PHYS 3031 Course Notes

Mathematical Methods in Physics II

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MATH METHODS IN PHYSICS

PHYS 3031 Mathematical Methods in Physics II

THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY

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Contents

1 Series

1.1 Convergence Condition for Positive Series $\sum_{n=1}^{\infty} a_n$

Necessary condition: $\lim_{N\to\infty} a_N = 0$

Hierarchy: $N! > a^N > N^b > \ln N$

Stirling's Formula $\ln N! \approx N\ln N - N \approx N\ln N$

Comparison Test 1 $\sum a_n < \sum b_n$, b converges $\to$ a converges

Comparison Test 2 (Integral Test) $\sum a_n \& \int a(n)\mathrm{d}n$ share the same fate

Ratio Test $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$, $\rho > 1 \to$ Diverges, $\rho < 1 \to$ Converges

Extended (Special) Comparison Test $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, then $\sum a_n \& \sum b_n$ share the same fate

1.2 Convergence Condition for Alternating Series $\sum_{n=1}^{\infty} (-1)^n a_n$

If $a_n > 0, \lim_{n\to\infty} a_n = 0$, this series may diverge.

(1) Absolute Convergence: If $\sum a_n$ converges, then $\sum(-1)^n a_n$ converges

(2) Convergence Condition: $\lim_{n\to\infty}$ and $a_n > a_{n+1}$

(3) Diverge: If $a_n < a_{n+1}$, then the series diverges

1.3 Power Series $\sum_{n=0}^{\infty} a_n(x-x_0)^n \to f(x)$

Convergent condition for x: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}(x-x_0)\right| < 1$

1.4 Asymptotic Series 渐近级数

For functions $f(z)$ and $\phi(z) \neq 0$ defined in $\mathring{U}(z_0)$, we say that $f(z) = O(\phi(z))$ at $z \to z_0$ if $f(z)/\phi(z)$ is bounded, and that $f(z) = o(\phi(z))$ at $z \to z_0$ if $f(z)/\phi(z) \to 0$.

If for $\forall m$, when $z \to z_0$,

$$f(z) - \sum_{n=0}^{m} a_n \phi_n(z) = o(\phi_m(z))$$

we say that $\sum_{n=0}^{m} a_n \phi_n(z)$ is an asymptotic series for $f(z)$, even though the series may not converge:

$$f(z) \sim \sum_{n=0}^{m} a_n \phi_n(z)$$

2 Taylor Expansion $\sum_{n=0}^{\infty} a_n(x-x_0)^n$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \qquad \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots \qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\arctan x = \int_0^x \sum_{n=0}^{\infty} (-t^2)^n \mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \text{ where } |x| < 1$$

2.1 Leibniz Rule

$$\frac{\mathrm{d}^{(M)}}{\mathrm{d}x^M}(u \cdot v) = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\partial}{\partial u} + \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\partial}{\partial v}\right)^M (u \cdot v) = \sum_{n=0}^{M} C_M^n \left(\frac{\mathrm{d}^{(M-n)}u}{\mathrm{d}x^{(M-n)}}\right)\left(\frac{\mathrm{d}^{(n)}v}{\mathrm{d}x^n}\right)$$

2.2 Error Estimation when N Terms are Kept

$$f(x) \approx \sum_{n=0}^{N}(-1)^n a_n (x-x_0)^n \qquad b_n \equiv a_n(x-x_0)^n > 0$$

2.2.1 Alternating Series $S = \sum_{n=0}^{\infty}(-1)^n b_n$

Maximum possible error for $f(x)$ is

$$b_{N+1} = a_{N+1}\left|x - x_0\right|^{N+1}$$

2.2.2 "Positive" Series $S = \sum_{n=0}^{\infty} a_n(x-x_0)^n, a_n(x-x_0)^n > 0$

If it converges when $\left|x - x_0\right| < 1$, and $\left|a_{n+1}\right| < \left|a_n\right|$, then

$$S - S_N < \frac{\left|a_{N+1}\right|\left|x - x_0\right|^{N+1}}{1 - \left|x - x_0\right|}$$

Note: In practice, Taylor Expansion is useful when $\left|x - x_0\right| << 1$, and an upper limit of error ϵ to be tolerated is given, even if the series converges for any value of $(x - x_0)$.

2.3 L'Ĥopital's Rule

Theorem 1: $\lim_{x\to x_0}\frac{f(x)}{g(x)} \xrightarrow[g(x_0)=0]{f(x_0)=0} \frac{0}{0} \implies \lim_{x\to x_0}\frac{f(x)}{g(x)} \longrightarrow \frac{f'(x)}{g'(x)}$

Theorem 2: $\lim_{x\to x_0}\frac{f(x)}{g(x)} \xrightarrow[g(x_0)=0]{f(x_0)=0} \frac{\infty}{\infty} \implies \lim_{x\to x_0}\frac{f(x)}{g(x)} \longrightarrow \frac{f'(x)}{g'(x)}$(proved by the inverse of the fraction)

3 Complex Analysis

Convergence of the Complex Series $\sum^{\infty} c_n = \sum^{\infty} a_n + i\sum^{\infty} b_n \implies a_n$and b_n both converges.

Complex Power Series $\sum^{\infty} c_n z^n = f(z)$ with convergence region $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}z\right| < 1$.

Euler's Formula: $e^z = \cos z + i \sin z$ help solving the inverse trigonometric functions.

3.1 Complex Functions $f(z) = f(x + iy) = u(x, y) + iv(x, y)$

Analytic Function

Property: $f'(z) = \lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\delta z}$ is **unique** regardless how $\Delta z \to 0$.

Necessary and Sufficient Conditions:

Cauchy-Riemann Conditions in Cartesian Coordinate: $\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v$

Cauchy-Riemann Conditions in Polar Coordinate: $\partial_\theta u = -\rho\partial_\rho v, \quad \partial_\theta v = \rho\partial_\rho u$

Isolated Zeros 孤立零点

If f is analytic at z_0, then f has a zeo of order $m \geq 1$ at z_0 if

$$f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$$

and $f^{(m)}(z_0) \neq 0$. Note that f becomes a **branch point** if m is not an integer, and that f is not analytic at that point.

Theorem: If $z = a$ is a zero of $f(z)$ which is not a constant at $\mathring{U}(a)$, then $\exists \rho > 0$, $f(z)$ doesn't have any zeros in the region $0 < |z - a| < \rho$.

3.2 Line Integrals

With the substitution of line $c : y = g(x)$, $\mathrm{d}y = g'(x)\mathrm{d}x$

$$\lim_{\delta z_n \to 0} \sum_{z_n \in c} f(z_n)\Delta z_n = \int_c f(z)\mathrm{d}z = \int_c f(x+iy)(\mathrm{d}x + i\mathrm{d}y) = \int_a^b f(x+ig(x))(1+ig'(x))\mathrm{d}x \implies \int_c f(z)\mathrm{d}z = -\int_{-c} f(z)\mathrm{d}z$$

Cauchy's Theorem for Analytic Functions:

$$\oint_C f(z)\mathrm{d}z = 0$$

Two Foundation Lemmas - Jordan's Lemma

• Small Arc Lemma (小圆弧引理):

If $f(z)$ is continuous in $\mathring{U}(a)$, and $(z-a)f(z)$ approaches k consistently as $|z-a| \to 0$ within $\theta_1 \leq \arg(z-a) \leq \theta_2$, then

$$\lim_{\delta \to 0} \int_{C_\delta} f(z)\mathrm{d}z = ik(\theta_2 - \theta_1)$$

• Big Arc Lemma (大圆弧引理):

If $f(z)$ is continuous in $\mathring{U}(\infty)$, and $zf(z)$ approaches K consistently as $z \to \infty$ within $\theta_1 \leq \arg(z-a) \leq \theta_2$, then

$$\lim_{R \to \infty} \int_{C_R} f(z)\mathrm{d}z = iK(\theta_2 - \theta_1)$$

Cauchy's Integral Formula: $f(z)$ is analytic inside and on the contour, then for $\forall z_0$ inside the contour,

$$\implies f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)\mathrm{d}z}{z - z_0} \implies f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})\mathrm{d}\theta \text{ (Mean Value Theorem 均值定理)}$$

i.e. Get full information inside by the information on the boundary only.

Note: If z_0 were outside the contour, then

• If f is analytic inside C, then $\frac{1}{2\pi i}\oint_C \frac{f(z)\mathrm{d}z}{z-z_0} = 0$

• If f is analytic outside C and $\lim_{z\to\infty} f(z) = K$, then $\frac{1}{2\pi i}\oint_C \frac{f(z)\mathrm{d}z}{z-z_0} = f(z_0) - K$

• The above two lemmas are NOT contradictory. In fact, if a complex function is analytic and bounded within $U(\infty)$, then it must be a constant function.

Liouville Theorem (in Complex Analysis) 刘维尔定理

Every bounded entire function must be constant. That is, every holomorphic function f for which there exists a positive number M such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$ is constant. Equivalently, non-constant holomorphic functions on $\mathbb{C}$ have unbounded images.

Poisson's Formula

Idea: If $f(z = x + iy) = u(x, y) + iv(x, y)$ is analytic on the upper-half plane and that we only know the value of $u(x, 0)$ or $v(x, 0)$, we can first get the value of $f(x \in \mathbb{R})$, then apply the Cauchy's Integral Formula to get all the complex value on the upper-half plane:

$$f(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{u(\xi, 0)}{\xi - (x + iy)} \mathrm{d}\xi = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{v(\xi, 0)}{\xi - (x + iy)} \mathrm{d}\xi$$

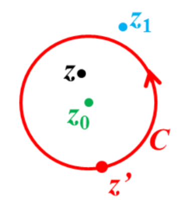
$$f(z) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)}{(\xi - x)^2 + y^2} \mathrm{d}\xi = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{(\xi - x) f(\xi)}{(\xi - x)^2 + y^2} \mathrm{d}\xi$$

3.3 Taylor Series

Derivative of $f(z)$

$$f^{(n)}(a) \equiv \frac{\mathrm{d}^{(n)} f}{\mathrm{d}a^n} = \frac{n!}{2\pi i}\oint_C \frac{f(z)\mathrm{d}z}{(z - a)^{n+1}}$$

Taylor Series $f(z) = \sum a_n (z - z_0)^n$



Suppose $f(z)$ has a singular point at z_1, we can expand $f(z)$ at z_0:

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z')\mathrm{d}z'}{z' - z} = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z - z_0)^n \oint_C \frac{f(z')\mathrm{d}z'}{(z' - z_0)^{n+1}} = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$a_n \equiv \frac{1}{2\pi i}\oint_C \frac{f(z')\mathrm{d}z'}{(z' - z_0)^{n+1}} = \frac{1}{n!}\frac{\mathrm{d}^n f(z_0)}{\mathrm{d}z^n}$$

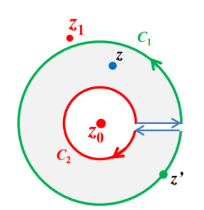
3.4 Lauren Series

Suppose $f(z)$ has a pole at z_0, define the hole as order $N \geq 1$ at z_0 if $\lim_{z \to z_0}(z - z_0)^N f(z)$ is finite and non-zero. ("Essential Pole" if such $N \to \infty$ like $e^{1/z}$ at $z = 0$)

$f(z)$ can be expressed as

$$f(z) = \sum_{m=0}^{\infty} a_m (z - z_0)^m + \sum_{n=1}^{N} \frac{b_m}{(z - z_0)^n} \text{ as } \lim_{z \to z_0}(z - z_0)^N f(z) = b_N$$

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')\mathrm{d}z'}{z' - z} + \oint_{C_2} \frac{f(z')\mathrm{d}z'}{z' - z} = I_1 + I_2$$



We have

$$a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')\mathrm{d}z'}{(z' - z_0)^{n+1}} \quad b_1 = \frac{1}{2\pi i}\oint_{C_2} f(z')\mathrm{d}z' \quad b_{n+1} = \frac{1}{2\pi i}\oint_{C_2} f(z')(z' - z_0)^n \mathrm{d}z' \quad \text{(Not Useful)}$$

Note that when $f(z)$ is analytic at z_0, a_n becomes the same as the coefficient in Taylor Series, and $b_i \equiv 0$ for $\forall i$.

3.5 Analytic Continuation 解析延拓

Suppose $f_1(z)$ is analytic in region g_1, $f_2(z)$ is analytic in region g_2, such that $g_1 \cap g_2 \neq \emptyset$. If $f_1(z) \equiv f_2(z)$ in $g_1 \cap g_2$, then $f_2(z)$ is the analytic continuation for $f_1(z)$ in region g_2.

3.6 Residue Theorem 留数定理

We want to evaluate $\oint_C f(z)\mathrm{d}z$ around the pole. By applying the Lauren Series, one can prove that

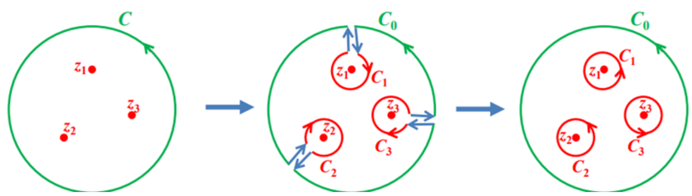
$$\oint_C f(z)\mathrm{d}z = 2\pi i b_1$$

To find b_1, notice that $\lim_{z\to z_0}(z-z_0)^N f(z)$ is finite and all the other terms in the Lauren Series would disappear by taking $(N-1)$ times derivatives and taking $z=z_0$:

$$b_1 = \lim_{z\to z_0}\frac{g^{(M-1)}(z)}{(M-1)!},\ \text{where } g(z)\equiv(z-z_0)^M f(z),\ M\geq N$$

(”$\geq$” to overkill the denominator, theoretically taking $M=N$ is enough)

We then define the coefficient b_1 of the Lauren Series at the pole z_0 as $b_1(z_0)\equiv R(z_0)$, and refer as the **residue** of $f(z)$ at z_0



If $f(z)$ has singular points $z_1, z_2, \cdots, z_n$ inside contour C, then

$$\oint_C f(z)\mathrm{d}z = 2\pi i\sum_{n=1}^{N} R(z_n)$$

Residue at Infinity

If ∞ is NOT a non-isolated singularity, define

$$R(f(\infty)) = \frac{1}{2\pi i}\oint_{C'} f(z)\mathrm{d}z$$

where C' is a closed curve **clockwise** around a point at infinity.

Note that

$$\begin{aligned} R(f(\infty)) &= \frac{1}{2\pi i}\oint_{C'} f(z)\mathrm{d}z = -\frac{1}{2\pi i}\oint_{C'}\frac{f(1/t)}{t^2}\mathrm{d}t \\ &= -\frac{f(1/t)}{t^2} \quad t^{-1}\text{'s coefficient expanding at } t=0 \\ &= -f(1/t) \quad t^{1}\text{'s coefficient expanding at } t=0 \\ &= -f(z) \quad z^{-1}\text{'s coefficient expanding at } z=\infty \end{aligned}$$

Note: $R(f(\infty))$ may NOT be zero even if $f(z)$ is analytic at $z=\infty$.

3.6.1 Rational Trignometric Function $\int_0^{2\pi} f(\sin\theta,\cos\theta)\mathrm{d}\theta$

By applying the transformation below,

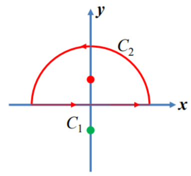
$$\sin\theta = \frac{z^2-1}{2iz} \quad \cos\theta = \frac{z^2+1}{2z} \quad \mathrm{d}\theta = \frac{\mathrm{d}z}{iz}$$

$$\int_0^{2\pi} f(\sin\theta,\cos\theta)\mathrm{d}\theta = \oint_{|z|=1} f(\frac{z^2-1}{2iz},\frac{z^2+1}{2z})\frac{\mathrm{d}z}{iz}$$

Example:

$$\begin{aligned} \int_{-\pi}^{\pi}\frac{1}{1+\epsilon\cos\theta}\mathrm{d}z\ (|\epsilon|<1) &= \oint_{|z|=1}\frac{2}{\epsilon z^2+2z+\epsilon}\frac{\mathrm{d}z}{i} = 2\pi\sum_{|z|<1} R\left(\frac{2}{\epsilon z^2+2z+\epsilon}\right) \\ &= 2\pi\frac{2}{\frac{\mathrm{d}}{\mathrm{d}z}(\epsilon z^2+2z+\epsilon)\big|_{z=(-1+\sqrt{1-\epsilon^2})/\epsilon}} = \frac{2\pi}{\sqrt{1-\epsilon^2}} \end{aligned}$$

3.6.2 Improper Intergral (over $\mathbb{R}$) $\int_{-\infty}^{\infty} f(x)\mathrm{d}x$



Example:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\mathrm{d}x = \oint_{C_1+C_2} \frac{1}{1+z^2}\mathrm{d}z = 2\pi i R(i) = \pi$$

Note: This requires the integral around the infinite point to be exist, so that when the radius of the integral path $C_2 \to \infty$, the integral $\to 0$.

Properties:

In general,

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\mathrm{d}x = 2\pi i \sum_{i=1}^{N} R(z_n)$$

where $P(x)$ and $Q(x)$ are polynomials and that $Q(x)$ **is at least two orders higher than** $P(x)$**, and have no real roots** (i.e. No extra poles on the x-axis).

Such order difference is required because

$$\int_{C_2} \frac{P(z)}{Q(z)}\mathrm{d}z \text{ as } C_2 \to \infty \text{ requires } \lim_{\rho\to\infty} \frac{P(\rho e^{i\theta})}{Q(\rho e^{i\theta})} i\rho e^{i\theta} \sim \lim_{\rho\to\infty} \rho^{1-M} = 0 \implies \text{Integer } M \geq 2$$

To deal with the poles on the x-axis when computing the (improper) integral over the real numbers, one can make use of the Jordan's Lemma (See Page 4) to compute the integral over a semi-circle to get the **Principal Value (PV)** at that point.

Properties for Poles on the Real Axis (Corollary of Jordan's Lemma):

Make substitutions: $z - z_0 = \rho e^{i\theta},\ \mathrm{d}z = i\rho e^{i\theta}\mathrm{d}\theta$

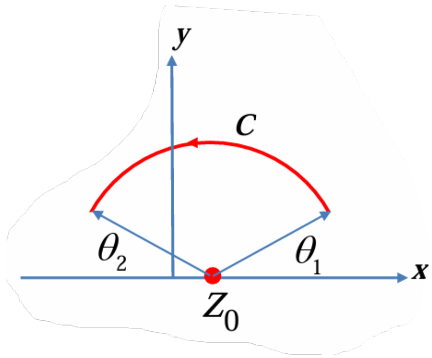
$$PV(z_0) = \int_{C(z_0)} f(z)\mathrm{d}z = \lim_{\rho\to 0} \int_{C_\rho(z_0)} f(z)\mathrm{d}z = i \lim_{\rho\to 0} \int_{\theta_1}^{\theta_2} f(z_0 + \rho e^{i\theta})\rho e^{i\theta}\mathrm{d}\theta$$

Suppose z_0 is a pole of order 1 on the real axis. According to the Lauren Series:

$$f(z) = \sum_{m=0}^{\infty} a_m (z-z_0)^m + \frac{b_1}{z-z_0}$$

$$\implies \int_{C(z_0)} (z-z_0)^m \mathrm{d}z = \lim_{\rho\to 0} \int_{C(z_0)} \rho^m e^{im\theta} i\rho e^{i\theta}\mathrm{d}\theta = i \lim_{\rho\to 0} \int_{\theta_1}^{\theta_2} \rho^{m+1} e^{i(m+1)\theta}\mathrm{d}\theta = 0$$

$$\implies PV = i \lim_{\rho\to 0} \int_{\theta_1}^{\theta_2} f(z_0 + \rho e^{i\theta})\rho e^{i\theta}\mathrm{d}\theta = ib_1 \lim_{\rho\to 0} \int_{\theta_1}^{\theta_2} \frac{\rho e^{i\theta}}{\rho e^{i\theta}}\mathrm{d}\theta = ib_1(\theta_2 - \theta_1) = i(\theta_2 - \theta_1)R(z_0)$$



Example: PV of the Integeal of $\frac{1}{x}$

In general,

$$\int_{x_1}^{x_2} \frac{\mathrm{d}x}{x} = \ln\left|\frac{x_2}{x_1}\right|$$

where, when $x_1 < 0, x_2 > 0$,

$$PV(\int_{x_1}^{x_2} \frac{\mathrm{d}x}{x}) = \lim_{\epsilon\to 0}\left[\int_{x_1}^{-\epsilon} \frac{\mathrm{d}x}{x} + \int_{\epsilon}^{x_2} \frac{\mathrm{d}x}{x}\right] = \lim_{\epsilon\to 0}\left[\ln\left|\frac{-\epsilon}{x_1}\right| + \ln\left|\frac{x_2}{\epsilon}\right|\right] = \ln\left|\frac{x_2}{x_1}\right|$$

3.6.3 Other Possible Integral Path

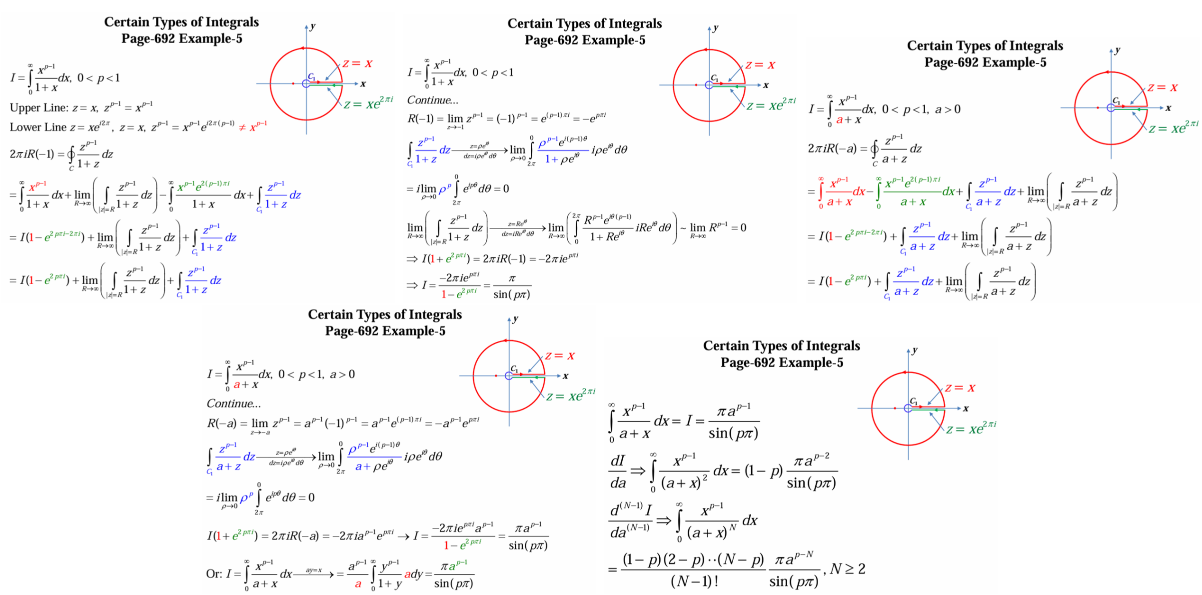


Figure 1: HKUST PHYS3031, 2023, Complex Analysis Page 93-97

3.7 Argument (Phase) Principle

Theorem 1:

If $f(z)$ has a root z_1 of order n_1 within the contour C, then

$$I = \oint_C \frac{f'(z)}{f(z)} \mathrm{d}z = 2\pi i n_1$$

Proof goes as $G(z) = \frac{\mathrm{d}}{\mathrm{d}z} \ln(f(z))$ has a pole of order 1 at z_1 with residue n_1.

Theorem 2

If $f(z)$ has a pole z_1 of order p_1 within the contour C, then

$$I = \oint_C \frac{f'(z)}{f(z)} \mathrm{d}z = -2\pi i p_1$$

Proof goes according to the Lauren Series of $f(z)$.

Corollary

If $f(z)$ is analytic everywhere, for a very large contour C that contains all roots,

Nouns	Explanations
Analytic (Holomorphic) Point	A point which the function has a derivative at and in a neighborhood around that point
Branch Point 分枝点	A point such that all of its neighborhoods contain a point that has more than n values
Regular Point	A point in the function’s domain where the function is differentiable
Singularity 奇点	Essential Singularity 本性奇点: $\lim_{z\to z_0}(z-z_0)^N f(z)$ is always infinite Isolated Singularity 孤立奇点: One that has no other singularities close to it Pole 极点: Lauren Series contains finitely many negative power terms Removable Singularity 可去奇点: Lauren Series doesn’t contain term with negative power terms

Table 1: Explanation of Important Nouns

4 Special Functions

5 Partial Differential Equations

Typical Types of PDE

$$\text{Laplace Equation: } \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$
$$\text{Diffusion Equation: } \alpha^2 \nabla^2 u = \frac{\partial u}{\partial t}$$
$$\text{Wave Equation: } \nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

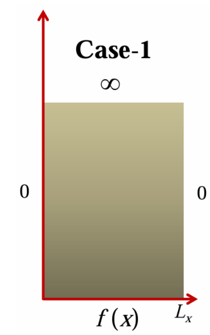
5.1 2D Laplace Equation

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Case 1

Seperation of variables: $T(x,y) = X(x)Y(y)$

$$\implies \frac{1}{X}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = -\frac{1}{Y}\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} = \lambda$$



Raises two sets of solution for $\lambda > 0$ and $\lambda < 0$.

For the given boundary condition, we have $\lambda = K^2 < 0$, then

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = -K^2 X \implies X(x) = A\cos Kx + B\sin Kx$$
$$\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} = K^2 Y \implies Y(y) = C\cosh Ky + D\sinh Ky$$

With the boundary condition, the general solution is given by

$$T(x,y) = \sum_{n=1}^{\infty} a_n T_n(x,y) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L_x} e^{-\frac{n\pi y}{L_x}}$$

Compare with te remaining boundary condition

$$T(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L_x}$$
$$\implies a_n = \frac{2}{L_x}\int_0^{L_x} f(x)\sin\frac{n\pi x}{L_x}\mathrm{d}x = \frac{2}{L_x}\langle f(x), \sin\frac{n\pi x}{L_x}\rangle$$

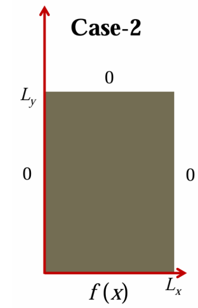
For constant function $f(x) = C$,

$$a_n = \frac{4C}{n\pi} \text{ for odd } m, 0 \text{ otherwise}$$

Case 2

This also requires x-direction to be sinusoidal, and y-direction to be exponential.

$$
\begin{aligned}
T(x,y) &= \sum_{n=1}^{\infty} a_n T_n(x,y) = \sum_{n=1}^{\infty} a_n \sin K_n x \sinh[K_n(L_y - y)] \\
\text{with } a_n &= \frac{2}{L_x \sinh(K_n L_y)} \int_0^{L_x} f(x) \sin K_n x \mathrm{d}x
\end{aligned}
$$



Linearity of the Solution

If $T_1(x,y)$ and $T_2(x,y)$ are solutions to the Laplace equation, then $T(x,y) = aT_1(x,y) + bT_2(x,y)$ is also a solution.

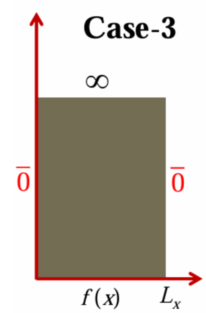
Uniqueness Theorem

Because of the orthogonality of the eigenfunctions $\sin(K_n x)$, if two functions T_1 and T_2 both satisfies the Laplace equation and the boundary condition, then the difference of these two functions raise $a_n \equiv 0 \implies T_1 = T_2$. Thus, the solution to the Laplace equation is unique.

Case 3

$\bar{0}$ refers to the boundary condition of $\partial_x T = 0$

$$
\begin{cases}
X(x) = A \sin Kx + B \cos Kx \text{ with } X'(x) = 0 \text{ at boundaries } & \implies K_n = \frac{n\pi}{L_x} \\
Y(y) = Ce^{Ky} + De^{-Ky} \text{ with finite value at } y \to \infty & \implies Y_n = e^{-K_n y}
\end{cases}
$$



For $K = 0$, $T(x,y) = A_0 + A_1 x + A_2 y + A_3 xy$ with boundary consitions $\implies A_1 = A_2 = A_3 = 0$

$$
\implies T(x,y) = A_0 + \sum_{n=1}^{\infty} B_n e^{-K_n y} \cos(K_n x)
$$

Applying Fourier Transformation,

$$
T(x,y) = \frac{L_x}{2} - \frac{4L_x}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} e^{-K_{2n+1} y} \cos(K_{2n+1} x)
$$

Usefal Integrals:

$$
\begin{aligned}
\int_0^{L_x} x \cos(K_n x) \mathrm{d}x &= -\frac{2}{K_n^2} = -\frac{2L_x^2}{n^2\pi^2} \\
\int_0^{L_x} x \sin(K_n x) \mathrm{d}x &= \frac{(-1)^{n+1} L_x^2}{n\pi}
\end{aligned}
$$