7.5.1

ANS: $\because (Cu)^\wedge (Cv) = C(u^\wedge v) \quad \forall u, v \in so(3)$

$\therefore (Cu)^\wedge v = (Cu)^\wedge (CC^T v) = C(u^\wedge C^T v) = (Cu^\wedge C^T) v$

$\forall u, v \in so(3)$

$\therefore (Cu)^\wedge \equiv Cu^\wedge C^T$

$\therefore$ Q.E.D

7.5.2

$$(Cu)^\wedge \equiv u^\wedge(\mathrm{tr}C\cdot I - C) - C^T u^\wedge$$

$$= u^\wedge(\mathrm{tr}C\cdot I) - u^\wedge C - C^T u^\wedge$$

$$= (\mathrm{tr}C)u^\wedge - u^\wedge C - C^T u^\wedge$$

$\because \mathrm{tr}C = 2\cos\theta + 1$

$$\therefore (Cu)^\wedge = (2\cos\theta + 1)u^\wedge - u^\wedge C - C^T u^\wedge$$

$\therefore$ Q. E. D.

7.5.3

$\because \exp(Cu)^{\wedge} = \sum_{n=0}^{+\infty} \frac{1}{n!} (Cu)^{\wedge n}$

$\because (Cu)^{\wedge} = Cu^{\wedge}C^{T}$

$\therefore \exp(Cu)^{\wedge} = \sum_{n=0}^{+\infty} \frac{1}{n!} (Cu^{\wedge}C^{T})^{n}$

$= C \sum_{n=0}^{+\infty} \frac{1}{n!} (u^{\wedge})^{n} C^{T}$

$= C \exp(u^{\wedge}) C^{T}$

$\therefore$ Q.E.D

7.5.4

For $T \in Ad(SE(3))$ $x, y \in se(3)$

$\because (Tx)^{\curlywedge} Ty = T(x^{\curlywedge} y)$

$$\therefore (Tx)^{\curlywedge} y = (Tx)^{\curlywedge}(TT^{-1}y)$$
$$= T(x^{\curlywedge} T^{-1} y)$$
$$= (T x^{\curlywedge} T^{-1}) y$$

$\therefore (Tx)^{\curlywedge} = T x^{\curlywedge} T^{-1}$

$\therefore$ Q.E.D

7.5.5:

$$\exp[(J\alpha)^{\ell}] = \sum_{n=0}^{+\infty} \frac{1}{n!} (J\alpha)^{\ell n}$$

$$= \sum_{n=0}^{+\infty} \frac{1}{n!} (J\alpha^{\ell}J^{-1})^{n}$$

$$= J\sum_{n=0}^{+\infty} \frac{1}{n!} \alpha^{\ell} J^{-1}$$

$$= J \exp \alpha^{\ell} J^{-1}$$

$\therefore$ Q.E.D

7.5.7.

$$\because \alpha^{\wedge} p = \begin{bmatrix} \phi^{\times} & \rho \\ 0^T & 0 \end{bmatrix} \begin{bmatrix} \varepsilon \\ \eta \end{bmatrix} = \begin{bmatrix} \phi^{\times}\varepsilon + \eta\rho \\ 0 \end{bmatrix}$$

$$\because p^{\odot} \alpha = \begin{bmatrix} \eta I & -\varepsilon^{\wedge} \\ 0^T & 0^T \end{bmatrix} \begin{bmatrix} \rho \\ \phi \end{bmatrix} = \begin{bmatrix} \eta\rho - \varepsilon^{\wedge}\phi \\ 0 \end{bmatrix} = \begin{bmatrix} \phi^{\times}\varepsilon + \eta\rho \\ 0 \end{bmatrix}$$

$$\therefore \alpha^{\wedge} p = p^{\odot} \alpha$$

$\therefore$ Q.E.D

7.5.8.

$$p^T \alpha^\wedge = [\varepsilon^T \ \eta] \begin{bmatrix} \phi^\wedge & \rho \\ 0^T & 0 \end{bmatrix} = [\varepsilon^T \phi^\wedge \ , \ \varepsilon^T \rho]$$

$$\alpha^T p^{\odot} = [\rho^T, \phi^T] \begin{bmatrix} 0^T & \varepsilon \\ -\varepsilon^\wedge & 0 \end{bmatrix} = [-\phi^T \varepsilon^\wedge, \rho^T \varepsilon]$$

$$= [\varepsilon^T \phi^\wedge, \rho^T \varepsilon]$$

$$\therefore p^T \alpha^\wedge = \alpha^T p^{\odot}$$

$\therefore$ Q.E.D.

7.5. 11.

$$\because (Tp)^{\odot} = \left(\begin{bmatrix} C & r \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \varepsilon \\ \eta \end{bmatrix}\right)^{\odot}$$

$$= \begin{bmatrix} C\varepsilon + \eta r \\ \eta \end{bmatrix}^{\odot} = \begin{bmatrix} \eta I & -(C\varepsilon+\eta r)^{\wedge} \\ 0^T & 0^T \end{bmatrix}$$

$$\because T p^{\odot} T^{-1} = \begin{bmatrix} C & r \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \eta I & -\varepsilon^{\wedge} \\ 0^T & 0^T \end{bmatrix}\begin{bmatrix} C^T & (-C^T r)^{\wedge} C^T \\ 0 & C^T \end{bmatrix}$$

$$= \begin{bmatrix} \eta C & -C\varepsilon^{\wedge} \\ 0^T & 0^T \end{bmatrix}\begin{bmatrix} C^T & (-C^T r)^{\wedge} C^T \\ 0 & C^T \end{bmatrix}$$

$$= \begin{bmatrix} \eta C C^T & -\eta C (C^T r)^{\wedge} C^T - C\varepsilon^{\wedge} C^T \\ 0^T & 0^T \end{bmatrix}$$

$$= \begin{bmatrix} \eta I & -\eta C C^T r^{\wedge} C C^T - (C\varepsilon)^{\wedge} \\ 0^T & 0^T \end{bmatrix}$$

$$= \begin{bmatrix} \eta I & -\eta r^{\wedge} - (C\varepsilon)^{\wedge} \\ 0^T & 0^T \end{bmatrix} = \begin{bmatrix} \eta I & -(\eta r + C\varepsilon)^{\wedge} \\ 0^T & 0^T \end{bmatrix}$$

$\therefore$ Q.E.D.

7.5.12. 6×4

$$\because (Tp)^{\odot T}(Tp)^{\odot} = \begin{bmatrix} \eta I & 0 \\ (C\varepsilon+\eta r)^{\wedge} & 0 \end{bmatrix}\begin{bmatrix} \eta I & -(C\varepsilon+\eta r)^{\wedge} \\ 0^T & 0^T \end{bmatrix}$$

$$= \begin{bmatrix} \eta^2 I , & -\eta(C\varepsilon+\eta r)^{\wedge} \\ \eta(C\varepsilon+\eta r)^{\wedge} , & -(C\varepsilon+\eta r)^{\wedge}(C\varepsilon+\eta r)^{\wedge} \end{bmatrix}$$

$$\because p^{\odot} T^{-1} = \begin{bmatrix} \eta I & -\varepsilon^{\wedge} \\ 0^T & 0^T \end{bmatrix}\begin{bmatrix} C^T & (-C^T r)^{\wedge} C^T \\ 0 & C^T \end{bmatrix}$$

$$= \begin{bmatrix} \eta C^T , & -\eta C^T r^{\wedge} - \varepsilon^{\wedge} C^T \\ 0^T , & 0^T \end{bmatrix}$$

$$\therefore T^{-T} p^{\odot T} p^{\odot} T^{-1} = \begin{bmatrix} \eta C , & 0 \\ \eta r^{\wedge} C + C\varepsilon^{\wedge} , & 0 \end{bmatrix}\begin{bmatrix} \eta C^T & -\eta C^T r^{\wedge} - \varepsilon^{\wedge} C^T \\ 0^T & 0^T \end{bmatrix}$$

$$= \begin{bmatrix} \eta^2 C C^T , & -\eta^2 r^{\wedge} - \eta C\varepsilon^{\wedge} C^T \\ \eta^2 r^{\wedge} + \eta C\varepsilon^{\wedge} C^T , & (\eta r^{\wedge} C + C\varepsilon^{\wedge})(\eta C^T r^{\wedge} + \varepsilon^{\wedge} C^T) \end{bmatrix}$$

$$\because -\eta^2 r^{\wedge} - \eta C\varepsilon^{\wedge} C^T = -\eta^2 r^{\wedge} - \eta (C\varepsilon)^{\wedge} = -\eta(\eta r + C\varepsilon)^{\wedge}$$

$$\because \eta^2 r^{\wedge} + \eta C\varepsilon C^T = \eta^2 r^{\wedge} + \eta (C\varepsilon)^{\wedge} = \eta(\eta r + C\varepsilon)^{\wedge}$$

$$\therefore (Tp)^{\odot T}(Tp)^{\odot} = T^{-T} p^{\odot T} p^{\odot} T^{-1} \quad \therefore Q.E.D$$