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# PHYS 3031 Course Notes

## Mathematical Methods in Physics II

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*MATH METHODS IN PHYSICS*

PHYS 3031 Mathematical Methods in Physics II



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# 1 Series

## 1.1 Convergence Condition for Positive Series $\sum_{n=1}^{\infty} a_n$

Necessary condition:  $\lim_{N \rightarrow \infty} a_N = 0$

Hierarchy:  $N! > a^N > N^b > \ln N$

**Stirling's Formula**  $\ln N! \approx N \ln N - N \approx N \ln N$

**Comparison Test 1**  $\sum a_n < \sum b_n$ ,  $b$  converges  $\rightarrow a$  converges

**Comparison Test 2 (Integral Test)**  $\sum a_n$  &  $\int a(n)dn$  share the same fate

**Ratio Test**  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \rho$ ,  $\rho > 1 \rightarrow$  Diverges,  $\rho < 1 \rightarrow$  Converges

**Extended (Special) Comparison Test**  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1$ , then  $\sum a_n$  &  $\sum b_n$  share the same fate

## 1.2 Convergence Condition for Alternating Series $\sum_{n=1}^{\infty} (-1)^n a_n$

If  $a_n > 0, \lim_{n \rightarrow \infty} a_n = 0$ , this series may diverge.

(1) Absolute Convergence: If  $\sum a_n$  converges, then  $\sum (-1)^n a_n$  converges

(2) Convergence Condition:  $\lim_{n \rightarrow \infty} a_n = 0$  and  $a_n > a_{n+1}$

(3) Diverge: If  $a_n < a_{n+1}$ , then the series diverges

## 1.3 Power Series $\sum_{n=0}^{\infty} a_n (x - x_0)^n \rightarrow f(x)$

Convergent condition for  $x$ :  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} (x - x_0) \right| < 1$

## 1.4 Asymptotic Series 渐近级数

For functions  $f(z)$  and  $\phi(z) \neq 0$  defined in  $\dot{U}(z_0)$ , we say that  $f(z) = O(\phi(z))$  at  $z \rightarrow z_0$  if  $f(z)/\phi(z)$  is bounded, and that  $f(z) = o(\phi(z))$  at  $z \rightarrow z_0$  if  $f(z)/\phi(z) \rightarrow 0$ .

If for  $\forall m$ , when  $z \rightarrow z_0$ ,

$$f(z) - \sum_{n=0}^m a_n \phi_n(z) = o(\phi_m(z))$$

we say that  $\sum_{n=0}^m a_n \phi_n(z)$  is an asymptotic series for  $f(z)$ , even though the series may not converge:

$$f(z) \sim \sum_{n=0}^m a_n \phi_n(z)$$

## 2 Taylor Expansion $\sum_{n=0}^{\infty} a_n (x - x_0)^n$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \dots \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\arctan x = \int_0^x \sum_{n=0}^{\infty} (-t^2)^n dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \text{ where } |x| < 1$$

## 2.1 Leibniz Rule

$$\frac{d^{(M)}}{dx^M}(u \cdot v) = \left( \frac{du}{dx} \frac{\partial}{\partial u} + \frac{dv}{dx} \frac{\partial}{\partial v} \right)^M (u \cdot v) = \sum_{n=0}^M C_M^n \left( \frac{d^{(M-n)}u}{dx^{(M-n)}} \right) \left( \frac{d^{(n)}v}{dx^n} \right)$$

## 2.2 Error Estimation when N Terms are Kept

$$f(x) \approx \sum_{n=0}^N (-1)^n a_n (x - x_0)^n \quad b_n \equiv a_n (x - x_0)^n > 0$$

### 2.2.1 Alternating Series $S = \sum_{n=0}^{\infty} (-1)^n b_n$

Maximum possible error for  $f(x)$  is

$$b_{N+1} = a_{N+1} |x - x_0|^{N+1}$$

### 2.2.2 "Positive" Series $S = \sum_{n=0}^{\infty} a_n (x - x_0)^n, a_n (x - x_0)^n > 0$

If it converges when  $|x - x_0| < 1$ , and  $|a_{n+1}| < |a_n|$ , then

$$S - S_N < \frac{|a_{N+1}| |x - x_0|^{N+1}}{1 - |x - x_0|}$$

Note: In practice, Taylor Expansion is useful when  $|x - x_0| \ll 1$ , and an upper limit of error  $\epsilon$  to be tolerated is given, even if the series converges for any value of  $(x - x_0)$ .

## 2.3 L'Hôpital's Rule

$$\text{Theorem 1: } \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} \xrightarrow{\frac{f(x_0)=0}{g(x_0)=0}} \frac{0}{0} \implies \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} \rightarrow \frac{f'(x)}{g'(x)}$$

$$\text{Theorem 2: } \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} \xrightarrow{\frac{f(x_0)=0}{g(x_0)=0}} \frac{\infty}{\infty} \implies \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} \rightarrow \frac{f'(x)}{g'(x)} \text{ (proved by the inverse of the fraction)}$$

## 3 Complex Analysis

Convergence of the Complex Series  $\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} a_n + i \sum_{n=0}^{\infty} b_n \implies a_n$  and  $b_n$  both converges.

Complex Power Series  $\sum_{n=0}^{\infty} c_n z^n = f(z)$  with convergence region  $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| z < 1$ .

Euler's Formula:  $e^z = \cos z + i \sin z$  help solving the inverse trigonometric functions.

### 3.1 Complex Functions $f(z) = f(x + iy) = u(x, y) + iv(x, y)$

#### Analytic Function

Property:  $f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$  is **unique** regardless how  $\Delta z \rightarrow 0$ .

Necessary and Sufficient Conditions:

Cauchy-Riemann Conditions in Cartesian Coordinate:  $\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v$

Cauchy-Riemann Conditions in Polar Coordinate:  $\partial_\theta u = -\rho \partial_\rho v, \quad \partial_\theta v = \rho \partial_\rho u$

### Isolated Zeros 孤立零点

If  $f$  is analytic at  $z_0$ , then  $f$  has a zero of order  $m \geq 1$  at  $z_0$  if

$$f(z_0) = f'(z_0) = \dots = f^{(m-1)}(z_0) = 0$$

and  $f^{(m)}(z_0) \neq 0$ . Note that  $f$  becomes a **branch point** if  $m$  is not an integer, and that  $f$  is not analytic at that point.

**Theorem:** If  $z = a$  is a zero of  $f(z)$  which is not a constant at  $\dot{U}(a)$ , then  $\exists \rho > 0$ ,  $f(z)$  doesn't have any zeros in the region  $0 < |z - a| < \rho$ .

## 3.2 Line Integrals

With the substitution of line  $c : y = g(x)$ ,  $dy = g'(x)dx$

$$\lim_{\delta z_n \rightarrow 0} \sum_{z_n \in c} f(z_n) \Delta z_n = \int_c f(z) dz = \int_c f(x + iy)(dx + idy) = \int_a^b f(x + ig(x))(1 + ig'(x))dx \implies \int_c f(z) dz = - \int_{-c} f(z) dz$$

**Cauchy's Theorem for Analytic Functions:**

$$\oint_C f(z) dz = 0$$

### Two Foundation Lemmas - Small & Big Arc Lemma

- Small Arc Lemma (小圆弧引理):

If  $f(z)$  is continuous in  $\dot{U}(a)$ , and  $(z - a)f(z)$  approaches  $k$  consistently as  $|z - a| \rightarrow 0$  within  $\theta_1 \leq \arg(z - a) \leq \theta_2$ , then

$$\lim_{\delta \rightarrow 0} \int_{C_\delta} f(z) dz = ik(\theta_2 - \theta_1)$$

- Big Arc Lemma (大圆弧引理):

If  $f(z)$  is continuous in  $\dot{U}(\infty)$ , and  $zf(z)$  approaches  $K$  consistently as  $z \rightarrow \infty$  within  $\theta_1 \leq \arg(z - a) \leq \theta_2$ , then

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = iK(\theta_2 - \theta_1)$$

**Cauchy's Integral Formula:**  $f(z)$  is analytic inside and on the contour, then for  $\forall z_0$  inside the contour,

$$\implies f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{z - z_0} \implies f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta}) d\theta \text{ (Mean Value Theorem 均值定理)}$$

i.e. Get full information inside by the information on the boundary only.

**Note:** If  $z_0$  were outside the contour, then

- If  $f$  is analytic inside  $C$ , then  $\frac{1}{2\pi i} \oint_C \frac{f(z) dz}{z - z_0} = 0$
- If  $f$  is analytic outside  $C$  and  $\lim_{z \rightarrow \infty} f(z) = K$ , then  $\frac{1}{2\pi i} \oint_C \frac{f(z) dz}{z - z_0} = f(z_0) - K$
- The above two lemmas are NOT contradictory. In fact, if a complex function is analytic and bounded within  $U(\infty)$ , then it must be a constant function.

### Liouville Theorem (in Complex Analysis) 刘维尔定理

Every bounded entire function must be constant. That is, every holomorphic function  $f$  for which there exists a positive number  $M$  such that  $|f(z)| \leq M$  for all  $z \in \mathbb{C}$  is constant. Equivalently, non-constant holomorphic functions on  $\mathbb{C}$  have unbounded images.

### Poisson's Formula

Idea: If  $f(z = x + iy) = u(x, y) + iv(x, y)$  is analytic on the upper-half plane and that we only know the value of  $u(x, 0)$  or  $v(x, 0)$ , we can first get the value of  $f(x \in \mathbb{R})$ , then apply the Cauchy's Integral Formula to get all the complex value on the upper-half plane:

$$f(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{u(\xi, 0)}{\xi - (x + iy)} d\xi = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v(\xi, 0)}{\xi - (x + iy)} d\xi$$

$$f(z) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{(\xi - x)^2 + y^2} d\xi = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{(\xi - x)f(\xi)}{(\xi - x)^2 + y^2} d\xi$$

### 3.3 Taylor Series

#### Derivative of $f(z)$

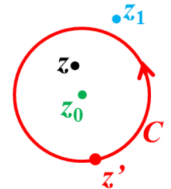
$$f^{(n)}(a) \equiv \frac{d^{(n)}f}{da^n} = \frac{n!}{2\pi i} \oint_C \frac{f(z)dz}{(z - a)^{n+1}}$$

**Taylor Series**  $f(z) = \sum a_n(z - z_0)^n$

Suppose  $f(z)$  has a singular point at  $z_1$ , we can expand  $f(z)$  at  $z_0$ :

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')dz'}{z' - z} = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_C \frac{f(z')dz'}{(z' - z_0)^{n+1}} = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

$$a_n \equiv \frac{1}{2\pi i} \oint_C \frac{f(z')dz'}{(z' - z_0)^{n+1}} = \frac{1}{n!} \frac{d^n f(z_0)}{dz^n}$$



### 3.4 Lauren Series

Suppose  $f(z)$  has a pole at  $z_0$ , define the hole as order  $N \geq 1$  at  $z_0$  if  $\lim_{z \rightarrow z_0} (z - z_0)^N f(z)$  is finite and non-zero. ("Essential Pole" if such  $N \rightarrow \infty$  like  $e^{1/z}$  at  $z = 0$ )

$f(z)$  can be expressed as

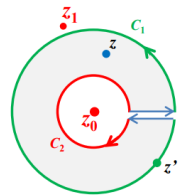
$$f(z) = \sum_{m=0}^{\infty} a_m(z - z_0)^m + \sum_{n=1}^N \frac{b_n}{(z - z_0)^n} \text{ as } \lim_{z \rightarrow z_0} (z - z_0)^N f(z) = b_N$$

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')dz'}{z' - z} + \oint_{C_2} \frac{f(z')dz'}{z' - z} = I_1 + I_2$$

We have

$$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')dz'}{(z' - z_0)^{n+1}} \quad b_1 = \frac{1}{2\pi i} \oint_{C_2} f(z')dz' \quad b_{n+1} = \frac{1}{2\pi i} \oint_{C_2} f(z')(z' - z_0)^n dz' \quad (\text{Not Useful})$$

Note that when  $f(z)$  is analytic at  $z_0$ ,  $a_n$  becomes the same as the coefficient in Taylor Series, and  $b_i \equiv 0$  for  $\forall i$ .



### 3.5 Analytic Continuation 解析延拓

Suppose  $f_1(z)$  is analytic in region  $g_1$ ,  $f_2(z)$  is analytic in region  $g_2$ , such that  $g_1 \cap g_2 \neq \emptyset$ . If  $f_1(z) \equiv f_2(z)$  in  $g_1 \cap g_2$ , then  $f_2(z)$  is the analytic continuation for  $f_1(z)$  in region  $g_2$ .

### 3.6 Residue Theorem 留数定理

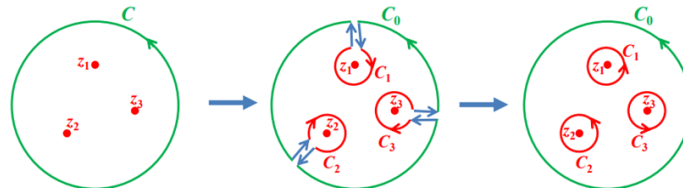
We want to evaluate  $\oint_C f(z)dz$  around the pole. By applying the Lauren Series, one can prove that

$$\oint_C f(z)dz = 2\pi i b_1$$

To find  $b_1$ , notice that  $\lim_{z \rightarrow z_0} (z - z_0)^N f(z)$  is finite and all the other terms in the Laurent Series would disappear by taking  $(N - 1)$  times derivatives and taking  $z = z_0$ :

$$b_1 = \lim_{z \rightarrow z_0} \frac{g^{(M-1)}(z)}{(M-1)!}, \text{ where } g(z) \equiv (z - z_0)^M f(z), M \geq N \text{ (Usually taking } M = N)$$

We then define the coefficient  $b_1$  of the Laurent Series at the pole  $z_0$  as  $b_1(z_0) \equiv R(z_0)$ , and refer as the **residue** of  $f(z)$  at  $z_0$



If  $f(z)$  has singular points  $z_1, z_2, \dots, z_n$  inside contour  $C$ , then

$$\oint_C f(z) dz = 2\pi i \sum_{n=1}^N R(z_n)$$

**Residue at Infinity**

Noun	Explanation
Analytic (Holomorphic) Point	A point which the function has a derivative at and in a neighborhood around that point
Branch Point 分枝点	A point such that all of its neighborhoods contain a point that has more than $n$ values
Pole	A point of a function $f$ if it is a zero of the function $1/f$ which is analytic
Regular Point	A point in the function's domain where the function is differentiable
Singularity 奇点	Essential Singularity: $\lim_{z \rightarrow z_0} (z - z_0)^N f(z)$ is always infinite Isolated Singularity: One that has no other singularities close to it Movable Singularity, Removable Singularity

Table 1: Explanation of Important Nouns