A fully connected feedforward neural network

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Consider a three-layer fully connected feedforward neural network with three inputs x_i^l and the same number of target variables y_i.

We write the considered neural network in mathematical form:

$$A^n = A^n(W^l A^l(W^l X^l)),$$

where X^l – matrix of input values, W^l – weight matrix, A^l – activation function matrix, l – layer number, n – number of layers in the neural network.

Forward Signal Propagation

Let's write down the equations for direct signal passing through the neural network

$$A^l = f(X^l)$$

$$a_1^2 = f(x_1^2) = f(w_{11}^1 x_1^1 + w_{12}^1 x_2^1 + w_{13}^1 x_3^1 + b_1^2),$$

$$a_2^2 = f(x_2^2) = f(w_{21}^1 x_1^1 + w_{22}^1 x_2^1 + w_{23}^1 x_3^1 + b_2^2),$$

$$a_3^2 = f(x_3^2) = f(w_{31}^1 x_1^1 + w_{32}^1 x_2^1 + w_{33}^1 x_3^1 + b_3^2);$$

$$a_1^3 = f(x_1^3) = f(w_{11}^2 a_1^2 + w_{12}^2 a_2^2 + w_{13}^2 a_3^2 + b_1^3),$$

$$a_2^3 = f(x_2^3) = f(w_{21}^2 a_1^2 + w_{22}^2 a_2^2 + w_{23}^2 a_3^2 + b_2^3),$$

$$a_3^3 = f(x_3^3) = f(w_{31}^2 a_1^2 + w_{32}^2 a_2^2 + w_{33}^2 a_3^2 + b_3^3)$$

and cost function[1]

$$Cost(y_i, a_i^l) = \frac{1}{n}\sum_{i=1}^{n}(y_i - a_i^l)^2 \rightarrow min,$$

where l – neural network layer number, i – number of output value on layer l, n – number of output values of the last layer, i and j – row and column number[2] W^l.

[1] You can meet other names, for example, «loss function», «objective function», «error function», «J».

[2] The matrix product AB is defined only if the number of columns of A is equal to the number of rows of B. Thus, the number of columns j of the matrix W_{ij}^l is equal to the number of rows i of the vectors X^l and A^l (vector is a special case of a matrix with only one row or one column).

Thus, the cost function for our neural network in expanded form:

$$C(y_i, a_i^3) = \frac{1}{3}((y_1 - a_1^3) + (y_2 - a_2^3) + (y_3 - a_3^3))$$

Backpropagation and update

Let's find partial derivatives with respect to all elements of the matrix W^2:

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{ij}^2} = \frac{1}{n}\sum_{i=1}^{n}\frac{\partial (y_i - a_i^3)^2}{\partial w_{ij}^2}$$

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{ij}^2} = \frac{2}{n}\sum_{i=1}^{n}(y_i - a_i^3)\frac{\partial (y_i - a_i^3)}{\partial w_{ij}^2}$$

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{ij}^2} = \frac{2}{n}\sum_{i=1}^{n}(y_i - a_i^3)\left(\frac{\partial y_i}{\partial w_{ij}^2} - \frac{\partial a_i^3}{\partial w_{ij}^2}\right)$$

Since y_i is a constant, then $\frac{\partial y_i}{\partial w_{ij}^2} = 0$.

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{ij}^2} = -\frac{2}{n}\sum_{i=1}^{n}(y_i - a_i^3)\frac{\partial a_i^3}{\partial w_{ij}^2}$$

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{ij}^2} = -\frac{2}{n}\sum_{i=1}^{n}(y_i - a_i^3)\frac{\partial a_i^3}{\partial x_i^3}\frac{\partial x_i^3}{\partial w_{ij}^2}$$

Let's transform the sigmoid and find its derivative:

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{ij}^2} = -\frac{2}{n}\sum_{i=1}^{n}(y_i - a_i^3)\left(\frac{1}{1+e^{-x_i^3}}\right)'_{x_i^3}\frac{\partial x_i^3}{\partial w_{ij}^2}$$

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{ij}^2} = -\frac{2}{n}\sum_{i=1}^{n}(y_i - a_i^3)\left(\frac{\frac{1}{1+e^{x_i^3}}}{e^{x_i^3}}\right)'_{x_i^3}\frac{\partial x_i^3}{\partial w_{ij}^2}$$

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{ij}^2} = -\frac{2}{n}\sum_{i=1}^{n}(y_i - a_i^3)\left(\frac{e^{x_i^3}}{1+e^{x_i^3}}\right)'_{x_i^3}\frac{\partial x_i^3}{\partial w_{ij}^2}$$

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{ij}^2} = -\frac{2}{n}\sum_{i=1}^{n}(y_i - a_i^3)\left(\frac{\left(e^{x_i^3}\right)'_{x_i^3}\left(1+e^{x_i^3}\right) - e^{x_i^3}\left(1+e^{x_i^3}\right)'_{x_i^3}}{\left(1+e^{x_i^3}\right)^2}\right)\frac{\partial x_i^3}{\partial w_{ij}^2}$$

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{ij}^2} = -\frac{2}{n}\sum_{i=1}^{n}(y_i - a_i^3)\frac{e^{x_i^3}}{\left(1+e^{x_i^3}\right)^2}\frac{\partial x_i^3}{\partial w_{ij}^2}$$

Expand the sum for the variable w_{11}^2 of the matrix W^2:

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{11}^2} = -\frac{2}{3}((y_1 - a_1^3)\frac{e^{x_1^3}}{\left(1+e^{x_1^3}\right)^2}\frac{\partial x_1^3}{\partial w_{11}^2} + (y_2 - a_2^3)\frac{e^{x_2^3}}{\left(1+e^{x_2^3}\right)^2}\frac{\partial x_2^3}{\partial w_{11}^2} + (y_3 - a_3^3)\frac{e^{x_3^3}}{\left(1+e^{x_3^3}\right)^2}\frac{\partial x_3^3}{\partial w_{11}^2})$$

Let's find the partial derivative with respect to the variable w_{11}^2. Since

$$x_1^3 = f(w_{11}^2 a_1^2 + w_{12}^2 a_2^2 + w_{13}^2 a_3^2 + b_1^3),$$

$$x_2^3 = f(w_{21}^2 a_1^2 + w_{22}^2 a_2^2 + w_{23}^2 a_3^2 + b_2^3),$$

$$x_3^3 = f(w_{31}^2 a_1^2 + w_{32}^2 a_2^2 + w_{33}^2 a_3^2 + b_3^3):$$

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{11}^2} = -\frac{2}{3}((y_1 - a_1^3)\frac{e^{x_1^3}}{\left(1+e^{x_1^3}\right)^2} a_1^2 + 0 + 0)$$

We transform the sigmoid and obtain the final form of the expression for $\frac{\partial C(y_1,a_1^3)}{\partial w_{11}^2}$:

$$\frac{\partial C(y_1,a_1^3)}{\partial w_{11}^2} = -\frac{2}{3}((y_1 - a_1^3)\frac{e^{x_1^3}}{\left(1+e^{x_1^3}\right)}\left(1 - \frac{e^{x_1^3}}{1+e^{x_1^3}}\right) a_1^2)$$

$$\text{или } \frac{\partial C(y_1,a_1^3)}{\partial w_{11}^2} = -\frac{2}{3}((y_1 - a_1^3)a_1^3(x_1^3)\big(1 - a_1^3(x_1^3)\big)a_1^2)$$

In the same way, for the variables w_{12}^2 and w_{13}^2 we get:

$$\frac{\partial C(y_1,a_1^3)}{\partial w_{12}^2} = -\frac{2}{3}((y_1 - a_1^3)a_1^3(x_1^3)\big(1 - a_1^3(x_1^3)\big)a_2^2),$$

$$\frac{\partial C(y_1,a_1^3)}{\partial w_{13}^2} = -\frac{2}{3}((y_1 - a_1^3)a_1^3(x_1^3)\big(1 - a_1^3(x_1^3)\big)a_3^2)$$

Let's find new values (updated weights) for the variables w_{11}^2, w_{12}^2 and w_{13}^2:

$$w_{11}^2 = w_{11}^2 - \eta(-\frac{2}{3}((y_1 - a_1^3)a_1^3(x_1^3)\big(1 - a_1^3(x_1^3)\big)a_1^2)),$$

$$w_{12}^2 = w_{12}^2 - \eta(-\frac{2}{3}((y_1 - a_1^3)a_1^3(x_1^3)\big(1 - a_1^3(x_1^3)\big)a_2^2)),$$

$$w_{13}^2 = w_{13}^2 - \eta(-\frac{2}{3}((y_1 - a_1^3)a_1^3(x_1^3)\big(1 - a_1^3(x_1^3)\big)a_3^2))$$

Let's find the remaining partial derivatives for the matrix W^2. Expand the sum for the variable w_{21}^2:

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{21}^2} = -\frac{2}{3}((y_1 - a_1^3)\frac{e^{x_1^3}}{\left(1+e^{x_1^3}\right)^2}\frac{\partial x_1^3}{\partial w_{21}^2} + (y_2 - a_2^3)\frac{e^{x_2^3}}{\left(1+e^{x_2^3}\right)^2}\frac{\partial x_2^3}{\partial w_{21}^2} + (y_3 - a_3^3)\frac{e^{x_3^3}}{\left(1+e^{x_3^3}\right)^2}\frac{\partial x_3^3}{\partial w_{21}^2})$$

Let's find the partial derivative with respect to the variable w_{21}^2:

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{21}^2} = -\frac{2}{3}(0 + (y_2 - a_2^3)\frac{e^{x_2^3}}{\left(1+e^{x_2^3}\right)^2} a_1^2 + 0)$$

We transform the sigmoid and obtain the final form of the expression for $\frac{\partial C(y_i,a_i^3)}{\partial w_{21}^2}$:

$$\frac{\partial C(y_2,a_2^3)}{\partial w_{21}^2} = -\frac{2}{3}((y_2 - a_2^3)\frac{e^{x_2^3}}{\left(1+e^{x_2^3}\right)}\left(1 - \frac{e^{x_2^3}}{1+e^{x_2^3}}\right)a_1^2)$$

$$\text{or } \frac{\partial C(y_2,a_2^3)}{\partial w_{21}^2} = -\frac{2}{3}((y_2 - a_2^3)a_2^3(x_2^3)\left(1 - a_2^3(x_2^3)\right)a_1^2)$$

In the same way for variables w_{22}^2 and w_{23}^2 we get:

$$\frac{\partial C(y_2,a_2^3)}{\partial w_{22}^2} = -\frac{2}{3}((y_2 - a_2^3)a_2^3(x_2^3)\left(1 - a_2^3(x_2^3)\right)a_2^2),$$

$$\frac{\partial C(y_2,a_2^3)}{\partial w_{23}^2} = -\frac{2}{3}((y_2 - a_2^3)a_2^3(x_2^3)\left(1 - a_2^3(x_2^3)\right)a_3^2)$$

Let's find new values (updated weights) for the variables w_{21}^2, w_{22}^2 and w_{23}^2:

$$w_{21}^2 = w_{21}^2 - \eta(-\frac{2}{3}((y_2 - a_2^3)a_2^3(x_2^3)\left(1 - a_2^3(x_2^3)\right)a_1^2)),$$

$$w_{22}^2 = w_{22}^2 - \eta(-\frac{2}{3}((y_2 - a_2^3)a_2^3(x_2^3)\left(1 - a_2^3(x_2^3)\right)a_2^2)),$$

$$w_{23}^2 = w_{23}^2 - \eta(-\frac{2}{3}((y_2 - a_2^3)a_2^3(x_2^3)\left(1 - a_2^3(x_2^3)\right)a_3^2))$$

Expand the sum for the variable w_{31}^2:

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{31}^2} = -\frac{2}{3}((y_1 - a_1^3)\frac{e^{x_1^3}}{\left(1+e^{x_1^3}\right)^2}\frac{\partial x_1^3}{\partial w_{31}^2} + (y_2 - a_2^3)\frac{e^{x_2^3}}{\left(1+e^{x_2^3}\right)^2}\frac{\partial x_2^3}{\partial w_{31}^2} + (y_3 - a_3^3)\frac{e^{x_3^3}}{\left(1+e^{x_3^3}\right)^2}\frac{\partial x_3^3}{\partial w_{31}^2})$$

Let's find the partial derivative with respect to the variable w_{31}^2:

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{31}^2} = -\frac{2}{3}(0 + 0 + (y_3 - a_3^3)\frac{e^{x_3^3}}{\left(1+e^{x_3^3}\right)^2}a_1^2)$$

We transform the sigmoid and obtain the final form of the expression for $\frac{\partial C(y_i,a_i^3)}{\partial w_{31}^2}$:

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{31}^2} = -\frac{2}{3}((y_3 - a_3^3)\frac{e^{x_3^3}}{\left(1+e^{x_3^3}\right)}\left(1 - \frac{e^{x_3^3}}{1+e^{x_3^3}}\right)a_1^2)$$

$$\text{или } \frac{\partial C(y_i,a_i^3)}{\partial w_{31}^2} = -\frac{2}{3}((y_3 - a_3^3)a_3^3(x_3^3)\left(1 - a_3^3(x_3^3)\right)a_1^2)$$

In the same way for variables w_{32}^2 and w_{33}^2 we get:

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{32}^2} = -\frac{2}{3}((y_3 - a_3^3)a_3^3(x_3^3)\left(1 - a_3^3(x_3^3)\right)a_2^2),$$

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{33}^2} = -\frac{2}{3}((y_3 - a_3^3)a_3^3(x_3^3)\big(1 - a_3^3(x_3^3)\big)a_3^2)$$

Let's find new values (updated weights) for the variables w_{31}^2, w_{32}^2 and w_{33}^2:

$$w_{31}^2 = w_{31}^2 - \eta(-\frac{2}{3}((y_3 - a_3^3)a_3^3(x_3^3)\big(1 - a_3^3(x_3^3)\big)a_1^2)),$$

$$w_{32}^2 = w_{32}^2 - \eta(-\frac{2}{3}((y_3 - a_3^3)a_3^3(x_3^3)\big(1 - a_3^3(x_3^3)\big)a_2^2)),$$

$$w_{33}^2 = w_{33}^2 - \eta(-\frac{2}{3}((y_3 - a_3^3)a_3^3(x_3^3)\big(1 - a_3^3(x_3^3)\big)a_3^2))$$

Now let's find the partial derivatives with respect to b_i^3:

$$\frac{\partial C(y_i,a_i^3)}{\partial b_i^3} = -\frac{2}{3}\sum_{i=1}^{n}\big(y_i - a_i^3\big)\frac{e^{x_i^3}}{\left(1+e^{x_i^3}\right)^2}\frac{\partial x_i^3}{\partial b_i^3}$$

Let's find the partial derivative with respect to b_1^3:

$$\frac{\partial C(y_i,a_i^3)}{\partial b_1^3} = -\frac{2}{3}((y_1 - a_1^3)\frac{e^{x_1^3}}{\left(1+e^{x_1^3}\right)^2}\frac{\partial x_1^3}{\partial b_1^3} + (y_2 - a_2^3)\frac{e^{x_2^3}}{\left(1+e^{x_2^3}\right)^2}\frac{\partial x_2^3}{\partial b_1^3} + (y_3 - a_3^3)\frac{e^{x_3^3}}{\left(1+e^{x_3^3}\right)^2}\frac{\partial x_3^3}{\partial b_1^3})$$

$$\frac{\partial C(y_i,a_i^3)}{\partial b_1^3} = -\frac{2}{3}((y_1 - a_1^3)\frac{e^{x_1^3}}{\left(1+e^{x_1^3}\right)^2}1 + 0 + 0)$$

We transform the sigmoid and obtain the final form of the expression for $\frac{\partial C(y_1,a_1^3)}{\partial b_1^3}$:

$$\frac{\partial C(y_i,a_i^3)}{\partial b_1^3} = -\frac{2}{3}((y_1 - a_1^3)\frac{e^{x_1^3}}{\left(1+e^{x_1^3}\right)}\left(1 - \frac{e^{x_1^3}}{1+e^{x_1^3}}\right)1)$$

or $\frac{\partial C(y_i,a_i^3)}{\partial b_1^3} = -\frac{2}{3}((y_1 - a_1^3)a_1^3(x_1^3)\big(1 - a_1^3(x_1^3)\big)1)$

Let's find a new value for the bias b_1^3:

$$b_1^3 = b_1^3 - \eta(-\frac{2}{3}((y_1 - a_1^3)a_1^3(x_1^3)\big(1 - a_1^3(x_1^3)\big)1))$$

Let's find the partial derivative with respect to b_2^3 and b_3^3:

$$\frac{\partial C(y_i,a_i^3)}{\partial b_2^3} = -\frac{2}{3}((y_1 - a_1^3)\frac{e^{x_1^3}}{\left(1+e^{x_1^3}\right)^2}\frac{\partial x_1^3}{\partial b_2^3} + (y_2 - a_2^3)\frac{e^{x_2^3}}{\left(1+e^{x_2^3}\right)^2}\frac{\partial x_2^3}{\partial b_2^3} + (y_3 - a_3^3)\frac{e^{x_3^3}}{\left(1+e^{x_3^3}\right)^2}\frac{\partial x_3^3}{\partial b_2^3})$$

$$\frac{\partial C(y_i,a_i^3)}{\partial b_2^3} = -\frac{2}{3}((0 + \big(y_2 - a_2^3\big)\frac{e^{x_2^3}}{\left(1+e^{x_2^3}\right)^2}\frac{\partial x_2^3}{\partial b_2^3}1 + 0))$$

$$\frac{\partial C(y_i,a_i^3)}{\partial b_2^3} = -\frac{2}{3}\left((y_2 - a_2^3)a_2^3(x_2^3)\left(1 - a_2^3(x_2^3)\right)1\right)$$

$$\frac{\partial C(y_i,a_i^3)}{\partial b_3^3} = -\frac{2}{3}\left((y_1 - a_1^3)\frac{e^{x_1^3}}{\left(1+e^{x_1^3}\right)^2}\frac{\partial x_1^3}{\partial b_3^3} + (y_2 - a_2^3)\frac{e^{x_2^3}}{\left(1+e^{x_2^3}\right)^2}\frac{\partial x_2^3}{\partial b_3^3} + (y_3 - a_3^3)\frac{e^{x_3^3}}{\left(1+e^{x_3^3}\right)^2}\frac{\partial x_3^3}{\partial b_3^3}\right)$$

$$\frac{\partial C(y_i,a_i^3)}{\partial b_3^3} = -\frac{2}{3}\left(0 + 0 + \left(y_3 - a_3^3\right)\frac{e^{x_3^3}}{\left(1+e^{x_3^3}\right)^2}\frac{\partial x_3^3}{\partial b_3^3}1\right)$$

$$\frac{\partial C(y_i,a_i^3)}{\partial b_3^3} = -\frac{2}{3}\left((y_3 - a_3^3)a_3^3(x_3^3)\left(1 - a_3^3(x_3^3)\right)1\right)$$

Let's find a new value for b_2^3 and b_3^3:

$$b_2^3 = b_2^3 - \eta\left(-\frac{2}{3}\left((y_2 - a_2^3)a_2^3(x_2^3)\left(1 - a_2^3(x_2^3)\right)1\right)\right),$$

$$b_3^3 = b_3^3 - \eta\left(-\frac{2}{3}\left((y_3 - a_3^3)a_3^3(x_3^3)\left(1 - a_3^3(x_3^3)\right)1\right)\right)$$

Let's find partial derivatives with respect to all elements of the matrix W^1. Expand the sum for the variable w_{11}^1 of the matrix W^1. Since

$$x_1^3 = f(w_{11}^2 a_1^2 + w_{12}^2 a_2^2 + w_{13}^2 a_3^2 + b_1^3),$$

$$x_2^3 = f(w_{21}^2 a_1^2 + w_{22}^2 a_2^2 + w_{23}^2 a_3^2 + b_2^3),$$

$$x_3^3 = f(w_{31}^2 a_1^2 + w_{32}^2 a_2^2 + w_{33}^2 a_3^2 + b_3^3),$$

in its turn,

$$a_1^2 = f(x_1^2) = f(w_{11}^1 x_1^1 + w_{12}^1 x_2^1 + w_{13}^1 x_3^1 + b_1^2),$$

$$a_2^2 = f(x_2^2) = f(w_{21}^1 x_1^1 + w_{22}^1 x_2^1 + w_{23}^1 x_3^1 + b_2^2),$$

$$a_3^2 = f(x_3^2) = f(w_{31}^1 x_1^1 + w_{32}^1 x_2^1 + w_{33}^1 x_3^1 + b_3^2),$$

then the sum for the variable w_{11}^1 matrix W^1:

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{ij}^1} = -\frac{2}{3}\sum_{i=1}^{n}\left(y_i - a_i^3\right)\frac{e^{x_i^3}}{\left(1+e^{x_i^3}\right)^2}\frac{\partial x_1^3}{\partial a_1^2}\frac{\partial a_1^2}{\partial x_1^2}\frac{\partial x_1^2}{\partial w_{11}^1}$$

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{11}^1} = -\frac{2}{3}\left((y_1 - a_1^3)\frac{e^{x_1^3}}{\left(1+e^{x_1^3}\right)^2}\frac{\partial x_1^3}{\partial a_1^2}\frac{\partial a_1^2}{\partial x_1^2}\frac{\partial x_1^2}{\partial w_{11}^1} + (y_2 - a_2^3)\frac{e^{x_2^3}}{\left(1+e^{x_2^3}\right)^2}\frac{\partial x_2^3}{\partial a_1^2}\frac{\partial a_1^2}{\partial x_1^2}\frac{\partial x_1^2}{\partial w_{11}^1} + \right.$$

$$\left.(y_3 - a_3^3)\frac{e^{x_3^3}}{\left(1+e^{x_3^3}\right)^2}\frac{\partial x_3^3}{\partial a_1^2}\frac{\partial a_1^2}{\partial x_1^2}\frac{\partial x_1^2}{\partial w_{11}^1}\right)$$

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{11}^1} = -\frac{2}{3}\Big((y_1 - a_1^3)\frac{e^{x_1^3}}{\left(1+e^{x_1^3}\right)^2} w_{11}^2 \frac{e^{x_1^2}}{\left(1+e^{x_1^2}\right)}\left(1 - \frac{e^{x_1^2}}{1+e^{x_1^2}}\right) x_1^1 + (y_2 - a_2^3)\frac{e^{x_2^3}}{\left(1+e^{x_2^3}\right)^2} w_{21}^2 \frac{e^{x_1^2}}{\left(1+e^{x_1^2}\right)}\left(1 - \frac{e^{x_1^2}}{1+e^{x_1^2}}\right) x_1^1 + (y_3 - a_3^3)\frac{e^{x_3^3}}{\left(1+e^{x_3^3}\right)^2} w_{31}^2 \frac{e^{x_1^2}}{\left(1+e^{x_1^2}\right)}\left(1 - \frac{e^{x_1^2}}{1+e^{x_1^2}}\right) x_1^1\Big)$$

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{11}^1} = -\frac{2}{3}\Big((y_1 - a_1^3) a_1^3(x_1^3)\left(1 - a_1^3(x_1^3)\right) w_{11}^2 a_1^2(x_1^2)\left(1 - a_1^2(x_1^2)\right) x_1^1 + (y_2 - a_2^3) a_2^3(x_2^3)\left(1 - a_2^3(x_2^3)\right) w_{21}^2 a_1^2(x_1^2)\left(1 - a_1^2(x_1^2)\right) x_1^1 + (y_3 - a_3^3) a_3^3(x_3^3)\left(1 - a_3^3(x_3^3)\right) w_{31}^2 a_1^2(x_1^2)\left(1 - a_1^2(x_1^2)\right) x_1^1\Big)$$

Let's find a new value (updated weight) for a variable w_{11}^1:

$$w_{11}^1 = w_{11}^1 - \eta \frac{\partial C(y_i,\ a_i^3)}{\partial w_{11}^1}$$

Let's find the remaining partial derivatives and their new values for the matrix W^1.

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{12}^1} = -\frac{2}{3}\Big((y_1 - a_1^3) a_1^3(x_1^3)\left(1 - a_1^3(x_1^3)\right) w_{11}^2 a_1^2(x_1^2)\left(1 - a_1^2(x_1^2)\right) x_2^1 + (y_2 - a_2^3) a_2^3(x_2^3)\left(1 - a_2^3(x_2^3)\right) w_{21}^2 a_1^2(x_1^2)\left(1 - a_1^2(x_1^2)\right) x_2^1 + (y_3 - a_3^3) a_3^3(x_3^3)\left(1 - a_3^3(x_3^3)\right) w_{31}^2 a_1^2(x_1^2)\left(1 - a_1^2(x_1^2)\right) x_2^1\Big)$$

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{13}^1} = -\frac{2}{3}\Big((y_1 - a_1^3) a_1^3(x_1^3)\left(1 - a_1^3(x_1^3)\right) w_{11}^2 a_1^2(x_1^2)\left(1 - a_1^2(x_1^2)\right) x_3^1 + (y_2 - a_2^3) a_2^3(x_2^3)\left(1 - a_2^3(x_2^3)\right) w_{21}^2 a_1^2(x_1^2)\left(1 - a_1^2(x_1^2)\right) x_3^1 + (y_3 - a_3^3) a_3^3(x_3^3)\left(1 - a_3^3(x_3^3)\right) w_{31}^2 a_1^2(x_1^2)\left(1 - a_1^2(x_1^2)\right) x_3^1\Big)$$

$$w_{12}^1 = w_{12}^1 - \eta \frac{\partial C(y_i,a_i^3)}{\partial w_{12}^1},\ w_{13}^1 = w_{13}^1 - \eta \frac{\partial C(y_i,a_i^3)}{\partial w_{13}^1}$$

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{21}^1} = -\frac{2}{3}\Big((y_1 - a_1^3) a_1^3(x_1^3)\left(1 - a_1^3(x_1^3)\right) w_{12}^2 a_2^2(x_2^2)\left(1 - a_2^2(x_2^2)\right) x_1^1 + (y_2 - a_2^3) a_2^3(x_2^3)\left(1 - a_2^3(x_2^3)\right) w_{22}^2 a_2^2(x_2^2)\left(1 - a_2^2(x_2^2)\right) x_1^1 + (y_3 - a_3^3) a_3^3(x_3^3)\left(1 - a_3^3(x_3^3)\right) w_{32}^2 a_2^2(x_2^2)\left(1 - a_2^2(x_2^2)\right) x_1^1\Big)$$

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{22}^1} = -\frac{2}{3}((y_1 - a_1^3)a_1^3(x_1^3)\left(1 - a_1^3(x_1^3)\right)w_{12}^2 a_2^2(x_2^2)\left(1 - a_2^2(x_2^2)\right)x_2^1 + (y_2 - a_2^3)a_2^3(x_2^3)\left(1 - a_2^3(x_2^3)\right)w_{22}^2 a_2^2(x_2^2)\left(1 - a_2^2(x_2^2)\right)x_2^1 + (y_3 - a_3^3)a_3^3(x_3^3)\left(1 - a_3^3(x_3^3)\right)w_{32}^2 a_2^2(x_2^2)\left(1 - a_2^2(x_2^2)\right)x_2^1)$$

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{23}^1} = -\frac{2}{3}((y_1 - a_1^3)a_1^3(x_1^3)\left(1 - a_1^3(x_1^3)\right)w_{12}^2 a_2^2(x_2^2)\left(1 - a_2^2(x_2^2)\right)x_3^1 + (y_2 - a_2^3)a_2^3(x_2^3)\left(1 - a_2^3(x_2^3)\right)w_{22}^2 a_2^2(x_2^2)\left(1 - a_2^2(x_2^2)\right)x_3^1 + (y_3 - a_3^3)a_3^3(x_3^3)\left(1 - a_3^3(x_3^3)\right)w_{32}^2 a_2^2(x_2^2)\left(1 - a_2^2(x_2^2)\right)x_3^1)$$

$$w_{21}^1 = w_{21}^1 - \eta\frac{\partial C(y_i,a_i^3)}{\partial w_{21}^1},\ w_{22}^1 = w_{22}^1 - \eta\frac{\partial C(y_i,a_i^3)}{\partial w_{22}^1},\ w_{23}^1 = w_{23}^1 - \eta\frac{\partial C(y_i,a_i^3)}{\partial w_{23}^1}$$

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{31}^1} = -\frac{2}{3}((y_1 - a_1^3)a_1^3(x_1^3)\left(1 - a_1^3(x_1^3)\right)w_{13}^2 a_3^2(x_3^2)\left(1 - a_3^2(x_3^2)\right)x_1^1 + (y_2 - a_2^3)a_2^3(x_2^3)\left(1 - a_2^3(x_2^3)\right)w_{23}^2 a_3^2(x_3^2)\left(1 - a_3^2(x_3^2)\right)x_1^1 + (y_3 - a_3^3)a_3^3(x_3^3)\left(1 - a_3^3(x_3^3)\right)w_{33}^2 a_3^2(x_3^2)\left(1 - a_3^2(x_3^2)\right)x_1^1)$$

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{32}^1} = -\frac{2}{3}((y_1 - a_1^3)a_1^3(x_1^3)\left(1 - a_1^3(x_1^3)\right)w_{13}^2 a_3^2(x_3^2)\left(1 - a_3^2(x_3^2)\right)x_2^1 + (y_2 - a_2^3)a_2^3(x_2^3)\left(1 - a_2^3(x_2^3)\right)w_{23}^2 a_3^2(x_3^2)\left(1 - a_3^2(x_3^2)\right)x_2^1 + (y_3 - a_3^3)a_3^3(x_3^3)\left(1 - a_3^3(x_3^3)\right)w_{33}^2 a_3^2(x_3^2)\left(1 - a_3^2(x_3^2)\right)x_2^1)$$

$$\frac{\partial C(y_i,a_i^3)}{\partial w_{33}^1} = -\frac{2}{3}((y_1 - a_1^3)a_1^3(x_1^3)\left(1 - a_1^3(x_1^3)\right)w_{13}^2 a_3^2(x_3^2)\left(1 - a_3^2(x_3^2)\right)x_3^1 + (y_2 - a_2^3)a_2^3(x_2^3)\left(1 - a_2^3(x_2^3)\right)w_{23}^2 a_3^2(x_3^2)\left(1 - a_3^2(x_3^2)\right)x_3^1 + (y_3 - a_3^3)a_3^3(x_3^3)\left(1 - a_3^3(x_3^3)\right)w_{33}^2 a_3^2(x_3^2)\left(1 - a_3^2(x_3^2)\right)x_3^1)$$

$$w_{31}^1 = w_{31}^1 - \eta\frac{\partial C(y_i,a_i^3)}{\partial w_{31}^1},\ w_{32}^1 = w_{32}^1 - \eta\frac{\partial C(y_i,a_i^3)}{\partial w_{32}^1},\ w_{33}^1 = w_{33}^1 - \eta\frac{\partial C(y_i,a_i^3)}{\partial w_{33}^1}$$

Now let's find the partial derivatives with respect to b_i^2. Expand the sum for the variable b_1^2:

$$\frac{\partial C(y_i,a_i^3)}{\partial b_1^2} = -\frac{2}{3}\sum_{i=1}^{n}(y_i - a_i^3)\frac{e^{x_i^3}}{\left(1+e^{x_i^3}\right)^2}\frac{\partial x_1^3}{\partial a_1^2}\frac{\partial a_1^2}{\partial x_1^2}\frac{\partial x_1^2}{\partial b_1^2}$$

$$\frac{\partial C(y_i,a_i^3)}{\partial b_1^2} = -\frac{2}{3}\left(\left(y_1 - a_1^3\right)\frac{e^{x_1^3}}{\left(1+e^{x_1^3}\right)^2}\frac{\partial x_1^3}{\partial a_1^2}\frac{\partial a_1^2}{\partial x_1^2}\frac{\partial x_1^2}{\partial b_1^2} + \left(y_2 - a_2^3\right)\frac{e^{x_2^3}}{\left(1+e^{x_2^3}\right)^2}\frac{\partial x_2^3}{\partial a_1^2}\frac{\partial a_1^2}{\partial x_1^2}\frac{\partial x_1^2}{\partial b_1^2} + \left(y_3 - a_3^3\right)\frac{e^{x_3^3}}{\left(1+e^{x_3^3}\right)^2}\frac{\partial x_3^3}{\partial a_1^2}\frac{\partial a_1^2}{\partial x_1^2}\frac{\partial x_1^2}{\partial b_1^2}\right)$$

$$\frac{\partial C(y_i,a_i^3)}{\partial b_1^2} = -\frac{2}{3}\left(\left(y_1 - a_1^3\right)\frac{e^{x_1^3}}{\left(1+e^{x_1^3}\right)^2}w_{11}^2\frac{e^{x_1^2}}{\left(1+e^{x_1^2}\right)}\left(1-\frac{e^{x_1^2}}{1+e^{x_1^2}}\right)1 + \left(y_2 - a_2^3\right)\frac{e^{x_2^3}}{\left(1+e^{x_2^3}\right)^2}w_{21}^2\frac{e^{x_1^2}}{\left(1+e^{x_1^2}\right)}\left(1-\frac{e^{x_1^2}}{1+e^{x_1^2}}\right)1 + \left(y_3 - a_3^3\right)\frac{e^{x_3^3}}{\left(1+e^{x_3^3}\right)^2}w_{31}^2\frac{e^{x_1^2}}{\left(1+e^{x_1^2}\right)}\left(1-\frac{e^{x_1^2}}{1+e^{x_1^2}}\right)1\right)$$

$$\frac{\partial C(y_i,a_i^3)}{\partial b_1^2} = -\frac{2}{3}\left(\left(y_1 - a_1^3\right)a_1^3(x_1^3)\left(1-a_1^3(x_1^3)\right)w_{11}^2a_1^2(x_1^2)\left(1-a_1^2(x_1^2)\right)1 + \left(y_2 - a_2^3\right)a_2^3(x_2^3)\left(1-a_2^3(x_2^3)\right)w_{21}^2a_1^2(x_1^2)\left(1-a_1^2(x_1^2)\right)1 + \left(y_3 - a_3^3\right)a_3^3(x_3^3)\left(1-a_3^3(x_3^3)\right)w_{31}^2a_1^2(x_1^2)\left(1-a_1^2(x_1^2)\right)1\right)$$

Let's find a new value (updated weight) for a variable b_1^2:

$$b_1^2 = b_1^2 - \eta\frac{\partial C\left(y_i,\ a_i^3\right)}{\partial b_1^2}$$

Let's find the rest of the partial derivatives for b_i^2:

$$\frac{\partial C(y_i,a_i^3)}{\partial b_2^2} = -\frac{2}{3}\left(\left(y_1 - a_1^3\right)a_1^3(x_1^3)\left(1-a_1^3(x_1^3)\right)w_{12}^2a_2^2(x_2^2)\left(1-a_2^2(x_2^2)\right)1 + \left(y_2 - a_2^3\right)a_2^3(x_2^3)\left(1-a_2^3(x_2^3)\right)w_{22}^2a_2^2(x_2^2)\left(1-a_2^2(x_2^2)\right)1 + \left(y_3 - a_3^3\right)a_3^3(x_3^3)\left(1-a_3^3(x_3^3)\right)w_{32}^2a_2^2(x_2^2)\left(1-a_2^2(x_2^2)\right)1\right)$$

$$\frac{\partial C(y_i,a_i^3)}{\partial b_3^2} = -\frac{2}{3}\left(\left(y_1 - a_1^3\right)a_1^3(x_1^3)\left(1-a_1^3(x_1^3)\right)w_{13}^2a_3^2(x_3^2)\left(1-a_3^2(x_3^2)\right)1 + \left(y_2 - a_2^3\right)a_2^3(x_2^3)\left(1-a_2^3(x_2^3)\right)w_{23}^2a_3^2(x_3^2)\left(1-a_3^2(x_3^2)\right)1 + \left(y_3 - a_3^3\right)a_3^3(x_3^3)\left(1-a_3^3(x_3^3)\right)w_{33}^2a_3^2(x_3^2)\left(1-a_3^2(x_3^2)\right)1\right)$$

Let's find new values (updated weight) for variables b_2^2 and b_3^2:

$$b_2^2 = b_2^2 - \eta\frac{\partial C\left(y_i,\ a_i^3\right)}{\partial b_2^2}$$

$$b_3^2 = b_3^2 - \eta\frac{\partial C\left(y_i,\ a_i^3\right)}{\partial b_3^2}$$