Q1: Derivation of Classic EKF

Given prior $x_{k-1} \sim N(\hat{x}_{k-1}, \hat{P}_{k-1})$

Prediction:

for motion eq. $x_k = f(x_{k-1}, v_k, w_k)$

linearize on $x_{k-1} := \hat{x}_{k-1}$ $\quad w_k := 0$

$$\therefore x_k \doteq f(\hat{x}_{k-1}, v_k, 0) + \frac{\partial f}{\partial x_{k-1}}(x_{k-1} - \hat{x}_{k-1}) + \frac{\partial f}{\partial w_k} w_k$$

$$= \check{x}_k + F_{k-1}(x_{k-1} - \hat{x}_{k-1}) + w_k'$$

$\therefore x_k \sim N(\check{x}_k, \check{P}_k)$ in which:

$$\check{x}_k := f(\hat{x}_{k-1}, v_k, 0)$$

$$\check{P}_k := F_{k-1} \hat{P}_{k-1} F_{k-1}^T + Q_k'$$

$$Q_k' = \frac{\partial f}{\partial w_k} Q_k \frac{\partial f}{\partial w_k}^T$$

Given $x_k \sim N(\check{x}_k, \check{P}_k)$

Correction:

for observation eq. $y_k = g(x_k, n_k)$

linearize on $x_k := \check{x}_k \quad n_k := 0$

$$\therefore y_k \doteq g(\check{x}_k, 0) + \frac{\partial g}{\partial x_k}(x_k - \check{x}_k) + \frac{\partial g}{\partial n_k} n_k$$

$$= \mu_{y_k} + G_k(x_k - \check{x}_k) + n_k'$$

$\therefore y_k \sim N(\mu_{y_k}, \Sigma_{y_k})$ in which

$$\mu_{y_k} := g(\check{x}_k, 0)$$

$$\Sigma_{y_k} := G_k \check{P}_k G_k^T + R_k'$$

$$R_k' = \frac{\partial g}{\partial n_k} R_k \frac{\partial g}{\partial n_k}^T$$

For joint distribution $[x_k, y_k]$

$$\because x_k \sim N(\check{x}_k, \check{P}_k)$$

$$y_k \sim N(\mu_{y_k}, \Sigma_{y_k})$$

$$\therefore \mu_x := \check{x}_k = f(\hat{x}_{k-1}, v_k, 0)$$

$$\mu_y = g(\check{x}_k, 0)$$

$$\Sigma_{xx} := \check{P}_k = F_{k-1} \hat{P}_{k-1} F_{k-1}^T + Q_k'$$

$$\Sigma_{yy} := G_k \check{P}_k G_k^T + R_k'$$

$$\Sigma_{xy} := \check{P}_k G_k^T$$

$$\Sigma_{yx} := G_k \check{P}_k$$

$$\therefore \mu_{x|y} = \mu_x + \Sigma_{xy} \Sigma_{yy}^{-1} (y_k - \mu_y)$$

$$\Sigma_{x|y} = \Sigma_{xx} - \Sigma_{xy} \Sigma_{yy}^{-1} \Sigma_{yx}$$

So:

define $K := \Sigma_{xy}\Sigma_{yy}^{-1} = \check{P}_k G_k^T (G_k \check{P}_k G_k^T + R_k')^{-1}$

$\therefore$ $\hat{x}_k = \mu_{x|y} = \check{x}_k + K(y_k - g(\check{x}_k, 0))$

$\hat{P}_k = \Sigma_{x|y} = \check{P}_k - K G_k \check{P}_k$

$= (I - K G_k)\check{P}_k$

$\therefore$ finally

$$\check{x}_k = f(\hat{x}_{k-1}, v_k, 0)$$

$$\check{P}_k = F_{k-1}\hat{P}_{k-1}F_{k-1}^T + Q_k'$$

$$K = \check{P}_k G_k^T (G_k \check{P}_k G_k^T + R_k')^{-1}$$

$$\hat{x}_k = \check{x}_k + K(y_k - g(\check{x}_k, 0))$$

$$\hat{P}_k = (I - K G_k)\check{P}_k$$

$\therefore$ Q.E.D

2. Weight of SP

$$\sum_{i=0}^{2L+1} \alpha_i = \frac{K}{L+K} + \frac{2L}{2(L+K)} = \frac{2K+2L}{2(L+K)} = 1$$

$\therefore$ $\{\alpha_i\}$ is a valid sample weight

$\therefore$ Q.E.D

3. Q1:

$$\check{x}_k = f(\hat{x}_{k-1}, v_k, 0)$$

$$= \begin{bmatrix} \hat{x}_{k-1} \\ \hat{y}_{k-1} \\ \hat{\theta}_{k-1} \end{bmatrix} + T \begin{bmatrix} \cos\hat{\theta}_{k-1} & 0 \\ \sin\hat{\theta}_{k-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_k \\ w_k \end{bmatrix}$$

$$\therefore F_{k-1} = \left.\frac{\partial f}{\partial x_{k-1}}\right|_{\hat{x}_{k-1}, v_k, 0} = I + \begin{bmatrix} 0 & 0 & -T\sin\hat{\theta}_{k-1} v_k \\ 0 & 0 & +T\cos\hat{\theta}_{k-1} v_k \\ 0 & 0 & 0 \end{bmatrix} \quad \cdots\cdots (1)$$

$$\therefore \left.\frac{\partial f}{\partial w_k}\right|_{\hat{x}_{k-1}, v_k, 0} = T \begin{bmatrix} \cos\hat{\theta}_{k-1} & 0 \\ \sin\hat{\theta}_{k-1} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore Q_k' = \left.\frac{\partial f}{\partial w_k}\right|_{\hat{x}_{k-1}, v_k, 0} Q \left.\frac{\partial f}{\partial w_k}\right|_{\hat{x}_{k-1}, v_k, 0} \quad \cdots\cdots (2)$$

$$\therefore G_k = \left.\frac{\partial g}{\partial x_k}\right|_{\check{x}_k, 0} = \begin{bmatrix} \frac{\check{x}_k}{\sqrt{\check{x}_k^2 + \check{y}_k^2}}, & \frac{\check{y}_k}{\sqrt{\check{x}_k^2 + \check{y}_k^2}}, & 0 \\ -\frac{\check{y}_k}{\check{x}_k^2 + \check{y}_k^2}, & \frac{\check{x}_k}{\check{x}_k^2 + \check{y}_k^2}, & -1 \end{bmatrix} \cdots (3)$$

$$\therefore \left.\frac{\partial g}{\partial n_k}\right|_{\check{x}_k, 0} = I \quad \cdots (4)$$

∴ Apply Eq. (1)···(4), then we have the desired EKF

∴ Q.E.D.