08 Pose Estimation.

Q1. Prove $r = p - q^+ y^+ q^{-1}$

in which: $y = \frac{1}{w}\sum_{j=1}^{M} w_j y_j$

$p = \frac{1}{w}\sum_{j=1}^{M} w_j p_j$

$w = \sum_{j=1}^{M} w_j$

ANS: In order to have $\frac{\partial J}{\partial r^T} = 0$

we have:

$$q^{-1\oplus}[\sum_j w_j y_j^{\oplus} - (\sum_j w_j p_j - \sum_j w_j r)^+]q = 0$$

divide both side by $w = \sum_j w_j$

$$\therefore q^{-1\oplus}[y^{\oplus} - (p-r)^+]q = 0$$

$$\therefore q^{-1\oplus} y^{\oplus} q = q^{-1\oplus}(p-r)^+ q$$

$$\therefore y^{\oplus} q = (p-r)^+ q = q^{\oplus}(p-r)$$

$$\therefore (q^{-1})^{\oplus} y^{\oplus} q = p - r$$

$$\because (q^{-1})^{\oplus} y^{\oplus} q = (q^{-1})^{\oplus}(q^+ y) = (q^+ y)^+ q^{-1} = q^+ y^+ q^{-1}$$

$$\therefore q^+ y^+ q^{-1} = p - r$$

$$\therefore r = p - q^+ y^+ q^{-1}$$

$\therefore$ Q.E.D

Q2 、Prove $\frac{1}{w}\sum_{j=1}^{M} w_j \, z_j^{\odot T} z_j^{\odot} = T_{op}^{-T} \left(\frac{1}{w}\sum_{j=1}^{M} w_j \, p_j^{\odot T} p_j^{\odot} \right) T_{op}^{-1}$

$\because \; z_j^{\odot} = (T_{op} p_j)^{\odot}$

$(T_{op} p_j)^{\odot T} (T_{op} p_j)^{\odot} = T_{op}^{-T} p_j^{\odot T} p_j^{\odot} T_{op}^{-1}$, from Exer. $\underline{\underline{7.5.12}}$

$$\therefore \frac{1}{w}\sum_{j=1}^{M} w_j \, z_j^{\odot T} z_j = T_{op}^{-T} \left(\frac{1}{w}\sum_{j=1}^{M} p_j^{\odot T} p_j^{\odot} \right) T_{op}^{-1}$$

$$\because \; p_j^{\odot T} p_j^{\odot} = \begin{bmatrix} 1 & 0 \\ p_j^{\wedge} & 0 \end{bmatrix} \begin{bmatrix} 1 & -p_j^{\wedge} \\ 0^T & 0^T \end{bmatrix} = \begin{bmatrix} 1 & -p_j^{\wedge} \\ p_j^{\wedge} & -p_j^{\wedge} p_j^{\wedge} \end{bmatrix}$$

$$\therefore \frac{1}{w}\sum_{j=1}^{M} w_j \, p_j^{\odot T} p_j^{\odot} = \begin{bmatrix} 1 & -p^{\wedge} \\ p^{\wedge} & -\frac{1}{w}\sum_{j=1}^{M} w_j p_j^{\wedge} p_j^{\wedge} \end{bmatrix} \qquad p = \frac{1}{w}\sum_{j=1}^{M} w_j p_j$$

$$\because \; M = \begin{bmatrix} 1 & p^{\wedge} \\ -p^{\wedge} & I - p^{\wedge} p^{\wedge} \end{bmatrix}$$

$$I = -\frac{1}{w}\sum_{j=1}^{M} w_j \left(p_j^{\wedge} p_j^{\wedge} - p_j^{\wedge} p^{\wedge} - p^{\wedge} p_j^{\wedge} + p^{\wedge} p^{\wedge} \right)$$

$$= -\frac{1}{w}\sum_{j=1}^{M} w_j \, p_j^{\wedge} p_j^{\wedge} + p^{\wedge} p^{\wedge}$$

$$\therefore I - p^{\wedge} p^{\wedge} = -\frac{1}{w}\sum_{j=1}^{M} w_j \, p_j^{\wedge} p_j^{\wedge}$$

$$\therefore \frac{1}{w}\sum_{j=1}^{M} w_j \, z_j^{\odot T} z_j^{\odot} = T_{op}^{-T} M T_{op}^{-1} , \; M = \begin{bmatrix} 1 & 0 \\ -p^{\wedge} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 1 & p^{\wedge} \\ 0 & 1 \end{bmatrix} :$$

$\therefore$ Q.E.D.

Q3、 Prove $\frac{1}{w}\sum_{j=1}^{M} w_j z_j^{\odot T}(y_j - z_j) = \begin{bmatrix} y - C_{op}(p - r_{op}) \\ b - y^\wedge C_{op}(p - r_{op}) \end{bmatrix}$

$\because z_j^{\odot} = (T_{op} p_j)^{\odot}$

$$= \left(\begin{bmatrix} C_{op} & -C_{op} r_{op} \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} p_j \\ 1 \end{bmatrix} \right)^{\odot} = \begin{bmatrix} C_{op}(p_j - r_{op}) \\ 1 \end{bmatrix}^{\odot} = \begin{bmatrix} I & -[C_{op}(p_j - r_{op})]^\wedge \\ 0^T & 0^T \end{bmatrix}$$

$$\therefore z_j^{\odot T}(y_j - z_j) = \begin{bmatrix} I & 0 \\ [C_{op}(p_j - r_{op})]^\wedge & 0 \end{bmatrix} \begin{bmatrix} y_j - C_{op}(p_j - r_{op}) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} y_j - C_{op}(p_j - r_{op}) \\ -y_j^\wedge C_{op}(p_j - r_{op}) \end{bmatrix}$$

$\because \frac{1}{w}\sum_{j=1}^{M} w_j [y_j - C_{op}(p_j - r_{op})] = \frac{1}{w}\sum_{j=1}^{M} w_j y_j - C_{op}(\frac{1}{w}\sum_{j=1}^{M} w_j p_j - r_{op}) = y - C_{op}(p - r_{op})$

$\because \frac{1}{w}\sum_{j=1}^{M} w_j - y_j^\wedge C_{op}(p_j - r_{op}) = -\frac{1}{w}\sum_{j=1}^{M} w_j (y_j - y)^\wedge C_{op}(p_j - p) - y^\wedge C_{op}(p - r_{op})$

$\because$ The i-th element of $-\frac{1}{w}\sum_{j=1}^{M} w_j (y_j - y)^\wedge C_{op}(p_j - p)$ is

$1_i^T - \frac{1}{w}\sum_{j=1}^{M} w_j (y_j - y)^\wedge C_{op}(p_j - p) = -\frac{1}{w}\sum_{j=1}^{M} w_j \operatorname{tr}\{C_{op}(p_j - p)[1_i^T (y_j - y)^\wedge]\}$

$\because 1_i^T (y_j - y) = -(y_j - y)^T 1_i$

$= \operatorname{tr}[C_{op} \frac{1}{w}\sum_{j=1}^{M} w_j (p_j - p)(y_j - y)^T 1_i^\wedge]$

$= \operatorname{tr}[1_i^\wedge C_{op} W^T], W^T = \frac{1}{w}\sum_{j=1}^{M} w_j (p_j - p)(y_j - y)^T$

$\therefore$ Q.E.D