PHYS 3031 Course Notes

Mathematical Methods in Physics II

LIN, Xuanyu

MATH METHODS IN PHYSICS

PHYS 3031 Mathematical Methods in Physics II

THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY

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Contents

1 Series

1.1 Convergence Condition for Positive Series $\sum_{n=1}^{\infty} a_n$

Necessary condition: $\lim_{N\to\infty} a_N = 0$

Hierarchy: $N! > a^N > N^b > \ln N$

Stirling's Formula $\ln N! \approx N\ln N - N \approx N \ln N$

Comparison Test 1 $\sum a_n < \sum b_n$, b converges $\to a$ converges

Comparison Test 2 (Integral Test) $\sum a_n \& \int a(n)\mathrm{d}n$ share the same fate

Ratio Test $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$, $\rho > 1 \to$ Diverges, $\rho < 1 \to$ Converges

Extended (Special) Comparison Test $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, then $\sum a_n \& \sum b_n$ share the same fate

1.2 Convergence Condition for Alternating Series $\sum_{n=1}^{\infty} (-1)^n a_n$

If $a_n > 0, \lim_{n\to\infty} a_n = 0$, this series may diverge.

(1) Absolute Convergence: If $\sum a_n$ converges, then $\sum (-1)^n a_n$ converges

(2) Convergence Condition: $\lim_{n\to\infty}$ and $a_n > a_{n+1}$

(3) Diverge: If $a_n < a_{n+1}$, then the series diverges

1.3 Power Series $\sum_{n=0}^{\infty} a_n(x-x_0)^n \to f(x)$

Convergent condition for x: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}(x-x_0)\right| < 1$

1.4 Asymptotic Series 渐近级数

For functions $f(z)$ and $\phi(z) \neq 0$ defined in $\mathring{U}(z_0)$, we say that $f(z) = O(\phi(z))$ at $z \to z_0$ if $f(z)/\phi(z)$ is bounded, and that $f(z) = o(\phi(z))$ at $z \to z_0$ if $f(z)/\phi(z) \to 0$.

If for $\forall m$, when $z \to z_0$,

$$f(z) - \sum_{n=0}^{m} a_n \phi_n(z) = o(\phi_m(z))$$

we say that $\sum_{n=0}^{m} a_n \phi_n(z)$ is an asymptotic series for $f(z)$, even though the series may not converge:

$$f(z) \sim \sum_{n=0}^{m} a_n \phi_n(z)$$

2 Taylor Expansion $\sum_{n=0}^{\infty} a_n(x-x_0)^n$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \qquad \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots \qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\arctan x = \int_0^x \sum_{n=0}^{\infty} (-t^2)^n \mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \text{ where } |x| < 1$$

2.1 Leibniz Rule

$$\frac{\mathrm{d}^{(M)}}{\mathrm{d}x^M}(u\cdot v)=\left(\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\partial}{\partial u}+\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\partial}{\partial v}\right)^M(u\cdot v)=\sum_{n=0}^{M}C_M^n\left(\frac{\mathrm{d}^{(M-n)}u}{\mathrm{d}x^{(M-n)}}\right)\left(\frac{\mathrm{d}^{(n)}v}{\mathrm{d}x^n}\right)$$

2.2 Error Estimation when N Terms are Kept

$$f(x)\approx\sum_{n=0}^{N}(-1)^n a_n(x-x_0)^n \qquad b_n\equiv a_n(x-x_0)^n>0$$

2.2.1 Alternating Series $S=\sum_{n=0}^{\infty}(-1)^n b_n$

Maximum possible error for $f(x)$ is

$$b_{N+1}=a_{N+1}\left|x-x_0\right|^{N+1}$$

2.2.2 "Positive" Series $S=\sum_{n=0}^{\infty}a_n(x-x_0)^n, a_n(x-x_0)^n>0$

If it converges when $|x-x_0|<1$, and $|a_{n+1}|<|a_n|$, then

$$S-S_N<\frac{|a_{N+1}|\left|x-x_0\right|^{N+1}}{1-|x-x_0|}$$

Note: In practice, Taylor Expansion is useful when $|x-x_0|<<1$, and an upper limit of error ϵ to be tolerated is given, even if the series converges for any value of $(x-x_0)$.

2.3 L'Ĥopital's Rule

Theorem 1: $\lim_{x\to x_0}\frac{f(x)}{g(x)}\overset{f(x_0)=0}{\underset{g(x_0)=0}{\longrightarrow}}\frac{0}{0}\implies\lim_{x\to x_0}\frac{f(x)}{g(x)}\longrightarrow\frac{f'(x)}{g'(x)}$

Theorem 2: $\lim_{x\to x_0}\frac{f(x)}{g(x)}\overset{f(x_0)=0}{\underset{g(x_0)=0}{\longrightarrow}}\frac{\infty}{\infty}\implies\lim_{x\to x_0}\frac{f(x)}{g(x)}\longrightarrow\frac{f'(x)}{g'(x)}$(proved by the inverse of the fraction)

3 Complex Analysis

Convergence of the Complex Series $\sum^{\infty}c_n=\sum^{\infty}a_n+i\sum^{\infty}b_n\implies a_n$and b_n both converges.

Complex Power Series $\sum^{\infty}c_nz^n=f(z)$ with convergence region $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}z\right|<1$.

Euler's Formula: $e^z=\cos z+i\sin z$ help solving the inverse trigonometric functions.

3.1 Complex Functions $f(z)=f(x+iy)=u(x,y)+iv(x,y)$

Analytic Function

Property: $f'(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\delta z}$ is **unique** regardless how $\Delta z\to 0$.

Necessary and Sufficient Conditions:

Cauchy-Riemann Conditions in Cartesian Coordinate: $\partial_x u=\partial_y v, \quad \partial_y u=-\partial_x v$

Cauchy-Riemann Conditions in Polar Coordinate: $\partial_\theta u=-\rho\partial_\rho v, \quad \partial_\theta v=\rho\partial_\rho u$

Isolated Zeros 孤立零点

If f is analytic at z_0, then f has a zeo of order $m \geq 1$ at z_0 if

$$f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$$

and $f^{(m)}(z_0) \neq 0$. Note that f becomes a **branch point** if m is not an integer, and that f is not analytic at that point.

Theorem: If $z = a$ is a zero of $f(z)$ which is not a constant at $\mathring{U}(a)$, then $\exists \rho > 0$, $f(z)$ doesn't have any zeros in the region $0 < |z - a| < \rho$.

3.2 Line Integrals

With the substitution of line $c : y = g(x)$, $\mathrm{d}y = g'(x)\mathrm{d}x$

$$\lim_{\delta z_n \to 0} \sum_{z_n \in c} f(z_n)\Delta z_n = \int_c f(z)\mathrm{d}z = \int_c f(x+iy)(\mathrm{d}x + i\mathrm{d}y) = \int_a^b f(x + ig(x))(1 + ig'(x))\mathrm{d}x \implies \int_c f(z)\mathrm{d}z = -\int_{-c} f(z)\mathrm{d}z$$

Cauchy's Theorem for Analytic Functions:

$$\oint_C f(z)\mathrm{d}z = 0$$

Two Foundation Lemmas - Small & Big Arc Lemma

• Small Arc Lemma (小圆弧引理):

If $f(z)$ is continuous in $\mathring{U}(a)$, and $(z - a)f(z)$ approaches k consistently as $|z - a| \to 0$ within $\theta_1 \leq \arg(z - a) \leq \theta_2$, then

$$\lim_{\delta \to 0} \int_{C_\delta} f(z)\mathrm{d}z = ik(\theta_2 - \theta_1)$$

• Big Arc Lemma (大圆弧引理):

If $f(z)$ is continuous in $\mathring{U}(\infty)$, and $zf(z)$ approaches K consistently as $z \to \infty$ within $\theta_1 \leq \arg(z - a) \leq \theta_2$, then

$$\lim_{R \to \infty} \int_{C_R} f(z)\mathrm{d}z = iK(\theta_2 - \theta_1)$$

Cauchy's Integral Formula: $f(z)$ is analytic inside and on the contour, then for $\forall z_0$ inside the contour,

$$\implies f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)\mathrm{d}z}{z - z_0} \implies f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})\mathrm{d}\theta \text{ (Mean Value Theorem 均值定理)}$$

i.e. Get full information inside by the information on the boundary only.

Note: If z_0 were outside the contour, then

• If f is analytic inside C, then $\frac{1}{2\pi i}\oint_C \frac{f(z)\mathrm{d}z}{z - z_0} = 0$

• If f is analytic outside C and $\lim_{z\to\infty} f(z) = K$, then $\frac{1}{2\pi i}\oint_C \frac{f(z)\mathrm{d}z}{z - z_0} = f(z_0) - K$

• The above two lemmas are NOT contradictory. In fact, if a complex function is analytic and bounded within $U(\infty)$, then it must be a constant function.

Liouville Theorem (in Complex Analysis) 刘维尔定理

Every bounded entire function must be constant. That is, every holomorphic function f for which there exists a positive number M such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$ is constant. Equivalently, non-constant holomorphic functions on $\mathbb{C}$ have unbounded images.

Poisson's Formula

Idea: If $f(z = x + iy) = u(x, y) + iv(x, y)$ is analytic on the upper-half plane and that we only know the value of $u(x, 0)$ or $v(x, 0)$, we can first get the value of $f(x \in \mathbb{R})$, then apply the Cauchy's Integral Formula to get all the complex value on the upper-half plane:

$$f(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{u(\xi, 0)}{\xi - (x + iy)} \mathrm{d}\xi = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{v(\xi, 0)}{\xi - (x + iy)} \mathrm{d}\xi$$

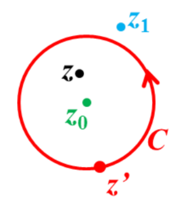
$$f(z) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)}{(\xi - x)^2 + y^2} \mathrm{d}\xi = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{(\xi - x)f(\xi)}{(\xi - x)^2 + y^2} \mathrm{d}\xi$$

3.3 Taylor Series

Derivative of $f(z)$

$$f^{(n)}(a) \equiv \frac{\mathrm{d}^{(n)}f}{\mathrm{d}a^n} = \frac{n!}{2\pi i} \oint_C \frac{f(z)\mathrm{d}z}{(z - a)^{n+1}}$$

Taylor Series $f(z) = \sum a_n (z - z_0)^n$



Suppose $f(z)$ has a singular point at z_1, we can expand $f(z)$ at z_0:

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')\mathrm{d}z'}{z' - z} = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_C \frac{f(z')\mathrm{d}z'}{(z' - z_0)^{n+1}} = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$a_n \equiv \frac{1}{2\pi i} \oint_C \frac{f(z')\mathrm{d}z'}{(z' - z_0)^{n+1}} = \frac{1}{n!} \frac{\mathrm{d}^n f(z_0)}{\mathrm{d}z^n}$$

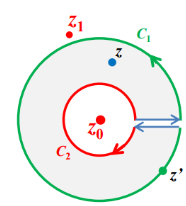
3.4 Lauren Series

Suppose $f(z)$ has a pole at z_0, define the hole as order $N \geq 1$ at z_0 if $\lim_{z \to z_0} (z - z_0)^N f(z)$ is finite and non-zero. ("Essential Pole" if such $N \to \infty$ like $e^{1/z}$ at $z = 0$)

$f(z)$ can be expressed as

$$f(z) = \sum_{m=0}^{\infty} a_m (z - z_0)^m + \sum_{n=1}^{N} \frac{b_m}{(z - z_0)^n} \text{ as } \lim_{z \to z_0} (z - z_0)^N f(z) = b_N$$

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')\mathrm{d}z'}{z' - z} + \oint_{C_2} \frac{f(z')\mathrm{d}z'}{z' - z} = I_1 + I_2$$



We have

$$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')\mathrm{d}z'}{(z' - z_0)^{n+1}} \quad b_1 = \frac{1}{2\pi i} \oint_{C_2} f(z')\mathrm{d}z' \quad b_{n+1} = \frac{1}{2\pi i} \oint_{C_2} f(z')(z' - z_0)^n \mathrm{d}z' \quad \text{(Not Useful)}$$

Note that when $f(z)$ is analytic at z_0, a_n becomes the same as the coefficient in Taylor Series, and $b_i \equiv 0$ for $\forall i$.

3.5 Analytic Continuation 解析延拓

Suppose $f_1(z)$ is analytic in region g_1, $f_2(z)$ is analytic in region g_2, such that $g_1 \cap g_2 \neq \emptyset$. If $f_1(z) \equiv f_2(z)$ in $g_1 \cap g_2$, then $f_2(z)$ is the analytic continuation for $f_1(z)$ in region g_2.

3.6 Residue Theorem 留数定理

We want to evaluate $\oint_C f(z)\mathrm{d}z$ around the pole. By applying the Lauren Series, one can prove that

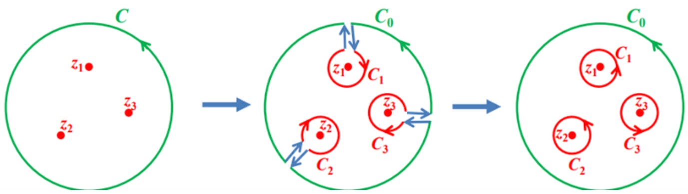
$$\oint_C f(z)\mathrm{d}z = 2\pi i b_1$$

To find b_1, notice that $\lim_{z\to z_0}(z-z_0)^N f(z)$ is finite and all the other terms in the Lauren Series would disappear by taking $(N-1)$ times derivatives and taking $z=z_0$:

$$b_1 = \lim_{z\to z_0} \frac{g^{(M-1)}(z)}{(M-1)!}, \text{ where } g(z) \equiv (z-z_0)^M f(z),\ M \geq N$$

("$\geq$" to overkill the denominator, theoretically taking $M=N$ is enough)

We then define the coefficient b_1 of the Lauren Series at the pole z_0 as $b_1(z_0) \equiv R(z_0)$, and refer as the **residue** of $f(z)$ at z_0



If $f(z)$ has singular points $z_1, z_2, \cdots, z_n$ inside contour C, then

$$\oint_C f(z)\mathrm{d}z = 2\pi i \sum_{n=1}^{N} R(z_n)$$

Residue at Infinity

If ∞ is NOT a non-isolated singularity, define

$$R(f(\infty)) = \frac{1}{2\pi i}\oint_{C'} f(z)\mathrm{d}z$$

where C' is a closed curve **clockwise** around a point at infinity.

Note that

$$\begin{aligned} R(f(\infty)) &= \frac{1}{2\pi i}\oint_{C'} f(z)\mathrm{d}z = -\frac{1}{2\pi i}\oint_{C'} \frac{f(1/t)}{t^2}\mathrm{d}t \\ &= -\frac{f(1/t)}{t^2} \quad t^{-1}\text{'s coefficient expanding at } t=0 \\ &= -f(1/t) \quad t^{1}\text{'s coefficient expanding at } t=0 \\ &= -f(z) \quad z^{-1}\text{'s coefficient expanding at } z=\infty \end{aligned}$$

Note: $R(f(\infty))$ may NOT be zero even if $f(z)$ is analytic at $z=\infty$.

3.6.1 Rational Trignometric Function $\int_0^{2\pi} f(\sin\theta,\cos\theta)\mathrm{d}\theta$

By applying the transformation below,

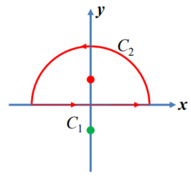
$$\sin\theta = \frac{z^2-1}{2iz} \qquad \cos\theta = \frac{z^2+1}{2z} \qquad \mathrm{d}\theta = \frac{\mathrm{d}z}{iz}$$

$$\int_0^{2\pi} f(\sin\theta,\cos\theta)\mathrm{d}\theta = \oint_{|z|=1} f(\frac{z^2-1}{2iz},\frac{z^2+1}{2z})\frac{\mathrm{d}z}{iz}$$

Example:

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{1}{1+\epsilon\cos\theta}\mathrm{d}z \ (|\epsilon|<1) &= \oint_{|z|=1} \frac{2}{\epsilon z^2+2z+\epsilon}\frac{\mathrm{d}z}{i} = 2\pi \sum_{|z|<1} R\left(\frac{2}{\epsilon z^2+2z+\epsilon}\right) \\ &= 2\pi \frac{2}{\frac{\mathrm{d}}{\mathrm{d}z}(\epsilon z^2+2z+\epsilon)\big|_{z=(-1+\sqrt{1-\epsilon^2})/\epsilon}} = \frac{2\pi}{\sqrt{1-\epsilon^2}} \end{aligned}$$

3.6.2 Improper Intergral (over $\mathbb{R}$) $\int_{-\infty}^{\infty} f(x)\mathrm{d}x$



Example:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\mathrm{d}x = \oint_{C_1+C_2} \frac{1}{1+z^2}\mathrm{d}z = 2\pi i R(i) = \pi$$

Note: This requires the integral around the infinite point to be exist, so that when the radius of the integral path $C_2 \to \infty$, the integral $\to 0$.

Properties:

In general,

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\mathrm{d}x = 2\pi i \sum_{i=1}^{N} R(z_n)$$

where $P(x)$ and $Q(x)$ are polynomials and that $Q(x)$ **is at least two orders higher than** $P(x)$**, and have no real roots**.

Such order difference is required because

$$\int_{C_2} \frac{P(z)}{Q(z)}\mathrm{d}z \text{ as } C_2 \to \infty \text{ requires } \lim_{\rho\to\infty} \frac{P(\rho e^{i\theta})}{Q(\rho e^{i\theta})} i\rho e^{i\theta} \sim \lim_{\rho\to\infty} \rho^{1-M} = 0 \implies M \geq 2$$

Nouns	Explanations
Analytic (Holomorphic) Point	A point which the function has a derivative at and in a neighborhood around that point
Branch Point 分枝点	A point such that all of its neighborhoods contain a point that has more than n values
Regular Point	A point in the function's domain where the function is differentiable
Singularity 奇点	Essential Singularity 本性奇点: $\lim_{z\to z_0}(z-z_0)^N f(z)$ is always infinite Isolated Singularity 孤立奇点: One that has no other singularities close to it Pole 极点: Lauren Series contains finitely many negative power terms Removable Singularity 可去奇点: Lauren Series doesn't contain term with negative power terms

Table 1: Explanation of Important Nouns