Notes written by Alun Stokes

# Course Notes on O-minimality and the Pila-Wilkie Theorem

Presented by Gareth Jones as part of the Fields Institute Graduate Course on O-minimality and Applications

May 12, 2022

Springer Nature

*As the instructor for this module, these lecture notes are dedicated to Dr Gareth Jones, whose presentation of the Pila-Wilkie Theorem and its proof was both illuminating and intellectually challenging for this young graduate student.*

# Preface

This collection of notes is the result of a series of 8 lectures given by Dr Gareth Jones at the Fields Institute over the course of 19 Jan to 11 Feb 2022. All the material presented here was given as part of this first module in the three-module course on O-minimality and Applications. Primarily, attempt was made to uniformize notation and formatting across the various lectures, and some extra details added where otherwise missing or left as an exercise to the viewer.

This current iteration of the notes are to be considered preliminary, and the style and particularities of their presentation by no means final. As the owner of the sole pair of eyes to have read this document, no assurances are made that it is free of mistakes or oversights – but to the best of my ability, I have tried best to minimize the sure-to-come lists of errata and corrigenda upon review.

The intent of this document is to act as an easily (to an early graduate student in mathematics) accessible introduction to o-minimality in the context of expansions of the real field and their application in proving the Pila-Wilkie Theorem. In particular, o-minimality is not discussed in full generality – rather the role the property of o-minimality of ordered fields plays in results on semi-algebraic sets, cell decompositions and parameterization by these decompositions, and of course, how this all comes together to prove the eponymous theorem of Pila and Wilkie. The ordering of content presented is not strictly adherent to the delineation given in the lectures, but it does broadly follow.

These notes are written, as most things are, from the perspective of the author – which is to say, someone with little to no background in model theory. As such, there may be at points a belabouring on ideas that another would find trivial, unnecessary, or otherwise not necessarily worth the space they take up on the page. The purpose in compiling these notes is not just to archive the lecture series given but also to make it somewhat more accessible by means of clarifying that which I myself had to, as I attended the course. This does not mean any significant amount of material is added above and beyond the course content itself – rather, more so that some 'one-liners' in the original presentation are afforded just a few more in these notes.

Where not otherwise cited, facts should be supposed to have been taken from the lectures — which themselves will periodically include references or suggestions for further reading material. For exteriorly sourced information, the citations are included along with the rest at the end of document.

McMaster University, *Alun Stokes*
April 2022 *MATH 712*

# Acknowledgements

I would like to acknowledge Dr Gareth Jones for his wonderful series on the content presented hence, along with Drs Ya'acov Peterzil and Jacob Tsimerman for jointly organizing the course. I would like to thank the Fields Institute for hosting this graduate course, and finally Dr Patrick Speissegger as the intructor of record for MATH 712 at McMaster University.

# Contents

# Acronyms

The following are used varyingly judiciously throughout.

CD  Cell Decomposition
PW  Pila-Wilkie
DLO  Dense Linear Order
FOL  First-Order Language
MT  Monotonicity Theorem
FT  Finiteness Theorem

# Part I
# O-minimality and Necessary Concepts

This introductory part will focus primarily on setting up the setting in which we find ourselves working for most of the course. Highlights include "*what even is o-minimality?*" and "*but how does that imply Pila-Wilkie?*" – and perhaps a favourite of mine: "*what is the Pila-Wilkie Theorem?*". While the latter two are answered in much more detail, later on, this part will take us quickly through some of the significant results we'll find ourselves needing in a bit. In particular, we prove the monotonicity theorem, define cell decompositions (CD), then prove the cell decomposition theorem, and top it off with a discussion of dimensionality and the agreement between the geometric and algebraic notions of the quantity.

The prepared reader should find themselves acquainted with preliminary ideas in mathematical logic (nothing further than one would find in an undergraduate class on the subject) and the basics of field theory. Again, nothing further than an undergraduate would necessarily be expected to have encountered.

A reader should leave this section feeling themselves reasonably well-acquainted with some of the tools they might see themselves using in general sorts of proofs about o-minimal structures and how they may go about proving or disproving something to be o-minimal. Some preliminary notions of how this all fits into counting points of bounded height on curves may be coming to surface by the end of this part, but the novice reader would be well-forgiven were that not the case. They should, however, feel comfortable identifying what is 'clearly' a definable set and be able to do some rudimentary reasoning on how we can use the finiteness and definability of one of a pair of complementary sets to say something about the definability of the other. The idea of cells and cell decompositions should be reasonably understood (at least *intuitively* if not in full technical detail), and the reader should have a map of sorts in their mind that sequentializes and connects the discussed matter in a reasonable and meaningful way – as these preliminary ideas form the basis for the larger lemmata and theorems to come.

# Chapter 1
# A Bit Before we Begin

**Abstract** We begin with just a few words in preparation for what is to come; some definitions, expectations of the experience (or lack thereof) on the part of the reader, and a general outline are given. The overly excited reader may feel free to skip right onto Chapter 2, but this section serves as just a bit of an *amuse-bouche* for those not so ready to jump right in.

## 1.1 Before the Lectures Proper

We feel somewhat compelled to address an aspect of this topic that we felt was slightly neglected in the curriculum. Had one seen the list of attendees to these lectures, the reason for skipping over such 'trivialities' as we are about to point out briefly is clear — with several attendees being former students of Pila or Wilkie themselves. Still, for the *not-even-amateur* logician, the following certainly bears some explicit mention.

In the absence of the results we come to find, o-minimality may appear a relatively unmotivated idea to study. Of course, as the pure mathematicians we are (or hope one day to be), why *shouldn't* the mere concept of further understanding be enough to compel our interest? Still, the progression from introductory mathematical logic and the importance and usefulness of quantifier elimination to o-minimality is one that was unapparent to this author until being elsewhere noted. We don't claim this to be a failure of the course so much as a failure in personal preparation, but should the reader find themselves similarly underprepared, then they will find themselves thankful for this little pretext.

To keep things brief, we will say just this: in a predicate logical system, we are interested in quantifier elimination. The result of the elimination of quantifiers is essentially the answer to the question a quantified statement asks. Perhaps the most famous example of this is the existence of real roots of quadratic equations. We ask the quantified 'question':

$$\exists x \in \mathbb{R}.(a \cdot x^2 + b \cdot x + c = 0 \wedge a \neq 0) \tag{1.1}$$

— that is, does there exist such an $x$? The quantifier eliminated equivalent form is

$$b^2 - 4 \cdot a \cdot c \geq 0 \wedge a \neq 0, \tag{1.2}$$

the first half of which should be recognized as the quadratic discriminant (and the second just to ensure non-degeneracy). Here, quantifier elimination gives the exact and deterministic characterization of the answer to the quantified statement – and it is this property that motivates its study. We trust at this point that the motivation has been sufficiently belaboured.

It is known that first-order theories with quantifier elimination (that is, decidability for the theory can be reduced to the question of satisfaction of quantifier-free sentence in the theory) are model complete. In the interest of not straying too far, we leave it to the reader to believe or convince themselves that this is a desirable property. While this is not the focus of how we define o-minimality in this course, it is true that a structure is o-minimal exactly if every formula given by no more than one free variable and some subset of $M$-parameters is equivalent to a quantifier-free formula defined only by these parameters, and the ordering on the structure [4]. Thus, for anyone finding themselves perhaps unconvinced upfront of the merit of some of the ideas explored here (outside the Pila-Wilkie Theorem), then we hope this motivates the sequence we are about to take on. And for everyone else, we hope that this section did not bore too thoroughly.

## 1.2 Preliminary Definitions

Throughout, we will be working with models $\mathcal{M} = (M, <)$ of the theory of dense linear orders (DLO) *without endpoints*. For now, $M$ will be fixed, but we will look at some specific instances later on. Perhaps then, one of the most important definitions, to begin with, is that of *definability*.

**Definition 1.1 (Definability of sets (without parameters))** For $n \in \mathbb{Z}_{\geq 1}$, we say a set $A \subseteq M^n$ is *definable without parameters* if there exists some formula in our model, $\phi$, satisfied exactly by the elements of $A$.

**Definition 1.2 (Definability of sets)** For $X \subseteq M$ and $n \in \mathbb{Z}_{\geq 1}$, then we say a set $A \subseteq M^n$ is *definable with parameters from* $X$ if there exists some formula in our model, $\phi$, and elements $b_1, \ldots, b_m$, such that $\phi$ is satisfied exactly by the elements of $A$ along with the parameters in $X$.

Notice then that definability without a parameter is simply the case of definability with parameters coming from the empty set. These definitions immediately and naturally induce the idea of definable functions and definable points. In particular, a function is definable in parameters if its graph is definable by those same parameters in $\mathcal{M} = (M, <)$. Similarly, an element $a$ is definable in $\mathcal{M}$ (with parameters) if the singleton $\{a\}$ is definable in $\mathcal{M}$ by those same parameters. This isn't something we will need to consider too extensively.

As ever, when introduced to a novel space, we are interested in what its open intervals look like. We have the following characterization:

**Definition 1.3 (Open Interval)** A set, $A \subset M$ is an open interval in $(M, <)$ if $A$ is of one of the following forms:

- $(a, b)$ with $a < b \in M$
- $(-\infty, a)$ with $a \in M$
- $(a, \infty)$ with $a \in M$

We say further that intervals of the first type — that is, those having finite bounds — are *bounded*. Easy to miss but important to note is that the endpoints must sit inside our domain. So, for example, in $(\mathbb{Q}, <)$, the set $(-5, \sqrt{7})$ is *not* an open interval.

We imbue $M$ with the order topology and $M^n$ with the product topology. We then define what it means to be an o-minimal expansion.

**Definition 1.4 (O-minimal expansion)** Taking $\mathcal{M} = (M, <, \ldots)$ an expansion of $(M, <)$, we say $\mathcal{M}$ is o-minimal if every definable (with parameters) subset of $M$ is given by a finite union of open intervals and points.

If we weaken the above and ask only for *convex* sets (which are a superset of our open intervals) in place of open intervals, then the above would define *weak o-minimality* — but that won't be a topic of discussion here.

For the etymologically inclined, it is noted that the 'o' in o-minimal comes from the shortening of 'order-minimality'. For more information on the history and development of the idea of o-minimality, one may reference **Tame Topology & O-minimal Structures** [2] or **Definable Sets in Ordered Structures I** [5] and **II** [3].

Some (arguably) simple examples of o-minimal structures are given by expansions of the real field. Consider, for example, $\overline{\mathbb{R}} = (\mathbb{R}, <, +, -, \cdot, 0, 1)$ and the further expansion $\mathbb{R}_{\exp} = (\overline{\mathbb{R}}, \exp)$, both of which are o-minimal. Mind not to mistake the use of 'simplicity' as an indication that these are trivial or did not require particular and considerable consideration — rather, just that they have a relatively simple-seeming form. For now and going forward, we fix $\mathcal{M}$ an o-minimal structure and move on to our first theorem.

# Chapter 2
# Setting it all up

**Abstract** We now begin properly with a from-the-basics definition of the objects at play: field expansions, monotonicity, cells and decompositions into them, semi-algebraicity and similarly fundamental ideas are each defined and contextualized. Note that we will not be discussing topological definitions in general. That is to say, the reader is assumed to be familiar with basic point-set topology, and the ordinary sorts of topologies we see cropping up (e.g. order, product) – not that topological ideas won't be discussed. As well, basic knowledge of mathematical logic is assumed; first-order languages (FOL), $\mathcal{L}-$Structures, relations, and satisfiability are all presumed familiarities. With definability now a part of our tool-set, we start by proving a few theorems fundamental to results to come later in this course.

## 2.1 On Monotonicity

What constitutes a 'nice' property of a function is generally non-contentious; injectivity and surjectivity are often useful – together even more so – and it would be the odd mathematician to turn their nose up at a function being bounded, supposing they weren't chasing a nasty counterexample or engaging in some other such endeavour. At present, we will focus on the property of *monotonicity*, and when we can determine a definable function to be monotonic in the context of open intervals. The following was proved in [5] by Pillay and Steinhorn:

**Theorem 2.1 (The Monotonicity Theorem)**
*Suppose $f\colon I \to M$ is a definable function for $I \subset M$ an open interval. Then there exist $a_1, \dots, a_k \in I$ such that on each adjacent interval, $(a_j, a_{j+1})$ (where $I = (a_0, a_{k+1})$) $f$ is either constant, or strictly monotonic and continuous. Further, if $f$ is definable over some $A \subseteq M$, then so too are $a_1, \dots, a_k$ definable over $A$.*

Hence, we will refer to this simply as the Monotonicity Theorem, abbreviated by MT. It is perhaps not immediately apparent why this should be true, or even that we should be interested that it is. The answer to the second point is that this piece-wise continuity and monotonicity of definable functions is a relatively rigid condition, and this (not just here but for structures in general) allows us to say a good bit about them. Observe as well that if we have some $X \subseteq M$ definable and infinite, then X must contain some open interval. This should be relatively intuitive, even if a proof doesn't come to you immediately, given what we've covered thus far. As for why the Monotonicity Theorem holds, we show this by piecing together three lemmata that should make the picture a bit more clear. Throughout, take $J \subset I$ as an open interval. To not get bogged down in the minutiae of their proofs as we go through — not that they are particularly challenging — but in any case, we will state all three and then prove them sequentially.

**Lemma 2.1** *There is an open interval, $J' \subseteq J$, on which $f$ is constant or injective.*

**Lemma 2.2** *If $f$ is injective on $J$, then there is an open interval, $J' \subseteq J$ on which $f$ is strictly monotonic.*

and finally,

**Lemma 2.3** *If $f$ is a strictly monotonic function on $J$, then there exists some open interval $J' \subseteq J$ on which $J$ is continuous.*

Taking these lemmata for granted, it is not terribly difficult to see how the Monotonicity Theorem falls out. The fun then is in proving these three facts — which is nice, as they are not terribly complicated.

We start where any sensible person would.

***Proof (Lemma 2.1)*** Suppose there is some $y \in M$ such that its preimage under $f$ intersected with $J$ is infinite. This necessarily implies the existence of $J' \subseteq J$ an open interval on which $f$ takes constant value, and so we can assume for any $y \in M$ that we have $f^{-1}(y) \cap J$ is finite. Then, we must have f(J) infinite, and so contains interior with subset $(a, b)$, for $a < b$. Taking

$$q\colon (a,b) \to J$$
$$q\colon y \mapsto \min\{\, x \in J \mid f(x) = y \,\},$$

we get q injective — and so this is an open interval $J' \subseteq q((a,b))$ on which $f$ is injective. □

***Proof (Lemma 2.2)*** This we can get quite quickly. Suppose such a strictly monotone function exists on $J$. Clearly, $f$ cannot be constant (else monotonicity would be non-strict), and so by o-minimality of $f$, we get that the image of $J$ under $f$ contains some open interval, $J' \subseteq \text{image}\,(f)$, on which we have preimage a sub-interval of $J$. We get monotonicity on this interval by Lemma 2.1 and non-constancy (and thus monotonicity) of $f$; this must be a bijection (either order-preserving or reversing, but bijective either way), and so we are finished. □

***Proof (Lemma 2.3)*** Write me. □

***Proof (Theorem 2.1)*** We now combine these three lemmata to get our result. Take $A$ the set of all $x \in I$ (coming from our original theorem statement) such that $f$ is both continuous and strictly monotone at $x$. We know that taking the restriction of $f$ to some open sub-interval on which $f$ is defined maintains both continuity and monotonicity by Lemmata 2 and 3 — and so taking the set difference of A from I, the original open interval, we cannot have *any* open intervals. There are then thus only finitely many points, and the theorem follows. □

Take note that the proof provided here is *not* precisely the one that was given in the lecture, but rather a bit more condensed, less roundabout method of achieving the result. The strategy is the same, however, differing only in presentation.

Two Exercises Lec1 pg 4

The following result is a special case in 2 dimensions of what is referred to as the *Finiteness Theorem*, abbreviated FT. We first prove this special case, and then take a brief detour to talk about cell decompositions before we can address the more general theorem.

### 2.1.1 The (Planar) Finiteness Theorem

**Theorem 2.2 (Finiteness Theorem in $M^2$)**

*Suppose $A \subseteq M^2$ and that for each $x \in M$, the fibre $A_x$ above $x$ — that is, the set of $y$ with $(x, y) \in A$ — is finite. Then, there exists some $N \in \mathbb{Z}_{\geq 1}$ such that $|A_x| \leq N$ for all $x \in M$*

***Proof (Finiteness Theorem in $M^2$ (Theorem 2.2))*** We define a point $(a, b) \in M^2$ to be *normal* if it sits in an open box, $I \times J$ satisfying

- $(I \times J) \cap A = \emptyset$
- $(a, b) \in A$
- There exists a continuous $f : I \to M$ such that $(I \times J) \cap A = \operatorname{graph}(f)$.

Similarly, for points with only one finite endpoint, we say some $(a, \infty)$ (resp. $(a, -\infty)$) is *normal* if there exists open interval $I$ such that $a \in I$ and some $b \in M$ such that

$$(I \times (b, \infty)) \cap A = \emptyset$$

and again, respectively taking $(b, -\infty)$ for the other case.

Supposing we take the set $\{\, (a, b) \in M^2 \mid (a, b) \text{ is normal} \,\}$, it easily follows that this set is definable, and similarly so for the $\{\pm\infty\}$ cases. We now define functions $f_1, f_2, \ldots, f_n$ by the property that

$$\mathrm{dom}\,(f_k) = \{\, x \in M \mid |A_x| \geq k \,\}.$$

That is, we have the property that $f_k(x)$ is the $k$-th element of $A_x$ — and so we get the definability of each $f_k$ by the finiteness of each fibre.

Fixing some $a \in M$ and taking $n \geq 0$ maximal such that all of $f_1, \ldots, f_n$ are defined and *continuous* on an open interval around $a$. We then say that $a$ is

- **good** if $a \notin \mathrm{cl}\,(\mathrm{dom}\,(f_{n+1}))$ and otherwise
- **bad** if $a$ *is* in this closure.

We partition into $G = \{\, a \in M \mid a \text{ is good} \,\}$ and $B = \{\, a \in M \mid a \text{ is bad} \,\}$. What we will now show is that $G$ is definable — which we do by showing that for any $a \in B$, there is a minimal $b \in M \cup \{\pm\infty\}$ such that $(a, b)$ is *not* normal.

Let $a \in B$. We use the following notation for convenience:

$$\lambda(a, -) = \begin{cases} \lim\limits_{x \to a^-} f_{n+1}(a) & : \ f_{n+1} \text{ defined on } (t, a) \text{ for some } t < a. \\ \infty & : \ \text{else} \end{cases}$$

$$\lambda(a, 0) = \begin{cases} f_{n+1}(a) & x \in \mathrm{dom}\,(f_{n+1}) \\ \infty & : \ \text{else} \end{cases}$$

$$\lambda(a, +) = \begin{cases} \lim\limits_{x \to a^+} f_{n+1}(a) & : \ f_{n+1} \text{ defined on } (a, t) \text{ for some } a < t. \\ \infty & : \ \text{else} \end{cases}$$

Take $\beta(a) = \min\{\lambda(a, -),\ \lambda(a, 0),\ \lambda(a, +)\}$. It is not difficult to see then that $\beta(a)$ is simply the least $b \in M \cup \{\pm\infty\}$ such that $(a, b)$ is not normal. Were we instead to take some $a \in G$, then $(a, b)$ must *always* be normal for any $b \in M \cup \{\pm\infty\}$. So, $B$ can be given as

$$B = \{\, a \in M \mid \exists b \in M \cup \{\pm\infty\} \text{ s.t. } (a, b) \text{ is not normal} \,\},$$

and as such, is definable.

If we take some $a \in G$, then $|A_x|$ is constant on an open interval about $a$ by definition of $G$. By showing that $B$ is finite, we get our desired result. Supposing $B$ to be *infinite*, we can partition $B$ into

$$\begin{aligned} B_+ &= \{\, a \in B \mid \exists y \text{ s.t. } y > \beta(a),\ (a, y) \in A \,\} \\ B_- &= \{\, a \in B \mid \exists y \text{ s.t. } y < \beta(a),\ (a, y) \in A \,\}, \end{aligned}$$

both evidently definable sets. By the infinitude of $B$, so too must at least one of $B_-, B_+$ be infinite — and further, so must one of

- $B_+ \cap B_-$

- $B_+ \setminus B_-$
- $B_- \setminus B_+$
- $B \setminus (B_+ \cup B_-)$.

We can then apply MT (Theorem 2.1) to each case to reach a contradiction by showing that assuming non-finiteness, we *should* be able to find a normal point with first coordinate $a$ – contradicting the 'badness' of any point in $B$. Thus, $B$ is *finite*, and so there must be some finite upper bound on the cardinality of all fibres, $A_x$, and our proof is complete. □

## 2.2 Cell Decompositions

We start with a few definitions, that should hopefully feel motivated in anticipation of the higher-dimensional analogues of what we have seen already.

**Definition 2.1 (Cells in $M^n$)** For a sequence $(i_1, \dots, i_n)$ for each $i_j \in \{0, 1\}$, we define $(i_1, \dots, i_n)$*-cells* of $M^n$ inductively as follows:

1. A 0-cell is a point in $M$, and a 1-cell an open interval (both in $M^1$).
2. Supposing $(i_1, \dots, i_n)$-cells are defined for $M^n$,

   a. we define an $(i_1, \dots, i_n, 0)$-cell to be a definable set given by graph $(f)$ for $f$ a continuous, definable function on an $(i_1, \dots, i_n)$-cell.
   b. Perhaps predictably then, we define an $(i_1, \dots, i_n, 1)$-cell to be a definable set of the form $(f, g)_C = \{ (x, y) \in C \times M \mid f(x) < y < g(x) \}$ for $f, g$ continuous, definable functions on an $(i_1, \dots, i_n)$-cell, with $C \subset M^n$. Note that we may also allow $f \equiv -\infty$ or $g \equiv \infty$.

As usual, we denote a projection map by $\pi$, and for any $(i_1, \dots, i_n)$-cell we can define the projection

$$\pi \colon M^n \to M^k$$

for $k$ the sum of $i_1, \dots, i_n$, such that the restriction of $\pi$ to our $(i_1, \dots, i_n)$-cell is a homeomorphism onto its image.

It is not hard to see that what we are doing here is just projecting away from the coordinate 0 parts of the cell. This can be thought of as a canonical coordinate projection that any cell comes naturally equipped with – which is quite a fine thing to have.

In what should hopefully be predictable at this point, we wish now to define what it means to *decompose* our space into cells. At some point, we will cease prefacing these definitions with 'as usual, we do so by induction' – but that point is yet to come. So, we proceed, as usual, by defining cell-decompositions by induction.

**Definition 2.2 (Cell Decomposition *of* $M$)** A *cell-decomposition* of $M$ is a finite set defined by some strictly increasing finite sequence $a_1, \dots, a_k$ that form the set

$$\{(-\infty, a_1), (a_1, a_2), \ldots, (a_k, \infty), \{a_1\}, \{a_2\}, \ldots \{a_k\}\}.$$

That is, all the sequential open intervals (including those with infinite endpoints) plus the singleton sets. Just for the sake of belabouring the point, this is a definitionally *definable* set.

As we have time and time before, we now up the dimension by induction to define general cell-decompositions:

**Definition 2.3 (Cell Decomposition *of* $M^{n+1}$)** A cell-decomposition of $M^{n+1}$ is a finite partition, $\mathfrak{D}$, of $M^{n+1}$ into cells, such that

$$\{\, \pi(C) \mid C \in \mathfrak{D} \,\}$$

is itself a decomposition of $M^n$, with each $\pi$ the respective projection as discussed above.

We trust the conjunction of these two definitions into an understanding of cell-decompositions in arbitrary dimensions is clear by induction. An idea that may seem a bit apropos until a bit later on is that of *compatibility* – but rest assured, dear reader, that this will all come together shortly.

**Definition 2.4 (Compatibility)** We call a cell-decomposition $\mathfrak{D}$ of $M^n$ *compatible* with a subset, $X \subseteq M^n$ if for each cell, $C \in \mathfrak{D}$, either $C \cap X$ is empty, or $C$ is a subset of $X$.

## 2.3 The Cell-Decomposition Theorem

The importance of this last point will be made clear in the theorem we have building up to: the Cell-Decomposition Theorem – which essentially says that these compatible decompositions actually exist, and are compatible with any finite collection of definable sets, and most importantly, that definable functions are continuous on each cell in such a decomposition (defined in the domain of the function, of course). Properly, and as proved by Knight, Pillay, and Steinhorn [3]:

**Theorem 2.3 (Cell-Decomposition)**

*Take $n \in \mathbb{Z}_{\geq 1}$.*

*Note the use (about to be made) of subscripts on statements $(I)_n$ and $(II)_n$ to denote the dimension of $M^n$ to which each statement refers. This is going to be notationally useful in the proof, but may seem a bit queer at present, and without introduction.*

*Then*

$(I)_n$ *Suppose $X_1, \ldots, X_k \subseteq M^n$ are definable sets. Then there is a cell-decomposition of $M^n$ compatible with each $X_j$.*

$(II)_n$ *If $f : X \to M$ is definable, then there is a cell-decomposition, $\mathfrak{D}$ of $M^n$ compatible with $X$ s.t. the restriction $f|_C$ is continuous for each $C \in \mathfrak{D}$.*

*Further, and in analogy to the Monotonicity Theorem, if our $X_1, \ldots, X_k$ or $f$ (depending on case $(I)$ or $(II)$) are definable over $A \subset M$, then we can take the cells in $\mathfrak{D}$ to be similarly definable over $A$; that is, with the* same *parameters.*

*This last point is perhaps a bit unfair to mention, as we will not be providing a proof for it – though for the sake of interest, it would feel incomplete to not at least analogize with Theorem 2.1. In truth, what follows is not a full proof of Cell-Decomposition, but a special case where we take $M$ to be $\mathbb{R}$, and use the yet unproven (or even stated) result of* uniform finiteness*. We take this result entirely for granted in the lectures, due to the oddity that a complete (unassuming) proof somewhat 'bootstraps' uniform finiteness into the induction we do on $(I)_j$ and $(II)_j$, proving it as we go along. This is because uniform finiteness is actually itself an immediate consequence of the Cell-Decomposition Theorem (which makes the proof a fun little oddity). For our purposes, we take it as assumedly true – in part due to the length of this proof even* with *that assumption – and trust that our dear intelligent reader sees plainly how we could fix this in absence of the assumption.*

Uniform finiteness is a generalization of the finiteness theorem we proved earlier (Theorem 2.2), but with potentially many parameters and in higher dimension. As with the argument for the Cell-Decomposition Theorem, we will similarly restrict our attention to the case where $M = \mathbb{R}$. This special-case is as follows:

**Proposition 2.1 (Uniform Finiteness (for $\mathbb{R}$))**

*Suppose $X \subset \mathbb{R}^{n+1}$ is definable with each fibre $X_x$ finite for $x \in \mathbb{R}^n$. Then there is some $N \in \mathbb{Z}_{\geq 1}$ such that $|X_x| \leq N$ for all $x \in \mathbb{R}^n$.*

The following proof is due to van den Dries [1] which, for no reason other than interest's sake, we mention went on to inspire the later work of Pillay and Steinhorn in [5].

***Proof (Cell-Decomposition (Theorem 2.3))*** We proceed by induction on parameter $n$. The base cases are both already done for us; $(I)_1$ is immediate from the definition of o-minimality, and $(II)_1$ is given by the Monotonicity Theorem. What we go on to show is two inductive facts that 'bounce off' one another in a sense, to allow us to prove both $(I)_n$ and $(II)_n$ for all $n$. These are

(a) Given $(I)_1, \ldots, (I)_n$ and $(II)_1, \ldots, (II)_{n-1}$, we can conclude $(II)_n$; and
(b) Given $(I)_1, \ldots, (I)_n$ and $(II)_1, \ldots, (II)_n$, we can conclude $(I)_{n+1}$.

That these two facts together give us the desired result should be clear. Getting there requires a bit more effort, and so we simply begin with (a). Thus we wish to prove $(II)_n$: that for a definable $f : X \longrightarrow M$, there is a cell-decomposition $\mathfrak{D}$ of $M^n$ compatible with $X$ and having continuity of $f|_C$ for each cell, $C \in \mathfrak{D}$.

Suppose $f : X \longrightarrow M$ is such a definable function. We are assuming (because we have already) $(I)_1$. By this, we may assume $X$ is a cell. If $X$ is not already an open cell, then recall that we can simply take its image under the canonical projection away from zero coordinates. Since we make no reference here to the dimension of $X$, we simply assume that it is open, or has been made so as described, and then use our inductive hypothesis to reach the conclusion. So, we suppose $X \in \mathfrak{D}$ is an open cell on which $f$ is continuous. Take

$$X' = \{\, x \in X \mid f \text{ is continuous and definable at } x \,\}.$$

Clearly, $X'$ is definable, and we are supposing that we know $X'$ to be open in $X$. Using inductive assumption $(I)_n$, we get a cell-decomposition, $\mathfrak{D}$ with $\mathbb{R}^n$ compatible with $X \setminus X'$ and with $X'$. If some $C \in \mathfrak{D}$ is an open cell contained in $X$, we get continuity of $f$ on $C$ by density; that is, $C \cap X' \neq \emptyset$, and so $C \subseteq X'$ and it follows that $f|_C$ is continuous. Supposing however that $C$ was *not* an open cell, we apply the aforementioned projection construction, and the argument just presented holds (up to a change in dimension).

This would be all well and good to end off (a) with, were it not predicated on the yet unjustified density of $X'$ in $X$ – and so we now prove this. Suppose $B \subseteq X$ is an open box. We will show that there must exist a point in $B$ at which $f$ is continuous. In analogy to our proof of monotonicity, we know that if $B'$ is an open box contained inside of $B$, then $f$ takes on infinitely-many values on $B'$ (following from $(I)_n$). This is the obvious case. Supposing otherwise, we proceed as follows:

Construct a sequence of open boxes, $(B_j)_{1\leq j\leq n}$ in $B$, and sequence $(I_j)_{1\leq j\leq n}$, of open intervals, each $I_j$ having length less than $\frac{1}{j}$, with the closure, $\mathrm{cl}\,(B_{n+1}) \subseteq B$, and $f(B_n) \subseteq I_n$. Then, by compactness. we get that the intersection of all $B_n$ is non-empty, and at some point in this intersection, $f$ is continuous. This is of course just our claim – we now go on to *prove* this by construction.

To get $I_1$, simply consider $f(B) \subseteq \mathbb{R}$ – meaning

$$f(B) = \bigcup_{p\in\mathbb{Z}_{\geq 1}} J_p \cup F$$

for $F$ a finite set, and $J_p$ a countable set of open intervals of length less than 1. Then, $B$ is given by

$$B = \left(\bigcup_{p\in\mathbb{Z}_{\geq 1}} f^{-1}(J_p) \cap B\right) \cup \left(\bigcup_{r\in F} f^{-1}(r) \cap B\right).$$

to each half of the middle cup, we can apply $(I)_n$ to determine the contents of each of the respective *big* cups toe be a finite union of cells – and so $B$ must be an *countable* union of cells, each of which is contained in one of these sets. Perhaps coming a bit out of left field, we apply the Baire Category Theorem to conclude that by openness of $B$, so too must be one of these cells be open.

Notice that this is one reason we restrict ourselves to working over $\mathbb{R}$ – the Baire Category Theorem simply does not hold in any DLO model, and so this argument could not be broadened beyond the reals (or compact spaces) as we are currently undertaking it.

This *cannot* be one of $f^{-1}(r) \cap B$, as it would then contain a box on which we took the value of $r$, an so this open cell must be in one of $f^{-1}(J_p) \cap B$ for some $p$. Taking $J_1$ to be that $J_p$, and $B_1$ to be an open box contained in $f^{-1}(J_1) \cap B$, with $\mathrm{cl}\,(B_1) \subseteq B$. As desired, we then have $f(B_1) \subset I_1$. Clearly the first step in an induction, we then (incompletely) note that, having $I_1, \ldots, I_n$, $B_1, \ldots, B_n$ constructed, we repeat exactly as above to finish the induction.

And with that we can give ourselves a *light* patting on the back – for as much as we've done so far, this is just the end of the proof of (a). To get the 'bounced-back' half of the induction, we now go on to prove (b); that is, given $(I)_1, \ldots, (I)_n$, $(II)_1, \ldots, (II)_n$, we may derive $(II)_{n+1}$.

For reasons of breaking up this lengthy proof into its two constituent sections, please enjoy the following horizontal line:

---

Try not to have too much fun with that, now. We move on to proving (b); recall our assumptions that $(I)_1, \ldots, (I)_n$ and $(II)_1, \ldots, (II)_n$ hold. We want now to prove $(II)_{n+1}$. First, we start with a small proposition.

**Proposition 2.2** *Suppose $\mathfrak{D}_1$, $\mathfrak{D}_2$ are cell-decompositions of $\mathbb{R}^{n+1}$ with a common refinement – that is, another cell-decomposition, $\mathfrak{D}$ of $R^{n+1}$ compatible with all cells in each of $\mathfrak{D}_1$ and $\mathfrak{D}_2$. Terminology-wise, we say that $\mathfrak{D}$* refines *$\mathfrak{D}_1$ and $\mathfrak{D}_2$, or that is is an* refinement *of $\mathfrak{D}_1$ and $\mathfrak{D}_2$.* □

A frustrated author's aside: please take no notice of the □ that sits just above this box and proceeds Proposition 2.2. Its presence is a mystery, and the method to remove it proves elusive, even after what would be not ungenerously called a *cursory* amount of investigation. We will simply pretend it does not exist, and trust that the honest, caring reader does so as well.

***Proof (Cell Refinement (Proposition 2.2))*** For the purpose of transparency, we not that this proof was left out of the lecture, and as an exercise to the interested (or obligated) viewer. The following takes inspiration from van den Dries [2]. We note that this could be made a bit cuter if we had the machinery of *dimension* that we will soon define, but in either case, this proof is relatively trivial. We have our $\mathfrak{D}_1$

and $\mathfrak{D}_2$ two decompositions of the trivially definable subset of, $\mathbb{R}^{n+1}$: $\mathbb{R}^{n+1}$ itself. We can then simply take a decomposition of the ambient space (which here is the *whole* space) containing our definable subset, and we have seen previously that we can take this decomposition to partition each cell of $\mathfrak{D}_1 \cup \mathfrak{D}_2$ – and the 'restriction' of this decomposition to our definable set (again, just to ensure this is sufficiently belaboured, this is not actually a restriction since our definable set is $\mathbb{R}^{n+1}$), we are left with our everywhere (on cells in $\mathfrak{D}_1 \cup \mathfrak{D}_2$) compatible decomposition. $\square$

A much less frustrated author's aside: let us all take a moment and appreciate the appropriately placed $\square$ above. We can move forth pretending all is well again, and we hope this has not caused the reader *too* much undue stress – beyond of course the normal, cursory amount.

Now, if some $A \subseteq \mathbb{R}$ is definable, we define its type, $\tau(A)$ as follows:

Let $a_1, \ldots, a_L$ strictly increasing be the points in the boundary of $A$. We let $\tau(A)$ then act as an indicator function on sequential intervals, $(a_j, a_{j+1})$, defined as the positive unit (1) when that interval sits inside $A$, and otherwise the negative unit $(-1)$. We set $a_0 = -\infty$ and $a_{L+1} = \infty$ (which is starting to become sort of an out-of-bounds norm), and define

$$\tau(A)(2j+1) = 1$$

if $(a_j, a_{j+1}) \subseteq A$ (and of course $-1$ otherwise). Note, of course, that this would then mean that the given interval is contained in the complement of $A$. For even numbers, we have

$$\tau(A)(2j) = 1$$

if $a_j \in A$ and naturally, $-1$ if $a_j \notin A$.

# References


1. Dries, L.P.D.v.d.: Remarks on Tarski's problem concerning (R, +, *, exp). In: G. Lolli, G. Longo, A. Marcja (eds.) Studies in Logic and the Foundations of Mathematics, *Logic Colloquium '82*, vol. 112, pp. 97–121. Elsevier (1984). DOI 10.1016/S0049-237X(08)71811-1. URL https://www.sciencedirect.com/science/article/pii/S0049237X08718111
2. Dries, L.P.D.v.d.: Tame topology and O-minimal structures. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge (1998). DOI 10.1017/CBO9780511525919. URL https://www.cambridge.org/core/books/tame-topology-and-ominimal-structures/7AC940248AF2B05DA4D33E4FB05C97A2
3. Knight, J.F., Pillay, A., Steinhorn, C.: Definable sets in ordered structures. II. Transactions of the American Mathematical Society **295**(2), 593–605 (1986). DOI 10.2307/2000053. URL https://www.jstor.org/stable/2000053. Publisher: American Mathematical Society
4. Marker, D.: Model theory, *Graduate Texts in Mathematics*, vol. 217. Springer-Verlag, New York (2002). DOI 10.1007/b98860. URL http://link.springer.com/10.1007/b98860
5. Pillay, A., Steinhorn, C.: Definable sets in ordered structures. I. Transactions of the American Mathematical Society **295**(2), 565–592 (1986). DOI 10.2307/2000052. URL https://www.jstor.org/stable/2000052. Publisher: American Mathematical Society