Implicit leap frog approach with solver

Mit drei Zeitschritten

$$\left| \begin{array}{lcl} \frac{u_0^{n+1}-u_0^n}{dt} + u_0^n \frac{u_1^{n+1}-u_0^{n+1}}{dx} & = & 0 \\ \frac{u_1^{n+1}-u_1^n}{dt} + u_1^n \frac{u_2^{n+1}-u_0^{n+1}}{2\,dx} & = & 0 \\ \frac{u_2^{n+1}-u_2^n}{dt} + u_2^n \frac{u_2^{n+1}-u_1^{n+1}}{dx} & = & 0 \end{array} \right| \tag{1}$$

$$\left[\begin{array}{lcl} u_0^{n+1} & = & \frac{2dxu_0^n\left(-dtu_1^n+dtu_2^n+dx\right)}{dt^2u_0^nu_1^n-2dt^2u_0^nu_2^n+dt^2u_1^nu_2^n-2dtdxu_0^n+2dtdxu_2^n+2dx^2} \\ u_1^{n+1} & = & \frac{dxu_1^n\left(-dtu_0^n+dtu_2^n+2dx\right)}{dt^2u_0^nu_1^n-2dt^2u_0^nu_2^n+dt^2u_1^nu_2^n-2dtdxu_0^n+2dtdxu_2^n+2dx^2} \\ u_2^{n+1} & = & \frac{2dxu_2^n\left(-dtu_0^n+dtu_1^n+dx\right)}{dt^2u_0^nu_1^n-2dt^2u_0^nu_2^n+dt^2u_1^nu_2^n-2dtdxu_0^n+2dtdxu_2^n+2dx^2} \end{array} \right] \tag{2}$$

$$\left[\begin{array}{lcl} u_0^{n+1} & = & \frac{2u_0^n\left(2dt^2u_1^nu_2^n+dtdxu_1^n+dtdxu_2^n+dx^2\right)}{dt^2u_0^nu_1^n-2dt^2u_0^nu_2^n+dt^2u_1^nu_2^n-2dtdxu_0^n+2dtdxu_2^n+2dx^2} \\ u_1^{n+1} & = & -\frac{u_1^n\left(4dt^2u_0^nu_2^n+3dtdxu_0^n-3dtdxu_2^n-2dx^2\right)}{dt^2u_0^nu_1^n-2dt^2u_0^nu_2^n+dt^2u_1^nu_2^n-2dtdxu_0^n+2dtdxu_2^n+2dx^2} \\ u_2^{n+1} & = & \frac{2u_2^n\left(2dt^2u_0^nu_1^n-dtdxu_0^n-dtdxu_1^n+dx^2\right)}{dt^2u_0^nu_1^n-2dt^2u_0^nu_2^n+dt^2u_1^nu_2^n-2dtdxu_0^n+2dtdxu_2^n+2dx^2} \end{array} \right] \tag{3}$$

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$$\left| \begin{array}{lcl} \frac{u_0^{n+1}-u_0^n}{dt} + u_0^n \frac{u_1^{n+1}-u_0^{n+1}}{dx} & = & 0 \\ \frac{u_1^{n+1}-u_1^n}{dt} + u_1^n \frac{u_2^{n+1}-u_0^{n+1}}{2\,dx} & = & 0 \\ \frac{u_2^{n+1}-u_2^n}{dt} + u_2^n \frac{u_3^{n+1}-u_1^{n+1}}{2\,dx} & = & 0 \\ \frac{u_3^{n+1}-u_3^n}{dt} + u_3^n \frac{u_3^{n+1}-u_2^{n+1}}{dx} & = & 0 \end{array} \right| \tag{4}$$

$$\left[\begin{array}{lcl} u_0^{n+1} & = & \frac{dxu_0^n\left(dt^3u_1^nu_2^nu_3^n-2dt^2dxu_2^nu_3^n-3dt^2u_1^nu_2^n\left(dtu_3^n+dx\right)+2dtu_1^n\left(dt^2u_2^nu_3^n+2dx\left(dtu_3^n+dx\right)\right)-4dx^2\left(dtu_3^n+dx\right)\right)}{-dt^4u_0^nu_1^nu_2^nu_3^n-2dt^2dxu_0^nu_1^n\left(dtu_3^n+dx\right)+2dt^2dxu_2^nu_3^n\left(dtu_0^n-dx\right)+dt^2u_1^nu_2^n\left(dtu_0^n-dx\right)\left(dtu_3^n+dx\right)+4dx^2\left(dtu_0^n-dx\right)\left(dtu_3^n+dx\right)} \\ u_1^{n+1} & = & \frac{dxu_1^n\left(dt^2u_2^nu_3^n\left(dtu_0^n-dx\right)-dtu_0^n\left(dt^2u_2^nu_3^n+2dx\left(dtu_3^n+dx\right)\right)-2dtu_2^n\left(dtu_0^n-dx\right)\left(dtu_3^n+dx\right)+2\left(dtu_0^n-dx\right)\left(dt^2u_2^nu_3^n+2dx\left(dtu_3^n+dx\right)\right)\right)}{-dt^4u_0^nu_1^nu_2^nu_3^n-2dt^2dxu_0^nu_1^n\left(dtu_3^n+dx\right)+2dt^2dxu_2^nu_3^n\left(dtu_0^n-dx\right)+dt^2u_1^nu_2^n\left(dtu_0^n-dx\right)\left(dtu_3^n+dx\right)+4dx^2\left(dtu_0^n-dx\right)\left(dtu_3^n+dx\right)} \\ u_2^{n+1} & = & \frac{dxu_2^n\left(-dt^2u_0^nu_1^n\left(dtu_3^n+dx\right)+2dtu_1^n\left(dtu_0^n-dx\right)\left(dtu_3^n+dx\right)+dtu_3^n\left(dt^2u_0^nu_1^n-2dx\left(dtu_0^n-dx\right)\right)-2\left(dtu_3^n+dx\right)\left(dt^2u_0^nu_1^n-2dx\left(dtu_0^n-dx\right)\right)\right)}{-dt^4u_0^nu_1^nu_2^nu_3^n-2dt^2dxu_0^nu_1^n\left(dtu_3^n+dx\right)+2dt^2dxu_2^nu_3^n\left(dtu_0^n-dx\right)+dt^2u_1^nu_2^n\left(dtu_0^n-dx\right)\left(dtu_3^n+dx\right)+4dx^2\left(dtu_0^n-dx\right)\left(dtu_3^n+dx\right)} \\ u_3^{n+1} & = & \frac{dxu_3^n\left(-dt^3u_0^nu_1^nu_2^n-2dt^2dxu_0^nu_1^n+3dt^2u_1^nu_2^n\left(dtu_0^n-dx\right)-2dtu_2^n\left(dt^2u_0^nu_1^n-2dx\left(dtu_0^n-dx\right)\right)+4dx^2\left(dtu_0^n-dx\right)\right)}{-dt^4u_0^nu_1^nu_2^nu_3^n-2dt^2dxu_0^nu_1^n\left(dtu_3^n+dx\right)+2dt^2dxu_2^nu_3^n\left(dtu_0^n-dx\right)+dt^2u_1^nu_2^n\left(dtu_0^n-dx\right)\left(dtu_3^n+dx\right)+4dx^2\left(dtu_0^n-dx\right)\left(dtu_3^n+dx\right)} \end{array} \right] \tag{5}$$

Implicit leap frog analytical approach

$$\frac{u_i^{n+1} - u_i^n}{dt} + u_i^n \, \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\,dx} = 0 \tag{6}$$

$$\frac{u_i^{n+1} - u_i^n}{dt} = -u_i^n \, \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\,dx} \tag{7}$$

$$u_i^{n+1} - u_i^n = -u_i^n \, dt \, \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\,dx} \tag{8}$$

$$u_i^{n+1} = -u_i^n \, dt \, \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\,dx} + u_i^n \tag{9}$$

$$2\,dx\,u_i^{n+1} = -u_i^n \, dt \, \left(u_{i+1}^{n+1} - u_{i-1}^{n+1}\right) + 2\,dx\,u_i^n \tag{10}$$

$$2\,dx\,u_i^{n+1} + u_i^n \, dt \, u_{i+1}^{n+1} - u_i^n \, dt \, u_{i-1}^{n+1} = 2\,dx\,u_i^n \tag{11}$$

Vektorschreibweise:

$$- u_i^n \, dt \, u_{i-1}^{n+1} + 2 \, dx \, u_i^{n+1} + u_i^n \, dt \, u_{i+1}^{n+1} = 2 \, dx \, u_i^n \tag{12}$$

$$\begin{bmatrix} dx - u_1^n \, dt & u_1^n \, dt & 0 & & 0 \\ -u_2^n \, dt & 2 \, dx & u_2^n \, dt & & 0 \\ 0 & -u_3^n \, dt & 2 \, dx & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & u_{M-1}^n \, dt \\ 0 & 0 & & -u_M^n \, dt & dx + u_1^M \, dt \end{bmatrix} \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ \vdots \\ u_M^{n+1} \end{bmatrix} = \begin{bmatrix} dx \, u_1^n \\ 2 \, dx \, u_2^n \\ 2 \, dx \, u_3^n \\ \vdots \\ dx \, u_M^n \end{bmatrix} \tag{13}$$

$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i \tag{14}$$

$$\begin{bmatrix} b_1 & c_1 & & & 0 \\ a_2 & b_2 & c_2 & & \\ & a_3 & b_3 & \ddots & \\ & & \ddots & \ddots & c_{n-1} \\ 0 & & & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{bmatrix} \tag{15}$$

Algorithm 1 Tridiagonal matrix algorithm (Thomas algorithm)

function THOMAS($\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$) ▷ Vectors

  $\hat{c}_1 \leftarrow \dfrac{c_1}{b_1}$

  $\hat{d}_1 \leftarrow \dfrac{d_1}{b_1}$

  for $i = 2, 3, \ldots, n-1$ **do** ▷ Forward sweep

    $\hat{c}_i \leftarrow \dfrac{c_i}{b_i - a_i \, \hat{c}_{i-1}}$

    $\hat{d}_i \leftarrow \dfrac{d_i - a_i \, \hat{d}_{i-1}}{b_i - a_i \, \hat{c}_{i-1}}$

  $x_n \leftarrow \hat{d}_n$

  for $i = n-1, n-2, \ldots, 1$ **do** ▷ Backwards substitution

    $x_i \leftarrow \hat{d}_i - \hat{c}_i \, x_{i+1}$

  return x