Q1:

for the system we have:

$$A_k = A = [1], \quad k = 0, \ldots, K$$

$$C_k = C = [1], \quad k = 0, \ldots, K$$

for $K = 5$, we have:

$$H = \left[\begin{array}{cccccc} -A & I & & & & \\ & -A & I & & & \\ & & -A & I & & \\ & & & -A & I & \\ & & & & -A & I \\ \hline C & & & & & \\ & C & & & & \\ & & C & & & \\ & & & C & & \\ & & & & C & \\ & & & & & C \end{array}\right] = \left[\begin{array}{cccccc} -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & 1 & & \\ & & & -1 & 1 & \\ & & & & -1 & 1 \\ \hline & & & & & \\ & & & I_6 & & \\ & & & & & \end{array}\right]$$

$$W = \left[\begin{array}{c} Q I_5 \\ \hline R I_6 \end{array}\right]$$

$$z = [v_1, \cdots, v_5, y_0, \cdots, y_5]^T$$

$$\therefore H^T W^{-1} H = \begin{bmatrix} Q^{-1}+R^{-1} & -Q^{-1} & & & & \\ -Q^{-1} & 2Q^{-1}+R^{-1} & -Q^{-1} & & & \\ & -Q^{-1} & 2Q^{-1}+R^{-1} & -Q^{-1} & & \\ & & -Q^{-1} & 2Q^{-1}+R^{-1} & -Q^{-1} & \\ & & & -Q^{-1} & 2Q^{-1}+R^{-1} & -Q^{-1} \\ & & & & -Q^{-1} & Q^{-1}+R^{-1} \end{bmatrix}$$

$$H^T W^{-1} z = \begin{bmatrix} -Q^{-1}v_1 + R^{-1}y_0 \\ Q^{-1}(v_1 - v_2) + R^{-1}y_1 \\ Q^{-1}(v_2 - v_3) + R^{-1}y_2 \\ Q^{-1}(v_3 - v_4) + R^{-1}y_3 \\ Q^{-1}(v_4 - v_5) + R^{-1}y_4 \\ Q^{-1}v_5 + R^{-1}y_5 \end{bmatrix}$$

$$\therefore \hat{x} = (H^T W^{-1} H)^{-1} H^T W^{-1} z$$

for observability:

$$O = [C] = [1] \qquad \text{rank } O = 1$$

$\therefore$ there is a unique solution to the system.

$\therefore$ Q.E.D

2.

when $Q = R = I$

use the expression from Q1:

$$H^T W^{-1} H = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 3 & -1 & & & \\ & -1 & 3 & -1 & & \\ & & -1 & 3 & -1 & \\ & & & -1 & 3 & -1 \\ & & & & -1 & 2 \end{bmatrix}$$

the expression of L can be solved as follows:

$L_0 L_0^T = L_0^2 = 2 \qquad \therefore L_0 = \sqrt{2}$

$L_0 L_{10} = -1 \qquad \therefore L_{10} = -\frac{1}{\sqrt{2}}$

$L_{10} L_{10}^T + L_1 L_1^T = \frac{1}{2} + L_1^2 = 3 \qquad \therefore L_1 = \sqrt{\frac{5}{2}}$

$L_1 L_{21} = -1 \qquad \therefore L_{21} = -\sqrt{\frac{2}{5}}$

$L_{21} L_{21}^T + L_2 L_2^T = \frac{2}{5} + L_2^2 = 3 \qquad \therefore L_2 = \sqrt{\frac{13}{5}}$

$L_2 L_{32} = -1 \qquad \therefore L_{32} = -\sqrt{\frac{5}{13}}$

$L_{32} L_{32}^T + L_3 L_3^T = \frac{5}{13} + L_3^2 = 3 \qquad \therefore L_3 = \sqrt{\frac{34}{13}}$

$L_3 L_{43} = -1 \qquad \therefore L_{43} = -\sqrt{\frac{13}{34}}$

$L_{43} L_{43}^T + L_4 L_4^T = \frac{13}{34} + L_4^2 = 3 \qquad L_4 = \sqrt{\frac{89}{34}}$

$L_4 L_{54} = -1 \qquad \therefore L_{54} = -\sqrt{\frac{34}{89}}$

$L_{54} L_{54}^T + L_5 L_5^T = \frac{34}{89} + L_5^2 = 2 \qquad L_5 = \sqrt{\frac{144}{89}}$

$$\therefore\ L = \begin{bmatrix} \sqrt{2} & & & & & \\ -\frac{1}{\sqrt{2}} & \sqrt{5/2} & & & & \\ & -\sqrt{2/5} & \sqrt{\frac{13}{5}} & & & \\ & & -\sqrt{5/13} & \sqrt{34/13} & & \\ & & & -\sqrt{13/34} & \sqrt{89/34} & \\ & & & & -\sqrt{\frac{34}{89}} & \sqrt{\frac{144}{89}} \end{bmatrix}$$

6: for each row of , $i = 1, \cdots, k-1$

$$M = \begin{bmatrix} I & & & & \\ A & I & & & \\ A^2 & A & I & & \\ \cdots & \cdots & \ddots & \ddots & \\ A^{k-1} & A^{k-2} & \cdots & A & I \end{bmatrix}$$

it can be transformed to the block-row of I

by:

$$M_{row,k} - A \cdot M_{row,k-1} = I_k$$

express this operation in matrix form:

$$\therefore \begin{bmatrix} I & & & & \\ -A & I & & & \\ & -A & I & & \\ & \ddots & \ddots & \ddots & \\ & & & -A & I \end{bmatrix} \begin{bmatrix} I & & & & \\ A & I & & & \\ A^2 & A & I & & \\ \cdots & \cdots & \cdots & \cdots & \\ A^{k-1} & A^{k-2} & \cdots & A & I \end{bmatrix} = I$$

$$\therefore \begin{bmatrix} I & & & \\ A & I & & \\ \cdots & \cdots & \ddots & \\ A^{k-1} & \cdots & \cdots & I \end{bmatrix}^{-1} = \begin{bmatrix} I & & & \\ -A & I & & \\ & \ddots & \ddots & \\ & & -A & I \end{bmatrix}$$

$\therefore$ Q.E.D