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Introduction

- In this project we discuss about mathematical models for the future proliferation in low earth orbit (LEO) of space debris that is created by collisions of small objects with artificial satellites or other "big" orbiting bodies.
- The model predicts that such collisional generation of fragments will be the dominant source of debris in a few decades. Subsequently, if the satellite launch rate remains comparable with the current one, then the number of satellites will reach a peak in about 150 years and then rapidly decline by about a factor of 10.
- On the other hand, space junk will continue to grow, eventually creating a debris belt around the earth which would affect space activities and the use of satellites in specific orbital ranges.
- Some alternative choices of model parameters (for example, to account for uncertainty in the projectile-to-target mass ratio required for breakup and the rate of future injection of material into orbit and intentional generation of debris) show that this evolution may be anticipated or delayed, but not qualitatively modified, unless ad hoc measures to avoid or limit collisions are adopted.
- The modified model predicts that the population of debris and satellites will reach a certain steady value after some years. Then mitigation techniques can be employed to clean up the orbit.

The Standard Mathematical Model

We assume that there are just two populations of orbiting bodies: ***N*** satellites, objects of cross section on the order of a few square meters and of mass of hundreds of kilograms; and ***n*** fragments, i.e., small bodies capable of causing catastrophic breakup when impacting a satellite. In our baseline case, the fragment population is characterized by typical sizes of order of 1 centimetre and masses of a few grams (as we mentioned above, at a collision velocity of 10 km s^{-1} a projectile-to-target mass ratio of 10^{-5} is enough to shatter most natural solid targets.)

To study the interaction of these two populations, we adopt a collision rate proportional (through a constant coefficient ***x***) to the product ***nN***; when a collision occurs, one satellite is destroyed and new "collisional" fragments are created. Finally, we assume that every year ***A*** new satellites enter orbit (***A*** should be interpreted as the difference between the number of satellites launched and the number re-entering the atmosphere); at the same time, "primary" orbiting fragments are created during the launch and satellite delivery operations. These assumptions lead to the pair of first-order differential equations:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = A - xnN \tag{1}$$

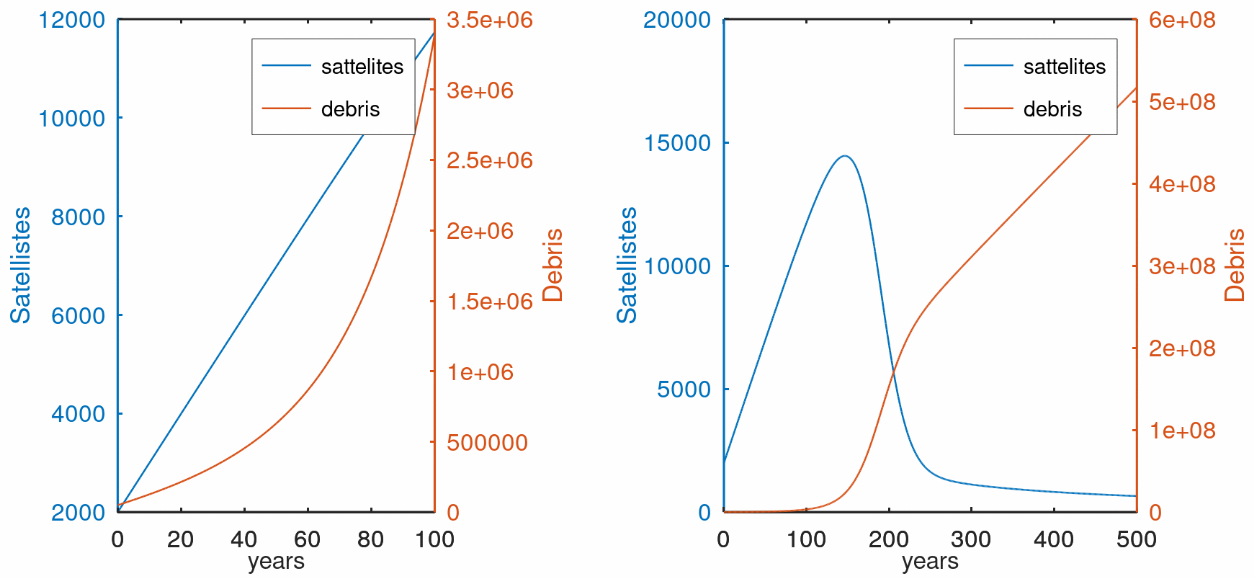
$$\frac{\mathrm{d}n}{\mathrm{d}t} = \beta A + \alpha xnN \tag{2}$$

Defining the parameters:

1) **$A=100$**. Current rate of insertion of satellites into the orbit.
2) **$x= 3x10^{-10}$**. (Collision rate constant) We assume that at a collision velocity 10km/s, objects of size ≥ 1cm can cause damage.
3) **$\alpha=10^{4}$**. This follows from the typical mass distribution of fragments generated in hypervelocity impacts.
4) **$\beta=70$**. Most of the "primary" fragments are generated in explosions during or after launches, involving rockets or second/third stages. The vast majority of the existing debris probably has this origin.
5) **$N(0)=2x10^{3}$**. It includes rockets/satellites in low earth orbit at time t=0.
6) **$n(0)=5x10^{4}$**. Number of fragments of size ≥ 1cm at time t=0.

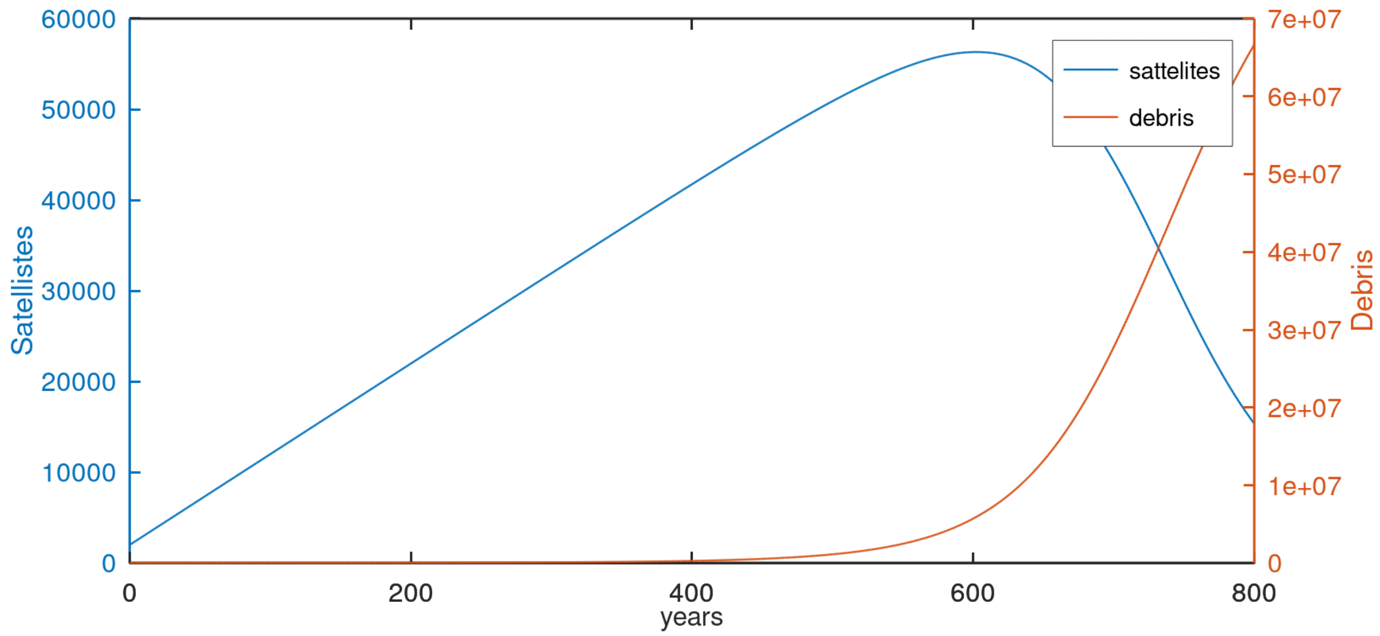
Observations

- At the beginning ***xnN<<A*** and ***xnN<< βA*** so collisions are not important and the population growth is almost linear for both satellites and fragments.
- When ***nN*** ≥ **(β*A*/ *αx*)** which happens at t ≈ 40 years, the generation of collisional fragments exceeds primary ones and the growth of fragments becomes exponential.
- At about t=150 years, ***nN ≈ A/x*** , and the number of satellites is maximum(N ≈ 15000). Before this ***dN/dt*** was positive indicating an increase in number of satellites. But after t=150 years it becomes negative and the number of satellites falls rapidly implying that more satellites are shattered than launched. It starts decaying exponentially.
- Finally at t ≈ 300 years, abundance of satellites stabilizes around 10^3 (but with a slow, continuous decline) and after this any material launched into orbit is totally converted into fragments by collisions. On the other hand, the number of fragments reaches $3\text{x}10^8$.



Variation in the Model

- Let's say a new policy is made for space missions such that the satellites are modified in a way that no new primary fragments are inserted are into the orbit, i.e. **β=0.**
- We also test a hypothesis that satellites can be collision-ally disrupted only by projectiles larger and more massive than we have assumed so far. **n(0)=0**, **$\alpha=10^4$**



Conclusions

- In the coming years the number of fragments in the orbit will increase to an alarming amount which could disrupt space activities.
- After a certain period of time all the satellites (even the newly launched) will completely be converted to debris.
- With the new policy in which no new primary fragments are inserted, we are able to delay the decline in number of satellites and the exponential growth of debris but still we are not able to prevent it.
- So, unless and until new measures are not adopted, space missions will become impossible in the coming years.

Limitations of the Model

1) The value of rate of insertion of satellites A does not consider the cyclic nature because of the 11 year solar cycle and hence needs correction.
2) The model doesn't consider the damage caused to the satellites because of their mutual collision.
3) The model doesn't consider the mutual collisions of the fragments.
4) Few parameters, like the initial conditions etc have uncertainty by about a factor of 2.

Modification of the Standard Model

$$\frac{dN}{dt} = \left(a + b\sin(ct+d)\right) - \frac{N}{f + g\sin(ht+k)} - xnN - 2yN^2 \tag{3}$$

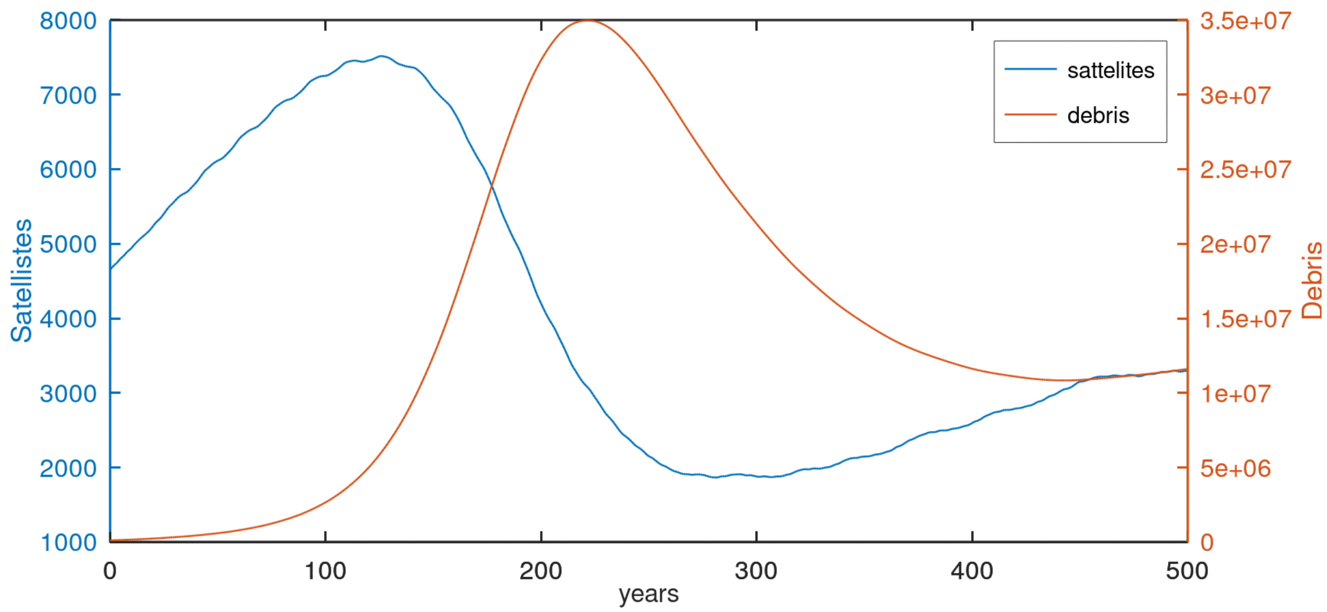
$$\frac{dn}{dt} = \beta\left(a + b\sin(ct+d)\right) - \frac{n}{p + q\sin(ht+k)} + \alpha xnN + \gamma yN^2 - 2zn^2 \tag{4}$$

- The ***(a+bsin(ct+d))*** term is the sinusoidally varying equivalent to the ***A*** term from the standard model. It indicates the global satellite launch rate to LEO.
- In the above equations we see negative terms proportional to **N** and **n** respectively. These account for annual re-entry of objects from orbit. The periodic variation is because of the 11-year solar cycle.
- Eq.(3) contains an extra term **2yN²**, representing the number of intact satellites lost due to the collision between them. Eq.(4) contains the corresponding term **γyN²**, representing the fact that such collisions produce debris fragments.
- Eq.(4) contains an additional term **2zn²**, for the rate of destruction of debris fragments due to collision between themselves.

Coefficient	Point Estimate	Units	Coefficient	Point Estimate	Units
a	31.41	satellites / year	x	6.895×10^{-10}	year $^{-1}$ · fragment $^{-1}$
b	7.794	satellites / year	y	1.369×10^{-9}	year $^{-1}$ · satellite $^{-1}$
c	1.935	radians / year	z	2.869×10^{-14}	year $^{-1}$ · fragment $^{-1}$
d	0.1680	radians	α	10,000	fragments / satellite
f	14,420	years	β	70	fragments / satellite
g	-10,430	years	γ	56,000	fragments / satellite
h	0.5712	radians / year			
k	-0.9996	radians	N_0	4,650	satellites
p	184.9	years	n_0	110,400	fragments
q	-137.9	years			

Observations

- For the initial years the results are very similar to the standard model.
- at t ≈ 120 years the number of satellites reaches its maximum value which is approximately N ≈ 7000. After this the number of satellites decrease rapidly.
- One very important observation is that number of satellites stabilizes around 1100 satellites.
- The population of debris grows till 250 years and start declining afterwards and also attains a stable value of around 41.12 million fragments.
- Once both the debris and satellites attain their stable values, they neither increase nor decrease with time.



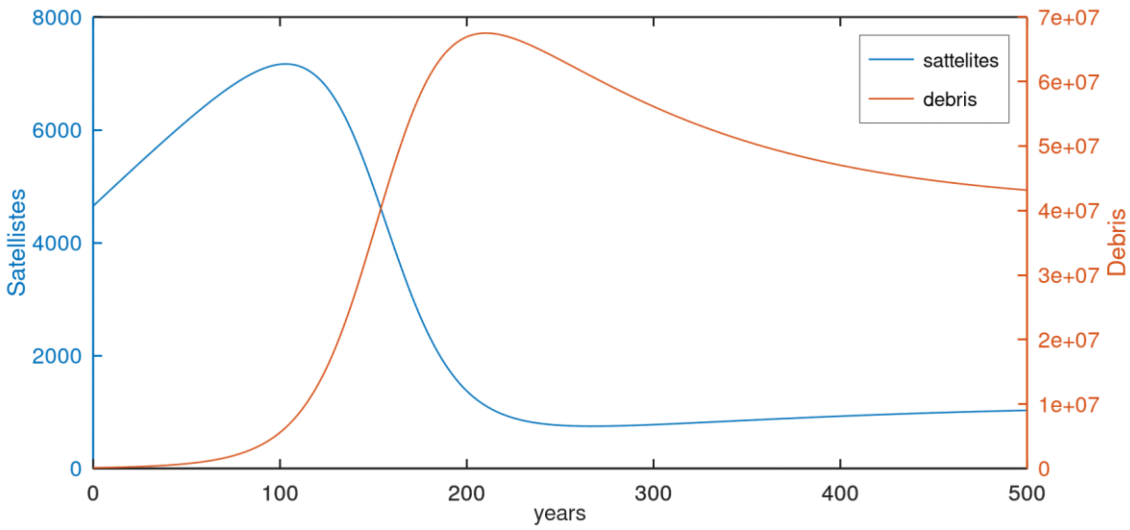
Steady State Analysis

To find the equilibrium solutions we will do Jacobian analysis. The previous set of differential equations is non-autonomous and hence finding its equilibrium solution is complicated. So instead, we modify the differential equations to produce a system of autonomous equations:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = a - xnN = f(N, n)$$

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \beta a - \frac{n}{p} + \alpha xnN = g(N, n)$$

Plot of the new system of equations:



The value of p (average time constant for orbital debris re-entry) is changed to 130 years. All other parameters are unchanged.

The equilibrium solution is obtained by putting f(N,n) = 0 and g(N,n) = 0.

We obtain them as:

n= ap(α+ β)=1107.9

N= 1/xp(α+ β)=41.12x10^{6}

The Jacobian matrix is given by:

$$\begin{bmatrix} N' \\ n' \end{bmatrix} = \begin{bmatrix} \partial f/\partial N & \partial f/\partial n \\ \partial g/\partial N & \partial g/\partial n \end{bmatrix} \begin{bmatrix} N \\ n \end{bmatrix}$$

$$\begin{bmatrix} N' \\ n' \end{bmatrix} = \begin{bmatrix} -xap(\alpha+\beta) & \frac{-1}{p(\alpha+\beta)} \\ \alpha xap(\alpha+\beta) & \frac{1}{p}(\frac{\alpha}{\alpha+\beta}-1) \end{bmatrix} \begin{bmatrix} N \\ n \end{bmatrix}$$

Substituting the values in the Jacobian matrix we get this:

```
 -2.8351e-02  -7.6388e-07
  2.8351e+02  -5.3472e-05
```

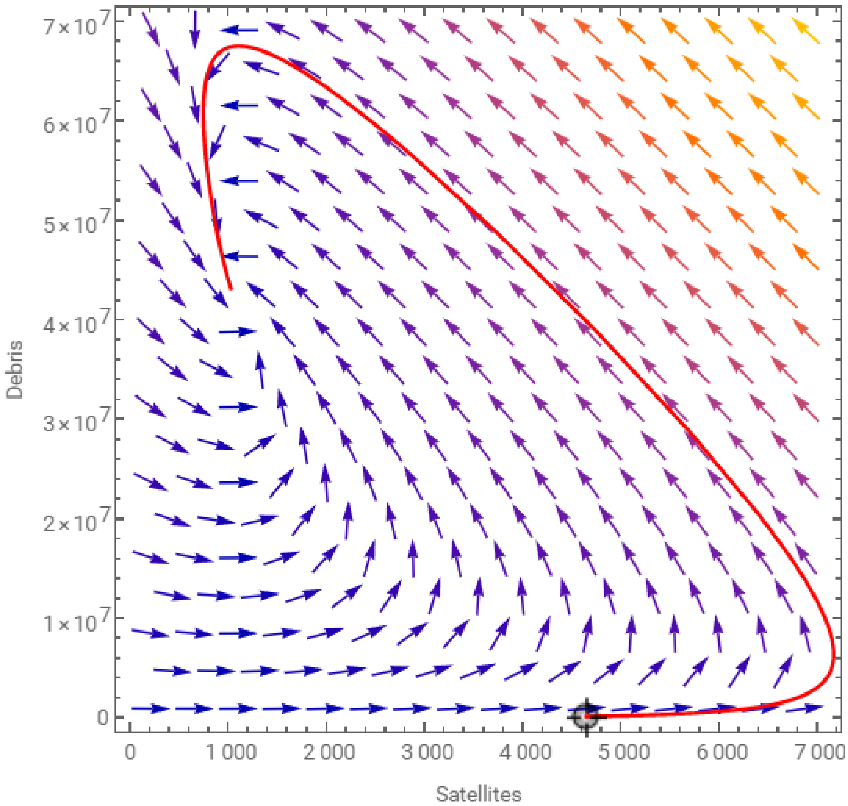
The eigenvalues of the matrix are:

```
eigenvalues =

  -0.014202 + 0.004047i
  -0.014202 - 0.004047i
```

Clearly the eigenvalues are complex with a negative real part. Hence the equilibrium point is a spiral sink.

The phase portrait with vector field and the trajectory is drawn as :



Clearly, as expected we obtain a spiral sink. The red line is the trajectory of the solution of the system of differential equations with the initial conditions specified already. The trajectory ends up at the equilibrium point (at the sink) which is in accordance to the calculations and the graphs obtained.

Conclusions

- The new model shows that the number of satellites in the orbit will reach a certain stable value. This means in future if we were to increase the number of satellites we won't be able to do so. This is a matter of concern and appropriate measures should to be taken.
- Restricting the insertion of primary fragments can only delay but not prevent the proliferation of the debris fragments.
- However, one good thing is that the number of debris fragments will also decrease and reach a certain stable value and won't increase further. But still its population is large enough that it would impact future space missions.
- So in order to conduct space missions without worrying about the threat posed by debris we would want to decrease its population.
- Common proposals for mitigation strategies involve (1) launching "space tugs" to deorbit intact but inoperative spacecraft or (2) launching giant "nets" of aerogel or similar material to catch or slow debris fragments. The first strategy indirectly slows the formation of fragments by removing one of their sources, while the second strategy has a direct effect on removing fragments from orbit.

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4) https://en.wikipedia.org/wiki/Kessler_syndrome

5) https://en.wikipedia.org/wiki/Stability_theory

6) https://www.jirka.org/diffyqs/