1 Maximum Likelihood Estimator

Three properties of maximum likelihood estimator:

1. θ_{ML} is consistent. $\theta_{ML} \to \theta_0$ when $n \to \infty$.
2. θ_{ML} is asymptotically normal. $\sqrt{n}(\theta_{ML} - \theta_0) \sim \mathcal{N}(0, I_n(\theta_0)$ when $n \to \infty$ and $I_n(\theta_0)$ is the fisher information.
3. θ_{ML} is asymptotically efficient. θ_{ML} minimizes $\mathbb{E}(\theta - \theta_0)^2$ when $n \to \infty$ because the asymptotic variance equals the Rao-Cramer bound (MLE is asymptotically unbiased). Note: when n is finite, θ_{ML} is not necessarily efficient, e.g., Stein estimator is universally more efficient for single sample.

Rao-Cramer bound: for any *unbiased* estimator $\hat{\theta}$ of θ_0, $\mathbb{E}(\hat{\theta} - \theta_0)^2 \geq 1/I_n(\theta_0)$, where $I_n(\theta) = -\mathbb{E}(\frac{\partial^2}{\partial\theta^2} \log f(X;\theta) \mid \theta) = \mathbb{E}(\frac{\partial}{\partial\theta} \log f(X;\theta) \mid \theta)^2$ is the fisher information.

Sketch of Proof: define $\Lambda = \frac{\partial \log P(X;\theta)}{\partial\theta}$. Cauchy-Schwarz says $\mathbf{Cov}^2(\Lambda, \hat{\theta}) \leq \mathbf{Var}(\Lambda)\mathbf{Var}(\hat{\theta}) = \mathbb{E}(\Lambda^2)\mathbf{Var}(\hat{\theta})$ because $\mathbb{E}\Lambda = 0$. Note that $\mathbf{Cov}(\Lambda, \hat{\theta}) = \mathbb{E}(\Lambda\hat{\theta}) = \int_X \hat{\theta}(x) \frac{\partial}{\partial\theta} f(x;\theta) dx = \frac{\partial}{\partial\theta} \int_X \hat{\theta}(x) f(x;\theta) dx = \frac{\partial}{\partial\theta} \mathbb{E}\hat{\theta} = 1$. Therefore, $\mathbf{Var}(\hat{\theta}) \geq 1/\mathbb{E}(\Lambda^2)$.

However, when the dimension of problem goes to infinity while keeping the data-dim ratio fixed, MLE is biased and the p-values are unreliable.

2 Regression

Bias-Variance trade-off

Let D be the training dataset and $\hat{f}$ be the predictive function. $\mathbb{E}_D \mathbb{E}_{Y|X}(\hat{f}(X) - Y)^2 = \mathbb{E}_D \mathbb{E}_{Y|X}[(\hat{f}(X) - \mathbb{E}_{Y|X} Y)^2 + (\mathbb{E}_{Y|X} Y - Y)^2] = \mathbb{E}_D(\hat{f}(X) - \mathbb{E}(Y \mid X))^2 + \mathbb{E}_D(\mathbb{E}(Y \mid X) - Y)^2 = \mathbb{E}_D(\hat{f}(x) - \mathbb{E}_D \hat{f}(x))^2 + \left(\mathbb{E}_D \hat{f}(x) - \mathbb{E}(Y \mid X)\right)^2 + \mathbb{E}_D(\mathbb{E}(Y \mid X) - Y)^2$. It means that expected square error (training) = variance of prediction + squared bias + variance of noise.

The optimal trade-off is achieved by avoiding under-fitting (large bias) and over-fitting (large variance). Note that here the variance of output is computed by refitting the regressor on a new dataset.

Regularization

Ridge and Lasso can be viewed as MAP (maximum a posterior) estimation. A Gaussian prior on β is equivalent to Ridge and a Laplacian prior is equivalent to Lasso. Using SVD, we get Ridge has built-in model selection: $X\beta^{\text{Ridge}} = \sum_{j=1}^{d} [d_j^2/(d_j^2 + \lambda)] u_j u_j^T Y$ (each $u_j u_j^T Y$ can be viewed as a model). Lasso has more sparse estimations because the gradient of regularization does not shrink as in the case of Ridge.

3 BLR and GP

Bayesian Linear Regression

$Y = X\beta + \epsilon$, $\epsilon \sim \mathcal{N}(0, \sigma^2)$ and a prior $\beta \sim \mathcal{N}(0, \Lambda^{-1})$. By Bayesian, $\beta \mid X, Y \sim \mathcal{N}(\mu_\beta, \Sigma_\beta)$, where $\mu_\beta = (X^T X + \sigma^2 \Lambda)^{-1} X^T Y$ and $\Sigma_\beta = \sigma^2 (X^T X + \sigma^2 \Lambda)^{-1}$. MAP estimation of this prior (i.e., μ_β) is equivalent to Ridge regression given $\Lambda = \lambda I$ and $\sigma = 1$.

When $\Lambda = \lambda I$, under the prior, $Y \sim \mathcal{N}(0, \frac{1}{\lambda} X^T X + \sigma^2 I)$. Therefore, $\mathbf{Cov}(y_i, y_j) = \frac{1}{\lambda} x_i^T x_j$. It means a prior that closer samples is more similar, i.e., $\mathbf{Cov}(y_i, y_j)$ is large when $x_i^T x_j$ is large. The kernel $X^T \Lambda^{-1} X$ is thus called linear kernel. When a general kernel is used, Gaussian Process appears.

Gaussian Process

Kernel Function

A function is a kernel iff $k(x, x') = k(x', x)$ (symmetry) and $\int_\Omega k(x, x') f(x) f(x') dx dx' \geq 0$ for any $f \in L_2$ and $\Omega \in \mathcal{R}^d$ (semi-positiveness in continuous case) or $K(X)$ is a valid covariance matrix for any X (semi-positiveness in discrete case). The latter is equivalent to either (1) $\sum_{i,j} a_i a_j K(x_i, x_j) \geq 0$ for any $a_{i,j}$ and $k_{i,j}$, or (2) $k(x, x') = \phi(x)^T \phi(x')$ for some ϕ.

Assume $k_{1,2}$ are valid kernels, then the following are valid kernels:

1. $k(x, x') = k_1(x, x') + k_2(x, x')$.
2. $k(x, x') = k_1(x, x') \cdot k_2(x, x')$. Proof: let $V \sim \mathcal{N}(0, K_1)$, $W \sim \mathcal{N}(0, k_2)$ and is independent to V, then $\mathbf{Cov}(V_i W_i, V_j W_j) = \mathbf{Cov}(V_i, V_j)\mathbf{Cov}(W_i, W_j) = k_1 \cdot k_2(x_i, x_j)$.
3. $k(x, x') = c k_1(x, x')$ for constant $c > 0$.
4. $k(x, x') = f(k_1(x, x'))$ if f is a polynomial with positive coefficients or the exp. Proof: polynomial can be proved by applying the product, positive scaling and addition. Exp can be proved by taking limit on the polynomial.
5. $k(x, x') = f(x) k_1(x, x') f(x')$.
6. $k(x, x') = k_1(\phi(x), \phi(x'))$ for any function ϕ.

Example: RBF kernel $k(x, y) = \exp(-\frac{1}{2\sigma^2}\|x - y\|^2) = \exp(-\frac{1}{2\sigma^2}\|x\|^2) \exp(\frac{1}{\sigma^2} x^T y) \exp(-\frac{1}{2\sigma^2}\|y\|^2)$ is valid. Since $x^T y$ is the linear kernel and thus $\exp(\frac{1}{\sigma^2} x^T y)$ is a valid kernel, let $f(x) = \exp(-\frac{1}{2\sigma^2}\|x\|^2)$, we get the RBF function equals $f(x) k(x, y) f(y)$, which is a valid kernel.

Mercer's Theorem

Assume $k(x, x')$ is a valid kernel. Then there exists an orthogonal basis e_i and $\lambda_i \geq 0$, s.t. $k(x, x') = \sum_i \lambda_i e_i(x) e_i(x')$.

Conditional Gaussian

$\mathbb{E}(y_2 \mid y_1) = \mu_2 + \Sigma_{21} \Sigma_{11}^{-1} (y_1 - \mu_1)$, $\mathbf{Var}(y_2 \mid y_1) = \Sigma_{22} - \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12}$.

4 Linear Methods for Classification

Concept Comparison

1. Probabilistic Generative, modeling $p(x, y)$: (1) can create new samples, (2) outlier detection, (3) probability for prediction, (4) high computational cost and (5) high bias.
2. Probabilistic Discriminative, modeling $p(y \mid x)$: (1) probability for prediction, (2) medium computational cost and (3) medium bias.
3. Discriminative, modeling $y = f(x)$: (1) no probability for prediction, (2) low computational cost and (3) low bias.

Infer $p(x, y)$ for classification problems

Use $p(x, y) = p(y) p(x \mid y)$. Since y has finite states, model $p(y)$ and $p(x \mid y)$ for different y. The modeling requires to (1) guess a distribution family and (2) infer parameters by MLE.

Compute $p(y \mid x)$ by discriminant analysis (DA)

Linear DA

Goal: classify a sample into two Gaussian distribution with $\Sigma_0 = \Sigma_1$. After calculation, $p(y = 1 \mid x) = 1/(1 + \exp(-\log \frac{p(x|y=1)p(y=1)}{p(x|y=0)p(y=0)})) = 1/(1 + \exp(w_1^T x + w_0))$ since the quadratic term is eliminated due to $\Sigma_0 = \Sigma_1$.

Quadratic DA

Goal: classify a sample into two Gaussian distribution with $\Sigma_0 \neq \Sigma_1$. After calculation, $p(y = 1 \mid x) = 1/(1 + \exp(x^T W x + w_1^T x + w_0))$.

Optimization Methods

Optimal Learning Rate for Gradient Descent

Goal: find $\eta^* = \mathbf{argmin}_\eta L(w^k - \eta \cdot \nabla L(w^k))$.

By Taylor expansion of $L(w^{k+1})$ at w^k and solve for the optimal η, we get $\eta^* = \frac{\|\nabla L(w^k)\|^2}{\nabla L(w^k)^T H_L(w^k) \nabla L(w^k)}$.

However, naive gradient descent has two weaknesses: (1) it often has a zig-zag behavior, especially in a very narrow, long and slightly downward valley; (2) the gradient update is small near the stationary point. This can be mitigated by adding a momentum term in the update: $w^{k+1} = w^k - \eta \nabla L(w^k) + \mu^k (w^k - w^{k-1})$ which speeds the update towards the "common" direction.

Newton's Method

Taylor-expand $L(w)$ at w_k to derive the optimal w^{k+1}: $L(w) \approx L(w) + (w - w^k)^T \nabla L(w^k) + \frac{1}{2}(w - w^k)^T H_L(w^k)(w - w^k) \Rightarrow w^{k+1} = w^k - H_L^{-1}(w^k) \nabla L(w^k)$.

Pros: (1) better updates compared to GD since it uses the second Taylor term and (2) does not require learning rate.

Cons: requires H_L^{-1} which is expensive.

Bayesian Method

In most cases, the posterior is intractable. Use approximation of posterior instead.

Laplacian Method

Idea: approximate posterior near the MAP estimation with a Gaussian distribution. $p(w \mid X, Y) \propto p(w, X, Y) \propto \exp(-R(w))$, where $R(w) = -\log p(w, X, Y)$. Let $w^* = \mathbf{argmin}\, R(w)$ be the MAP estimation and Taylor-expand $R(w)$ at w^*: $R(w) \approx R(w^*) + \frac{1}{2}(w - w^*)^T H_R(w^*)(w - w^*)$. Therefore, $p(w \mid X, Y) \propto \exp(-R(w^*) - \frac{1}{2}(w - w^*)^T H_R(w^*)(w - w^*))$ and thus $(w \mid X, Y) \sim \mathcal{N}(w^*, H_R^{-1}(w^*))$.

AIC & BIC

- Define BIC $= k \log N - 2 \log \hat{L}$, where k is #parameters and $\hat{L}$ is the likelihood $p(x \mid w^*)$. A lower BIC means a better model.
- Define AIC $= 2k - 2 \log \hat{L}$. A lower AIC means a better model.

LDA by loss minimization

Perceptron

Goal: for $y_i \in \{0, 1\}$, find w, s.t. $y_i w^T x_i > 0$ for any i. The classification function is $c(x) = \mathbf{sgn}(w^T x)$.

$L(y, c(x)) = 0$ if $y w^T x > 0$ and $L(y, c(x)) = -y w^T x$ o.w. By gradient descent, the Perceptron is guaranteed to converge if (1) the data is linearly separable, (2) learning rate $\eta(k) > 0$, (3) $\sum_k \eta(k) \to +\infty$ and (4) $(\sum_k \eta(k)^2)/(\sum_k \eta(k))^2 \to 0$. However, there exists multiple solutions if the data is linearly separable.

Fisher's LDA

Idea: project the two distribution into one dimension and maximize the ratio of the variance between the classes and the variance within the classes, i.e., $\max(w^T u_1 - w^T u_0)^2/(w^T S w)$, where $S = \Sigma_0 + \Sigma_1$. Let gradient be zero and solve for w^*, we get $w^* \propto S^{-1}(u_1 - u_0)$.

We first compute w^* and fit distributions of the two-class projection. Then apply Bayesian decision theory to make classification.

5 Convex Optimization

Definition: the objective is convex and the feasible set is convex. The standard form is to minimize a convex function ($f(w)$ for convex f) under affine equality constraint ($g_i(w) = 0$ for affine g_i) and convex non-positive constraint ($h_i(w) \leq 0$ for convex h_i).

How to Solve

(1) Define Lagrangian: $L(w, \lambda, \alpha) = f(w) + \sum_i \lambda_i g_i(w) + \sum_j \alpha_j h_j(w)$ and $\alpha_j \geq 0$.

(2) Check whether strong duality holds using Slater's condition (sufficient but not necessary): there is a strictly feasible point, i.e., $\exists w_0$, s.t. $g_i(w_0) = 0$ and $h_i(w_0) < 0$.

(3) Solve for $dL(w) = 0$, $g_i(w) = 0$, $\alpha \geq 0$, $h_j(w) \leq 0$ and $\alpha_j h_j(w) = 0$ (complementary slackness).

(4) Weak duality always holds and when Slater's condition holds it equals $\min_w f(w)$: solve $\max_{\lambda,\alpha} \Theta(\lambda, \alpha)$ s.t. $\alpha \geq 0$, where $\Theta(\lambda, \alpha) = \min_w L(w, \lambda, \alpha)$. The optimal in the weak duality is a lower bound for $\min_w f(w)$ since $\Theta(\lambda, \alpha) \leq \min_w f(w)$ for any λ and α.

6 Support Vector Machine

Hard-Margin SVM

The optimization form is to minimize $\frac{1}{2}\|w\|^2$, s.t. $y_i(w^T x_i + w_0) \geq 1$ ($y_i \in \{-1, 1\}$). It is convex and the Slater's condition holds under the assumption of linear separability.

By solving $\min_{w,w_0} L(w, w_0, \alpha)$, the duality form is $\max_\alpha -\frac{1}{2}\sum_{i,j} \alpha_i \alpha_j y_i y_j x_i^T x_j + \sum_i \alpha_i$, s.t. $\alpha_i \geq 0$, $\sum_i \alpha_i y_i = 0$. $w^* = \sum_{i \in \text{support vec}} \alpha_i^* y_i x_i$.

Soft-Margin SVM & Kernel SVM

When the data is not linearly separable, one solution is to add slack variables: $\min_{\xi,w,w_0} \frac{1}{2}\|w\|^2 + C\sum_i \xi_i$, s.t. $y_i(w^T x_i + w_0) \geq 1 - \xi_i$ and $\xi_i \geq 0$. Note: when the data is linearly separable, soft-margin does not necessarily produce the same cut-plane as the hard-margin SVM, especially when the margin is small due to few support vectors. When $C \to \infty$, it becomes hard-margin SVM.

The only difference in the dual form is that we need an extra condition $0 \leq \alpha \leq C$. $\xi_i^* = \max(0, 1 - y_i(w^{*T} x_i + w_0))$.

Another solution, Kernel SVM, is to use $K(x_i, x_j) = \phi(x_i)^T \phi(x_j)$ to replace $x_i^T x_j$.

Multi-class SVM

Idea: use M := #class hyperplanes and maximize the generalized margin: $\min \sum_{i=1}^M w_i^T w_i$ s.t. $w_{y_i}^T x_i + w_{y_i,0} - \max_{y \neq y_i} w_y^T x_i + w_{y,0} \geq 1$.

Structural SVM

Goal: predict a structured output label, e.g., parsing trees.

Difficulty of predicting structures:

1. Compact representation of output space. Sol: use a score function and feature map $\hat{y} = \mathbf{argmax}_y\, w^T \Psi(x, y)$.
2. Efficient prediction (cannot use brute-force search to find argmax). Sol: assume structures such as decomposable output spaces.
3. Define prediction error (cannot use 0/1 errors). Sol: use a loss function $\Delta(y, \hat{y})$.
4. Efficient training (cannot evaluate all constraints). Sol: iteratively add new constraints to the training.

$\min_{\mathbf{w},\xi \geq 0} \frac{1}{2}\mathbf{w}^\top \mathbf{w} + C\sum_{i=1}^n \xi_i$, s.t. $\mathbf{w}^\top \Psi(z_i, \mathbf{y}_i) - \max_{z \neq z_i}[\Delta(z, z_i) + \mathbf{w}^\top \Psi(z, \mathbf{y}_i)] \geq -\xi_i, \forall \mathbf{y}_i \in \mathcal{Y}$.

7 Ensemble

Random Forest

Random feature selection induces regulation.

Adaboost

Classifier weight $\alpha_t = \frac{1}{\text{weighted err}_t} - 1$. Sample weight $w_{t+1} = \alpha_t w_t$ for mislabeled samples, o.w. unchanged.

Adaboost has following properties:

1. It minimizes exponential loss forwardly.
2. It trains max-margin classifiers.
3. It, as well as Random Forest, is spiky self-averaging interpolators, which localize the effect of noise.
4. It falls into the double descent regime: over-parameterized models can have better generalization.

8 Generative Models

Variational Autoencoder: use approximated probability to model $p(z \mid x)$ and the $p(x \mid z)$.

- Pro: can be used as feature representation.
- Con: generated images are blurrier and relatively low compared to GAN.

Generative Adversarial Network: use 2-player game to sample from $p(x)$.

- Pro: SOTA.
- Con: (1) unstable, (2) potential mode collapse and (3) cannot solve $p(x)$.

9 Non-parametric Bayesian Inference (BI)

Exact Conjugate Prior of Multivariate Gaussian

Data: $x_i \sim \mathcal{N}(\mu, \Sigma)$ i.i.d.. Inverse Wishart: $\Sigma \sim \mathcal{W}^{-1}(S, v) \propto |\Sigma|^{(v+p+1)/2} \exp(-\mathrm{Tr}(\Sigma^{-1} S)/2)$.

Normal Inverse Wishart as conjugate prior: $p(\mu, \Sigma | m_0, k_0, v_0, S_0) = \mathcal{N}(\mu | m, \Sigma/k_0)\mathcal{W}^{-1}(\Sigma | S_0, v_0)$.

Update rule: $m_p = (k_0 m_0 + N\overline{x})/(k_0 + N)$, $k_p = k_0 + N$, $v_p = v_0 + N$, $S_p = S_0 + k_0 m_0 m_0^\top - k_p m_p m_p^\top + \sum(x_i - \overline{x})(x_i - \overline{x})^\top$.

BI with Semi-Conjugate Prior

New prior: $\mu \sim \mathcal{N}(m_0, V_0)$, $\Sigma \sim \mathcal{W}^{-1}(S_0, v_0)$, then posterior $p(\mu, \Sigma | X)$ is intractable, but condition posterior is exact, $p(\mu | \Sigma, X) = \mathcal{N}(m_p, V_p)$, $V_p^{-1} = V_0^{-1} + N\Sigma^{-1}$, $V_p^{-1} m_p = V_0^{-1} m_0 + N\Sigma^{-1}\overline{x}$; $p(\Sigma | \mu, X) = \mathcal{W}^{-1}(S_p, v_p)$, $v_p = v_0 + N$, $S_p = S_0 + \sum x_i x_i^\top + N\mu\mu^\top - 2N\overline{x}\mu^\top$.

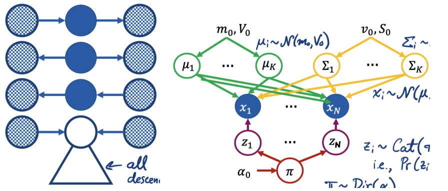
Gibbs sampling: random variable $p(z_1, \cdots, z_p)$ intractable, cyclically resample z_i according to tractable conditional distribution $p(z_i | z_{/i})$ n times, when $n \to \infty$, $(z_1, \cdots, z_p) \sim p(z_1, \cdots, z_p)$

Finally, replace posterior with MC sampling: $\mathbb{E}_{\theta|X} f(x|\theta) \approx \sum f(x|\theta_i)/N$

BI for Gaussian Mixture Model

Data model: latent K class variable $z_i \sim \mathrm{Cat}(\pi)$, observed $x_i \sim \mathcal{N}(\mu_{z_i}, \Sigma_{z_i})$. Prior: $\mu_k \sim \mathcal{N}(m_0, V_0)$, $\Sigma_k \sim \mathcal{W}^{-1}(S_0, v_0)$, $\pi \sim \mathrm{Dir}(\alpha) \propto \prod_k^K p_k^{\alpha_k - 1}$. Prior also intractable.

Goal Gibbs sampling for BI, but to simplify conditional distribution.



d-seperation: for verifying conditional independence. Given with observed variable set C, if every path from variable A to B is blocked on probability graph, then A and B are independent condition on C. By this thm: (1) z_i, z_j (2) μ, π (3) Σ, π all independent condition on other parameter. Sampling procedure: (1) $z^{(t)} \leftarrow p\left(\cdot | x, \mu^{(t-1)}, \Sigma^{(t-1)}\right)$, (2) $\mu^{(t)} \leftarrow p\left(\cdot | x, \Sigma^{(t-1)}, z^{(t)}\right)$, (3) $\Sigma^{(t)} \leftarrow p\left(\cdot | x, \mu^{(t)}, z^{(t)}\right)$, (4) $\pi^{(t)} \leftarrow p\left(\cdot | x, z^{(t)}\right)$

BI for Non-Parametric GMM

Goal: sample from infinite categorical distri.

Dirichlet Process (DP): Θ parameter space, H prior distri on Θ, $A_1, \cdots, A_r$ arbitrary partition of Θ. G a categorical distribution over $\{A_i\}$ is $G \sim \mathrm{DP}(\alpha, H)$ if $(G(A_1), \ldots, G(A_r)) \sim \mathrm{Dir}(\alpha H(A_1), \ldots, \alpha H(A_r))$.

Posterior: $G | \{\theta_i\}_{i=1}^n \sim \mathrm{DP}\left(\alpha + n, \frac{\alpha H + \sum_{i=1}^n \delta_{\theta_i}}{\alpha + n}\right)$

Condition on θ, **Margin over** G: $\theta_{n+1} \mid \theta_1, \ldots, \theta_n \sim \frac{1}{\alpha+n}\left(\alpha H + \sum_{i=1}^n \delta_{\theta_i}\right)$, Leads to CRP

Three Methods of Sampling from DP

In $K \to \infty$ GMM, θ in DP is z, G is π.

(1) Chinese Restaurant Process (CRP), sample z, marginalize over π:

$$p(z_n = k | \theta_{i<n}) = \begin{cases} n_k/(\alpha + n - 1), \text{existing } k \\ \alpha/(\alpha + n - 1), \text{new } k \end{cases}$$

Expect # of Class $\sum_{i=1}^n \frac{\alpha}{\alpha+i-1} \simeq \alpha \log\left(1 + \frac{n}{\alpha}\right)$

(2) Stick-breaking Construction samples π: $\beta_k \sim \mathrm{Beta}(1, \alpha)$, $\theta_k^* \sim H$, $\pi_k = \beta_k \prod_{l=1}^{k-1}(1 - \beta_l)$

(3) Marginalize over μ, Σ when sampling z (if intractable), less variance (Rao-Blackwall).

Exchangeability: $p(\{\theta_i\}) = \prod_{n=1}^N p(\theta_n | \{\theta_{i<n}\})$ unchanged after permuting sampling order.

DeFinetti's Thm any exchangeable distri is a mixture model $P(\{\theta_i\}) = \int \prod_{i=1}^n G(\theta_i)\, dP(G)$

10 PAC Learning

Definitions:

- A learning algorithm $\mathcal{A}$ can learn $c \in C$ if there is a poly(.,.), s.t. for (1) any distribution $\mathcal{D}$ on $\mathcal{X}$ and (2) $\forall 0 < \epsilon < \frac{1}{2}, 0 < \delta < \frac{1}{2}$, $\mathcal{A}$ outputs $\hat{c} \in \mathcal{H}$ given a sample of size at least poly($\frac{1}{\epsilon}$, $\frac{1}{\delta}$, size(c)) such that $P(\mathcal{R}(\hat{c}) - \inf_{c \in C} \mathcal{R}(c) \leq \epsilon) \geq 1 - \delta$.
- $\mathcal{A}$ is called an efficient PAC algorithm if it runs in polynomial of $\frac{1}{\epsilon}$ and $\frac{1}{\delta}$.
- C is (efficiently) PAC-learnable from $\mathcal{H}$ if there is an algorithm $\mathcal{A}$ that (efficiently) learns C from $\mathcal{H}$.

VC inequality:

- For an ERM $\hat{c}_n^*$, $\mathbf{P}(\mathcal{R}(\hat{c}_n^*) - \inf_{c \in \mathcal{C}} \mathcal{R}(c) > \epsilon) \leq 2|\mathcal{C}| \exp\left(-\frac{n\epsilon^2}{2}\right)$.
- Finite VC-dimension means PAC-learnable.

A Appendix

$\frac{\partial \vec{u}^T \vec{v}}{\partial x} = \frac{\partial \vec{u}}{\partial x}\vec{v} + \frac{\partial \vec{v}}{\partial x}\vec{u}$.

$\frac{\partial A\vec{u}}{\partial x} = \frac{\partial \vec{u}}{\partial x} A^T$.

$\mathcal{N}(\mu, \Sigma)$: $\frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} \exp(-\frac{1}{2}(x - \mu)^T \Sigma^{-1}(x - \mu))$.