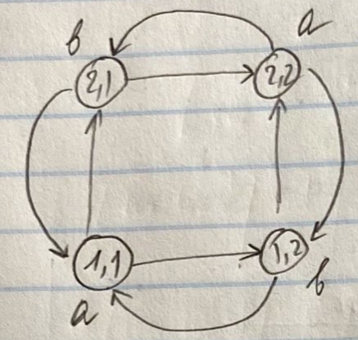
Final exam alena3

1) Gibbs sampling

a)



$$p(1,1)(1,2) = \frac{1}{d} \cdot \frac{p_{1,2}}{p_{1,1}+p_{1,2}} = \frac{1}{2}\,\frac{b}{a+b}$$

$$p(1,1)(2,1) = \frac{b}{2(a+b)}$$

$$p(1,1)(1,1) = \frac{1}{d}\,\frac{a}{a+b} + \frac{1}{d}\,\frac{a}{a+b} = \frac{a}{2(a+b)} + \frac{a}{2(a+b)}$$

$$= \frac{a}{a+b}$$

$$p(1,2)(1,2) = \frac{b}{(a+b)}$$

$$p(1,2)(1,1) = \frac{1}{2}\frac{a}{(a+b)}$$, and so on, so the matrix is

$$P = \begin{array}{c} \\ p(1,1) \\ p(1,2) \\ p(2,1) \\ p(2,2) \end{array} \begin{array}{c} \begin{array}{cccc} p(1,1) & p(1,2) & p(2,1) & p(2,2) \end{array} \\ \begin{bmatrix} \frac{a}{a+b} & \frac{b}{2(a+b)} & \frac{b}{2(a+b)} & 0 \\ \frac{a}{2(a+b)} & \frac{b}{a+b} & 0 & \frac{a}{2(a+b)} \\ \frac{a}{2(a+b)} & 0 & \frac{b}{a+b} & \frac{a}{2(a+b)} \\ 0 & \frac{b}{2(a+b)} & \frac{b}{2(a+b)} & \frac{a}{a+b} \end{bmatrix} \end{array}$$

$a+a+b+b=1 \Rightarrow 2(a+b)=1 \Rightarrow$ matrix is

$$P = \begin{array}{c} P_{11} \\ P_{12} \\ P_{21} \\ P_{22} \end{array} \begin{bmatrix} 2a & b & b & 0 \\ a & 2b & 0 & a \\ a & 0 & 2b & a \\ 0 & b & b & 2a \end{bmatrix}$$

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b) To verify the answer:

$\pi P = \pi$

$$[a \; b \; b \; a] \begin{bmatrix} 2a & b & b & 0 \\ a & 2b & 0 & a \\ a & 0 & 2b & a \\ 0 & b & b & 2a \end{bmatrix}$$

$$= [(2a^2 + 2ab) \; (2b^2 + 2ab) \; (2b^2 + 2ab) \; (2ab + 2a^2)] \; (=)$$

since $2(a+b) = 1 \Rightarrow$ $2a(a+b) = a$
$2b(a+b) = b$

$(=) \; [a \; b \; b \; a]$. So, $\pi P = \pi$ is verified.

2) Linear separability

a) $P = \{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \}$, $N = \{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \}$

Let w be $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, substituting values to the LP formulation

$$\begin{cases} [w_1 \; w_2] \begin{bmatrix} 0 \\ 1 \end{bmatrix} + b \geq 1 \\ [w_1 \; w_2] \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \geq 1 \\ [w_1 \; w_2] \begin{bmatrix} 0 \\ 0 \end{bmatrix} + b \leq -1 \\ [w_1 \; w_2] \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \leq -1 \end{cases} \Rightarrow \begin{cases} w_2 + b \geq 1 \\ w_1 + b \geq 1 \\ b \leq -1 \\ w_1 + w_2 + b \leq -1 \end{cases}$$

$\Downarrow$

$w_2 \geq 1 - b$
$w_1 \geq 1 - b$

$(1-b) + (1-b) + b \leq -1$

$1 - b + 1 - \cancel{b} + \cancel{b} \leq -1$

$3 - b \leq 0$

$b \geq 3$ contradicts $b \leq -1 \Rightarrow$ no solution.

Since the system of inequalities does not have a solution $\Rightarrow$ set is linearly inseparable.

b) $P = \left\{ \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$N = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

Let $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, substituting values to the LP:

$$\begin{cases} [w_1\ w_2] \begin{bmatrix} -1 \\ -1 \end{bmatrix} + b \geq 1 \\ [w_1\ w_2] \begin{bmatrix} 0 \\ 0 \end{bmatrix} + b \geq 1 \\ [w_1\ w_2] \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \geq 1 \\ [w_1\ w_2] \begin{bmatrix} -1 \\ 1 \end{bmatrix} + b \leq -1 \\ [w_1\ w_2] \begin{bmatrix} 1 \\ -1 \end{bmatrix} + b \leq -1 \end{cases} \Rightarrow \begin{cases} -w_1 - w_2 + b \geq 1 \\ b \geq 1 \\ w_1 + w_2 + b \geq 1 \\ -w_1 + w_2 + b \leq -1 \\ w_1 - w_2 + b \leq -1 \end{cases}$$

$w_1 \geq 1 - b - w_2$

$w_2 \geq 1 - b - w_1$

$-w_1 + 1 - \cancel{b} - w_1 + \cancel{b} \leq -1$

$-2w_1 \leq -2$

$w_1 \geq 1$

$-(w_1 + w_2) \geq 1 - b$

$w_1 + w_2 \leq b - 1$

$1 + w_2 \leq b - 1$

$w_2 \leq b - 2$

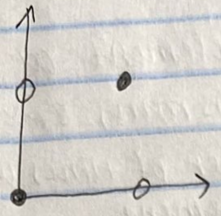
$1 - w_2 + b \leq -1$

$-w_2 \leq -2 - b$

contradicts each other $\begin{cases} w_2 \geq b + 2 \\ w_2 \leq b - 2 \end{cases}$

$\Rightarrow$ no solution $\Rightarrow$ set is not linearly separable

3) a) $P = \{(0,1)^T, (1,0)^T\}$, $N = \{(0,0)^T, (1,1)^T\}$
is not linearly separable in R^2



try to increase the dimensions: to R^3 s.t.

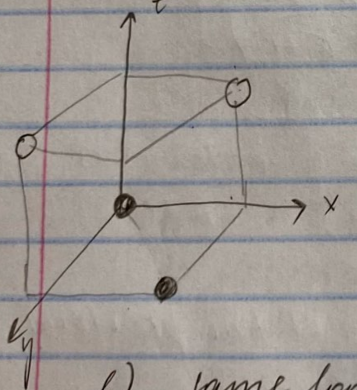
$$\begin{bmatrix} x_i \\ y_i \\ f(x_i, y_i) \end{bmatrix}$$

let $f(x_i, y_i)$ be a $\boxed{\sin\left((x_i + y_i) \times \frac{\pi}{2}\right)}$

$$\Rightarrow P = \left\{ \begin{bmatrix} 0 \\ 1 \\ \sin\left((0+1)\cdot\frac{\pi}{2}\right) = 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ \sin\left((1+0)\cdot\frac{\pi}{2}\right) = 1 \end{bmatrix} \right\}$$

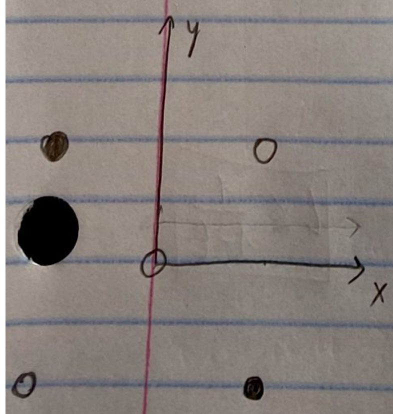
$$N = \left\{ \begin{bmatrix} 0 \\ 0 \\ \sin\left((0+0)\cdot\frac{\pi}{2}\right) = 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ \sin\left(2\cdot\frac{\pi}{2}\right) = \sin\pi = 0 \end{bmatrix} \right\}$$

new set in R^3 is $P = \{(0,1,1)^T, (1,0,1)^T\}$
$N = \{(0,0,0)^T, (1,1,0)^T\}$



now it is linearly separable by the plane $\parallel xy$, $z \in (0,1)$.
exactly, $z = 0.5$

b) same logic as in a, but let $f(x_i, y_i)$ be xy



$$P = \left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

$$N = \left(\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right)$$

Now, in R^3, the set is linearly separable by the plane $\parallel xy$, $z \in (0,-1)$
exactly, $z = -0.5$

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4) $\forall_j \; |f_j - \hat{f}_j| \leq \varepsilon n$, for $\varepsilon > 0$

$\hat{f}_j \geq f_j - \sum_t \delta_t$

δ_t - min element at time t, since δ_t is min, $\delta_t \approx 1$

t - number of times we delete an element.

$\Rightarrow \hat{f}_j \geq f_j - t$

$0 \leq \sum_j \hat{f}_j \leq \sum_t (1 -$

lower bound for $\sum_j \hat{f}_j$

each step, when we delete min element from each key, we empty at least one key

$\frac{l-1}{l}$ elements will lose that min number of keys.

$\Rightarrow 0 \leq \sum_j \hat{f}_j \leq \sum_t \left(1 - \frac{l-1}{l}\right)$

$= n - \frac{l-1}{l} t \geq 0$

$\frac{l-1}{l} t \geq n$

$\hat{f}_j - f_j \geq -t$

$|f_j - \hat{f}_j| \leq t$

$f_j - \hat{f}_j \leq t \leq \varepsilon n$

$t \geq \frac{n(l-1)}{l}$

$\varepsilon n \geq \frac{n(l-1)}{l}$

$\varepsilon \geq \frac{l-1}{l} \Rightarrow$

$\varepsilon l \geq l - 1$

$\varepsilon l - l \geq -1$

$l(\varepsilon - 1) \geq -1$

$-l(\varepsilon - 1) \leq 1$

$\boxed{l \leq \frac{1}{1-\varepsilon}}$

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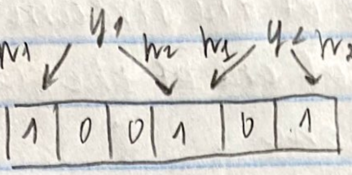
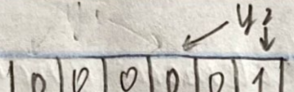
5. a) suppose we have $n = 6$ num of slots and
$k = 2$ hash functions

suppose we have y_1 and y_2 (elements)



so, h_2 hashes y_1 to the same slot as h_1 hashes y_2.

suppose we want to naively delete y_1, by zeroing out all it's hashes, so that,



Now, if we want to find y_2, it would be impssible since the value of $h_1(y_2) = 0$ (false). This will lead to the false negative error.

b) counting bloom filter counts how many times paritcular slot has been selected by hashing the elements. When we insert the element, the number of bits in slots where this element hashes to will be increased by 1. On contrast, when we delete an element, the number of bits in the slots where this element hashes to will be decremented.

c) The number of bits $= 4$ is sufficient for a counting bloom filter since the probability for specific slot to have 16 object is almost zero, and in order to store up to 16 objects we need only 4 bits

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Q6) Suppose we have s estimates which track some values from the stream.
We are at some point of the stream, say k+1.
If the $k+1^{th}$ value of the stream is picked, we drop the value of one of the old s estimators and track a $k+1^{th}$ one.
Before that, the probability of previous k-stream elements were $\frac{s}{k}$, since it was chosen uniformly.

Probability that $k+1^{th}$ element is picked: $\frac{s}{k+1}$

Probability that $k+1^{th}$ element is not picked: $1-\frac{s}{k+1}$

. One of the estimates is uniformly randomly chosen to track new element. $P(s \text{ is chosen}) = \frac{1}{s}$,
$P(s \text{ is not chosen}) = 1-\frac{1}{s}$.
Thus, probability of selecting k previous elements is

$$\frac{s}{k}\left[\left(1-\frac{s}{k+1}\right)+\left(1-\frac{1}{s}\right)\left(\frac{s}{k+1}\right)\right]$$

$$= \frac{s}{k}\left[\frac{k+1-s}{k+1} + \frac{s-1}{s}\,\frac{s}{k+1}\right] = \frac{s}{k}\left[\frac{k+1-\cancel{s}+\cancel{s}-\cancel{1}}{k+1}\right] = \frac{s}{k+1}$$

after observing $k+1^{th}$ element,
⇒ the algorithms picks any of the previous member with an equal probability $\left(\frac{s}{k+1}\right)$