PHYS 3031 Course Notes

Mathematical Methods in Physics II

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MATH METHODS IN PHYSICS

PHYS 3031 Mathematical Methods in Physics II

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1 Series

1.1 Convergence Condition for Positive Series $\sum_{n=1}^{\infty} a_n$

Necessary condition: $\lim_{N\to\infty} a_N = 0$

Hierarchy: $N! > a^N > N^b > \ln N$

Stirling's Formula $\ln N! \approx N \ln N - N \approx N \ln N$

Comparison Test 1 $\sum a_n < \sum b_n$, b converges $\to$ a converges

Comparison Test 2 (Integral Test) $\sum a_n \& \int a(n)\mathrm{d}n$ share the same fate

Ratio Test $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$, $\rho > 1 \to$ Diverges, $\rho < 1 \to$ Converges

Extended (Special) Comparison Test $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, then $\sum a_n \& \sum b_n$ share the same fate

1.2 Convergence Condition for Alternating Series $\sum_{n=1}^{\infty} (-1)^n a_n$

If $a_n > 0, \lim_{n\to\infty} a_n = 0$, this series may diverge.

(1) Absolute Convergence: If $\sum a_n$ converges, then $\sum(-1)^n a_n$ converges (2) Convergence Condition: $\lim_{n\to\infty}$ and $a_n > a_{n+1}$ (3) Diverge: If $a_n < a_{n+1}$, then the series diverges

1.3 Power Series $\sum_{n=0}^{\infty} a_n (x - x_0)^n \to f(x)$

Convergent condition for x: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}(x - x_0)\right| < 1$

2 Taylor Expansion $\sum_{n=0}^{\infty} a_n (x - x_0)^n$

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + ...$ $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\arctan x = \int_0^x \sum_{n=0}^{\infty} (-t^2)^n \mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$, where $|x| < 1$

2.1 Leibniz Rule

$$\frac{\mathrm{d}^{(M)}}{\mathrm{d}x^M}(u \cdot v) = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\partial}{\partial u} + \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\partial}{\partial v}\right)(u \cdot v) = \sum_{n=0}^{M} C_M^n \left(\frac{\mathrm{d}^{(M-n)}u}{\mathrm{d}x^{(M-n)}}\right)\left(\frac{\mathrm{d}^{(n)}v}{\mathrm{d}x^n}\right)$$

2.2 Error Estimation when N Terms are Kept

$$f(x) \approx \sum_{n=0}^{N} (-1)^n a_n (x - x_0)^n \qquad b_n \equiv a_n (x - x_0)^n > 0$$

2.2.1 Alternating Series $S = \sum_{n=0}^{\infty} (-1)^n b_n$

Maximum possible error for $f(x)$ is

$$b_{N+1} = a_{N+1} |x - x_0|^{N+1}$$

2.2.2 "Positive" Series $S = \sum_{n=0}^{\infty} a_n (x-x_0)^n, a_n(x-x_0)^n > 0$

If it converges when $|x - x_0| < 1$, and $|a_{n+1}| < |a_n|$, then

$$S - S_N < \frac{|a_{N+1}||x-x_0|^{N+1}}{1 - |x - x_0|}$$

Note: In practice, Taylor Expansion is useful when $|x - x_0| << 1$, and an upper limit of error ϵ to be tolerated is given, even if the series converges for any value of $(x - x_0)$.

2.3 L'Ĥopital's Rule

Theorem 1: $\lim_{x\to x_0} \frac{f(x)}{g(x)} \xrightarrow[g(x_0)=0]{f(x_0)=0} \frac{0}{0} \implies \lim_{x\to x_0} \frac{f(x)}{g(x)} \longrightarrow \frac{f'(x)}{g'(x)}$

Theorem 2: $\lim_{x\to x_0} \frac{f(x)}{g(x)} \xrightarrow[g(x_0)=0]{f(x_0)=0} \frac{\infty}{\infty} \implies \lim_{x\to x_0} \frac{f(x)}{g(x)} \longrightarrow \frac{f'(x)}{g'(x)}$(proved by the inverse of the fraction)

3 Complex Analysis

Convergence of the Complex Series $\sum^{\infty} c_n = \sum^{\infty} a_n + i \sum^{\infty} b_n \implies a_n$and b_n both converges.

Complex Power Series $\sum^{\infty} c_n z^n = f(z)$ with convergence region $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n} z\right| < 1$.

Euler's Formula: $e^z = \cos z + i \sin z$ help solving the inverse trigonometric functions.

3.1 Complex Functions $f(z) = f(x+iy) = u(x,y) + iv(x,y)$

Analytic Function

Property: $f'(z) = \lim_{\delta z\to 0} \frac{f(z_\delta z) - f(z)}{\delta z}$ is unique regardless how $\delta z \to 0$.

Sufficient Conditions: Cauchy-Riemann Conditions: $\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v$

Cauchy-Riemann Conditions in Polar Form: $\partial_\theta u = -\rho \partial_\rho v, \quad \partial_\theta v = \rho \partial_\rho u$

3.2 Line Integrals

With the substitution of line $c : y = g(x)$, $\mathrm{d}y = g'(x)\mathrm{d}x$

$$\lim_{\delta z_n \to 0} \sum_{z_n \in c} f(z_n)\Delta z_n = \int_c f(z)\mathrm{d}z = \int_c f(x+iy)(\mathrm{d}x + i\mathrm{d}y) = \int_a^b f(x + ig(x))(1 + ig'(x))\mathrm{d}x \implies \int_c f(z)\mathrm{d}z = -\int_{-c} f(z)\mathrm{d}z$$

Cauchy's Theorem for Analytic Functions: $\oint_C f(z)\mathrm{d}z = 0$

Cauchy's Integral Formula: $f(z)$ is analytic inside and on the contour $\implies f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)\mathrm{d}z}{z - z_0}$

Homework 1

Problem 1

Use the comparison test to prove the convergence of the following series:

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n+3^n}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n2^n}$

Solution.

(a) By considering the n-th term, we have

$$\frac{1}{2^n + 3^n} < \frac{1}{2^n + 2^n} = \frac{1}{2^{n+1}}$$

Since the series is positive and that

$$\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2} < \infty,$$

the original series converges.

(b) Consider the n-th term, we have

$$\frac{1}{n2^n} < \frac{1}{2^n}$$

for $n \geq 2$.

Since the series is positive and that

$$\sum_{n=1}^{\infty} \frac{1}{n2^n} < \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1}{2^n} = 1 < \infty,$$

the original sequence converges.

Problem 2

Test the following series for convergence using the comparison test:

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

(b) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

Solution.

(a) Consider the n-th term, we have

$$\frac{1}{\sqrt{n}} > \frac{1}{n}$$

for $n \geq 2$.

Since

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > 1 + \sum_{n=2}^{\infty} \frac{1}{n} \to \infty,$$

the series diverges by the comparison test.

(b) Since $\ln n < n$ for $\forall n \in \mathbb{N}$,

$$\sum_{n=2}^{\infty} \frac{1}{\ln n} > \sum_{n=2}^{\infty} \frac{1}{n} \to \infty$$

Thus, the series diverges by the comparison test.

Problem 3

Use the integral test to find whether the following series converge or diverge:

(a) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

(b) $\sum_{n=1}^{\infty} \frac{e^n}{e^{2n}+9}$

Solution.

(a) Consider the integral

$$\int_2^{\infty} \frac{1}{x \ln x} \mathrm{d}x = \int_{\ln 2}^{\infty} \frac{\mathrm{d} \ln x}{\ln x} = \ln \ln x|_2^{\infty} \to \infty$$

Thus, the original series diverges.

(b) Consider the integral

$$\int_2^{\infty} \frac{e^x}{e^{2x}+9} \mathrm{d}x = \int_2^{\infty} \frac{1}{e^x + \frac{9}{e^x}} \mathrm{d}x \geq \int_2^{\infty} \frac{1}{2e^x} = -\frac{1}{2} e^{-x}|_2^{\infty} = \frac{e^{-2}}{2} < \infty$$

Thus, the original series converges.

Problem 4

Use the ratio test to find whether the following series converge or diverge: $\sum_{n=0}^{\infty} \frac{(n!)^3 e^{3n}}{(3n)!}$

Solution.

The n-th term is given by

$$a_n = \frac{(n!)^3 e^{3n}}{(3n)!}$$

Thus,

$$\frac{a_{n+1}}{a_n} = \frac{[(n+1)!]^3 e^{3(n+1)}}{[3(n+1)]!} \frac{(3n)!}{(n!)^3 e^{3n}} = \frac{(n+1)^3 \cdot e^3}{(3n+1)(3n+2)(3n+3)} = e^3 \cdot \frac{n^3+3n^2+3n+1}{27n^3+54n^2+33n+6}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} e^3 \cdot \frac{n^3+3n^2+3n+1}{27n^3+54n^2+33n+6} = \lim_{n\to\infty} e^3 \cdot \frac{1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3}}{27+\frac{54}{n}+\frac{33}{n^2}+\frac{6}{n^3}} = \frac{e^3}{27} < 1$$

Thus, the series converges by the ratio test.

Problem 5

Use the special comparison test to find whether the following series converge or diverge:

(a) $\sum_{n=9}^{\infty} \frac{(2n+1)(3n-5)}{\sqrt{n^2-73}}$

(b) $\sum_{n=3}^{\infty} \frac{(n-\ln n)^2}{5n^4-3n^2+1}$

(c) $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+5n-1}}{n^2-\sin n^3}$

Solution.

(a)

$$a_n = \frac{(2n+1)(3n-5)}{\sqrt{n^2-73}}$$

Consider another sequence with $b_n = 6n$. Obviously, $\sum b_n$ diverges.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(2n+1)(3n-5)}{6n\sqrt{n^2-73}} = \lim_{n\to\infty} \frac{6-\frac{7}{n}-\frac{5}{n^2}}{6\sqrt{1-\frac{73}{n^2}}} = 1$$

Thus, $\sum a_n$ also diverges.

(b)

$$a_n = \frac{(n-\ln n)^2}{5n^4-3n^2+1}$$

Consider another sequence with $b_n = \frac{1}{5n^2}$. Obviously, $\sum b_n$ converges.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{5n^2(n-\ln n)^2}{5n^4-3n^2+1} = \lim_{n\to\infty} \frac{(1-\frac{\ln n}{n})^2}{1-\frac{3}{5n^2}+\frac{1}{5n^4}} = 1$$

Thus, $\sum a_n$ also converges.

(c)

$$a_n = \frac{\sqrt{n^3+5n-1}}{n^2-\sin n^3}$$

Consider another sequence with $b_n = \frac{1}{n^{\frac{1}{2}}}$. Then $\sum b_n$ diverges from the previous problem.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n^4+5n^2-n}}{n^2-\sin n^3} = \lim_{n\to\infty} \frac{\sqrt{1+\frac{5}{n^2}-\frac{1}{n^3}}}{1-\frac{\sin n^3}{n^2}} = 1$$

as

$$0 = \lim_{n\to\infty} -\frac{1}{n^2} \le \lim_{n\to\infty} \frac{\sin n^3}{n^2} \le \lim_{n\to\infty} \frac{1}{n^2} = 0$$

Thus, $\sum a_n$ also diverges.

Problem 6

Test the following series for convergence:

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

(b) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$

Solution.

(a) Since

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\sqrt{n}}{\sqrt{n+1}} < 1$$

and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n}} = 0$$

Thus, the original series converges.

(b)

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} -\frac{2n^2}{(n+1)^2} = \lim_{n\to\infty} -\frac{2}{(\frac{n+1}{n})^2} = -2$$

Thus, the original series diverges.

Homework 2

Problem 7

Find the interval of convergence of the following power series; be sure to investigate the endpoints of the interval: $\sum_{n=1}^{\infty} \frac{(-1)^n(x+1)^n}{n}$

Solution.

$$a_n = \frac{(-1)^n(x+1)^n}{n} \implies \rho_n = \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(x+1)^{n+1}}{n+1}\frac{n}{(-1)^n(x+1)^n}\right| = \frac{n}{n+1}\left|x+1\right|$$

Thus,

$$\rho = \lim_{n\to\infty} \rho_n = \left|x+1\right|$$

For the interval of convergence, we have

$$\rho < 1 \implies \left|x+1\right| < 1 \implies x \in (-2, 0)$$

When $x = -2$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n(x+1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \text{ Diverges}$$

When $x = 0$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n(x+1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ Converges}$$

Thus, the interval of convergence is $(-2, 0]$.

Problem 8

The following series are not power series, but you can transform each one into a power series by a change of variable and so find out where it converges:

(a) $\sum_{n=2}^{\infty} \frac{(-1)^n x^{n/2}}{n \ln n}$

(b) $\sum_{n=0}^{\infty} (\sqrt{x^2+1})^n \frac{2^n}{3^n+n^3}$

(c) $\sum_{n=0}^{\infty} (\sin x)^n (-1)^n 2^n$

Solution.

(a) Let $y = x^{1/2}$, then $a_n = \frac{(-1)^n y^n}{n \ln n}$

$$\rho = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{(-1)^{n+1} y^{n+1}}{(n+1)\ln(n+1)} \frac{n \ln n}{(-1)^n y^n}| = \lim_{n\to\infty} \frac{n \ln n}{(n+1)\ln(n+1)} |y| = \lim_{n\to\infty} \frac{\ln n + 1}{\ln(n+1)+1} |y| = |y|$$

For the interval of convergence, we have

$$\rho < 1 \implies |y| < 1 \implies y \in (-1,1) \implies x \in [0,1)$$

When $x = 1$,

$$\sum_{n=2}^{\infty} \frac{(-1)^n x^{n/2}}{n \ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$$

It converges since it's an alternating series with $|a_n| > |a_{n+1}|$.

Thus, the interval of convergence is $[0,1]$.

(b) Let $y = \sqrt{x^2+1}$, then $a_n = y^n \frac{2^n}{3^n+n^3}$

$$\rho = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{y^{n+1}}{y^n} \frac{2^{n+1}}{3^{n+1}+(n+1)^3} \frac{3^n+n^3}{2^n}| = \lim_{n\to\infty} |2y \frac{1+\frac{n^3}{3^n}}{3+\frac{(n+1)^3}{3^n}}| = \frac{2}{3}y$$

For the interval of convergence, we have

$$\rho < 1 \implies |\frac{2}{3}y| < 1 \implies y \in (-\frac{3}{2}, \frac{3}{2}) \implies x \in \left[0, \frac{\sqrt{5}}{2}\right)$$

When $x = \frac{\sqrt{5}}{2}$,

$$\sum_{n=0}^{\infty} (\sqrt{x^2+1})^n \frac{2^n}{3^n+n^3} = \sum_{n=0}^{\infty} (\frac{3}{2})^n \frac{2^n}{3^n+n^3} = \sum_{n=0}^{\infty} \frac{1}{1+\frac{n^3}{3^n}} > \sum_{n=0}^{\infty} \frac{1}{1+n} \text{Diverges}$$

(c) Let $y = \sin x$, then $a_n = (-2y)^n$

$$\rho = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 2|y|$$

For the interval of convergence, we have

$$\rho < 1 \implies y \in (-\frac{1}{2}, \frac{1}{2}) \implies x \in \left(-\frac{\pi}{6} + k\pi, \frac{\pi}{6} + k\pi\right)$$

When $x = -\frac{\pi}{6} + k\pi$ or $\frac{\pi}{6} + k\pi$, the series doesn't converge.

Problem 9

Find the first few terms of the Maclaurin series for the following functions and check your results by computer: $\frac{e^x}{1-x}$

Solution.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...$$

$$\implies \frac{e^x}{1-x} = 1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + ...$$

Problem 10

Show that $\ln(1-x) = -x$ with an error less than 0.0056 for $|x| < 0.1$.

Solution.

According to Taylor Series,

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - ...$$

$$\implies \left|\ln(1-x) - (-x)\right| \leq \frac{|x|^2}{2} + \frac{|x|^3}{3} + \frac{|x|^4}{4} + \cdots \leq \frac{0.1^2}{2} + \frac{0.1^3}{3} + \frac{0.1^4}{4} + \cdots \leq 0.005 + 0.0004 + 0.000111... < 0.0056$$

Problem 11

Show that $2/\sqrt{4-x} = 1 + \frac{1}{8}x$ with an error less than $\frac{1}{32}$ for $0 < x < 1$.

Solution.

According to the Taylor Series,

$$\frac{2}{\sqrt{4-x}} = 1 + \frac{x}{8} + \frac{3x^2}{128} + \frac{5x^3}{1024} + \cdots$$

Error:

$$S - S_N < \frac{|a_{N+1}||x-x_0|^{N+1}}{1-|x-x_0|} = \frac{|a_2||x|^2}{1-|x|} = \frac{3x^2}{128(1-x)} < \frac{3}{128} < \frac{1}{32}$$

Problem 12

Use power series to evaluate the function at the given point. Compare with computer results, using the computer to find the series, and also to do the problem without series. Resolve any disagreement in results: $e^{\arcsin x} + \ln(\frac{1-x}{e})$ at $x = 0.0003$

Solution.

According to Taylor Series at $x = 0$,

$$\begin{aligned} e^{\arcsin x} + \ln(\frac{1-x}{e}) &\approx e^{x+\frac{x^3}{6}} - (1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots) \\ &\approx [1 + x + \frac{x^3}{6} + \frac{1}{2}(x + \frac{x^3}{6})^2 + \frac{1}{6}(x + \frac{x^3}{6})^3 + \cdots] - [1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots] \\ &\approx [1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{5}{24}x^4] - [1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}] \\ &= -\frac{1}{24}x^4 = -\frac{1}{24}(3 \times 10^{-4})^4 \approx 3 \times 10^{-16} \end{aligned}$$

Problem 13

Use Maclaurin series to evaluate each of the following. Although you could do them by computer, you can probably do them in your head faster than you can type them into the computer. So use these to practice quick and skillful use of basic series to make simple calculations: $\frac{\mathrm{d}^3}{\mathrm{d}x^3}(\frac{x^2e^x}{1-x})$ at $x=0$

Solution.

$$\frac{x^2e^x}{1-x} = x^2 \cdot (1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots) \cdot (1 + x + x^2 + x^3 + \ldots) = x^2 + 2x^3 + \frac{5}{2}x^4 + \cdots$$

Thus,

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}(\frac{x^2e^x}{1-x})\bigg|_{x=0} = \frac{\mathrm{d}^3}{\mathrm{d}x^3}(x^2 + 2x^3 + \frac{5}{2}x^4 + \cdots)\bigg|_{x=0} = 12$$

Problem 14

Find the sum of the following series by recognizing it as the Maclaurin series for a function evaluated at a point: $\sum_{n=1}^{\infty} \frac{1}{n2^n}$

Solution.

Maclaurin series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Consider the function:

$$f(x) = \log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

When $x = -\frac{1}{2}$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (-\frac{1}{2})^n = -\sum_{n=1}^{\infty} \frac{1}{n2^n} = \log(1-\frac{1}{2}) = -\log 2$$

$$\implies \sum_{n=1}^{\infty} \frac{1}{n2^n} = \log 2$$

Problem 15

Evaluate the following indeterminate forms by using L'Hopital's rule and check your results by computer. (Note that Maclaurin series would not be useful here because x does not tend to zero, or because a function (ln x, for example) is not expandable in a Maclaurin series.): $\lim_{x\to\pi}\frac{x\sin x}{x-\pi}$

Solution.

By L'Hopital's Rule,

$$\lim_{x\to\pi}\frac{x\sin x}{x-\pi} = \left.\frac{\sin x + x\cos x}{1}\right|_{x=\pi} = -\pi$$

Homework 3

Problem 16

Test each of the following series for convergence:

(a) $\sum e^{in\pi/6}$

(b) $\sum \frac{i^n}{n}$

(c) $\sum \left(\frac{1+i}{1-i\sqrt{3}}\right)^n$

Solution.

(a)

(b)

(c)

Problem 17

Solution.

Problem 18

Solution.

Problem 19

Solution.

Problem 20

Solution.