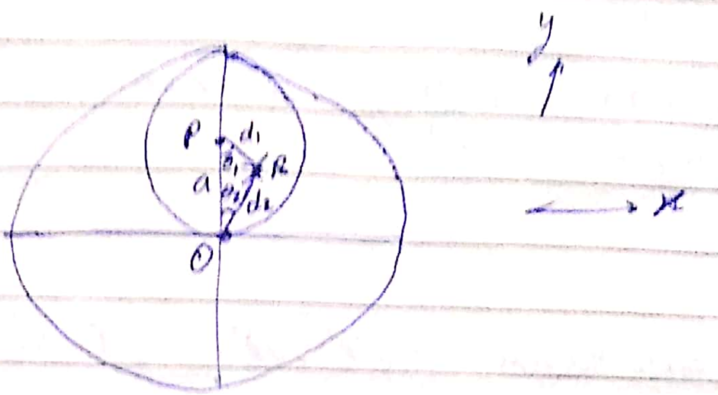
Assignment - 2

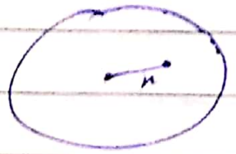
Q13



$\vec{F_1}$ be the force due to big sphere before carving out the sphere at R
$\vec{F_2}$ be the force due to small sphere at R

Final force at R = $\vec{F_1} - \vec{F_2}$

$|E| \cdot 4\pi r^2 = \frac{\rho \frac{4}{3}\pi r^3}{\epsilon_0}$ (Gauss law)



ρ = volume density

$|E| = \frac{\rho r}{3\epsilon_0}$

$\therefore \vec{F_1} = \frac{\rho}{3\epsilon_0}\left(d_2\cos\theta_2\,\hat{j} + d_2\sin\theta_2\,\hat{i}\right)$

$\vec{F_2} = \frac{\rho}{3\epsilon_0}\left(-d_1\cos\theta_1\,\hat{j} + d_1\sin\theta_1\,\hat{i}\right)$

$\vec{F_1} - \vec{F_2} = \frac{\rho}{3\epsilon_0}\left(\left(d_2\cos\theta_2 + d_1\cos\theta_1\right)\hat{j} + \left(d_2\sin\theta_2 - d_1\sin\theta_1\right)\hat{i}\right)$

$d_1\cos\theta_1 + d_2\cos\theta_2 = a$, $d_1\sin\theta_1 = d_2\sin\theta_2$

$$\boxed{\vec{F_1} - \vec{F_2} = \frac{\rho a}{3\epsilon_0}}$$

Q12. Let square plate has Area = A

$C_1 = \frac{\epsilon_0 A\, dx\, K_1}{\cancel{a}\,d\cancel{x}}$, $C_2 = \frac{\epsilon_0 A\, dx\, K_2}{a\,d(1-\frac{x}{a})}$ they are in ~~parallel~~ series

$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{d}{\epsilon_0 A\, dx\, a K_2 K_1} \times (aK_1 + (K_2 - K_1)x)$

$C = \int_0^a \frac{\epsilon_0 A\, dx\, K_2 K_1}{d\left(aK_1 + (K_2 - K_1)x\right)}$ as in parallel

$$C = \int_0^a \frac{\epsilon_0 A\, dx\, K_2 K_1}{d\left(aK_1 + (K_2 - K_1)x\right)}$$

$$= \frac{\epsilon_0 A K_2 K_1}{d(K_2 - K_1)} \ln\left(\frac{K_2}{K_1}\right) \quad \text{for } K_2 \neq K_1$$

$$C = \frac{\epsilon_0 A K}{d} \quad \text{for } K_2 = K_1 = K$$

Q.11 From Stokes' theorem $\int (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} = \oint \vec{E} \cdot d\vec{l}$

As electrostatic field is conservative, so $\oint \vec{E} \cdot d\vec{l} = 0$

as ~~[illegible]~~ $\int_{\text{along }(1)} \vec{E} \cdot d\vec{l} = \int_{\text{along }(2)} \vec{E} \cdot d\vec{l}$, so curl of $\vec{E}$ should be $\vec{\text{zero}}$

1) $\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -y & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + \left(\frac{-\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\hat{k}$

$= \vec{0}$

2) $\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + \left(\frac{\partial x}{\partial x} - \frac{\partial y}{\partial y}\right)\hat{k}$

$= \vec{0}$

3) $\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + \left(\frac{-\partial x}{\partial x} - \frac{\partial y}{\partial y}\right)\hat{k}$

$= -2\hat{k} \neq \vec{0}$

So only 1), 2) describe an electric field

Q6. i) $|E| \cdot 2\pi x l = \frac{\sigma 2\pi r l}{\epsilon_0}$ for $x \geq r$, r is the radius of cylinder, l is length of long cylinder

$|E| = \frac{\sigma r}{x \epsilon_0}$, $x \geq r$

$|E| \cdot 2\pi x l = 0$ $x < r$

$|E| = 0$ for $x < r$

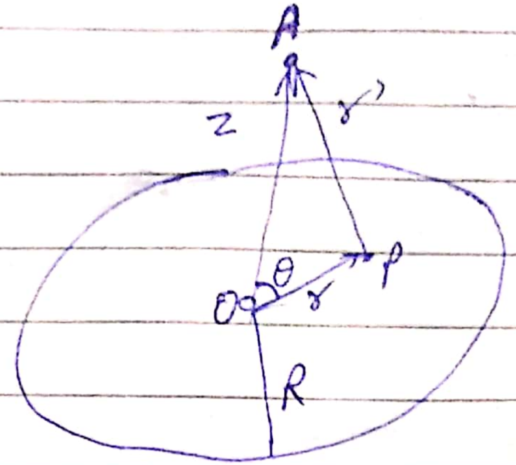
$$E = \begin{cases} \frac{\sigma r}{x \epsilon_0} \hat{n} & x \geq r \\ 0 & x < r \end{cases}$$

, $\hat{n}$ is the unit vector $\perp$ to axis of cylinder

ii) $\frac{1}{2} \frac{1}{\cancel{4}\pi\epsilon_0} \int_0^R \frac{\rho \cancel{4}\pi x^2 \, dx}{x} = W$, $\rho = \frac{3Q}{4\pi R^3}$

$\frac{1}{2\epsilon_0} \frac{\rho R^2}{2} = \frac{3Q R^2}{4\pi R^3 \cdot 4\epsilon_0} = \boxed{\frac{3Q}{16\pi R \epsilon_0}} = W$

4.1)



$$\rho = \frac{3q}{4\pi R^3}$$

$$r' = z^2 + (r)^2 - 2zr\cos\theta$$

$$\int \frac{1}{4\pi\epsilon_0} \frac{\rho \, d\tau}{r'} = \int \frac{1}{4\pi\epsilon_0} \frac{\rho \, 2\pi r^2 \sin\theta d\theta \, dr}{\sqrt{z^2 + (r)^2 - 2zr\cos\theta}}$$

$$\cancel{= \frac{1}{4\pi\epsilon_0} \int \rho \, 4\pi \int \frac{r^2 dr}{\sqrt{z^2 + r^2 - 2zr\cos\theta}}} \quad \cancel{= \frac{\rho}{\epsilon_0} \int \frac{r^2 dr}{\sqrt{z^2 + r^2 - 2zr\cos\theta}}}$$

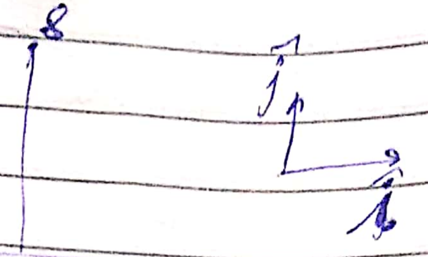
$$= \int \frac{\rho \cdot 2\pi r^2 dr}{4\pi\epsilon_0} \int \frac{\sin\theta d\theta}{\sqrt{z^2 + r^2 - 2zr\cos\theta}} = \frac{\rho r^2 dr}{2\epsilon_0} \frac{2}{2zr} \left(|z+r| - |r-z| \right)$$

$$= \int_0^z \frac{\rho r^2 dr}{2\epsilon_0 \cdot z} \cdot 2r + \int_z^R \frac{\rho r \, dr}{\epsilon_0}$$

$$= \frac{\rho}{\epsilon_0} \frac{(3R^2 - z^2)}{6} = \boxed{\frac{Q}{8\pi R^3 \epsilon_0} \left(3R^2 - z^2\right)}$$

for $0 \le z \le R$

4.2)



Using Gauss theorem

$$|E| \cdot 2\pi s l = \frac{\lambda l}{\epsilon_0}$$

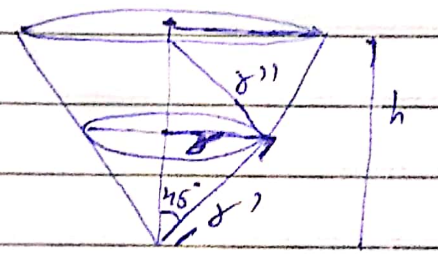
$$|E| = \frac{\lambda}{2\pi s \epsilon_0} , \quad \vec{E} = \frac{\lambda}{2\pi s \epsilon_0} \hat{j}$$

$$V(s) - V(1) = -\int_1^s \frac{\vec{\lambda} \cdot d\vec{l}}{2\pi \epsilon_0} = -\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{s}{1}\right)$$

$$= \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{1}{s}\right)$$

$$\boxed{V(s) = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{1}{s}\right) + V(1)}$$

4.3) $V_A = \int \frac{1}{4\pi\epsilon_0} \frac{\sigma}{\sqrt{2}r} 2\pi\sigma dr'$



$r' = \sqrt{2}r$

$$= \int \frac{1}{4\pi\epsilon_0} \frac{\sigma 2\pi r \sqrt{2} dr}{\sqrt{2} r}$$

$$= \frac{\sigma h}{2\epsilon_0}$$

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$$V_B = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma\, 2\pi r\, dr'}{r''}$$

$$r'' = \sqrt{h^2 + r'^2 - \sqrt{2} r' h}$$

$$= \frac{\sigma}{4\pi\epsilon_0} \int_0^{\frac{1}{\sqrt{2}}h} \frac{2\pi r' dr'}{\sqrt{h^2 + r'^2 - \sqrt{2} r' h}} = \frac{\sigma}{2\sqrt{2}\epsilon_0}\int_0^{\sqrt{2}h} \frac{r' dr'}{\sqrt{h^2 + r'^2 - \sqrt{2} r' h}}$$

$$\int_0^{\sqrt{2}h} \frac{r' dr'}{\sqrt{h^2 + r'^2 - \sqrt{2} r' h}} = \int_0^{\sqrt{2}h} \frac{x\,dx}{\sqrt{ax^2 + bx + c}}$$

$$a = 1, \quad b = -\sqrt{2}h, \quad c = h^2$$

$$= \frac{\sqrt{ax^2+bx+c}}{a}\Bigg|_0^{\sqrt{2}h} - \frac{b}{2a^{3/2}} \ln\left(2\sqrt{a}\sqrt{ax^2+bx+c} + 2ax + b\right)\Bigg|_0^{\sqrt{2}h}$$

$$= h - h + \frac{\sqrt{2}h}{2}\ln\left(2h + 2\sqrt{2}h - \sqrt{2}h\right) - \frac{\sqrt{2}h}{2}\ln\left(2h - \sqrt{2}h\right)$$

$$= \frac{h}{\sqrt{2}} \ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right)$$

$$V_B = \frac{\sigma h}{4\epsilon_0}\ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right)$$

$$\Delta V_{AB} = V_A - V_B = \frac{\sigma h}{2\epsilon_0}\left[1 - \frac{1}{2}\ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right)\right]$$

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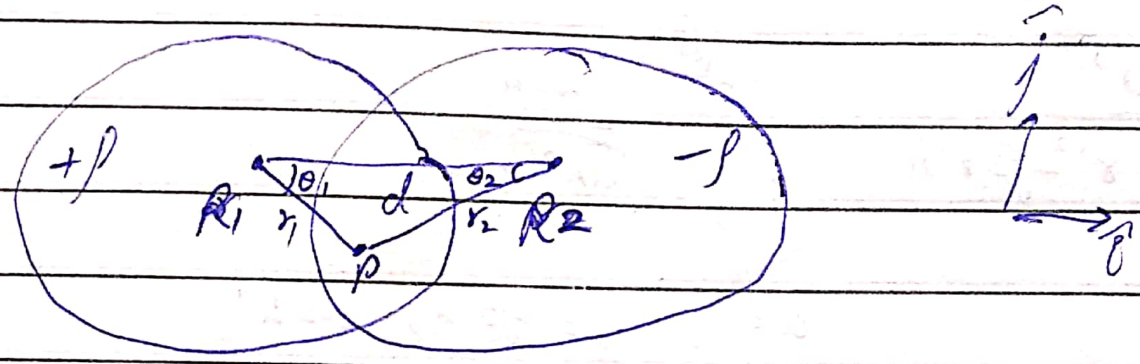
Q3. 1) $|E| \cdot 4\pi r^2 = \frac{\rho \frac{4}{3}\pi r^3}{\epsilon_0}$

$|E| = \frac{\rho r}{3\epsilon_0}, \quad 0 \le r \le R$

$$\boxed{E = \frac{\rho r}{3\epsilon_0}\hat{r}}$$

2)



$\vec{F_1}$ due to first sphere at $P = \frac{\rho}{3\epsilon_0}(r_1\cos\theta_1\,\hat{i} - r_1\sin\theta_1\,\hat{j})$

$\vec{F_2}$ at $P = \frac{\rho}{3\epsilon_0}(r_2\cos\theta_2\,\hat{i} + r_2\sin\theta_2\,\hat{j})$

$r_1\cos\theta_1 + r_2\cos\theta_2 = d$ and $r_1\sin\theta_1 = r_2\sin\theta_2$

$\therefore \vec{F_1} + \vec{F_2} = \frac{\rho d}{3\epsilon_0}$, d is the distance b/w the centres

(3) $\rho = \frac{K}{r^3}$ $\quad a \le r \le b$

- $|E| \cdot 4\pi r^2 = \frac{0}{\epsilon_0}$

  $\vec{E} = 0\,\hat{r}$ for $r < a$

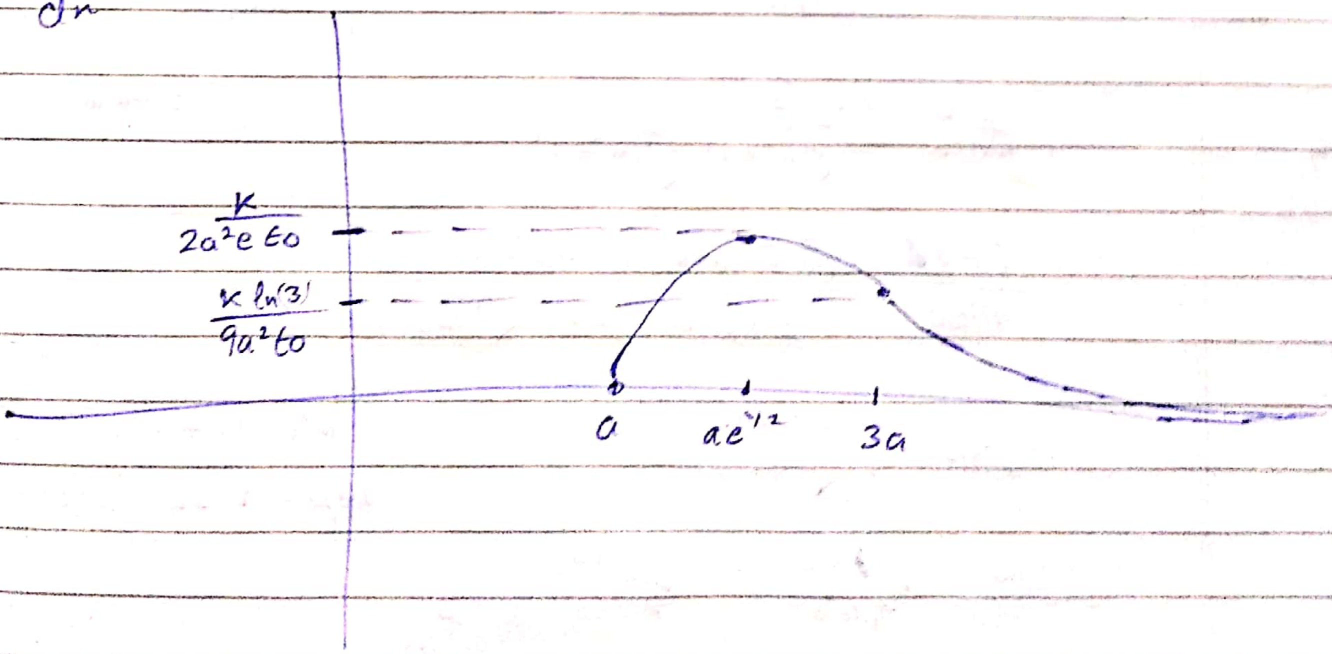
- $|E| \cdot 4\pi r^2 = \frac{\int_a^r \frac{K}{r^3} 4\pi r^2 \, dr}{\epsilon_0} = \frac{4\pi K \ln\left(\frac{r}{a}\right)}{\epsilon_0}$

  $\vec{E} = \frac{K \ln\left(\frac{r}{a}\right)}{r^2 \epsilon_0}\,\hat{r}$ for $a < r < b$

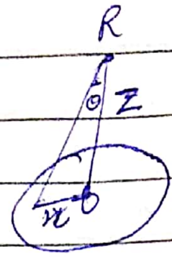
- $|E| \cdot 4\pi r^2 = \frac{4\pi K \ln\left(\frac{b}{a}\right)}{\epsilon_0}$

  $\vec{E} = \frac{K \ln\left(\frac{b}{a}\right)}{r^2 \epsilon_0}\,\hat{r}$ for $r > b$

$\frac{d\left(\frac{K \ln\left(\frac{r}{a}\right)}{r^2 \epsilon_0}\right)}{dr} = \frac{K}{\epsilon_0} \frac{1 - 2\ln\left(\frac{r}{a}\right)}{r^3}$ for $a < r < b$



2. ii) Circular



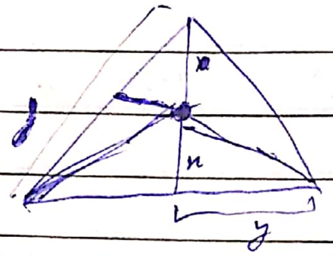
, σ

$$\vec{E} = \int \frac{1}{4\pi\epsilon_0} \frac{\sigma\, 2\pi x\, dx\, z}{(x^2+z^2)^{3/2}}$$

$x = z\tan\theta$

$$= \int \frac{\sigma \cancel{z}}{2\epsilon_0} \frac{\cancel{z}\tan\theta\, \cancel{z}\sec^2\theta d\theta}{\cancel{z^3}\sec^3\theta} = \boxed{\frac{\sigma}{2\epsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right)\hat{j}}$$

3)



$x = \frac{y}{\sqrt{3}}$

$$\vec{E} = \int \frac{6}{4\pi\epsilon_0} \frac{\sigma\, dy\, dx\, z}{(y^2+x^2+z^2)^{3/2}}$$

$d = \sqrt{x^2+z^2}\,/\tan\theta$

y from 0 to $\sqrt{3}x$

$$= \int \frac{6}{4\pi\epsilon_0} \frac{\sigma z\, dx\, \sqrt{3}x}{(x^2+z^2)\sqrt{4x^2+z^2}}$$

$x = z\tan\theta$

$$= \int \frac{6\sigma \cancel{z}}{4\pi\epsilon_0} \frac{\sqrt{3}\, \cancel{z}\tan\theta\ \cancel{z}\sec^2\theta d\theta}{\cancel{z^2}\cancel{\sec^2\theta}\ \cancel{z}\cdot \sqrt{4\tan^2\theta+1}}$$

$$= \frac{3\sqrt{3}\sigma}{2\pi\epsilon_0} \frac{\tan\theta\, d\theta}{\sqrt{4\tan^2\theta+1}}$$

$p^2 = 4\tan^2\theta + 1$

$p\,dp = 4\tan\theta\sec^2\theta \cdot d\theta$

$$= \frac{3\sqrt{3}\sigma}{2\pi\epsilon_0} \frac{dp}{4\sec^2\theta} = \frac{3\sqrt{3}\sigma}{2\pi\epsilon_0 (p^2+3)}$$

$$= \frac{\sqrt{3}\sigma}{2\pi\epsilon_0} \tan^{-1}\left(\frac{p}{\sqrt{3}}\right)\Bigg|_1^{\sqrt{a^2/3z^2+1}}$$

$$= \frac{\sqrt{3}\sigma}{2\pi\epsilon_0}\left[\tan^{-1}\left(\sqrt{\frac{a^2}{9z^2}+\frac{1}{3}}\right) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{\sqrt{3}\sigma}{2\pi\epsilon_0}\left[\tan^{-1}\left(\sqrt{\frac{a^2}{9z^2}+\frac{1}{3}}\right) - \frac{\pi}{6}\right]$$

$$= \boxed{\frac{3\sigma}{2\pi\epsilon_0}\left[\tan^{-1}\left(\frac{\sqrt{a^2+3z^2}}{3z}\right) - \frac{\pi}{6}\right]}$$

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1) Ignore

$$\boxed{y_1 \boxed{x.}}$$

$$\vec{E} = \int \frac{8}{4\pi\epsilon_0} \frac{\sigma\, dy\, dx\, z}{(x^2+y^2+z^2)^{3/2}} \qquad y = \sqrt{x^2+z^2}\tan\theta$$

$$= \int \frac{2\sigma z}{\pi\epsilon_0} \frac{dx\, x}{(x^2+z^2)\sqrt{2x^2+z^2}} \qquad x = z\tan\theta$$

$$dx = z\sec^2\theta\, d\theta$$

$$= \int \frac{2\sigma z}{\pi\epsilon_0} \frac{z^2 \sec^2\theta \tan\theta\, d\theta}{z^2\sec^2\theta\; z\sqrt{2\tan^2\theta + 1}} = \frac{2\sigma}{\pi\epsilon_0} \frac{\tan\theta\, d\theta}{\sqrt{2\tan^2\theta + 1}}$$

$$p^2 = 2\tan^2\theta + 1$$

$$p\,dp = 2\tan\theta \sec^2\theta\, d\theta$$

$$\int \frac{2\sigma}{\pi\epsilon_0} \frac{p\,dp}{(p^2+1).p} = \int \frac{2\sigma}{\pi\epsilon_0} \frac{dp}{(p^2+1)} = \frac{2\sigma}{\pi\epsilon_0} \tan^{-1}(p)$$

$$= \frac{2\sigma}{\pi\epsilon_0} \left[\tan^{-1}\left(\sqrt{\frac{2a^2}{2z^2} + 1}\right) - \tan^{-1}(1) \right]$$

$$= \frac{2\sigma}{\pi\epsilon_0} \left[\tan^{-1}\sqrt{\frac{a^2}{2z^2} + 1} - \frac{\pi}{4} \right]$$

9. 1) $\sigma_b = \vec{P}.\hat{r} = K\vec{r}.\hat{r} = Kr$

$\rho_b = -\vec{\nabla}.\vec{P} = -\frac{1}{r^2} \frac{\partial}{\partial r}(r^2 Kr) = -3K$

2) $|E|\, 4\pi r^2 = \frac{3k\, \frac{4}{3}\pi r^3}{\epsilon_0}$ $\qquad r < R$

$$\boxed{\vec{E} = \frac{Kr}{\epsilon_0}\hat{r} \qquad r < R}$$

$$|E|\, 4\pi r^2 = \frac{k 4\pi R^3 - k 4\pi R^3}{\epsilon_0} = 0$$

$$\boxed{\vec{E} = 0\,\hat{r} \qquad r > R}$$

Q. 3) $V = \int -\vec{E} \cdot d\vec{l}$

$$V(r) - V(s) = \int_s^r \frac{k r\, dr}{\epsilon_0} = \frac{k_1}{2}(r^2 - R^2) \quad \text{for } r < R$$

$$V(r) = \frac{k}{2}(r^2 - R^2) + V(s), \quad s \text{ is at the point on the surface}$$

$$V(r) - V(s) = -\int_s^r \frac{k r\, dr}{\epsilon_0} = \frac{k}{2}(r^2 - R^2) \quad \text{for } r \geq R$$

$$\therefore \boxed{V(r) = \frac{k}{2}(r^2 - R^2) + V(s)} \quad \text{for } 0 \leq r$$

1. $\nabla \cdot \vec{E} = 0$ — ①

$\nabla \cdot \vec{H} = 0$ — ②

$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}$ — ③

$\nabla \times \vec{H} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$ — ④

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - (\nabla^2 E)$$

$\nabla \cdot \vec{E} = 0$

$$\vec{\nabla} \times \left(-\frac{1}{c}\frac{\partial H}{\partial t}\right) = -\frac{1}{c}\frac{\partial \vec{\nabla} \times \vec{H}}{\partial t}$$

$$= -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial E}{\partial t}\right)$$

$$= -\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

So

$$\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \nabla^2 E \quad \text{— ①}$$

Similarity

$$\nabla \times (\nabla \times \vec{H}) = \nabla (\cancelto{0}{\nabla \cdot \vec{H}}) - \nabla^2 H$$

$$= \nabla \times \left(\frac{1}{c} \frac{\partial \varepsilon}{\partial t} \right) = \frac{1}{c} \frac{\partial (\nabla \times \varepsilon)}{\partial t}$$

$$= \frac{1}{c} \frac{\partial}{\partial t} \left(\frac{-1}{c} \frac{\partial H}{\partial t} \right)$$

$$= \frac{-1}{c^2} \frac{\partial^2 H}{\partial t^2}$$

$$\frac{1}{c^2} \frac{\partial^2 H}{\partial t^2} = \nabla^2 H \quad \text{—} (2)$$

So we generalize this from (1) & (2)

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$$

Q.5. 1) for two charges

Energy conservation :-

$$\frac{1}{4\pi\epsilon_0} \cdot \frac{q_A q_B}{a} = \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2 \quad \text{--- (1)}$$

Linear momentum conservation :-

$$m_A v_A = m_B v_B \quad \text{--- (2)}$$

from (1) & (2)

$$\frac{1}{4\pi\epsilon_0} \frac{q_A q_B}{a} = \frac{1}{2} m_A v_A^2 + \frac{1}{2} \frac{m_A^2 v_A^2}{m_B}$$

$$v_A^2 = \frac{1}{4\pi\epsilon_0} \frac{q_A q_B}{a} \frac{2 m_B}{m_A (m_A + m_B)}$$

$$v_A = \sqrt{\frac{q_A q_B}{2\pi\epsilon_0 a} \frac{m_B}{m_A (m_A + m_B)}}$$

$$v_B = \sqrt{\frac{q_A q_B}{2\pi\epsilon_0 a} \frac{m_A}{m_B (m_A + m_B)}}$$

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5(ii) $\longleftarrow \; +q \underbrace{\quad}_{a} -q \underbrace{\quad}_{a} +q \underbrace{\quad}_{a} -q \; \cdots \longrightarrow$

$$W = \frac{1}{2} \pm qV = \frac{\pm q}{2} \frac{2}{4\pi\epsilon_0} \mp \frac{q}{a} \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \right)$$

$$= -\frac{q}{2} \cdot \frac{2}{4\pi\epsilon_0} \frac{q}{a} \ln(2) = \frac{-q^2 \ln(2)}{4\pi\epsilon_0 a} = \frac{q^2 \ln\left(\frac{1}{2}\right)}{4\pi\epsilon_0 a}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

5.1) For Three charges :-

Energy conservation — (1)

$$\frac{1}{4\pi\epsilon_0}\left(\frac{q_A q_B + q_B q_C + q_A q_C}{a}\right) = \frac{1}{2} m_A V_A^2 + \frac{1}{2} m_B V_B^2 + \frac{1}{2} m_C V_C^2$$

Momentum Conservation — (2)

2 eqn. but 3 variables

Hence not possible