GT lecture notes

Graph Theory

These are the notes for graph theory course. The professor says it'll be a simple course, let's see about that. I am using Obsidian and this is an amazing markdown editor! It has a lot of community plugins. Anyways, study now... xD

Here is a somewhat detailed overview.

1. GT/Lecture 1 : Introduction to the course and grading. Defining graphs
2. GT/Lecture 2 : Something more here

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Overview

This was the first lecture of graph theory. The professor is super old and kind lol. Course summary: Proof based and fun questions. Cutoffs are absolute. For A, aim for 80%+ (cutoff at 75%). The remaining grades are out of question anyways. A very interesting thing which came up during the meet is the page rank algorithm.

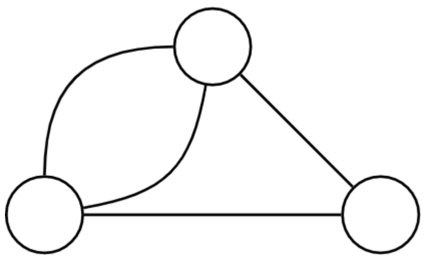
Defining graphs

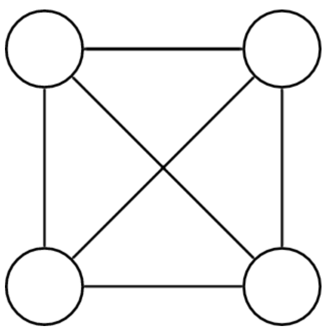
- The normal way: A graph G can be defined as a pair (V, E), where V is a set of vertices, and E is a set of edges between the vertices $E \subseteq \{(u, v) | u, v \in V\}$. If the graph is undirected, the adjacency relation defined by the edges is *symmetric*, or $E \subseteq \{\{u, v\} | u, v \in V\}$ (sets of vertices rather than ordered pairs). If the graph does not allow *self-loops*, adjacency is *irreflexive*.
- The formal way: A graph G is a triplet consisting of:
    - A vertex set $V(G)$ (non empty)

– An edge set $E(G)$, disjoint from the vertex set (could be empty)
– A relation between an edge and a pair of vertices General notation: $|V(G)| = n, |E(G)| = m$. Please adhere to this xD.

A bit of imagery. Note, "we" define graphs such that the vertex set can never be empty. Also, for the scope of the course, our graph shall be finite.

- Loop: An edge whose endpoints are equal
- Multiple edges: Edges having same pair of endpoints
- Simple graph: No loop or multiple edges.





- Finite Graph: a graph whose vertex set and edge set are finite
- Null Graph: a graph whose vertex set and edge set are empty
- Adjacent vertices: v_j and v_k are adjacent if edge e_i is incident upon both v_j and v_i.
- Degree: of a vertex v_k is the number of edges incident upon it. Denoted using $d(v_k)$

Proposition 1:

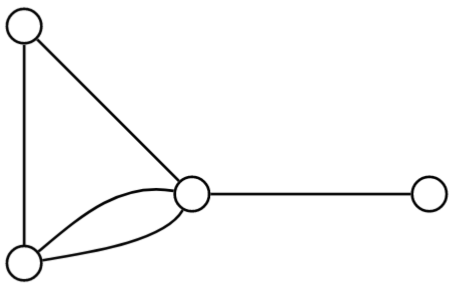
Let G be a graph. Then, the sum of degrees of the vertices is twice the number of edges, i.e.

$$\sum d(v) = 2|E(G)|, v \in V(G)$$

Proof: Each edge contributes 2 degrees (to the vertices of G). To apply it on any graph, we declare that a loop contributes 2 degrees to the vertex.

Adjacency Matrix

- Let $G = (V, E), |V| = n$ and $|E| = m$
- The adjacency matrix of G written $A(G)$, is the $n \times n$ in which entry $a_{i,j}$ is the number of edges in G with endpoints $\{v_i, v_j\}$.



$$\begin{array}{c c} & \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{array}{c} w \\ x \\ y \\ z \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

G a simple graph then, adjacency matrix $A(G)$ is symmetric $(0, 1)$ matrix. An **important result** for real symmetric matrices, they are orthogonally diagonalizable over $\mathbb{R}$.

Incidence Matrix

Let $G = (V, E), |V| = n$ and $|E| = m$. The incidence matrix $M(G)$ is $n \times m$ matrix in which entry $m_{i,j}$ is 1 if v_i is an endpoint of e_i and otherwise 0.

$$\begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} w \\ x \\ y \\ z \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

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Complement (Simple Graph)

Complement of G: The complement G' of a simple graph G:

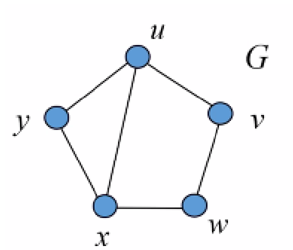
- $V(G') = V(G)$
- $E(G') = \{uv | uv \notin E(G)\}$: Every edge is determined by it's endpoints (u, v).

Subgraph

- $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and,
- The assignment of endpoints to edges in H is the same as in G.
- Induced subgraph: Whenever the vertices of $H(G)$ have **all** the edges in $E(G)$ corresponding to all vertices $H(G)$.

Clique and Independent Set

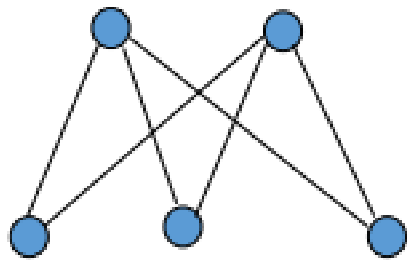
- Complete Graph: A simple graph whose vertices are pairwise adjacent.
- We use the notation K_n to denote a complete graph of n vertices.
  - $\binom{n}{2}$ edges
  - Complement of K_n is trivial (has no edges)
  - Induced subgraph: smaller or equal complete graph
- A **Clique** in a graph is a set of pairwise adjacent vertices (a complete graph). (more like a complete subgraph)
- An **Independent set** in a graph: a set of pairwise non-adjacent vertices.
- Example:
  - $\{x, y, u\}$ is a clique in G



  - $\{u, w\}, \{v, y\}$ is an independent set.
  - Question: The largest possible independent set? [Later]
  - Question: The largest clique in a graph? (Not an interesting problem though lol)

Bipartite Graphs

- A graph G is bipartite if $V(G)$ is the union of two disjoint independent sets called partite sets of G.
- Also: The vertices can be partitioned into two sets such that each set is independent
- The Matching Problem
- The Job Assignment Problem
- Complete bipartite graph (biclique) is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets.



A complete bipartite graph with partite sets of size r and s is denoted by $K_{r,s}$.
A graph G is k-partite if $V(G)$ is a union of k independent sets.

Path and Cycle

- Path: A sequence of distinct vertices such that two consecutive vertices are adjacent. E.g. (a,b,c,d,e)
- Cycle: A closed path (Only repeated vertex is the first which is same as last) E.g. (a,d,c,b,e,a)

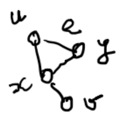
Walk and Trail

- A walk of length k is sequence of $v_0, e_1, v_1, e_2 \ldots, e_k, v_k$ of vertices and edges such that $e_i = v_{i-1}v_i$ for all i.
  - A trail is a walk with NO repeated edge
  - A path is a walk with no repeated vertex
  - A U, V-walk or U, V-trail has first vertex U and last vertex V and these are the endpoints
- A walk is closed if it has length at least one and its endpoints are equal.
  - A cycle is closed trail in which "first=last" is the only repetition
  - A loop is a cycle of length one

Proposition 2

Every u, v-walk contains a u, v-path. Proof by strong induction:

- Use induction on the length of a u, v-walk W.
- Basis step: $l = 0$
  - Having no edge, W consists of a single vertex ($u = v$)
  - This vertex is a u, v-path of length 0.
- Induction step : $l \geq 1$
  - Suppose that the claim holds for walks of length less than l
  - If W has no repeated vertex, then its vertices and edges form a u, v-path. Here's an illustration



$u\,x\,y\,\overset{e}{u}\,\overset{e}{y}\,u\,x\,y\,u\,x\,v \to W$

$u\,x\,y\,u \text{ - - - - - } \to W'$

$\ell(W') < \ell(W) \quad \ell - 2$

- If W has a repeated vertex w, then deleting the edges and vertices between appearances of w (leaving one copy of w) yields a shorter u, v-walk W' contained in W.
  - By the inductive hypothesis, W' contains a u, v-path P, and this path P is contained in W.