Assignment 9.

N1. Let $Z=[\xi, \eta, \zeta]^T \sim N(\bar{\mu}, \Sigma)$, where $\bar{\mu}=\begin{bmatrix}1\\0\\-2\end{bmatrix}$, $\Sigma=\begin{bmatrix}2&3&-1\\3&6&-1\\-1&-1&1\end{bmatrix}$

a) Since the vector Z is multivariate normal, its PDF is defined as follows:

$$f_Z(w,x,y)=\frac{1}{\sqrt{(2\pi)^3\det\Sigma}}e^{-\frac{1}{2}\langle\begin{bmatrix}w\\x\\y\end{bmatrix}-\bar{\mu},\ \Sigma^{-1}\cdot(\begin{bmatrix}w\\x\\y\end{bmatrix}-\bar{\mu})\rangle}$$

$$\det\Sigma=2\cdot(6-1)-3\cdot(3-1)-(-3+6)=1,\quad \Sigma^{-1}=\frac{1}{1}\cdot\begin{bmatrix}6-1&1-3&-3+6\\1-3&2-1&2-3\\6-3&2-3&12-9\end{bmatrix}=\begin{bmatrix}5&-2&3\\-2&1&-1\\3&-1&3\end{bmatrix}$$

$$\begin{bmatrix}w-1&x&y+2\end{bmatrix}\begin{bmatrix}5&-2&3\\-2&1&-1\\3&-1&3\end{bmatrix}\begin{bmatrix}w-1\\x\\y+2\end{bmatrix}=5w^2+x^2+3y^2-4xw+6yw-2xy+2w+6y+5$$

Therefore, $f_Z(w,x,y)=\frac{1}{\sqrt{(2\pi)^3}}e^{-\frac{1}{2}(5w^2+x^2+3y^2-4xw+6yw-2xy+2w+6y+5)}$

Since Z is multivariate normal, its components are normally distributed:

$\xi\sim N(1;2)$, $\eta\sim N(0;6)$, $\zeta\sim N(-2;1)$, thus their PDFs are trivially obtainable.

$$f_{\xi,\zeta}(w,y)=\int_{-\infty}^{+\infty}f_{\xi,\eta,\zeta}(w,x,y)dx=\frac{1}{\sqrt{(2\pi)^3}}e^{-\frac{1}{2}(w^2+2y^2+2yw+2w+6y+5)}\cdot\int_{-\infty}^{+\infty}e^{-\frac{1}{2}(x^2+4w^2+y^2-4xw-2xy+4yw)}dx=$$

$$=\frac{1}{\sqrt{(2\pi)^3}}e^{-\frac{1}{2}(w^2+2y^2+2yw+2w+6y+5)}\cdot\underbrace{\int_{-\infty}^{+\infty}e^{-\frac{1}{2}(x-2w-q)^2}d(x-2w-q)}_{\sqrt{2\pi},\text{ from a PDF of a st. normal dist.}}=\frac{1}{2\pi}e^{-\frac{1}{2}(w^2+2y^2+2yw+2w+6y+5)}$$

b) Since Z is multivariate normal, any linear combination of its components is normally distributed:

$$2\xi-3\eta-\zeta\sim N(2\cdot1-3\cdot0-1\cdot(-2);\ [2\ -3\ -1]\cdot\Sigma\cdot\begin{bmatrix}2\\-3\\-1\end{bmatrix})=N(4;25)=5N(0;1)+4$$

$$P(2\xi-3\eta-\zeta<9)=\Phi(\tfrac{9-4}{5})=\Phi(1)$$

$$2\eta-5\zeta\sim N(0\cdot1+2\cdot0-5\cdot(-2);\ [0\ 2\ -5]\cdot\Sigma\cdot\begin{bmatrix}0\\2\\-5\end{bmatrix})=N(10;69)=\sqrt{69}N(0;1)+10$$

$$P(|2\eta-5\zeta|<16)=P(-16<2\eta-5\zeta<16)=\Phi(\tfrac{16-10}{\sqrt{69}})-\Phi(\tfrac{-16-10}{\sqrt{69}})$$

N2. As shown in N1, $2\xi - 3\eta - \varsigma \sim N(4; 25)$, $2\eta - 5\varsigma \sim N(10; 69)$

Thus, $f_{2\xi-3\eta-\varsigma}(u) = \frac{1}{\sqrt{2\pi}\cdot 5} e^{-\frac{1}{2}\frac{(u-4)^2}{25}}$, $\varphi_{2\xi-3\eta-\varsigma}(t) = e^{4it - \frac{25}{2}t^2}$

Let $P(2\eta - 5\varsigma < x) = F(x)$, then $P(2\eta - 5\varsigma - 7 < x) = F(x+7)$

$f_{2\eta-5\varsigma-7}(v) = (F(v+7))'_v = \frac{1}{\sqrt{2\pi\cdot 69}} e^{-\frac{1}{2}\frac{(v+7-10)^2}{69}} \cdot \underbrace{(v+7)'_v}_{1}$ PDF of $N(3; 69)$

Therefore, $2\eta - 5\varsigma - 7 \sim N(3; 69)$, $\varphi_{2\eta-5\varsigma-7}(t) = e^{3it - \frac{69}{2}t^2}$

$\underline{N3}$. $f_{\xi_1,\xi_2,\xi_3} = ae^{-\frac{1}{2}(x_1+3)^2-(x_2-2)^2-3(x_3+1)^2+(x_1+3)(x_2-2)+2(x_1+3)(x_3+1)-3(x_2-2)(x_3+1)}$

Property 1: $\int\limits_{-\infty}^{+\infty} e^{-\frac{1}{2}(ax+C)^2}dx = \frac{\sqrt{2\pi}}{a}$, where a, C are constants w.r.t. x

$\square$ $\int\limits_{-\infty}^{+\infty} e^{-\frac{1}{2}(ax+C)^2}dx = \frac{1}{a}\int\limits_{-\infty}^{+\infty} e^{-\frac{1}{2}(ax+C)^2}d(ax+C) = \frac{\sqrt{2\pi}}{a}\int\limits_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt = \frac{\sqrt{2\pi}}{a}$ (normal PDF) $\blacksquare$

Let's denote the power of the exponent in f_{ξ_1,ξ_2,ξ_3} as $g(x_1, x_2, x_3)$:

$$g(x_1,x_2,x_3) = -\frac{1}{2}(x_1+3)^2-(x_2-2)^2-3(x_3+1)^2+(x_1+3)(x_2-2)+2(x_1+3)(x_3+1)-3(x_2-2)(x_3+1)$$

$$f_{\xi_1,\xi_2}(x_1,x_2) = \int\limits_{-\infty}^{+\infty} f_{\xi_1,\xi_2,\xi_3}(x_1,x_2,x_3)dx_3$$

To integrate x_3 out of f_{ξ_1,ξ_2,ξ_3}, one needs to express $g(x_1,x_2,x_3)$ as $-\frac{1}{2}(Bx_3+C(x_1,x_2))^2+h(x_1,x_2)$
for some B, C, h

$$g(x_1,x_2,x_3) = -\frac{1}{2}((x_1+3)^2+2(x_2-2)^2+\overset{B^2}{\overset{\shortparallel}{6}}(x_3+1)^2-2(x_1+3)(x_2-2)-4(x_1+3)(x_3+1)+6(x_2-2)(x_3+1)) =$$

$$= -\frac{1}{2}((\sqrt{6}(x_3+1))^2+(x_1+3)^2+2(x_2-2)^2+2\cdot(-\tfrac{2}{\sqrt{6}}(x_1+3))(\tfrac{3}{\sqrt{6}}(x_2-2))+2\cdot(\sqrt{6}(x_3+1))(-\tfrac{2}{\sqrt{6}}(x_1+3))+2(\sqrt{6}(x_3+1))(\tfrac{3}{\sqrt{6}}(x_2-2)))$$

$$= -\frac{1}{2}((\sqrt{6}(x_3+1)-\tfrac{2}{\sqrt{6}}(x_1+3)+\tfrac{3}{\sqrt{6}}(x_2-2))^2-\tfrac{4}{6}(x_1+3)^2-\tfrac{9}{6}(x_2-2)^2+(x_1+3)^2+2(x_2-2)^2)$$

$$= -\frac{1}{2}(\sqrt{6}x_3+\underbrace{\sqrt{6}-\tfrac{2}{\sqrt{6}}(x_1+3)+\tfrac{3}{\sqrt{6}}(x_2-2)}_{C(x_1,x_2)})^2\underbrace{-\tfrac{1}{6}(x_1+3)^2-\tfrac{1}{4}(x_2-2)^2}_{h(x_1,x_2)}$$

Then $f_{\xi_1,\xi_2}(x_1,x_2) = ae^{-\frac{1}{6}(x_1+3)^2-\frac{1}{4}(x_2-2)^2}\cdot\sqrt{\frac{2\pi}{6}}$, according to property 1.

$$f_{\xi_2,\xi_3}(x_2,x_3)=\int_{-\infty}^{+\infty} f_{\xi_1,\xi_2,\xi_3}(x_1,x_2,x_3)\,dx_1$$

$B^2=1$

$$g(x_1,x_2,x_3)=-\frac12\left((x_1+3)^2+2(x_2-2)^2+6(x_3+1)^2-2(x_1+3)(x_2-2)-4(x_1+3)(x_3+1)+6(x_2-2)(x_3+1)\right)=$$

$$=-\frac12\left((x_1+3)^2+2(x_2-2)^2+6(x_3+1)^2+2(x_1+3)(-(x_2-2))+2(x_1+3)(-2(x_3+1))+2(-(x_2-2))(-2(x_3+1))+2(x_2-2)(x_3+1)\right)=$$

$$=-\frac12\left((x_1+3-(x_2-2)-2(x_3+1))^2-(x_2-2)^2-4(x_3+1)^2+2(x_2-2)^2+6(x_3+1)^2+2(x_2-2)(x_3+1)\right)=$$

$$=-\frac12(x_1+\underbrace{3-(x_2-2)-2(x_3+1)}_{C(x_2,x_3)})^2\underbrace{-\frac12(x_2-2)^2-(x_3+1)^2-(x_2-2)(x_3+1)}_{h(x_2,x_3)}$$

Then $f_{\xi_2,\xi_3}(x_2,x_3)=ae^{-\frac12(x_2-2)^2-(x_3+1)^2-(x_2-2)(x_3+1)}\cdot\sqrt{2\pi}$, according to property 1.

$$f_{\xi_1,\xi_3}(x_1,x_3)=\int_{-\infty}^{+\infty} f_{\xi_1,\xi_2,\xi_3}(x_1,x_2,x_3)\,dx_2$$

B

$$g(x_1,x_2,x_3)=-\frac12\left((x_1+3)^2+(\sqrt2(x_2-2))^2+6(x_3+1)^2-2(x_1+3)(x_2-2)-4(x_1+3)(x_3+1)+6(x_2-2)(x_3+1)\right)=$$

$$=-\frac12\left((\sqrt2(x_2-2))^2+(x_1+3)^2+6(x_3+1)^2+2(\sqrt2(x_2-2))(-\tfrac1{\sqrt2}(x_1+3))+2(-\tfrac1{\sqrt2}(x_1+3))(\tfrac3{\sqrt2}(x_3+1))-(x_1+3)(x_3+1)+\right.$$
$$\left.+2(\sqrt2(x_2-2))(\tfrac3{\sqrt2}(x_3+1))\right)=$$

$$=-\frac12\left((\sqrt2x_2-2\sqrt2-\tfrac1{\sqrt2}(x_1+3)+\tfrac3{\sqrt2}(x_3+1))^2-\frac12(x_1+3)^2-\frac92(x_3+1)^2+(x_1+3)^2+6(x_3+1)^2-(x_1+3)(x_3+1)\right)^2=$$

$$=-\frac12(\sqrt2x_2\underbrace{-2\sqrt2-\tfrac1{\sqrt2}(x_1+3)+\tfrac3{\sqrt2}(x_3+1)}_{C(x_1,x_3)})^2\underbrace{-\frac14(x_1+3)^2-\frac34(x_3+1)^2+\frac12(x_1+3)(x_3+1)}_{h(x_1,x_3)}$$

Then $f_{\xi_1,\xi_3}(x_1,x_3)=ae^{-\frac14(x_1+3)^2-\frac34(x_3+1)^2+\frac12(x_1+3)(x_3+1)}\cdot\frac{\sqrt{2\pi}}{\sqrt2}$, according to property 1.

$$f_{\xi_1}(x_1)=\int_{-\infty}^{+\infty} f_{\xi_1,\xi_2}(x_1,x_2)\,dx_2=a\cdot\sqrt{\frac\pi3}e^{-\frac16(x_1+3)^2}\cdot\int_{-\infty}^{+\infty} e^{-\frac12(\frac1{\sqrt2}x_2-\frac2{\sqrt2})^2}dx_2=a\sqrt{\frac\pi3}e^{-\frac16(x_1+3)^2}\cdot2\sqrt\pi \text{ (property 1)}$$

$$f_{\xi_2}(x_2)=\int_{-\infty}^{+\infty} f_{\xi_1,\xi_2}(x_1,x_2)\,dx_1=a\cdot\sqrt{\frac\pi3}e^{-\frac14(x_2-2)^2}\cdot\int_{-\infty}^{+\infty} e^{-\frac12(\frac1{\sqrt3}x_1+\frac3{\sqrt3})}dx_1=a\sqrt{\frac\pi3}e^{-\frac14(x_2-2)^2}\cdot\sqrt{6\pi} \text{ (property 1)}$$

$$f_{\xi_3}(x_3)=\int_{-\infty}^{+\infty} f_{\xi_2,\xi_3}(x_2,x_3)\,dx_2=a\sqrt{2\pi}\int_{-\infty}^{+\infty} e^{-\frac12((x_2-2)^2+2(x_2-2)(x_3+1)+2(x_3+1)^2)}dx_2=$$

$$=a\sqrt{2\pi}e^{-\frac12(x_3+1)^2}\cdot\int_{-\infty}^{+\infty} e^{-\frac12(x_2-2+x_3+1)^2}dx_2=a\sqrt{2\pi}e^{-\frac12(x_3+1)^2}\cdot\sqrt{2\pi} \text{ (property 1)}$$

$$\int_{-\infty}^{+\infty} f_{\xi_3}(x_3)\,dx_3=1 \Leftrightarrow a\cdot2\pi\cdot\int e^{-\frac12(x_3+1)^2}dx_3=1 \Leftrightarrow \boxed{a=\frac1{\sqrt{(2\pi)^3}}}$$

Then $f_{\xi_1}(x_1)=\frac1{\sqrt{2\pi}\cdot\sqrt3}e^{-\frac{(x_1+3)^2}{2\cdot3}}$, $f_{\xi_2}(x_2)=\frac1{\sqrt{2\pi}\cdot\sqrt2}e^{-\frac{(x_2-2)^2}{2\cdot2}}$, $f_{\xi_3}(x_3)=\frac1{\sqrt{2\pi}}e^{-\frac{(x_3+1)^2}{2}}$

$\Downarrow$ $\Downarrow$ $\Downarrow$

$\xi_1\sim N(-3;3)$, $E\xi_1=-3$, $Var\,\xi_1=3$; $\xi_2\sim N(2;2)$, $E\xi_2=2$, $Var\,\xi_2=2$; $\xi_3\sim N(-1;1)$, $E\xi_3=-1$, $Var\,\xi_3=1$

$$E(\xi_1\xi_2)=\int_{-\infty}^{+\infty}dx_1\int_{-\infty}^{+\infty}x_1x_2f_{\xi_1,\xi_2}(x_1,x_2)\,dx_2=\frac1{2\pi\sqrt6}\int_{-\infty}^{+\infty}x_1e^{-\frac16(x_1+3)^2}dx_1\cdot\int_{-\infty}^{+\infty}x_2e^{-\frac14(x_2-2)^2}dx_2=E(\xi_1)\cdot E(\xi_2)$$

$$E(\xi_2\xi_3)=\int_{-\infty}^{+\infty}dx_2\int_{-\infty}^{+\infty}x_2x_3f_{\xi_2,\xi_3}(x_2,x_3)\,dx_3=\frac1{2\pi}\int_{-\infty}^{+\infty}x_2dx_2\int_{-\infty}^{+\infty}x_3e^{-\frac12(x_2-2)^2-(x_3+1)^2-(x_2-2)(x_3+1)}dx_3$$

$$\left(e^{-\frac12(x_2-2)^2-(x_3+1)^2-(x_2-2)(x_3+1)}\right)'_{x_3}=e^{-\frac12(x_2-2)^2-(x_3+1)^2-(x_2-2)(x_3+1)}\cdot(-2(x_3+1)-(x_2-2))$$

$$E(\xi_2\xi_3)=\frac1{2\pi}\int_{-\infty}^{+\infty}x_2dx_2\left(-\frac12\right)\int_{-\infty}^{+\infty}(-2x_3-x_2+x_2)e^{-\frac12(x_2-2)^2-(x_3+1)^2-(x_2-2)(x_3+1)}dx_3=$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} x_2\, dx_2 \left(-\tfrac{1}{2}\right)\left(x_2 \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x_2-2)^2-(x_3+1)^2-(x_2-2)(x_3+1)}dx_3 + \underbrace{\int_{-\infty}^{+\infty}\left(e^{-\frac{1}{2}(x_2-2)^2-(x_3+1)^2-(x_2-2)(x_3+1)}\right)'_{x_3}dx_3}_{0}\right) =$$

$$= -\frac{1}{4\pi}\int_{-\infty}^{+\infty} x_2^2\, dx_2 \int_{-\infty}^{+\infty} e^{-\frac{1}{2}((x_2-2)^2+2(x_3+1)^2+2(x_2-2)(x_3+1))}dx_3 = -\frac{1}{4\pi}\int_{-\infty}^{+\infty} x_2^2 e^{-\frac{1}{4}(x_2-2)^2}dx_2 \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(\sqrt{2}(x_3+1)+\frac{1}{\sqrt{2}}(x_2-2))^2}dx_3 =$$

$$= -\frac{1}{4\pi}\int_{-\infty}^{+\infty} x_2^2 e^{-\frac{1}{4}(x_2-2)^2}\cdot \frac{1}{\sqrt{2}}\sqrt{2\pi}\, dx_2 = E(\xi_2\xi_3)$$

reminder

$$\left(e^{-\frac{1}{4}(x_2-2)^2}\right)'_{x_2} = e^{-\frac{1}{4}(x_2-2)^2}\cdot\left(-\tfrac{1}{4}\cdot 2(x_2-2)\right) = \left(-\tfrac{1}{2}x_2+1\right)e^{-\frac{1}{4}(x_2-2)^2}$$

$$E(\xi_2\xi_3) = -\frac{1}{4\sqrt{\pi}}(-2)\int_{-\infty}^{+\infty} x_2\cdot\left(-\tfrac{1}{2}x_2+1-1\right)\cdot e^{-\frac{1}{4}(x_2-2)^2}dx_2 = \frac{1}{2\sqrt{\pi}}\left(\underbrace{-\int_{-\infty}^{+\infty} x_2 e^{-\frac{1}{4}(x_2-2)^2}dx_2}_{-2\sqrt{\pi}E(\xi_2)=-4\sqrt{\pi}} + \int_{-\infty}^{+\infty} x_2\left(e^{-\frac{1}{4}(x_2-2)^2}\right)'_{x_2}dx_2\right) =$$

$$= -2 + \frac{1}{2\sqrt{\pi}}\left(x_2 e^{-\frac{1}{4}(x_2-2)^2}\Big|_{x_2=-\infty}^{+\infty} - \int_{-\infty}^{+\infty} e^{-\frac{1}{4}(x_2-2)^2}dx_2\right) = -2 + \frac{1}{2\sqrt{\pi}}\left(2\lim_{x_2\to\infty} x_2 e^{-\frac{1}{4}(x_2-2)^2} - 2\sqrt{\pi}\right)$$

$$\lim_{x_2\to\infty} x_2 e^{-\frac{1}{4}(x_2-2)^2} = \lim_{x_2\to\infty}\frac{(x_2)'}{\left(e^{\frac{1}{4}(x_2-2)^2}\right)'} = \lim_{x_2\to\infty}\frac{1}{\left(\frac{1}{2}x_2-1\right)e^{\frac{1}{4}(x_2-2)^2}} = 0 \Rightarrow E(\xi_2\xi_3) = -3$$

$$E(\xi_1\xi_3) = \int_{-\infty}^{+\infty} dx_1\int_{-\infty}^{+\infty} x_1x_3 f_{\xi_1\xi_3}(x_1,x_3)dx_3 = \frac{1}{2\pi\sqrt{2}}\int_{-\infty}^{+\infty} x_1\, dx_1\int_{-\infty}^{+\infty} x_3 e^{-\frac{1}{4}(x_1+3)^2-\frac{3}{4}(x_3+1)^2+\frac{1}{2}(x_1+3)(x_3+1)}dx_3$$

$$\left(e^{-\frac{1}{4}(x_1+3)^2-\frac{3}{4}(x_3+1)^2+\frac{1}{2}(x_1+3)(x_3+1)}\right)'_{x_3} = e^{-\frac{1}{4}(x_1+3)^2-\frac{3}{4}(x_3+1)^2+\frac{1}{2}(x_1+3)(x_3+1)}\cdot\left(-\tfrac{3}{4}\cdot 2\cdot(x_3+1)+\tfrac{1}{2}(x_1+3)\right)$$

$$E(\xi_1\xi_3) = \frac{1}{2\pi\sqrt{2}}\int_{-\infty}^{+\infty} x_1\, dx_1\left(-\tfrac{2}{3}\right)\int_{-\infty}^{+\infty}\left(-\tfrac{3}{2}x_3+\tfrac{1}{2}x_1-\tfrac{1}{2}x_1\right)e^{-\frac{1}{4}(x_1+3)^2-\frac{3}{4}(x_3+1)^2+\frac{1}{2}(x_1+3)(x_3+1)}dx_3 =$$

$$= \frac{1}{2\pi\sqrt{2}}\int_{-\infty}^{+\infty} x_1\, dx_1\left(-\tfrac{2}{3}\right)\left(-\tfrac{1}{2}x_1\int_{-\infty}^{+\infty} e^{-\frac{1}{4}(x_1+3)^2-\frac{3}{4}(x_3+1)^2+\frac{1}{2}(x_1+3)(x_3+1)}dx_3 + \underbrace{\int_{-\infty}^{+\infty}\left(e^{-\frac{1}{4}(x_1+3)^2-\frac{3}{4}(x_3+1)^2+\frac{1}{2}(x_1+3)(x_3+1)}\right)'_{x_3}dx_3}_{0}\right) =$$

$$= \frac{1}{6\pi\sqrt{2}}\int_{-\infty}^{+\infty} x_1^2\, dx_1\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\frac{1}{2}(x_1+3)^2+\frac{3}{2}(x_3+1)^2-(x_1+3)(x_3+1)\right)}dx_3 = \frac{1}{6\pi\sqrt{2}}\int_{-\infty}^{+\infty} x_1^2 e^{-\frac{1}{6}(x_1+3)^2}dx_1\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\sqrt{\frac{3}{2}}(x_3+1)-\frac{1}{\sqrt{6}}(x_1+3)\right)^2}dx_3 =$$

$$= \frac{1}{6\pi\sqrt{2}}\int_{-\infty}^{+\infty} x_1^2 e^{-\frac{1}{6}(x_1+3)^2}\cdot 2\sqrt{\frac{\pi}{3}}\, dx_1$$

$$\left(e^{-\frac{1}{6}(x_1+3)^2}\right)'_{x_1} = e^{-\frac{1}{6}(x_1+3)^2}\cdot\left(-\tfrac{1}{6}\cdot 2(x_1+3)\right) = \left(-\tfrac{1}{3}x_1-1\right)e^{-\frac{1}{6}(x_1+3)^2}$$

$$E(\xi_1\xi_3) = \frac{1}{3\sqrt{6\pi}}(-3)\int_{-\infty}^{+\infty} x_1\cdot\left(-\tfrac{1}{3}x_1-1+1\right)e^{-\frac{1}{6}(x_1+3)^2}dx_1 = -\frac{1}{\sqrt{6\pi}}\left(\underbrace{\int_{-\infty}^{+\infty} x_1 e^{-\frac{1}{6}(x_1+3)^2}dx_1}_{\sqrt{6\pi}E(\xi_1)=-3\sqrt{6\pi}} + \int_{-\infty}^{+\infty} x_1\cdot\left(e^{-\frac{1}{6}(x_1+3)^2}\right)'_{x_1}dx_1\right) =$$

$$= 3 - \frac{1}{\sqrt{6\pi}}\left(\underbrace{x_1 e^{-\frac{1}{6}(x_1+3)^2}}_{0}\Big|_{x_1=-\infty}^{+\infty} - \int_{-\infty}^{+\infty} e^{-\frac{1}{6}(x_1+3)^2}dx_1\right) = 4$$

Therefore,
$$Cov(\xi_1,\xi_2) = E(\xi_1\xi_2) - E(\xi_1)E(\xi_2) = 0$$
$$Cov(\xi_1,\xi_3) = E(\xi_1\xi_3) - E(\xi_1)E(\xi_3) = 4-3 = 1$$
$$Cov(\xi_2,\xi_3) = E(\xi_2\xi_3) - E(\xi_2)E(\xi_3) = -3+2 = -1$$

a) $a = \frac{1}{(2\pi)^{\frac{3}{2}}}$, $\mu_{1,2,3} = [-3\ \ 2\ \ -1]^T$, $\Sigma_{1,2,3} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

б) Let $\eta_1 = \xi_1 + 2\xi_2 - 3\xi_3$, $\eta_2 = 3\xi_1 + 2\xi_2 - \xi_3$

One can verify that $f_{\xi_1,\xi_2,\xi_3}(x_1, x_2, x_3) = \frac{1}{(\sqrt{2\pi})^3\sqrt{\det\Sigma}} e^{-\frac{1}{2}<\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}-\bar{\mu},\ \Sigma^{-1}\cdot(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}-\bar{\mu})>}$

Therefore, $\xi = \begin{bmatrix}\xi_1\\\xi_2\\\xi_3\end{bmatrix}$ is multivariate normal $\Rightarrow$ any linear combination of its components is normal.

$\eta_1 \sim N([1\ 2\ -3]\times\bar{\mu}_{1,2,1};\ [1\ 2\ -3]\times\Sigma_{1,2,1}\times\begin{bmatrix}1\\2\\-3\end{bmatrix}) = N(4; 26) \Rightarrow \eta_1 = \sqrt{26}\,\eta_0 + 4$

for $\eta_0 \sim N(0;1)$

$\eta_2 \sim N([3\ 2\ -1]\times\bar{\mu}_{1,2,1};\ [3\ 2\ -1]\times\Sigma_{1,2,1}\times\begin{bmatrix}3\\2\\-1\end{bmatrix}) = N(-4, 34) \Rightarrow \eta_2 \Rightarrow \sqrt{34}\,\eta_0 - 4$

$P(0<\eta_1<8) = P(0<\sqrt{26}\,\eta_0+4<8) = P(-\frac{4}{\sqrt{26}}<\eta_0<\frac{4}{\sqrt{26}}) = \Phi(\frac{4}{\sqrt{26}}) - \Phi(-\frac{4}{\sqrt{26}})$

$P(-10<\eta_2<2) = P(-10<\sqrt{34}\,\eta_0-4<2) = P(-\frac{6}{\sqrt{34}}<\eta_0<\frac{6}{\sqrt{34}}) = \Phi(\frac{6}{\sqrt{34}}) - \Phi(-\frac{6}{\sqrt{34}})$

<u>N4.</u> $f_{\xi_1,\xi_2,\xi_3}(x_1, x_2, x_3) = a e^{-2(x_1+1)^2 - 3(x_2-1)^2 - 9x_3^2 + 4(x_1+1)(x_2-1) + 2(x_2-1)x_3}$

Following the same procedure as in N3: // three pages later

$a = \frac{4}{\sqrt{\pi^3}}$, $\xi_1 \sim N(-1; \frac{13}{16})$, $\xi_2 \sim N(1; \frac{9}{16})$, $\xi_3 \sim N(0; \frac{1}{16})$

$Cov(\xi_1, \xi_2) = -\frac{7}{16} + 1 = \frac{9}{16}$

$Cov(\xi_1, \xi_3) = \frac{1}{16} - 0 = \frac{1}{16}$

$Cov(\xi_2, \xi_3) = \frac{1}{16} - 0 = \frac{1}{16}$

$\Rightarrow \bar{\mu} = \begin{bmatrix}-1\\1\\0\end{bmatrix},\ \Sigma = \begin{bmatrix}\frac{13}{16} & \frac{9}{16} & \frac{1}{16}\\ \frac{9}{16} & \frac{9}{16} & \frac{1}{16}\\ \frac{1}{16} & \frac{1}{16} & \frac{1}{16}\end{bmatrix}$

One can verify that $f_{\xi_1,\xi_2,\xi_3}(x_1, x_2, x_3) = \frac{1}{(\sqrt{2\pi})^3\sqrt{\det\Sigma}} e^{-\frac{1}{2}<\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}-\bar{\mu},\ \Sigma^{-1}\times(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}-\bar{\mu})>}$

Therefore, $\xi = \begin{bmatrix}\xi_1\\\xi_2\\\xi_3\end{bmatrix}$ is multivariate normal $\Rightarrow$ any linear combination of its components is normal.

Let $\eta = 2\xi_1 - \xi_2 - 3\xi_3$, $\eta_0 \sim N(0;1)$

$\eta \sim N([2\ -1\ -3]\times\bar{\mu};\ [2\ -1\ -3]\times\Sigma\times\begin{bmatrix}2\\-1\\-3\end{bmatrix}) = N(-3; \frac{7}{4}) \Rightarrow \eta = \sqrt{\frac{7}{4}}\,\eta_0 - 3$

$P(\eta<\lambda) = P(\sqrt{\frac{7}{4}}\,\eta_0 - 3 < \lambda) = \Phi(\frac{\lambda+3}{\sqrt{\frac{7}{4}}}) = \frac{1}{4}$

$\lambda = \sqrt{\frac{7}{4}}\,\Phi^{-1}(\frac{1}{4}) - 3$

<u>N5.</u> $Z = \begin{bmatrix}\xi\\\eta\end{bmatrix} \sim N(\mu; K)$, $\mu = \begin{bmatrix}-3\\1\end{bmatrix}$, $K = \begin{bmatrix}1 & -1\\-1 & 4\end{bmatrix}$

Since Z is bivariate normal, we know that $(\eta|\xi=k) \sim N(\mu_\eta + \rho_{\xi,\eta}\cdot(\frac{\sigma_\eta}{\sigma_\xi})(k-\mu_\xi);\ \sigma_\eta^2(1-\rho_{\xi,\eta}^2))$

Therefore, $(\eta|\xi=0) \sim N(-2; 3)$, $(\eta|\xi=1) \sim N(-3; 3)$, $(\eta|\xi=2) \sim N(-4; 3)$

a) $P(\eta<3|\xi=0) = P(\sqrt{3}\,\eta_0 - 2 < 3) = \Phi(\frac{5}{\sqrt{3}})$

б) $P(\eta<0|\xi=1) = P(\sqrt{3}\,\eta_0 - 3 < 0) = \Phi(\sqrt{3})$

for $\eta_0 \sim N(0;1)$

c) $P(|\eta+4|<2|\xi=2) = P(-6<\sqrt{3}\,\eta_0 - 4 < -2) = \Phi(\frac{2}{\sqrt{3}}) - \Phi(-\frac{2}{\sqrt{3}})$

<u>N6.</u> $Z = \begin{bmatrix}\xi\\\eta\\\zeta\end{bmatrix} \sim N(\mu; K)$, $\mu = \begin{bmatrix}1\\0\\-2\end{bmatrix}$, $K = \begin{bmatrix}2 & 3 & -1\\3 & 6 & -1\\-1 & -1 & 1\end{bmatrix}$

Since Z is multivariate normal, partitioning it into $\begin{bmatrix}Z_1\\Z_2\end{bmatrix}$ with $Z_1 \sim N(\mu_1; \Sigma_{11})$, $Z_2 \sim N(\mu; \Sigma_{22})$,

we know that $(Z_1|Z_2=x) \sim N(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x-\mu_2);\ \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$

a) Let's introduce the partition $\begin{bmatrix}\eta\\ \zeta\end{bmatrix}$ and $[\xi]$, then $\mu_1=\begin{bmatrix}0\\-2\end{bmatrix}$, $\mu_2=[1]$

$\Sigma_{11}=\begin{bmatrix}6 & -1\\ -1 & 1\end{bmatrix}$, $\Sigma_{22}=[2]$, $\Sigma_{12}=\begin{bmatrix}3\\-1\end{bmatrix}$, $\Sigma_{21}=[3\ \ -1]$

Therefore, $\left(\begin{bmatrix}\eta\\ \zeta\end{bmatrix} \Big| \xi=x\right)\sim N\left(\begin{bmatrix}\frac{3}{2}x-\frac{3}{2}\\ -\frac{1}{2}x+\frac{5}{2}\end{bmatrix};\begin{bmatrix}\frac{3}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}\right)$

$$P(|\zeta|>1\,|\,\xi=1)=1-P(|\zeta|<1\,|\,\xi=1)=1-P(\zeta<1\,|\,\xi=1)+P(\zeta<-1\,|\,\xi=1)$$

$$f_{\eta,\zeta}(u,v\,|\,\xi=x)=\frac{1}{2\pi\sqrt{\left|\frac{1}{2}\begin{bmatrix}3 & 1\\ 1 & 1\end{bmatrix}\right|}}e^{-\frac{1}{2}\left\langle\begin{bmatrix}u-\frac{3}{2}x+\frac{3}{2}\\ v+\frac{1}{2}x-\frac{5}{2}\end{bmatrix},\left(\frac{1}{2}\begin{bmatrix}3 & 1\\ 1 & 1\end{bmatrix}\right)^{-1}\times\begin{bmatrix}u-\frac{3}{2}x+\frac{3}{2}\\ v+\frac{1}{2}x-\frac{5}{2}\end{bmatrix}\right\rangle}=$$

$$=\frac{\sqrt{2}}{2\pi}e^{-\frac{1}{2}(u^2+3v^2-4xu+6xv-2vu+8u-18v+\frac{9}{2}x^2-21x+\frac{57}{2})}$$

$$f_{\zeta}(v\,|\,\xi=x)=\int_{-\infty}^{+\infty}f_{\eta,\zeta}(u,v)\,du=\frac{\sqrt{2}}{2\pi}\int_{-\infty}^{+\infty}e^{-\frac{1}{2}((u+4-2x-v)^2+2v^2+2xv-10v+\frac{1}{2}x^2-5x+\frac{25}{2})}du=$$

$$=\frac{\sqrt{2}}{2\pi}e^{-v^2-xv+5v-\frac{1}{4}x^2+\frac{5}{2}x+\frac{25}{4}}\cdot\int_{-\infty}^{+\infty}e^{-\frac{1}{2}(u+4-2x-v)^2}d(u+4-2x-v)=\frac{1}{\sqrt{\pi}}e^{-v^2-xv+5v-\frac{1}{4}x^2+\frac{5}{2}x+\frac{25}{4}}$$

$$P(\zeta<v\,|\,\xi=x)=F_{\zeta}(v\,|\,\xi=x)=\int_{-\infty}^{v}f_{\zeta}(t\,|\,\xi=x)\,dt=\int_{-\infty}^{v}\frac{1}{\sqrt{\pi}}e^{-t^2-xt+5t-\frac{1}{4}x^2+\frac{5}{2}x+\frac{25}{4}}dt=$$

$$=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{v}e^{-\frac{1}{2}((\sqrt{2}t-\frac{5}{\sqrt{2}})^2+2xt+\frac{1}{2}x^2-5x)}dt=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{v}e^{-\frac{1}{2}(\sqrt{2}t-\frac{5}{\sqrt{2}}+\frac{x}{\sqrt{2}})^2}dt=\Phi\left(\sqrt{2}v-\frac{5}{\sqrt{2}}+\frac{x}{\sqrt{2}}\right)$$

Then $P(|\zeta|>1\,|\,\xi=1)=1-\Phi(-\sqrt{2})+\Phi(-3\sqrt{2})=1-\Phi(3\sqrt{2})+\Phi(\sqrt{2})$

b) Let's introduce another partition, $\begin{bmatrix}\xi\\ \zeta\end{bmatrix}$ and $[\eta]$, then $\mu_1=\begin{bmatrix}1\\-2\end{bmatrix}$, $\mu_2=[0]$

$\Sigma_{11}=\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}$, $\Sigma_{22}=[6]$, $\Sigma_{12}=\begin{bmatrix}3\\-1\end{bmatrix}$, $\Sigma_{21}=[3\ \ -1]$

Therefore, $\left(\begin{bmatrix}\xi\\ \zeta\end{bmatrix} \Big| \eta=x\right)\sim N\left(\begin{bmatrix}\frac{1}{2}x+1\\ -\frac{1}{6}x-2\end{bmatrix};\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{5}{6}\end{bmatrix}\right)$

$$f_{\xi,\zeta}(w,v\,|\,\eta=x)=\frac{1}{2\pi\sqrt{\left|\frac{1}{2}\begin{bmatrix}1 & -1\\ -1 & \frac{5}{3}\end{bmatrix}\right|}}e^{-\frac{1}{2}\left\langle\begin{bmatrix}w-\frac{1}{2}x-1\\ v+\frac{1}{6}x+2\end{bmatrix},\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{5}{6}\end{bmatrix}^{-1}\times\begin{bmatrix}w-\frac{1}{2}x-1\\ v+\frac{1}{6}x+2\end{bmatrix}\right\rangle}=$$

$$=\frac{\sqrt{6}}{2\pi}e^{-\frac{1}{2}(3v^2+6vw-2vx+6v+5w^2-4wx+2w+\frac{5}{6}x^2+5)}$$

$$f_{\zeta}(v\,|\,\eta=x)=\int_{-\infty}^{+\infty}f_{\xi,\zeta}(w,v)\,dw=\frac{\sqrt{6}}{2\pi}\int_{-\infty}^{+\infty}e^{-\frac{1}{2}((\sqrt{5}w+\frac{1}{\sqrt{5}}+\frac{3}{\sqrt{5}}v-\frac{2}{\sqrt{5}}x)^2+\frac{24}{5}v+\frac{4}{5}x+\frac{2}{5}vx+\frac{6}{5}v^2+\frac{1}{30}x^2+\frac{24}{5})}=$$

$$=\frac{\sqrt{6}}{2\pi}e^{-\frac{1}{2}(\frac{24}{5}v+\frac{4}{5}x+\frac{2}{5}vx+\frac{6}{5}v^2+\frac{1}{30}x^2+\frac{24}{5})}\cdot\frac{1}{\sqrt{5}}\underbrace{\int_{-\infty}^{+\infty}e^{-\frac{1}{2}(\sqrt{5}w+\frac{1}{\sqrt{5}}+\frac{3}{\sqrt{5}}v-\frac{2}{\sqrt{5}}x)^2}d(\sqrt{5}w+\frac{1}{\sqrt{5}}+\frac{3}{\sqrt{5}}-\frac{2}{\sqrt{5}}x)}_{\sqrt{2\pi}}$$

$$P(\zeta<v\,|\,\eta=x)=F_{\zeta}(v\,|\,\eta=x)=\int_{-\infty}^{v}f_{\zeta}(t\,|\,\eta=x)\,dt=\sqrt{\frac{6}{5}}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{v}e^{-\frac{1}{2}(\frac{6}{5}t^2+\frac{2}{5}tx+\frac{24}{5}t+\frac{4}{5}x+\frac{1}{30}x^2+\frac{24}{5})}dt=$$

$$=\sqrt{\frac{6}{5}}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{v}e^{-\frac{1}{2}((\sqrt{\frac{6}{5}}t+2\sqrt{\frac{6}{5}})^2+\frac{2}{5}tx+\frac{4}{5}x+\frac{1}{30}x^2)}dt=\Phi\left(\sqrt{\frac{6}{5}}v+2\sqrt{\frac{6}{5}}+\frac{x}{\sqrt{30}}\right)$$

Then $P(|\zeta+3|<1\,|\,\eta=6)=\Phi\left(\frac{6}{\sqrt{30}}\right)-\Phi\left(-\frac{6}{\sqrt{30}}\right)$

N7. Let $\xi_1, \xi_2, \dots, \xi_{100} \sim^{iid} N(0;1)$, $\eta_1 = \sum_{i=1}^{40} \xi_i$, $\eta_2 = \sum_{i=1}^{100} \xi_i$

Since all ξ_i are independent, $\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_{100} \end{bmatrix}$ is multivariate normal ($\mu = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$)

Therefore, any linear combination of components of ξ is normally distributed.

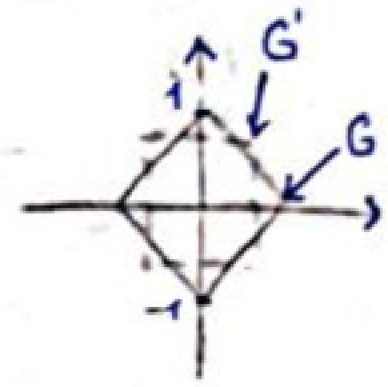
$$\eta_1 \sim N([\underbrace{1 \cdots 1}_{40} \underbrace{0 \cdots 0}_{60}] \times \mu \,;\, [1 \cdots 1\, 0 \cdots 0] \times \Sigma \times \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}) = N(0;\, 40)$$

Similarly, $\eta_2 \sim N(0;\, 100)$

We also know that any vector consisting of linear combinations of components of ξ is also multivariate normal:

$$\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \sim N(\begin{bmatrix} 0 \\ 0 \end{bmatrix}; \begin{bmatrix} 40 & 40 \\ 40 & 100 \end{bmatrix}) \Rightarrow f_{\eta_1, \eta_2}(u, v) = \frac{1}{40\sqrt{6}} e^{-\frac{1}{2}(\frac{u^2}{24} - \frac{uv}{30} + \frac{v^2}{60})}$$

N8. $\xi, \eta \sim^{iid} N(0;1)$



$P(\begin{bmatrix} \xi \\ \eta \end{bmatrix} \notin G) = P(\begin{bmatrix} U \\ V \end{bmatrix} \notin G')$, where $\begin{matrix} \xi = x \\ \eta = y \end{matrix}$, $\begin{matrix} U = u \\ V = v \end{matrix}$, $\begin{matrix} u = \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y \\ v = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y \end{matrix}$

Then $\begin{matrix} x = \frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v \\ y = -\frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v \end{matrix}$, $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left\| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right\| = \left\| \begin{matrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{matrix} \right\| = 1$

Since ξ and η are independent, $f_{\xi,\eta}(x,y) = f_\xi(x) \cdot f_\eta(y)$

$$f_{U,V}(u,v) = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| f_{\xi,\eta}(x(u,v), y(u,v)) = \frac{1}{2\pi} e^{-\frac{u^2}{2} - \frac{v^2}{2}}$$

$$P(1 \leq |x| + |y|) = P(\begin{bmatrix} u \\ v \end{bmatrix} \notin G') = 1 - \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} e^{-\frac{u^2}{2}} du \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} e^{-\frac{v^2}{2}} dv = 1 - (2\Phi(\tfrac{\sqrt{2}}{2}) - 1)^2$$

Let G_2 be a shape twice as large as G, G_2' be G_2 turned $45°$ ccw

$$P(1 \leq |x| + |y| \leq 2) = P(\begin{bmatrix} u \\ v \end{bmatrix} \in G_2') - P(\begin{bmatrix} u \\ v \end{bmatrix} \in G') = (2\Phi(\sqrt{2}) - 1)^2 - (2\Phi(\tfrac{\sqrt{2}}{2} - 1) - 1)^2$$

$P(\sqrt{x^2+y^2} \leq 2) = P(\begin{bmatrix} \xi \\ \eta \end{bmatrix} \in C((0;0), 2))$, where $C((0;0), 2)$ is a circle of radius 2 centered at $(0;0)$.

$P(\begin{bmatrix} \xi \\ \eta \end{bmatrix} \in C((0;0), 2)) = P(F \in [0; 2\pi), R \in [0; 2])$, where $\begin{matrix} \xi = x \\ \eta = y \end{matrix}$, $\begin{matrix} F = \varphi \\ R = r \end{matrix}$, $\begin{matrix} r = \sqrt{x^2+y^2} \\ \varphi = \operatorname{arctg}(\frac{y}{x}) \end{matrix}$

Then $\begin{matrix} x = r\cos\varphi \\ y = r\sin\varphi \end{matrix}$, $\left|\frac{\partial(x,y)}{\partial(r,\varphi)}\right| = r$

$$f_{R,F}(r, \varphi) = \left|\frac{\partial(x,y)}{\partial(r,\varphi)}\right| f_{\xi,\eta}(x(r,\varphi), y(r,\varphi)) = \frac{r}{2\pi} e^{-\frac{r^2}{2}}$$

$$P(\sqrt{x^2+y^2} \leq 2) = \int_0^2 dr \int_0^{2\pi} \frac{r}{2\pi} e^{-\frac{r^2}{2}} d\varphi = 1 - e^{-2}$$

N9. Let $\xi_1, \xi_2 \sim^{iid} N(0;1)$

Let $\begin{matrix} U = \xi_1 - 3\xi_2 \\ V = \xi_1 \end{matrix}$, $\begin{matrix} \xi_1 = x \\ \xi_2 = y \end{matrix}$, $\begin{matrix} U = u \\ V = v \end{matrix}$, $\begin{matrix} u = x - 3y \\ v = x \end{matrix} \Rightarrow \begin{matrix} x = v \\ y = \frac{v-u}{3} \end{matrix}$, $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left\| \begin{matrix} 0 & 1 \\ -\frac{1}{3} & \frac{1}{3} \end{matrix} \right\| = \frac{1}{3}$

$$f_{U,V}(u,v) = \frac{1}{3} f_{\xi_1,\xi_2}(x(u,v), y(u,v)) = \frac{1}{3} \cdot \frac{1}{2\pi} e^{-\frac{v^2}{2} - \frac{(v-u)^2}{18}}$$

$$f_U(u) = \int_{-\infty}^{+\infty} f_{U,V}(u,v)\, dv = \frac{1}{6\pi} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(\frac{10}{9}v^2 - \frac{2}{9}vu + \frac{1}{9}u^2)} dv = \frac{1}{6\pi} e^{-\frac{1}{20}u^2} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(\frac{\sqrt{10}}{3}v - \frac{1}{3\sqrt{10}}u)^2} dv = \frac{1}{\sqrt{20\pi}} e^{-\frac{u^2}{20}}$$

$P(\xi_1 < 3\xi_2) = \int_{-\infty}^{0} f_U(u)\, du = \Phi(0) = 0{,}5$, since $U \sim N(0; 10)$, $U = \sqrt{10}\,\eta_0$, where $\eta_0 \sim N(0;1)$

B) Let $U=\sqrt{3}\xi_2-\xi_1$, $V=\sqrt{3}\xi_2+\xi_1$, $\xi_1=x$, $\xi_2=y$, $U=u$, $V=v$, $u=\sqrt{3}y-x$, $v=\sqrt{3}y+x$ $\Rightarrow$ $x=\frac{v-u}{2}$, $y=\frac{v+u}{2\sqrt{3}}$, $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\left\|\begin{matrix}-\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}}\end{matrix}\right\|=\frac{1}{2\sqrt{3}}$

$$\sqrt{3}|\xi_2|>|\xi_1| \Leftrightarrow 3\xi_2^2>\xi_1^2 \Leftrightarrow (\sqrt{3}\xi_2-\xi_1)(\sqrt{3}\xi_2+\xi_1)>0 \Leftrightarrow \left[\begin{array}{l}\begin{cases}U>0\\V>0\end{cases}\\ \begin{cases}U<0\\V<0\end{cases}\end{array}\right.$$

disjoint events

$$f_{U,V}(u,v)=\left|\frac{\partial(x,y)}{\partial(u,v)}\right| f_{\xi_1,\xi_2}(x(u,v),y(u,v))=\frac{1}{2\sqrt{3}}\cdot\frac{1}{2\pi}e^{-\frac{1}{2}\left(\frac{(v-u)^2}{4}+\frac{(v+u)^2}{12}\right)}$$

$$P(U<u,\ V<v)=\int_{-\infty}^{u}dt_u\int_{-\infty}^{v}\frac{1}{4\pi\sqrt{3}}e^{-\frac{1}{2}\left(\frac{t_v^2-t_ut_v+t_v^2}{3}\right)}dt_v$$

$$P(\sqrt{3}|\xi_2|>|\xi_1|)=P((U>0,\ V>0)+(U<0,\ V<0))\overset{\text{disjoint}}{=}P(U>0,V>0)+P(U<0,V<0)=$$

$$=1-P(U<0,\ V<+\infty)-P(U<+\infty,\ V<0)+2P(U<0,V<0)$$

$$P(U<0,\ V<0)=\int_{-\infty}^{0}dt_u\int_{-\infty}^{0}\frac{1}{4\pi\sqrt{3}}e^{-\frac{1}{2}\left(\frac{t_v^2}{3}-\frac{2t_vt_u}{2\cdot 3}+\frac{t_u^2}{12}-\frac{t_u^2}{12}+\frac{t_u^2}{3}\right)}dt_v=\int_{-\infty}^{0}e^{-\frac{t_u^2}{8}}dt_u\int_{-\infty}^{0}\frac{1}{4\pi\sqrt{3}}e^{-\frac{1}{2}\left(\frac{t_v}{\sqrt{3}}-\frac{t_u}{2\sqrt{3}}\right)^2}dt_v=$$

$$=\int_{-\infty}^{0}\frac{1}{2\sqrt{2\pi}}e^{-\frac{t_u^2}{8}}dt_u\int_{-\infty}^{-\frac{t_u}{2\sqrt{3}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{t_v}{\sqrt{3}}-\frac{t_u}{2\sqrt{3}}\right)^2}d\left(\frac{t_v}{\sqrt{3}}-\frac{t_u}{2\sqrt{3}}\right)=\int_{-\infty}^{0}\frac{1}{2\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{t_u}{2}\right)^2}\Phi\left(-\frac{t_u}{2\sqrt{3}}\right)dt_u=$$

Wikipedia: integrals of Gaussian functions

$$=\frac{1}{2}\int_{-\infty}^{0}\varphi\left(\frac{t_u}{2}\right)\Phi\left(-\frac{t_u}{2\sqrt{3}}\right)dt_u=\frac{1}{2}\cdot\frac{1}{2\pi\cdot\frac{1}{2}}\left(\frac{\pi}{2}-\operatorname{arctg}\left(-\frac{2}{2\sqrt{3}}\right)\right)=\frac{1}{3}$$

$$P(U<0,\ V<+\infty)=\int_{-\infty}^{0}dt_u\int_{-\infty}^{+\infty}\frac{1}{4\pi\sqrt{3}}e^{-\frac{1}{2}\left(\frac{t_v}{\sqrt{3}}-\frac{t_u}{2\sqrt{3}}\right)^2-\frac{t_u^2}{8}}dt_v=\int_{-\infty}^{0}\frac{1}{2\sqrt{2\pi}}e^{-\frac{t_u^2}{8}}dt_u=\frac{1}{2}$$

Similarly, $P(U<+\infty,\ V<0)=\frac{1}{2}$

Therefore, $P(\sqrt{3}|\xi_2|>|\xi_1|)=1-1+\frac{2}{3}=\frac{2}{3}$

<u>N10.</u> $\xi,\eta\sim N(0;1)$ $\quad \frac{\operatorname{Cov}(\xi,\eta)}{\sqrt{\operatorname{Var}\xi\cdot\operatorname{Var}\eta}}=\rho$

It is known (though not mentioned in the problem statement) that ξ and η are bivariate normal.

$\operatorname{Var}\xi=\operatorname{Var}\eta=1 \Rightarrow \frac{\operatorname{Cov}(\xi,\eta)}{\sqrt{\operatorname{Var}\xi\cdot\operatorname{Var}\eta}}=\operatorname{Cov}(\xi,\eta)$

Since ξ and η are bivariate normal, $f_{\xi,\eta}(x,y)=\frac{1}{2\pi\sqrt{\det\Sigma}}e^{-\frac{1}{2}\langle\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]-\bar{\mu},\ \Sigma^{-1}\times\left(\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]-\bar{\mu}\right)\rangle}$,

where $\mu=\begin{bmatrix}0\\0\end{bmatrix}$, $\Sigma=\begin{bmatrix}1&\rho\\\rho&1\end{bmatrix}$

$\det\Sigma=1-\rho^2$, $\Sigma^{-1}=\frac{1}{\det\Sigma}\begin{bmatrix}1&-\rho\\-\rho&1\end{bmatrix}\Rightarrow f_{\xi,\eta}(x,y)=\frac{1}{2\pi\sqrt{1-\rho^2}}e^{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}}$

$$E(\xi^3\eta^3)=\int_{-\infty}^{+\infty}x^3dx\underbrace{\int_{-\infty}^{+\infty}y^3\cdot\frac{1}{2\pi\sqrt{1-\rho^2}}e^{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}}dy}_{(1)},\quad \text{let } g(x,y)=\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}$$

$$(1)=\frac{\sqrt{1-\rho^2}}{2\pi}\int_{-\infty}^{+\infty}y^2\cdot\frac{y}{1-\rho^2}e^{-g(x,y)}dy=\frac{\sqrt{1-\rho^2}}{2\pi}\left(\underbrace{-\int_{-\infty}^{+\infty}y^2\cdot\frac{\rho x-y}{1-\rho^2}\cdot e^{-g(x,y)}dy}_{(2)}+\underbrace{\int_{-\infty}^{+\infty}y^2\frac{\rho x}{1-\rho^2}e^{-g(x,y)}dy}_{(3)}\right)$$

$$(2)=-\int_{-\infty}^{+\infty}y^2\cdot\left(e^{-g(x,y)}\right)'_y dy=-\left(\underbrace{y^2\cdot e^{-g(x,y)}\Big|_{y=-\infty}^{+\infty}}_{(4)}\underbrace{-2\int_{-\infty}^{+\infty}ye^{-g(x,y)}dy}_{(5)}\right)$$

$$(3)=\int_{-\infty}^{+\infty} y^2\cdot\frac{\rho x}{1-\rho^2}\cdot e^{-g(x,y)}dy=-\rho x\int_{-\infty}^{+\infty} y\cdot\frac{-y}{1-\rho^2}e^{-g(x,y)}dy=-\rho x\Big(\underbrace{\int_{-\infty}^{+\infty} y\cdot\frac{\rho x-y}{1-\rho^2}e^{-g(x,y)}dy}_{(6)}-\underbrace{\int_{-\infty}^{+\infty} y\cdot\frac{\rho x}{1-\rho^2}e^{-g(x,y)}dy}_{\frac{\rho x}{1-\rho^2}\cdot(5)}\Big)$$

$$(4)=y^2\cdot e^{-g(x,y)}\Big|_{y=-\infty}^{+\infty}=\lim_{y\to+\infty}y^2e^{-g(x,y)}-\lim_{y\to-\infty}y^2e^{-g(x,y)}$$

$$\lim_{y\to+\infty}y^2e^{-g(x,y)}=\lim_{y\to+\infty}\frac{y^2}{e^{g(x,y)}}\overset{*}{=}\lim_{y\to+\infty}\frac{2y(1-\rho^2)}{e^{g(x,y)}(y-\rho x)}\overset{*}{=}\lim_{y\to+\infty}\frac{2(1-\rho)^2}{e^{g(x,y)}\cdot\left(\frac{(y-\rho x)^2}{1-\rho^2}-\rho x\right)}=0$$

*L'Hôpital's rule

$$\lim_{y\to-\infty}y^2e^{-g(x,y)}=\lim_{y\to-\infty}\frac{y^2e^{\frac{2\rho xy}{1-\rho^2}}}{e^{\frac{x^2+y^2}{1-\rho^2}}}=\frac{\left(\lim_{y\to-\infty}ye^{\frac{\rho xy}{1-\rho^2}}\right)^2}{\lim_{y\to-\infty}e^{\frac{x^2+y^2}{1-\rho^2}}}$$

$$\lim_{y\to-\infty}ye^{\frac{\rho xy}{1-\rho^2}}=\lim_{y\to-\infty}\frac{y}{e^{\frac{-\rho xy}{1-\rho^2}}}\overset{*}{=}\lim_{y\to-\infty}\frac{1}{e^{\frac{-\rho xy}{1-\rho^2}}\cdot\left(\frac{-\rho x}{1-\rho^2}\right)}=0\quad,\qquad\lim_{y\to-\infty}e^{\frac{x^2+y^2}{1-\rho^2}}=\infty$$

Therefore, $\lim_{y\to-\infty}y^2e^{-g(x,y)}=\frac{0}{\infty}=0\implies(4)=0$

$$(5)=\int_{-\infty}^{+\infty}ye^{-g(x,y)}dy=-(1-\rho^2)\int_{-\infty}^{+\infty}\frac{-y}{1-\rho^2}e^{-g(x,y)}dy=-(1-\rho^2)\Big(\int_{-\infty}^{+\infty}\frac{\rho x-y}{1-\rho^2}e^{-g(x,y)}dy-\int_{-\infty}^{+\infty}\frac{\rho x}{1-\rho^2}e^{-g(x,y)}dy\Big)=$$

$$=-(1-\rho^2)\Big(\underbrace{e^{-g(x,y)}\Big|_{y=-\infty}^{+\infty}}_{0}-\frac{\rho x}{1-\rho^2}\int_{-\infty}^{+\infty}e^{-\frac{1}{2}\left(\frac{x^2-2\rho xy+y^2}{1-\rho^2}\right)}dy=$$

$$=\rho x\cdot e^{-\frac{x^2}{2}}\int_{-\infty}^{+\infty}e^{-\frac{1}{2}\left(\frac{y}{\sqrt{1-\rho^2}}-\frac{\rho x}{\sqrt{1-\rho^2}}\right)^2}dy=\rho\sqrt{2\pi(1-\rho^2)}\,xe^{-\frac{x^2}{2}}$$

$$(6)=\int_{-\infty}^{+\infty}y\cdot\left(e^{-g(x,y)}\right)'_y dy=\Big(\underbrace{ye^{-g(x,y)}\Big|_{y=-\infty}^{+\infty}}_{0}-\int_{-\infty}^{+\infty}e^{-g(x,y)}dy\Big)=-\sqrt{2\pi(1-\rho^2)}\cdot e^{-\frac{x^2}{2}}$$

Therefore, $(3)=-\rho x\left(-\sqrt{2\pi(1-\rho^2)}\cdot e^{-\frac{x^2}{2}}-\frac{\rho^2}{1-\rho^2}\sqrt{2\pi(1-\rho^2)}\cdot x^2\cdot e^{-\frac{x^2}{2}}\right)$

$(2)=-\left(0-2\rho\sqrt{2\pi(1-\rho^2)}\,xe^{-\frac{x^2}{2}}\right)$

$(1)=\frac{\sqrt{1-\rho^2}}{2\pi}\left(2\rho\sqrt{2\pi(1-\rho^2)}\,xe^{-\frac{x^2}{2}}+\frac{\sqrt{2\pi}}{\sqrt{1-\rho^2}}\rho xe^{-\frac{x^2}{2}}(1-\rho^2(1-x^2))\right)$

$$E(\xi^3\eta^3)=\int_{-\infty}^{+\infty}x^3\cdot\frac{1-\rho^2}{\sqrt{2\pi}}\left(2\rho xe^{-\frac{x^2}{2}}+\frac{\rho}{1-\rho^2}xe^{-\frac{x^2}{2}}(1-\rho^2(1-x^2))\right)dx=\int_{-\infty}^{+\infty}\frac{\rho x^4e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}(3-3\rho^2+\rho^2x^2)dx=$$

$$=\frac{3\rho}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}x^4e^{-\frac{x^2}{2}}dx-\frac{3\rho^3}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}x^4e^{-\frac{x^2}{2}}dx+\frac{\rho^3}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}x^6e^{-\frac{x^2}{2}}dx$$

Integrating by parts yields:

$$\int_{-\infty}^{+\infty}x^4e^{-\frac{x^2}{2}}dx=3\sqrt{2\pi}\,,\qquad\int_{-\infty}^{+\infty}x^6e^{-\frac{x^2}{2}}dx=15\sqrt{2\pi}$$

Therefore, $E(\xi^3\eta^3)=9\rho-9\rho^3+15\rho^3=9\rho+6\rho^3$

N12. Let $\xi_1, \xi_2, \ldots,$ $\xi_i \sim \begin{pmatrix} -\sqrt{i} & 0 & \sqrt{i} \\ \frac{1}{2i} & 1-\frac{1}{i} & \frac{1}{2i} \end{pmatrix}$, $S_n = \sum_{i=1}^{n} \xi_i$ $ES_n = \sum_{i=1}^{n} E\xi_i = 0 \cdot n = 0$

LLN: $P(|\frac{S_n}{n}| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0 \quad \forall \varepsilon > 0$

$Var\, S_n \underset{\uparrow}{=} \sum_{i=1}^{n} Var\, \xi_i = \sum_{i=1}^{n} (E\xi_i^2 - (E\xi_i)^2) = \sum_{i=1}^{n} 1 = n \Rightarrow Var\frac{S_n}{n} = \frac{1}{n^2} Var\, S_n = \frac{1}{n}$

ξ_i are indep.

According to Chebyshev's inequality, $P(|\frac{S_n}{n}| \geq \varepsilon) \leq \frac{Var\frac{S_n}{n}}{\varepsilon^2}$, $\frac{Var\frac{S_n}{n}}{\varepsilon^2} = \frac{1}{n\varepsilon^2} \xrightarrow[n\to\infty]{} 0$

According to the properties of probability, $P(|\frac{S_n}{n}| \geq \varepsilon) \geq 0$

Therefore, according to the squeezing theorem, $\lim_{n\to\infty} P(|\frac{S_n}{n}| \geq \varepsilon) = 0 \quad \forall \varepsilon > 0$

N13. Let $\xi_1, \xi_2, \ldots,$ $\xi_i \sim \begin{pmatrix} -i & 0 & i \\ 2^{-i} & 1-2^{-i+1} & 2^{-i} \end{pmatrix}$, $S_n = \sum_{i=1}^{n} \xi_i$, $ES_n = \sum_{i=1}^{n} E\xi_i = 0 \cdot n = 0$

LLN: $P(|\frac{S_n}{n}| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0 \quad \forall \varepsilon > 0$

$$Var\, S_n \underset{\uparrow}{=} \sum_{i=1}^{n} Var\, \xi_i = \sum_{i=1}^{n} (E\xi_i^2 - (E\xi_i)^2) = \sum_{i=1}^{n} \frac{2i^2}{2^i} = \sum_{i=0}^{n} \frac{2(i+1)^2}{2^{i+1}} = \underbrace{\sum_{i=0}^{n} \frac{i^2}{2^i}}_{= \sum_{i=1}^{n} \frac{i^2}{2^i}} + \sum_{i=0}^{n} \frac{4i}{2^{i+1}} + \sum_{i=0}^{n} \frac{1}{2^i} \Rightarrow$$

ξ_i are indep.

$$\Rightarrow 2\sum_{i=1}^{n} \frac{i^2}{2^i} = \sum_{i=1}^{n} \frac{i^2}{2^i} + 2\sum_{i=1}^{n} \frac{i}{2^i} + \sum_{i=0}^{n} \frac{1}{2^i} \qquad \left(\text{similarly, } \sum_{i=1}^{n} \frac{i}{2^i} = 2\sum_{i=1}^{n} \frac{1}{2^i} = \sum_{i=0}^{n} \frac{1}{2^i}\right)$$

Therefore, $\sum_{i=1}^{n} \frac{i^2}{2^i} = 3\sum_{i=0}^{n} \frac{1}{2^i} = \frac{Var\, S_n}{2} \Rightarrow Var\, S_n = 6\sum_{i=0}^{n} \frac{1}{2^i} \Rightarrow Var\frac{S_n}{n} = \frac{1}{n^2} Var\, S_n = \frac{6}{n^2}\sum_{i=0}^{n} \frac{1}{2^i}$

Using Chebyshev's inequality: $0 \leq P(|\frac{S_n}{n}| \geq \varepsilon) \leq \frac{Var\frac{S_n}{n}}{\varepsilon^2} = \frac{6}{n^2\varepsilon^2} \underbrace{\sum_{i=0}^{n} \frac{1}{2^i}}_{\xrightarrow[n\to\infty]{} 2} \xrightarrow[n\to\infty]{} 0$

Therefore, according to the squeezing theorem, $\lim_{n\to\infty} P(|\frac{S_n}{n}| \geq \varepsilon) = 0 \quad \forall \varepsilon > 0$

N11. The problem statement is incomplete and, possibly, incorrect, so let's assume the following:

- $E\xi_i = \mu \quad \forall i$
- $\frac{Var\, S_n}{n} \xrightarrow[n\to\infty]{} 0$, as opposed to $\frac{Var\, \xi_n}{n} \xrightarrow[n\to\infty]{} 0$ $\qquad (S_n = \sum_{i=1}^{n} \xi_i)$
- We need to prove the LLN in the form $P(|\frac{S_n}{n} - \mu| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0$

□ $ES_n = E\sum_{i=1}^{n} \xi_i = \sum_{i=1}^{n} E\xi_i = n\mu$, $\quad E\frac{S_n}{n} = \frac{1}{n} ES_n = \mu$

$Var\, S_n \underset{\nearrow}{=} \sum_{i=1}^{n} Var\, \xi_i$, $\quad Var\frac{S_n}{n} = \frac{1}{n^2} Var\, S_n$

ξ_i are indep.

According to Chebyshev's inequality, $P(|\frac{S_n}{n} - \mu| \geq \varepsilon) \leq \frac{Var\frac{S_n}{n}}{\varepsilon^2} = \frac{Var\, S_n}{n \cdot n\varepsilon^2} \xrightarrow[n\to\infty]{} 0$

Since $P(x) \geq 0 \ \forall x$, we can use the squeezing theorem to show that $\lim_{n\to\infty} P(|\frac{S_n}{n} - \mu| \geq \varepsilon) = 0 \quad \forall \varepsilon > 0$