
PHYS 3031 Course Notes

Mathematical Methods in Physics II

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MATH METHODS IN PHYSICS

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1 Series

1.1 Convergence Condition for Positive Series $\sum_{n=1}^{\infty} a_n$

Necessary condition: $\lim_{N \rightarrow \infty} a_N = 0$

Hierarchy: $N! > a^N > N^b > \ln N$

Stirling's Formula $\ln N! \approx N \ln N - N \approx N \ln N$

Comparison Test 1 $\sum a_n < \sum b_n$, b converges $\rightarrow a$ converges

Comparison Test 2 (Integral Test) $\sum a_n$ & $\int a(n)dn$ share the same fate

Ratio Test $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \rho$, $\rho > 1 \rightarrow$ Diverges, $\rho < 1 \rightarrow$ Converges

Extended (Special) Comparison Test $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1$, then $\sum a_n$ & $\sum b_n$ share the same fate

1.2 Convergence Condition for Alternating Series $\sum_{n=1}^{\infty} (-1)^n a_n$

If $a_n > 0$, $\lim_{n \rightarrow \infty} a_n = 0$, this series may diverge.

(1) Absolute Convergence: If $\sum a_n$ converges, then $\sum (-1)^n a_n$ converges

(2) Convergence Condition: $\lim_{n \rightarrow \infty} a_n = 0$ and $a_n > a_{n+1}$

(3) Diverge: If $a_n < a_{n+1}$, then the series diverges

1.3 Power Series $\sum_{n=0}^{\infty} a_n (x - x_0)^n \rightarrow f(x)$

Convergent condition for x : $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} (x - x_0) \right| < 1$

1.4 Asymptotic Series 渐近级数

For functions $f(z)$ and $\phi(z) \neq 0$ defined in $\mathring{U}(z_0)$, we say that $f(z) = O(\phi(z))$ at $z \rightarrow z_0$ if $f(z)/\phi(z)$ is bounded, and that $f(z) = o(\phi(z))$ at $z \rightarrow z_0$ if $f(z)/\phi(z) \rightarrow 0$.

If for $\forall m$, when $z \rightarrow z_0$,

$$f(z) - \sum_{n=0}^m a_n \phi_n(z) = o(\phi_m(z))$$

we say that $\sum_{n=0}^m a_n \phi_n(z)$ is an asymptotic series for $f(z)$, even though the series may not converge:

$$f(z) \sim \sum_{n=0}^m a_n \phi_n(z)$$

2 Taylor Expansion $\sum_{n=0}^{\infty} a_n (x - x_0)^n$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \dots \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\arctan x = \int_0^x \sum_{n=0}^{\infty} (-t^2)^n dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \text{ where } |x| < 1$$

2.1 Leibniz Rule

$$\frac{d^{(M)}}{dx^M}(u \cdot v) = \left(\frac{du}{dx} \frac{\partial}{\partial u} + \frac{dv}{dx} \frac{\partial}{\partial v} \right)^M (u \cdot v) = \sum_{n=0}^M C_M^n \left(\frac{d^{(M-n)}u}{dx^{(M-n)}} \right) \left(\frac{d^{(n)}v}{dx^n} \right)$$

2.2 Error Estimation when N Terms are Kept

$$f(x) \approx \sum_{n=0}^N (-1)^n a_n (x - x_0)^n \quad b_n \equiv a_n (x - x_0)^n > 0$$

2.2.1 Alternating Series $S = \sum_{n=0}^{\infty} (-1)^n b_n$

Maximum possible error for $f(x)$ is

$$b_{N+1} = a_{N+1} |x - x_0|^{N+1}$$

2.2.2 "Positive" Series $S = \sum_{n=0}^{\infty} a_n (x - x_0)^n, a_n (x - x_0)^n > 0$

If it converges when $|x - x_0| < 1$, and $|a_{n+1}| < |a_n|$, then

$$S - S_N < \frac{|a_{N+1}| |x - x_0|^{N+1}}{1 - |x - x_0|}$$

Note: In practice, Taylor Expansion is useful when $|x - x_0| \ll 1$, and an upper limit of error ϵ to be tolerated is given, even if the series converges for any value of $(x - x_0)$.

2.3 L'Hôpital's Rule

$$\text{Theorem 1: } \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} \xrightarrow{\frac{f(x_0)=0}{g(x_0)=0}} \frac{0}{0} \implies \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} \rightarrow \frac{f'(x)}{g'(x)}$$

$$\text{Theorem 2: } \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} \xrightarrow{\frac{f(x_0)=0}{g(x_0)=0}} \frac{\infty}{\infty} \implies \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} \rightarrow \frac{f'(x)}{g'(x)} \text{ (proved by the inverse of the fraction)}$$

3 Complex Analysis

Convergence of the Complex Series $\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} a_n + i \sum_{n=0}^{\infty} b_n \implies a_n$ and b_n both converges.

Complex Power Series $\sum_{n=0}^{\infty} c_n z^n = f(z)$ with convergence region $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| z < 1$.

Euler's Formula: $e^z = \cos z + i \sin z$ help solving the inverse trigonometric functions.

3.1 Complex Functions $f(z) = f(x + iy) = u(x, y) + iv(x, y)$

Analytic Function

Property: $f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$ is **unique** regardless how $\Delta z \rightarrow 0$.

Necessary and Sufficient Conditions:

Cauchy-Riemann Conditions in Cartesian Coordinate: $\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v$

Cauchy-Riemann Conditions in Polar Coordinate: $\partial_\theta u = -\rho \partial_\rho v, \quad \partial_\theta v = \rho \partial_\rho u$

Isolated Zeros 孤立零点

If f is analytic at z_0 , then f has a zero of order $m \geq 1$ at z_0 if

$$f(z_0) = f'(z_0) = \dots = f^{(m-1)}(z_0) = 0$$

and $f^{(m)}(z_0) \neq 0$. Note that f becomes a **branch point** if m is not an integer, and that f is not analytic at that point.

Theorem: If $z = a$ is a zero of $f(z)$ which is not a constant at $\dot{U}(a)$, then $\exists \rho > 0$, $f(z)$ doesn't have any zeros in the region $0 < |z - a| < \rho$.

3.2 Line Integrals

With the substitution of line $c : y = g(x)$, $dy = g'(x)dx$

$$\lim_{\delta z_n \rightarrow 0} \sum_{z_n \in c} f(z_n) \Delta z_n = \int_c f(z) dz = \int_c f(x + iy)(dx + idy) = \int_a^b f(x + ig(x))(1 + ig'(x))dx \implies \int_c f(z) dz = - \int_{-c} f(z) dz$$

Cauchy's Theorem for Analytic Functions:

$$\oint_C f(z) dz = 0$$

Two Foundation Lemmas - Small & Big Arc Lemma

- Small Arc Lemma (小圆弧引理):

If $f(z)$ is continuous in $\dot{U}(a)$, and $(z - a)f(z)$ approaches k consistently as $|z - a| \rightarrow 0$ within $\theta_1 \leq \arg(z - a) \leq \theta_2$, then

$$\lim_{\delta \rightarrow 0} \int_{C_\delta} f(z) dz = ik(\theta_2 - \theta_1)$$

- Big Arc Lemma (大圆弧引理):

If $f(z)$ is continuous in $\dot{U}(\infty)$, and $zf(z)$ approaches K consistently as $z \rightarrow \infty$ within $\theta_1 \leq \arg(z - a) \leq \theta_2$, then

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = iK(\theta_2 - \theta_1)$$

Cauchy's Integral Formula: $f(z)$ is analytic inside and on the contour, then for $\forall z_0$ inside the contour,

$$\implies f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{z - z_0} \implies f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta}) d\theta \text{ (Mean Value Theorem 均值定理)}$$

i.e. Get full information inside by the information on the boundary only.

Note: If z_0 were outside the contour, then

- If f is analytic inside C , then $\frac{1}{2\pi i} \oint_C \frac{f(z) dz}{z - z_0} = 0$

- If f is analytic outside C and $\lim_{z \rightarrow \infty} f(z) = K$, then $\frac{1}{2\pi i} \oint_C \frac{f(z) dz}{z - z_0} = f(z_0) - K$

• The above two lemmas are NOT contradictory. In fact, if a complex function is analytic and bounded within $U(\infty)$, then it must be a constant function.

Liouville Theorem (in Complex Analysis) 刘维尔定理

Every bounded entire function must be constant. That is, every holomorphic function f for which there exists a positive number M such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$ is constant. Equivalently, non-constant holomorphic functions on \mathbb{C} have unbounded images.

Poisson's Formula

Idea: If $f(z = x + iy) = u(x, y) + iv(x, y)$ is analytic on the upper-half plane and that we only know the value of $u(x, 0)$ or $v(x, 0)$, we can first get the value of $f(x \in \mathbb{R})$, then apply the Cauchy's Integral Formula to get all the complex value on the upper-half plane:

$$f(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{u(\xi, 0)}{\xi - (x + iy)} d\xi = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v(\xi, 0)}{\xi - (x + iy)} d\xi$$

$$f(z) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{(\xi - x)^2 + y^2} d\xi = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{(\xi - x)f(\xi)}{(\xi - x)^2 + y^2} d\xi$$

3.3 Taylor Series

Derivative of $f(z)$

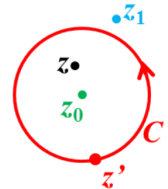
$$f^{(n)}(a) \equiv \frac{d^{(n)}f}{da^n} = \frac{n!}{2\pi i} \oint_C \frac{f(z)dz}{(z-a)^{n+1}}$$

Taylor Series $f(z) = \sum a_n(z - z_0)^n$

Suppose $f(z)$ has a singular point at z_1 , we can expand $f(z)$ at z_0 :

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')dz'}{z' - z} = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_C \frac{f(z')dz'}{(z' - z_0)^{n+1}} = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

$$a_n \equiv \frac{1}{2\pi i} \oint_C \frac{f(z')dz'}{(z' - z_0)^{n+1}} = \frac{1}{n!} \frac{d^n f(z_0)}{dz^n}$$



3.4 Lauren Series

Suppose $f(z)$ has a pole at z_0 , define the hole as order $N \geq 1$ at z_0 if $\lim_{z \rightarrow z_0} (z - z_0)^N f(z)$ is finite and non-zero. ("Essential Pole" if such $N \rightarrow \infty$ like $e^{1/z}$ at $z = 0$)

$f(z)$ can be expressed as

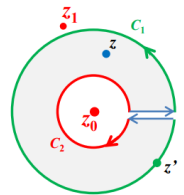
$$f(z) = \sum_{m=0}^{\infty} a_m(z - z_0)^m + \sum_{n=1}^N \frac{b_n}{(z - z_0)^n} \text{ as } \lim_{z \rightarrow z_0} (z - z_0)^N f(z) = b_N$$

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')dz'}{z' - z} + \oint_{C_2} \frac{f(z')dz'}{z' - z} = I_1 + I_2$$

We have

$$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')dz'}{(z' - z_0)^{n+1}} \quad b_1 = \frac{1}{2\pi i} \oint_{C_2} f(z')dz' \quad b_{n+1} = \frac{1}{2\pi i} \oint_{C_2} f(z')(z' - z_0)^n dz' \quad (\text{Not Useful})$$

Note that when $f(z)$ is analytic at z_0 , a_n becomes the same as the coefficient in Taylor Series, and $b_i \equiv 0$ for $\forall i$.



3.5 Analytic Continuation 解析延拓

Suppose $f_1(z)$ is analytic in region g_1 , $f_2(z)$ is analytic in region g_2 , such that $g_1 \cap g_2 \neq \emptyset$. If $f_1(z) \equiv f_2(z)$ in $g_1 \cap g_2$, then $f_2(z)$ is the analytic continuation for $f_1(z)$ in region g_2 .

3.6 Residue Theorem 留数定理

We want to evaluate $\oint_C f(z)dz$ around the pole. By applying the Lauren Series, one can prove that

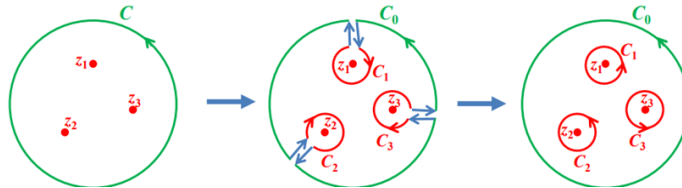
$$\oint_C f(z)dz = 2\pi i b_1$$

To find b_1 , notice that $\lim_{z \rightarrow z_0} (z - z_0)^N f(z)$ is finite and all the other terms in the Laurent Series would disappear by taking $(N - 1)$ times derivatives and taking $z = z_0$:

$$b_1 = \lim_{z \rightarrow z_0} \frac{g^{(M-1)}(z)}{(M-1)!}, \text{ where } g(z) \equiv (z - z_0)^M f(z), \quad M \geq N$$

("≥" to overkill the denominator, theoretically taking $M = N$ is enough)

We then define the coefficient b_1 of the Laurent Series at the pole z_0 as $b_1(z_0) \equiv R(z_0)$, and refer as the **residue** of $f(z)$ at z_0



If $f(z)$ has singular points z_1, z_2, \dots, z_n inside contour C , then

$$\oint_C f(z) dz = 2\pi i \sum_{n=1}^N R(z_n)$$

Residue at Infinity

If ∞ is NOT a non-isolated singularity, define

$$R(f(\infty)) = \frac{1}{2\pi i} \oint_{C'} f(z) dz$$

where C' is a closed curve **clockwise** around a point at infinity.

Note that

$$\begin{aligned} R(f(\infty)) &= \frac{1}{2\pi i} \oint_{C'} f(z) dz = -\frac{1}{2\pi i} \oint_{C'} \frac{f(1/t)}{t^2} dt \\ &= -\frac{f(1/t)}{t^2} \quad t^{-1}\text{'s coefficient expanding at } t = 0 \\ &= -f(1/t) \quad t^1\text{'s coefficient expanding at } t = 0 \\ &= -f(z) \quad z^{-1}\text{'s coefficient expanding at } z = \infty \end{aligned}$$

Note: $R(f(\infty))$ may NOT be zero even if $f(z)$ is analytic at $z = \infty$.

3.6.1 Rational Trigonometric Function $\int_0^{2\pi} f(\sin \theta, \cos \theta) d\theta$

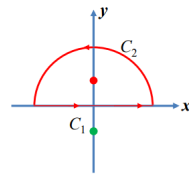
By applying the transformation below,

$$\begin{aligned} \sin \theta &= \frac{z^2 - 1}{2iz} \quad \cos \theta = \frac{z^2 + 1}{2z} \quad d\theta = \frac{dz}{iz} \\ \int_0^{2\pi} f(\sin \theta, \cos \theta) d\theta &= \oint_{|z|=1} f\left(\frac{z^2 - 1}{2iz}, \frac{z^2 + 1}{2z}\right) \frac{dz}{iz} \end{aligned}$$

Example:

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{1}{1 + \epsilon \cos \theta} dz \quad (|\epsilon| < 1) &= \oint_{|z|=1} \frac{2}{\epsilon z^2 + 2z + \epsilon} \frac{dz}{i} = 2\pi \sum_{|z|<1} R\left(\frac{2}{\epsilon z^2 + 2z + \epsilon}\right) \\ &= 2\pi \frac{2}{\frac{d}{dz}(\epsilon z^2 + 2z + \epsilon) \Big|_{z=(-1+\sqrt{1-\epsilon^2})/\epsilon}} = \frac{2\pi}{\sqrt{1-\epsilon^2}} \end{aligned}$$

3.6.2 Improper Integral (over \mathbb{R}) $\int_{-\infty}^{\infty} f(x)dx$



Example:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \oint_{C_1+C_2} \frac{1}{1+z^2} dz = 2\pi i R(i) = \pi$$

Note: This requires the integral around the infinite point to be exist, so that when the radius of the integral path $C_2 \rightarrow \infty$, the integral $\rightarrow 0$.

Properties:

In general,

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum_{i=1}^N R(z_n)$$

where $P(x)$ and $Q(x)$ are polynomials and that $Q(x)$ is at least two orders higher than $P(x)$, and have no real roots.

Such order difference is required because

$$\int_{C_2} \frac{P(z)}{Q(z)} dz \text{ as } C_2 \rightarrow \infty \text{ requires } \lim_{\rho \rightarrow \infty} \frac{P(\rho e^{i\theta})}{Q(\rho e^{i\theta})} i \rho e^{i\theta} \sim \lim_{\rho \rightarrow \infty} \rho^{1-M} = 0 \implies M \geq 2$$

Nouns	Explanations
Analytic (Holomorphic) Point	A point which the function has a derivative at and in a neighborhood around that point
Branch Point 分枝点	A point such that all of its neighborhoods contain a point that has more than n values
Regular Point	A point in the function's domain where the function is differentiable
Singularity 奇点	Essential Singularity 本性奇点: $\lim_{z \rightarrow z_0} (z - z_0)^N f(z)$ is always infinite
	Isolated Singularity 孤立奇点: One that has no other singularities close to it
	Pole 极点: Lauren Series contains finitely many negative power terms
	Removable Singularity 可去奇点: Lauren Series doesn't contain term with negative power terms

Table 1: Explanation of Important Nouns