Prime Number Theorem And ...

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Chapter 1

The project

The project github page is `https://github.com/AlexKontorovich/PrimeNumberTheoremAnd`.

The project docs page is `https://alexkontorovich.github.io/PrimeNumberTheoremAnd/docs`.

The first main goal is to prove the Prime Number Theorem in Lean. (This remains one of the outstanding problems on Wiedijk's list of 100 theorems to formalize.) Note that PNT has been formalized before, first by Avigad et al in Isabelle, `https://arxiv.org/abs/cs/0509025` following the Selberg / Erdos method, then by Harrison in HOL Light `https://www.cl.cam.ac.uk/$\sim$jrh13/papers/mikefest.html` via Newman's proof. Carniero gave another formalization in Metamath of the Selberg / Erdos method: `https://arxiv.org/abs/1608.02029`, and Eberl-Paulson gave a formalization of Newman's proof in Isabelle: `https://www.isa-afp.org/entries/Prime_Number_Theorem.html`

Continuations of this project aim to extend this work to primes in progressions (Dirichlet's theorem), Chebotarev's density theorem, etc etc.

There are (at least) three approaches to PNT that we may want to pursue simultaneously. The quickest, at this stage, is likely to follow the "Euler Products" project by Michael Stoll, which has a proof of PNT missing only the Wiener-Ikehara Tauberian theorem.

The second develops some complex analysis (residue calculus on rectangles, argument principle, Mellin transforms), to pull contours and derive a PNT with an error term which is stronger than any power of log savings.

The third approach, which will be the most general of the three, is to: (1) develop the residue calculus et al, as above, (2) add the Hadamard factorization theorem, (3) use it to prove the zero-free region for zeta via Hoffstein-Lockhart+Goldfeld-Hoffstein-Liemann (which generalizes to higher degree L-functions), and (4) use this to prove the prime number theorem with exp-root-log savings.

A word about the expected "rate-limiting-steps" in each of the approaches.

(*) In approach (1), I think it will be the fact that the Fourier transform is a bijection on the Schwartz class. There is a recent PR (`https://github.com/leanprover-community/mathlib4/pull/9773`) with David Loeffler and Heather Macbeth making the first steps in that direction, just computing the (Frechet) derivative of the Fourier transform. One will need to iterate on that to get arbitrary derivatives, to conclude that the transform of a Schwartz function is Schwartz. Then to get the bijection, we need an inversion formula. We can derive Fourier inversion *from* Mellin inversion! So it seems that the most important thing to start is Perron's formula.

(*) In approach (2), there are two rate-limiting-steps, neither too serious (in my estimation). The first is how to handle meromorphic functions on rectangles. Perhaps in this project, it should not be done in any generality, but on a case by case basis. There are two simple poles whose residues need to be computed in the proof of the Perron formula, and one simple pole in the log-derivative of zeta, nothing too complicated, and maybe we shouldn't get bogged down in the general case. The other is the fact that the ϵ-smoothed Chebyshev function differs from the unsmoothed by ϵX (and not $\epsilon X \log X$, as follows from a trivial bound). This needs a Brun-Titchmarsh type theorem, perhaps can be done even more easily in this case with a Selberg sieve, on which there is (partial?) progress in Mathlib.

(*) In approach (3), it's obviously the Hadamard factorization, which needs quite a lot of nontrivial mathematics. (But after that, the math is not hard, on top of things in approach (2) – and if we're getting the strong error term, we can afford to lose $\log X$ in the Chebyshev discussion above...).

Chapter 2

First approach: Wiener-Ikehara Tauberian theorem.

2.1 A Fourier-analytic proof of the Wiener-Ikehara theorem

The Fourier transform of an absolutely integrable function $\psi : \mathbb{R} \to \mathbb{C}$ is defined by the formula

$$\hat{\psi}(u) := \int_{\mathbb{R}} e(-tu)\psi(t)\ dt$$

where $e(\theta) := e^{2\pi i\theta}$.

Let $f : \mathbb{N} \to \mathbb{C}$ be an arithmetic function such that $\sum_{n=1}^{\infty} \frac{|f(n)|}{n^\sigma} < \infty$ for all $\sigma > 1$. Then the Dirichlet series

$$F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

is absolutely convergent for $\sigma > 1$.

Lemma 1 (first$_f$*ourier*). If $\psi : \mathbb{R} \to \mathbb{C}$ is integrable and $x > 0$, then for any $\sigma > 1$

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^\sigma} \hat{\psi}(\frac{1}{2\pi} \log \frac{n}{x}) = \int_{\mathbb{R}} F(\sigma + it)\psi(t)x^{it}\ dt.$$

Proof. By the definition of the Fourier transform, the left-hand side expands as

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}} \frac{f(n)}{n^\sigma} \psi(t) e(-\frac{1}{2\pi} t \log \frac{n}{x})\ dt$$

while the right-hand side expands as

$$\int_{\mathbb{R}} \sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma+it}} \psi(t) x^{it}\ dt.$$

Since

$$\frac{f(n)}{n^\sigma} \psi(t) e(-\frac{1}{2\pi} t \log \frac{n}{x}) = \frac{f(n)}{n^{\sigma+it}} \psi(t) x^{it}$$

the claim then follows from Fubini's theorem. □

Lemma 2 (second$_fourier$). If $\psi : \mathbb{R} \to \mathbb{C}$ is continuous and compactly supported and $x > 0$, then for any $\sigma > 1$

$$\int_{-\log x}^{\infty} e^{-u(\sigma-1)} \hat{\psi}(\frac{u}{2\pi})\ du = x^{\sigma-1} \int_{\mathbb{R}} \frac{1}{\sigma+it-1} \psi(t) x^{it}\ dt.$$

Proof. The left-hand side expands as

$$\begin{array}{c} \int_{-\log x}^{\infty} \int_{\mathbb{R}} e^{-u(\sigma-1)} \psi(t) e(-\frac{tu}{2\pi})\ dt\ du \\ ? = x^{\sigma-1} \int_{\mathbb{R}} \frac{1}{\sigma+it-1} \psi(t) x^{it}\ dt \end{array}$$

so by Fubini's theorem it suffices to verify the identity

$$\begin{aligned} \int_{-\log x}^{\infty} e^{-u(\sigma-1)} e(-\frac{tu}{2\pi})\ du &= \int_{-\log x}^{\infty} e^{(it-\sigma+1)u}\ du \\ &= \frac{1}{it-\sigma+1} e^{(it-\sigma+1)u}\ \Big|_{-\log x}^{\infty} \\ &= x^{\sigma-1} \frac{1}{\sigma+it-1} x^{it} \end{aligned}$$

□

Now let $A \in \mathbb{C}$, and suppose that there is a continuous function $G(s)$ defined on $\mathrm{Re}s \geq 1$ such that $G(s) = F(s) - \frac{A}{s-1}$ whenever $\mathrm{Re}s > 1$. We also make the Chebyshev-type hypothesis

$$\sum_{n \leq x} |f(n)| \ll x \tag{2.1}$$

for all $x \geq 1$ (this hypothesis is not strictly necessary, but simplifies the arguments and can be obtained fairly easily in applications).

Lemma 3 (Preliminary decay bound I). If $\psi : \mathbb{R} \to \mathbb{C}$ is absolutely integrable then

$$|\hat{\psi}(u)| \leq \|\psi\|_1$$

for all $u \in \mathbb{R}$. where C is an absolute constant.

Proof. Immediate from the triangle inequality. □

Lemma 4 (Preliminary decay bound II). If $\psi : \mathbb{R} \to \mathbb{C}$ is absolutely integrable and of bounded variation, and ψ' is bounded variation, then

$$|\hat{\psi}(u)| \leq \|\psi\|_{TV}/2\pi|u|$$

for all non-zero $u \in \mathbb{R}$.

Proof. By integration by parts we will have

$$2\pi i u \hat{\psi}(u) = \int_{\mathbb{R}} e(-tu) \psi'(t)\ dt$$

and the claim then follows from the triangle inequality. □

Lemma 5 (Preliminary decay bound III)**.** If $\psi : \mathbb{R} \to \mathbb{C}$ is absolutely integrable, absolutely continuous, and ψ' is of bounded variation, then

$$|\hat{\psi}(u)| \leq \|\psi'\|_{TV}/(2\pi|u|)^2$$

for all non-zero $u \in \mathbb{R}$.

Proof. Should follow from previous lemma. ☐

Lemma 6 (Decay bound, alternate form)**.** If $\psi : \mathbb{R} \to \mathbb{C}$ is absolutely integrable, absolutely continuous, and ψ' is of bounded variation, then

$$|\hat{\psi}(u)| \leq (\|\psi\|_1 + \|\psi'\|_{TV}/(2\pi)^2)/(1 + |u|^2)$$

for all $u \in \mathbb{R}$.

Proof. Should follow from previous lemmas. ☐

It should be possible to refactor the lemma below to follow from Lemma **??** instead.

Lemma 7 (Decay bounds)**.** If $\psi : \mathbb{R} \to \mathbb{C}$ is C^2 and obeys the bounds

$$|\psi(t)|, |\psi''(t)| \leq A/(1 + |t|^2)$$

for all $t \in \mathbb{R}$, then

$$|\hat{\psi}(u)| \leq CA/(1 + |u|^2)$$

for all $u \in \mathbb{R}$, where C is an absolute constant.

Proof. From two integration by parts we obtain the identity

$$(1 + u^2)\hat{\psi}(u) = \int_{\mathbf{R}} (\psi(t) - \frac{u}{4\pi^2}\psi''(t))e(-tu)\ dt.$$

Now apply the triangle inequality and the identity $\int_{\mathbf{R}} \frac{dt}{1+t^2}\ dt = \pi$ to obtain the claim with $C = \pi + 1/4\pi$. ☐

Lemma 8 (Limiting Fourier identity)**.** If $\psi : \mathbb{R} \to \mathbb{C}$ is C^2 and compactly supported and $x \geq 1$, then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \hat{\psi}(\frac{1}{2\pi} \log \frac{n}{x}) - A \int_{-\log x}^{\infty} \hat{\psi}(\frac{u}{2\pi})\ du = \int_{\mathbb{R}} G(1 + it)\psi(t)x^{it}\ dt.$$

Proof. By Lemma **??** and Lemma **??**, we know that for any $\sigma > 1$, we have

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^\sigma} \hat{\psi}(\frac{1}{2\pi} \log \frac{n}{x}) - Ax^{1-\sigma} \int_{-\log x}^{\infty} e^{-u(\sigma-1)}\hat{\psi}(\frac{u}{2\pi})\ du = \int_{\mathbb{R}} G(\sigma + it)\psi(t)x^{it}\ dt.$$

Now take limits as $\sigma \to 1$ using dominated convergence together with (**??**) and Lemma **??** to obtain the result. ☐

Corollary 1 (Corollary of limiting identity)**.** With the hypotheses as above, we have

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \hat{\psi}(\frac{1}{2\pi} \log \frac{n}{x}) = A \int_{-\infty}^{\infty} \hat{\psi}(\frac{u}{2\pi})\ du + o(1)$$

as $x \to \infty$.

Proof. Immediate from the Riemann-Lebesgue lemma, and also noting that $\int_{-\infty}^{-\log x} \hat{\psi}(\frac{u}{2\pi})\ du = o(1)$. □

Lemma 9 (Smooth Urysohn lemma)**.** If I is a closed interval contained in an open interval J, then there exists a smooth function $\Psi : \mathbb{R} \to \mathbb{R}$ with $1_I \leq \Psi \leq 1_J$.

Proof. A standard analysis lemma, which can be proven by convolving 1_K with a smooth approximation to the identity for some interval K between I and J. Note that we have "SmoothBumpFunction"s on smooth manifolds in Mathlib, so this shouldn't be too hard... □

Lemma 10 (Limiting identity for Schwartz functions)**.** The previous corollary also holds for functions ψ that are assumed to be in the Schwartz class, as opposed to being C^2 and compactly supported.

Proof. For any $R > 1$, one can use a smooth cutoff function (provided by Lemma **??** to write $\psi = \psi_{\leq R} + \psi_{>R}$, where $\psi_{\leq R}$ is C^2 (in fact smooth) and compactly supported (on $[-R, R]$), and $\psi_{>R}$ obeys bounds of the form

$$|\psi_{>R}(t)|, |\psi''_{>R}(t)| \ll R^{-1}/(1 + |t|^2)$$

where the implied constants depend on ψ. By Lemma **??** we then have

$$\hat{\psi}_{>R}(u) \ll R^{-1}/(1 + |u|^2).$$

Using this and (**??**) one can show that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \hat{\psi}_{>R}(\frac{1}{2\pi} \log \frac{n}{x}), A \int_{-\infty}^{\infty} \hat{\psi}_{>R}(\frac{u}{2\pi})\ du \ll R^{-1}$$

(with implied constants also depending on A), while from Lemma **??** one has

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \hat{\psi}_{\leq R}(\frac{1}{2\pi} \log \frac{n}{x}) = A \int_{-\infty}^{\infty} \hat{\psi}_{\leq R}(\frac{u}{2\pi})\ du + o(1).$$

Combining the two estimates and letting R be large, we obtain the claim. □

Lemma 11 (Bijectivity of Fourier transform)**.** The Fourier transform is a bijection on the Schwartz class. [Note: only surjectivity is actually used.]

Proof. This is a standard result in Fourier analysis. It can be proved here by appealing to Mellin inversion, Theorem **??**. In particular, given f in the Schwartz class, let $F : \mathbb{R}_+ \to \mathbb{C} : x \mapsto f(\log x)$ be a function in the "Mellin space"; then the Mellin transform of F on the imaginary axis $s = it$ is the Fourier transform of f. The Mellin inversion theorem gives Fourier inversion. □

Corollary 2 (Smoothed Wiener-Ikehara)**.** If $\Psi : (0, \infty) \to \mathbb{C}$ is smooth and compactly supported away from the origin, then,

$$\sum_{n=1}^{\infty} f(n) \Psi(\frac{n}{x}) = Ax \int_0^{\infty} \Psi(y)\ dy + o(x)$$

as $x \to \infty$.

Proof. By Lemma **??**, we can write

$$y\Psi(y) = \hat{\psi}(\frac{1}{2\pi}\log y)$$

for all $y > 0$ and some Schwartz function ψ. Making this substitution, the claim is then equivalent after standard manipulations to

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \hat{\psi}(\frac{1}{2\pi}\log\frac{n}{x}) = A\int_{-\infty}^{\infty} \hat{\psi}(\frac{u}{2\pi})\ du + o(1)$$

and the claim follows from Lemma **??**. ☐

Now we add the hypothesis that $f(n) \geq 0$ for all n.

Proposition 1 (Wiener-Ikehara in an interval)**.** For any closed interval $I \subset (0, +\infty)$, we have

$$\sum_{n=1}^{\infty} f(n)1_I(\frac{n}{x}) = Ax|I| + o(x).$$

Proof. Use Lemma **??** to bound 1_I above and below by smooth compactly supported functions whose integral is close to the measure of $|I|$, and use the non-negativity of f. ☐

Corollary 3 (Wiener-Ikehara theorem)**.** We have

$$\sum_{n\leq x} f(n) = Ax + o(x).$$

Proof. Apply the preceding proposition with $I = [\varepsilon, 1]$ and then send ε to zero (using (**??**) to control the error). ☐

2.2 Weak PNT

Theorem 1 (WeakPNT)**.** We have

$$\sum_{n\leq x} \Lambda(n) = x + o(x).$$

Proof. Already done by Stoll, assuming Wiener-Ikehara. ☐

2.3 Removing the Chebyshev hypothesis

In this section we do *not* assume the bound (**??**), but instead derive it from the other hypotheses.

Lemma 12 (limiting$_f ourier_v ariant$). If $\psi : \mathbb{R} \to \mathbb{C}$ is C^2 and compactly supported with f and $\hat{\psi}$ non-negative, and $x \geq 1$, then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \hat{\psi}(\frac{1}{2\pi}\log\frac{n}{x}) - A\int_{-\log x}^{\infty} \hat{\psi}(\frac{u}{2\pi})\ du = \int_{\mathbb{R}} G(1+it)\psi(t)x^{it}\ dt.$$

Proof. Repeat the proof of Lemma **??**, but use monotone convergence instead of dominated convergence. (The proof should be simpler, as one no longer needs to establish domination for the sum.) ☐

Corollary 4 (crude$_{u}$*pper*$_{b}$*ound*). If $\psi : \mathbb{R} \to \mathbb{C}$ is C^2 and compactly supported with f and $\hat{\psi}$ non-negative, then there exists a constant B such that

$$|\sum_{n=1}^{\infty} \frac{f(n)}{n} \hat{\psi}(\frac{1}{2\pi} \log \frac{n}{x})| \leq B$$

for all $x > 0$.

Proof. For $x \geq 1$, this readily follows from the previous lemma and the triangle inequality. For $x < 1$, only a bounded number of summands can contribute and the claim is trivial. ☐

Corollary 5 (auto$_{c}$*heby*). One has

$$\sum_{n \leq x} f(n) = O(x)$$

for all $x \geq 1$.

Proof. By applying Corollary **??** for a specific compactly supported function ψ, one can obtain a bound of the form $\sum_{(1-\varepsilon)x<n\leq x} f(n) = O(x)$ for all x and some absolute constant ε (which can be made explicit). If C is a sufficiently large constant, the claim $|\sum_{n\leq x} f(n)| \leq Cx$ can now be proven by strong induction on x, as the claim for $(1-\varepsilon)x$ implies the claim for x by the triangle inequality (and the claim is trivial for $x < 1$).

Corollary 6 (WienerIkeharaTheorem”)**.** We have

$$\sum_{n \leq x} f(n) = Ax + o(x).$$

Proof. Use Corollary **??** to remove the Chebyshev hypothesis in Theorem **??**. ☐

2.4 The prime number theorem in arithmetic progressions

Lemma 13 (WeakPNT$_{c}$*haracter*). If $q \geq 1$ and a is coprime to q, and $\mathrm{Re} s > 1$, we have

$$\sum_{n:n=a\ (q)} \frac{\Lambda(n)}{n^s} = -\frac{1}{\varphi(q)} \sum_{\chi\ (q)} \overline{\chi(a)} \frac{L'(s,\chi)}{L(s,\chi)}.$$

Proof. From the Fourier inversion formula on the multiplicative group $(\mathbb{Z}/q\mathbb{Z})^\times$, we have

$$1_{n=a\ (q)} = \frac{\varphi(q)}{q} \sum_{\chi\ (q)} \overline{\chi(a)} \chi(n).$$

On the other hand, from standard facts about L-series we have for each character χ that

$$\sum_{n} \frac{\Lambda(n)\chi(n)}{n^s} = -\frac{L'(s,\chi)}{L(s,\chi)}.$$

Combining these two facts, we obtain the claim. ☐

Proposition 2 (WeakPNT$_A P_p relim$). If $q \geq 1$ and a is coprime to q, the Dirichlet series $\sum_{n \leq x : n = a\ (q)} \Lambda(n) n^s$ converges for $\mathrm{Re}(s) > 1$ to $\frac{1}{\varphi(q)} \frac{1}{s-1} + G(s)$ where G has a continuous extension to $\mathrm{Re}(s) = 1$.

Proof. We expand out the left-hand side using Lemma **??**. The contribution of the non-principal characters χ extend continuously to $\mathrm{Re}(s) = 1$ thanks to the non-vanishing of $L(s, \chi)$ on this line (which should follow from another component of this project), so it suffices to show that for the principal character χ_0, that

$$-\frac{L'(s, \chi_0)}{L(s, \chi_0)} - \frac{1}{s-1}$$

also extends continuously here. But we already know that

$$-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$$

extends, and from Euler product machinery one has the identity

$$\frac{L'(s, \chi_0)}{L(s, \chi_0)} = \frac{\zeta'(s)}{\zeta(s)} + \sum_{p|q} \frac{\log p}{p^s - 1}.$$

Since there are only finitely many primes dividing q, and each summand $\frac{\log p}{p^s-1}$ extends continuously, the claim follows. ☐

Theorem 2 (WeakPNT$_A P$). If $q \geq 1$ and a is coprime to q, we have

$$\sum_{n \leq x : n = a\ (q)} \Lambda(n) = \frac{x}{\varphi(q)} + o(x).$$

Proof. Apply Theorem **??** (or Theorem **??**) to Proposition **??**. (The Chebyshev bound follows from the corresponding bound for Λ.) ☐

2.5 The Chebotarev density theorem: the case of cyclotomic extensions

In this section, K is a number field, $L = K(\mu_m)$ for some natural number m, and $G = Gal(K/L)$.

The goal here is to prove the Chebotarev density theorem for the case of cyclotomic extensions.

Lemma 14 (Dedekind$_f actor$). We have

$$\zeta_L(s) = \prod_{\chi} L(\chi, s)$$

for $\Re(s) > 1$, where χ runs over homomorphisms from G to $\mathbb{C}^\times$ and L is the Artin L-function.

Proof. See Propositions 7.1.16, 7.1.19 of https://www.math.ucla.edu/ sharifi/algnum.pdf . ☐

Lemma 15 (Simple pole)**.** ζ_L has a simple pole at $s = 1$.

Proof. See Theorem 7.1.12 of https://www.math.ucla.edu/ sharifi/algnum.pdf . □

Lemma 16 (Dedekind$_n$*onvanishing*). For any non-principal character χ of $Gal(K/L)$, $L(\chi, s)$ does not vanish for $\Re(s) = 1$.

Proof. For $s = 1$, this will follow from Lemmas **??**, **??**. For the rest of the line, one should be able to adapt the arguments for the Dirichlet L-function. □

2.6 The Chebotarev density theorem: the case of abelian extensions

(Use the arguments in Theorem 7.2.2 of https://www.math.ucla.edu/ sharifi/algnum.pdf to extend the previous results to abelian extensions (actually just cyclic extensions would suffice))

2.7 The Chebotarev density theorem: the general case

(Use the arguments in Theorem 7.2.2 of https://www.math.ucla.edu/ sharifi/algnum.pdf to extend the previous results to arbitrary extensions

Lemma 17 (PNT for one character)**.** For any non-principal character χ of $Gal(K/L)$,

$$\sum_{N\mathfrak{p} \leq x} \chi(\mathfrak{p}) \log N\mathfrak{p} = o(x).$$

Proof. This should follow from Lemma **??** and the arguments for the Dirichlet L-function. (It may be more convenient to work with a von Mangoldt type function instead of $\log N\mathfrak{p}$). □

Chapter 3

Second approach

3.1 Residue calculus on rectangles

This files gathers definitions and basic properties about rectangles.

The border of a rectangle is the union of its four sides.

Definition 1 (RectangleBorder)**.** A Rectangle's border, given corners z and w is the union of the four sides.

Definition 2 (RectangleIntegral)**.** A RectangleIntegral of a function f is one over a rectangle determined by z and w in $\mathbb{C}$. We will sometimes denote it by $\int_z^w f$. (There is also a primed version, which is $1/(2\pi i)$ times the original.)

It is very convenient to define integrals along vertical lines in the complex plane, as follows.

Definition 3 (VerticalIntegral)**.** Let f be a function from $\mathbb{C}$ to $\mathbb{C}$, and let σ be a real number. Then we define

$$\int_{(\sigma)} f(s)ds = \int_{\sigma-i\infty}^{\sigma+i\infty} f(s)ds.$$

We also have a version with a factor of $1/(2\pi i)$.

Theorem 3 (existsDifferentiableOn$_o f_b ddAbove$)**.** If f is differentiable on a set s except at $c \in s$, and f is bounded above on $s \setminus \{c\}$, then there exists a differentiable function g on s such that f and g agree on $s \setminus \{c\}$.

Proof. This is the Riemann Removable Singularity Theorem, slightly rephrased from what's in Mathlib. (We don't care what the function g is, just that it's holomorphic.) □

Theorem 4 (HolomorphicOn.vanishesOnRectangle)**.** If f is holomorphic on a rectangle z and w, then the integral of f over the rectangle with corners z and w is 0.

Proof. This is in a Mathlib PR. □

The next lemma allows to zoom a big rectangle down to a small square, centered at a pole.

Lemma 18 (RectanglePullToNhdOfPole)**.** If f is holomorphic on a rectangle z and w except at a point p, then the integral of f over the rectangle with corners z and w is the same as the integral of f over a small square centered at p.

Proof. Chop the big rectangle with two vertical cuts and two horizontal cuts into smaller rectangles, the middle one being the desired square. The integral over each of the outer rectangles vanishes, since f is holomorphic there. (The constant c being "small enough" here just means that the inner square is strictly contained in the big rectangle.)

□

Lemma 19 (ResidueTheoremAtOrigin)**.** The rectangle (square) integral of $f(s) = 1/s$ with corners $-1 - i$ and $1 + i$ is equal to $2\pi i$.

Proof. This is a special case of the more general result above. □

Lemma 20 (ResidueTheoremOnRectangleWithSimplePole)**.** Suppose that f is a holomorphic function on a rectangle, except for a simple pole at p. By the latter, we mean that there is a function g holomorphic on the rectangle such that, $f = g + A/(s - p)$ for some $A \in \mathbb{C}$. Then the integral of f over the rectangle is A.

Proof. Replace f with $g + A/(s-p)$ in the integral. The integral of g vanishes by Lemma **??**. To evaluate the integral of $1/(s-p)$, pull everything to a square about the origin using Lemma **??**, and rescale by c; what remains is handled by Lemma **??**. □

3.2 Perron Formula

In this section, we prove the Perron formula, which plays a key role in our proof of Mellin inversion.

The following is preparatory material used in the proof of the Perron formula, see Lemma **??**.

TODO : Move to general section

Lemma 21 (limitOfConstant)**.** Let $a : \mathbb{R} \to \mathbb{C}$ be a function, and let $\sigma > 0$ be a real number. Suppose that, for all $\sigma, \sigma' > 0$, we have $a(\sigma') = a(\sigma)$, and that $\lim_{\sigma \to \infty} a(\sigma) = 0$. Then $a(\sigma) = 0$.

Proof.

$= 0$

□

Lemma 22 (limitOfConstantLeft)**.** Let $a : \mathbb{R} \to \mathbb{C}$ be a function, and let $\sigma < -3/2$ be a real number. Suppose that, for all $\sigma, \sigma' > 0$, we have $a(\sigma') = a(\sigma)$, and that $\lim_{\sigma \to -\infty} a(\sigma) = 0$. Then $a(\sigma) = 0$.

Proof.

$$= 0$$

□

Lemma 23 (tendsto$_rpow_atTop_nhds_zero_of_norm_lt_one$)**.** Let $x > 0$ and $x < 1$. Then

$$\lim_{\sigma \to \infty} x^\sigma = 0.$$

Proof. Standard.

□

Lemma 24 (tendsto$_rpow_atTop_nhds_zero_of_norm_gt_one$)**.** Let $x > 1$. Then

$$\lim_{\sigma \to -\infty} x^\sigma = 0.$$

Proof. Standard. □

Lemma 25 (isHolomorphicOn)**.** Let $x > 0$. Then the function $f(s) = x^s/(s(s+1))$ is holomorphic on the half-plane $\{s \in \mathbb{C} : \Re(s) > 0\}$.

Proof. Composition of differentiabilities.

□

Lemma 26 (integralPosAux)**.** The integral

$$\int_{\mathbb{R}} \frac{1}{|(1+t^2)(2+t^2)|^{1/2}} dt$$

is positive (and hence convergent - since a divergent integral is zero in Lean, by definition).

Proof. This integral is between $\frac{1}{2}$ and 1 of the integral of $\frac{1}{1+t^2}$, which is π.

□

Lemma 27 (vertIntBound)**.** Let $x > 0$ and $\sigma > 1$. Then

$$\left| \int_{(\sigma)} \frac{x^s}{s(s+1)} ds \right| \le x^\sigma \int_{\mathbb{R}} \frac{1}{|(1+t^2)(2+t^2)|^{1/2}} dt.$$

Proof. Triangle inequality and pointwise estimate. □

Lemma 28 (vertIntBoundLeft)**.** Let $x > 1$ and $\sigma < -3/2$. Then

$$\left| \int_{(\sigma)} \frac{x^s}{s(s+1)} ds \right| \le x^\sigma \int_{\mathbb{R}} \frac{1}{|(1/4+t^2)(2+t^2)|^{1/2}} dt.$$

Proof. Triangle inequality and pointwise estimate. □

Lemma 29 (isIntegrable)**.** Let $x > 0$ and $\sigma \in \mathbb{R}$. Then

$$\int_{\mathbb{R}} \frac{x^{\sigma+it}}{(\sigma+it)(1+\sigma+it)} dt$$

is integrable.

Proof. By **??**, f is continuous, so it is integrable on any interval.
Also, $|f(x)| = \Theta(x^{-2})$ as $x \to \infty$,
and $|f(-x)| = \Theta(x^{-2})$ as $x \to \infty$.
Since $g(x) = x^{-2}$ is integrable on $[a, \infty)$ for any $a > 0$, we conclude. □

Lemma 30 ($\text{tendsto}_z ero_L ower$). Let $x > 0$ and $\sigma', \sigma'' \in \mathbb{R}$. Then

$$\int_{\sigma'}^{\sigma''} \frac{x^{\sigma+it}}{(\sigma+it)(1+\sigma+it)} d\sigma$$

goes to 0 as $t \to -\infty$.

Proof. The numerator is bounded and the denominator tends to infinity. □

Lemma 31 ($\text{tendsto}_z ero_U pper$). Let $x > 0$ and $\sigma', \sigma'' \in \mathbb{R}$. Then

$$\int_{\sigma'}^{\sigma''} \frac{x^{\sigma+it}}{(\sigma+it)(1+\sigma+it)} d\sigma$$

goes to 0 as $t \to \infty$.

Proof. The numerator is bounded and the denominator tends to infinity. □

We are ready for the first case of the Perron formula, namely when $x < 1$:

Lemma 32 (formulaLtOne)**.** For $x > 0$, $\sigma > 0$, and $x < 1$, we have

$$\frac{1}{2\pi i} \int_{(\sigma)} \frac{x^s}{s(s+1)} ds = 0.$$

Proof. Let $f(s) = x^s/(s(s+1))$. Then f is holomorphic on the half-plane $\{s \in \mathbb{C} : \Re(s) > 0\}$. The rectangle integral of f with corners $\sigma - iT$ and $\sigma + iT$ is zero. The limit of this rectangle integral as $T \to \infty$ is $\int_{(\sigma')} - \int_{(\sigma)}$. Therefore, $\int_{(\sigma')} = \int_{(\sigma)}$.
But we also have the bound $\int_{(\sigma')} \leq x^{\sigma'} * C$, where
$C = \int_{\mathbb{R}} \frac{1}{|(1+t)(1+t+1)|} dt$.
Therefore $\int_{(\sigma')} \to 0$ as $\sigma' \to \infty$. □

The second case is when $x > 1$. Here are some auxiliary lemmata for the second case. TODO: Move to more general section

Lemma 33 (keyIdentity)**.** Let $x \in \mathbb{R}$ and $s \neq 0, -1$. Then

$$\frac{x^\sigma}{s(1+s)} = \frac{x^\sigma}{s} - \frac{x^\sigma}{1+s}$$

Proof. By ring. ☐

Lemma 34 (diffBddAtZero)**.** Let $x > 0$. Then for $0 < c < 1/2$, we have that the function

$$s \mapsto \frac{x^s}{s(s+1)} - \frac{1}{s}$$

is bounded above on the rectangle with corners at $-c - i * c$ and $c + i * c$ (except at $s = 0$).

Proof. Applying Lemma **??**, the function $s \mapsto x^s/s(s+1) - 1/s = x^s/s - x^0/s - x^s/(1+s)$. The last term is bounded for s away from -1. The first two terms are the difference quotient of the function $s \mapsto x^s$ at 0; since it's differentiable, the difference remains bounded as $s \to 0$. ☐

Lemma 35 (diffBddAtNegOne)**.** Let $x > 0$. Then for $0 < c < 1/2$, we have that the function

$$s \mapsto \frac{x^s}{s(s+1)} - \frac{-x^{-1}}{s+1}$$

is bounded above on the rectangle with corners at $-1 - c - i * c$ and $-1 + c + i * c$ (except at $s = -1$).

Proof. Applying Lemma **??**, the function $s \mapsto x^s/s(s+1) - x^{-1}/(s+1) = x^s/s - x^s/(s+1) - (-x^{-1})/(s+1)$. The first term is bounded for s away from 0. The last two terms are the difference quotient of the function $s \mapsto x^s$ at -1; since it's differentiable, the difference remains bounded as $s \to -1$. ☐

Lemma 36 (residueAtZero)**.** Let $x > 0$. Then for all sufficiently small $c > 0$, we have that

$$\frac{1}{2\pi i} \int_{-c-i*c}^{c+i*c} \frac{x^s}{s(s+1)} ds = 1.$$

Proof. For $c > 0$ sufficiently small,

$x^s/(s(s+1))$ is equal to $1/s$ plus a function, g, say, holomorphic in the whole rectangle (by Lemma **??**).

Now apply Lemma **??**.

☐

Lemma 37 (residueAtNegOne)**.** Let $x > 0$. Then for all sufficiently small $c > 0$, we have that

$$\frac{1}{2\pi i} \int_{-c-i*c-1}^{c+i*c-1} \frac{x^s}{s(s+1)} ds = -\frac{1}{x}.$$

Proof. Compute the integral. ☐

Lemma 38 (residuePull1)**.** For $x > 1$ (of course $x > 0$ would suffice) and $\sigma > 0$, we have

$$\frac{1}{2\pi i} \int_{(\sigma)} \frac{x^s}{s(s+1)} ds = 1 + \frac{1}{2\pi i} \int_{(-1/2)} \frac{x^s}{s(s+1)} ds.$$

Proof. We pull to a square with corners at $-c - i * c$ and $c + i * c$ for $c > 0$ sufficiently small. By Lemma **??**, the integral over this square is equal to 1. ☐

Lemma 39 (residuePull2)**.** For $x > 1$, we have

$$\frac{1}{2\pi i}\int_{(-1/2)}\frac{x^s}{s(s+1)}ds = -1/x + \frac{1}{2\pi i}\int_{(-3/2)}\frac{x^s}{s(s+1)}ds.$$

Proof. Pull contour from $(-1/2)$ to $(-3/2)$. □

Lemma 40 (contourPull3)**.** For $x > 1$ and $\sigma < -3/2$, we have

$$\frac{1}{2\pi i}\int_{(-3/2)}\frac{x^s}{s(s+1)}ds = \frac{1}{2\pi i}\int_{(\sigma)}\frac{x^s}{s(s+1)}ds.$$

Proof. Pull contour from $(-3/2)$ to (σ). □

Lemma 41 (formulaGtOne)**.** For $x > 1$ and $\sigma > 0$, we have

$$\frac{1}{2\pi i}\int_{(\sigma)}\frac{x^s}{s(s+1)}ds = 1 - 1/x.$$

Proof. Let $f(s) = x^s/(s(s+1))$. Then f is holomorphic on $\mathbb{C} \setminus 0, -1$.

First pull the contour from (σ) to $(-1/2)$, picking up a residue 1 at $s = 0$.

Next pull the contour from $(-1/2)$ to $(-3/2)$, picking up a residue $-1/x$ at $s = -1$.

Then pull the contour all the way to (σ') with $\sigma' < -3/2$.

For $\sigma' < -3/2$, the integral is bounded by $x^{\sigma'}\int_{\mathbb{R}}\frac{1}{|(1+t^2)(2+t^2)|^{1/2}}dt$.

Therefore $\int_{(\sigma')} \to 0$ as $\sigma' \to \infty$.

□

The two together give the Perron formula. (Which doesn't need to be a separate lemma.) For $x > 0$ and $\sigma > 0$, we have

$$\frac{1}{2\pi i}\int_{(\sigma)}\frac{x^s}{s(s+1)}ds = \begin{cases} 1 - \frac{1}{x} & \text{if } x > 1 \\ 0 & \text{if } x < 1 \end{cases}.$$

3.3 Mellin transforms

Lemma 42 (PartialIntegration)**.** Let f, g be once differentiable functions from $\mathbb{R}_{>0}$ to $\mathbb{C}$ so that fg' and $f'g$ are both integrable, and $f \cdot g(x) \to 0$ as $x \to 0^+, \infty$. Then

$$\int_0^\infty f(x)g'(x)dx = -\int_0^\infty f'(x)g(x)dx.$$

Proof. Partial integration. □

In this section, we define the Mellin transform (already in Mathlib, thanks to David Loeffler), prove its inversion formula, and derive a number of important properties of some special functions and bumpfunctions.

Def: (Already in Mathlib) Let f be a function from $\mathbb{R}_{>0}$ to $\mathbb{C}$. We define the Mellin transform of f to be the function $\mathcal{M}(f)$ from $\mathbb{C}$ to $\mathbb{C}$ defined by

$$\mathcal{M}(f)(s) = \int_0^\infty f(x)x^{s-1}dx.$$

[Note: My preferred way to think about this is that we are integrating over the multiplicative group $\mathbb{R}_{>0}$, multiplying by a (not necessarily unitary!) character $|\cdot|^s$, and integrating with respect to the invariant Haar measure dx/x. This is very useful in the kinds of calculations carried out below. But may be more difficult to formalize as things now stand. So we might have clunkier calculations, which "magically" turn out just right - of course they're explained by the aforementioned structure...]

Definition 4 (MellinTransform)**.** Let f be a function from $\mathbb{R}_{>0}$ to $\mathbb{C}$. We define the Mellin transform of f to be the function $\mathcal{M}(f)$ from $\mathbb{C}$ to $\mathbb{C}$ defined by

$$\mathcal{M}(f)(s) = \int_0^\infty f(x)x^{s-1}dx.$$

[Note: already exists in Mathlib, with some good API.]

Definition 5 (MellinInverseTransform)**.** Let F be a function from $\mathbb{C}$ to $\mathbb{C}$. We define the Mellin inverse transform of F to be the function $\mathcal{M}^{-1}(F)$ from $\mathbb{R}_{>0}$ to $\mathbb{C}$ defined by

$$\mathcal{M}^{-1}(F)(x) = \frac{1}{2\pi i}\int_{(\sigma)} F(s)x^{-s}ds,$$

where σ is sufficiently large (say $\sigma > 2$).

Lemma 43 (PerronInverseMellin$_l t$)**.** Let $0 < t < x$ and $\sigma > 0$. Then the inverse Mellin transform of the Perron function

$$F : s \mapsto t^s/s(s+1)$$

is equal to

$$\frac{1}{2\pi i}\int_{(\sigma)} \frac{t^s}{s(s+1)}x^{-s}ds = 0.$$

Proof. This is a straightforward calculation. □

Lemma 44 (PerronInverseMellin$_g t$)**.** Let $0 < x < t$ and $\sigma > 0$. Then the inverse Mellin transform of the Perron function is equal to

$$\frac{1}{2\pi i}\int_{(\sigma)} \frac{t^s}{s(s+1)}x^{-s}ds = 1 - x/t.$$

Proof. This is a straightforward calculation. □

Theorem 5 (MellinInversion)**.** Let f be a twice differentiable function from $\mathbb{R}_{>0}$ to $\mathbb{C}$, and let σ be sufficiently large. Then

$$f(x) = \frac{1}{2\pi i}\int_{(\sigma)} \mathcal{M}(f)(s)x^{-s}ds.$$

Proof. The proof is from [Goldfeld-Kontorovich 2012]. Integrate by parts twice (assuming f is twice differentiable, and all occurring integrals converge absolutely, and boundary terms vanish).

$$\mathcal{M}(f)(s) = \int_0^\infty f(x)x^{s-1}dx = -\int_0^\infty f'(x)x^s\frac{1}{s}dx = \int_0^\infty f''(x)x^{s+1}\frac{1}{s(s+1)}dx.$$

We now have at least quadratic decay in s of the Mellin transform. Inserting this formula into the inversion formula and Fubini-Tonelli (we now have absolute convergence!) gives:

$$RHS = \frac{1}{2\pi i}\left(\int_{(\sigma)}\int_0^\infty f''(t)t^{s+1}\frac{1}{s(s+1)}dt\right)x^{-s}ds$$

$$= \int_0^\infty f''(t)t\left(\frac{1}{2\pi i}\int_{(\sigma)}(t/x)^s\frac{1}{s(s+1)}ds\right)dt.$$

Apply the Perron formula to the inside:

$$= \int_x^\infty f''(t)t\left(1-\frac{x}{t}\right)dt = -\int_x^\infty f'(t)dt = f(x),$$

where we integrated by parts (undoing the first partial integration), and finally applied the fundamental theorem of calculus (undoing the second). ☐

Finally, we need Mellin Convolutions and properties thereof.

Definition 6 (MellinConvolution)**.** Let f and g be functions from $\mathbb{R}_{>0}$ to $\mathbb{C}$. Then we define the Mellin convolution of f and g to be the function $f * g$ from $\mathbb{R}_{>0}$ to $\mathbb{C}$ defined by

$$(f*g)(x) = \int_0^\infty f(y)g(x/y)\frac{dy}{y}.$$

Let us start with a simple property of the Mellin convolution.

Lemma 45 (MellinConvolutionSymmetric)**.** Let f and g be functions from $\mathbb{R}_{>0}$ to $\mathbb{R}$ or $\mathbb{C}$, for $x \neq 0$,

$$(f*g)(x) = (g*f)(x).$$

Proof. By Definition **??**,

$$(f*g)(x) = \int_0^\infty f(y)g(x/y)\frac{dy}{y}$$

in which we change variables to $z = x/y$:

$$(f*g)(x) = \int_0^\infty f(x/z)g(z)\frac{dz}{z} = (g*f)(x).$$

☐

The Mellin transform of a convolution is the product of the Mellin transforms.

Theorem 6 (MellinConvolutionTransform)**.** Let f and g be functions from $\mathbb{R}_{>0}$ to $\mathbb{C}$ such that

$$(x,y) \mapsto f(y)\frac{g(x/y)}{y}x^{s-1} \tag{3.1}$$

is absolutely integrable on $[0,\infty)^2$. Then

$$\mathcal{M}(f*g)(s) = \mathcal{M}(f)(s)\mathcal{M}(g)(s).$$

Proof. By Definitions **??** and **??**

$$\mathcal{M}(f * g)(s) = \int_0^\infty \int_0^\infty f(y)g(x/y)x^{s-1}\frac{dy}{y}dx$$

By (**??**) and Fubini's theorem,

$$\mathcal{M}(f * g)(s) = \int_0^\infty \int_0^\infty f(y)g(x/y)x^{s-1}dx\frac{dy}{y}$$

in which we change variables from x to $z = x/y$:

$$\mathcal{M}(f * g)(s) = \int_0^\infty \int_0^\infty f(y)g(z)y^{s-1}z^{s-1}dzdy$$

which, by Definition **??**, is

$$\mathcal{M}(f * g)(s) = \mathcal{M}(f)(s)\mathcal{M}(g)(s).$$

□

Let ν be a bumpfunction.

Theorem 7 (SmoothExistence)**.** There exists a smooth (once differentiable would be enough), nonnegative "bumpfunction" ν, supported in $[1/2, 2]$ with total mass one:

$$\int_0^\infty \nu(x)\frac{dx}{x} = 1.$$

Proof. Same idea as Urysohn-type argument. □

The ν function has Mellin transform $\mathcal{M}(\nu)(s)$ which is entire and decays (at least) like $1/|s|$.

Theorem 8 (MellinOfPsi)**.** The Mellin transform of ν is

$$\mathcal{M}(\nu)(s) = O\left(\frac{1}{|s|}\right),$$

as $|s| \to \infty$ with $\sigma_1 \leq \Re(s) \leq 2$.

[Of course it decays faster than any power of $|s|$, but it turns out that we will just need one power.]

Proof. Integrate by parts:

$$\left|\int_0^\infty \nu(x)x^s\frac{dx}{x}\right| = \left|-\int_0^\infty \nu'(x)\frac{x^s}{s}dx\right|$$
$$\leq \frac{1}{|s|}\int_{1/2}^2 |\nu'(x)|x^{\Re(s)}dx.$$

Since $\Re(s)$ is bounded, the right-hand side is bounded by a constant times $1/|s|$. □

We can make a delta spike out of this bumpfunction, as follows.

Definition 7 (DeltaSpike)**.** Let ν be a bumpfunction supported in $[1/2, 2]$. Then for any $\epsilon > 0$, we define the delta spike ν_ϵ to be the function from $\mathbb{R}_{>0}$ to $\mathbb{C}$ defined by

$$\nu_\epsilon(x) = \frac{1}{\epsilon}\nu\left(x^{\frac{1}{\epsilon}}\right).$$

This spike still has mass one:

Lemma 46 (DeltaSpikeMass)**.** For any $\epsilon > 0$, we have

$$\int_0^\infty \nu_\epsilon(x)\frac{dx}{x} = 1.$$

Proof. Substitute $y = x^{1/\epsilon}$, and use the fact that ν has mass one, and that dx/x is Haar measure. □

The Mellin transform of the delta spike is easy to compute.

Theorem 9 (MellinOfDeltaSpike)**.** For any $\epsilon > 0$, the Mellin transform of ν_ϵ is

$$\mathcal{M}(\nu_\epsilon)(s) = \mathcal{M}(\nu)(\epsilon s).$$

Proof. Substitute $y = x^{1/\epsilon}$, use Haar measure; direct calculation. □

In particular, for $s = 1$, we have that the Mellin transform of ν_ϵ is $1 + O(\epsilon)$.

Corollary 7 (MellinOfDeltaSpikeAt1)**.** For any $\epsilon > 0$, we have

$$\mathcal{M}(\nu_\epsilon)(1) = \mathcal{M}(\nu)(\epsilon).$$

Proof. This is immediate from the above theorem. □

Lemma 47 (MellinOfDeltaSpikeAt1$_a$*symp*)**.** As $\epsilon \to 0$, we have

$$\mathcal{M}(\nu_\epsilon)(1) = 1 + O(\epsilon).$$

Proof. By Lemma **??**,

$$\mathcal{M}(\nu_\epsilon)(1) = \mathcal{M}(\nu)(\epsilon)$$

which by Definition **??** is

$$\mathcal{M}(\nu)(\epsilon) = \int_0^\infty \nu(x)x^{\epsilon-1}dx.$$

Since $\nu(x)x^{\epsilon-1}$ is integrable (because ν is continuous and compactly supported),

$$\mathcal{M}(\nu)(\epsilon) - \int_0^\infty \nu(x)\frac{dx}{x} = \int_0^\infty \nu(x)(x^{\epsilon-1} - x^{-1})dx.$$

By Taylor's theorem,

$$x^{\epsilon-1} - x^{-1} = O(\epsilon)$$

so, since ν is absolutely integrable,

$$\mathcal{M}(\nu)(\epsilon) - \int_0^\infty \nu(x)\frac{dx}{x} = O(\epsilon).$$

We conclude the proof using Theorem **??**. □

Let $1_{(0,1]}$ be the function from $\mathbb{R}_{>0}$ to $\mathbb{C}$ defined by

$$1_{(0,1]}(x) = \begin{cases} 1 & \text{if } x \le 1 \\ 0 & \text{if } x > 1 \end{cases}.$$

This has Mellin transform

Theorem 10 (MellinOf1)**.** The Mellin transform of $1_{(0,1]}$ is

$$\mathcal{M}(1_{(0,1]})(s) = \frac{1}{s}.$$

[Note: this already exists in mathlib]

Proof. This is a straightforward calculation. □

What will be essential for us is properties of the smooth version of $1_{(0,1]}$, obtained as the Mellin convolution of $1_{(0,1]}$ with ν_ϵ.

Definition 8 (Smooth1)**.** Let $\epsilon > 0$. Then we define the smooth function $\widetilde{1_\epsilon}$ from $\mathbb{R}_{>0}$ to $\mathbb{C}$ by

$$\widetilde{1_\epsilon} = 1_{(0,1]} * \nu_\epsilon.$$

Proof. Let $c := 2^\epsilon > 1$, in terms of which we wish to prove

$$-1 < c \log c - c.$$

Letting $f(x) := x \log x - x$, we can rewrite this as $f(1) < f(c)$. Since

$$\frac{d}{dx} f(x) = \log x > 0,$$

f is monotone increasing on $[1, \infty)$, and we are done. □

In particular, we have the following two properties.

Lemma 48 (Smooth1Properties$_b elow$). Fix $\epsilon > 0$. There is an absolute constant $c > 0$ so that: If $0 < x \le (1 - c\epsilon)$, then

$$\widetilde{1_\epsilon}(x) = 1.$$

Proof. Opening the definition, we have that the Mellin convolution of $1_{(0,1]}$ with ν_ϵ is

$$\int_0^\infty 1_{(0,1]}(y) \nu_\epsilon(x/y) \frac{dy}{y} = \int_0^1 \nu_\epsilon(x/y) \frac{dy}{y}.$$

The support of ν_ϵ is contained in $[1/2^\epsilon, 2^\epsilon]$, so it suffices to consider $y \in [1/2^\epsilon x, 2^\epsilon x]$ for nonzero contributions. If $x < 2^{-\epsilon}$, then the integral is the same as that over $(0, \infty)$:

$$\int_0^1 \nu_\epsilon(x/y) \frac{dy}{y} = \int_0^\infty \nu_\epsilon(x/y) \frac{dy}{y},$$

in which we change variables to $z = x/y$ (using $x > 0$):

$$\int_0^\infty \nu_\epsilon(x/y) \frac{dy}{y} = \int_0^\infty \nu_\epsilon(z) \frac{dz}{z},$$

which is equal to one by Lemma **??**. We then choose

$$c := \log 2,$$

which satisfies

$$c > \frac{1 - 2^{-\epsilon}}{\epsilon}$$

by Lemma **??**, so

$$1 - c\epsilon < 2^{-\epsilon}.$$

□

Lemma 49 (Smooth1Properties$_a$*bove*)**.** Fix $0 < \epsilon < 1$. There is an absolute constant $c > 0$ so that: if $x \geq (1 + c\epsilon)$, then

$$\widetilde{1_\epsilon}(x) = 0.$$

Proof. Again the Mellin convolution is

$$\int_0^1 \nu_\epsilon(x/y)\frac{dy}{y},$$

but now if $x > 2^\epsilon$, then the support of ν_ϵ is disjoint from the region of integration, and hence the integral is zero. We choose

$$c := 2\log 2.$$

By Lemma **??**,

$$c > 2\frac{1 - 2^{-\epsilon}}{\epsilon} > 2^\epsilon\frac{1 - 2^{-\epsilon}}{\epsilon} = \frac{2^\epsilon - 1}{\epsilon},$$

so

$$1 + c\epsilon > 2^\epsilon.$$

□

Lemma 50 (Smooth1Nonneg)**.** If ν is nonnegative, then $\widetilde{1_\epsilon}(x)$ is nonnegative.

Proof. By Definitions **??**, **??** and **??**

$$\widetilde{1_\epsilon}(x) = \int_0^\infty 1_{(0,1]}(y)\frac{1}{\epsilon}\nu((x/y)^{\frac{1}{\epsilon}})\frac{dy}{y}$$

and all the factors in the integrand are nonnegative. □

Lemma 51 (Smooth1LeOne)**.** If ν is nonnegative and has mass one, then $\widetilde{1_\epsilon}(x) \leq 1$, $\forall x > 0$.

Proof. By Definitions **??**, **??** and **??**

$$\widetilde{1_\epsilon}(x) = \int_0^\infty 1_{(0,1]}(y)\frac{1}{\epsilon}\nu((x/y)^{\frac{1}{\epsilon}})\frac{dy}{y}$$

and since $1_{(0,1]}(y) \leq 1$, and all the factors in the integrand are nonnegative,

$$\widetilde{1_\epsilon}(x) \leq \int_0^\infty \frac{1}{\epsilon}\nu((x/y)^{\frac{1}{\epsilon}})\frac{dy}{y}$$

(because in mathlib the integral of a non-integrable function is 0, for the inequality above to be true, we must prove that $\nu((x/y)^{\frac{1}{\epsilon}})/y$ is integrable; this follows from the computation below). We then change variables to $z = (x/y)^{\frac{1}{\epsilon}}$:

$$\widetilde{1_\epsilon}(x) \leq \int_0^\infty \nu(z) \frac{dz}{z}$$

which by Theorem **??** is 1. ☐

Combining the above, we have the following three Main Lemmata of this section on the Mellin transform of $\widetilde{1_\epsilon}$.

Lemma 52 (MellinOfSmooth1a)**.** Fix $\epsilon > 0$. Then the Mellin transform of $\widetilde{1_\epsilon}$ is

$$\mathcal{M}(\widetilde{1_\epsilon})(s) = \frac{1}{s} \left(\mathcal{M}(\nu) \left(\epsilon s \right) \right).$$

Proof. By Definition **??**,

$$\mathcal{M}(\widetilde{1_\epsilon})(s) = \mathcal{M}(1_{(0,1]} * \nu_\epsilon)(s).$$

We wish to apply Theorem **??**. To do so, we must prove that

$$(x, y) \mapsto 1_{(0,1]}(y)\nu_\epsilon(x/y)/y$$

is integrable on $[0, \infty)^2$. It is actually easier to do this for the convolution: $\nu_\epsilon * 1_{(0,1]}$, so we use Lemma **??**: for $x \neq 0$,

$$1_{(0,1]} * \nu_\epsilon(x) = \nu_\epsilon * 1_{(0,1]}(x).$$

Now, for $x = 0$, both sides of the equation are 0, so the equation also holds for $x = 0$. Therefore,

$$\mathcal{M}(\widetilde{1_\epsilon})(s) = \mathcal{M}(\nu_\epsilon * 1_{(0,1]})(s).$$

Now,

$$(x, y) \mapsto \nu_\epsilon(y)1_{(0,1]}(x/y)\frac{x^{s-1}}{y}$$

has compact support that is bounded away from $y = 0$ (specifically $y \in [2^{-\epsilon}, 2^\epsilon]$ and $x \in (0, y]$), so it is integrable. We can thus apply Theorem **??** and find

$$\mathcal{M}(\widetilde{1_\epsilon})(s) = \mathcal{M}(\nu_\epsilon)(s)\mathcal{M}(1_{(0,1]})(s).$$

By Lemmas **??** and **??**,

$$\mathcal{M}(\widetilde{1_\epsilon})(s) = \frac{1}{s}\mathcal{M}(\nu)(\epsilon s).$$

☐

Lemma 53 (MellinOfSmooth1b)**.** Given $0 < \sigma_1 \leq \sigma_2$, for any s such that $\sigma_1 \leq \mathcal{R}e(s) \leq \sigma_2$, we have

$$\mathcal{M}(\widetilde{1_\epsilon})(s) = O\left(\frac{1}{\epsilon |s|^2}\right).$$

Proof. Use Lemma **??** and the bound in Lemma **??**. ☐

Lemma 54 (MellinOfSmooth1c)**.** At $s = 1$, we have

$$\mathcal{M}(\widetilde{1_\epsilon})(1) = 1 + O(\epsilon)).$$

Proof. Follows from Lemmas **??**, **??** and **??**. □

Lemma 55 (Smooth1ContinuousAt)**.** Fix a nonnegative, continuously differentiable function F on $\mathbb{R}$ with support in $[1/2, 2]$. Then for any $\epsilon > 0$, the function $x \mapsto \int_{(0,\infty)} x^{1+it}\widetilde{1_\epsilon}(x)dx$ is continuous at any $y > 0$.

Proof. Use Lemma **??** to write $\widetilde{1_\epsilon}(x)$ as an integral over an integral near 1, in particular avoiding the singularity at 0. The integrand may be bounded by $2^\epsilon \nu_\epsilon(t)$ which is independent of x and we can use dominated convergence to prove continuity. □

3.4 Zeta Bounds

We record here some prelimiaries about the zeta function and general holomorphic functions.

Theorem 11 (ResidueOfTendsTo)**.** If a function f is holomorphic in a neighborhood of p and $\lim_{s\to p}(s-p)f(s) = A$, then $f(s) = \frac{A}{s-p} + O(1)$ near p.

Proof. The function $(s-p)\cdot f(s)$ bounded, so by Theorem **??**, there is a holomorphic function, g, say, so that $(s-p)f(s) = g(s)$ in a neighborhood of $s = p$, and $g(p) = A$. Now because g is holomorphic, near $s = p$, we have $g(s) = A + O(s-p)$. Then when you divide by $(s-p)$, you get $f(s) = A/(s-p) + O(1)$. □

Theorem 12 (riemannZetaResidue)**.** The Riemann zeta function $\zeta(s)$ has a simple pole at $s = 1$ with residue 1. In particular, the function

$$\zeta(s) - \frac{1}{s-1}$$

is bounded in a neighborhood of $s = 1$.

Proof. From ‘riemannZeta$_r$*esidue*$_o$*ne*‘(*inMathlib*), *weknowthat*(s-1) (s)*goesto*1*as*s$\to$1.*NowapplyTheorem***??**.(*Thi*
below, which is expressed as $1/(s-1)$ plus things that are holomorphic for $\Re(s) > 0$...) □

Theorem 13 (nonZeroOfBddAbove)**.** If a function f has a simple pole at a point p with residue $A \neq 0$, then f is nonzero in a punctured neighborhood of p.

Proof. We know that $f(s) = \frac{A}{s-p} + O(1)$ near p, so we can write

$$f(s) = \left(f(s) - \frac{A}{s-p}\right) + \frac{A}{s-p}.$$

The first term is bounded, say by M, and the second term goes to ∞ as $s \to p$. Therefore, there exists a neighborhood V of p such that for all $s \in V \setminus \{p\}$, we have $f(s) \neq 0$. □

Theorem 14 (logDerivResidue)**.** If f is holomorphic in a neighborhood of p, and there is a simple pole at p, then f'/f has a simple pole at p with residue -1:

$$\frac{f'(s)}{f(s)} = \frac{-1}{s-p} + O(1).$$

Proof. Using Theorem **??**, there is a function g holomorphic near p, for which $f(s) = A/(s-p) + g(s) = h(s)/(s-p)$. Here $h(s) := A + g(s)(s-p)$ which is nonzero in a neighborhood of p (since h goes to A which is nonzero). Then $f'(s) = (h'(s)(s-p) - h(s))/(s-p)^2$, and we can compute the quotient:

$$\frac{f'(s)}{f(s)} + 1/(s-p) = \frac{h'(s)(s-p) - h(s)}{h(s)} \cdot \frac{1}{(s-p)} + 1/(s-p) = \frac{h'(s)}{h(s)}.$$

Since h is nonvanishing near p, this remains bounded in a neighborhood of p. □

Theorem 15 (BddAbove$_t o_I sBigO$)**.** If f is bounded above in a punctured neighborhood of p, then f is $O(1)$ in that neighborhood.

Proof. Elementary... □

Let's also record that if a function f has a simple pole at p with residue A, and g is holomorphic near p, then the residue of $f \cdot g$ is $A \cdot g(p)$.

Theorem 16 (ResidueMult)**.** If f has a simple pole at p with residue A, and g is holomorphic near p, then the residue of $f \cdot g$ at p is $A \cdot g(p)$. That is, we assume that

$$f(s) = \frac{A}{s-p} + O(1)$$

near p, and that g is holomorphic near p. Then

$$f(s) \cdot g(s) = \frac{A \cdot g(p)}{s-p} + O(1).$$

Proof. Elementary calculation.

$$f(s) * g(s) - \frac{A * g(p)}{s-p} = \left(f(s) * g(s) - \frac{A * g(s)}{s-p}\right) + \left(\frac{A * g(s) - A * g(p)}{s-p}\right).$$

The first term is $g(s)(f(s) - \frac{A}{s-p})$, which is bounded near p by the assumption on f and the fact that g is holomorphic near p. The second term is A times the log derivative of g at p, which is bounded by the assumption that g is holomorphic. □

As a corollary, the log derivative of the Riemann zeta function has a simple pole at $s = 1$:

Theorem 17 (riemannZetaLogDerivResidue)**.** The log derivative of the Riemann zeta function $\zeta(s)$ has a simple pole at $s = 1$ with residue -1:

$$-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} = O(1).$$

Proof. This follows from Theorem **??** and Theorem **??**. □

Definition 9 (riemannZeta0)**.** For any natural $N \geq 1$, we define

$$\zeta_0(N, s) := \sum_{1 \leq n \leq N} \frac{1}{n^s} + \frac{-N^{1-s}}{1-s} + \frac{-N^{-s}}{2} + s \int_N^\infty \frac{\lfloor x \rfloor + 1/2 - x}{x^{s+1}}\, dx$$

Lemma 56 (sum$_e q_i nt_d eriv_a ux$). Let $k \le a < b \le k+1$, with k an integer, and let ϕ be continuously differentiable on $[a, b]$. Then

$$\sum_{a<n\le b} \phi(n) = \int_a^b \phi(x)\,dx + \Big(\lfloor b\rfloor + \frac{1}{2} - b\Big)\,\phi(b) - \Big(\lfloor a\rfloor + \frac{1}{2} - a\Big)\,\phi(a) - \int_a^b \Big(\lfloor x\rfloor + \frac{1}{2} - x\Big)\,\phi'(x)\,dx.$$

Proof. Partial integration. □

Lemma 57 (sum$_e q_i nt_d eriv$). Let $a < b$, and let ϕ be continuously differentiable on $[a, b]$. Then

$$\sum_{a<n\le b} \phi(n) = \int_a^b \phi(x)\,dx + \Big(\lfloor b\rfloor + \frac{1}{2} - b\Big)\,\phi(b) - \Big(\lfloor a\rfloor + \frac{1}{2} - a\Big)\,\phi(a) - \int_a^b \Big(\lfloor x\rfloor + \frac{1}{2} - x\Big)\,\phi'(x)\,dx.$$

Proof. Apply Lemma **??** in blocks of length ≤ 1. □

Lemma 58 (ZetaSum$_a ux1$). Let $0 < a < b$ be natural numbers and $s \in \mathbb{C}$ with $s \ne 1$ and $s \ne 0$. Then

$$\sum_{a<n\le b} \frac{1}{n^s} = \frac{b^{1-s} - a^{1-s}}{1-s} + \frac{b^{-s} - a^{-s}}{2} + s\int_a^b \frac{\lfloor x\rfloor + 1/2 - x}{x^{s+1}}\,dx.$$

Proof. Apply Lemma **??** to the function $x \mapsto x^{-s}$. □

Lemma 59 (ZetaBnd$_a ux1a$). For any $0 < a < b$ and $s \in \mathbb{C}$ with $\sigma = \Re(s) > 0$,

$$\int_a^b \left|\frac{\lfloor x\rfloor + 1/2 - x}{x^{s+1}}\,dx\right| \le \frac{a^{-\sigma} - b^{-\sigma}}{\sigma}.$$

Proof. Apply the triangle inequality

$$\left|\int_a^b \frac{\lfloor x\rfloor + 1/2 - x}{x^{s+1}}\,dx\right| \le \int_a^b \frac{1}{x^{\sigma+1}}\,dx,$$

and evaluate the integral. □

Lemma 60 (ZetaSum$_a ux2$). Let N be a natural number and $s \in \mathbb{C}$, $\Re(s) > 1$. Then

$$\sum_{N<n} \frac{1}{n^s} = \frac{-N^{1-s}}{1-s} + \frac{-N^{-s}}{2} + s\int_N^\infty \frac{\lfloor x\rfloor + 1/2 - x}{x^{s+1}}\,dx.$$

Proof. Apply Lemma **??** with $a = N$ and $b \to \infty$. □

Lemma 61 (ZetaBnd$_a ux1b$). For any $N \ge 1$ and $s = \sigma + tI \in \mathbb{C}$, $\sigma > 0$,

$$\left|\int_N^\infty \frac{\lfloor x\rfloor + 1/2 - x}{x^{s+1}}\,dx\right| \le \frac{N^{-\sigma}}{\sigma}.$$

Proof. Apply Lemma **??** with $a = N$ and $b \to \infty$. □

Lemma 62 (ZetaBnd$_a ux1$). For any $N \ge 1$ and $s = \sigma + tI \in \mathbb{C}$, $\sigma =\in (0, 2], 2 < |t|$,

$$\left|s\int_N^\infty \frac{\lfloor x\rfloor + 1/2 - x}{x^{s+1}}\,dx\right| \le 2|t|\frac{N^{-\sigma}}{\sigma}.$$

Proof. Apply Lemma ?? and estimate $|s| \ll |t|$. $\square$

Big-Oh version of Lemma ??.

Lemma 63 (ZetaBnd$_aux1p$). For any $N \geq 1$ and $s = \sigma + tI \in \mathbb{C}$, $\sigma =\in (0, 2], 2 < |t|$,

$$\left| s \int_N^\infty \frac{\lfloor x \rfloor + 1/2 - x}{x^{s+1}} \, dx \right| \ll |t| \frac{N^{-\sigma}}{\sigma}.$$

Proof. Apply Lemma ?? and estimate $|s| \ll |t|$. $\square$

Lemma 64 (HolomorphicOn$_Zeta0$). For any $N \geq 1$, the function $\zeta_0(N, s)$ is holomorphic on $\{s \in \mathbb{C} \mid \Re(s) > 0 \land s \neq 1\}$.

Proof. The function $\zeta_0(N, s)$ is a finite sum of entire functions, plus an integral that's absolutely convergent on $\{s \in \mathbb{C} \mid \Re(s) > 0 \land s \neq 1\}$ by Lemma ??. $\square$

Lemma 65 (isPathConnected$_aux$). The set $\{s \in \mathbb{C} \mid \Re(s) > 0 \land s \neq 1\}$ is path-connected.

Proof. Construct explicit paths from 2 to any point, either a line segment or two joined ones. $\square$

Lemma 66 (Zeta0EqZeta)**.** For $\Re(s) > 0$, $s \neq 1$, and for any N,

$$\zeta_0(N, s) = \zeta(s).$$

Proof. Use Lemma ?? and the Definition ??. $\square$

Lemma 67 (ZetaBnd$_aux2$). Given $n \leq t$ and σ with $1 - A/\log t \leq \sigma$, we have that

$$|n^{-s}| \leq n^{-1} e^A.$$

Proof. Use $|n^{-s}| = n^{-\sigma} = e^{-\sigma \log n} \leq \exp(-\left(1 - \frac{A}{\log t}\right) \log n) \leq n^{-1} e^A$, since $n \leq t$. $\square$

Lemma 68 (ZetaUpperBnd)**.** For any $s = \sigma + tI \in \mathbb{C}$, $1/2 \leq \sigma \leq 2, 3 < |t|$ and any $0 < A < 1$ sufficiently small, and $1 - A/\log |t| \leq \sigma$, we have

$$|\zeta(s)| \ll \log t.$$

Proof. First replace $\zeta(s)$ by $\zeta_0(N, s)$ for $N = \lfloor |t| \rfloor$. We estimate:

$$|\zeta_0(N, s)| \ll \sum_{1 \leq n \leq |t|} |n^{-s}| + \frac{-|t|^{1-\sigma}}{|1 - s|} + \frac{-|t|^{-\sigma}}{2} + |t| \cdot |t|^{-\sigma}/\sigma$$

$$\ll e^A \sum_{1 \leq n < |t|} n^{-1} + |t|^{1-\sigma}$$

, where we used Lemma ?? and Lemma ??. The first term is $\ll \log |t|$. For the second term, estimate

$$|t|^{1-\sigma} \leq |t|^{1-(1-A/\log |t|)} = |t|^{A/\log |t|} \ll 1.$$

$\square$

Lemma 69 (DerivUpperBnd$_a ux7$). For any $s = \sigma + tI \in \mathbb{C}$, $1/2 \le \sigma \le 2, 3 < |t|$, and any $0 < A < 1$ sufficiently small, and $1 - A/\log|t| \le \sigma$, we have

$$\left\| s \cdot \int_N^\infty \left(\lfloor x \rfloor + \frac{1}{2} - x \right) \cdot x^{-s-1} \cdot (-\log x) \right\| \le 2 \cdot |t| \cdot N^{-\sigma}/\sigma \cdot \log|t|.$$

Proof. Estimate $|s| = |\sigma + tI|$ by $|s| \le 2 + |t| \le 2|t|$ (since $|t| > 3$). Estimating $|\lfloor x \rfloor + 1/2 - x|$ by 1, and using $|x^{-s-1}| = x^{-\sigma-1}$, we have

$$\left\| s \cdot \int_N^\infty \left(\lfloor x \rfloor + \frac{1}{2} - x \right) \cdot x^{-s-1} \cdot (-\log x) \right\| \le 2 \cdot |t| \int_N^\infty x^{-\sigma} \cdot (\log x).$$

For the last integral, integrate by parts, getting:

$$\int_N^\infty x^{-\sigma-1} \cdot (\log x) = \frac{1}{\sigma} N^{-\sigma} \cdot \log N + \frac{1}{\sigma^2} \cdot N^{-\sigma}.$$

Now use $\log N \le \log|t|$ to get the result. □

Lemma 70 (ZetaDerivUpperBnd)**.** For any $s = \sigma + tI \in \mathbb{C}$, $1/2 \le \sigma \le 2, 3 < |t|$, there is an $A > 0$ so that for $1 - A/\log t \le \sigma$, we have

$$|\zeta'(s)| \ll \log^2 t.$$

Proof. First replace $\zeta(s)$ by $\zeta_0(N, s)$ for $N = \lfloor |t| \rfloor$. Differentiating term by term, we get:

$$\zeta'(s) = -\sum_{1 \le n < N} n^{-s} \log n + \frac{N^{1-s}}{(1-s)^2} + \frac{N^{1-s} \log N}{1-s} + \frac{N^{-s} \log N}{2} + \int_N^\infty \frac{\lfloor x \rfloor + 1/2 - x}{x^{s+1}}\, dx - s \int_N^\infty \log x \frac{\lfloor x \rfloor + 1/2 - x}{x^{s+1}}\, dx$$

Estimate as before, with an extra factor of $\log|t|$. □

Lemma 71 (ZetaNear1BndFilter)**.** As $\sigma \to 1^+$,

$$|\zeta(\sigma)| \ll 1/(\sigma - 1).$$

Proof. Zeta has a simple pole at $s = 1$. Equivalently, $\zeta(s)(s - 1)$ remains bounded near 1. Lots of ways to prove this. Probably the easiest one: use the expression for $\zeta_0(N, s)$ with $N = 1$ (the term $N^{1-s}/(1 - s)$ being the only unbounded one). □

Lemma 72 (ZetaNear1BndExact)**.** There exists a $c > 0$ such that for all $1 < \sigma \le 2$,

$$|\zeta(\sigma)| \le c/(\sigma - 1).$$

Proof. Split into two cases, use Lemma **??** for σ sufficiently small and continuity on a compact interval otherwise. □

Lemma 73 (ZetaInvBound1)**.** For all $\sigma > 1$,

$$1/|\zeta(\sigma + it)| \le |\zeta(\sigma)|^{3/4} |\zeta(\sigma + 2it)|^{1/4}$$

Proof. The identity

$$1 \le |\zeta(\sigma)|^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)|$$

for $\sigma > 1$ is already proved by Michael Stoll in the EulerProducts PNT file. □

Lemma 74 (ZetaInvBound2). For $\sigma > 1$ (and $\sigma \leq 2$),

$$1/|\zeta(\sigma + it)| \ll (\sigma - 1)^{-3/4}(\log |t|)^{1/4},$$

as $|t| \to \infty$.

Proof. Combine Lemma **??** with the bounds in Lemmata **??** and **??**. ☐

Lemma 75 (Zeta$_e q_i nt_d erivZeta$). For any $t \neq 0$ (so we don't pass through the pole), and $\sigma_1 < \sigma_2$,

$$\int_{\sigma_1}^{\sigma_2} \zeta'(\sigma + it) dt = \zeta(\sigma_2 + it) - \zeta(\sigma_1 + it).$$

Proof. This is the fundamental theorem of calculus. ☐

Lemma 76 (Zeta$_d iff_B nd$). For any $A > 0$ sufficiently small, there is a constant $C > 0$ so that whenever $1 - A/\log t \leq \sigma_1 < \sigma_2 \leq 2$ and $3 < |t|$, we have that:

$$|\zeta(\sigma_2 + it) - \zeta(\sigma_1 + it)| \leq C(\log |t|)^2(\sigma_2 - \sigma_1).$$

Proof. Use Lemma **??** and estimate trivially using Lemma **??**. ☐

Lemma 77 (ZetaInvBnd). For any $A > 0$ sufficiently small, there is a constant $C > 0$ so that whenever $1 - A/\log^9 |t| \leq \sigma < 1$ and $3 < |t|$, we have that:

$$1/|\zeta(\sigma + it)| \leq C \log^7 |t|.$$

Proof. Let σ be given in the prescribed range, and set $\sigma' := 1 + A/\log^9 |t|$. Then

$$|\zeta(\sigma + it)| \geq |\zeta(\sigma' + it)| - |\zeta(\sigma + it) - \zeta(\sigma' + it)| \geq C(\sigma' - 1)^{3/4} \log |t|^{-1/4} - C \log^2 |t|(\sigma' - \sigma)$$

$$\geq CA^{3/4} \log |t|^{-7} - C \log^2 |t|(2A/\log^9 |t|),$$

where we used Lemma **??** and Lemma **??**. Now by making A sufficiently small (in particular, something like $A = 1/16$ should work), we can guarantee that

$$|\zeta(\sigma + it)| \geq \frac{C}{2}(\log |t|)^{-7},$$

as desired. ☐

Annoyingly, it is not immediate from this that ζ doesn't vanish there! That's because $1/0 = 0$ in Lean. So we give a second proof of the same fact (refactor this later), with a lower bound on ζ instead of upper bound on $1/\zeta$.

Lemma 78 (ZetaLowerBnd). For any $A > 0$ sufficiently small, there is a constant $C > 0$ so that whenever $1 - A/\log^9 |t| \leq \sigma < 1$ and $3 < |t|$, we have that:

$$|\zeta(\sigma + it)| \geq C \log^7 |t|.$$

Proof. Follow same argument. ☐

Now we get a zero free region.

Lemma 79 (ZetaZeroFree)**.** There is an $A > 0$ so that for $1 - A/\log^9 |t| \le \sigma < 1$ and $3 < |t|$,
$$\zeta(\sigma + it) \neq 0.$$

Proof. Apply Lemma **??**. $\square$

Lemma 80 (LogDerivZetaBnd)**.** There is an $A > 0$ so that for $1 - A/\log^9 |t| \le \sigma < 1$ and $3 < |t|$,
$$|\frac{\zeta'}{\zeta}(\sigma + it)| \ll \log^9 |t|.$$

Proof. Combine the bound on $|\zeta'|$ from Lemma **??** with the bound on $1/|\zeta|$ from Lemma **??**. $\square$

Lemma 81 (LogDerivZetaBndUniform)**.** There is an $A > 0$ so that for $1-A/\log^9 T \le \sigma < 1$ and $3 < |t| \le T$,
$$|\frac{\zeta'}{\zeta}(\sigma + it)| \ll \log^9 T.$$

Proof. This Lemma **??**, but uniform in t. The point is that the upper bound on ζ'/ζ and the lower bound on σ only improve as $|t|$ increases. $\square$

Theorem 18 (ZetaNoZerosOn1Line)**.** The zeta function does not vanish on the 1-line.

Proof. This fact is already proved in Stoll's work. $\square$

Then, since ζ doesn't vanish on the 1-line, there is a $\sigma < 1$ (depending on T), so that the box $[\sigma, 1] \times_{\mathbb{C}} [-T, T]$ is free of zeros of ζ.

Lemma 82 (ZetaNoZerosInBox)**.** For any $T > 0$, there is a constant $\sigma < 1$ so that
$$\zeta(\sigma' + it) \neq 0$$
for all $|t| < T$ and $\sigma' \ge \sigma$.

Proof. Assume not. Then there is a sequence $|t_n| \le T$ and $\sigma_n \to 1$ so that $\zeta(\sigma_n + it_n) = 0$. By compactness, there is a subsequence $t_{n_k} \to t_0$ along which $\zeta(\sigma_{n_k} + it_{n_k}) = 0$. If $t_0 \neq 0$, use the continuity of ζ to get that $\zeta(1 + it_0) = 0$; this is a contradiction. If $t_0 = 0$, ζ blows up near 1, so can't be zero nearby. $\square$

We now prove that there's an absolute constant σ_0 so that ζ'/ζ is holomorphic on a rectangle $[\sigma_2, 2] \times_{\mathbb{C}} [-3, 3] \setminus \{1\}$.

Lemma 83 (LogDerivZetaHolcSmallT)**.** There is a $\sigma_2 < 1$ so that the function
$$\frac{\zeta'}{\zeta}(s)$$
is holomorphic on $\{\sigma_2 \le \Re s \le 2, |\Im s| \le 3\} \setminus \{1\}$.

Proof. The derivative of ζ is holomorphic away from $s = 1$; the denominator $\zeta(s)$ is nonzero in this range by Lemma **??**. $\square$

Lemma 84 (LogDerivZetaHolcLargeT)**.** There is an $A > 0$ so that for all $T > 3$, the function $\frac{\zeta'}{\zeta}(s)$ is holomorphic on $\{1 - A/\log^9 T \le \Re s \le 2, |\Im s| \le T\} \setminus \{1\}$.

Proof. The derivative of ζ is holomorphic away from $s = 1$; the denominator $\zeta(s)$ is nonzero in this range by Lemma **??**. $\square$

It would perhaps (?) be better to refactor this entire file so that we're not using explicit constants but instead systematically using big Oh notation... The punchline would be:

Lemma 85 (LogDerivZetaBndAlt)**.** There is an $A > 0$ so that for $1 - A/\log^9 |t| \leq \sigma < 1$ and $|t| \to \infty$,

$$|\frac{\zeta'}{\zeta}(\sigma + it)| \ll \log^9 |t|.$$

(Same statement but using big-Oh and filters.)

Proof. Same as above. $\square$

3.5 Proof of Medium PNT

The approach here is completely standard. We follow the use of $\mathcal{M}(\widetilde{1_\epsilon})$ as in [Kontorovich 2015].

Definition 10. The (second) Chebyshev Psi function is defined as

$$\psi(x) := \sum_{n \leq x} \Lambda(n),$$

where $\Lambda(n)$ is the von Mangoldt function.

It has already been established that zeta doesn't vanish on the 1 line, and has a pole at $s = 1$ of order 1. We also have the following.

Theorem 19 (LogDerivativeDirichlet)**.** We have that, for $\Re(s) > 1$,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

Proof. Already in Mathlib. $\square$

The main object of study is the following inverse Mellin-type transform, which will turn out to be a smoothed Chebyshev function.

Definition 11 (SmoothedChebyshev)**.** Fix $\epsilon > 0$, and a bumpfunction supported in $[1/2, 2]$. Then we define the smoothed Chebyshev function ψ_ϵ from $\mathbb{R}_{>0}$ to $\mathbb{C}$ by

$$\psi_\epsilon(X) = \frac{1}{2\pi i} \int_{(\sigma)} \frac{-\zeta'(s)}{\zeta(s)} \mathcal{M}(\widetilde{1_\epsilon})(s) X^s ds,$$

where we'll take $\sigma = 1 + 1/\log X$.

Lemma 86 (SmoothedChebyshevDirichlet$_a ux_i ntegrable$)**.** Fix a nonnegative, continuously differentiable function F on $\mathbb{R}$ with support in $[1/2, 2]$, and total mass one, $\int_{(0,\infty)} F(x)/x dx = 1$. Then for any $\epsilon > 0$, and $\sigma \in (1, 2]$, the function

$$x \mapsto \mathcal{M}(\widetilde{1_\epsilon})(\sigma + ix)$$

is integrable on $\mathbb{R}$.

Proof. By Lemma **??** the integrand is $O(1/t^2)$ as $t \to \infty$ and hence the function is integrable. ☐

Lemma 87 (SmoothedChebyshevDirichlet$_a ux_t sum_i ntegral$)**.** Fix a nonnegative, continuously differentiable function F on $\mathbb{R}$ with support in $[1/2, 2]$, and total mass one, $\int_{(0,\infty)} F(x)/x dx = 1$. Then for any $\epsilon > 0$ and $\sigma \in (1, 2]$, the function $x \mapsto \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma+it}} \mathcal{M}(\widetilde{1_\epsilon})(\sigma + it)x^{\sigma+it}$ is equal to $\sum_{n=1}^{\infty} \int_{(0,\infty)} \frac{\Lambda(n)}{n^{\sigma+it}} \mathcal{M}(\widetilde{1_\epsilon})(\sigma + it)x^{\sigma+it}$.

Proof. Interchange of summation and integration. ☐

Inserting the Dirichlet series expansion of the log derivative of zeta, we get the following.

Theorem 20 (SmoothedChebyshevDirichlet)**.** We have that

$$\psi_\epsilon(X) = \sum_{n=1}^{\infty} \Lambda(n)\widetilde{1_\epsilon}(n/X).$$

Proof. We have that

$$\psi_\epsilon(X) = \frac{1}{2\pi i} \int_{(2)} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \mathcal{M}(\widetilde{1_\epsilon})(s)X^s ds.$$

We have enough decay (thanks to quadratic decay of $\mathcal{M}(\widetilde{1_\epsilon})$) to justify the interchange of summation and integration. We then get

$$\psi_\epsilon(X) = \sum_{n=1}^{\infty} \Lambda(n) \frac{1}{2\pi i} \int_{(2)} \mathcal{M}(\widetilde{1_\epsilon})(s)(n/X)^{-s} ds$$

and apply the Mellin inversion formula (Theorem **??**). ☐

The smoothed Chebyshev function is close to the actual Chebyshev function.

Theorem 21 (SmoothedChebyshevClose)**.** We have that

$$\psi_\epsilon(X) = \psi(X) + O(\epsilon X \log X).$$

Proof. Take the difference. By Lemma **??** and **??**, the sums agree except when $1 - c\epsilon \leq n/X \leq 1 + c\epsilon$. This is an interval of length $\ll \epsilon X$, and the summands are bounded by $\Lambda(n) \ll \log X$. ☐

Returning to the definition of ψ_ϵ, fix a large T to be chosen later, and set $\sigma_0 = 1 + 1/logX$, $\sigma_1 = 1 - A/\log T^9$, and $\sigma_2 < \sigma_1$ a constant. Pull contours (via rectangles!) to go from $\sigma_0 - i\infty$ up to $\sigma_0 - iT$, then over to $\sigma_1 - iT$, up to $\sigma_1 - 3i$, over to $\sigma_2 - 3i$, up to $\sigma_2 + 3i$, back over to $\sigma_1 + 3i$, up to $\sigma_1 + iT$, over to $\sigma_0 + iT$, and finally up to $\sigma_0 + i\infty$.

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In the process, we will pick up the residue at $s = 1$. We will do this in several stages. Here the interval integrals are defined as follows:

Definition 12 (I).

$$I_1(\nu, \epsilon, X, T) := \frac{1}{2\pi i} \int_{-\infty}^{-T} \left(\frac{-\zeta'}{\zeta}(\sigma_0 + ti) \right) \mathcal{M}(\tilde{1}_\epsilon)(\sigma_0 + ti) X^{\sigma_0 + ti} \; i \; dt$$

Definition 13 (I).

$$I_2(\nu, \epsilon, X, T, \sigma_1) := \frac{1}{2\pi i} \int_{\sigma_1}^{\sigma_0} \left(\frac{-\zeta'}{\zeta}(\sigma - iT) \right) \mathcal{M}(\tilde{1}_\epsilon)(\sigma - iT) X^{\sigma - iT} \; d\sigma$$

Definition 14 (I).

$$I_{37}(\nu, \epsilon, X, T, \sigma_1) := \frac{1}{2\pi i} \int_{-T}^{T} \left(\frac{-\zeta'}{\zeta}(\sigma_1 + ti) \right) \mathcal{M}(\tilde{1}_\epsilon)(\sigma_1 + ti) X^{\sigma_1 + ti} \; i \; dt$$

Definition 15 (I).

$$I_8(\nu, \epsilon, X, T, \sigma_1) := \frac{1}{2\pi i} \int_{\sigma_1}^{\sigma_0} \left(\frac{-\zeta'}{\zeta}(\sigma + Ti) \right) \mathcal{M}(\tilde{1}_\epsilon)(\sigma + Ti) X^{\sigma + Ti} \; d\sigma$$

Definition 16 (I).

$$I_9(\nu, \epsilon, X, T) := \frac{1}{2\pi i} \int_{T}^{\infty} \left(\frac{-\zeta'}{\zeta}(\sigma_0 + ti) \right) \mathcal{M}(\tilde{1}_\epsilon)(\sigma_0 + ti) X^{\sigma_0 + ti} \; i \; dt$$

Definition 17 (I).

$$I_3(\nu, \epsilon, X, T, \sigma_1) := \frac{1}{2\pi i} \int_{-T}^{-3} \left(\frac{-\zeta'}{\zeta}(\sigma_1 + ti) \right) \mathcal{M}(\tilde{1}_\epsilon)(\sigma_1 + ti) X^{\sigma_1 + ti} \; i \; dt$$

Definition 18 (I).

$$I_7(\nu, \epsilon, X, T, \sigma_1) := \frac{1}{2\pi i} \int_3^T \left(\frac{-\zeta'}{\zeta}(\sigma_1 + ti) \right) \mathcal{M}(\tilde{1}_\epsilon)(\sigma_1 + ti) X^{\sigma_1 + ti} \ i \ dt$$

Definition 19 (I).

$$I_4(\nu, \epsilon, X, \sigma_1, \sigma_2) := \frac{1}{2\pi i} \int_{\sigma_2}^{\sigma_1} \left(\frac{-\zeta'}{\zeta}(\sigma - 3i) \right) \mathcal{M}(\tilde{1}_\epsilon)(\sigma - 3i) X^{\sigma - 3i} \ d\sigma$$

Definition 20 (I).

$$I_6(\nu, \epsilon, X, \sigma_1, \sigma_2) := \frac{1}{2\pi i} \int_{\sigma_2}^{\sigma_1} \left(\frac{-\zeta'}{\zeta}(\sigma + 3i) \right) \mathcal{M}(\tilde{1}_\epsilon)(\sigma + 3i) X^{\sigma + 3i} \ d\sigma$$

Definition 21 (I).

$$I_5(\nu, \epsilon, X, \sigma_2) := \frac{1}{2\pi i} \int_{-3}^{3} \left(\frac{-\zeta'}{\zeta}(\sigma_2 + ti) \right) \mathcal{M}(\tilde{1}_\epsilon)(\sigma_2 + ti) X^{\sigma_2 + ti} \ i \ dt$$

Lemma 88 (dlog$_riemannZeta_bdd_on_vertical_lines'$). For $\sigma_0 > 1$, there exists a constant $C > 0$ such that

$$\forall t \in \mathbb{R}, \quad \left\| \frac{\zeta'(\sigma_0 + ti)}{\zeta(\sigma_0 + ti)} \right\| \leq C.$$

Proof. Write as Dirichlet series and estimate trivially using Theorem **??**. ☐

Lemma 89 (SmoothedChebyshevPull1$_aux_integrable$). The integrand

$$\zeta'(s)/\zeta(s)\mathcal{M}(\widetilde{1_\epsilon})(s)X^s$$

is integrable on the contour $\sigma_0 + ti$ for $t \in \mathbb{R}$ and $\sigma_0 > 1$.

Proof. The $\zeta'(s)/\zeta(s)$ term is bounded, as is X^s, and the smoothing function $\mathcal{M}(\widetilde{1_\epsilon})(s)$ decays like $1/|s|^2$ by Theorem **??**. Actually, we already know that $\mathcal{M}(\widetilde{1_\epsilon})(s)$ is integrable from Theorem **??**, so we should just need to bound the rest. ☐

Lemma 90 (BddAboveOnRect). Let $g : \mathbb{C} \to \mathbb{C}$ be a holomorphic function on a rectangle, then g is bounded above on the rectangle.

Proof. Use the compactness of the rectangle and the fact that holomorphic functions are continuous. ☐

Theorem 22 (SmoothedChebyshevPull1). We have that

$$\psi_\epsilon(X) = \mathcal{M}(\widetilde{1_\epsilon})(1)X^1 + I_1 - I_2 + I_{37} + I_8 + I_9.$$

Proof. Pull rectangle contours and evaluate the pole at $s = 1$. ☐

Next pull contours to another box.

Lemma 91 (SmoothedChebyshevPull2). We have that

$$I_{37} = I_3 - I_4 + I_5 + I_6 + I_7.$$

Proof. Mimic the proof of Lemma **??**. □

We insert this information in ψ_ϵ. We add and subtract the integral over the box $[1-\delta, 2] \times_{\mathbb{C}} [-T, T]$, which we evaluate as follows

Theorem 23 (ZetaBoxEval)**.** For all $\epsilon > 0$ sufficiently close to 0, the rectangle integral over $[1-\delta, 2] \times_{\mathbb{C}} [-T, T]$ of the integrand in ψ_ϵ is

$$\frac{X^1}{1}\mathcal{M}(\widetilde{1_\epsilon})(1) = X(1 + O(\epsilon)),$$

where the implicit constant is independent of X.

Proof. Unfold the definitions and apply Lemma **??**. □

It remains to estimate all of the integrals.
This auxiliary lemma is useful for what follows.

Lemma 92 (IBound$_a ux1$)**.** Given $k > 0$, there exists $C > 0$ so that for all $T > 3$,

$$\log T^k \leq C \cdot T.$$

Proof. Elementary. Use 'isLittleO$_l og_r pow_r pow_a tTop$'$inMathlib$. □

Lemma 93 (I1Bound)**.** We have that

$$|I_1(\nu, \epsilon, X, T)\ | \ll \frac{X}{\epsilon T}.$$

Same with I_9.

Proof. Unfold the definitions and apply the triangle inequality.

$$|I_1(\nu, \epsilon, X, T)| = \left| \frac{1}{2\pi i} \int_{-\infty}^{-T} \left(\frac{-\zeta'}{\zeta}(\sigma_0 + ti) \right) \mathcal{M}(\tilde{1}_\epsilon)(\sigma_0 + ti) X^{\sigma_0 + ti}\ i\ dt \right|$$

By Theorem **??** (once fixed!!), $\zeta'/\zeta(\sigma_0 + ti)$ is bounded by $\zeta'/\zeta(\sigma_0)$, and Theorem **??** gives $\ll 1/(\sigma_0 - 1)$ for the latter. This gives:

$$\leq \frac{1}{2\pi} \left| \int_{-\infty}^{-T} C \log X \cdot \frac{C'}{\epsilon |\sigma_0 + ti|^2} X^{\sigma_0}\ dt \right|,$$

where we used Theorem **??**. Continuing the calculation, we have

$$\leq \log X \cdot C'' \frac{X^{\sigma_0}}{\epsilon} \int_{-\infty}^{-T} \frac{1}{t^2}\ dt\ \leq\ C''' \frac{X \log X}{\epsilon T},$$

where we used that $\sigma_0 = 1 + 1/\log X$, and $X^{\sigma_0} = X \cdot X^{1/\log X} = e \cdot X$. □

Lemma 94 (I2Bound)**.** We have that

$$|I_2(\nu, \epsilon, X, T)| \ll \frac{X}{\epsilon T}.$$

Same with I_8.

Proof. Unfold the definitions and apply the triangle inequality.

$$|I_2(\nu, \epsilon, X, T, \sigma_1)| = \left| \frac{1}{2\pi i} \int_{\sigma_1}^{\sigma_0} \left(\frac{-\zeta'}{\zeta}(\sigma - Ti) \right) \mathcal{M}(\tilde{1}_\epsilon)(\sigma - Ti) X^{\sigma - Ti} \, d\sigma \right|$$

$$\leq \frac{1}{2\pi} \int_{\sigma_1}^{\sigma_0} C \cdot \log T^9 \frac{C'}{\epsilon|\sigma - Ti|^2} X^{\sigma_0} \, d\sigma \leq C'' \cdot \frac{X \log T^9}{\epsilon T^2},$$

where we used Theorems ?? and ??, and the fact that $X^\sigma \leq X^{\sigma_0} = X \cdot X^{1/\log X} = e \cdot X$. Since $T > 3$, we have $\log T^9 \leq C''' T$. □

Lemma 95 (I3Bound)**.** We have that

$$|I_3(\nu, \epsilon, X, T)| \ll \frac{X}{\epsilon} X^{-\frac{A}{(\log T)^9}}.$$

Same with I_7.

Proof. Unfold the definitions and apply the triangle inequality.

$$|I_3(\nu, \epsilon, X, T, \sigma_1)| = \left| \frac{1}{2\pi i} \int_{-T}^{3} \left(\frac{-\zeta'}{\zeta}(\sigma_1 + ti) \right) \mathcal{M}(\tilde{1}_\epsilon)(\sigma_1 + ti) X^{\sigma_1 + ti} \, i \, dt \right|$$

$$\leq \frac{1}{2\pi} \int_{-T}^{3} C \cdot \log t^9 \frac{C'}{\epsilon|\sigma_1 + ti|^2} X^{\sigma_1} \, dt,$$

where we used Theorems ?? and ??. Now we estimate $X^{\sigma_1} = X \cdot X^{-A/\log T^9}$, and the integral is absolutely bounded. □

Lemma 96 (I4Bound)**.** We have that

$$|I_4(\nu, \epsilon, X, \sigma_1, \sigma_2)| \ll \frac{X}{\epsilon} X^{-\frac{A}{(\log T)^9}}.$$

Same with I_6.

Proof. The analysis of I_4 is similar to that of I_2, (in Lemma ??) but even easier. Let C be the sup of $-\zeta'/\zeta$ on the curve $\sigma_2 + 3i$ to $1 + 3i$ (this curve is compact, and away from the pole at $s = 1$). Apply Theorem ?? to get the bound $1/(\epsilon|s|^2)$, which is bounded by C'/ϵ. And X^s is bounded by $X^{\sigma_1} = X \cdot X^{-A/\log T^9}$. Putting these together gives the result. □

Lemma 97 (I5Bound)**.** We have that

$$|I_5(\nu, \epsilon, X, \sigma_2)| \ll \frac{X^{\sigma_2}}{\epsilon}.$$

Proof. Here ζ'/ζ is absolutely bounded on the compact interval $\sigma_2 + i[-3, 3]$, and X^s is bounded by X^{σ_2}. Using Theorem ?? gives the bound $1/(\epsilon|s|^2)$, which is bounded by C'/ϵ. Putting these together gives the result. □

3.6 MediumPNT

Theorem 24 (MediumPNT)**.** We have

$$\sum_{n \leq x} \Lambda(n) = x + O(x \exp(-c(\log x)^{1/10})).$$

Proof. Evaluate the integrals. □

3.7 Strong PNT

This section has been removed.

Chapter 4

Elementary Corollaries

Lemma 98 (finsum$_r$*ange*$_e$*q*$_s$*um*$_r$*ange*). For any arithmetic function f and real number x, one has
$$\sum_{n \le x} f(n) = \sum_{n \le \lfloor x \rfloor_+} f(n)$$
and
$$\sum_{n < x} f(n) = \sum_{n < \lceil x \rceil_+} f(n).$$

Proof. Straightforward. □

Theorem 25 (chebyshev$_a$*symptotic*). One has
$$\sum_{p \le x} \log p = x + o(x).$$

Proof. From the prime number theorem we already have
$$\sum_{n \le x} \Lambda(n) = x + o(x)$$
so it suffices to show that
$$\sum_{j \ge 2} \sum_{p^j \le x} \log p = o(x).$$
Only the terms with $j \le \log x / \log 2$ contribute, and each j contributes at most $\sqrt{x} \log x$ to the sum, so the left-hand side is $O(\sqrt{x} \log^2 x) = o(x)$ as required. □

Corollary 8 (primorial$_b$*ounds*). We have
$$\prod_{p \le x} p = \exp(x + o(x))$$

Proof. Exponentiate Theorem **??**. □

Theorem 26 (pi$_a$*symp*). There exists a function $c(x)$ such that $c(x) = o(1)$ as $x \to \infty$ and
$$\pi(x) = (1 + c(x)) \int_2^x \frac{dt}{\log t}$$
for all x large enough.

Proof. We have the identity

$$\pi(x) = \frac{1}{\log x}\sum_{p\leq x}\log p + \int_2^x (\sum_{p\leq t}\log p)\frac{dt}{t\log^2 t}$$

as can be proven by interchanging the sum and integral and using the fundamental theorem of calculus. For any ϵ, we know from Theorem **??** that there is x_ϵ such that $\sum_{p\leq t}\log p = t + O(\epsilon t)$ for $t \geq x_\epsilon$, hence for $x \geq x_\epsilon$

$$\pi(x) = \frac{1}{\log x}(x + O(\epsilon x)) + \int_{x_\epsilon}^x (t + O(\epsilon t))\frac{dt}{t\log^2 t} + O_\epsilon(1)$$

where the $O_\epsilon(1)$ term can depend on x_ϵ but is independent of x. One can evaluate this after an integration by parts as

$$\pi(x) = (1 + O(\epsilon))\int_{x_\epsilon}^x \frac{dt}{\log t} + O_\epsilon(1)$$

$$= (1 + O(\epsilon))\int_2^x \frac{dt}{\log t}$$

for x large enough, giving the claim. □

Corollary 9 (pi$_a lt$). One has

$$\pi(x) = (1 + o(1))\frac{x}{\log x}$$

as $x \to \infty$.

Proof. An integration by parts gives

$$\int_2^x \frac{dt}{\log t} = \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{dt}{\log^2 t}.$$

We have the crude bounds

$$\int_2^{\sqrt{x}} \frac{dt}{\log^2 t} = O(\sqrt{x})$$

and

$$\int_{\sqrt{x}}^x \frac{dt}{\log^2 t} = O(\frac{x}{\log^2 x})$$

and combining all this we obtain

$$\int_2^x \frac{dt}{\log t} = \frac{x}{\log x} + O(\frac{x}{\log^2 x})$$

$$= (1 + o(1))\frac{x}{\log x}$$

and the claim then follows from Theorem **??**. □

Let p_n denote the n^{th} prime.

Proposition 3 (pn$_a$*symptotic*). One has

$$p_n = (1 + o(1))n \log n$$

as $n \to \infty$.

Proof. Use Corollary **??** to show that for any $\epsilon > 0$, and for x sufficiently large, the number of primes up to $(1-\epsilon)n \log n$ is less than n, and the number of primes up to $(1+\epsilon)n \log n$ is greater than n. □

Corollary 10 (pn$_p$n$_p$*lus*$_o$*ne*). We have $p_{n+1} - p_n = o(p_n)$ as $n \to \infty$.

Proof. Easy consequence of preceding proposition. □

Corollary 11 (prime$_b$*etween*). For every $\epsilon > 0$, there is a prime between x and $(1+\epsilon)x$ for all sufficiently large x.

Proof. Use Corollary **??** to show that $\pi((1+\epsilon)x) - \pi(x)$ goes to infinity as $x \to \infty$. □

Proposition 4. We have $|\sum_{n \le x} \frac{\mu(n)}{n}| \le 1$.

Proof. From Möbius inversion $1_{n=1} = \sum_{d|n} \mu(d)$ and summing we have

$$1 = \sum_{d \le x} \mu(d) \lfloor \frac{x}{d} \rfloor$$

for any $x \ge 1$. Since $\lfloor \frac{x}{d} \rfloor = \frac{x}{d} - \epsilon_d$ with $0 \le \epsilon_d < 1$ and $\epsilon_x = 0$, we conclude that

$$1 \ge x \sum_{d \le x} \frac{\mu(d)}{d} - (x-1)$$

and the claim follows. □

Proposition 5 (Möbius form of prime number theorem)**.** We have $\sum_{n \le x} \mu(n) = o(x)$.

Proof. From the Dirichlet convolution identity

$$\mu(n) \log n = -\sum_{d|n} \mu(d) \Lambda(n/d)$$

and summing we obtain

$$\sum_{n \le x} \mu(n) \log n = -\sum_{d \le x} \mu(d) \sum_{m \le x/d} \Lambda(m).$$

For any $\epsilon > 0$, we have from the prime number theorem that

$$\sum_{m \le x/d} \Lambda(m) = x/d + O(\epsilon x/d) + O_\epsilon(1)$$

(divide into cases depending on whether x/d is large or small compared to ϵ). We conclude that

$$\sum_{n \le x} \mu(n) \log n = -x \sum_{d \le x} \frac{\mu(d)}{d} + O(\epsilon x \log x) + O_\epsilon(x).$$

Applying (**??**) we conclude that

$$\sum_{n\leq x}\mu(n)\log n = O(\epsilon x\log x) + O_\epsilon(x).$$

and hence

$$\sum_{n\leq x}\mu(n)\log x = O(\epsilon x\log x) + O_\epsilon(x) + O(\sum_{n\leq x}(\log x - \log n)).$$

From Stirling's formula one has

$$\sum_{n\leq x}(\log x - \log n) = O(x)$$

thus

$$\sum_{n\leq x}\mu(n)\log x = O(\epsilon x\log x) + O_\epsilon(x)$$

and thus

$$\sum_{n\leq x}\mu(n) = O(\epsilon x) + O_\epsilon(\frac{x}{\log x}).$$

Sending $\epsilon \to 0$ we obtain the claim. $\square$

Proposition 6. We have $\sum_{n\leq x}\lambda(n) = o(x)$.

Proof. From the identity

$$\lambda(n) = \sum_{d^2|n}\mu(n/d^2)$$

and summing, we have

$$\sum_{n\leq x}\lambda(n) = \sum_{d\leq\sqrt{x}}\sum_{n\leq x/d^2}\mu(n).$$

For any $\epsilon > 0$, we have from Proposition **??** that

$$\sum_{n\leq x/d^2}\mu(n) = O(\epsilon x/d^2) + O_\epsilon(1)$$

and hence on summing in d

$$\sum_{n\leq x}\lambda(n) = O(\epsilon x) + O_\epsilon(x^{1/2}).$$

Sending $\epsilon \to 0$ we obtain the claim. $\square$

Proposition 7 (Alternate Möbius form of prime number theorem)**.** We have $\sum_{n\leq x}\mu(n)/n = o(1)$.

Proof. As in the proof of Theorem **??**, we have

$$1 = \sum_{d\leq x}\mu(d)\lfloor\frac{x}{d}\rfloor$$

$$= x\sum_{d\leq x}\frac{\mu(d)}{d} - \sum_{d\leq x}\mu(d)\{\frac{x}{d}\}$$

so it will suffice to show that
$$\sum_{d\leq x}\mu(d)\{\frac{x}{d}\} = o(x).$$
Let N be a natural number. It suffices to show that
$$\sum_{d\leq x}\mu(d)\{\frac{x}{d}\} = O(x/N).$$
if x is large enough depending on N. We can split the left-hand side as the sum of
$$\sum_{d\leq x/N}\mu(d)\{\frac{x}{d}\}$$
and
$$\sum_{j=1}^{N-1}\sum_{x/(j+1)<d\leq x/j}\mu(d)(x/d-j).$$
The first term is clearly $O(x/N)$. For the second term, we can use Theorem **??** and summation by parts (using the fact that $x/d-j$ is monotone and bounded) to find that
$$\sum_{x/(j+1)<d\leq x/j}\mu(d)(x/d-j) = o(x)$$
for any given j, so in particular
$$\sum_{x/(j+1)<d\leq x/j}\mu(d)(x/d-j) = O(x/N^2)$$
for all $j = 1,\dots,N-1$ if x is large enough depending on N. Summing all the bounds, we obtain the claim. □

4.1 Consequences of the PNT in arithmetic progressions

Theorem 27 (Prime number theorem in AP)**.** If $a\ (q)$ is a primitive residue class, then one has
$$\sum_{p\leq x:p=a\ (q)}\log p = \frac{x}{\phi(q)} + o(x).$$

Proof. This is a routine modification of the proof of Theorem **??**. □

Corollary 12 (Dirichlet's theorem)**.** Any primitive residue class contains an infinite number of primes.

Proof. If this were not the case, then the sum $\sum_{p\leq x:p=a\ (q)}\log p$ would be bounded in x, contradicting Theorem **??**. □

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4.2 Consequences of the Chebotarev density theorem

Lemma 99 (Cyclotomic Chebotarev)**.** For any a coprime to m,

$$\sum_{N\mathfrak{p}\leq x; N\mathfrak{p}=a\ (m)} \log N\mathfrak{p} = \frac{1}{|G|} \sum_{N\mathfrak{p}\leq x} \log N\mathfrak{p}.$$

Proof. This should follow from Lemma **??** by a Fourier expansion. $\square$