

Mathematical Formulae  
A Book of High School and Engineering Common Core  
Mathematical Formulae

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# Part I

## Algebra

# Chapter 1

## Logarithm

### 1.1 Basic Formulae

For  $a^x = b$ :

$$\log_a x, \forall x \leq 0 \text{ is undefined} \quad (1.1)$$

$$\log_a b = x, \text{ if } a^x = b, a \neq 1 \quad (1.2)$$

$$\log_b a^m = m \log_b a, \text{ for } a^m = b \quad (1.3)$$

$$a^{\log_a x} = x \quad (1.4)$$

$$a^{\log_b c} = c^{\log_b a} \quad (1.5)$$

$$\frac{1}{\log_a b} = \log_b a \quad (1.6)$$

$$\log_c(ab) = \log_c a + \log_c b \quad (1.7)$$

$$\log_c \left( \frac{a}{b} \right) = \log_c a - \log_c b \quad (1.8)$$

$$|\log_a x| = \begin{cases} -\log_a x, & \text{if } 0 < x < 1 \\ \log_a x, & \text{if } 1 \leq x < \infty \end{cases} \quad (1.9)$$

### 1.2 Series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \infty = \lim_{n \rightarrow \infty} \sum_{i=0}^n \frac{x^i}{i!} \quad (1.10)$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \infty = \lim_{n \rightarrow \infty} \sum_{i=1}^n (-1)^{(i-1)} \frac{x^i}{i} \quad (1.11)$$

## Chapter 2

### Complex Numbers

### 2.1 Basic Formulae

For  $z = x + iy$ ,

$$|z| = \sqrt{x^2 + y^2} \quad (2.1)$$

$$\tan \theta = \frac{y}{x} \quad (2.2)$$

$$\bar{z} = x - iy \quad (2.3)$$

### 2.2 Arithmetic Operation of Complex Number

For two complex numbers  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$ :

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \quad (2.4)$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \quad (2.5)$$

$$|z_1 z_2| = |z_1| \cdot |z_2| \quad (2.6)$$

$$\frac{z_1}{z_2} = \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{a_2^2 + b_2^2} \quad (2.7)$$

$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \quad (2.8)$$

### 2.3 Euler's Formula

$$z = r e^{i\theta}, \text{ where} \quad (2.9)$$

$$r = |z| \quad (2.10)$$

$$e^{i\theta} = \cos \theta + i \sin \theta \quad (2.11)$$

$$\theta = \arctan \left( \frac{y}{x} \right) \quad (2.12)$$

## 2.4 Trigonometric Ratios in Complex Form

$$e^{i\theta} + e^{-i\theta} = 2 \cos \theta \quad (2.13)$$

$$\Rightarrow \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad (2.14)$$

$$e^{i\theta} - e^{-i\theta} = 2 \sin \theta \quad (2.15)$$

$$\Rightarrow \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2} \quad (2.16)$$

## 2.5 De Moivre's Formula

According to DeMoivre's Formula:

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta) \quad (2.17)$$

**Proof**

$$\begin{aligned} \cos \theta + i \sin \theta &= e^{i\theta} \\ \Rightarrow (\cos \theta + i \sin \theta)^n &= e^{n(i\theta)} \\ &= \cos(n\theta) + i \sin(n\theta) \\ &\text{Q.E.D.} \end{aligned}$$

## 2.6 Application of Euler's and De Moivre's Formula

For  $z_1 = |r_1| e^{i\theta_1}$  and  $z_2 = |r_2| e^{i\theta_2}$

$$z_1 z_2 = (r_1 r_2) e^{i(\theta_1 + \theta_2)} \quad (2.18)$$

$$\frac{z_1}{z_2} = \left( \frac{r_1}{r_2} \right) e^{i(\theta_1 - \theta_2)} \quad (2.19)$$

## 2.7 Roots of Unity

$$\sqrt[n]{1} = e^{i \frac{2k\pi}{n}}, \text{ where } k \in [0, n-1] \quad (2.20)$$

## 2.8 Important Relations of Complex Numbers

$$|z_1 + z_2| \leq |z_1| + |z_2| \quad (2.21)$$

$$|z_1 - z_2| \leq |z_1| + |z_2| \quad (2.22)$$

$$|z_1 - z_2| \geq ||z_1| - |z_2|| \quad (2.23)$$

$$|z_1 + z_2| \geq ||z_1| - |z_2|| \quad (2.24)$$

$$|z_1 + z_2|^2 = 2 \left( |z_1|^2 + |z_2|^2 \right) \quad (2.25)$$

## Chapter 3

### Progression

### 3.1 Arithmetic Progression (A.P.)

An arithmetic sequence is  $a, a + n, a + 2n, \dots \infty$  or  $t_n = a + (n - 1)d$ , where  $a$  is the first term,  $d$  is the common difference, and  $n$  is the  $n^{th}$ -term.

An arithmetic series is  $a + (a + d) + (a + 2d) + \dots \infty$ .

#### 3.1.1 Sum of A.P. Series

$$\begin{aligned} S_n &= a + (a + d) + \dots + (a + \overline{n - 2}d) + (a + \overline{n - 1}d) \\ S_n &= (a + \overline{n - 1}d) + (a + \overline{n - 2}d + \dots + (a + d) + a \\ &\Rightarrow 2S_n = n(2a + \overline{n - 1}d) \\ &\Rightarrow S_n = \frac{n}{2}(2a + \overline{n - 1}d) \end{aligned} \tag{3.1}$$

#### 3.1.2 Important Relation

If the three terms  $a, b, c$  are in A.P., then

$$2b = a + c \tag{3.2}$$

### 3.2 Geometric Progression (G.P.)

An geometric sequence is  $a, ar, ar^2, \dots \infty$  or  $t_n = ar^{n-1}$ , where  $a$  is the first term,  $r$  is the common ratio, and  $n$  is the  $n^{th}$ -term. An geometric series is  $a + ar + ar^2 + \dots \infty$ .

#### 3.2.1 The Value of 'r'

$$r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \dots = \frac{t_n}{t_{n-1}} \tag{3.3}$$



### 3.2.2 Sum of a G.P. Series

For a definite G.P. series, where there are  $n$  terms in the series, the sum of the series is:

$$S_n = \frac{a|r^n - 1|}{|r - 1|} \quad (3.4)$$

For an infite G.P. series the sum of the series is defined for  $r < 1$ . Sum of such a series is:

$$S_\infty = \frac{a}{1 - r} \quad (3.5)$$

### 3.2.3 Important relations

If the three terms  $a, b, c$  are in G.P., then:

$$b^2 = ac \quad (3.6)$$

## 3.3 Harmonic Progression (H.P.)

If  $a, b, c$  are terms of an H.P. then  $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$  are in A.P.

$$\frac{2}{b} = \frac{1}{a} + \frac{1}{c} \quad (3.7)$$

$$\Rightarrow b = \frac{2ac}{a + c} \quad (3.8)$$

## 3.4 Arithmetico-Geometric Progression (A.G.P.)

Sequence  $a, (a+d)r, (a+2d)r^2, \dots, (a+\overline{n-1}d)r^{n-1}$ , where  $a \rightarrow$  first term of A.G.P.,  $d \rightarrow$  common difference, and  $r \rightarrow$  common ratio.

### 3.4.1 Sum of A.G.P.:

For an infinite A.G.P. series, the sum is defined for  $r < 1$ :

$$S_\infty = \frac{a}{1 - r} + \frac{dr}{(1 - r)^2} \quad (3.9)$$

## 3.5 Special Series

For  $n \in \mathbb{N}$

$$1 + 2 + 3 + \dots + (n - 1) + n = \frac{n(n - 1)}{2} \quad (3.10)$$

$$1^2 + 2^2 + 3^2 + \dots + (n - 1)^2 + n^2 = \frac{n(n + 1)(2n + 1)}{6} \quad (3.11)$$

$$1^3 + 2^3 + 3^3 + \dots + (n - 1)^3 + n^3 = \left[ \frac{n(n - 1)}{2} \right]^2 \quad (3.12)$$

### 3.5.1 Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (3.13)$$

### 3.5.2 Riemann's Infinite Series as an Integration

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=r_1}^{r_2} f\left(\frac{i}{n}\right) = \int_{\frac{r_1}{n}}^{\frac{r_2}{n}} f(x) dx \quad (3.14)$$

## Chapter 4

### Test of Convergence of Infinite Series

If  $a_1, a_2, a_3, \dots, a_n$  is a sequence by  $a_n$  and their sum of series is  $S_n$ , then the following apply.

#### 4.1 Definition

If

$$\lim_{n \rightarrow \infty} S_n = l$$

where  $l$  is a finite value, the series  $S_n$  is said to converge. A non-convergent series is called a divergent series.

#### 4.2 Tests of Convergence

##### 4.2.1 Comparison Test

If  $u_n$  and  $v_n$  are two positive series, then:

1. (a)  $v_n$  converges  
(b)  $u_n \leq v_n \forall n$  Then  $u_n$  converges.
2. (a)  $v_n$  diverges  
(b)  $u_n \geq v_n \forall n$  Then  $u_n$  diverges.

##### 4.2.2 Limit Form

If

$$\lim_{x \rightarrow \infty} \frac{u_n}{v_n} = l$$

where  $l$  is a finite quantity  $\neq 0$ , then  $u_n$  and  $v_n$  converge and diverge together.

##### 4.2.3 Integral Test or Maclaurin-Cauchy Test

For a series

$$\sum_{i=N}^{\infty} f(x), \text{ where } N \in \mathbb{Z} \quad (4.1)$$

will only converge if the improper integral

$$\int_N^{\infty} f(x)dx \quad (4.2)$$

is finite.

If the improper integral is finite, the upper and lower limit of the infinite series is given by:

$$\int_N^{\infty} f(x)dx \leq \sum_{i=N}^{\infty} f(x) \leq f(N) + \int_N^{\infty} f(x)dx \quad (4.3)$$

#### 4.2.4 Ratio Test

If, for two series  $\sum u_n$  and  $\sum v_n$ :

1. (a)  $\sum v_n$  converges  
 (b) from or after a particular term  $\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$ , then  $u_n$  converges.
2. (a)  $\sum v_n$  diverges  
 (b) from or after a particular term  $\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$ , then  $u_n$  diverges.

#### 4.2.5 D'Alembert's Ratio Test

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lambda \quad (4.4)$$

- series converges if  $\lambda < 1$
- series diverges if  $\lambda > 1$
- fails if  $\lambda = 1$

#### 4.2.6 Rabbe's Test

$$\lim_{n \rightarrow \infty} n \left[ \frac{u_n}{u_{n+1}} - 1 \right] = \kappa \quad (4.5)$$

- series converges if  $\kappa < 1$
- series diverges if  $\kappa > 1$
- fails if  $\kappa = 1$

#### 4.2.7 Cauchy's Root Test

$$\lim_{n \rightarrow \infty} |u_n|^{\frac{1}{n}} = \lambda \quad (4.6)$$

- series converges for  $\lambda < 1$
- series diverges for  $\lambda > 1$
- test fails for  $\lambda = 1$

### 4.2.8 Logarithmic Test

$$\lim_{n \rightarrow \infty} n \log \left( \frac{u_n}{u_{n+1}} \right) = \kappa \quad (4.7)$$

- series converges for  $\kappa < 1$
- series diverges for  $\kappa > 1$
- test fails for  $\kappa = 1$

## Chapter 5

### Determinants

## 5.1 Definition

For a determinant:

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \quad (5.1)$$

### 5.1.1 Minor and Cofactor

For a third order determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

**Minor**

$$M(a_{11}) = M_{11} = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \quad (5.2)$$

i.e., all the terms of determinant except those in the same row and columns as the one of which the minor is being calculated.

**Cofactor**

$$C_{ij} = (-1)^{i+j} M_{ij} \quad (5.3)$$

## 5.2 Properties of Determinants

1. Transposing a determinant does not alter its value.

$$\Delta = \Delta^T \quad (5.4)$$

2. If rows and columns are interchanges  $m$  times, the value of the new determinant is

$$\Delta' = (-1)^m \Delta \quad (5.5)$$

3. If two parallel lines are equal, then  $\Delta = 0$

4. For  $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$  and  $\Delta_1 = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ , then  $\Delta_1 = k\Delta$

5. For  $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$  and  $\Delta_1 = \begin{vmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{vmatrix}$ , then  $\Delta_1 = k\Delta$

6. For  $C_n \rightarrow k_1 C_l + k_2 C_m + k_3 C_n$  or  $R_n \rightarrow k_1 R_l + k_2 R_m + k_3 R_n$ ,  $\Delta' = \Delta$

### 5.3 Cramer's Rule

For a system of equations:

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

the following determinants are defined:

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

The solution for the system of equations is:

$$x = \frac{D_x}{D} \quad (5.6)$$

$$y = \frac{D_y}{D} \quad (5.7)$$

$$z = \frac{D_z}{D} \quad (5.8)$$

### 5.3.1 Consistency Test

1. If  $D \neq 0$ , the system is consistent and has unique solutions.
2. If  $D = D_x = D_y = D_z = 0$ , the system may or may not be consistent and it will have infinite solutions and the system will be dependent.
3. If  $D = 0$  and at least one of  $D_x, D_y, D_z$  is non zero, the system is inconsistent



## Chapter 6

### Matrices

For a matrix,

$$A_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

and where  $I_p$  is an identity matrix of the  $p^{th}$  order, the following relations are applicable.

#### 6.1 Sum of Two Matrices

$$A_{m \times n} + B_{m \times n} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{21} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix} \quad (6.1)$$

#### 6.2 Multiplication of Two Matrices

If

$$C_{m \times p} = A_{m \times n} \cdot B_{n \times p} \quad (6.2)$$

then,

$$c_{ik} = \sum_{j=1}^n a_{ij} \cdot b_{jk} \quad (6.3)$$

##### 6.2.1 Multiplicative Properties

1. Multiplication of matrices is associative, hence  $(AB)C = A(BC)$ .
2.  $AI = A$

3.  $A \cdot A^{-1} = I$
4.  $A \cdot (\text{adj } A) = (\text{adj } A) \cdot A = |A|I$
5.  $A^{-1} = \frac{1}{|A|}(\text{adj } A)^t$
6.  $(AB)^t = B^t A^t$

### 6.3 Adjoint of a Matrix

$$\text{adj } A = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m1} & M_{m2} & \cdots & M_{mn} \end{bmatrix}^t, \text{ where } M_{ij} \text{ is the minor of } a_{ij} \quad (6.4)$$

### 6.4 Martin's Rule

For a system of equation,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n \end{aligned}$$

The system can be written as:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad (6.5)$$

$$\Rightarrow AX = B \quad (6.6)$$

$$\Rightarrow X = A^{-1}B \quad (6.7)$$

## Chapter 7

### Binomial Theorem

For a binomial expansion  $(a + b)^n$ , there are  $(n + 1)$  terms and  $(a + b + c)^n$  has  $\frac{(n + 1)(n + 2)}{2}$  terms.

### 7.1 Expansion of a binomial expression

$$\begin{aligned}
 (a + b)^n &= {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \cdots + {}^nC_n a^0 b^n \\
 &= \sum_{i=0}^n {}^nC_i a^{n-i} b^i \\
 \forall n \in \mathbb{N}
 \end{aligned} \tag{7.1}$$

$$\begin{aligned}
 (a + b)^n &= a^n b^0 + n a^{n-1} b + \frac{n(n-1)}{2!} a^{n-2} b^2 + \cdots + \frac{n(n-1) \cdots 3 \cdot 2 \cdot 1}{n!} a^0 b^n + \cdots \infty \\
 \forall n \in \mathbb{R}
 \end{aligned} \tag{7.2}$$

### 7.2 Trinomial Expansion

For  $(a + b + c)^n$ :

$$\begin{aligned}
 (a + b + c)^n &= \sum \frac{n!}{i!j!k!} a^i b^j c^k \\
 \forall (i + j + k) &= n; i, j, k, n \in \mathbb{N}
 \end{aligned} \tag{7.3}$$

### 7.3 Properties of Coefficients

$$\text{Sum of Co-efficients: } C_0 + C_1 + C_2 + \cdots + C_{n-1} + C_n = 2^n \tag{7.4}$$

$$\text{Sum of Odd Co-efficients: } C_0 + C_2 + C_4 + \cdots + C_{2n-3} + C_{2n-1} = 2^{n-1} \tag{7.5}$$

$$C_0 - C_1 + C_2 - \cdots + C_{2n-1} - C_{2n} = 0 \tag{7.6}$$

## 7.4 Pascal's Rule

For  $1 \leq k \leq n$  and  $k, n \in \mathbb{N}$ :

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k} \quad (7.7)$$

# Chapter 8

## Boolean Algebra

Let  $B$  be a set of  $a, b, c$  with operations sum  $(+)$  and product  $(\cdot)$ .  
Then  $B$  is said to belong to the Boolean Structure if the following conditions are satisfied:

Table 8.1: Properties of Boolean Algebraic Structure

Property	Name of Property
$a + b \in B$ $a \cdot b \in B$	Closure Property
$a + b = b + a$ $a \cdot b = b \cdot a$	Associative Law
$a(b + c) = ab + ac$ $a + bc = (a + b)(a + c)$	Commutative Law
$\{0, 1\} \in B$ $a + 0 = a$ $a + 1 = 1$ $a \cdot 0 = 0$ $a \cdot 1 = a$	Laws of 1 and 0
$a + ab = a$ $a(a + b) = a$	Absorption Law
$(a + b)' = (a'b')$	De'Morgan's Law

## Chapter 9

### Remainder Theorems

### 9.1 Remainder Theorem

If a function  $f(x)$  is divided by a binomial  $x - a$ , then the remainder is provided by  $f(a)$ .

$$\frac{f(x)}{x - a} \equiv f(a) \pmod{(x - a)} \quad (9.1)$$

#### Worked Example

Find the remainder when  $f(x) = x^3 - 4x^2 - 7x + 10$  is divided by  $(x - 2)$ .  
The remainder:

$$R = (x^3 - 4x^2 - 7x + 10) \pmod{(x - 2)}$$

is given by:

$$\begin{aligned} R = f(2) &= (2)^3 - 4(2)^2 - 7(2) + 10 \\ &= 8 - 16 - 14 + 10 = -12 \end{aligned}$$

### 9.2 Euler's Remainder Theorem

According to Euler's Remainder Theorem, if  $x$  and  $n$  are two co-prime numbers:

$$x^{\varphi(n)} \equiv 1 \pmod{n}, x, n \in \mathbb{Z}^+ \quad (9.2)$$

where,  $\varphi(n)$  is Euler's totient function.

#### 9.2.1 Euler's Totient Function

For a number defined as:

$$n = \prod_{i=1}^r a_i^{b_i} \quad (9.3)$$

then Euler's totient function is defined as:

$$\begin{aligned}\varphi(n) &= n \cdot \left[ \left(1 - \frac{1}{a_1}\right) \cdot \left(1 - \frac{1}{a_2}\right) \cdot \left(1 - \frac{1}{a_3}\right) \cdots \right] \\ &= n \prod_{i=1}^r \left(1 - \frac{1}{a_i}\right)\end{aligned}\tag{9.4}$$

### Worked Example

Find the remainder if  $3^{76}$  is divided by 35.

Since:

$$35 = 5^1 \times 7^1$$

Hence the totient quotient of 35 is:

$$\begin{aligned}\varphi(35) &= 35 \cdot \left(1 - \frac{1}{5}\right) \cdot \left(1 - \frac{1}{7}\right) \\ &= 35 \times \frac{4}{5} \times \frac{6}{7} \\ &= 24\end{aligned}$$

Hence Euler's Theorem yields:

$$\begin{aligned}3^{24} &\equiv 1 \pmod{35} \\ 3^{76} &\equiv 3^{24 \times 3 + 4} \\ &\equiv (3^{24})^3 \times 3^4 \pmod{35} \\ &\equiv (1)^3 \times 3^4 \pmod{35} \\ &\equiv 81 \pmod{35} \\ &\equiv 11 \pmod{35}\end{aligned}$$

The remainder when  $3^{76}$  is divided by 35 is 11.

## 9.3 Wilson Theorem

According to Wilson Theorem:

$$(n-1)! \equiv -1 \pmod{n}\tag{9.5}$$

**Worked Example**

Find the remainder when  $28!$  is divided by 31.

By Wilson's Theorem:

$$\begin{array}{ll}
 30! & \equiv -1 \pmod{31} \\
 \Rightarrow 30 \cdot 29 \cdot 28! & \equiv -1 \pmod{31} \\
 \text{Let } 28! \pmod{31} & = x \\
 \Rightarrow (-1) \cdot (-2) \cdot x & \equiv 30 \pmod{31} \\
 \Rightarrow 2x & = 30 \\
 \Rightarrow x & = 15
 \end{array}$$

The remainder when  $28!$  is divided by 31 is 15.



# Part II

## Co-ordinate Geometry

## Chapter 10

### 2-D Co-ordinate Geometry

For the ordered pairs,  $A(x_1, y_1)$  and  $B(x_2, y_2)$ :

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \quad (10.1)$$

$$\text{Mid point of AB} = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \quad (10.2)$$

$$\text{Point C, which divides AB in the ratio } m : n = \left( \frac{nx_1 + mx_2}{m + n}, \frac{ny_1 + my_2}{m + n} \right) \quad (10.3)$$

## Chapter 11

### Triangles

For a triangle defined with three vertices  $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$  and corresponding sides of length  $a, b, c$ , then:

$$\text{Centroid of } \triangle ABC = \left( \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right) \quad (11.1)$$

$$\text{Area of } \triangle \quad (11.2)$$

$$ABC = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad (11.3)$$

For a triangle, the semi-perimeter,  $s$ , is defined as:

$$s = \frac{a + b + c}{2}$$

Then the radius,  $r$ , and centre of incircle,  $o$ , is:

$$o = \left( \frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c} \right) \quad (11.4)$$

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \quad (11.5)$$

$$(11.6)$$

The radius,  $R$ , and centre,  $O$ , of circumcircle is defined as:

$$O = \left( \frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C} \right) \quad (11.7)$$

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad (11.8)$$

## Chapter 12

### Straight Line

A straight line can be defined as:

$$y = mx + c \quad (12.1)$$

$$\frac{x}{a} + \frac{y}{b} = 1, \text{ where } a \text{ and } b \text{ are the intercepts at x and y axes respectively} \quad (12.2)$$

$$x \cos \alpha + y \sin \alpha = p \text{ (Normal Form)} \quad (12.3)$$

$$Ax + By + C = 0 \text{ (General Form)} \quad (12.4)$$

### 12.1 Equation of Straight Line Passing Through $(x_0, y_0)$ and Slope $m$

$$(y - y_0) = m(x - x_0) \quad (12.5)$$

### 12.2 Distance Between Two Points on a Line

$$\frac{y_1 - y_2}{\sin \theta} = \frac{x_1 - x_2}{\cos \theta} = \gamma \quad (12.6)$$

$$\theta = \tan^{-1} m \quad (12.7)$$

### 12.3 Angle Between Two Lines

For two lines with slopes  $m_1, m_2$ , the angle between them,  $\theta$ :

$$\theta = \arctan \left( \frac{m_1 - m_2}{1 + m_1 m_2} \right) \quad (12.8)$$

#### Distance of a Point from a Line

Line:  $ax + by + c = 0$  Point:  $(g, h)$

$$S = \frac{ag + bh + c}{\sqrt{a^2 + b^2}} \quad (12.9)$$

**Angle Bisector of a Line** For the two lines:  $a_1x + b_1y + c_1 = 0$  and  $a_2x + b_2y + c_2 = 0$ , the angle bisector is:

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \quad (12.10)$$

If the sign of  $c_1$  and  $c_2$  is the same, then the equation obtained is the internal bisector.

### Equation of a Straight Line Passing through the Intersection of Two Lines

$$(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0 \quad \forall k \in \mathbb{R} \quad (12.11)$$

**Relative Position of Points w.r.t. a Line** For the points  $(x_1, y_1)$  and  $(x_2, y_2)$ :

$$k_1 = ax_1 + by_1 + c$$

$$k_2 = ax_2 + by_2 + c$$

If  $k_1$  and  $k_2$  have the same sign, they are on the same side of a line, otherwise on opposite sides.

**Ratio of Division of Line Segment** For any line,  $f(x, y) = 0$ , the ratio in which it divides  $(x_1, y_1)$  and  $(x_2, y_2)$  is given by:

$$r = -\frac{f(x_1, y_1)}{f(x_2, y_2)} \quad (12.12)$$

If  $\begin{cases} r > 0, \text{ then division is internal} \\ r < 0, \text{ then division is external} \end{cases}$ .