# Phys3031 Mid-term Examination

**(3 pm – 5 pm, Oct. 8, 2016, Room 4503)**

**(Total 70 points)**

**Q1 (8 points)** Determine if each of the following series is convergent or divergent. (2 points each)

(a) $\sum_{n=1}^{\infty}\frac{(3n)!}{3^n(n!)^2}$; (b) $\sum_{n=1}^{\infty}\frac{(-1)^n}{3\sqrt{n}+2}$; (c) $\sum_{n=1}^{\infty}\frac{3n-\sqrt{n}}{n^2+1}$; (d) $\sum_{n=0}^{\infty}\left(\frac{i+1}{\sqrt{3}}\right)^n$.

**Q2 (6 points)** Determine (and draw a sketch of) the convergence range of the following power series. (3 points each)

(a) $\sum_{n=1}^{\infty}\frac{(-3)^n x^n}{n}$; (b) $\sum_{n=1}^{\infty}\frac{(z-i)^n}{n^2}$.

**Q3 (12 points)** Find the value of the following functions with the required accuracy by Taylor series expansion. (4 points each)

(a) $f(x)=\ln(1+xe^x)$ at $x=0.01$ with accuracy of $10^{-6}$.

(b) $f(x)=1-\tan x$ at $x=\frac{\pi}{4}-10^{-5}$ with accuracy of $10^{-4}$.

(c) $\int_0^{0.01}\sin(\sqrt{x})dx$, with accuracy of $10^{-4}$.

**Q4 (15 points)** Use Cauchy-Riemann conditions to determine whether the following functions are analytic. You may use the theorem that if *f*(*z*) and *g*(*z*) are analytic, then *f*(*z*)**g*(*z*) is also analytic. (5 points each)

(a) $\frac{1}{z+1}$, $z\neq-1$; (b) $e^z$; (c) $\ln(z^z)$, $z>0$.

**Q5 (24 points)** Find the poles, their orders, and their residues of each of the following functions, and find the line integral of each function on the contour $|z| = 1.5$. (4 points each for (a) and (b), 8 points each for (c) and (d))

(a) $\frac{z}{(z-2)(1-2z)}$; (b) $\frac{z}{(z^3+1)}$; (c) $\frac{z}{\cos z-1}$; (d) $\frac{e^z-1}{\sin^2(z)}$.

**Q6 (5 points)** Find the coefficients of the power terms of $z^0$ and $z^{-2}$ in the Lauren series of $f(z)=\frac{e^z-1}{\sin^2(z)}$.

**That's all, folks.**

**Fall 2020 Physics 3031 Final Take-Home Examination**

**(2 pages, 6 out of 7 questions, 90 points in total)**

19-Dec-2020, 16:30 - 19:30 Hong Kong Time

**Q1 (4 points)** Choose ***two*** of the following series and determine if each of them is convergent or divergent. If you do more than two, the best two answers will be counted. (2 points each)

(a) $\sum_{n=1}^{\infty}\frac{(3n)!}{2^n(n!)^2}$; (b) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{0.25}}$; (c) $\sum_{n=1}^{\infty}\frac{3n-\sqrt{n}}{4n^2+5}$; (d) $\sum_{n=0}^{\infty}\left(\frac{i+3}{3.3}\right)^n$.

**Q2 (8 points)** Choose ***one*** of the following functions and calculate its value with the required accuracy by Taylor series expansion. If you do more than one, the best answer will be counted.

(a) $f(x)=\ln(1+xe^{2x})$ at $x=0.01$ with accuracy of $10^{-6}$. (8 points)

(b) $f(x)=1-\cos(x^2)$ at $x=0.20$ with accuracy of $10^{-4}$. (8 points)

**Q3 (12 points)**

Calculate the following integrals. Use Beta and/or Gamma functions for integrals (c) and (d).

(a) $\int_{-2}^{1} x\cdot\delta(\sin(x))dx$; (b) $\int_{-4}^{4} x\delta(2x-1)dx$;

(c) $\int_{0}^{\infty}\frac{x^{1/2}dx}{(x+2)^2}$; (d) $\int_{0}^{\infty}\frac{xdx}{(x+3)^3}$.

**Q4 (20 points)**

Find the values of the following integrals by using complex path integral and residue theorem. Find the principal value(s) if necessary.

(a) $\int_{0}^{2\pi}\frac{\sin\theta}{2-\cos\theta}d\theta$; (b) $\int_{0}^{\infty}\frac{x^2+1}{x^4-1}dx$; (c) $\int_{0}^{\infty}\frac{\cos x}{1+x^2}dx$;

(d) $\int_{-\infty}^{\infty}\frac{1}{(x+1)^2-1}dx$; (e) $\int_{0}^{\infty}\frac{x^{1/2}dx}{(x+2)^2}$.

Choose two out of the three questions (Q5, Q6, and Q7) below. If you do all three, only the two that you get the highest scores will be counted. The maximum scores you will get from these questions are 46 points.

**Q5 (20 points)**

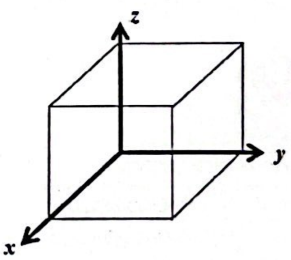
Consider a square plate of side length $L$. The bottom edge of the plate is kept at temperature $T_0$, the right edge is *insulating*, and the other two edges are kept at 0 (zero). Find the steady temperature field within the plate.

**Q6 (22 points)**

You may skip the process of separation of variables, and use the general solutions as the starting point to solve this problem. Consider a thin conductor spherical shell of radius $R$. The upper half of the shell is held at voltage $V_1$, and the lower half is kept at $V_2$. Find the electric potential inside and outside the spherical shell. In your answers, the coefficients for Legendre polynomials $P_0(\cos\theta)$, $P_1(\cos\theta)$, and $P_2(\cos\theta)$ should be given in explicit algebraic expressions. The higher order coefficients can be expressed in integral forms without carrying out the actual integrals, but the ones with value '0' (if there are any) should be pointed out explicitly.

**Q7 (24 points)**

Consider an empty cubic box of side length $L$ with impermeable walls. Choose the X-Y-Z axes along the three edges of the box that are perpendicular to one another. At time $t$ = 0, $N$ small particles are injected at the corner of the box with coordinate ($L$, $L$, 0). The initial particle density function is therefore $N \cdot \delta(x-L) \cdot \delta(y-L) \cdot \delta(z)$. The diffusion equation is $\alpha^2\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}\right)-\frac{\partial U}{\partial t}=0$, where $\alpha$ is a constant.




(a) Find the particle number density as a function of space and time afterwards. (12 points)
(b) After very long time, the wall at $x = 0$ suddenly becomes totally absorbing, such that at $x = 0$ the particle density is always zero. Reset the clock to $t = 0$. Find the particle number density as a function of space and time afterwards. (12 points)

**That's all, folks.**

# Phys3031 Mid-term Examination

**(7 pm – 9 pm, Oct. 10, 2014, Room 2404)**

**(Total 70 points plus 5 bonus points)**

**Q1 (8 points)** Determine if the each of the following series is convergent or divergent. (2 points each)

(a) $\sum_{n=0}^{\infty}\frac{(3n)!}{3^n(2n!)^2}$; (b) $\sum_{n=1}^{\infty}\frac{3n-\sqrt{n}}{4n^2+1}$; (c) $\sum_{n=1}^{\infty}\frac{(-1)^n}{3n+2}$; (d) $\sum_{n=0}^{\infty}\left(\frac{i+1}{2}\right)^n$.

**Q2 (6 points)** Determine (and draw a sketch of) the convergence range of the power series. (3 points each)

(a) $\sum_{n=0}^{\infty}\frac{(-3)^n x^n}{n}$; (b) $\sum_{n=0}^{\infty}n^2(z-4i)^n$.

**Q3 (8 points)** Find the value of the following functions with the required accuracy by Taylor series expansion. (4 points each)

(a) $f(x)=\ln(1+x^2e^x)$ at $x=0.01$ with accuracy of $10^{-6}$.

(b) $f(x)=1-\sin x$ at $x=\frac{\pi}{2}-10^{-4}$ with accuracy of $10^{-4}$.

**Q4 (15 points)** Determine whether the following functions are analytical. (5 points each)

(a) $\frac{1}{z^2+1}$, $z\neq\pm i$; (b) $\sin z$; (c) $\sqrt{\ln z}$, $z\neq 0$.

**Q5 (5 points)** Find the sum $S$ of the infinite series

$$S=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\ldots=\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}.$$

**Q6 (28 points)** Find the poles, their orders, and their residues of each of the following functions, and find the line integral of each function on the loop $|z|=1$. (5 points each for (a) and (b), 9 points each for (c) and (d))

(a) $\frac{z}{(z-2)(2-4z)}$; (b) $\frac{z}{4z^2+i}$; (c) $\frac{1}{z(e^{iz}-1)}$; (d) $\frac{1}{\sin(z^2)}$.

**Bonus (5 points)**

Find the coefficients of the power terms $z^0$ and $z^{-2}$ in the Lauren series of $f(z)=\frac{1}{z(e^{iz}-1)}$.

**That's all, folks.**

**Fall 2014 Physics 3031 Final Examination**

**(3 pages, 7 questions, and 110 points in total)**

9-Dec-14, 8:30 – 11:30, LG1027

**Recursion relations of Legendre Polynomials and Bessel Functions**

(5.8)

(a) $lP_l(x) = (2l-1)xP_{l-1}(x) - (l-1)P_{l-2}(x)$,

(b) $xP_l'(x) - P_{l-1}'(x) = lP_l(x)$,

(c) $P_l'(x) - xP_{l-1}'(x) = lP_{l-1}(x)$,

(d) $(1-x^2)P_l'(x) = lP_{l-1}(x) - lxP_l(x)$,

(e) $(2l+1)P_l(x) = P_{l+1}'(x) - P_{l-1}'(x)$,

(f) $(1-x^2)P_{l-1}'(x) = lxP_{l-1}(x) - lP_l(x)$.

.1) $\frac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x)$,

.2) $\frac{d}{dx}[x^{-p} J_p(x)] = -x^{-p} J_{p+1}(x)$,

.3) $J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x)$,

.4) $J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x)$,

.5) $J_p'(x) = -\frac{p}{x} J_p(x) + J_{p-1}(x) = \frac{p}{x} J_p(x) - J_{p+1}(x)$.

**Examination Questions**

**Q1 (10 points)**

A function $f(x)$ can be expanded by the Legendre polynomials $P_l(x)$ in $x \in [-1, 1]$, i. e., $f(x) = \sum_{l=0}^{\infty} a_l P_l(x)$. Find the first three expansion coefficients, namely $a_0$, $a_1$, and $a_2$ for the following functions.

(a) $f(x) = \begin{cases} 1, & |x| < \frac{1}{2} \\ 0, & |x| \geq \frac{1}{2} \end{cases}$.  (b) $f(x) = \delta(\sin x)$.

Hint: $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$.

**Q2 (40 points)**

Find the values of the following integrals by using complex path integral and residue theorem. Find the principal value(s) if necessary.

(a) $\int_0^{2\pi} \frac{d\theta}{1+\cos\alpha\sin\theta}$; (b) $\int_0^{\infty} \frac{x^2 dx}{4x^4+1}$; (c) $\int_0^{\infty} \frac{\cos \pi x}{1+x^4} dx$; (d) $\int_{-\infty}^{\infty} \frac{x dx}{x^4-1}$; (e) $\int_0^{\infty} \frac{x^{1/3} dx}{(x+1)^2}$.

**Q3 (12 points)**

Judge whether an integral is zero or not by using the orthogonal property of eigenfunctions. Brief reason(s) should be given. $P_l(x)$ stands for Legendre polynomial. $J_P(x)$ stands for Bessel function.

(a) $\int_{-\pi}^{\pi} P_l(\cos\theta) P_{l+3}(\cos\theta)\cos\theta d\theta$.

(b) $\int_0^1 J_1(k_1 x) x J_2(k_2 x) dx$, where $k_1 \neq k_2$, and $J_1(k_1) = J_2(k_2) = 0$.

(c) $\int_0^2 J_1(k_1 x) x J_1(\frac{k_2 x}{2}) dx$, where $k_1 \neq k_2$, and $J_1(k_1) = J_1(k_2) = 0$.

(d) $\int_0^1 x J_1(k_1 x) J_1(k_2 x) dx$, where $k_1 \neq k_2$, and $J_1(k_1) = J_1(k_2) = 0$.

**Q4 (14 points)**

Find the solution of the following differential equations in terms of Bessel Function.

(Hint: For an equation in the form of $y'' + \frac{1-2a}{x} y' + \left[ (bcx^{c-1})^2 + \frac{a^2 - p^2 c^2}{x^2} \right] y = 0$, the solution is $y = x^a J_p(bx^c)$, where $a$, $b$, $c$, and $p$ are constants.)

(a) $y'' + (K^2 - \frac{3}{4x^2}) y = 0$, where $K$ is a constant.

(b) $y'' + \frac{1}{x} y' + (x - \frac{1}{x^2}) y = 0$.

**Q5 (11 points)**

Consider a temperature field $T(x,t)$ in $x \in [0,\ L]$ that satisfies the equation

$$\alpha^2 \frac{\partial^2 T}{\partial x^2} - \frac{\partial T}{\partial t} = 0,$$

where $t$ is time and $\alpha$ is a known constant.

(a) Find the ***steady*** temperature field when the temperature at $x = 0$ is held at constant $C_1$ at all time, and at $x = L$ there is no heat flow.

(b) At $t = 0$ the insulation end ($x = L$) is suddenly changed to and maintained at temperature $C_2$, while at the other end ($x = 0$) there is no heat flow. Find the time dependent temperature field for $t > 0$.

**Q6 (11 points)**

The vibration displacement field $u(x,t)$ of an elastic string in $x \in [0,\ L]$ satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial^2 t} = 0,$$

where $t$ is time and $v$ is a known constant. The two ends of the string are fixed so the displacement there is zero. The string is stationary and without displacement when $t < 0$. At $t = 0$, an impulse is applied to the center of the string so its velocity distribution is $\frac{\partial u}{\partial t}|_{t=0} = a\delta(x - \frac{L}{2})$.

Find the vibration field afterwards.

**Q7 (12 points)**

Consider a square plate of side length $L$. The bottom edge of the plate is kept at temperature $C_1$, and the left edge is kept at temperature $C_2$. The other two edges of the plate are perfectly insulated. Find the steady temperature field within the plate.