$ and $f''(x) < 0$,

then $f(c)$ is the local maxima point of the function $f(x)$.

29.4.2 Minima

For the local minima of a function $f(x)$:

1. $f'(c) = 0$ and

$$\lim_{\epsilon \to c^-} f'(\epsilon) < 0$$
$$\lim_{\epsilon \to c^+} f'(\epsilon) > 0$$

OR

2. $f'(c) = 0$ and $f''(x) > 0$,

then $f(c)$ is the local minima point of the function $f(x)$.

29.5 Taylor's Theorem

For a function which is differentiable n times:

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n}{x!}R_n \quad (29.1)$$

where R_n is the remainder term.

29.5.1 Remainder Term

LeGrange's Form

$$R_n = f^n(a + \theta h), \theta \in (0, 1) \quad (29.2)$$

Cauchy's Form

$$R_n = n(1-\theta)^{n-1} f^n(a+\theta h), \theta \in (0,1) \tag{29.3}$$

29.5.2 Conditions for Validity of Expansion

For validity of Taylor Expansion,the condition

$$\lim_{n\to\infty} R_n = 0 \tag{29.4}$$

needs to be satisfied either where R_n is the remainder term in either LeGrange's Form or Cauchy's Form. If the condition is satisfied in a certain domain, then the expansion is valid within that domain only.

29.5.3 Taylor's Theorem for Two Variables

$$\begin{aligned} f(a+x,b+y) = \quad & f(x,y) + \left(a\frac{\delta}{\delta x} + b\frac{\delta}{\delta y}\right) f(x,y) + \\ & \frac{1}{2!}\left(a^2\frac{\delta^2}{\delta x^2} + b^2\frac{\delta^2}{\delta y^2}\right) f(x,y) + \cdots + \\ & \frac{1}{n!}\left(a^n\frac{\delta^n}{\delta x^n} + b^n\frac{\delta^n}{\delta y^n}\right) f(x+\theta a, y+\theta b), \theta \in (0,1) \end{aligned} \tag{29.5}$$

29.6 Maclaurin's Series

$$\begin{aligned} f(x) = \quad & f(0) + xf'(0) + \frac{1}{2!}x^2 f''(0) + \frac{1}{3!}x^3 f'''(0) + \cdots \infty \\ = \quad & \sum_{i=0}^{\infty} \frac{1}{i!} x^i f^i(0) \end{aligned} \tag{29.6}$$

29.6.1 Maclaurin's Series with Two Variables

$$\begin{aligned} f(a,b) = \quad & f(0,0) + \left(a\frac{\delta}{\delta x} + b\frac{\delta}{\delta x}\right) f(0,0) + \\ & \frac{1}{2!}\left(a^2\frac{\delta^2}{\delta x^2} + b^2\frac{\delta^2}{\delta x^2}\right) f(0,0) + \cdots \infty \\ = \quad & \sum_{i=0}^{\infty} \frac{1}{n!}\left(a^i\frac{\delta^i}{\delta x^i} + b^i\frac{\delta^i}{\delta x^i}\right) f(0,0) \end{aligned} \tag{29.7}$$

29.7 Curvature

Curvature is the rate of change of direction w.r.t. arc. Mathematically:

$$\text{Curvature} = \frac{d(\text{direction})}{d(\text{arc})}$$

$$\lim_{\Delta s \to 0} \frac{\Delta \psi}{\Delta s} = \frac{d\psi}{ds} \tag{29.8}$$

29.7.1 Radius of Curvature

Cartesian Form

For a curve $y = f(x)$:

$$\rho = \frac{(1 + y'^2)^{\frac{3}{2}}}{y''} \tag{29.9}$$

However, this formula fails for $y' \to \infty$.

Parametric Form

For a curve defined as $x = \phi(t)$ and $y = \psi(t)$:

$$\rho = \frac{(\ddot{x}^2 + \ddot{y}^2)^{\frac{3}{2}}}{x\ddot{y} - y\ddot{x}} \tag{29.10}$$

29.7.2 Newton's Formula

1. If the curve passes through origin, and the tangent at origin is the x-axis:
$$\rho = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2}{2y} \tag{29.11}$$
2. If the curve passes through origin, and the tangent at origin is the y-axis:
$$\rho = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{y^2}{2x} \tag{29.12}$$
3. If the curve passes through origin and $ax + by + c = 0$ is the tangent at origin:
$$\rho(0,0) = \frac{1}{2}\sqrt{a^2 + b^2} \lim_{\substack{x \to 0 \\ y \to 0}} \frac{a^2 + y^2}{ax + by} \tag{29.13}$$

29.7.3 Tangent at Origin

For a curve

$$\sum c_i x^j y^k = 0, i \in \mathbb{N} \text{ and } j, k \in \mathbb{Z} - \{0\} \tag{29.14}$$

The curve passes through origin $\because c = 0$. Then the lowest degree term equated to x gives the tangent at origin.

29.8 Asymptotes

If the distance between a line P and a curve $f(x)$, s is such that $s \to 0$, as $x \to \infty$, then P is the asymptote of $f(x)$. For asymptotes not parallel to x-axis:
Let $y = mx + c$ be the asymptote of the function $y = f(x)$, then:

$$m = \lim_{x \to \infty} \frac{y}{x} \tag{29.15}$$

$$c = \lim_{x \to \infty} (y - mx) \tag{29.16}$$

29.8.1 Asymptote of Algebraic Curves

For an algebraic curve, passing through origin, defined as:

$$\begin{aligned}(a_0x^n + a_1x^{n-1}y^1 + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n)+\\ (b_0x^{n-1} + b_1x^{n-2}y^1 + b_2x^{n-3}y^2 + \cdots + b_{n-1}xy^{n-2} + a_ny^{n-1})+\\ \cdots = 0\end{aligned}$$

$$\Rightarrow x^n\phi_n\left(\frac{y}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right) + \cdots + x\phi_1\left(\frac{y}{x}\right) = 0$$

The asymptote(s) defined as $y = mx + c$,

1. m is the solution for the equation

$$\phi_n(m) = 0 \tag{29.17}$$

2.

$$c = -\frac{\phi_{n-1}(m)}{\phi_n(m)} \tag{29.18}$$

where c is a finite value.

Chapter 30

Integration

30.1 General Formulae

[1]

$$\int nx^{n-1}dx = x^n + A \tag{30.1}$$

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + A \tag{30.2}$$

$$\int e^x dx = e^x + A \tag{30.3}$$

$$\int \frac{1}{x}dx = \ln x + A \tag{30.4}$$

$$\int \ln x dx = x(\ln x - 1) + A \tag{30.5}$$

[1] A is the constant of integration in all cases

30.2 Circular Trigonometric Functions

$$\int \sin x dx = -\cos x + A \tag{30.6}$$

$$\int \cos x dx = \sin x + A \tag{30.7}$$

$$\int \sec^2 x dx = \tan x + A \tag{30.8}$$

$$\int \csc^2 x dx = -\cot x + A \tag{30.9}$$

$$\int \sec x \tan x dx = \sec x + A \tag{30.10}$$

$$\int \csc x \cot x dx = -\csc x + A \tag{30.11}$$

$$\int \sec x dx = \ln(\sec x + \tan x) + A \tag{30.12}$$

$$\int \csc x dx = -\ln(\csc x + \cot x) + A \tag{30.13}$$

$$\begin{aligned} \int \tan x dx &= -\ln(\cos x) + A \\ &= \ln(\sec x) + A \end{aligned} \tag{30.14}$$

$$\int \cot x dx = \ln(\sin x) + A \tag{30.15}$$

30.3 Inverse Circular Trigonometric Function

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + A \tag{30.16}$$

$$\int \frac{-1}{\sqrt{1-x^2}} dx = \cos^{-1} x + A \tag{30.17}$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + A \tag{30.18}$$

$$\int \frac{-1}{1+x^2} dx = \cot^{-1} x + A = -\tan^{-1} x + A \tag{30.19}$$

$$\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + A = -\csc^{-1} x + A \tag{30.20}$$

$$\int \frac{-1}{x\sqrt{x^2-1}} dx = \csc^{-1} x + A = -\sec^{-1} x + A \tag{30.21}$$

30.4 Standard Integrals

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + A \tag{30.22}$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2}) + A \tag{30.23}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + A \tag{30.24}$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln(x + \sqrt{x^2 - a^2}) + A \tag{30.25}$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + A \tag{30.26}$$

$$\int \sqrt{a^2 - x^2}dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + A \tag{30.27}$$

$$\int \sqrt{a^2 + x^2}dx = \frac{x\sqrt{a^2 + x^2}}{2} + \frac{a^2}{2}\ln(x + \sqrt{x^2 + a^2}) + A \tag{30.28}$$

$$\int \sqrt{x^2 - a^2}dx = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2}\ln(x + \sqrt{a^2 - x^2}) + A \tag{30.29}$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x - a}{x + a}\right| + A \tag{30.30}$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a + x}{a - x}\right| + A \tag{30.31}$$

[2]

30.5 Special Forms

For a function $f(x)$:

$$\int [f(x)]^n f'(x)dx = \begin{cases} \dfrac{[f(x)]^{n+1}}{n+1} + A, n \neq 1 \\ \ln|f(x)| + A, n = 1 \end{cases} \tag{30.32}$$

[2] a is a constant $\in \mathbb{R}$

30.5.1 Integration by Part

For two functions $u(x)$ and $v(x)$:

$$\int u(x)v(x)dx = u(x)\left[\int v(x)dx\right] - \int\left[\frac{du(x)}{dx}\left(\int v(x)dx\right)dx\right] \quad (30.33)$$

Chapter 31

Definite Integral

31.1 Definition

For a function $f(x)$ for which $\int f(x)dx = F(x) + A$,

$$\int_a^b f(x)dx = F(b) - F(a) \tag{31.1}$$

31.2 Properties of Definite Integration

$$\int_a^b f(x)dx = \int_a^b f(t)dt \tag{31.2}$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx \tag{31.3}$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \tag{31.4}$$

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx \tag{31.5}$$

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx \tag{31.6}$$

$$\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, \; f(2a-x) = f(x) \\ 0, \; f(2a-x) = f-(x) \end{cases} \tag{31.7}$$

$$\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx, \; f(x) \text{ is even} \\ 0, \; f(x) \text{ is odd} \end{cases} \tag{31.8}$$

31.3 Approximation

$$f(a)(b-a) \leq \int_a^b f(x)dx \leq f(b)(b-a) \tag{31.9}$$

31.4 Sum of Infinite Series as a Definite Integral

Refer to 3.5.2.

Chapter 32

Reduction Formulae

$$\begin{aligned}\int \sin^n x dx = & -\frac{1}{n}\sin^{n-1} x \cos x dx + \\ & \frac{n-1}{n}\int \sin^{n-2} x dx\end{aligned} \tag{32.1}$$

$$\begin{aligned}\int \cos^n x dx = & -\frac{1}{n}\cos^{n-1} x \sin x dx + \\ & \frac{n-1}{n}\int \cos^{n-2} x dx\end{aligned} \tag{32.2}$$

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx = \begin{cases} \dfrac{(n-1)\cdot(n-3)\cdots 3\cdot 1}{n\cdot(n-2)\cdots 4\cdot 2}\left(\dfrac{\pi}{2}\right), n \to \text{even} \\ \dfrac{(n-1)\cdot(n-3)\cdots 4\cdot 2}{n\cdot(n-2)\cdots 3\cdot 1}, n \to \text{odd} \end{cases} \tag{32.3}$$

$$\int \sin^m x \cos^n x dx = \frac{-\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x dx \tag{32.4}$$

For $I(m,n) = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx$:
When m and n are both even:

$$I(m,n) = \frac{[(m-1).(m-3)\cdots 3.1][(n-1).(n-3)\cdots 3.1]}{(m+n).(m+n-1)\cdots(4).(2)} \cdot \frac{\pi}{2} \tag{32.5}$$

Otherwise:

$$I(m,n) = \frac{[(m-1).(m-3)\cdots(2 \text{ or } 1)][(n-1).(n-3)\cdots()(2 \text{ or } 1)]}{(m+n).(m+n-1)\cdots(2 \text{ or } 1))} \tag{32.6}$$

$$I_n = \int \tan^n x dx$$

$$\Rightarrow I_n = \frac{\tan^{n-2} x}{n-1} - I_{n-2} \tag{32.7}$$

$$I_n = \int \cot^n x dx$$

$$\Rightarrow I_n = -\frac{\cot^{n-2} x}{n-1} - I_{n-2} \tag{32.8}$$

$$I_n = \int \sec^n x dx$$

$$\Rightarrow I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2} \tag{32.9}$$

$$I_n = \int \csc^n x dx$$

$$\Rightarrow I_n = -\frac{\csc^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2} \tag{32.10}$$

$$I_n = \int x^n e^{ax} dx \tag{32.11}$$

$$\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-2} \tag{32.12}$$

$$I(m,n) = \int x^m (\ln x)^n dx \tag{32.13}$$

$$\int x^m (\ln x)^n dx = \frac{x^{m+1}}{m+1} (\ln x)^n - \frac{n}{m+1} I_{m,n-1} \tag{32.14}$$

Chapter 33

Multiple Integrals

33.1 Two Variables

For

$$I = \iint_R f(x, y)dxdy \tag{33.1}$$

The following substitution are made:

$$x = g(r, \theta) \tag{33.2}$$

$$y = h(r, \theta) \tag{33.3}$$

$$\therefore dxdy = |J|drd\theta \tag{33.4}$$

Where J is the Jacobian, defined as:

$$J = \begin{vmatrix} \frac{\delta x}{\delta r} & \frac{\delta y}{\delta r} \\ \frac{\delta x}{\delta \theta} & \frac{\delta y}{\delta \theta} \end{vmatrix} \tag{33.5}$$

The equivalent integral is:

$$I = \iint_{R_1} f(g(r, \theta), h(r, \theta))|J|drd\theta \tag{33.6}$$

33.2 Three Variables

For

$$I = \iiint_R f(x, y, z)dxdydz \tag{33.7}$$

The following substitution are made:

$$x = g(r, \theta, \phi) \tag{33.8}$$
$$y = h(r, \theta, \phi) \tag{33.9}$$
$$z = k(r, \theta, \phi) \tag{33.10}$$

$$\therefore dxdydz = |J| dr d\theta d\phi \tag{33.11}$$

Where J is the Jacobian, defined as:

$$J = \begin{vmatrix} \frac{\delta x}{\delta r} & \frac{\delta y}{\delta r} & \frac{\delta z}{\delta r} \\ \frac{\delta x}{\delta \theta} & \frac{\delta y}{\delta \theta} & \frac{\delta z}{\delta \theta} \\ \frac{\delta x}{\delta \phi} & \frac{\delta y}{\delta \phi} & \frac{\delta z}{\delta \phi} \end{vmatrix} \tag{33.12}$$

The equivalent integral is:

$$I = \iiint_{R_1} f(g(r, \theta, \phi), h(r, \theta, \phi), k(r, \theta, \phi)) |J| dr d\theta d\phi \tag{33.13}$$

Chapter 34

Differential Equation

34.1 1^{st} Order, 1^{st} Degree Differential Equation

For the equation:

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{34.1}$$

Then an Integral Function (I.F.) is defined as:

$$I.F. = e^{\int P(x)dx} \tag{34.2}$$

Then the solution of the equation 34.1 is given by:

$$y(I.F.) = \int Q(I.F.)dx \tag{34.3}$$

34.2 2^{nd} Order, 1^{st} Degree Differential Equation

For the equation:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0 \tag{34.4}$$

OR

$$y'' + ay' + by = 0 \tag{34.5}$$

By substituting $y = e^{\lambda x}$, the equation obtained is:

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} + be^{\lambda x} = 0$$

$$\because e^{\lambda x} \neq 0$$

$$\Rightarrow \lambda^2 + a\lambda + b = 0 \tag{34.6}$$

If α and β are the solutions of the equation 34.6, then the solution of 34.4 can be:

1. If $\alpha = \beta$ and $\alpha, \beta \in \mathbb{R}$:
$$y = (c_1 + c_2 x)e^{\alpha x} \tag{34.7}$$
2. If $\alpha \neq \beta$ and $\alpha, \beta \in \mathbb{R}$:
$$y = c_1 e^{\alpha x} + c_2 e^{\beta x} \tag{34.8}$$
3. If $\lambda = \alpha + i\beta$:
$$y = e^{\alpha x}\left[A\cos(\beta x) + B\sin(\beta x)\right] \tag{34.9}$$

34.3 Special Cases of Differential Equation

34.3.1 Definition of Inverse Operator

The operator D is equivalent to $\dfrac{d}{dx}$. If $Df(x) = X$, then $f(x) = \dfrac{1}{D}X = \int X dx$.

34.3.2 Special Cases

1.
$$f(x) = \frac{1}{D-a}X = e^{ax}\int Xe^{-ax}dx \tag{34.10}$$
2.
$$\frac{1}{f(D)}e^{ax} = \begin{cases} \dfrac{e^{ax}}{f(a)}, f(a) \neq 0 \\ x\dfrac{e^{ax}}{f'(a)}, f(x) = 0 \text{ and } f'(a) \neq 0 \\ x^2\dfrac{e^{ax}}{f''(a)}, f(x) = 0 \text{ and } f'(a) = 0 \end{cases} \tag{34.11}$$
3.
$$\frac{1}{f(D)}x^m = [f(D)]^{-1}x^m \tag{34.12}$$
$[f(D)]^{-1}$ is expanded and arranged in terms of ascending powers of D and operated on x^m.
4. (a)
$$\begin{aligned} \frac{1}{f(D)}\sin(ax) = & \quad \frac{1}{\phi(D^2)}\sin(ax) \\ = & \quad \frac{1}{\phi(-a^2)}\sin(ax) \end{aligned} \tag{34.13}$$

(b)

$$\frac{1}{f(D)}\cos(ax) = \frac{1}{\phi(D^2)}\cos(ax) = \frac{1}{\phi(-a^2)}\cos(ax) \tag{34.14}$$

5. (a)

$$\frac{1}{f(D)}\sin(ax) = \frac{1}{\phi(D^2, D)}\sin(ax) = \frac{1}{\phi(-a^2, D)}\sin(ax) \tag{34.15}$$

(b)

$$\frac{1}{f(D)}\cos(ax) = \frac{1}{\phi(D^2, D)}\cos(ax) = \frac{1}{\phi(-a^2, D)}\cos(ax) \tag{34.16}$$

6. (a)

$$\frac{1}{f(D)}\sin(ax) = \frac{\psi(D)}{\phi(D^2)}\sin(ax) = \frac{\psi(D)}{\phi(-a^2)}\sin(ax) \tag{34.17}$$

(b)

$$\frac{1}{f(D)}\cos(ax) = \frac{\psi(D)}{\phi(D^2)}\cos(ax) = \frac{\psi(D)}{\phi(-a^2)}\cos(ax) \tag{34.18}$$

7. (a)

$$\frac{1}{f(D)}\sin(ax) = x\frac{1}{f'(D)}\sin(ax) \tag{34.19}$$

(b)

$$\frac{1}{f(D)}\cos(ax) = x\frac{1}{f'(D)}\cos(ax) \tag{34.20}$$

34.4 Method of Variation of Parameters

If the equation is of the form:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f \tag{34.21}$$

where a, b, f are functions of x. The solution for 34.21 is obtained by solving for:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0 \tag{34.22}$$

If y_1 and y_2 are the two independent solution of equation 34.22.
Then the general solution of the equation is:

$$y = c_1y_1 + c_2y_2 \tag{34.23}$$

where c_1 and c_2 are the constants.
The particular solution of equation 34.22 will be:

$$y = y_1\left(\int \frac{y_2(-f)}{W}dx\right) + y_2\left(\int \frac{y_1 f}{W}dx\right) \tag{34.24}$$

W is the Wronskian, which is defined by:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{34.25}$$

34.5 Singular and Ordinary Point

For a differential equation:

$$P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_{n-1}\frac{dy}{dx} + P_ny = R(x) \tag{34.26}$$

where $P_0 \cdots P_n$ are functions of x.
If at a point $x = x_0$:

1. $P_0(x_0) \neq 0$, x_0 is an ordinary point.
2. $P_0(x_0) = 0$, x_0 is an singular point:
   (a)
   $$\lim_{x\to x_0}(x - x_0)P_1(x) = c_1 \tag{34.27}$$
   $$\lim_{x\to x_0}(x - x_0)^2P_2(x) = c_2 \tag{34.28}$$
   $$\tag{34.29}$$
   where c_1 and c_2 are both finite quantities x_0 is a regular singular point.
   (b) otherwise it is an irregular singular point.

Chapter 35

Beta and Gamma Functions

For $m, n > 0$:

$$\begin{aligned}\beta(m,n) = & \int_0^1 x^{m-1}(1-x)^{n-1}dx \\ = & \ 2\int_0^{\frac{\pi}{2}} \sin^{2m-1} x \cos^{2n-1} x dx\end{aligned} \tag{35.1}$$

$$\Gamma(n) = \int_0^\infty e^{-1}x^{n-1}dx \tag{35.2}$$

35.1 Important Relations between $\beta(m,n)$ and $\Gamma(n)$ Functions

$$\Gamma(n) = \frac{\Gamma(n+1)}{n} \tag{35.3}$$

$$\Gamma(1) = 1 \tag{35.4}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \approx 1.772 \tag{35.5}$$

$$\Gamma(n+1) = n!, n \in \mathbb{N} \tag{35.6}$$

$$\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \sqrt{\pi}\Gamma(2m) \tag{35.7}$$

$$\Gamma(m)\Gamma(m-1) = \pi\csc(m\pi) \tag{35.8}$$

$$\beta(m,n) = \beta(n,m) \tag{35.9}$$

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \tag{35.10}$$

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \pi \tag{35.11}$$

$$\int_0^{\frac{\pi}{2}} \sin^p x \cos^q x = \frac{1}{2}\frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{\Gamma(\frac{p+2}{2})} \tag{35.12}$$

$$\tag{35.13}$$

Chapter 36

Laplace Transformations

The Laplace Transformation of a function $f(t)$ is defined as:

$$F(s) = \mathcal{L}\{f(t)\} = \lim_{x \to \infty} \int_0^x e^{-st} f(t) dt \tag{36.1}$$

36.1 Basic Transformations

$f(t)$	F(s)
$af(t) + bg(t)$	$aF(s) + bG(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$

Table 36.1: Table of Laplace Transformations

36.2 Important Relations

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a) \tag{36.2}$$

$$\mathcal{L}\{tf(t)\} = -F'(s) \tag{36.3}$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^n(s) \tag{36.4}$$

$$\mathcal{L}\{\frac{f(t)}{t}\} = \lim_{x\to\infty}\int_s^x F(u)du \tag{36.5}$$

$$\mathcal{L}\{\frac{f(t)}{t^n}\} = \lim_{x\to\infty}\int_1\int_2\cdots\int_s^x F(u)du\cdots du \tag{36.6}$$

36.3 Convolution

For two functions $f(t)$ and $g(t)$ be given such that their Laplace transforms are $F(s)$ and $G(s)$, then:

$$\mathcal{L}\{f(t)\star g(t)\} = F(s)G(s) \tag{36.7}$$

where $f(t)\star g(t)$ is defined as:

$$\int_0^t f(u)g(t-u)du \tag{36.8}$$

36.4 Laplace Transforms of Differentials

If the Laplace Transform of $f(t)$ is $F(s)$[1]:

$$\mathcal{L}\{f'(t)\} = sF(s) - y(0) \tag{36.9}$$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - [sy(0) + y'(0)] \tag{36.10}$$

$$\vdots$$

$$\mathcal{L}\{f^n(t)\} = s^nF(s) - [\sum_{i=0}^{n-1} s^{n-i}y^i(0)] \tag{36.11}$$

[1] Used in initial value problems

Part VI

Operations Research

Chapter 37

Linear Programming Problems

37.1 Basic Feasible Solution

The standard LPP problem has an objective function and conditions.

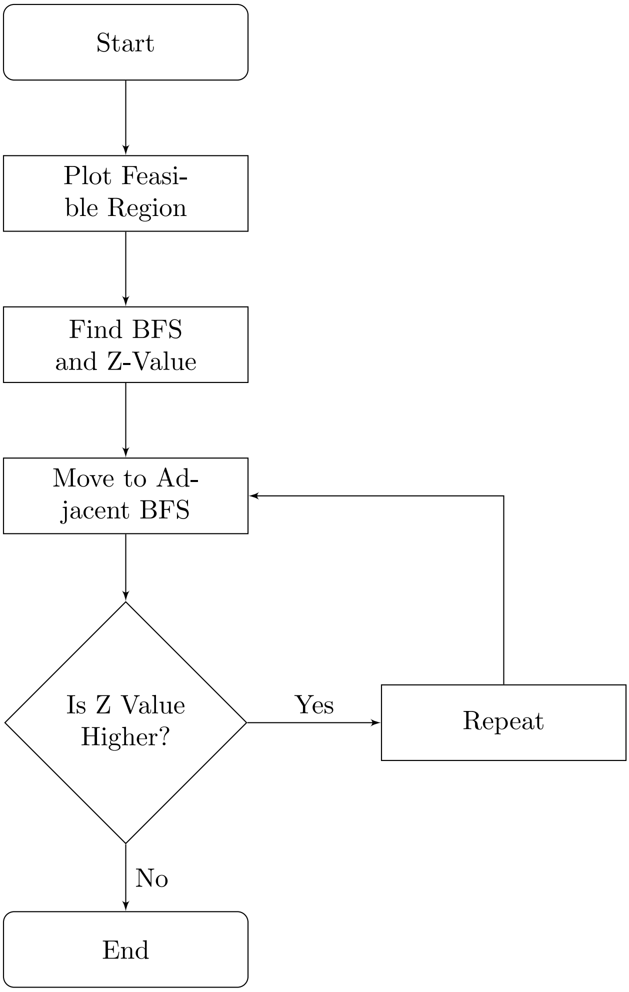
$$
\begin{aligned}
Z = a_1x_1 + a_2x_2 + \cdots + a_nx_n \\
\text{Subject to:} \\
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m
\end{aligned}
$$

For a system with n variables and m conditions, the number of basic solutions are: $\binom{n}{k}$. For any n - m system there are $n - m$ non-basic variables (NBV) and m basic variables (BV).
For the above system, the basic solutions are obtained by:

NBV	BV	BFS
$x_1, x_2, \cdots, x_{n-m} = 0$ $\vdots$	$x_{n-m+1} = c_1, \cdots, x_n = c_n$ $\vdots$	If $x_{n-m+1,\cdots,x_n,} \geq 0$ then it is a basic feasible solution.

37.1.1 Adjacent Basic Feasible Solutions

If two adjacent BFS share $m-1$ BV then they are called adjacent varibales. The optimal solution is always a extreme point. Thus, graphically:



37.2 Simplex Method