

This is a part of SFQED-Loops script collection
developed for calculating loop processes in
Strong-Field Quantum Electrodynamics.

The scripts are available on <https://github.com/ArsenyMironov/SFQED-Loops>

If you use this script in your research, please, consider
citing our papers:

- A. A. Mironov, S. Meuren, and A. M. Fedotov, PRD 102, 053005 (2020),
<https://doi.org/10.1103/PhysRevD.102.053005>

- A. A. Mironov, A. M. Fedotov, arXiv:2109.00634 (2021)

If you have any questions, please, don't hesitate to contact:
mironov.hep@gmail.com

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$$\text{wavy line with shaded circle} = \text{wavy line} + \text{wavy line with } 1\text{PI} \text{ circle} + \text{wavy line with } 1\text{PI } 1\text{PI circles} + \dots$$

```
In[1]:= NotebookEvaluate[NotebookOpen[NotebookDirectory[] <> "definitions.nb"]]
```

FeynCalc 9.3.1 (stable version). For help, use the documentation center, check out the wiki or visit the forum.

To save your and our time, please check our FAQ for answers to some common FeynCalc questions. See also the supplied examples. If you use FeynCalc in your research, please cite

- V. Shtabovenko, R. Mertig and F. Orellana, Comput.Phys.Commun. 256 (2020) 107478, arXiv:2001.04407.
- V. Shtabovenko, R. Mertig and F. Orellana, Comput.Phys.Commun. 207 (2016) 432–444, arXiv:1601.01167.
- R. Mertig, M. Böhm, and A. Denner, Comput. Phys. Commun. 64 (1991) 345–359.

Exact photon propagator in momentum representation

$$D^c_{\mu\nu}(\mathbf{l}) = D_0(\mathbf{l}^2, \chi_{\mathbf{l}}) g_{\mu\nu} + D_1(\mathbf{l}^2, \chi_{\mathbf{l}}) \epsilon_{\mu}^{(1)}(\mathbf{l}) \epsilon_{\nu}^{(1)}(\mathbf{l}) + D_2(\mathbf{l}^2, \chi_{\mathbf{l}}) \epsilon_{\mu}^{(2)}(\mathbf{l}) \epsilon_{\nu}^{(2)}(\mathbf{l});$$

\mathbf{l}^{μ} – the photon propagator 4 – momentum;

$$\chi_{\mathbf{l}} = \frac{e}{m^3} \sqrt{-(F_{\mu\nu} \mathbf{l}^{\nu})^2};$$

$$\epsilon_{\mu}^{(1)}(\mathbf{l}) = \frac{e F_{\mu\nu} \mathbf{l}^{\nu}}{m^3 \chi_{\mathbf{l}}};$$

$$\epsilon_{\mu}^{(2)}(\mathbf{l}) = \frac{e F_{\mu\nu}^* \mathbf{l}^{\nu}}{m^3 \chi_{\mathbf{l}}}; \quad F^{*\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\delta\lambda} F_{\delta\lambda};$$

$$(\epsilon^{(i)}(\mathbf{l}))^2 = -1;$$

$$D_0(\mathbf{l}^2, \chi_{\mathbf{l}}) = \frac{-i}{\mathbf{l}^2 - \mathbf{l}^2 \hat{\Pi}}, \quad D_{1,2}(\mathbf{l}^2, \chi_{\mathbf{l}}) = \frac{i \Pi_{1,2}}{(\mathbf{l}^2 - \mathbf{l}^2 \hat{\Pi})(\mathbf{l}^2 - \mathbf{l}^2 \hat{\Pi} - \Pi_{1,2})};$$

$$\mathbf{l}^2 \hat{\Pi} = \mathbf{l}^2 \hat{\Pi}(\mathbf{l}^2, \chi_{\mathbf{l}}),$$

$\Pi_{1,2} = \Pi_{1,2}(\mathbf{l}^2, \chi_{\mathbf{l}})$ – polarization operator eigenfunctions;

Our goal : exact photon propagator in coordinate representation

$$D^c_{\mu\nu}(x) = \frac{\Lambda^{4-D}}{(2\pi)^D} \int d^D \mathbf{l} D^c_{\mu\nu}(\mathbf{l}) e^{-i\mathbf{l}x};$$

Let's define photon momentum and the coordinate variables

```
In[2]:= NewMomentum["l"]
NewCoordinate["x"]
```

$$\left\{ l^\alpha, l^\beta, k \cdot l, F l^\alpha, F F l^\alpha, F D l^\alpha, a \cdot l, 0, 0, 0, -a^2 (k \cdot l), 0, 0, -\frac{m^6 \chi l^2}{e^2}, -\frac{m^6 \chi l^2}{e^2}, \frac{m^6 \chi l^2}{e^2}, 0, 0, 0, 0, 0, 0 \right\}$$

$$\left\{ x^\alpha, x^2, k \cdot x, a \cdot x, F x^\alpha, F F x^\alpha, F D x^\alpha, k \cdot x, 0, 0, 0, -a^2 (k \cdot x), 0, 0, -\frac{m^2 \xi^2 (k \cdot x)^2}{e^2}, -\frac{m^2 \xi^2 (k \cdot x)^2}{e^2}, \frac{m^2 \xi^2 (k \cdot x)^2}{e^2}, 0, 0, 0, 0, 0, 0 \right\}$$

Eigenvectors and tensor structures

We intentionally leave tensors and vectors uncontracted

```
In[4]:=  $\epsilon 1[\mu_, \nu_] = e \text{ Ft}[\mu, \nu] \times \text{lv}[\nu] / m^3 / \chi$ 
 $\epsilon 2[\mu_, \nu_] = e \text{ FDt}[\mu, \nu] \times \text{lv}[\nu] / m^3 / \chi$ 
 $T0[\mu_, \nu_] = \text{MTD}[\mu, \nu]$ 
 $T1[\mu_, \nu_] = \epsilon 1[\mu, \alpha] \times \epsilon 1[\nu, \beta]$ 
 $T2[\mu_, \nu_] = \epsilon 2[\mu, \alpha] \times \epsilon 2[\nu, \beta]$ 
Contract[Contract[T1[\mu, \mu]] /. FieldSubstitutions]
Contract[Contract[T2[\mu, \mu]] /. FieldSubstitutions]
```

Out[4]=
$$\frac{e l^\nu F(\mu, \nu)}{m^3 \chi}$$

Out[5]=
$$\frac{e l^\nu \text{FD}(\mu, \nu)}{m^3 \chi}$$

Out[6]=
$$g^{\mu \nu}$$

Out[7]=
$$\frac{e^2 l^\alpha l^\beta F(\alpha, \mu) F(\beta, \nu)}{m^6 \chi^2}$$

Out[8]=
$$\frac{e^2 l^\alpha l^\beta \text{FD}(\alpha, \mu) \text{FD}(\beta, \nu)}{m^6 \chi^2}$$

Out[9]= -1

Out[10]= -1

We write $D_{\mu\nu}^c$ in the following form

$$\int d^D l \text{ [Coeff * Matrix * Exp (i Phase)]},$$

where

Coeff – is a general multiplier for all terms,

Matrix – tensor part,

Phase – total phase of the expression,

We assume

$$Dk = Dk(l^2, \chi_l)$$

In[11]:= **Coeff** = $\Lambda^{(4-D)/(2\pi)^D}$

Matrix = $D0 * T0[\mu, \nu] + D1 * T1[\mu, \nu] + D2 * T2[\mu, \nu]$

Phase = $-\text{Contract}[xv[\alpha] \times lv[\alpha]]$

Out[11]= $(2\pi)^{-D} \Lambda^{4-D}$

$$\text{Out[12]} = \frac{D1 \, e^2 \, l^\alpha \, l^\beta \, F(\alpha, \mu) \, F(\beta, \nu)}{m^6 \, \chi l^2} + \frac{D2 \, e^2 \, l^\alpha \, l^\beta \, FD(\alpha, \mu) \, FD(\beta, \nu)}{m^6 \, \chi l^2} + D0 \, g^{\mu\nu}$$

Out[13]= $-(l \cdot x)$

We need to calculate the integrals of two types

$$\int d^D l \, D_0(l^2, \chi_l) e^{-i l x};$$

and

$$\int d^D l \, l_\alpha l_\beta D_{1,2}(l^2, \chi_l) e^{-i l x} = i \frac{\partial}{\partial x^\alpha} i \frac{\partial}{\partial x^\beta} \int d^D l \, D_{1,2}(l^2, \chi_l) e^{-i l x};$$

Symbol $d_{\alpha,\beta}$ means that we need to differentiate the expression later

Let us now change the variables

$$l_m = l_- = l_0 - l_3;$$

$$l_p = l_+ = \frac{l_0 + l_3}{2};$$

$$l_t = l_{\perp} \text{ - transverse components of } l \text{ (in } D=4 \text{ } l_{\perp} = (l_1, l_2));$$

$$l^2 = l^2$$

$$l^2 = 2 l_- l_+ - l_{\perp}^2;$$

Proper time

$$s = x_- / 2 l_- = kx / 2 kl;$$

$$l_- = x_- / 2 s;$$

$$l_+ = (l^2 + l_{\perp}^2) / 2 l_- = s (l^2 + l_{\perp}^2) / x_- \text{ -- expressed via } l^2;$$

Hereinafter

$$x_m = x_- = x_0 - x_3;$$

$$x_p = x_+ = \frac{x_0 + x_3}{2};$$

$$x_t = x_{\perp}$$

Change of variables

$$l^{\mu} \rightarrow \{l_-, l_+, l_{\perp}\} = \left\{ \frac{x_-}{2s}, \frac{s}{x_-} (l^2 + l_{\perp}^2), l_{\perp} \right\} \rightarrow \{s, l^2, l_{\perp}\}$$

New integration measure

$$d^D l \dots = \frac{d^4 s}{2 |s|} d l^2 d^{D-2} l_{\perp}$$

In[14]:= **(*Checking Jacobian*)**

D[{xm / 2 / s, s (lv2 + lt2) / xm}, {{s, lv2}}]

Jac = Abs[Det[%]]

Out[14]=
$$\begin{pmatrix} -\frac{x_m}{2 s^2} & 0 \\ \frac{l^2 + l_t^2}{x_m} & -\frac{s}{x_m} \end{pmatrix}$$

Out[15]=
$$\frac{1}{2 |s|}$$

```
In[16]:= Coeff1 = Coeff * Jac
Matrix1 = Matrix /. {lv[α_] → dα}
Phase1 = Phase;

Out[16]= 
$$\frac{2^{-D-1} \pi^{-D} \Lambda^{4-D}}{|s|}$$


Out[17]= 
$$\frac{D1 e^2 d_\alpha d_\beta F(\alpha, \mu) F(\beta, \nu)}{m^6 \chi l^2} + \frac{D2 e^2 d_\alpha d_\beta FD(\alpha, \mu) FD(\beta, \nu)}{m^6 \chi l^2} + D0 g^{\mu \nu}$$

```

Integration over

$\int d^{D-2} l_\perp \dots$

$$I_0 = \int d^{D-2} l_\perp \text{Exp}[-I A l_\perp^2 + I (J \cdot l_\perp)] =$$

$$= \text{Exp}\left[-I \frac{\pi}{2} \frac{D-2}{2}\right] \pi^{\frac{D-2}{2}} (\det A)^{-\frac{1}{2}} \text{Exp}\left[I \frac{1}{4} J \cdot A^{-1} \cdot J\right]$$

where

$$A = s,$$

$$J = x_\perp,$$

$$\det A = s^{D-2},$$

$$A^{-1} = 1/s$$

In effect,

integration results into multiplication of Coeff by I_0 and changing Phase

```
In[19]:= Coeff2 = Coeff1;
Matrix2 = Matrix1;
Phase2 = Expand[Phase1 /. {Pair[Momentum[x, D], Momentum[l, D]] → xm lp + xp lm - xt lt} /.
  {lm → xm / 2 / s, lp → s (lv2 + lt^2) / xm} /. {xp → (xv2 + xt^2) / 2 / xm}]

Out[21]= 
$$-l^2 s + lt^2 (-s) + lt xt - \frac{x^2}{4 s} - \frac{xt^2}{4 s}$$


In[22]:= Amatr = -Coefficient[Phase2, lt^2]
J = Coefficient[Phase2, lt]
CI0 = Exp[-I Pi / 2 (D / 2 - 1)] Pi^(D / 2 - 1) / Amatr^(D / 2 - 1)

Out[22]= s

Out[23]= xt

Out[24]= 
$$e^{-\frac{1}{2} i \pi \left(\frac{D}{2}-1\right)} \pi^{\frac{D}{2}-1} s^{1-\frac{D}{2}}$$

```

```
In[25]:= Coeff3 = Coeff2 * CI0
Matrix3 = Matrix2;
Phase3 = Expand[(((Phase2 /. {lt -> 0}) + 1 / 4 J^2 / Amatr)]
```

$$\text{Out[25]} = \frac{2^{-D-1} e^{-\frac{1}{2} i \pi \left(\frac{D-1}{2}\right)} \pi^{-\frac{D-1}{2}} \Lambda^{4-D} s^{1-\frac{D}{2}}}{|s|}$$

$$\text{Out[27]} = l^2(-s) - \frac{x^2}{4s}$$

The integrals

$$\int_{-\infty}^{\infty} \frac{d\tau}{2|\tau|} e^{-i \frac{\tau^2}{4s}} \int_{-\infty}^{\infty} d\mathbf{l}^2 D_k(\mathbf{l}^2, \chi\mathbf{l}) e^{-i \mathbf{l}^2 s}$$

will remain

Let us introduce

$$J_k = J_k(s) =$$

$$-i \int_{-\infty}^{\infty} d\mathbf{l}^2 D_k(\mathbf{l}^2, \chi\mathbf{l}) e^{-i \mathbf{l}^2 s} \quad \text{-- note that this integral is dimensionless}$$

It can be shown that $J_k(s \leq 0) = 0$

```
In[28]:= Coeff4 = Simplify[Coeff3 * I, Assumptions -> {s > 0}]
Matrix4 = Matrix3 /. {D0 -> J0, D1 -> J1, D2 -> J2}
Phase4 = Phase3 /. {lv2 -> 0}
```

$$\text{Out[28]} = -2^{-D-1} e^{-\frac{1}{4} i \pi D} \pi^{-\frac{D-1}{2}} \Lambda^{4-D} s^{-D/2}$$

$$\text{Out[29]} = \frac{e^2 J_1 d_\alpha d_\beta F(\alpha, \mu) F(\beta, \nu)}{m^6 \chi l^2} + \frac{e^2 J_2 d_\alpha d_\beta \text{FD}(\alpha, \mu) \text{FD}(\beta, \nu)}{m^6 \chi l^2} + J_0 g^{\mu\nu}$$

$$\text{Out[30]} = -\frac{x^2}{4s}$$

Calculation of the tensor structure

Let us perform the remaining differentiation
and contract the resulting vectors and tensors

```
In[31]:= Expand[
  Simplify[I FourDivergence[I FourDivergence[Exp[I Phase4], xv[α]], xv[β]] Exp[-I Phase4]]]
Matrix4 /. {dα dβ → %}
Matrix5 =
  Collect[Contract[Contract[%] /. FieldSubstitutions /. FieldSubstitutions], {J0, J1, J2}]
Coeff5 = Coeff4
Phase5 = Phase4
```

$$\text{Out[31]} = \frac{x^\alpha x^\beta}{4 s^2} + \frac{i g^{\alpha\beta}}{2 s}$$

$$\text{Out[32]} = \frac{e^2 J1 F(\alpha, \mu) F(\beta, \nu) \left(\frac{x^\alpha x^\beta}{4 s^2} + \frac{i g^{\alpha\beta}}{2 s} \right)}{m^6 \chi l^2} + \frac{e^2 J2 \text{FD}(\alpha, \mu) \text{FD}(\beta, \nu) \left(\frac{x^\alpha x^\beta}{4 s^2} + \frac{i g^{\alpha\beta}}{2 s} \right)}{m^6 \chi l^2} + J0 g^{\mu\nu}$$

$$\text{Out[33]} = J2 \left(\frac{e^2 \text{FD}x^\mu \text{FD}x^\nu}{4 m^6 s^2 \chi l^2} - \frac{i e^2 \text{FF}(\mu, \nu)}{2 m^6 s \chi l^2} \right) + J1 \left(\frac{e^2 \text{F}x^\mu \text{F}x^\nu}{4 m^6 s^2 \chi l^2} - \frac{i e^2 \text{FF}(\mu, \nu)}{2 m^6 s \chi l^2} \right) + J0 g^{\mu\nu}$$

$$\text{Out[34]} = -2^{-D-1} e^{-\frac{1}{4} i \pi D} \pi^{-\frac{D}{2}-1} \Lambda^{4-D} s^{-D/2}$$

$$\text{Out[35]} = -\frac{x^2}{4 s}$$

Change to the dimensionless proper time

```
In[36]:= Coeff6 = Simplify[Coeff5 / m^2 /. {s → t / m^2}, Assumptions → {m > 0}]
Matrix6 =
  Matrix5 /. {χ l → ξ k l / m^2} /. {k l → k x / 2 / s} /. {k x → ϕ} /. {FFt[μ, ν] → -av2 kv[μ] × kv[ν]} /.
  {av2 → -m^2 ξ^2 / e^2} /. {s → t / m^2}
Phase6 = Phase5 /. {s → t / m^2}
```

$$\text{Out[36]} = -2^{-D-1} e^{-\frac{1}{4} i \pi D} \pi^{-\frac{D}{2}-1} \Lambda^{4-D} m^{D-2} t^{-D/2}$$

$$\text{Out[37]} = J2 \left(\frac{e^2 \text{FD}x^\mu \text{FD}x^\nu}{m^2 \xi^2 \phi^2} - \frac{2 i t k^\mu k^\nu}{m^2 \phi^2} \right) + J1 \left(\frac{e^2 \text{F}x^\mu \text{F}x^\nu}{m^2 \xi^2 \phi^2} - \frac{2 i t k^\mu k^\nu}{m^2 \phi^2} \right) + J0 g^{\mu\nu}$$

$$\text{Out[38]} = -\frac{m^2 x^2}{4 t}$$

Final result

$$\begin{aligned}
D^c_{\mu\nu}(x) &= \frac{\Lambda^{4-D}}{(2\pi)^D} \int d^D l \, D^c_{\mu\nu}(l) e^{-ilx} = \\
&= \text{Exp} \left[-i \frac{\pi}{2} \left(\frac{D}{2} - 2 \right) \right] \frac{\Lambda^{4-D}}{2^{D+1} \pi^{D/2+1}} \\
&\quad \int_0^\infty \frac{ds}{s^{D/2}} e^{-i \frac{s^2}{4s}} \left\{ g_{\mu\nu} J_0(s) - i \frac{1}{2s m^6 \chi_l^2} (e^2 (F^2)_{\mu\nu} + i \frac{1}{2s} e^2 Fx_\mu Fx_\nu) J_1(s) \right. \\
&\quad \left. - i \frac{1}{2s m^6 \chi_l^2} (e^2 (F^2)_{\mu\nu} + i \frac{1}{2s} e^2 FDx_\mu FDx_\nu) J_2(s) \right\} = \\
&= \text{Exp} \left[-i \frac{\pi}{2} \left(\frac{D}{2} - 2 \right) \right] \frac{1}{2^{D+1} \pi^{D/2+1}} \frac{\Lambda^{4-D}}{m^{2-D}} \\
&\quad \int_0^\infty \frac{dt}{t^{D/2}} e^{-i \frac{m^2 t^2}{4t}} \left\{ g_{\mu\nu} J_0(m^{-2} t) + \left(-2 i t \frac{k_\mu k_\nu}{m^2 \phi^2} + \frac{e^2 Fx_\mu Fx_\nu}{m^2 \xi^2 \phi^2} \right) J_1(m^{-2} t) \right. \\
&\quad \left. + \left(-2 i t \frac{k_\mu k_\nu}{m^2 \phi^2} + \frac{e^2 FDx_\mu FDx_\nu}{m^2 \xi^2 \phi^2} \right) J_2(m^{-2} t) \right\};
\end{aligned}$$

$$J_k(s, \chi_l) = -i \int_{-\infty}^\infty d\mathbf{l}^2 D_k(\mathbf{l}^2, \chi_l) e^{-i\mathbf{l}^2 s};$$

$$\phi = kx;$$

$$\chi_l = \xi k l / m^2 = \xi \phi / 2 m^2 s = \xi \phi / 2 t;$$

$$\xi^2 = -e^2 a^2 / m^2;$$

$$Fx_\mu = F_{\mu\nu} x^\nu;$$

$$FDx_\mu = F^*_{\mu\nu} x^\nu;$$