State Estimation

01 Intro

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Q1:

Define: $$I_x = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\cdot\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2}x'^2\right) dx', \quad x' = \frac{x-\mu}{\sigma}$$

$$\therefore I_x^2 = I_x \cdot I_y$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{2\pi} \cdot \exp\left[-\frac{1}{2}(x'^2+y'^2)\right] dx'dy'$$

$$= \frac{1}{2\pi} \cdot \int_0^{2\pi} \cdot \int_0^{+\infty} \exp\left(-\frac{1}{2}\rho^2\right)\rho \, d\rho \, d\theta$$

$$= \int_0^{+\infty} \exp(-\tau) d\tau$$

$$= -\exp(-\tau)\Big|_0^{+\infty}$$

$$= 1$$

$$\therefore I_x = \sqrt{1} = 1$$

$\therefore$ Q.E.D

Q2: For $u, v \in R^d$

$$tr(vu^T) = \sum_{k=1}^{d} v_k u_k = \sum_{k=1}^{d} u_i \cdot v_i = u^T v$$

$\therefore$ Q.E.D

Q3: For $\Sigma = U\Lambda U^T = (U\Lambda^{\frac{1}{2}}U^T)(U\Lambda^{\frac{1}{2}}U^T)$

Define: $y = U\Lambda^{-\frac{1}{2}}U^T(x-u)$

$$\therefore x = U\Lambda^{\frac{1}{2}}U^T y + u$$

$$\therefore dy = \frac{1}{|\Sigma|^{\frac{1}{2}}} dx$$

$$\therefore \int x p(x) dx = U\Lambda^{\frac{1}{2}}U^T \int \frac{1}{(2\pi)^{d/2}} y \exp(-\tfrac{1}{2}y^T y) dy$$

$$+ u \int \frac{1}{(2\pi)^{d/2}} \exp(-\tfrac{1}{2}y^T y) dy$$

$$= u$$

$\therefore$ Q.E.D

Q4: Use the definitions in Q3:

$$\therefore \quad x-\mu = U\Lambda^{1/2}U^T y$$

$$\therefore \quad (x-\mu)(x-\mu)^T = U\Lambda^{1/2}U^T y y^T U\Lambda^{1/2}U^T$$

$$\therefore \quad \int (x-\mu)(x-\mu)^T p(x)\, dx$$

$$= U\Lambda^{1/2}U^T \int \frac{1}{(2\pi)^{d/2}} y \cdot y^T \cdot \exp(-\tfrac{1}{2} y^T y)\, dy \cdot U\Lambda^{1/2}U^T$$

$$= U\Lambda^{1/2}U^T U\Lambda^{1/2}U^T$$

$$= U\Lambda U^T$$

$$= \Sigma$$

$\therefore$ Q.E.D.

Q5: For $\{x_1, \cdots, x_k\}$

$$x-\mu = \begin{bmatrix} x_1-\mu_1 \\ \cdots \\ x_k-\mu_k \end{bmatrix}$$

$$\Sigma^{-1} = \mathrm{diag}(\Sigma_1^{-1}, \cdots, \Sigma_k^{-1})$$

$$\therefore \exp\left[-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right]$$

$$= \exp\left[-\frac{1}{2}\left[(x_1-\mu_1)^T, \cdots, (x_k-\mu_k)^T\right]\mathrm{diag}(\Sigma_1^{-1}, \cdots, \Sigma_k^{-1})\begin{bmatrix} x_1-\mu_1 \\ \cdots \\ x_k-\mu_k \end{bmatrix}\right]$$

$$= \exp\left[-\frac{1}{2}\sum_{i=1}^{k}(x_i-\mu_i)^T\Sigma_i^{-1}(x_i-\mu_i)\right]$$

$$= \prod_{i=1}^{k}\exp\left[-\frac{1}{2}(x_i-\mu_i)^T\Sigma_i^{-1}(x_i-\mu_i)\right]$$

$$\therefore \eta = \frac{|\Sigma|^{d/2}}{\prod_{i=1}^{k}|\Sigma_i|^{d_i/2}} \qquad \therefore Q.E.D.$$