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Abstract

The abstract paragraph should be indented ½ inch (3 picas) on both the left- and right-hand margins. Use 10 point type, with a vertical spacing (leading) of 11 points. The word **Abstract** must be centered, bold, and in point size 12. Two line spaces precede the abstract. The abstract must be limited to one paragraph.

1 Introduction

2 Preliminaries

2.1 Markov Decision Processes

We define a Markov decision process (MDP) as a tuple $\mathcal{M} = \langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, p_0, \gamma \rangle$, where $\mathcal{S}$ is the state-space, $\mathcal{A}$ is a finite set of actions, $\mathcal{P}(\cdot|s,a)$ is the distribution of the next state s' given that action a is taken in state s, $\mathcal{R} : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ is the reward function, p_0 is the initial-state distribution, and $\gamma \in [0,1)$ is the discount factor. We assume the reward function to be uniformly bounded by a constant $R_{max} > 0$. A deterministic policy $\pi : \mathcal{S} \to \mathcal{A}$ is a mapping from states to actions. At the beginning of each episode of interaction, the initial state s_0 is drawn from p_0. Then, the agent takes the action $a_0 = \pi(s_0)$, receives a reward $\mathcal{R}(s_0, a_0)$, transitions to the next state $s_1 \sim \mathcal{P}(\cdot|s_0, a_0)$, and the process is repeated. The goal is to find the policy maximizing the long-term return over a possibly infinite horizon: $\max_\pi J(\pi) \triangleq \mathbb{E}[\sum_{t=0}^{\infty} \gamma^t r_t \mid \mathcal{M}, \pi]$. To this end, we define the optimal value function $Q^*(s,a)$ as the expected return obtained by taking action a in state s and following an optimal policy thereafter. Then, an optimal policy π^* is a policy that is greedy with respect to the optimal value function, i.e., $\pi^*(s) = \operatorname{argmax}_a Q^*(s,a)$ for all states s. It can be shown (e.g., [1]) that Q^* is the unique fixed-point of the optimal Bellman operator T defined by $TQ(s,a) = \mathcal{R}(s,a) + \gamma \mathbb{E}_{\mathcal{P}}[\max_{a'} Q(s', a')]$ for any value function Q. From now on, we adopt the term Q-function to denote any plausible value function, i.e., any function $Q : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ uniformly bounded by $\frac{R_{max}}{1-\gamma}$.

When learning the optimal value function, a quantity of interest is how close a given Q-function is to the fixed-point of the Bellman operator. This is given by its Bellman residual, defined by $B(Q) \triangleq TQ - Q$. Notice that Q is optimal if, and only if, $B(Q)(s,a) = 0$ for all s, a. Furthermore, if we assume the existence of a distribution ν over $\mathcal{S} \times \mathcal{A}$, the squared Bellman error of Q is defined as the expected squared Bellman residual of Q under ν, $\|B(Q)\|_\nu^2 = \mathbb{E}_\mu \left[B^2(Q)\right]$. Although minimizing the empirical Bellman error is an appealing objective, it is well-known that an unbiased estimator requires two independent samples of the next state s' of each s, a (e.g., []). In practice, cite Maillard the empirical Bellman error is typically replaced by the TD error, which approximates the former using a single transition sample. Given a dataset of N samples, the TD error is computed as $\|B(Q)\|_D^2 = \frac{1}{N} \sum_{i=1}^{N} (r_i + \gamma \max_{a'} Q(s_i', a') - Q(s_i, a_i))^2$.

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2.2 Variational Inference

When working with Bayesian approaches, the posterior distribution of hidden variables $\boldsymbol{w} \in \mathbb{R}^K$ given data D,

$$p(\boldsymbol{w}|D) = \frac{p(D|\boldsymbol{w})p(\boldsymbol{w})}{p(D)} = \frac{p(D|\boldsymbol{w})p(\boldsymbol{w})}{\int_{\boldsymbol{w}} p(D|\boldsymbol{w})p(\boldsymbol{w})}, \tag{1}$$

is typically intractable for many models of interest (e.g., when working with deep neural networks) due to difficulties in computing the integral of Eq. (1). The main intuition behind variational inference [] is to approximate the intractable posterior $p(\boldsymbol{w}|D)$ with a simpler distribution $q_{\boldsymbol{\xi}}(\boldsymbol{w})$. The latter is chosen in a parametric family, with variational parameters $\boldsymbol{\xi}$, as the minimizer of the Kullback-Leibler (KL) divergence w.r.t. p:

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$$\min_{\boldsymbol{\xi}} KL\left(q_{\boldsymbol{\xi}}(\boldsymbol{w}) \mid\mid p(\boldsymbol{w} \mid D)\right) \tag{2}$$

It is well-known that minimizing the KL divergence is equivalent to maximizing the so-called *evidence lower bound* (ELBO), which is defined as:

$$\mathrm{ELBO}(\boldsymbol{\xi}) = \mathbb{E}_{\boldsymbol{w} \sim q_{\boldsymbol{\xi}}}\left[\log p(D|\boldsymbol{w})\right] - KL\left(q_{\boldsymbol{\xi}}(\boldsymbol{w}) \mid\mid p(\boldsymbol{w})\right) \tag{3}$$

Intuitively, the best approximation is the one that maximizes the expected log-likelihood of the data, while minimizing the KL divergenge w.r.t. the prior $p(\boldsymbol{w})$.

3 Variational Transfer Learning

3.1 Algorithm

3.2 Gaussian Variational Transfer

3.3 Mixture of Gaussian Variational Transfer

4 Theoretical Analysis

In this section, we theoretically analyze our variational transfer algorithm...

A first important question that we need to answer is whether replacing max with mellow-max in the Bellman operator constitutes a strong approximation or not. It has been proved [] that the mellow Bellman operator is a contraction under the L_∞-norm and, thus, has a unique fixed-point. However, how such fixed-point differs from the one of the optimal Bellman operator remains an open question. Since mellow-max monotonically converges to max as $\kappa \to \infty$, it would be desirable if the corresponding operator also monotonically converged to the optimal one. We confirm that this property actually holds in the following theorem.

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Theorem 1. *Let V be the fixed-point of the optimal Bellman operator T, and Q the corresponding action-value function. Define the action-gap function $g(s)$ as the difference between the value of the best action and the second best action at each state s. Let $\widetilde{V}$ be the fixed-point of the mellow Bellman operator $\widetilde{T}$ with parameter $\kappa > 0$ and denote by $\beta > 0$ the inverse temperature of the induced Boltzmann distribution (as in []). Let ν be a probability measure over the state-space. Then, for any $p \geq 1$:*

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$$\left\| V - \widetilde{V} \right\|_{\nu,p}^{p} \leq \frac{2R_{max}}{(1-\gamma)^2} \left\| 1 - \frac{1}{1 + |\mathcal{A}|\, e^{-\beta g}} \right\|_{\nu,p}^{p} \tag{4}$$

5 Related Works

6 Experiments

6.1 Gridworld

6.2 Classic Control

6.3 Maze Navigation

7 Conclusion

References

[1] Martin L. Puterman. *Markov Decision Processes: Discrete Stochastic Dynamic Programming*. John Wiley & Sons, Inc., New York, NY, USA, 1994.

A Proofs

Theorem 1. *Let V be the fixed-point of the optimal Bellman operator T, and Q the corresponding action-value function. Define the action-gap function $g(s)$ as the difference between the value of the best action and the second best action at each state s. Let $\widetilde{V}$ be the fixed-point of the mellow Bellman operator $\widetilde{T}$ with parameter $\kappa > 0$ and denote by $\beta > 0$ the inverse temperature of the induced Boltzmann distribution (as in []). Let ν be a probability measure over the state-space. Then, for any $p \geq 1$:* Cite MM

$$\left\| V - \widetilde{V} \right\|_{\nu,p}^p \leq \frac{2R_{max}}{(1-\gamma)^2} \left\| 1 - \frac{1}{1 + |\mathcal{A}|\, e^{-\beta g}} \right\|_{\nu,p}^p \tag{4}$$

Proof. We begin by noticing that:

$$\begin{aligned}
\left\| V - \widetilde{V} \right\|_{\nu,p}^p &= \left\| TV - \widetilde{T}\widetilde{V} \right\|_{\nu,p}^p \\
&= \left\| TV - \widetilde{T}V + \widetilde{T}V - \widetilde{T}\widetilde{V} \right\|_{\nu,p}^p \\
&\leq \left\| TV - \widetilde{T}V \right\|_{\nu,p}^p + \left\| \widetilde{T}V - \widetilde{T}\widetilde{V} \right\|_{\nu,p}^p \\
&\leq \left\| TV - \widetilde{T}V \right\|_{\nu,p}^p + \gamma \left\| V - \widetilde{V} \right\|_{\nu,p}^p
\end{aligned}$$

where the first inequality follows from Minkowsky's inequality and the second one from the contraction property of the mellow Bellman operator. This implies that:

$$\left\| V - \widetilde{V} \right\|_{\nu,p}^p \leq \frac{1}{1-\gamma} \left\| TV - \widetilde{T}V \right\|_{\nu,p}^p \tag{5}$$

Let us bound the norm on the right-hand side separately. In order to do that, we will bound the function $\left| TV(s) - \widetilde{T}V(s) \right|$ point-wisely for any state s. By applying the definition of the optimal and mellow Bellman operators, we obtain:

$$\begin{aligned}
\left| TV(s) - \widetilde{T}V(s) \right| &= \left| \max_a \{R(s,a) + \gamma \mathbb{E}\left[V(s')\right]\} - \operatorname*{mm}_a \{R(s,a) + \gamma \mathbb{E}\left[V(s')\right]\} \right| \\
&= \left| \max_a Q(s,a) - \operatorname*{mm}_a Q(s,a) \right|
\end{aligned}$$

Recall that applying the mellow-max is equivalent to computing an expectation under a Boltzmann distribution with inverse temperature β induced by κ []. Thus, we can write: Cite MM

$$\begin{aligned}
\left| \max_a Q(s,a) - \operatorname*{mm}_a Q(s,a) \right| &= \left| \sum_a \pi^*(a|s) Q(s,a) - \sum_a \pi_\beta(a|s) Q(s,a) \right| \\
&= \left| \sum_a Q(s,a) \left(\pi^*(a|s) - \pi_\beta(a|s) \right) \right| \\
&\leq \sum_a |Q(s,a)| \left| \pi^*(a|s) - \pi_\beta(a|s) \right| \\
&\leq \frac{R_{max}}{1-\gamma} \sum_a \left| \pi^*(a|s) - \pi_\beta(a|s) \right|
\end{aligned} \tag{6}$$

where π^* is the optimal (deterministic) policy w.r.t. Q and π_β is the Boltzmann distribution induced by Q with inverse temperature β:

$$\pi_\beta(a|s) = \frac{e^{\beta Q(s,a)}}{\sum_{a'} e^{\beta Q(s,a')}}$$

Denote by $a_1(s)$ the optimal action for state s under Q. We can then write:

$$\begin{aligned}\sum_a |\pi^*(a|s) - \pi_\beta(a|s)| &= |\pi^*(a_1(s)|s) - \pi_\beta(a_1(s)|s)| + \sum_{a \neq a_1(s)} |\pi^*(a|s) - \pi_\beta(a|s)| \\ &= |1 - \pi_\beta(a_1(s)|s)| + \sum_{a \neq a_1(s)} |\pi_\beta(a|s)| \\ &= 2\,|1 - \pi_\beta(a_1(s)|s)| \end{aligned} \tag{7}$$

Finally, let us bound this last term:

$$\begin{aligned}|1 - \pi_\beta(a_1(s)|s)| &= \left|1 - \frac{e^{\beta Q(s,a_1(s))}}{\sum_{a'} e^{\beta Q(s,a')}}\right| \\ &= \left|1 - \frac{e^{\beta(Q(s,a_1(s)) - Q(s,a_2(s)))}}{\sum_{a'} e^{\beta(Q(s,a') - Q(s,a_2(s)))}}\right| \\ &= \left|1 - \frac{e^{\beta g(s)}}{\sum_{a'} e^{\beta(Q(s,a') - Q(s,a_2(s)))}}\right| \\ &= \left|1 - \frac{e^{\beta g(s)}}{e^{\beta g(s)} + \sum_{a' \neq a_1(s)} e^{\beta(Q(s,a') - Q(s,a_2(s)))}}\right| \\ &\leq \left|1 - \frac{e^{\beta g(s)}}{e^{\beta g(s)} + |\mathcal{A}|}\right| \\ &= \left|1 - \frac{1}{1 + |\mathcal{A}|\, e^{-\beta g(s)}}\right| \end{aligned} \tag{8}$$

Combining Eq. (6), (7), and (8), we obtain:

$$\left|\max_a Q(s,a) - \operatorname*{mm}_a Q(s,a)\right| \leq \frac{2R_{max}}{1-\gamma} \left|1 - \frac{1}{1 + |\mathcal{A}|\, e^{-\beta g(s)}}\right|$$

Taking the norm and plugging this into Eq. (5) concludes the proof. $\square$

Lemma 1. *Let p and ν denote probability measures over Q-functions and state-action pairs, respectively. Assume Q^* is the unique fixed-point of the optimal Bellman operator T. Then, for any $\delta > 0$, with probability at least $1-\delta$ over the choice of a Q-function Q, the following holds:*

$$\|Q - Q^*\|_\nu^2 \leq \frac{\mathbb{E}_p\left[\|B(Q)\|_\nu^2\right]}{(1-\gamma)\delta} \tag{9}$$

Proof. First notice that:

$$\begin{aligned}\|Q - Q^*\| &= \|Q + TQ - TQ - TQ^*\| \\ &\leq \|Q - TQ\| + \|TQ - TQ^*\| \\ &\leq \|Q - TQ\| + \gamma\,\|Q - Q^*\| \\ &= \|B(Q)\| + \gamma\,\|Q - Q^*\|\end{aligned}$$

which implies that:

$$\|Q - Q^*\| \leq \frac{1}{1-\gamma}\,\|B(Q)\|$$

Then we can write:

$$P\left(\|Q - Q^*\| > \epsilon\right) \leq P\left(\|B(Q)\| > \epsilon(1-\gamma)\right) \leq \frac{\mathbb{E}_p\left[\|B(Q)\|_\nu^2\right]}{(1-\gamma)\epsilon}$$

Settings the right-hand side equal to δ and solving for ϵ concludes the proof. $\square$

Corollary 1. *Let p and ν denote probability measures over Q-functions and state-action pairs, respectively. Assume $\widetilde{Q}$ is the unique fixed-point of the mellow Bellman operator $\widetilde{T}$. Then, for any $\delta > 0$, with probability at least $1-\delta$ over the choice of a Q-function Q, the following holds:*

$$\left\| Q - \widetilde{Q} \right\|_{\nu}^{2} \leq \frac{\mathbb{E}_p \left[\left\| \widetilde{B}(Q) \right\|_{\nu}^{2} \right]}{(1-\gamma)\delta} \tag{10}$$

Lemma 2. *Assume Q-functions belong to a parametric space of functions bounded by $\frac{R_{max}}{1-\gamma}$. Let p and q be arbitrary distributions over the parameter space $\mathcal{W}$, and ν be a probability measure over $\mathcal{S} \times \mathcal{A}$. Consider a dataset D of N samples and define $v(\boldsymbol{w}) \triangleq \mathbb{E}_{\nu}\left[Var_{\mathcal{P}}\left[b(\boldsymbol{w})\right]\right]$. Then, for any $\delta > 0$, with probability at least $1-\delta$, the following two inequalities hold simultaneously:*

$$\mathbb{E}_q \left[\|B(\boldsymbol{w})\|_{\nu}^{2} \right] \leq \mathbb{E}_q \left[\|B(\boldsymbol{w})\|_{D}^{2} \right] - \mathbb{E}_q \left[v(\boldsymbol{w}) \right] + \frac{\lambda}{N} KL(q||p) + 4 \frac{R_{max}^2}{(1-\gamma)^2} \sqrt{\frac{\log \frac{2}{\delta}}{2N}} \tag{11}$$

$$\mathbb{E}_q \left[\|B(\boldsymbol{w})\|_{D}^{2} \right] \leq \mathbb{E}_q \left[\|B(\boldsymbol{w})\|_{\nu}^{2} \right] + \mathbb{E}_q \left[v(\boldsymbol{w}) \right] + \frac{\lambda}{N} KL(q||p) + 4 \frac{R_{max}^2}{(1-\gamma)^2} \sqrt{\frac{\log \frac{2}{\delta}}{2N}} \tag{12}$$

Proof. From Hoeffding's inequality we have:

$$P\left(\left| \mathbb{E}_{\nu,\mathcal{P}} \left[\|B(\boldsymbol{w})\|_{D}^{2} \right] - \|B(\boldsymbol{w})\|_{D}^{2} \right| > \epsilon \right) \leq 2exp\left(- \frac{2N\epsilon^2}{\left(2\frac{R_{max}}{1-\gamma}\right)^4} \right)$$

which implies that, for any $\delta > 0$, with probability at least $1-\delta$:

$$\left| \mathbb{E}_{\nu,\mathcal{P}} \left[\|B(\boldsymbol{w})\|_{D}^{2} \right] - \|B(\boldsymbol{w})\|_{D}^{2} \right| \leq 4 \frac{R_{max}^2}{(1-\gamma)^2} \sqrt{\frac{\log \frac{2}{\delta}}{2N}}$$

Under independence assumptions, the expected TD error can be re-written as:

$$\begin{aligned}
\mathbb{E}_{\nu,\mathcal{P}} \left[\|B(\boldsymbol{w})\|_{D}^{2} \right] &= \mathbb{E}_{\nu,\mathcal{P}} \left[\frac{1}{N} \sum_{i=1}^{N} (r_i + \gamma \operatorname*{mm}_{a'} Q_{\boldsymbol{w}}(s_i', a') - Q_{\boldsymbol{w}}(s_i, a_i))^2 \right] \\
&= \mathbb{E}_{\nu,\mathcal{P}} \left[(R(s,a) + \gamma \operatorname*{mm}_{a'} Q_{\boldsymbol{w}}(s', a') - Q_{\boldsymbol{w}}(s, a))^2 \right] \\
&= \mathbb{E}_{\nu} \left[\mathbb{E}_{\mathcal{P}} \left[b(\boldsymbol{w})^2 \right] \right] \\
&= \mathbb{E}_{\nu} \left[Var_{\mathcal{P}} \left[b(\boldsymbol{w}) \right] + \mathbb{E}_{\mathcal{P}} \left[b(\boldsymbol{w}) \right]^2 \right] \\
&= v(\boldsymbol{w}) + \|B(\boldsymbol{w})\|_{\nu}^{2}
\end{aligned}$$

where $v(\boldsymbol{w}) \triangleq \mathbb{E}_{\nu}\left[Var_{\mathcal{P}}\left[b(\boldsymbol{w})\right]\right]$. Thus:

$$\left| \|B(\boldsymbol{w})\|_{\nu}^{2} + v(\boldsymbol{w}) - \|B(\boldsymbol{w})\|_{D}^{2} \right| \leq 4 \frac{R_{max}^2}{(1-\gamma)^2} \sqrt{\frac{\log \frac{2}{\delta}}{2N}} \tag{13}$$

From the change of measure inequality [], we have that, for any measurable function $f(\boldsymbol{w})$ and any two probability measures p and q: Find a reference for this

$$\log \mathbb{E}_p \left[e^{f(\boldsymbol{w})} \right] \geq \mathbb{E}_q \left[f(\boldsymbol{w}) \right] - KL(q||p)$$

Thus, multiplying both sides of (13) by $\lambda^{-1}N$ and applying the change of measure inequality with $f(\boldsymbol{w}) = \lambda^{-1}N \left| \|B(\boldsymbol{w})\|_{\nu}^{2} + v(\boldsymbol{w}) - \|B(\boldsymbol{w})\|_{D}^{2} \right|$, we obtain:

$$\mathbb{E}_q \left[f(\boldsymbol{w}) \right] - KL(q||p) \leq \log \mathbb{E}_p \left[e^{f(\boldsymbol{w})} \right] \leq 4 \frac{R_{max}^2 \lambda^{-1} N}{(1-\gamma)^2} \sqrt{\frac{\log \frac{2}{\delta}}{2N}}$$

where the second inequality holds since the right-hand side of (13) does not depend on $\boldsymbol{w}$. Finally, we can explicitly write:

$$\mathbb{E}_q \left[\left| \|B(\boldsymbol{w})\|_\nu^2 + v(\boldsymbol{w}) - \|B(\boldsymbol{w})\|_D^2 \right| \right] \leq \frac{\lambda}{N} KL(q||p) + 4 \frac{R_{max}^2}{(1-\gamma)^2} \sqrt{\frac{\log \frac{2}{\delta}}{2N}}$$

from which the lemma follows straightforwardly. $\square$

Lemma 3. *Let p be a prior distribution over the parameter space $\mathcal{W}$, and ν be a probability measure over $\mathcal{S} \times \mathcal{A}$. Assume $\widehat{\xi}$ is the minimizer of $ELBO(\xi) = \mathbb{E}_{q_\xi} \left[\|B(\boldsymbol{w})\|_D^2 \right] + \frac{\lambda}{N} KL(q_\xi||p)$ for a dataset D of N samples. Define $v(\boldsymbol{w}) \triangleq \mathbb{E}_\nu \left[Var_{\mathcal{P}} \left[b(\boldsymbol{w}) \right] \right]$. Then, for any $\delta > 0$, with probability at least $1 - \delta$:*

$$\mathbb{E}_{q_{\widehat{\xi}}} \left[\|B(\boldsymbol{w})\|_\nu^2 \right] \leq \inf_{\xi \in \Xi} \left\{ \mathbb{E}_{q_\xi} \left[\|B(\boldsymbol{w})\|_\nu^2 \right] + \mathbb{E}_{q_\xi} \left[v(\boldsymbol{w}) \right] + 2 \frac{\lambda}{N} KL(q_\xi||p) \right\} + 2 \frac{R_{max}^2}{(1-\gamma)^2} \sqrt{\frac{\log \frac{2}{\delta}}{N}}$$

Proof. Let us use Lemma 2 for the specific choice $q = q_{\widehat{\xi}}$. From Eq. (11), we have:

$$\begin{aligned}
\mathbb{E}_{q_{\widehat{\xi}}} \left[\|B(\boldsymbol{w})\|_\nu^2 \right] &\leq \mathbb{E}_{q_{\widehat{\xi}}} \left[\|B(\boldsymbol{w})\|_D^2 \right] - \mathbb{E}_{q_{\widehat{\xi}}} \left[v(\boldsymbol{w}) \right] + \frac{\lambda}{N} KL(q_{\widehat{\xi}}||p) + 4 \frac{R_{max}^2}{(1-\gamma)^2} \sqrt{\frac{\log \frac{2}{\delta}}{2N}} \\
&\leq \mathbb{E}_{q_{\widehat{\xi}}} \left[\|B(\boldsymbol{w})\|_D^2 \right] + \frac{\lambda}{N} KL(q_{\widehat{\xi}}||p) + 4 \frac{R_{max}^2}{(1-\gamma)^2} \sqrt{\frac{\log \frac{2}{\delta}}{2N}} \\
&= \inf_{\xi \in \Xi} \left\{ \mathbb{E}_{q_\xi} \left[\|B(\boldsymbol{w})\|_D^2 \right] + \frac{\lambda}{N} KL(q_\xi||p) \right\} + 4 \frac{R_{max}^2}{(1-\gamma)^2} \sqrt{\frac{\log \frac{2}{\delta}}{2N}}
\end{aligned}$$

where the second inequality holds since $v(\boldsymbol{w}) > 0$, while the equality holds from the definition of $\widehat{\xi}$. We can now use Eq. (12) to bound $\mathbb{E}_{q_\xi} \left[\|B(\boldsymbol{w})\|_D^2 \right]$, thus obtaining:

$$\mathbb{E}_{q_{\widehat{\xi}}} \left[\|B(\boldsymbol{w})\|_\nu^2 \right] \leq \inf_{\xi \in \Xi} \left\{ \mathbb{E}_{q_\xi} \left[\|B(\boldsymbol{w})\|_\nu^2 \right] + \mathbb{E}_{q_\xi} \left[v(\boldsymbol{w}) \right] + 2 \frac{\lambda}{N} KL(q_\xi||p) \right\} + 2 \frac{R_{max}^2}{(1-\gamma)^2} \sqrt{\frac{\log \frac{2}{\delta}}{N}}$$

This concludes the proof. $\square$