ROCKET THERMAL ANALYSIS		
	Tau Rocket Team	
	Project Phase: PDR	Subsystem: Propulsion
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Abstract

For decades rocket groups have been using the program THERMCAS as a mean of initial estimations for the thermal analysis of a Solid Rocket Motor. Aiming to provide a better and more intuitive program, a new software was developed, Rocket Thermal Analysis (RTA). By applying the Finite Difference Method in analytical equations for convection and conduction, models for the casing and bulkhead heat distribution were successfully implemented inside RTA. The outputs showed good accordance with the expected values and the program is available for download.

Contents

1 Program Purpose

The idea of developing RTA was brought to light by the Propulsion subsystem at the Latin America Space Challenge Cooperation Program, in 2020. It was proposed a new software to replace the old THERMCAS, created by Richard Nakka. During the RTA development it was always in our mind that the software needed to be easy to execute and run a simulation and should have more features than THERMCAS does. This is why in RTA a bulkhead analysis is able to be conducted as well as two Finite Difference Methods for the casing thermal distribution. It is planned the insertion of many material properties for the user to choose, since now only EPDM and Aluminium 6061-T6 are available.

2 Program Explanation

2.1 Casing

For a thin cylinder of lenght to radius ratio of about 10, an infinite cylinder having only one dimension for conduction can be assume as an approximation [1], since the only considerable heat conduction path is radially. Solid Rocket Motors usually have a lenght to diameter ratio of 10 [2], so it is suitable to apply such an idealization in these applications.

Inside the rocket chamber the insulator area most exposed to hot gases are in the sections between grains. These areas will be receiving heat for a longer time than any other along the casing, and by having so it is the critical point for heat insulation. Having said that, the heat exposure time will be set as the combustion time and an even thickness will be assumed throughout the insulator.

At first, heat will enter the system through convection by the hot gases provenient from the propellant burning. In order to have an estimation of the convective heat coefficent, an expression for convection in tube with turbulent flow can be used [1], it is shown in the Annex I. The convective heat will then be calculated utilizing Newton's Law of Cooling, considering the chamber and wall temperature.

After that, the boundary condition is set by applying the Energy Balance Equation at the first node, expression present in Annex I. The heats entering are from conduction from the posterior node and convection by the gases.

Then, conduction is taken within the materials. For that, a one-dimensional transient Fourier Law is applied, giving the care to change between the properties of the insulator and the casing structural material. The initial temperature of the materials composing the wall are set as the ambient temperature through all of its extent.

2.2 Bulkhead

In order to tackle the bulkhead temperature distribution a two-dimensional model was addopted, it was based on cylindrical coordinates assuming symmetry along the angular coordinate. The bulkhead is considered here as a cylinder that is receiving heat on one of its circular surfaces. At the bulkhead side that is within the chamber an insulator of radius equal to that of the chamber is adopted.

The same equations used for the casing convective heat were used for the bulkhead scheme, see Annex II. For conduction the equations were adapted to two dimensions, going radially from zero to the outer radius and in height from zero to the bulkhead thickness, see Annex II.

2.3 Solution Methods

The method chosen to solve the partial differential equations was the Finite Difference Method because of its simplicity and fairly good results. For the casing an explicit and implicit form of the method was developed, but for the bulkhead only the explicit version was succesfully implemented. The reason for this is that to solve the equation implicitly it was required to solve simultaneously for nodes varying in r and z, which could not be done by the ease it was done in the one-dimension scheme.

In the Explicit Finite Difference Method the future temperature of a node was only dependent on previously known temperatures in its surroundings, so it was possible to program the code to solve the temperature for each node at a given time, see Annex I and II. This process is computationaly costly and is able to not converge which is a downside, but it can do the work once one is capable of accepting these faults.

In the other hand the Implicit Finite Difference Method created a system of algebraic equations that gave the previous temperature at the node based on the future temperature of the node and its surroundings, see Annex I. In order to solve this system the inverse matrix of the coefficients multiplying the new temperatures was calculated and then multiplied with the known temperatures vector to achieve the new temperature vector.

3 Stability Analysis

Solving a Partial Differential Equation by the Finite Difference Method and utilizing an Explicit approach requires a stability criterion to be respected for the solution to converge. This criterion is, according to the Von Neumann necessary condition for stability, that the coefficient of the associated node of interest at the previous time is greater than or equal to zero.

For the 1-D solution, applied to the casing model, this condition gives the following relation between the finite step in time and the finite step in space.

$$\Delta t \leq \frac{(\Delta r)^2}{2\alpha} \tag{1}$$

This was implemented at the code to always get a time step from a spatial step that assures the solution convergence. The same was done for the 2-D solution for the bulkhead model. In that case, the expression is

$$\Delta t \leq \frac{(\Delta z \Delta r)^2}{2\alpha(\Delta r^2 + \Delta z^2)} \tag{2}$$

An exemple of how the solution behaves when the time and spatial steps are changed can be seen in Figure 1. Fixing the number of time steps in $n_t = 120$ in the right hand side and fixig the spatial steps in $n_r = 120$ for the left hand side.

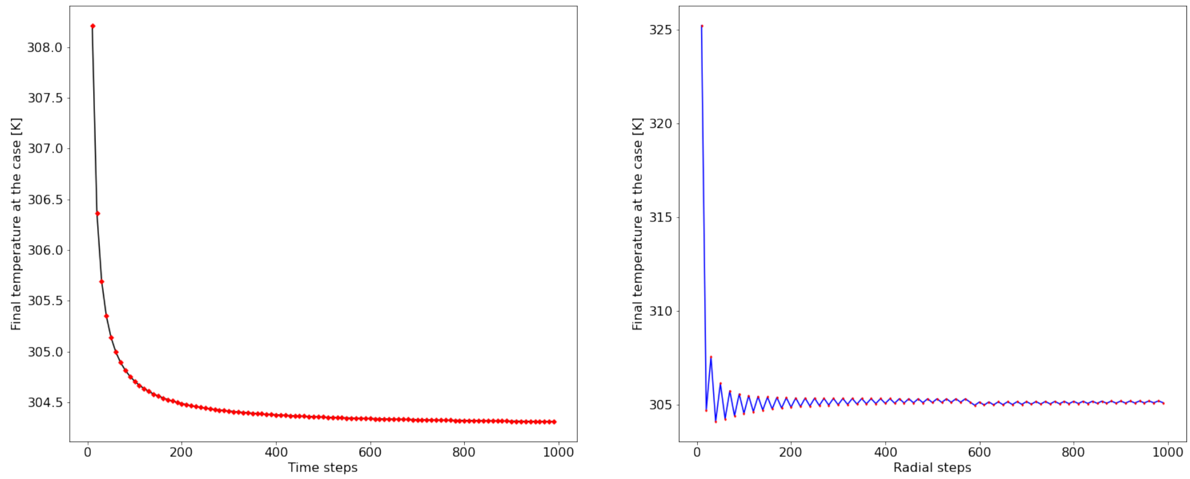

Figure 1: Convergence of the solution varying time and space steps.

4 Results Comparison

In order to have a comparison between the explicit, implicit and THERMCAS outputs, a simulation was conducted in the three of them, all with the same values. Because THERMCAS has a limit of 10 radial points, the same amount was used in the RTA. The values used are listed in the table below.

Variable	Description	Insulator	Casing
C_p	Specific Heat		
ρ	Specific Mass		
k	Thermal Conductivity		
Δr	Radial step		
Δt	Time step		
r_0	Outer radius		
r_i	Inner radius		
t	Total time		
n_t	Number of time steps		
n_r	Number of radial points		
h_m	Convection Coefficient		

Table 1: Values used in the simulations.

From the simulation run the plots shown below were obtained. In it, one can identify the clear difference between THERMCAS and RTA, both explicit and implicit methods showed good accordance, but in the other way, THERMCAS got smaller values.

5 Conclusion

References

[1] T. L. B. A. S. L. Frank P. Incropera, David P. DeWitt. *Fundamentals of Heat and Mass Transfer*. John Wiley & Sons, 6^{th} edition, 2006.

[2] O. B. George Paul Sutton. *Rocket Propulsion Elements*. John Wiley & Sons, 9^{th} edition, 2010.

Annex I - Casing Model

Analytical Equations

1-D Cylindrical Heat Cunduction Equation

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}$$

1-D Conservation of Energy Equation with Convection

$$\rho c_p \frac{\partial T}{\partial t} = k \left.\frac{\partial T}{\partial r}\right|_{back} + h_m(T_c - T_w)$$

Newton's Law of cooling

$$q\text{''} = h_m(T_c - T_W)$$

Convective Heat Coefficent in Circular Tube with Turbulent Flow

$$h_m = 0.023\frac{k}{D}Re_D^{0.8}Pr^{0.3}$$

Discrete Equations

Implicit Discretization

First r node

$$\rho c_p \left(\frac{T_j^{i+1} - T_j^i}{\Delta t}\right) = k\left(\frac{T_{j+1}^{i+1} - T_j^{i+1}}{\Delta r}\right) + h_m(T_c - T_j^i)$$

$$\left(1 - \frac{h_m \Delta t}{\rho c_p}\right)T_j^i + \frac{h_m \Delta t T_c}{\rho c_p} = \left(1 + \frac{k\Delta t}{\rho c_p \Delta r}\right)T_j^{i+1} + \left(\frac{-k\Delta t}{\rho c_p \Delta r}\right)T_{j+1}^{i+1}$$

Inner r node

$$\frac{1}{\alpha}\frac{T_j^{i+1} - T_j^i}{\Delta t} = \frac{T_{j+1}^{i+1} + T_{j-1}^{i+1} - 2T_j^{i+1}}{(\Delta r)^2} + \frac{T_{j+1}^{i+1} - T_{j-1}^{i+1}}{2r\Delta r}$$

$$T_j^i = \left(1 + \frac{2\alpha\Delta t}{(\Delta r)^2}\right)T_j^{i+1} + \left(\frac{\alpha\Delta t}{2r\Delta r} - \frac{\alpha\Delta t}{(\Delta r)^2}\right)T_{j-1}^{i+1} + \left(\frac{-\alpha\Delta t}{2r\Delta r} - \frac{\alpha\Delta t}{(\Delta r)^2}\right)T_{j+1}^{i+1}$$

Last r node

$$\frac{1}{\alpha}\frac{T_j^{i+1}-T_j^i}{\Delta t}=\frac{T_{j+2}^{i+1}+T_j^{i+1}-2T_{j+1}^{i+1}}{(\Delta r)^2}+\frac{T_{j+1}^{i+1}-T_j^{i+1}}{r\Delta r}$$

$$T_j^i=\left(1-\frac{\alpha\Delta t}{(\Delta r)^2}+\frac{\alpha\Delta t}{r\Delta r}\right)T_j^{i+1}+\left(\frac{-\alpha\Delta t}{(\Delta r)^2}\right)T_{j+2}^{i+1}+\left(\frac{2\alpha\Delta t}{(\Delta r)^2}-\frac{\alpha\Delta t}{r\Delta r}\right)T_{j+1}^{i+1}$$

Explicit Discretization

First r node

$$\rho c_p\left(\frac{T_j^{i+1}-T_j^i}{\Delta t}\right)=k\left(\frac{T_{j+1}^i-T_j^i}{\Delta r}\right)+h(T_c-T_j^i)$$

$$T_j^{i+1}=\left(1-\frac{k\Delta t}{\rho c_p\Delta r}-\frac{h_m\Delta t}{\rho c_p}\right)T_j^i+\left(\frac{k\Delta t}{\rho c_p\Delta r}\right)T_{j+1}^i+\frac{h_mT_c\Delta t}{\rho c_p}$$

Inner r node

$$\frac{1}{\alpha}\frac{T_j^{i+1}-T_j^i}{\Delta t}=\frac{T_{j+1}^i+T_{j-1}^i-2T_j^i}{(\Delta r)^2}+\frac{T_{j+1}^i-T_{j-1}^i}{2r\Delta r}$$

$$T_j^{i+1}=\left(1-\frac{2\alpha\Delta t}{(\Delta r)^2}\right)T_j^i+\left(\frac{\alpha\Delta t}{(\Delta r)^2}-\frac{\alpha\Delta t}{2r\Delta r}\right)T_{j-1}^i+\left(\frac{\alpha\Delta t}{2r\Delta r}+\frac{\alpha\Delta t}{(\Delta r)^2}\right)T_{j+1}^i$$

Last r node

$$\frac{1}{\alpha}\frac{T_j^{i+1}-T_j^i}{\Delta t}=\frac{T_{j+2}^i+T_j^i-2T_{j+1}^i}{(\Delta r)^2}+\frac{T_{j+1}^i-T_j^i}{r\Delta r}$$

$$T_j^{i+1}=\left(1+\frac{\alpha\Delta t}{(\Delta r)^2}-\frac{\alpha\Delta t}{r\Delta r}\right)T_j^i+\left(\frac{\alpha\Delta t}{(\Delta r)^2}\right)T_{j+2}^i+\left(\frac{\alpha\Delta t}{r\Delta r}-\frac{2\alpha\Delta t}{(\Delta r)^2}\right)T_{j+1}^i$$

Annex II - Bulkhead Model

Analytical Equations

2-D Cylindrical Heat Conduction Equation

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}$$

2-D Conservation of Energy Equation with Convection

$$\frac{\rho c_p \Delta r \Delta z}{2}\frac{\partial T}{\partial t} = \frac{k\Delta z}{2}\left.\frac{\partial T}{\partial r}\right|_{up} + \frac{k\Delta z}{2}\left.\frac{\partial T}{\partial r}\right|_{down} + \frac{k\Delta r}{2}\left.\frac{\partial T}{\partial z}\right|_{back} + h\Delta r(T_c - T_w)$$

Discrete Equations

Explicit Discretization

First z and first r node

$$\frac{\rho c_p \Delta r \Delta z}{2}\left(\frac{T_{j,l}^{i+1} - T_{j,l}^{i}}{\Delta t}\right) = \frac{k\Delta z}{2}\left(\frac{T_{j+1,l}^{i} - T_{j,l}^{i}}{\Delta r}\right) + \frac{k\Delta r}{2}\left(\frac{T_{j,l+1}^{i} - T_{j,l}^{i}}{\Delta z}\right) + h_m \Delta r(T_c - T_{j,l}^{i})$$

$$T_{j,l}^{i+1} = \frac{2h_m\Delta t(T_c)}{\rho c_p \Delta z} + \left(1 - \frac{k\Delta t}{\rho c_p \Delta r^2} - \frac{k\Delta t}{\rho c_p \Delta z^2} - \frac{2h_m \Delta t}{\rho c_p \Delta z}\right)T_{j,l}^{i} + \left(\frac{k\Delta t}{\rho c_p \Delta r^2}\right)T_{j+1,l}^{i} + \left(\frac{k\Delta t}{\rho c_p \Delta z^2}\right)T_{j,l+1}^{i}$$

First z and inner r node

$$\frac{\rho c_p \Delta r \Delta z}{2}\left(\frac{T_{j,l}^{i+1} - T_{j,l}^{i}}{\Delta t}\right) = \frac{k\Delta z}{2}\left(\frac{T_{j+1,l}^{i} - T_{j,l}^{i}}{\Delta r}\right) + \frac{k\Delta z}{2}\left(\frac{T_{j,l}^{i} - T_{j-1,l}^{i}}{\Delta r}\right) + \frac{k\Delta r}{2}\left(\frac{T_{j,l+1}^{i} - T_{j,l}^{i}}{\Delta z}\right) + h_m dr(T_c - T_{j,l}^{i})$$

$$T_{j,l}^{i+1} = \frac{2h_m\Delta t(T_c)}{\rho c_p \Delta z} + \left(1 - \frac{k\Delta t}{\rho c_p \Delta z^2} - \frac{2h_m \Delta t}{\rho c_p \Delta z}\right)T_{j,l}^{i} + \left(\frac{k\Delta t}{\rho c_p \Delta r^2}\right)T_{j+1,l}^{i} + \left(\frac{-k\Delta t}{\rho c_p \Delta r^2}\right)T_{j-1,l}^{i} + \left(\frac{k\Delta t}{\rho c_p \Delta z^2}\right)T_{j,l+1}^{i}$$

First z and last r node

$$\frac{\rho c_p \Delta r \Delta z}{2}\left(\frac{T_{j,l}^{i+1}-T_{j,l}^{i}}{\Delta t}\right)=\frac{k\Delta z}{2}\left(\frac{T_{j,l}^{i}-T_{j-1,l}^{i}}{\Delta r}\right)+\frac{k\Delta r}{2}\left(\frac{T_{j,l+1}^{i}-T_{j,l}^{i}}{\Delta z}\right)+h_m\Delta r(T_c-T_{j,l}^{i})$$

$$T_{j,l}^{i+1}=\frac{2h_m\Delta t(T_c)}{\rho c_p\Delta z}+\left(1+\frac{k\Delta t}{\rho c_p\Delta r^2}-\frac{k\Delta t}{\rho c_p\Delta z^2}-\frac{2h_m\Delta t}{\rho c_p\Delta z}\right)T_{j,l}^{i}+\left(\frac{-k\Delta t}{\rho c_p\Delta r^2}\right)T_{j-1,l}^{i}$$

$$\left(\frac{k\Delta t}{\rho c_p\Delta z^2}\right)T_{j,l+1}^{i}$$

Inner z and first r node

$$\frac{1}{\alpha}\frac{T_{j,l}^{i+1}-T_{j,l}^{i}}{\Delta t}=\frac{T_{j+2,l}^{i}+T_{j,l}^{i}-2T_{j+1,l}^{i}}{(\Delta r)^2}+\frac{T_{j+1,l}^{i}-T_{j,l}^{i}}{r\Delta r}+\frac{T_{j,l+1}^{i}+T_{j,l-1}^{i}-2T_{j,l}^{i}}{(\Delta z)^2}$$

$$T_{j,l}^{i+1}=\left(1-\frac{\alpha\Delta t}{r\Delta r}+\frac{\alpha\Delta t}{(\Delta r)^2}-\frac{2\alpha\Delta t}{(\Delta z)^2}\right)T_{j,l}^{i}+\left(\frac{\alpha\Delta t}{r\Delta r}-\frac{2\alpha\Delta t}{(\Delta r)^2}\right)T_{j+1,l}^{i}+\left(\frac{\alpha\Delta t}{(\Delta r)^2}\right)T_{j+2,l}^{i}+$$

$$\left(\frac{\alpha\Delta t}{(\Delta z)^2}\right)T_{j,l-1}^{i}+\left(\frac{\alpha\Delta t}{(\Delta z)^2}\right)T_{j,l+1}^{i}$$

Inner z and inner r node

$$\frac{1}{\alpha}\frac{T_{j,l}^{i+1}-T_{j,l}^{i}}{\Delta t}=\frac{T_{j+1,l}^{i}+T_{j-1,l}^{i}-2T_{j,l}^{i}}{(\Delta r)^2}+\frac{T_{j+1,l}^{i}-T_{j-1,l}^{i}}{2r\Delta r}+\frac{T_{j,l+1}^{i}+T_{j,l-1}^{i}-2T_{j,l}^{i}}{(\Delta z)^2}$$

$$T_{j,l}^{i+1}=\left(1-\frac{2\alpha\Delta t}{(\Delta r)^2}-\frac{2\alpha\Delta t}{(\Delta z)^2}\right)T_{j,l}^{i}+\left(\frac{\alpha\Delta t}{(\Delta r)^2}-\frac{\alpha\Delta t}{2r\Delta r}\right)T_{j-1,l}^{i}+\left(\frac{\alpha\Delta t}{2r\Delta r}+\frac{\alpha\Delta t}{(\Delta r)^2}\right)T_{j+1,l}^{i}+$$

$$\left(\frac{\alpha\Delta t}{(\Delta z)^2}\right)T_{j,l-1}^{i}+\left(\frac{\alpha\Delta t}{(\Delta z)^2}\right)T_{j,l+1}^{i}$$

Inner z and last r node

$$\frac{1}{\alpha}\frac{T_{j,l}^{i+1}-T_{j,l}^{i}}{\Delta t}=\frac{T_{j-2,l}^{i}+T_{j,l}^{i}-2T_{j-1,l}^{i}}{(\Delta r)^2}+\frac{T_{j,l}^{i}-T_{j-1,l}^{i}}{r\Delta r}+\frac{T_{j,l+1}^{i}+T_{j,l-1}^{i}-2T_{j,l}^{i}}{(\Delta z)^2}$$

$$T_{j,l}^{i+1}=\left(1+\frac{\alpha\Delta t}{r\Delta r}+\frac{\alpha\Delta t}{(\Delta r)^2}-\frac{2\alpha\Delta t}{(\Delta z)^2}\right)T_{j,l}^{i}+\left(\frac{-2\alpha\Delta t}{(\Delta r)^2}-\frac{\alpha\Delta t}{r\Delta r}\right)T_{j-1,l}^{i}+\left(\frac{\alpha\Delta t}{(\Delta r)^2}\right)T_{j-2,l}^{i}+$$
$$\left(\frac{\alpha\Delta t}{(\Delta z)^2}\right)T_{j,l-1}^{i}+\left(\frac{\alpha\Delta t}{(\Delta z)^2}\right)T_{j,l+1}^{i}$$

Final z and first r node

$$\frac{1}{\alpha}\frac{T_{j,l}^{i+1}-T_{j,l}^{i}}{\Delta t}=\frac{T_{j+2,l}^{i}+T_{j,l}^{i}-2T_{j+1,l}^{i}}{(\Delta r)^2}+\frac{T_{j+1,l}^{i}-T_{j,l}^{i}}{r\Delta r}+\frac{T_{j,l-2}^{i}+T_{j,l}^{i}-2T_{j,l-1}^{i}}{(\Delta z)^2}$$

$$T_{j,l}^{i+1}=\left(1-\frac{\alpha\Delta t}{r\Delta r}+\frac{\alpha\Delta t}{(\Delta r)^2}+\frac{\alpha\Delta t}{(\Delta z)^2}\right)T_{j,l}^{i}+\left(\frac{\alpha\Delta t}{r\Delta r}-\frac{2\alpha\Delta t}{(\Delta r)^2}\right)T_{j+1,l}^{i}+\left(\frac{\alpha\Delta t}{(\Delta r)^2}\right)T_{j+2,l}^{i}+$$
$$\left(\frac{-2\alpha\Delta t}{(\Delta z)^2}\right)T_{j,l-1}^{i}+\left(\frac{\alpha\Delta t}{(\Delta z)^2}\right)T_{j,l-2}^{i}$$

Final z and inner r node

$$\frac{1}{\alpha}\frac{T_{j,l}^{i+1}-T_{j,l}^{i}}{\Delta t}=\frac{T_{j+1,l}^{i}+T_{j-1,l}^{i}-2T_{j,l}^{i}}{(\Delta r)^2}+\frac{T_{j+1,l}^{i}-T_{j-1,l}^{i}}{2r\Delta r}+\frac{T_{j,l-2}^{i}+T_{j,l}^{i}-2T_{j,l-1}^{i}}{(\Delta z)^2}$$

$$T_{j,l}^{i+1}=\left(1-\frac{2\alpha\Delta t}{(\Delta r)^2}+\frac{\alpha\Delta t}{(\Delta z)^2}\right)T_{j,l}^{i}+\left(\frac{\alpha\Delta t}{(\Delta r)^2}-\frac{\alpha\Delta t}{2r\Delta r}\right)T_{j-1,l}^{i}+\left(\frac{\alpha\Delta t}{2r\Delta r}+\frac{\alpha\Delta t}{(\Delta r)^2}\right)T_{j+1,l}^{i}+$$
$$\left(\frac{-2\alpha\Delta t}{(\Delta z)^2}\right)T_{j,l-1}^{i}+\left(\frac{\alpha\Delta t}{(\Delta z)^2}\right)T_{j,l-2}^{i}$$

Final z and last r node

$$\frac{1}{\alpha}\frac{T_{j,l}^{i+1}-T_{j,l}^{i}}{\Delta t}=\frac{T_{j-2,l}^{i}+T_{j,l}^{i}-2T_{j-1,l}^{i}}{(\Delta r)^2}+\frac{T_{j,l}^{i}-T_{j-1,l}^{i}}{r\Delta r}+\frac{T_{j,l-2}^{i}+T_{j,l}^{i}-2T_{j,l-1}^{i}}{(\Delta z)^2}$$

$$T_{j,l}^{i+1}=\left(1+\frac{\alpha\Delta t}{r\Delta r}+\frac{\alpha\Delta t}{(\Delta r)^2}+\frac{\alpha\Delta t}{(\Delta z)^2}\right)T_{j,l}^{i}+\left(\frac{-2\alpha\Delta t}{(\Delta r)^2}-\frac{\alpha\Delta t}{r\Delta r}\right)T_{j-1,l}^{i}+\left(\frac{\alpha\Delta t}{(\Delta r)^2}\right)T_{j-2,l}^{i}+$$
$$\left(\frac{-2\alpha\Delta t}{(\Delta z)^2}\right)T_{j,l-1}^{i}+\left(\frac{\alpha\Delta t}{(\Delta z)^2}\right)T_{j,l-2}^{i}$$