ADA lecture notes

Analysis and design of algorithms

These are the notes for analysis and design of algorithms course. The professor says it'll be an interesting course, let's see about that. I am using Obsidian and this is an amazing markdown editor! It has a lot of community plugins. Anyways, study now... xD

Here is a somewhat detailed overview.

1. ADA/Lecture 1 : Introduction to the course and grading.
2. ADA/Lecture 2 : Mastering Master Theorem
3. ADA/Lecture 3 : DAC
4. ADA/Lecture 4 : DAC continued
5. ADA/Lecture 5 : Last DAC

Lecture→1

It's like DSA version 2 (in terms of management). Here's the link for previous year: ADA2020, ADA2022. Solutions and questions in this course are made by the instructor and hence making it public is not a good idea. So, these notes with stick around the lectures and maybe sometimes touching things but WILL NOT quote.

Evaluation

- Quizzes : 15% (n-1)
- Homework Assignments (Theory) : 15% (group of two)
- Programming Assignments : 10% (Foobar, No lab hours, Individual)
- Midsem : 30%
- Endsem : 30% Both theory

Multiplying large integers

Input : Two n-digit numbers A and B Output: Product $A \times B$ Primitive Ops: Add/Multiply two single digit integers (recall digital circuits adder)

- Classical pen-paper approach:
  - At max $2n$ operations per partial product, since n, we have $2n^2$

- Summation of them, $2n^2$
- Net $4n^2$

- Doing it differently: (Main idea: $\frac{n}{2}$ digits for each a, b, c, d)
  - $\overbrace{56}^{a}\overbrace{78}^{b} \times \overbrace{12}^{c}\overbrace{34}^{d}$
    1. Compute $a.c = 672$
    2. Compute $b.d = 2652$
    3. Compute $(a + b)(c + d) = 6164$
    4. Compute $\boxed{3. - 2. - 1.} = 2840$
    5. Put it all together $6720000 + 2652 + 284000$ (Notice the padding)
    6. Do it all recursively
  - Here's the recursive implementation, where $A = 10^{\frac{n}{2}} \cdot a + b, B = 10^{\frac{n}{2}} \cdot c + d$ and $A \times B = 10^n ac + 10^{\frac{n}{2}}(ad + bc) + bd$
    1. Recursively compute $a.c$
    2. Recursively compute $b.d$
    3. Recursively compute $(a + b) \cdot (c + d)$ (Karatsuba method, otherwise 4 recursive calls)
    4. Compute $\boxed{3. - 2. - 1.}$ for each call
    5. Pad and add!

Lecture→2

Analysis: The recurrence method

$T(n)$ = Runtime of Algorithm 1 for multiplying two n-digit numbers
Base Case : $n = 1, T(n) = c$: Multiplying two single digit numbers
Recurrence (for $n > 1$) : Express $T(n)$ as the runtime of recursive calls + additional work which maybe done in that call.

Recursively compute each ac, bd, bc, ad, work in this step

$$\begin{aligned} T(n) &= \overbrace{4T\left(\frac{n}{2}\right)}^{\text{computing ac,bd,bc,bd}} + \overbrace{c_1 \cdot n}^{\text{Adding 4 n/2 (+padded) digit no.s}} \\ T(1) &= \underbrace{c}_{\text{Base case}} \end{aligned}$$

Karatsuba's Algorithm (more like optimization) ($a + b$ may have $\frac{n}{2} + 1$ digits)

$$\begin{aligned} T(n) &= \overbrace{3T\left(\frac{n}{2}\right)}^{\text{n/2+1 but ignore}} + \overbrace{(c_2 + c_3) \cdot n}^{\text{Adding a+b, multiplying each other}} \\ T(1) &= \underbrace{c}_{\text{Base case}} \end{aligned}$$

Master Method / Master Theorem

A 'Black-box' method to solve many common recurrences in Algorithm design (especially DAC)

Assumption: All the recursive calls are made on subproblems of equal size. (If not, use the proof we will do now)

Assumption (for proof): Both constants c are equal.

$$\begin{aligned} T(n) &= aT\left(\frac{n}{b}\right) + c \cdot n^d \\ T(1) &\leq c \end{aligned}$$

$$a \geq 1, b \geq 1, c, d \geq 0$$

$a =$ Number of recursive calls
$b =$ Shrinkage factor of subproblem size
$d =$ Affects the runtime of the additional work (outside recursion)

Master Theorem (Simpler Version): Prof. says no need to *remember*

$$T(n) = \begin{cases} \mathcal{O}(n^d \log n), & \text{if } a = b^d \\ \mathcal{O}(n^d), & \text{if } a < b^d \\ \mathcal{O}(n^{\log_b a}), & \text{if } a > b^d \end{cases}$$

Example: Mergesort, $T(n) = 2T(n/2) + c \cdot n$ so $a = 2, b = 2, d = 1$, so case 1. $T(n) = \mathcal{O}(n \log n)$

Example: Binary search, $T(n) = T(n/2) + c \cdot n^0$, so $a = 1, b = 2, d = 0$, so case 1. $T(n) = \mathcal{O}(\log n)$

Example: Multiplication algorithm1, $T(n) = 4T(n/2) + c_1 n$, so $a = 4, b = 2, d = 1$, so case 3. $\mathcal{O}(n^{\log_4 2})$

Example: Multiplication algorithm (Karatsuba), $T(n) = 3T(n/2) + c_1 n$, so $a = 3, b = 2, d = 1$, so case 3. $\mathcal{O}(n^{\log_3 2})$

- The calculator uses Strassen Schonenhage: $\mathcal{O}(n \log n \log \log n)$
- New proposed solution: $\mathcal{O}(n \log n)$

Proof of master theorem (Simpler version)

Assumtions:
1. n is a power of b (the shrinkage factor)
2. Base case: $T(1) = c$ (same as n^d)

Main technique:

- Recursion Trees (eww)
  - Levels: $\boxed{\log_b n + 1}$
  - Subproblems at level j: $\boxed{a^j}$
  - Subproblem size at level j: $\boxed{\dfrac{n}{b^j}}$
  - *Total* work done outside recursive calls at level j: $\boxed{a^j \cdot \left(\dfrac{n}{b^j}\right)^d \cdot c} = n^d \left(\frac{a}{b^d}\right)^j \cdot c$

    Should be intuitive from the above equation, Good $= a$, bad $= b^d$, in the end we just sum it all.

– So, work is sum of total work across all levels

$$\sum_{j=0}^{\log_b n+1} n^d \left(\frac{a}{b^d}\right)^j \cdot c$$

Lecture→3

Divide and Conquer Algorithms

1. Divide (break into several parts)
2. Conquer (Solve the smallest solvable)
3. Combine (subproblems)

Counting Inversions in an Array

Input: $1, 3, 5, 2, 4, 6$
Output: 3
Inversion pairs: $(3, 2), (5, 2), (5, 4)$ {kind of like bubble sort}
Golden Benchmark to get inversions: Sorted array (ascending)
Trivial algorithm $= \mathcal{O}(n^2)$
Today: $\mathcal{O}(n \log n)$
Q. Can we output all inversions in same time above? No, total $O(n^2)$ possible invs.

- Key Ideas
  - Suppose A is divided in to X and Y (possibly in the middle)
  - An inversion pair (i, j) is :
    1. Left inversion : Both (i, j) in X
    2. Right inversion : Both (i, j) in Y
    3. Split inversion : i in X and j in Y
  - Using recursion get $(1), (2)$ and after you get the results, count split inversions.
  - Here's the pseudo code

```
CountInv(array& A, length n):
    if n==1 return 0
    X = A[1,2,...n/2], Y=A[n/2+1,...,n]
```

```
    x = CountInv(X,n/2)
    y = CountInv(Y,n/2)
    x = CountSplitInv(X,n/2)
    return x+y+x
Copy
```

- Now, since split inversions wont be affected if we sort each X and Y (along the recursive calls) it'll get easier to count inversions. We will use this to count split inversions while merging. `count += (n/2 - i + 1)` where i is the iterator of X and we are using the combine/merge function. This takes advantage of sorting.

Lecture→4

Closest pair of points in 2D

Input: A set of points with 2 coordinates
Distance (d) between two points is *Euclidean distance* in $2D$
Output: $(a, b) : d(a, b)$ is minimum
Assumption: All points have x and y coordinates (Non distinct left as exercise)

The 1-D case:
Sort the given points, and linearly traverse. So complexity $\mathcal{O}(n \log n)$

Back to 2D:
P_x be the set sorted by x-coordinate
P_y be the set sorted by y-coordinate (independent of other coordinate)
Now we choose the median using P_x. Then we have two sets, Q,R on left and right of the median (assume median on Q).

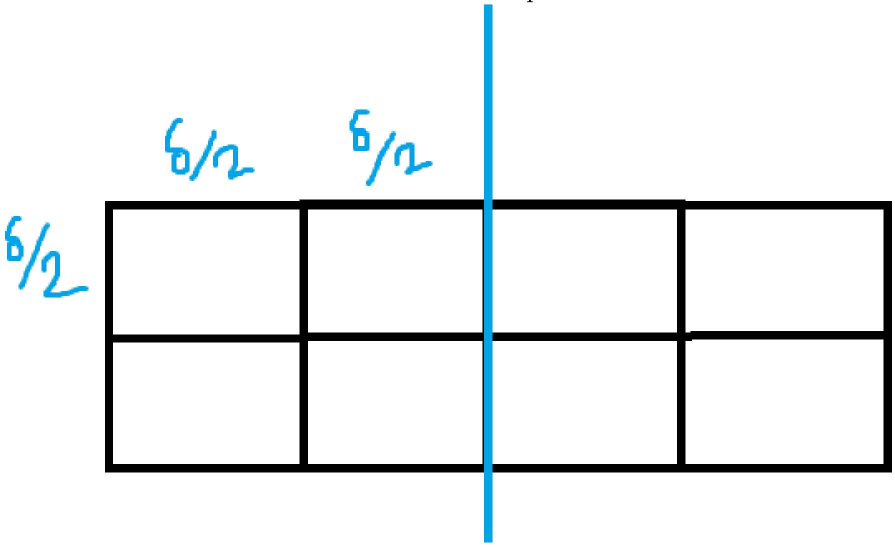
Recursion:

1. Compute Q_x, Q_y, R_x, R_y
2. $(p_1, q_1) = ClosestPair(Q_x, Q_y)$
3. $(p_2, q_2) = ClosestPair(R_x, R_y)$
4. Generate the sets Q_x, Q_y, R_x, R_y in $\mathcal{O}(n)$ time [Exercise]
5. $(p_3, q_3) = ClosestSplitPair(Q, R)$
6. get minimum of all 3

ClosestSplitPair:

1. Our search space will be restricted to $\pm\delta$ where
   $\delta = \min(ClosestPair(Q), ClosestPair(R))$
2. Note that this may not return a correct answer if the closest split pair does not lie in our restricted search space.
3. Calculations, we'll make it $\mathcal{O}(n)$; now assume the set S is the set of points contained in the predefined region.
4. Compute S_y (sorted by y coordinate) in linear time.
5. Traverse the list and apply the 1D algorithm but lookahead **seven** points instead of **one**. So complexity $\mathcal{O}(7n) \subseteq \mathcal{O}(n)$.
   - Proof of correctness of **seven**
   - We use the fact that our search space is restricted by $\pm\delta$ and what is δ

- Therefore each box below will have at most one point



- And also the box height is restricted by delta.
- bdmtish, drumroll, so, we only need to compare with points which may be in these boxes. Therefore a linear algo.

Lecture→5

The selection problem

Input: An array (arbitrary), an integer i
Goal: Get i^{th} smallest number or i^{th} order statistic.

We'll see a linear time selection algo.

```
Select(A, length n, i)
    base: do something idk
    else:
        1. Choose any element as pivot = p
        2. Partition A around p (just like quicksort)
        3. Let j: position of p now
        4. if j==i then return P
        5. if j>i: recurse: return Select(A[1..j-1],j-1,i)
        6. if j<i: recurse: return Select(A[j+1..n], n-j, i-j)
Copy
```

Runtime:
We will analyze the worst case (i.e. we recurse on the larger subarray)

- $T(n) = c.n + T[\max(n-j, j-1)]$
- Worst case: $c \cdot n + c \cdot (n-1) + c \cdot (n-2) \cdots \in \mathcal{O}(cn^2)$

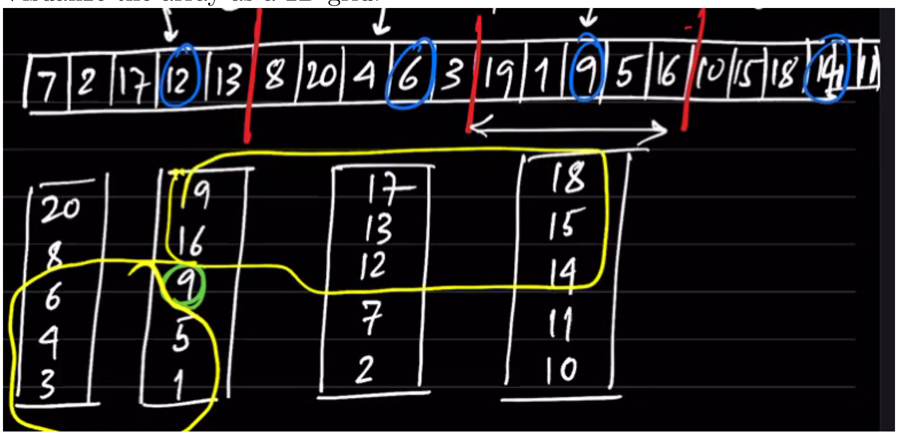
So... `epic algorithm fail`.

How to choose a good pivot fast?

Here are two choices

- Median (Best!) : But ...aren't you solving the median problem xD
- [30%-70%] (Good enough)
  - Randomization: Choose pivot as any element *uniformly at random*
  - Theorem: RSelect works in *expected* $\mathcal{O}(n)$ time
  - Deterministic pivot selection:
    * Break array A into buckets of 5 elements (i.e. $K = n/5$ groups)
    * Find median of each group $= e$ where $|e| = K$
    * pivot $p = Median(e)$
    * Let's re-write the algo:

      ```
      DSelect(A,length n, i)
          base: idk
          else:
              1. Group A into chunks of 5
              2. e = set of medians from each
              3. p = DSelect( e, n/5 = K, K/2 )
              4. ...same as before
      ```
      `Copy`
    * Runtime analysis: We have $T(n) = c \cdot n + T(n/5) + T(?)$
      - Key lemma: Median of e is bigger than (and also smaller than) atleast 30% of the set
      - Proof:
        Visualize the array as a 2D grid.

        Here, pivot is 9. now, let's generalize.

      We have the recurrence $T(n) \leq c \cdot n + T(n/5) + T(0.7 \cdot n)$