Problem Set 1.1

1 (a) 56

ξ = number of rolls needed if the last number is 5; η = number of rolls needed if the last number is not 5. Then

$$E\xi = \underbrace{\frac{1}{20}\cdot 1}_{6 \text{ is rolled}} + \underbrace{\frac{1}{20}(1+E\xi)}_{5 \text{ is rolled}} + \underbrace{\frac{18}{20}\cdot(1+E\eta)}_{\text{any other number is rolled}}$$

$$E\eta = \underbrace{\frac{1}{20}(1+E\xi)}_{5 \text{ is rolled}} + \underbrace{\frac{19}{20}(1+E\eta)}_{\text{any other number is rolled}}$$

$\Rightarrow E\xi = 380, \ E\eta = 400$ //.

(b) 55

$$E\xi = \frac{1}{20}\cdot 1 + \frac{19}{20}\cdot(1+E\eta)$$

$$E\eta = \frac{1}{20}(1+E\xi) + \frac{19}{20}(1+E\eta)$$

$\Rightarrow E\xi = 400, \ E\eta = 420$ //

So we get a different answer.

x = the number on the first die,
y = the number on the second die

Independence $\Longleftrightarrow P(A_i B_j) = P(A_i)\,P(B_j) \Longleftrightarrow$

$$P(x+y=i \;\&\; y=j) = P(x+y=i)\cdot P(y=j) \Longleftrightarrow$$

$$\Longleftrightarrow \underbrace{P(x=i-j \;\&\; y=j)}_{=\frac{1}{36}\cdot I(1\le i-j\le 6)} = P(x+y=i)\cdot \underbrace{P(y=j)}_{=\frac{1}{6}}$$

↓ if the indicator $=0$, the equality is impossible.

3.1

Therefore, $P(x+y=i) = \frac{1}{6} \Longrightarrow i = 7 \Longrightarrow 1 \le j \le 6.$

Answer: $i=7$ & $1 \le j \le 6$ (or when at least one of the events is impossible)

①

7 ● 16 ○ $\xrightarrow{1}$ 6 ● 3 ○ 7 ● 1 ○

$P = \frac{7}{23}$ 7 ● 3 ○

$P = \frac{16}{23}$ 6 ● 4 ○

(situation in the second urn)

8 ● 1 ○ $P = \frac{7}{23} \cdot \frac{7}{10} + \frac{16}{23} \cdot \frac{6}{10} = \frac{145}{230} = \frac{29}{46}$

7 ● 2 ○ $P = \frac{7}{23} \cdot \frac{3}{10} + \frac{16}{23} \cdot \frac{4}{10} = \frac{85}{230} = \frac{17}{46}$

(situation in the third urn)

$P(\text{white}) = \frac{1}{9} \cdot \frac{29}{46} + \frac{2}{9} \cdot \frac{17}{46} = \frac{63}{9 \cdot 46} = \frac{7}{46}$ //

4.1

2. **PROBLEM SET 2.** Eleven people enter the lift at the ground floor of a 12-storey house. Each of them can get off the lift at all the floors starting with the first one with equal probabilities and independently of each other. Let ξ be equal to the quantity of floors the lift does not reach at all. (For example, if the last person exits the lift at the seventh floor, it implies that the lift does not reach floors 8, 9, etc.) Find the expected value of ξ.

$P(\xi \geq 1) = P(\text{nobody goes out at } 11^{th} \text{ floor}) = \left(\frac{10}{11}\right)^{11},$

$P(\xi \geq 2) = P(\text{nobody goes out at } 10^{th} \text{ \& } 11^{th}) = \left(\frac{9}{11}\right)^{11},$

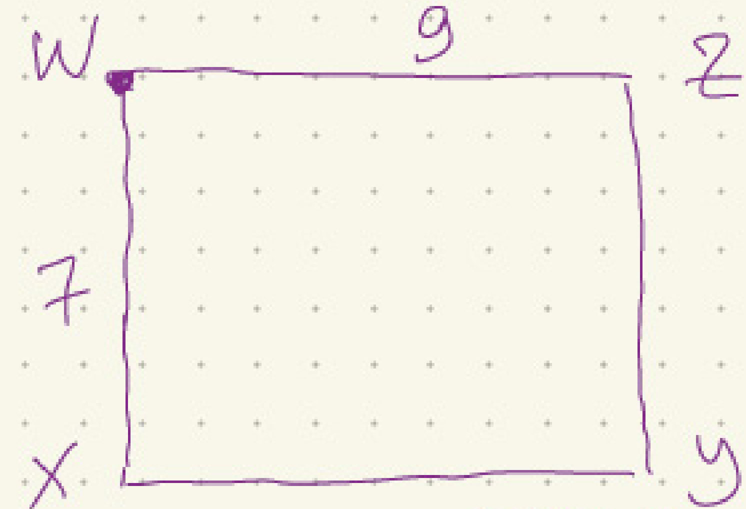
$P(\xi \geq 3) = \left(\frac{8}{11}\right)^{11} \ldots$

$E\xi = \left(\frac{10}{11}\right)^{11} + \left(\frac{9}{11}\right)^{11} + \ldots + \left(\frac{1}{11}\right)^{11}.$

5.2

6. A taxi driver moves between four settlements W, X, Y, Z located at the vertices of a rectangle. Four highways connecting the settlements run along the sides of the rectangle, and $WX = 7$ kilometers, $WZ = 9$ kilometers. Having delivered a customer the driver waits for the next taxi call. Then he goes to the settlement the call came from, collects a customer, and goes to a new destination. A new call can emerge from any settlement with probability 0.25, and the customer chooses the settlement to travel to with probability $\frac{1}{3}$. Travelling between adjacent settlements, the driver always chooses the shortest route. The distances inside each of the settlements are very small (i.e. they may be neglected). Let ζ be the distance the driver needs to travel in order to collect and deliver a customer. Find the distribution of random variable ζ. Determine $E\zeta$ and $\operatorname{Var}\zeta$.

(3-6) $WX = 7$, $WZ = 9$



Let the driver be in W.
Outcome AB means
that a passenger calls
for a taxi from A and goes to B. All these outcomes
have equal probabilities ($\frac{1}{12}$). The corresponding
distances are listed below:

WX	7
WZ	9
WY	16
XW	14
XZ	23
XY	16
YW	32
YX	25
YZ	23
ZX	25
ZY	16
ZW	18

$$\xi \sim \begin{pmatrix} 7 & 9 & 14 & 16 & 18 & 23 & 25 & 32 \\ 1/12 & 1/12 & 1/12 & 3/12 & 1/12 & 2/12 & 2/12 & 1/12 \end{pmatrix}$$

$$E\xi = \frac{224}{12} = \frac{56}{3}$$

$$E\xi^2 = \frac{4750}{12} = \frac{2375}{6}$$

$$\operatorname{Var}\xi = \frac{2375}{6} - \left(\frac{56}{3}\right)^2 = \frac{853}{18}$$

6.3

Random variable X is uniformly distributed on $[0;4]$. Find the probability density of $Y = \frac{9-2X}{2+X}$.

3-7 $X \sim U[0,4]$, $y = \dfrac{9-2X}{2+X}$

$$\underbrace{f_y(y) = f_X(x(y)) \cdot \left|\frac{\partial x(y)}{\partial y}\right|}$$

$2y + xy = 9 - 2x$, $x(y+2) = 9 - 2y$, $x = \dfrac{9-2y}{2+y}$

$$\frac{\partial x}{\partial y} = \frac{-2(2+y) - (9-2y)}{(2+y)^2} = \frac{-13}{(2+y)^2}$$

$$f_y(y) = \frac{1}{4} \cdot I\left(0 \le \frac{9-2y}{2+y} \le 4\right) \cdot \frac{13}{(2+y)^2} =$$

$$= \frac{13}{4(y+2)^2} \cdot I\left(\frac{1}{6} \le y \le \frac{9}{2}\right).$$

7.3

8. Variables $y_1, y_2, \ldots, y_7$ can take only positive integer values. One of the solutions of the equation $y_1 + y_2 + \ldots + y_7 = 18$ is chosen at random. Determine the probability that $y_5 = 2$.

$\underbrace{y_1 + y_2 + \ldots + y_7 = 18}$, $P(y_5 = 2) = ?$

the quantity of solutions $= \binom{17}{6}$ /stars and bars problem/

If $y_5 = 2$ then we have $y_1 + y_2 + y_3 + y_4 + y_6 + y_7 = 16$,

and the quantity of solutions is $\binom{15}{5}$.

8.3

Hence, the probability in question is

$$\frac{\binom{15}{5}}{\binom{17}{6}} = \frac{15!\cdot 6!\cdot 11!}{5!\,10!\,17!} = \frac{6\cdot 11}{16\cdot 17} = \frac{33}{136}$$ //