qFunctions Package usage in the “Factorial Basis Method for q -Series Applications” Paper

In[1]:= `<< qFunctions.m`

`In order to use the full functionality you have to load the HolonomicFunctions package.`

qFunctions by Jakob Ablinger and Ali K. Uncu – RISC – Version 1.1 (03 / 11 / 23) Help

Example 5.2 (Identity 3.38 of Sills [21])

Guessing - Proving a shorter recurrence operator for a' (k) :

First we input the annihilating operator applied to a’(k) in Mathematica (a’(k) is denoted by a[k] here):

In qFunctions all recurrences are implicitly set equal to 0.

In[2]:= recAkPrimeApplied :=

$$\left(-q^{5+2k}+q^{6+3k}+q^{7+3k}-q^{8+4k}\right) a[k] + \left(q^{6+2k}+q^{7+2k}-q^{8+3k}-q^{9+3k}\right) a[1+k] + \left(q-q^{2+k}-q^{3+k}+q^{4+2k}-q^{8+2k}\right) a[2+k] + \left(-1-q+q^{2+k}+q^{3+k}\right) a[3+k] + a[4+k]$$

List 20 initial values:

In[3]:= list20InitialValues =

```
qREToList[recAkPrimeApplied, a[k], {0, {1, q, q^4, q^7, q^12}}, 20]
```

Out[3]= $\{1, q, q^4, q^7, q^{12}, q^{17}, q^{24}, q^{31}, q^{40}, q^{49}, q^{60}, q^{71}, q^{84}, q^{97}, q^{112}, q^{127}, q^{144}, q^{161}, q^{180}, q^{199}\}$

Guess a recurrence with order 2 by ansatz:

(One can see free variables arise if they bounds 2,{2,4} are changed for larger values.)

In[4]:= GuessedRecFromInitialConditions =

```
GuessqRecurrence[list20InitialValues, a[k], 2, {2, 4}]
```

Out[4]= $q^{4+2k} a[k] - a[2+k]$

We calculate 20 initial values of this sequence with the same initial values as the original a[k]:

In[5]:=
```
list20InitialValuesFromGuessed = qREToList[recAkPrimeApplied, a[k], {0, {1, q}}, 20]
```

Out[5]= $\{1, q, q^{4}, q^{7}, q^{12}, q^{17}, q^{24}, q^{31}, q^{40}, q^{49}, q^{60}, q^{71}, q^{84}, q^{97}, q^{112}, q^{127}, q^{144}, q^{161}, q^{180}, q^{199}\}$

And show that it is equal to the previously calculated initial values:

In[6]:=
```
list20InitialValues == list20InitialValuesFromGuessed
```

Out[6]= True

Prove that the GCD of the high-order (from Factorial basis method) and low-order (guessed) recurrences is the guessed recurrence.

In[7]:=
```
qREGCD[recAkPrimeApplied, GuessedRecFromInitialConditions, a[k]]
```

Out[7]= $-q^{4+2k}\, a[k] + a[2+k]$

There is a sign of the guessed recurrence and the GCD calculated here, but this is unimportant. Since all recurrences are implicitly equal to 0, we can conclude they are the same recurrence relation for a[k].

Example 5.5 (Proof of Theorem 1.4)

Guessing - Proving a shorter recurrence operator for \hat{c}(k) :

First we input the annihilating operator applied to \hat{c}(k) in Mathematica (a’(k) is denoted by c[k] here):

In qFunctions all recurrences are implicitly set equal to 0.

In[8]:=

$$\begin{aligned}\text{recCHatIPrimeApplied} := {} & \left(-q^{2+3k}+q^{3+4k}+q^{4+4k}-q^{5+5k}\right) c[k] + \\ & \left(-q^{2+2k}+q^{3+3k}+2\,q^{4+3k}+q^{5+3k}-q^{5+4k}-q^{6+4k}-q^{7+4k}\right) c[1+k] + \\ & \left(q-q^{2+k}-q^{3+k}+q^{3+2k}+2\,q^{4+2k}-q^{5+3k}-q^{6+3k}-q^{7+3k}\right) c[2+k] + \\ & \left(-1-q+q^{2+k}+q^{3+k}-q^{5+2k}\right) c[3+k] + c[4+k]\end{aligned}$$

List 20 initial values:

In[9]:=
```
list20InitialValues =
  qREToList[recCHatIPrimeApplied, c[k], {0, {1, -1, 0, -q^4, -q^9}}, 20];
```

Guess a recurrence with order 2 by ansatz:

(One can see free variables arise if they bounds 2,{3,1} are changed for larger values.)
We check up to the q^101 precision.

In[10]:=
```
GuessedRecFromInitialConditions =
 GuessqRecurrence[list20InitialValues, c[k], 2, {3, 1}, ExpansionOrder → 100]
```

Out[10]= $-q^{1+3k}\, c[k] - q^{1+2k}\, c[1+k] + c[2+k]$

We calculate 20 initial values of this sequence with the same initial values as the original c[k]:

In[11]:=
```
list20InitialValuesFromGuessed =
  qREToList[recCHatIPrimeApplied, c[k], {0, {1, -1}}, 20];
```

And show that it is equal to the previously calculated initial values:

In[12]:=
```
list20InitialValues == list20InitialValuesFromGuessed
```

Out[12]= True

Prove that the GCD of the high-order (from Factorial basis method) and low-order (guessed) recurrences is the guessed recurrence.

In[13]:=
```
qREGCD[recCHatIPrimeApplied, GuessedRecFromInitialConditions, c[k]]
```

Out[13]= $-q^{1+3k}\, c[k] - q^{1+2k}\, c[1+k] + c[2+k]$

There is a sign of the guessed recurrence and the GCD calculated here, but this is unimportant. Since all recurrences are implicitly equal to 0, we can conclude they are the same recurrence relation for c[k].