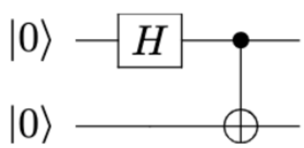
# Quantum Computing: An Applied Approach

## Chapter 3 Problems: Qubits, Operators and Measurement

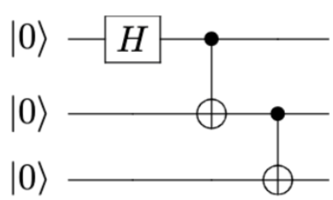
### 1

#### (a)



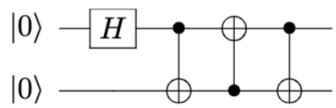
Qubit states transform as $|00\rangle \rightarrow \frac{|00\rangle+|11\rangle}{\sqrt{2}} \rightarrow \boxed{\frac{|00\rangle+|11\rangle}{\sqrt{2}}}$

#### (b)



Qubit states transform as $|000\rangle \rightarrow \frac{|000\rangle+|100\rangle}{\sqrt{2}} \rightarrow \frac{|000\rangle+|110\rangle}{\sqrt{2}} \rightarrow \boxed{\frac{|000\rangle+|111\rangle}{\sqrt{2}}}$

### (c)



Qubit states transform as $|00\rangle \rightarrow \frac{|00\rangle+|10\rangle}{\sqrt{2}} \rightarrow \frac{|00\rangle+|11\rangle}{\sqrt{2}} \rightarrow \frac{|00\rangle+|01\rangle}{\sqrt{2}} \rightarrow \boxed{\dfrac{|00\rangle + |01\rangle}{\sqrt{2}}}$

## 2

### (a)

$|0\rangle \sim \boxed{(0, 0, 1)}$

### (b)

$|0\rangle \sim \boxed{(0, 0, -1)}$

### (c)

$\frac{|0\rangle+|1\rangle}{\sqrt{2}} \sim \boxed{(1, 0, 0)}$

### (d)

$\frac{|0\rangle+e^{i\phi}|1\rangle}{\sqrt{2}} \sim \boxed{(cos(\phi), sin(\phi), 0)}$
For the given parameterizations of $\phi$:

$$\phi = 0 : \boxed{(1, 0, 0)}$$
$$\phi = \frac{\pi}{2} : \boxed{(0, 1, 0)}$$
$$\phi = \pi : \boxed{(-1, 0, 0)}$$
$$\phi = \frac{3\pi}{2} : \boxed{(0, -1, 0)}$$

### (e)

$\frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle \sim \boxed{(0.96, 0, -0.28)}$

**Note**:
The general expression for a pure qubit state is $|\psi\rangle = cos(\frac{\theta}{2}) + e^{i\phi}sin(\frac{\theta}{2})|1\rangle$, while the corresponding qubit state can be represented by the point on the point $(sin(\theta)cos(\phi), sin(\theta)sin(\phi), cos(\theta))$ on the Bloch sphere. For a given qubit state of the form $|\psi\rangle = a|0\rangle + b|1\rangle$ with $a, b \in \mathbb{C}$, the angle $\theta$ satisfies $\theta = 2\text{Arctan}(\frac{a}{b})$ while $\phi = Arg(b)$.

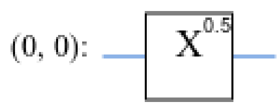
For the state $\frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle$, $\theta = 2\text{Arctan}(\frac{3}{4}) = 1.855$ radians and $\phi = 0$ so that $(x, y, z) = (0.96, 0, -0.28)$.

# 3

## (a)

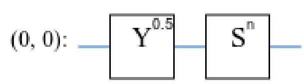
The state $\frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle$ lies in the xz-plane, so it can be reached by rotating an initialized $|0\rangle$ state about the y-axis.

As calculated in part 2(e), this corresponds to a rotation angle $\theta = 0.59\pi$ so that this circuit can be modeled as:




## (b)

The state $\frac{|0\rangle + e^{i\phi}|1\rangle}{\sqrt{2}}$ where $\phi \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ lies on the equator of the Bloch sphere, at a rotation of $\phi$ radians about the z-axis. To form this state from an initialization of a qubit in the $|0\rangle$ state, rotate the $|0\rangle$ state onto the equator using a square root-Y gate, staying in the xz-plane, then perform an additional azimuthal rotation in increments of $\frac{\pi}{2}$ using successive S gates:




where $n = \frac{2\phi}{\pi}$.

# 4

## (a)

No, this is a subset of the Clifford group of operators that, by the Gottesman-Knill Theorem, generates only the set of quantum algorithms efficiently simulatable on a classic computer.

### (b)

No, this is equivalent to the generators of the Clifford group.

### (c)

Yes, this is a superset of the universal gate set {CNOT,T,H}.

### (d)

Yes, this is equal to the universal gate set {CNOT,T,H}.

### (e)

No, the CZ gate just swaps between basis elements and $H$ and $S$ are generators of the Clifford group. In fact, the CZ gate can be formed by inserting a Hadamard gate before and after the X operation on the target qubit line. This gate set can then be generated by the Clifford group.

### (f)

This gate set is universal, as a CNOT can likewise be formed from a CZ gate by inserting a Hadamard before and after the X gate on the target line. The universal generating set {CNOT,T,H} can then be formed from these gates, from which any unitary can be generated.

### (g)

Yes, CNOT gates with arbitrary single-qubit unitary rotations form a universal gate set.

### (h)

Yes; the Hadamard is included in the set of unitary single-qubit rotations, and when placed before and after the $Z$ gate in the target line of the CZ can form a CNOT gate. Then the set {CNOT, U} generates {CZ, U} so that this gate set is also universal.

## 5

Let $\sigma \in \{X, Y, Z\}$.

$$e^{i\theta\sigma} = \sum_{n=0}^{\infty} \frac{(i\theta\sigma)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (\theta\sigma)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^n (\theta\sigma)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(\theta)^{2n}}{(2n)!} I + i \sum_{n=0}^{\infty} \frac{(-1)^n(\theta)^{2n+1}\sigma}{(2n+1)!} \text{ (by the property } \sigma^2 = I) =$$

$$\boxed{cos(\theta)I + i\sin(\theta)\sigma}$$

# 6

$$R_x(\theta) \coloneqq e^{\frac{i\theta X}{2}} = \cos(\frac{\theta}{2})I + i\sin(\frac{\theta}{2})X = \begin{pmatrix} \cos(\frac{\theta}{2}) & i\sin(\frac{\theta}{2}) \\ i\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}$$

$$R_y(\theta) \coloneqq e^{\frac{i\theta Y}{2}} = \cos(\frac{\theta}{2})I + i\sin(\frac{\theta}{2})Y = \begin{pmatrix} \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \\ -\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}$$

$$R_z(\theta) \coloneqq e^{\frac{i\theta Z}{2}} = \cos(\frac{\theta}{2})I + i\sin(\frac{\theta}{2})Z = \begin{pmatrix} \cos(\frac{\theta}{2}) + i\sin(\frac{\theta}{2}) & 0 \\ 0 & \cos(\frac{\theta}{2}) - i\sin(\frac{\theta}{2}) \end{pmatrix}$$

# 7

## (a)

$$R_x(\theta_2)R_x(\theta_1) = \begin{pmatrix} \cos(\frac{\theta_2}{2}) & i\sin(\frac{\theta_2}{2}) \\ i\sin(\frac{\theta_2}{2}) & \cos(\frac{\theta_2}{2}) \end{pmatrix} \cdot \begin{pmatrix} \cos(\frac{\theta_1}{2}) & i\sin(\frac{\theta_1}{2}) \\ i\sin(\frac{\theta_1}{2}) & \cos(\frac{\theta_1}{2}) \end{pmatrix} =$$

$$\begin{pmatrix} \cos(\frac{\theta_2}{2})\cos(\frac{\theta_1}{2}) - \sin(\frac{\theta_2}{2})\sin(\frac{\theta_1}{2}) & i[\cos(\frac{\theta_2}{2})\sin(\frac{\theta_1}{2}) + \cos(\frac{\theta_1}{2})\sin(\frac{\theta_2}{2})] \\ i[\cos(\frac{\theta_2}{2})\sin(\frac{\theta_1}{2}) + \cos(\frac{\theta_1}{2})\sin(\frac{\theta_2}{2})] & \cos(\frac{\theta_2}{2})\cos(\frac{\theta_1}{2}) - \sin(\frac{\theta_2}{2})\sin(\frac{\theta_1}{2}) \end{pmatrix} =$$

$$\begin{pmatrix} \cos(\frac{\theta_2+\theta_1}{2}) & i\sin(\frac{\theta_2+\theta_1}{2}) \\ i\sin(\frac{\theta_2+\theta_1}{2}) & \cos(\frac{\theta_2+\theta_1}{2}) \end{pmatrix} = R_x(\theta_1 + \theta_2)$$

## (b)

$$R_y(\theta_2)R_y(\theta_1) = \begin{pmatrix} \cos(\frac{\theta_2}{2}) & \sin(\frac{\theta_2}{2}) \\ -\sin(\frac{\theta_2}{2}) & \cos(\frac{\theta_2}{2}) \end{pmatrix} \cdot \begin{pmatrix} \cos(\frac{\theta_1}{2}) & \sin(\frac{\theta_1}{2}) \\ -\sin(\frac{\theta_1}{2}) & \cos(\frac{\theta_1}{2}) \end{pmatrix} =$$

$$\begin{pmatrix} \cos(\frac{\theta_2}{2})\cos(\frac{\theta_1}{2}) - \sin(\frac{\theta_2}{2})\sin(\frac{\theta_1}{2}) & \cos(\frac{\theta_2}{2})\sin(\frac{\theta_1}{2}) + \cos(\frac{\theta_1}{2})\sin(\frac{\theta_2}{2}) \\ -\cos(\frac{\theta_2}{2})\sin(\frac{\theta_1}{2}) - \cos(\frac{\theta_1}{2})\sin(\frac{\theta_2}{2}) & \cos(\frac{\theta_2}{2})\cos(\frac{\theta_1}{2}) - \sin(\frac{\theta_2}{2})\sin(\frac{\theta_1}{2}) \end{pmatrix} =$$

$$\begin{pmatrix} \cos(\frac{\theta_2+\theta_1}{2}) & \sin(\frac{\theta_2+\theta_1}{2}) \\ -\sin(\frac{\theta_2+\theta_1}{2}) & \cos(\frac{\theta_2+\theta_1}{2}) \end{pmatrix} = R_y(\theta_1 + \theta_2)$$

## (c)

$$R_z(\theta_2)R_z(\theta_1) = \begin{pmatrix} \cos(\frac{\theta_2}{2}) + i\sin(\frac{\theta_2}{2}) & 0 \\ 0 & \cos(\frac{\theta_2}{2}) - i\sin(\frac{\theta_2}{2}) \end{pmatrix} \cdot \begin{pmatrix} \cos(\frac{\theta_1}{2}) + i\sin(\frac{\theta_1}{2}) & 0 \\ 0 & \cos(\frac{\theta_1}{2}) - i\sin(\frac{\theta_1}{2}) \end{pmatrix} =$$

$$\begin{pmatrix} \begin{matrix} \cos(\frac{\theta_2}{2})\cos(\frac{\theta_1}{2}) + i\cos(\frac{\theta_2}{2})\sin(\frac{\theta_1}{2}) + \\ i\cos(\frac{\theta_1}{2})\sin(\frac{\theta_2}{2}) - \sin(\frac{\theta_2}{2})\sin(\frac{\theta_1}{2}) \end{matrix} & 0 \\ 0 & \begin{matrix} \cos(\frac{\theta_2}{2})\cos(\frac{\theta_1}{2}) - i\cos(\frac{\theta_2}{2})\sin(\frac{\theta_1}{2}) - \\ i\cos(\frac{\theta_1}{2})\sin(\frac{\theta_2}{2}) - \sin(\frac{\theta_2}{2})\sin(\frac{\theta_1}{2}) \end{matrix} \end{pmatrix} =$$

$$\begin{pmatrix} \cos(\frac{\theta_2+\theta_1}{2}) + i\sin(\frac{\theta_2+\theta_1}{2}) & 0 \\ 0 & \cos(\frac{\theta_2+\theta_1}{2}) - i\sin(\frac{\theta_2+\theta_1}{2}) \end{pmatrix} = R_z(\theta_1 + \theta_2)$$

# 8

The field of complex numbers $\mathbb{C}$ is one of the simplest unordered fields, ensuring that the gauge condition $|0\rangle + e^{i\phi}|1\rangle = |0\rangle + e^{i(\phi+2\pi)}|1\rangle$ holds. A real system does not suffice because there are only two roots of unity, as opposed to infinite in the complex plane. The Hilbert space of a five-qubit system can be represented as the product of five Hilbert spaces, and is isomorphic to $\mathbb{C}^5$.

# 9

Measuring in the standard basis, $|\langle 0|\psi\rangle|^2 = \boxed{0.36}$, $|\langle 1|\psi\rangle|^2 = \boxed{0.64}$

# 10

After measurement in the $|0\rangle$ state, $|\psi\rangle = |0\rangle$. The probability of measuring the $|+\rangle$ state is $|\langle 0|\frac{|0\rangle+|1\rangle}{\sqrt{2}}\rangle|^2 = \boxed{\frac{1}{2}}$, and the probability of measuring in the $|-\rangle$ state is $|\langle 0|\frac{|0\rangle-|1\rangle}{\sqrt{2}}\rangle|^2 = \boxed{\frac{1}{2}}$

# 11

The number of unique measurable states in a system of $n$ $d-$dits is $d^n$, so $4^n = 2^{10^6} \implies 2n = 10^6$, $n = \boxed{5 \cdot 10^5}$.

# 12

## (a)

$$HXH = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} =$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$$

**(b)**

$$HZH = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} =$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$$

**(c)**

$$HYH = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} =$$

$$\begin{pmatrix} \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -Y$$

**(d)**

$$H^2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

**(e)**

$$CNOT_{ij}CNOT_{ji}CNOT_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = SWAP_{ij}$$

**(f)**

$$R_{z,1}(\theta)CNOT_{1,2} = \begin{pmatrix} cos(\frac{\theta}{2}) + isin(\frac{\theta}{2}) & 0 & 0 & 0 \\ 0 & cos(\frac{\theta}{2}) - isin(\frac{\theta}{2}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} cos(\frac{\theta}{2}) + isin(\frac{\theta}{2}) & 0 & 0 & 0 \\ 0 & cos(\frac{\theta}{2}) - isin(\frac{\theta}{2}) & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} cos(\frac{\theta}{2}) + isin(\frac{\theta}{2}) & 0 & 0 & 0 \\ 0 & cos(\frac{\theta}{2}) - isin(\frac{\theta}{2}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = CNOT_{1,2}R_{z,1}(\theta)$$