Task 2 Perceptron

1) The classification function for the perceptron function is given by

$\hat{f}(x) = w^T x + b$

with x being training data $\in \mathbb{R}^d$ and w a weightvector assigning a weight to each variable of x

We can rewrite $\hat{f}(x)$ to

$\hat{f}(x) = w^T x$

by prepending the bias b to our weightvector (or appending) and adding a 1 at the respective position in x

For binary classification ($y \in \{-1, 1\}$) data is then classified as:

$$y_i = \begin{cases} 1 & \text{if } \hat{f}(x_i) > 0 \\ -1 & \text{if } \hat{f}(x_i) < 0 \end{cases}$$

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2) $w_{init} = [1 \ -1 \ 0{,}5]$ with $0{,}5$ being b

$\hat{f}(x) = (1 \ -1 \ 0{,}5) \cdot x$

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$\hat{f}(x_1) = (1 \ -1 \ 0{,}5) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0{,}5; \ 0{,}5 \cdot (-1) = -0{,}5 \quad (\hat{f}(x_i) y_i)$

→ wrongly classified

$w_{new} = w_{old} - 0{,}6 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \text{sign}\left(\hat{f}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)\right)$

$= \begin{pmatrix} 1 \\ -1 \\ 0{,}5 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0{,}6 \end{pmatrix} \cdot 1 = \begin{pmatrix} 1 \\ -1 \\ -0{,}1 \end{pmatrix}$

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$\hat{f}(x_1) = (1 \ -1 \ -0{,}1) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -0{,}1; \ -0{,}1 \cdot (-1) = 0{,}1$ ✓

$\hat{f}(x_2) = (1 \ -1 \ -0{,}1) \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = -1{,}1; \ -1{,}1 \cdot 1 = -1{,}1$

→ wrongly classified!

$w_{new} = \begin{pmatrix} 1 \\ -1 \\ -0{,}1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0{,}6 \\ 0{,}6 \end{pmatrix} \cdot (-1) = \begin{pmatrix} 1 \\ -0{,}4 \\ 0{,}5 \end{pmatrix}$

$$\hat{f}(x_1) = (1 \;\; -0{,}4 \;\; 0{,}5) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0{,}5 \;;\; 0{,}5 \cdot (-1) = -0{,}5$$

$\to$ wrongly classified

$$w_{new} = \begin{pmatrix} 1 \\ -0{,}4 \\ 0{,}5 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0{,}6 \end{pmatrix} \cdot 1 = \begin{pmatrix} 1 \\ -0{,}4 \\ -0{,}1 \end{pmatrix}$$

$$\hat{f}(x_1) = (1 \;\; -0{,}4 \;\; -0{,}1) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -0{,}1 \;;\; -0{,}1 \cdot (-1) = 0{,}1 \checkmark$$

$$\hat{f}(x_2) = (1 \;\; -0{,}4 \;\; -0{,}1) \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = -0{,}5 \;;\; -0{,}5 \cdot 1 = -0{,}5$$

$\to$ wrongly classified

$$w_{new} = \begin{pmatrix} 1 \\ -0{,}4 \\ -0{,}1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0{,}6 \\ 0{,}6 \end{pmatrix} \cdot (-1) = \begin{pmatrix} 1 \\ 0{,}2 \\ 0{,}5 \end{pmatrix}$$

$$\hat{f}(x_1) = (1 \;\; 0{,}2 \;\; 0{,}5) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0{,}5 \;;\; 0{,}5 \cdot (-1) = -0{,}5$$

$\to$ wrongly classified

$$w_{new} = \begin{pmatrix} 1 \\ 0{,}2 \\ 0{,}5 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0{,}6 \end{pmatrix} \cdot 1 = \begin{pmatrix} 1 \\ 0{,}2 \\ -0{,}1 \end{pmatrix}$$

$$\hat{f}(x_1) = (1 \;\; 0{,}2 \;\; -0{,}1) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -0{,}1 \;;\; -0{,}1 \cdot (-1) = 0{,}1 \checkmark$$

$$\hat{f}(x_2) = (1 \;\; 0{,}2 \;\; -0{,}1) \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0{,}1 \;;\; 0{,}1 \cdot 1 = 0{,}1 \checkmark$$

$$\hat{f}(x_3) = (1 \;\; 0{,}2 \;\; -0{,}1) \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0{,}9 \;;\; 0{,}9 \cdot 1 = 0{,}9 \checkmark$$

$$\hat{f}(x_4) = (1 \;\; 0{,}2 \;\; -0{,}1) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1{,}1 \;;\; 1{,}1 \cdot 1 = 1{,}1 \checkmark$$

$\Rightarrow$ no more wrong classifications

$$w_{final} = [1 \;\; 0{,}2 \;\; -0{,}1]$$

3) Assume there is a linear perception that classifies each x_i of the XOR function correctly. That classifier is of form
$\hat{f}(x_i) = w^T x_i$ (Let w_3 be bias b)
This yields the following 4 inequalities (one for each datapoint x_1 to x_4). (Remember $y = [-1\ 1\ 1\ -1]^T$)

$$w^T x_1 = w_3 \cdot 1 < 0 \qquad (1)$$
$$w^T x_2 = w_2 + w_3 > 0 \qquad (2)$$
$$w^T x_3 = w_1 + w_3 > 0 \qquad (3)$$
$$w^T x_4 = w_1 + w_2 + w_3 < 0 \qquad (4)$$

Now obviously (1) gives us $w_3 < 0$, plugging this in (2) and (3) gives $w_2 > 0$ and $w_1 > 0$ and also $w_2 > (-w_3)$ and $w_1 > (-w_3)$.
This means, that $w_1 + w_2 + w_3 < 0$ can not hold!
Such classifier for the XOR function can't exist!