Cubic Bézier Curve and Surface

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Bézier curve and surface are related. Therefore, first we will describe Bézier curve then Bézier surface.

0.1 Cubic Bézier Curve

A Bézier curve is a parametric curve that passes through its *end-control-points* while shape of the curve is controlled by its *middle-control-points*. Continuity of cubic Bézier is C^0. A cubic Bézier curve of degree 3 can be defined as follows:

$$Q(t) = \sum_{i=0}^{3} P_i B_{i,3}(t), \tag{1}$$

where P_0, P_1, P_2 and P_3 are control points of Bézier curve and $B_{0,3}(t), B_{1,3}(t), B_{2,3}(t)$ and $B_{3,3}(t)$ are Bernstein polynomials. A control point can be defined in Euclidean space R^n. For example in Euclidean space R^3 a control point has three components (x,y,z). Bernstein polynomials $B_{0,3}(t), B_{1,3}(t), B_{2,3}(t)$ and $B_{3,3}(t)$ can be defined as follows:

$$\begin{aligned} B_{0,3}(t) &= (1-t)^3 \\ B_{1,3}(t) &= 3t(1-t)^2 \\ B_{2,3}(t) &= 3t^2(1-t) \\ B_{3,3}(t) &= 3t^3 \end{aligned} \tag{2}$$

Figure 1 shows plot of cubic Bézier basis functions.

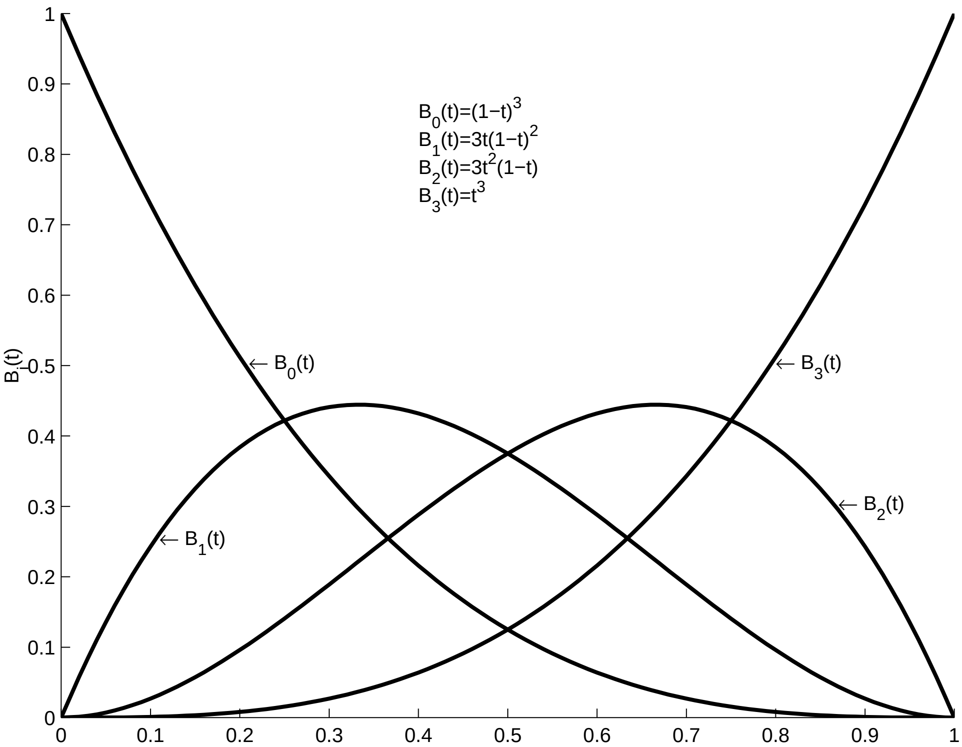

Figure 1: Cubic Bézier basis functions

By combining Equations (1) and (2) we can write equation of a cubic Bézier as follows:

$$Q(t) \quad = \quad (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t) P_2 + t^3 P_3. \tag{3}$$

P_0 and P_3 are called *end control points* (ECP) while P_1 and P_2 are called *middle control points* (MCP). $Q(t)$ is an interpolated point at parameter value t. To generate n points between first and last control points inclusive, the parameter t is divided into $n-1$ intervals between 0 and 1 inclusive and $Q(t)$ is evaluated at n values of t. Figures 2 and 3 show the cubic Bézier in 2D and 3D Euclidean space respectively.

In matrix form equation of a cubic Bézier curve can be written as follows:

$$Q(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} \tag{4}$$

Continuous multiple segment Bézier curve can be constructed by taking the first control point of current segment same as the last control point of the previous segment.

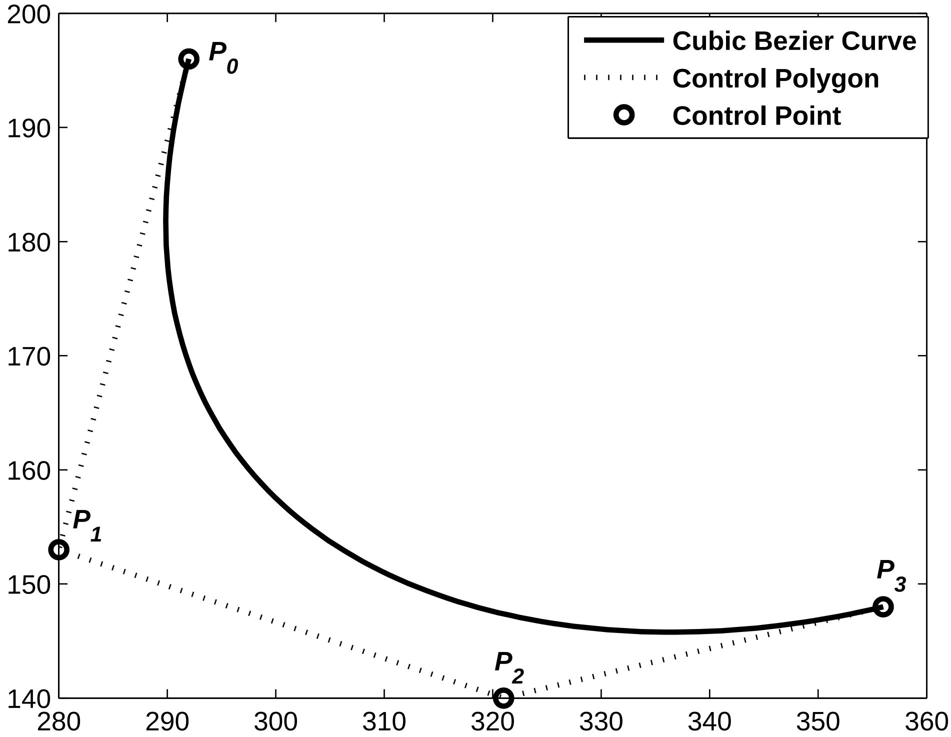

Figure 2: A cubic Bézier curve in Euclidean space R^2.

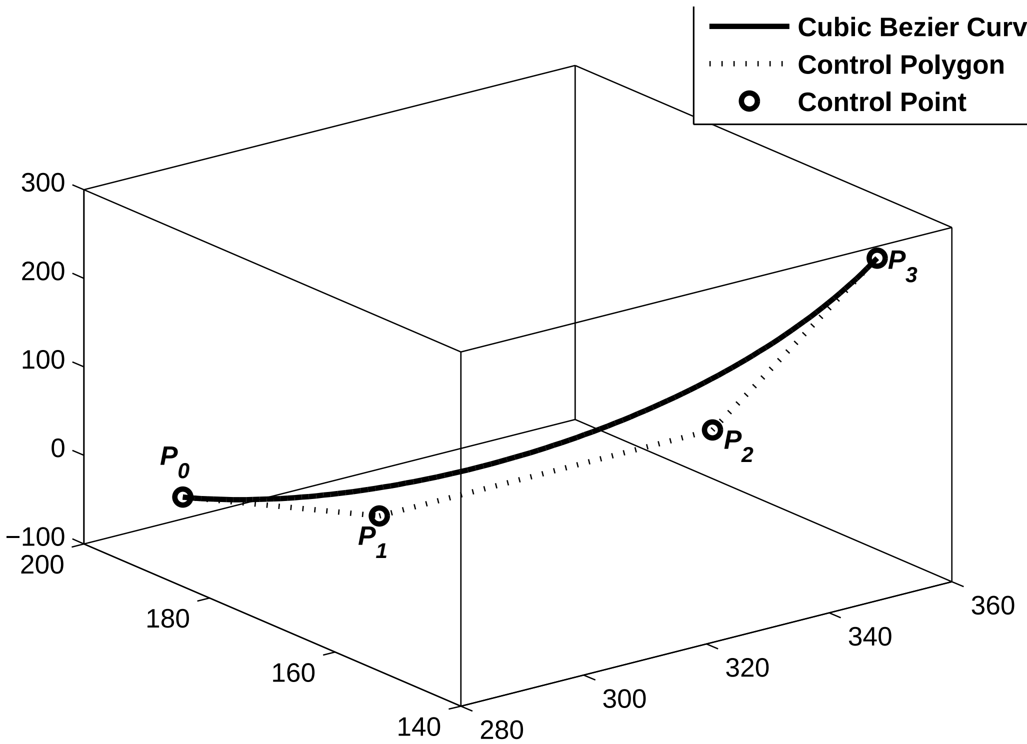

Figure 3: A cubic Bézier curve in Euclidean space R^3.

0.2 Cubic Bézier Surface

A cubic Bézier surface is composed of one or more cubic Bézier patches. A Bézier curve is a function of one variable and takes a sequence of control points. A Bézier patch is a function of two variables with an array of control points. Most of the methods for the patch are direct extensions of those for the curves. A cubic Bézier patch is constructed from an 4×4 array of control points: $P_{ij},\ 0 \leq i \leq 3,\ 0 \leq j \leq 3$.

A cubic Bézier patch is parameterized by two variables u and v is given by the equation:

$$Q(u,v) = \sum_{j=0}^{3} \sum_{i=0}^{3} P_{i,j}(u) B_{i,3}(u) B_{j,3}(v), \quad 0 \leq u \leq 1,\ 0 \leq v \leq 1. \tag{5}$$

The bi-variate Bernstein Polynomials $B_{i,3}(u)$ and $B_{j,3}(v)$ serving as the functions that blend the control points together. Figures 4 shows a cubic Bézier patch in 3D space. Figures 5 shows a cubic Bézier surface (teapot) in 3D space. This teapot surface is constructed using 32 cubic Bézier patches.

In matrix form equation of a cubic Bézier patch is written as follows:

$$Q(u,v) = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} \\ P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} \\ P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} \\ P_{3,0} & P_{3,1} & P_{3,2} & P_{3,3} \end{bmatrix} \begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix} \tag{6}$$

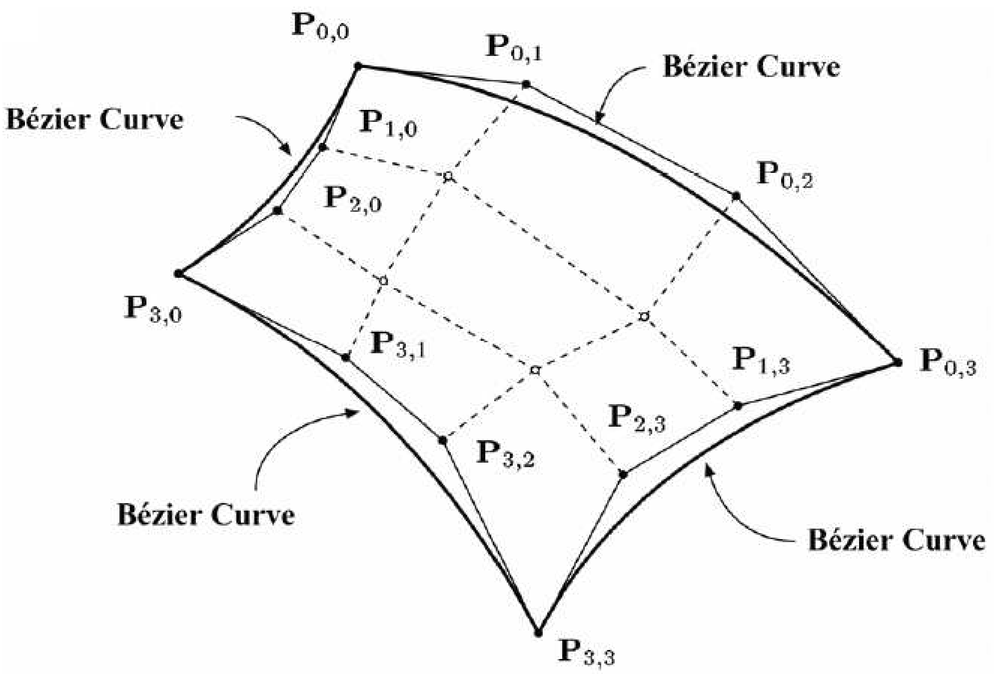

Figure 4: A cubic Bézier patch in Euclidean space R^3.

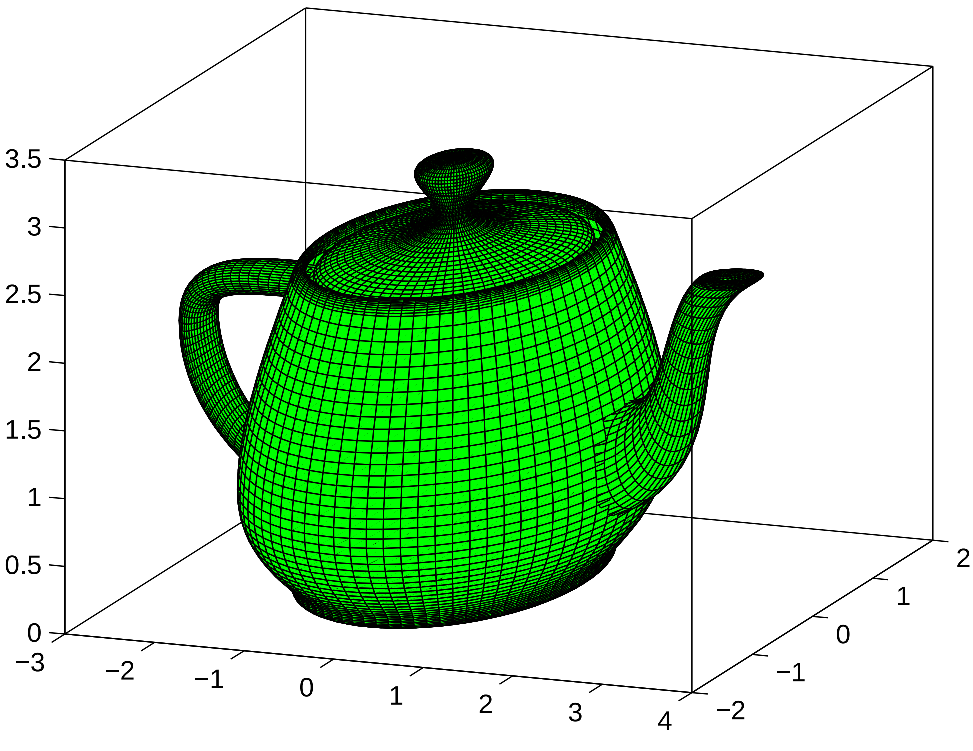

Figure 5: A cubic Bézier surface in Euclidean space R^3.