

This is a part of SFQED-Loops script collection
developed for calculating loop processes in
Strong-Field Quantum Electrodynamics.
The scripts are available on <https://github.com/ArsenyMironov/SFQED-Loops>

If you use this script in your research, please, consider
citing our papers:

- A. A. Mironov, S. Meuren, and A. M. Fedotov, PRD 102, 053005 (2020),
<https://doi.org/10.1103/PhysRevD.102.053005>

If you have any questions, please, don't hesitate to contact me:
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`NotebookEvaluate[NotebookOpen[NotebookDirectory[] <> "definitions.nb"]]`

FeynCalc 9.2.0. For help, use the documentation center, check out the wiki or write to the mailing list.

See also the supplied examples. If you use FeynCalc in your research, please cite

- V. Shtabovenko, R. Mertig and F. Orellana,
Comput. Phys. Commun., 207C, 432–444, 2016, arXiv:1601.01167
- R. Mertig, M. Böhm, and A. Denner, Comput. Phys. Commun., 64, 345–359, 1991.

Mass operator ($e > 0$)

$-iM(q, p) =$

$$(ie)^2 \Lambda^{2D-8} \int d^D x_1 d^D x_2 \bar{E}_q(x_2) \gamma^\mu S^c(x_2, x_1) \gamma^\nu E_p(x_1) D_{\mu\nu}^c(x_1 - x_2);$$

x_1^μ, x_2^μ – position of the left and right vertices of the diagram;

p^μ, q^μ – initial and final electron momenta;

$S^c, D^c_{\mu\nu}$ – electron and photon casual propagators;

The Ritus E_p – function

$$E_p(x_1) = \left[1 - \frac{e(\gamma k)(\gamma a)}{2(kp)}(kx_2) \right]$$

$$\text{Exp} \left[-i(p x_1) + i \frac{e(a p)}{2(k p)}(k x_1)^2 + i \frac{e^2 a^2}{6(k p)}(k x_1)^3 \right];$$

$$\bar{E}_q(x_2) = \gamma^0 E_q(x_2) \gamma^0;$$

Electron propagator in a CCF in D dimensions

$$S^c(x_2, x_1) = e^{i\eta} S^c_{\text{diag}}(x_2 - x_1) =$$

$$= e^{i\eta} e^{-i\frac{\pi}{2}(\frac{D}{2}-1)} \frac{\Lambda^{4-D}}{2^D \pi^{D/2}} m$$

$$\int_0^\infty \frac{ds}{s^{D/2}} \left[1 + \frac{(\gamma x)}{2ms} - \frac{e^2 s(\gamma F F x)}{3m} + \frac{i}{2} e s (\sigma^{\alpha\beta} F_{\alpha\beta}) + \frac{i e F_{\alpha\beta}^* x^\beta \gamma^\alpha \gamma^5}{2m} \right]$$

$$e^{-is - i\frac{x^2}{4s} + i\frac{s}{12} e^2 (Fx)^2}$$

$$= e^{i\eta} e^{-i\frac{\pi}{2}(\frac{D}{2}-1)} \frac{\Lambda^{4-D}}{2^D \pi^{D/2}} m \int_0^\infty \frac{ds}{s^{D/2}} \left[1 + \frac{(\gamma x)}{2ms} + \frac{e(\gamma a)(kx)}{2m} - \frac{e(\gamma k)(ax)}{2m} + \right.$$

$$\left. \frac{e(\gamma x)(\gamma a)(\gamma k)}{2m} + e s(\gamma a)(\gamma k) + \frac{e^2 a^2 s(\gamma k)(kx)}{3m} \right] e^{-is - i\frac{x^2}{4s} + i\frac{s}{12} e^2 (Fx)^2}$$

$$\eta = e(ax)(k, (x_1 + x_2)/2),$$

$$x = x_2 - x_1,$$

$$e > 0,$$

$$[\Lambda] = m - \text{mass scale},$$

$$\sigma^{\alpha\beta} = \frac{i}{2} (\gamma^\alpha \gamma^\beta - \gamma^\beta \gamma^\alpha),$$

$$\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3,$$

$$F_{\alpha\beta} = k_\alpha a_\beta - k_\beta a_\alpha,$$

$$F_{\alpha\beta} F^\beta_\lambda = -a^2 k_\alpha k_\lambda$$

$$e^{-i\frac{\pi}{2}(\frac{D}{2}-1)} \frac{\Lambda^{4-D}}{2^D \pi^{D/2}} m^{D-1} \rightarrow \frac{(-i) m^3}{16 \pi^2}, \quad D \rightarrow 4$$

Exact photon propagator in momentum representation

$$D^c_{\mu\nu}(l) =$$

$$D_0(l^2) g_{\mu\nu} + D_1(l^2, \chi_1) \epsilon_\mu^{(1)}(l) \epsilon_\nu^{(1)}(l) + D_2(l^2, \chi_1) \epsilon_\mu^{(2)}(l) \epsilon_\nu^{(2)}(l);$$

l^μ – the photon 4 – momentum;

$$\chi_l = \frac{e}{m^3} \sqrt{-(F_{\mu\nu} l^\nu)^2};$$

$$\epsilon_\mu^{(1)}(l) = \frac{e F_{\mu\nu} l^\nu}{m^3 \chi_l};$$

$$\epsilon_\mu^{(2)}(l) = \frac{e F_{\mu\nu}^* l^\nu}{m^3 \chi_l}; \quad F^{*\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\delta\lambda} F_{\delta\lambda};$$

$$\left(\epsilon^{(i)}(l)\right)^2 = -1;$$

$$D_0(l^2) = \frac{-i}{l^2 + i0}, \quad D_{1,2}(l^2, \chi_l) = \frac{i\Pi_{1,2}}{(l^2 + i0)(l^2 - \Pi_{1,2})};$$

$\Pi_{1,2} = \Pi_{1,2}(l^2, \chi_l)$ – polarization operator eigenfunctions;

Exact photon propagator in coordinate representation

$$\begin{aligned} D_{\mu\nu}^c(x) &= \frac{\Lambda^{4-D}}{(2\pi)^D} \int d^D l D_{\mu\nu}^c(l) e^{-ilx} = \\ &= \text{Exp} \left[-i \frac{\pi}{2} \left(\frac{D}{2} - 2 \right) \right] \frac{1}{2^D \pi^{D/2}} \frac{\Lambda^{4-D}}{m^{2-D}} \\ &\quad \int_0^\infty \frac{dt}{t^{D/2}} e^{-i \frac{m^2 x^2}{4t}} \left\{ g_{\mu\nu} J_0(t) + \left(-2i t \frac{k_\mu k_\nu}{m^2 \phi^2} + \frac{e^2 F x_\mu F x_\nu}{m^2 \xi^2 \phi^2} \right) J_1(t, \chi_l) \right. \\ &\quad \left. + \left(-2i t \frac{k_\mu k_\nu}{m^2 \phi^2} + \frac{e^2 F D x_\mu F D x_\nu}{m^2 \xi^2 \phi^2} \right) J_2(t, \chi_l) \right\}; \end{aligned}$$

$$J_k(t, \chi_l) = -i \int_0^\infty dl^2 D_k(l^2, \chi_l) e^{-il^2 t};$$

$$\chi_l = \xi k l / m^2 = \xi \phi / 2 m^2 t;$$

Preliminaries

Let us define momenta and coordinate variables

$$x = x_2 - x_1;$$

$$X = \frac{1}{2} (x_1 + x_2);$$

$$\phi = kx;$$

$$\Phi = kX;$$

$$\phi_1 = kx_1;$$

$$\phi_2 = kx_2;$$

The functions NewMomentum and NewCoordinate are predefined in the file definitions.nb

They provide the corresponding 4 - vector along with all possible contractions with the field tensor $F_{\mu\nu}$ and 4 - vectors a_μ, k_μ (e.g. $Fxv[\mu] == (Fx)_\mu = F_{\mu\nu} x^\nu$, see more details in the file definitions.nb)

```

NewMomentum["p"]
NewMomentum["q"]
NewMomentum["l"]
NewCoordinate["x1"]
NewCoordinate["x2"]
NewCoordinate["x"]
NewCoordinate["X"]
ScalarProduct[k, x] =  $\phi$ ;
ScalarProduct[k, X] =  $\bar{\Phi}$ ;
ScalarProduct[k, x1] =  $\phi_1$ ;
ScalarProduct[k, x2] =  $\phi_2$ ;
ScalarProduct[Fx, Fx] =  $-\xi^2 \phi^2 * m^2 / e^2$ ;
ScalarProduct[FDx, FDx] =  $-\xi^2 \phi^2 * m^2 / e^2$ ;
ScalarProduct[x, FFx] =  $\xi^2 \phi^2 * m^2 / e^2$ ;

```

$$\left\{ p^\alpha, p^2, k \cdot p, Fp^\alpha, FFp^\alpha, FDP^\alpha, a \cdot p, 0, 0, 0, -a^2 (k \cdot p), 0, 0, -\frac{m^6 \chi p^2}{e^2}, -\frac{m^6 \chi p^2}{e^2}, \frac{m^6 \chi p^2}{e^2}, 0, 0, 0, 0, 0 \right\}$$

$$\left\{ q^\alpha, q^2, k \cdot q, Fq^\alpha, FFq^\alpha, FDq^\alpha, a \cdot q, 0, 0, 0, -a^2 (k \cdot q), 0, 0, -\frac{m^6 \chi q^2}{e^2}, -\frac{m^6 \chi q^2}{e^2}, \frac{m^6 \chi q^2}{e^2}, 0, 0, 0, 0, 0 \right\}$$

$$\left\{ l^\alpha, l^2, k \cdot l, Fl^\alpha, FF l^\alpha, FDI^\alpha, a \cdot l, 0, 0, 0, -a^2 (k \cdot l), 0, 0, -\frac{m^6 \chi l^2}{e^2}, -\frac{m^6 \chi l^2}{e^2}, \frac{m^6 \chi l^2}{e^2}, 0, 0, 0, 0, 0 \right\}$$

$$\left\{ x1^\alpha, x1^2, k \cdot x1, a \cdot x1, Fx1^\alpha, FFx1^\alpha, FDx1^\alpha, k \cdot x1, 0, 0, 0, -a^2 (k \cdot x1), \right. \\ \left. 0, 0, -\frac{m^2 \xi^2 (k \cdot x1)^2}{e^2}, -\frac{m^2 \xi^2 (k \cdot x1)^2}{e^2}, \frac{m^2 \xi^2 (k \cdot x1)^2}{e^2}, 0, 0, 0, 0, 0 \right\}$$

$$\left\{ x2^\alpha, x2^2, k \cdot x2, a \cdot x2, Fx2^\alpha, FFx2^\alpha, FDx2^\alpha, k \cdot x2, 0, 0, 0, -a^2 (k \cdot x2), \right. \\ \left. 0, 0, -\frac{m^2 \xi^2 (k \cdot x2)^2}{e^2}, -\frac{m^2 \xi^2 (k \cdot x2)^2}{e^2}, \frac{m^2 \xi^2 (k \cdot x2)^2}{e^2}, 0, 0, 0, 0, 0 \right\}$$

$$\left\{ x^\alpha, x^2, k \cdot x, a \cdot x, Fx^\alpha, FFx^\alpha, FDx^\alpha, k \cdot x, 0, 0, 0, -a^2 (k \cdot x), \right. \\ \left. 0, 0, -\frac{m^2 \xi^2 (k \cdot x)^2}{e^2}, -\frac{m^2 \xi^2 (k \cdot x)^2}{e^2}, \frac{m^2 \xi^2 (k \cdot x)^2}{e^2}, 0, 0, 0, 0, 0 \right\}$$

$$\left\{ X^\alpha, X^2, k \cdot X, a \cdot X, FX^\alpha, FFX^\alpha, FDX^\alpha, k \cdot X, 0, 0, 0, -a^2 (k \cdot X), \right. \\ \left. 0, 0, -\frac{m^2 \xi^2 (k \cdot X)^2}{e^2}, -\frac{m^2 \xi^2 (k \cdot X)^2}{e^2}, \frac{m^2 \xi^2 (k \cdot X)^2}{e^2}, 0, 0, 0, 0, 0 \right\}$$

E_p - functions

The functions E_p and E_p^C are predefined in the file `definitions.nb`

They provide a list with two data

fields : the preexponent and the phase of the E_p –
function (or the adjoint \bar{E}_p – function)

$E_{px1} = E_p[x1, p]$

$E_{qBarx2} = E_p^C[x2, q]$

$$\left\{ 1 - \frac{e\phi 1 (\gamma \cdot k) (\gamma \cdot a)}{2 (k \cdot p)}, \frac{a^2 e^2 \phi 1^3}{6 (k \cdot p)} + \frac{e\phi 1^2 (a \cdot p)}{2 (k \cdot p)} - p \cdot x1 \right\}$$

$$\left\{ 1 - \frac{e\phi 2 (\gamma \cdot a) (\gamma \cdot k)}{2 (k \cdot q)}, -\frac{a^2 e^2 \phi 2^3}{6 (k \cdot q)} - \frac{e\phi 2^2 (a \cdot q)}{2 (k \cdot q)} + q \cdot x2 \right\}$$

Propagators in coordinate representation and proper times

s – electron proper time of dimension m^{-2} ;

t – photon proper time of dimension m^{-2}

The functions `DiracElectronPropagatorXRepr` and

`PhotonPropagatorExactXRepr` are predefined in the file `definitions.nb`

They provide a list with three data

fields : a γ –matrix (for S^C) or tensor (for D^C) preexponential,
a scalar prefactor and the phase of the exponent.

It is implied that there is an integration over the proper time from 0 to ∞

$S_c = \text{DiracElectronPropagatorXRepr}[x, X, m^2 s];$

$S_c[[2]] = \text{Simplify}[S_c[[2]] * m^2, \text{Assumptions} \rightarrow \{m > 0\}];$

S_c

$$\left\{ -\frac{e(a \cdot x) \gamma \cdot k}{2 m} + \frac{e(\gamma \cdot x) (\gamma \cdot a) (\gamma \cdot k)}{2 m} + e s (\gamma \cdot a) (\gamma \cdot k) + \frac{e \phi \gamma \cdot a}{2 m} - \frac{1}{3} m \xi^2 s \phi \gamma \cdot k + \frac{\gamma \cdot x}{2 m s} + 1, \right.$$

$$\left. i 2^{-D} e^{-\frac{1}{4} i \pi D} \pi^{-D/2} m \Lambda^{4-D} s^{-D/2}, e \Phi(a \cdot x) - \frac{1}{12} m^2 \xi^2 s \phi^2 - m^2 s - \frac{x^2}{4 s} \right\}$$

```
Dc = PhotonPropagatorExactXRepr[x, m^2 t, μ, ν];
Dc[[2]] = Simplify[Dc[[2]] * m^2, Assumptions → {m > 0}];
Dc
```

$$\left\{ J_2 \left(t, \frac{\xi \phi}{2 m^2 t} \right) \left(\frac{1}{m^2 \xi^2 \phi^2} e^2 (-k^\mu k^\nu (a \cdot x)^2 + a^2 \phi^2 g^{\mu\nu} - a^2 \phi k^\nu x^\mu - a^2 \phi k^\mu x^\nu + \right. \right. \\ \left. \left. \phi a^\nu k^\mu (a \cdot x) + \phi a^\mu k^\nu (a \cdot x) + a^2 x^2 k^\mu k^\nu + \phi^2 (-a^\mu) a^\nu \right) - \frac{2 i t k^\mu k^\nu}{\phi^2} \right) + \\ J_1 \left(t, \frac{\xi \phi}{2 m^2 t} \right) \left(\frac{1}{m^2 \xi^2 \phi^2} e^2 (k^\mu k^\nu (a \cdot x)^2 - \phi a^\nu k^\mu (a \cdot x) - \phi a^\mu k^\nu (a \cdot x) + \phi^2 a^\mu a^\nu) - \frac{2 i t k^\mu k^\nu}{\phi^2} \right) + \\ g^{\mu\nu} J_0 \left(t, \frac{\xi \phi}{2 m^2 t} \right), \\ -2^{-D-1} e^{-\frac{1}{4} i \pi D} \pi^{-\frac{D}{2}-1} \Lambda^{4-D} t^{-D/2}, \\ \left. -\frac{x^2}{4 t} \right\}$$

Calculation of M

The mass operator

$$M(q, p) = i \int d^D x_1 d^D x_2 \dots;$$

We will write the integrand in the following form

$$\text{Coeff} * \text{Tr}[\text{Matrix}] * \text{Exp}[i \text{Phase}],$$

where

Coeff – is a dimensional coefficient in

front of the expression (also depends on s and t),

Matrix – the γ – matrix factor,

Phase – the total phase of the exponential

We also introduce the notation

$$J_k = J_k \left[t, \frac{\xi \phi}{2 m^2 t} \right]$$

$$\text{Coeff} = \text{I} \left(\text{I} e \right)^2 \Lambda^{2D-8} \text{Sc}[[2]] \text{Dc}[[2]]$$

$$\text{Phase} = \text{EqBarx2}[[2]] + \text{Sc}[[3]] + \text{Dc}[[3]] + \text{Epx1}[[2]]$$

$$\text{Matrix} = \left(\text{EqBarx2}[[1]] \cdot \text{GAD}[\mu] \cdot \text{Sc}[[1]] \cdot \text{GAD}[\nu] \cdot \text{Epx1}[[1]] \text{Dc}[[1]] \right) / .$$

$$\{J0[t, \xi \phi / 2 / t / m^2] \rightarrow J0, J1[t, \xi \phi / 2 / t / m^2] \rightarrow J1, J2[t, \xi \phi / 2 / t / m^2] \rightarrow J2\}$$

$$-2^{-2D-1} e^{-\frac{1}{2} i \pi D} \pi^{-D-1} e^2 m s^{-D/2} t^{-D/2}$$

$$\frac{\alpha^2 e^2 \phi^3}{6(k \cdot p)} - \frac{\alpha^2 e^2 \phi^3}{6(k \cdot q)} + \frac{e \phi^2 (a \cdot p)}{2(k \cdot p)} - \frac{e \phi^2 (a \cdot q)}{2(k \cdot q)} + e \Phi(a \cdot x) - \frac{1}{12} m^2 \xi^2 s \phi^2 - m^2 s - p \cdot x_1 + q \cdot x_2 - \frac{x^2}{4s} - \frac{x^2}{4t}$$

$$\left(J2 \left(\frac{1}{m^2 \xi^2 \phi^2} e^2 (-k^\mu k^\nu (a \cdot x)^2 + \alpha^2 \phi^2 g^{\mu\nu} - \alpha^2 \phi k^\nu x^\mu - \alpha^2 \phi k^\mu x^\nu + \right. \right.$$

$$\left. \phi a^\nu k^\mu (a \cdot x) + \phi a^\mu k^\nu (a \cdot x) + \alpha^2 x^2 k^\mu k^\nu + \phi^2 (-a^\mu) a^\nu \right) - \frac{2 i t k^\mu k^\nu}{\phi^2} \Bigg) +$$

$$J1 \left(\frac{1}{m^2 \xi^2 \phi^2} e^2 (k^\mu k^\nu (a \cdot x)^2 - \phi a^\nu k^\mu (a \cdot x) - \phi a^\mu k^\nu (a \cdot x) + \phi^2 a^\mu a^\nu) - \frac{2 i t k^\mu k^\nu}{\phi^2} \right) +$$

$$J0 g^{\mu\nu} \Bigg) \left(1 - \frac{e \phi^2 (\gamma \cdot a) \cdot (\gamma \cdot k)}{2(k \cdot q)} \right) \cdot \gamma^\mu.$$

$$\left(-\frac{e(a \cdot x) \gamma \cdot k}{2m} + \frac{e(\gamma \cdot x) \cdot (\gamma \cdot a) \cdot (\gamma \cdot k)}{2m} + e s (\gamma \cdot a) \cdot (\gamma \cdot k) + \frac{e \phi \gamma \cdot a}{2m} - \frac{1}{3} m \xi^2 s \phi \gamma \cdot k + \frac{\gamma \cdot x}{2ms} + 1 \right).$$

$$\gamma^\nu \cdot \left(1 - \frac{e \phi^2 (\gamma \cdot k) \cdot (\gamma \cdot a)}{2(k \cdot p)} \right)$$

γ - matrix algebra

We perform simplifications of the γ

-matrix factor with the aid of FeynCalc functions DotSimplify,
DiracSimplify, and then Contract the Lorentz indices.

```
Coeff1 = Coeff;
```

```
Phase1 = Phase;
```

```
Contract[DotSimplify[Expand[Matrix]]];
```

```
Matrix1 = Collect[Contract[DiracSimplify[%]], {J0, J1, J2}]
```

$$\begin{aligned}
& J0 \left(-\frac{D\phi\phi1\gamma\cdot k a^2 e^2}{4m(k\cdot p)} + \frac{3\phi\phi1\gamma\cdot k a^2 e^2}{2m(k\cdot p)} + \frac{D\phi\phi2\gamma\cdot k a^2 e^2}{4m(k\cdot q)} - \frac{3\phi\phi2\gamma\cdot k a^2 e^2}{2m(k\cdot q)} + \frac{D\phi\phi1\phi2\gamma\cdot k a^2 e^2}{4ms(k\cdot p)(k\cdot q)} - \right. \\
& \quad \frac{\phi\phi1\phi2\gamma\cdot k a^2 e^2}{2ms(k\cdot p)(k\cdot q)} - \frac{D\phi\gamma\cdot a e}{2m} + \frac{\phi\gamma\cdot a e}{m} + Ds(\gamma\cdot a).(\gamma\cdot k)e - 4s(\gamma\cdot a).(\gamma\cdot k)e - \frac{(\gamma\cdot k).(\gamma\cdot a).(\gamma\cdot x)e}{m} - \\
& \quad \frac{D(\gamma\cdot x).(\gamma\cdot a).(\gamma\cdot k)e}{2m} + \frac{2(\gamma\cdot x).(\gamma\cdot a).(\gamma\cdot k)e}{m} + \frac{D\gamma\cdot k(a\cdot x)e}{2m} - \frac{\gamma\cdot k(a\cdot x)e}{m} - \frac{D\phi1(\gamma\cdot k).(\gamma\cdot a)e}{2(k\cdot p)} + \\
& \quad \frac{D\phi1(\gamma\cdot x).(\gamma\cdot k).(\gamma\cdot a)e}{4ms(k\cdot p)} - \frac{\phi1(\gamma\cdot x).(\gamma\cdot k).(\gamma\cdot a)e}{2ms(k\cdot p)} - \frac{D\phi2(\gamma\cdot a).(\gamma\cdot k)e}{2(k\cdot q)} + \frac{D\phi2(\gamma\cdot a).(\gamma\cdot k).(\gamma\cdot x)e}{4ms(k\cdot q)} - \\
& \quad \frac{\phi2(\gamma\cdot a).(\gamma\cdot k).(\gamma\cdot x)e}{2ms(k\cdot q)} + D + \frac{1}{3}Dms\xi^2\phi\gamma\cdot k - \frac{2}{3}ms\xi^2\phi\gamma\cdot k - \frac{D\gamma\cdot x}{2ms} + \frac{\gamma\cdot x}{ms} \Big) + \\
& J1 \left(\frac{\phi\phi1\gamma\cdot k a^4 e^4}{4m^3\xi^2(k\cdot p)} - \frac{\phi\phi2\gamma\cdot k a^4 e^4}{4m^3\xi^2(k\cdot q)} + \frac{\phi\phi1\phi2\gamma\cdot k a^4 e^4}{4m^3s\xi^2(k\cdot p)(k\cdot q)} + \frac{\phi\gamma\cdot a a^2 e^3}{2m^3\xi^2} + \frac{s(\gamma\cdot k).(\gamma\cdot a)a^2 e^3}{m^2\xi^2} - \right. \\
& \quad \frac{(\gamma\cdot x).(\gamma\cdot k).(\gamma\cdot a)a^2 e^3}{2m^3\xi^2} - \frac{\gamma\cdot k a^2(a\cdot x)e^3}{2m^3\xi^2} - \frac{\phi1(\gamma\cdot k).(\gamma\cdot a)a^2 e^3}{2m^2\xi^2(k\cdot p)} + \frac{\phi1(\gamma\cdot x).(\gamma\cdot k).(\gamma\cdot a)a^2 e^3}{4m^3s\xi^2(k\cdot p)} - \\
& \quad \frac{\phi2(\gamma\cdot a).(\gamma\cdot k)a^2 e^3}{2m^2\xi^2(k\cdot q)} + \frac{\phi2(\gamma\cdot k).(\gamma\cdot x).(\gamma\cdot a)a^2 e^3}{4m^3s\xi^2(k\cdot q)} - \frac{\phi2\gamma\cdot k a^2(a\cdot x)e^3}{2m^3s\xi^2(k\cdot q)} + \frac{\gamma\cdot k(a\cdot x)^2 e^2}{m^3s\xi^2\phi} + \\
& \quad \frac{s\phi\gamma\cdot k a^2 e^2}{3m} - \frac{\gamma\cdot x a^2 e^2}{2m^3s\xi^2} + \frac{a^2 e^2}{m^2\xi^2} + \frac{\gamma\cdot a(a\cdot x)e^2}{m^3s\xi^2} - \frac{(\gamma\cdot a).(\gamma\cdot k)(a\cdot x)e^2}{m^2\xi^2\phi} - \\
& \quad \frac{(\gamma\cdot k).(\gamma\cdot a)(a\cdot x)e^2}{m^2\xi^2\phi} - \frac{(\gamma\cdot a).(\gamma\cdot x).(\gamma\cdot k)(a\cdot x)e^2}{2m^3s\xi^2\phi} - \frac{(\gamma\cdot k).(\gamma\cdot x).(\gamma\cdot a)(a\cdot x)e^2}{2m^3s\xi^2\phi} - \frac{2it\gamma\cdot k}{ms\phi} \Big) + \\
& J2 \left(-\frac{D\phi\phi1\gamma\cdot k a^4 e^4}{4m^3\xi^2(k\cdot p)} + \frac{3\phi\phi1\gamma\cdot k a^4 e^4}{4m^3\xi^2(k\cdot p)} + \frac{D\phi\phi2\gamma\cdot k a^4 e^4}{4m^3\xi^2(k\cdot q)} - \frac{3\phi\phi2\gamma\cdot k a^4 e^4}{4m^3\xi^2(k\cdot q)} + \frac{D\phi\phi1\phi2\gamma\cdot k a^4 e^4}{4m^3s\xi^2(k\cdot p)(k\cdot q)} - \right. \\
& \quad \frac{3\phi\phi1\phi2\gamma\cdot k a^4 e^4}{4m^3s\xi^2(k\cdot p)(k\cdot q)} - \frac{D\phi\gamma\cdot a a^2 e^3}{2m^3\xi^2} + \frac{\phi\gamma\cdot a a^2 e^3}{2m^3\xi^2} + \frac{Ds(\gamma\cdot a).(\gamma\cdot k)a^2 e^3}{m^2\xi^2} - \\
& \quad \frac{4s(\gamma\cdot a).(\gamma\cdot k)a^2 e^3}{m^2\xi^2} - \frac{s(\gamma\cdot k).(\gamma\cdot a)a^2 e^3}{m^2\xi^2} - \frac{(\gamma\cdot a).(\gamma\cdot k).(\gamma\cdot x)a^2 e^3}{m^3\xi^2} - \frac{3(\gamma\cdot k).(\gamma\cdot a).(\gamma\cdot x)a^2 e^3}{2m^3\xi^2} - \\
& \quad \frac{D(\gamma\cdot x).(\gamma\cdot a).(\gamma\cdot k)a^2 e^3}{2m^3\xi^2} + \frac{3(\gamma\cdot x).(\gamma\cdot a).(\gamma\cdot k)a^2 e^3}{2m^3\xi^2} + \frac{(\gamma\cdot x).(\gamma\cdot k).(\gamma\cdot a)a^2 e^3}{2m^3\xi^2} + \frac{D\gamma\cdot k a^2(a\cdot x)e^3}{2m^3\xi^2} - \\
& \quad \frac{\gamma\cdot k a^2(a\cdot x)e^3}{2m^3\xi^2} - \frac{D\phi1(\gamma\cdot k).(\gamma\cdot a)a^2 e^3}{2m^2\xi^2(k\cdot p)} + \frac{3\phi1(\gamma\cdot k).(\gamma\cdot a)a^2 e^3}{2m^2\xi^2(k\cdot p)} + \frac{D\phi1(\gamma\cdot x).(\gamma\cdot k).(\gamma\cdot a)a^2 e^3}{4m^3s\xi^2(k\cdot p)} - \\
& \quad \frac{3\phi1(\gamma\cdot x).(\gamma\cdot k).(\gamma\cdot a)a^2 e^3}{4m^3s\xi^2(k\cdot p)} - \frac{D\phi2(\gamma\cdot a).(\gamma\cdot k)a^2 e^3}{2m^2\xi^2(k\cdot q)} + \frac{3\phi2(\gamma\cdot a).(\gamma\cdot k)a^2 e^3}{2m^2\xi^2(k\cdot q)} + \\
& \quad \frac{D\phi2(\gamma\cdot a).(\gamma\cdot k).(\gamma\cdot x)a^2 e^3}{4m^3s\xi^2(k\cdot q)} - \frac{\phi2(\gamma\cdot a).(\gamma\cdot k).(\gamma\cdot x)a^2 e^3}{2m^3s\xi^2(k\cdot q)} - \frac{\phi2(\gamma\cdot k).(\gamma\cdot x).(\gamma\cdot a)a^2 e^3}{4m^3s\xi^2(k\cdot q)} + \\
& \quad \frac{\phi2\gamma\cdot k a^2(a\cdot x)e^3}{2m^3s\xi^2(k\cdot q)} - \frac{\gamma\cdot k(a\cdot x)^2 e^2}{m^3s\xi^2\phi} + \frac{Ds\phi\gamma\cdot k a^2 e^2}{3m} - \frac{s\phi\gamma\cdot k a^2 e^2}{m} - \frac{D\gamma\cdot x a^2 e^2}{2m^3s\xi^2} + \frac{3\gamma\cdot x a^2 e^2}{2m^3s\xi^2} - \\
& \quad \frac{(\gamma\cdot k).(\gamma\cdot x)a^2 e^2}{m^2\xi^2\phi} - \frac{(\gamma\cdot x).(\gamma\cdot k)a^2 e^2}{m^2\xi^2\phi} + \frac{Da^2 e^2}{m^2\xi^2} - \frac{a^2 e^2}{m^2\xi^2} - \frac{\gamma\cdot a(a\cdot x)e^2}{m^3s\xi^2} + \frac{(\gamma\cdot a).(\gamma\cdot k)(a\cdot x)e^2}{m^2\xi^2\phi} + \\
& \quad \frac{(\gamma\cdot k).(\gamma\cdot a)(a\cdot x)e^2}{m^2\xi^2\phi} + \frac{(\gamma\cdot a).(\gamma\cdot x).(\gamma\cdot k)(a\cdot x)e^2}{2m^3s\xi^2\phi} + \frac{(\gamma\cdot k).(\gamma\cdot x).(\gamma\cdot a)(a\cdot x)e^2}{2m^3s\xi^2\phi} - \frac{2it\gamma\cdot k}{ms\phi} \Big)
\end{aligned}$$

Substitution of the variables

$$x1 = x - \frac{x}{2};$$

$$x2 = x + \frac{X}{2};$$

$$\phi1 = \phi - \frac{\Phi}{2};$$

$$\phi2 = \phi + \frac{\Phi}{2};$$

$$\phi = kx = m x_-;$$

$$\Phi = kX = m X_-;$$

The integration measure :

$$d^D x_1 d^D x_2 = d^D x d^D X$$

Coeff2 = Coeff1;

Phase2 = Expand[ExpandScalarProduct[

Phase1 /. { $\phi2 \rightarrow \bar{\phi} + \phi/2$, $\phi1 \rightarrow \bar{\phi} - \phi/2$, $av2 \rightarrow -m^2 \xi^2 / e^2$ } /.

{Momentum[x1, D] → Momentum[X, D] - Momentum[x, D] / 2,

Momentum[x2, D] → Momentum[X, D] + Momentum[x, D] / 2}

]]

Matrix2 = Collect[

Expand[

Matrix1 /. { $\phi2 \rightarrow \bar{\phi} + \phi/2$, $\phi1 \rightarrow \bar{\phi} - \phi/2$, $av2 \rightarrow -m^2 \xi^2 / e^2$ }

],

{J0, J1, J2}]

$$\begin{aligned} & \frac{e\Phi^2(a \cdot p)}{2(k \cdot p)} + \frac{e\phi^2(a \cdot p)}{8(k \cdot p)} - \frac{e\Phi\phi(a \cdot p)}{2(k \cdot p)} - \frac{e\Phi^2(a \cdot q)}{2(k \cdot q)} - \frac{e\phi^2(a \cdot q)}{8(k \cdot q)} - \frac{e\Phi\phi(a \cdot q)}{2(k \cdot q)} + \\ & e\Phi(a \cdot x) - \frac{m^2 \xi^2 \Phi^3}{6(k \cdot p)} + \frac{m^2 \xi^2 \phi^3}{48(k \cdot p)} - \frac{m^2 \xi^2 \Phi\phi^2}{8(k \cdot p)} + \frac{m^2 \xi^2 \Phi^2\phi}{4(k \cdot p)} + \frac{m^2 \xi^2 \Phi^3}{6(k \cdot q)} + \frac{m^2 \xi^2 \phi^3}{48(k \cdot q)} + \\ & \frac{m^2 \xi^2 \Phi\phi^2}{8(k \cdot q)} + \frac{m^2 \xi^2 \Phi^2\phi}{4(k \cdot q)} - \frac{1}{12} m^2 \xi^2 s \phi^2 - m^2 s + \frac{p \cdot x}{2} - p \cdot X + \frac{q \cdot x}{2} + q \cdot X - \frac{x^2}{4s} - \frac{x^2}{4t} \\ J1 & \left(-\frac{m\xi^2 \gamma \cdot k \phi^3}{16s(k \cdot p)(k \cdot q)} - \frac{m\xi^2 \gamma \cdot k \phi^2}{8(k \cdot p)} - \frac{m\xi^2 \gamma \cdot k \phi^2}{8(k \cdot q)} - \frac{e\gamma \cdot a \phi}{2m} - \frac{1}{3} m s \xi^2 \gamma \cdot k \phi + \frac{m\xi^2 \Phi \gamma \cdot k \phi}{4(k \cdot p)} - \right. \\ & \frac{e(\gamma \cdot k)(\gamma \cdot a) \phi}{4(k \cdot p)} + \frac{e(\gamma \cdot x)(\gamma \cdot k)(\gamma \cdot a) \phi}{8ms(k \cdot p)} - \frac{m\xi^2 \Phi \gamma \cdot k \phi}{4(k \cdot q)} + \frac{e(\gamma \cdot a)(\gamma \cdot k) \phi}{4(k \cdot q)} - \frac{e(\gamma \cdot k)(\gamma \cdot x)(\gamma \cdot a) \phi}{8ms(k \cdot q)} + \\ & \frac{e\gamma \cdot k(a \cdot x) \phi}{4ms(k \cdot q)} + \frac{m\xi^2 \Phi^2 \gamma \cdot k \phi}{4s(k \cdot p)(k \cdot q)} + \frac{\gamma \cdot x}{2ms} - es(\gamma \cdot k)(\gamma \cdot a) + \frac{e(\gamma \cdot x)(\gamma \cdot k)(\gamma \cdot a)}{2m} + \\ & \frac{e^2 \gamma \cdot a(a \cdot x)}{m^3 s \xi^2} + \frac{e\gamma \cdot k(a \cdot x)}{2m} + \frac{e\Phi(\gamma \cdot k)(\gamma \cdot a)}{2(k \cdot p)} - \frac{e\Phi(\gamma \cdot x)(\gamma \cdot k)(\gamma \cdot a)}{4ms(k \cdot p)} + \frac{e\Phi(\gamma \cdot a)(\gamma \cdot k)}{2(k \cdot q)} - \end{aligned}$$

$$\begin{aligned}
& \frac{e\Phi(\gamma \cdot k)(\gamma \cdot x)(\gamma \cdot a)}{4ms(k \cdot q)} + \frac{e\Phi\gamma \cdot k(a \cdot x)}{2ms(k \cdot q)} - 1 + \frac{e^2\gamma \cdot k(a \cdot x)^2}{m^3s\xi^2\phi} - \frac{2it\gamma \cdot k}{ms\phi} - \frac{e^2(\gamma \cdot a)(\gamma \cdot k)(a \cdot x)}{m^2\xi^2\phi} - \\
& \frac{e^2(\gamma \cdot k)(\gamma \cdot a)(a \cdot x)}{m^2\xi^2\phi} - \frac{e^2(\gamma \cdot a)(\gamma \cdot x)(\gamma \cdot k)(a \cdot x)}{2m^3s\xi^2\phi} - \frac{e^2(\gamma \cdot k)(\gamma \cdot x)(\gamma \cdot a)(a \cdot x)}{2m^3s\xi^2\phi} \Big) + \\
& \text{J0} \left(\frac{Dm\xi^2\gamma \cdot k\phi^3}{16s(k \cdot p)(k \cdot q)} - \frac{m\xi^2\gamma \cdot k\phi^3}{8s(k \cdot p)(k \cdot q)} - \frac{Dm\xi^2\gamma \cdot k\phi^2}{8(k \cdot p)} + \frac{3m\xi^2\gamma \cdot k\phi^2}{4(k \cdot p)} - \frac{Dm\xi^2\gamma \cdot k\phi^2}{8(k \cdot q)} + \frac{3m\xi^2\gamma \cdot k\phi^2}{4(k \cdot q)} - \right. \\
& \frac{De\gamma \cdot a\phi}{2m} + \frac{e\gamma \cdot a\phi}{m} + \frac{1}{3}Dms\xi^2\gamma \cdot k\phi - \frac{2}{3}ms\xi^2\gamma \cdot k\phi + \frac{Dm\xi^2\Phi\gamma \cdot k\phi}{4(k \cdot p)} - \frac{3m\xi^2\Phi\gamma \cdot k\phi}{2(k \cdot p)} + \\
& \frac{De(\gamma \cdot k)(\gamma \cdot a)\phi}{4(k \cdot p)} - \frac{De(\gamma \cdot x)(\gamma \cdot k)(\gamma \cdot a)\phi}{8ms(k \cdot p)} + \frac{e(\gamma \cdot x)(\gamma \cdot k)(\gamma \cdot a)\phi}{4ms(k \cdot p)} - \frac{Dm\xi^2\Phi\gamma \cdot k\phi}{4(k \cdot q)} + \frac{3m\xi^2\Phi\gamma \cdot k\phi}{2(k \cdot q)} - \\
& \frac{De(\gamma \cdot a)(\gamma \cdot k)\phi}{4(k \cdot q)} + \frac{De(\gamma \cdot a)(\gamma \cdot k)(\gamma \cdot x)\phi}{8ms(k \cdot q)} - \frac{e(\gamma \cdot a)(\gamma \cdot k)(\gamma \cdot x)\phi}{4ms(k \cdot q)} - \frac{Dm\xi^2\Phi^2\gamma \cdot k\phi}{4s(k \cdot p)(k \cdot q)} + \frac{m\xi^2\Phi^2\gamma \cdot k\phi}{2s(k \cdot p)(k \cdot q)} - \\
& D - \frac{D\gamma \cdot x}{2ms} + \frac{\gamma \cdot x}{ms} + Des(\gamma \cdot a)(\gamma \cdot k) - 4es(\gamma \cdot a)(\gamma \cdot k) - \frac{e(\gamma \cdot k)(\gamma \cdot a)(\gamma \cdot x)}{m} - \frac{De(\gamma \cdot x)(\gamma \cdot a)(\gamma \cdot k)}{2m} + \\
& \frac{2e(\gamma \cdot x)(\gamma \cdot a)(\gamma \cdot k)}{m} + \frac{De\gamma \cdot k(a \cdot x)}{2m} - \frac{e\gamma \cdot k(a \cdot x)}{m} - \frac{De\Phi(\gamma \cdot k)(\gamma \cdot a)}{2(k \cdot p)} + \frac{De\Phi(\gamma \cdot x)(\gamma \cdot k)(\gamma \cdot a)}{4ms(k \cdot p)} - \\
& \frac{e\Phi(\gamma \cdot x)(\gamma \cdot k)(\gamma \cdot a)}{2ms(k \cdot p)} - \frac{De\Phi(\gamma \cdot a)(\gamma \cdot k)}{2(k \cdot q)} + \frac{De\Phi(\gamma \cdot a)(\gamma \cdot k)(\gamma \cdot x)}{4ms(k \cdot q)} - \frac{e\Phi(\gamma \cdot a)(\gamma \cdot k)(\gamma \cdot x)}{2ms(k \cdot q)} \Big) + \\
& \text{J2} \left(-\frac{Dm\xi^2\gamma \cdot k\phi^3}{16s(k \cdot p)(k \cdot q)} + \frac{3m\xi^2\gamma \cdot k\phi^3}{16s(k \cdot p)(k \cdot q)} + \frac{Dm\xi^2\gamma \cdot k\phi^2}{8(k \cdot p)} - \frac{3m\xi^2\gamma \cdot k\phi^2}{8(k \cdot p)} + \frac{Dm\xi^2\gamma \cdot k\phi^2}{8(k \cdot q)} - \frac{3m\xi^2\gamma \cdot k\phi^2}{8(k \cdot q)} + \right. \\
& \frac{De\gamma \cdot a\phi}{2m} - \frac{e\gamma \cdot a\phi}{2m} - \frac{1}{3}Dms\xi^2\gamma \cdot k\phi + ms\xi^2\gamma \cdot k\phi - \frac{Dm\xi^2\Phi\gamma \cdot k\phi}{4(k \cdot p)} + \frac{3m\xi^2\Phi\gamma \cdot k\phi}{4(k \cdot p)} - \\
& \frac{De(\gamma \cdot k)(\gamma \cdot a)\phi}{4(k \cdot p)} + \frac{3e(\gamma \cdot k)(\gamma \cdot a)\phi}{4(k \cdot p)} + \frac{De(\gamma \cdot x)(\gamma \cdot k)(\gamma \cdot a)\phi}{8ms(k \cdot p)} - \frac{3e(\gamma \cdot x)(\gamma \cdot k)(\gamma \cdot a)\phi}{8ms(k \cdot p)} + \frac{Dm\xi^2\Phi\gamma \cdot k\phi}{4(k \cdot q)} - \\
& \frac{3m\xi^2\Phi\gamma \cdot k\phi}{4(k \cdot q)} + \frac{De(\gamma \cdot a)(\gamma \cdot k)\phi}{4(k \cdot q)} - \frac{3e(\gamma \cdot a)(\gamma \cdot k)\phi}{4(k \cdot q)} - \frac{De(\gamma \cdot a)(\gamma \cdot k)(\gamma \cdot x)\phi}{8ms(k \cdot q)} + \frac{e(\gamma \cdot a)(\gamma \cdot k)(\gamma \cdot x)\phi}{4ms(k \cdot q)} + \\
& \frac{e(\gamma \cdot k)(\gamma \cdot x)(\gamma \cdot a)\phi}{8ms(k \cdot q)} - \frac{e\gamma \cdot k(a \cdot x)\phi}{4ms(k \cdot q)} + \frac{Dm\xi^2\Phi^2\gamma \cdot k\phi}{4s(k \cdot p)(k \cdot q)} - \frac{3m\xi^2\Phi^2\gamma \cdot k\phi}{4s(k \cdot p)(k \cdot q)} - D + \frac{D\gamma \cdot x}{2ms} - \frac{3\gamma \cdot x}{2ms} - \\
& Des(\gamma \cdot a)(\gamma \cdot k) + 4es(\gamma \cdot a)(\gamma \cdot k) + es(\gamma \cdot k)(\gamma \cdot a) + \frac{e(\gamma \cdot a)(\gamma \cdot k)(\gamma \cdot x)}{m} + \frac{3e(\gamma \cdot k)(\gamma \cdot a)(\gamma \cdot x)}{2m} + \\
& \frac{De(\gamma \cdot x)(\gamma \cdot a)(\gamma \cdot k)}{2m} - \frac{3e(\gamma \cdot x)(\gamma \cdot a)(\gamma \cdot k)}{2m} - \frac{e(\gamma \cdot x)(\gamma \cdot k)(\gamma \cdot a)}{2m} - \frac{e^2\gamma \cdot a(a \cdot x)}{m^3s\xi^2} - \frac{De\gamma \cdot k(a \cdot x)}{2m} + \\
& \frac{e\gamma \cdot k(a \cdot x)}{2m} + \frac{De\Phi(\gamma \cdot k)(\gamma \cdot a)}{2(k \cdot p)} - \frac{3e\Phi(\gamma \cdot k)(\gamma \cdot a)}{2(k \cdot p)} - \frac{De\Phi(\gamma \cdot x)(\gamma \cdot k)(\gamma \cdot a)}{4ms(k \cdot p)} + \frac{3e\Phi(\gamma \cdot x)(\gamma \cdot k)(\gamma \cdot a)}{4ms(k \cdot p)} + \\
& \frac{De\Phi(\gamma \cdot a)(\gamma \cdot k)}{2(k \cdot q)} - \frac{3e\Phi(\gamma \cdot a)(\gamma \cdot k)}{2(k \cdot q)} - \frac{De\Phi(\gamma \cdot a)(\gamma \cdot k)(\gamma \cdot x)}{4ms(k \cdot q)} + \frac{e\Phi(\gamma \cdot a)(\gamma \cdot k)(\gamma \cdot x)}{2ms(k \cdot q)} + \\
& \frac{e\Phi(\gamma \cdot k)(\gamma \cdot x)(\gamma \cdot a)}{4ms(k \cdot q)} - \frac{e\Phi\gamma \cdot k(a \cdot x)}{2ms(k \cdot q)} + 1 - \frac{e^2\gamma \cdot k(a \cdot x)^2}{m^3s\xi^2\phi} - \frac{2it\gamma \cdot k}{ms\phi} + \frac{(\gamma \cdot k)(\gamma \cdot x)}{\phi} + \frac{(\gamma \cdot x)(\gamma \cdot k)}{\phi} + \\
& \frac{e^2(\gamma \cdot a)(\gamma \cdot k)(a \cdot x)}{m^2\xi^2\phi} + \frac{e^2(\gamma \cdot k)(\gamma \cdot a)(a \cdot x)}{m^2\xi^2\phi} + \frac{e^2(\gamma \cdot a)(\gamma \cdot x)(\gamma \cdot k)(a \cdot x)}{2m^3s\xi^2\phi} + \frac{e^2(\gamma \cdot k)(\gamma \cdot x)(\gamma \cdot a)(a \cdot x)}{2m^3s\xi^2\phi} \Big)
\end{aligned}$$

Integration over

$$\int d^{D-2} X_{\perp} dX_{+} \dots$$

Preexp does not depend on X_{\perp} and

X_{+} . Phase contains X_{\perp} and X_{+} in scalar products

$$pX = p_{-} X_{+} + p_{+} X_{-} - p_{\perp} X_{\perp};$$

$$qX = q_{-} X_{+} + q_{+} X_{-} - q_{\perp} X_{\perp};$$

so

$$\int d^{D-2} X_{\perp} dX_{+} e^{-i(p-q) \cdot X} \dots = (2\pi)^{D-1} \delta^{(D-2)}(p_{\perp} - q_{\perp}) \delta(p_{-} - q_{-}).$$

We will not write the δ - functions explicitly,
but we will assume that they are present;

Due to the conservation law $\delta^{(D-2)}(p_{\perp} - q_{\perp}) \delta(p_{-} - q_{-})$

$$kq \rightarrow kp$$

$$aq \rightarrow ap$$

Notations

$$pp = p_{+};$$

$$qp = q_{+};$$

The remaining integrals : $\int dX_{-} d^D x \dots$

$$\text{Coeff3} = \text{Coeff2} * (2\pi)^{(D-1)}$$

$$\text{Phase3} = \text{Phase2} /. \{\text{Pair}[\text{Momentum}[p, D], \text{Momentum}[X, D]] \rightarrow pp \, X_m, \\ \text{Pair}[\text{Momentum}[q, D], \text{Momentum}[X, D]] \rightarrow qp \, X_m\} /. \\ \{X_m \rightarrow \mathfrak{x}/m\} /. \{kq \rightarrow kp, aq \rightarrow ap\}$$

$$\text{Matrix3} = \text{Matrix2} /. \{kq \rightarrow kp, aq \rightarrow ap\}$$

$$-\frac{2^{-D-2} e^{-\frac{1}{2}i\pi D} e^2 m s^{-D/2} t^{-D/2}}{\pi^2}$$

$$-\frac{e\Phi\phi(a \cdot p)}{k \cdot p} + e\Phi(a \cdot x) + \frac{m^2 \xi^2 \phi^3}{24(k \cdot p)} + \frac{m^2 \xi^2 \Phi^2 \phi}{2(k \cdot p)} -$$

$$\frac{1}{12} m^2 \xi^2 s \phi^2 - m^2 s - \frac{pp \Phi}{m} + \frac{qp \Phi}{m} + \frac{p \cdot x}{2} + \frac{q \cdot x}{2} - \frac{x^2}{4s} - \frac{x^2}{4t}$$

$$J_0 \left(\frac{D m \xi^2 \gamma \cdot k \phi^3}{16 s (k \cdot p)^2} - \frac{m \xi^2 \gamma \cdot k \phi^3}{8 s (k \cdot p)^2} - \frac{D m \xi^2 \gamma \cdot k \phi^2}{4 (k \cdot p)} + \frac{3 m \xi^2 \gamma \cdot k \phi^2}{2 (k \cdot p)} - \frac{D e \gamma \cdot a \phi}{2 m} + \frac{e \gamma \cdot a \phi}{m} + \frac{1}{3} D m s \xi^2 \gamma \cdot k \phi - \right. \\ \left. \frac{2}{3} m s \xi^2 \gamma \cdot k \phi - \frac{D e (\gamma \cdot a) \cdot (\gamma \cdot k) \phi}{4 (k \cdot p)} + \frac{D e (\gamma \cdot k) \cdot (\gamma \cdot a) \phi}{4 (k \cdot p)} + \frac{D e (\gamma \cdot a) \cdot (\gamma \cdot k) \cdot (\gamma \cdot x) \phi}{8 m s (k \cdot p)} - \frac{e (\gamma \cdot a) \cdot (\gamma \cdot k) \cdot (\gamma \cdot x) \phi}{4 m s (k \cdot p)} - \right.$$

$$\begin{aligned}
& \frac{D e(\gamma \cdot x)(\gamma \cdot k)(\gamma \cdot a) \phi}{8 m s(k \cdot p)} + \frac{e(\gamma \cdot x)(\gamma \cdot k)(\gamma \cdot a) \phi}{4 m s(k \cdot p)} - \frac{D m \xi^2 \Phi^2 \gamma \cdot k \phi}{4 s(k \cdot p)^2} + \frac{m \xi^2 \Phi^2 \gamma \cdot k \phi}{2 s(k \cdot p)^2} + D - \frac{D \gamma \cdot x}{2 m s} + \\
& \frac{\gamma \cdot x}{m s} + D e s(\gamma \cdot a)(\gamma \cdot k) - 4 e s(\gamma \cdot a)(\gamma \cdot k) - \frac{e(\gamma \cdot k)(\gamma \cdot a)(\gamma \cdot x)}{m} - \frac{D e(\gamma \cdot x)(\gamma \cdot a)(\gamma \cdot k)}{2 m} + \\
& \frac{2 e(\gamma \cdot x)(\gamma \cdot a)(\gamma \cdot k)}{m} + \frac{D e \gamma \cdot k(a \cdot x)}{2 m} - \frac{e \gamma \cdot k(a \cdot x)}{m} - \frac{D e \Phi(\gamma \cdot a)(\gamma \cdot k)}{2(k \cdot p)} - \frac{D e \Phi(\gamma \cdot k)(\gamma \cdot a)}{2(k \cdot p)} + \\
& \frac{D e \Phi(\gamma \cdot a)(\gamma \cdot k)(\gamma \cdot x)}{4 m s(k \cdot p)} - \frac{e \Phi(\gamma \cdot a)(\gamma \cdot k)(\gamma \cdot x)}{2 m s(k \cdot p)} + \frac{D e \Phi(\gamma \cdot x)(\gamma \cdot k)(\gamma \cdot a)}{4 m s(k \cdot p)} - \frac{e \Phi(\gamma \cdot x)(\gamma \cdot k)(\gamma \cdot a)}{2 m s(k \cdot p)} \Big) + \\
& \text{J2} \left(-\frac{D m \xi^2 \gamma \cdot k \phi^3}{16 s(k \cdot p)^2} + \frac{3 m \xi^2 \gamma \cdot k \phi^3}{16 s(k \cdot p)^2} + \frac{D m \xi^2 \gamma \cdot k \phi^2}{4(k \cdot p)} - \frac{3 m \xi^2 \gamma \cdot k \phi^2}{4(k \cdot p)} + \frac{D e \gamma \cdot a \phi}{2 m} - \frac{e \gamma \cdot a \phi}{2 m} - \frac{1}{3} D m s \xi^2 \gamma \cdot k \phi + \right. \\
& m s \xi^2 \gamma \cdot k \phi + \frac{D e(\gamma \cdot a)(\gamma \cdot k) \phi}{4(k \cdot p)} - \frac{3 e(\gamma \cdot a)(\gamma \cdot k) \phi}{4(k \cdot p)} - \frac{D e(\gamma \cdot k)(\gamma \cdot a) \phi}{4(k \cdot p)} + \frac{3 e(\gamma \cdot k)(\gamma \cdot a) \phi}{4(k \cdot p)} - \\
& \frac{D e(\gamma \cdot a)(\gamma \cdot k)(\gamma \cdot x) \phi}{8 m s(k \cdot p)} + \frac{e(\gamma \cdot a)(\gamma \cdot k)(\gamma \cdot x) \phi}{4 m s(k \cdot p)} + \frac{e(\gamma \cdot k)(\gamma \cdot x)(\gamma \cdot a) \phi}{8 m s(k \cdot p)} + \frac{D e(\gamma \cdot x)(\gamma \cdot k)(\gamma \cdot a) \phi}{8 m s(k \cdot p)} - \\
& \frac{3 e(\gamma \cdot x)(\gamma \cdot k)(\gamma \cdot a) \phi}{8 m s(k \cdot p)} - \frac{e \gamma \cdot k(a \cdot x) \phi}{4 m s(k \cdot p)} + \frac{D m \xi^2 \Phi^2 \gamma \cdot k \phi}{4 s(k \cdot p)^2} - \frac{3 m \xi^2 \Phi^2 \gamma \cdot k \phi}{4 s(k \cdot p)^2} - D + \frac{D \gamma \cdot x}{2 m s} - \frac{3 \gamma \cdot x}{2 m s} - \\
& D e s(\gamma \cdot a)(\gamma \cdot k) + 4 e s(\gamma \cdot a)(\gamma \cdot k) + e s(\gamma \cdot k)(\gamma \cdot a) + \frac{e(\gamma \cdot a)(\gamma \cdot k)(\gamma \cdot x)}{m} + \frac{3 e(\gamma \cdot k)(\gamma \cdot a)(\gamma \cdot x)}{2 m} + \\
& \frac{D e(\gamma \cdot x)(\gamma \cdot a)(\gamma \cdot k)}{2 m} - \frac{3 e(\gamma \cdot x)(\gamma \cdot a)(\gamma \cdot k)}{2 m} - \frac{e(\gamma \cdot x)(\gamma \cdot k)(\gamma \cdot a)}{2 m} - \frac{e^2 \gamma \cdot a(a \cdot x)}{m^3 s \xi^2} - \frac{D e \gamma \cdot k(a \cdot x)}{2 m} + \\
& \frac{e \gamma \cdot k(a \cdot x)}{2 m} + \frac{D e \Phi(\gamma \cdot a)(\gamma \cdot k)}{2(k \cdot p)} - \frac{3 e \Phi(\gamma \cdot a)(\gamma \cdot k)}{2(k \cdot p)} + \frac{D e \Phi(\gamma \cdot k)(\gamma \cdot a)}{2(k \cdot p)} - \frac{3 e \Phi(\gamma \cdot k)(\gamma \cdot a)}{2(k \cdot p)} - \\
& \frac{D e \Phi(\gamma \cdot a)(\gamma \cdot k)(\gamma \cdot x)}{4 m s(k \cdot p)} + \frac{e \Phi(\gamma \cdot a)(\gamma \cdot k)(\gamma \cdot x)}{2 m s(k \cdot p)} + \frac{e \Phi(\gamma \cdot k)(\gamma \cdot x)(\gamma \cdot a)}{4 m s(k \cdot p)} - \frac{D e \Phi(\gamma \cdot x)(\gamma \cdot k)(\gamma \cdot a)}{4 m s(k \cdot p)} + \\
& \frac{3 e \Phi(\gamma \cdot x)(\gamma \cdot k)(\gamma \cdot a)}{4 m s(k \cdot p)} - \frac{e \Phi \gamma \cdot k(a \cdot x)}{2 m s(k \cdot p)} + 1 - \frac{e^2 \gamma \cdot k(a \cdot x)^2}{m^3 s \xi^2 \phi} - \frac{2 i t \gamma \cdot k}{m s \phi} + \frac{(\gamma \cdot k)(\gamma \cdot x)}{\phi} + \frac{(\gamma \cdot x)(\gamma \cdot k)}{\phi} + \\
& \frac{e^2(\gamma \cdot a)(\gamma \cdot k)(a \cdot x)}{m^2 \xi^2 \phi} + \frac{e^2(\gamma \cdot k)(\gamma \cdot a)(a \cdot x)}{m^2 \xi^2 \phi} + \frac{e^2(\gamma \cdot a)(\gamma \cdot x)(\gamma \cdot k)(a \cdot x)}{2 m^3 s \xi^2 \phi} + \frac{e^2(\gamma \cdot k)(\gamma \cdot x)(\gamma \cdot a)(a \cdot x)}{2 m^3 s \xi^2 \phi} \Big) + \\
& \text{J1} \left(-\frac{m \xi^2 \gamma \cdot k \phi^3}{16 s(k \cdot p)^2} - \frac{m \xi^2 \gamma \cdot k \phi^2}{4(k \cdot p)} - \frac{e \gamma \cdot a \phi}{2 m} - \frac{1}{3} m s \xi^2 \gamma \cdot k \phi + \frac{e(\gamma \cdot a)(\gamma \cdot k) \phi}{4(k \cdot p)} - \frac{e(\gamma \cdot k)(\gamma \cdot a) \phi}{4(k \cdot p)} - \right. \\
& \frac{e(\gamma \cdot k)(\gamma \cdot x)(\gamma \cdot a) \phi}{8 m s(k \cdot p)} + \frac{e(\gamma \cdot x)(\gamma \cdot k)(\gamma \cdot a) \phi}{8 m s(k \cdot p)} + \frac{e \gamma \cdot k(a \cdot x) \phi}{4 m s(k \cdot p)} + \frac{m \xi^2 \Phi^2 \gamma \cdot k \phi}{4 s(k \cdot p)^2} + \frac{\gamma \cdot x}{2 m s} - \\
& e s(\gamma \cdot k)(\gamma \cdot a) + \frac{e(\gamma \cdot x)(\gamma \cdot k)(\gamma \cdot a)}{2 m} + \frac{e^2 \gamma \cdot a(a \cdot x)}{m^3 s \xi^2} + \frac{e \gamma \cdot k(a \cdot x)}{2 m} + \frac{e \Phi(\gamma \cdot a)(\gamma \cdot k)}{2(k \cdot p)} + \frac{e \Phi(\gamma \cdot k)(\gamma \cdot a)}{2(k \cdot p)} - \\
& \frac{e \Phi(\gamma \cdot k)(\gamma \cdot x)(\gamma \cdot a)}{4 m s(k \cdot p)} - \frac{e \Phi(\gamma \cdot x)(\gamma \cdot k)(\gamma \cdot a)}{4 m s(k \cdot p)} + \frac{e \Phi \gamma \cdot k(a \cdot x)}{2 m s(k \cdot p)} - 1 + \frac{e^2 \gamma \cdot k(a \cdot x)^2}{m^3 s \xi^2 \phi} - \frac{2 i t \gamma \cdot k}{m s \phi} - \\
& \frac{e^2(\gamma \cdot a)(\gamma \cdot k)(a \cdot x)}{m^2 \xi^2 \phi} - \frac{e^2(\gamma \cdot k)(\gamma \cdot a)(a \cdot x)}{m^2 \xi^2 \phi} - \frac{e^2(\gamma \cdot a)(\gamma \cdot x)(\gamma \cdot k)(a \cdot x)}{2 m^3 s \xi^2 \phi} - \frac{e^2(\gamma \cdot k)(\gamma \cdot x)(\gamma \cdot a)(a \cdot x)}{2 m^3 s \xi^2 \phi} \Big)
\end{aligned}$$

Reordering the γ - matrix terms in the preexponent

We employ the equality

$$(\gamma a) (\gamma b) (\gamma c) \rightarrow -i \gamma^\beta \cdot \overline{\gamma}^5 \epsilon^{\beta \mu \nu \delta} a_\mu b_\nu c_\delta + (a b) (\gamma c) - (a c) (\gamma b) + (\gamma a) (b c)$$

to rewrite the terms with 3 gamma matrices

Then we recollect tensors $F_{\mu\nu}$,

$F^*_{\mu\nu}$ and $(F^2)_{\mu\nu}$ from the combinations of a_μ , k_μ and the antisymmetric tensor $\epsilon^{\alpha\beta\mu\nu}$

The scalar products (γa) and (γk) can be expressed as

$$(\gamma a) = \frac{1}{\phi} [(\gamma k) (ax) - (\gamma Fx)]$$

$$(\gamma k) = (\gamma k) (kx) / \phi = -a^2 k_\mu k_\nu \gamma^\mu x^\nu \frac{1}{-a^2 \phi} = \frac{e^2}{m^2 \xi^2 \phi} (\gamma F^2 x)$$

Then the result can be expressed as a linear

combination of following γ - matrix structures :

$$1,$$

$$(\gamma x),$$

$$(\gamma F^2 x),$$

$$(\sigma F) = \sigma_{\mu\nu} F^{\mu\nu},$$

$$\gamma^\beta \gamma^5 (F^* x)_\beta.$$

Also note that we treat $\overline{\gamma}^5$ as a 4 - dimensional object,

assuming that the terms incorporating it are finite

Coeff4 = Coeff3;

Phase4 = Phase3;

Matrix4 = Collect[

```
((Expand[Matrix3 /. TripleGamma]) /. EpsToF) /. FieldSubstitutions) /. akToF /.
{DiracGamma[Momentum[k, D], D].DiracGamma[Momentum[x, D], D] / phi ->
-DiracGamma[Momentum[x, D], D].DiracGamma[Momentum[k, D], D] / phi +
2}, {J0, J1, J2}]
```

$$\begin{aligned}
& \text{J1} \left(-\frac{m\xi^2 \gamma \cdot k \phi^3}{16s(k \cdot p)^2} - \frac{m\xi^2 \gamma \cdot k \phi^2}{4(k \cdot p)} - \frac{1}{3} m s \xi^2 \gamma \cdot k \phi + \frac{i e \sigma F \phi}{4(k \cdot p)} - \frac{e \Phi \gamma \cdot a \phi}{2 m s(k \cdot p)} - \frac{i e \gamma^\beta \cdot \bar{\gamma}^5 F D x^\beta \phi}{4 m s(k \cdot p)} + \right. \\
& \quad \left. \frac{m\xi^2 \Phi^2 \gamma \cdot k \phi}{4 s(k \cdot p)^2} + \frac{1}{2} i e s \sigma F + \frac{\gamma \cdot x}{2 m s} - \frac{i e \gamma^\beta \cdot \bar{\gamma}^5 F D x^\beta}{2 m} + \frac{e \Phi \gamma \cdot k(a \cdot x)}{2 m s(k \cdot p)} - 1 - \frac{2 i t \gamma \cdot k}{m s \phi} \right) + \\
& \text{J0} \left(\frac{D m \xi^2 \gamma \cdot k \phi^3}{16 s(k \cdot p)^2} - \frac{m \xi^2 \gamma \cdot k \phi^3}{8 s(k \cdot p)^2} - \frac{D m \xi^2 \gamma \cdot k \phi^2}{4(k \cdot p)} + \frac{3 m \xi^2 \gamma \cdot k \phi^2}{2(k \cdot p)} + \frac{1}{3} D m s \xi^2 \gamma \cdot k \phi - \right. \\
& \quad \frac{2}{3} m s \xi^2 \gamma \cdot k \phi - \frac{i D e \sigma F \phi}{4(k \cdot p)} + \frac{D e \Phi \gamma \cdot a \phi}{2 m s(k \cdot p)} - \frac{e \Phi \gamma \cdot a \phi}{m s(k \cdot p)} + \frac{i D e \gamma^\beta \cdot \bar{\gamma}^5 F D x^\beta \phi}{4 m s(k \cdot p)} - \\
& \quad \frac{i e \gamma^\beta \cdot \bar{\gamma}^5 F D x^\beta \phi}{2 m s(k \cdot p)} - \frac{D m \xi^2 \Phi^2 \gamma \cdot k \phi}{4 s(k \cdot p)^2} + \frac{m \xi^2 \Phi^2 \gamma \cdot k \phi}{2 s(k \cdot p)^2} + D + \frac{1}{2} i D e s \sigma F - 2 i e s \sigma F - \\
& \quad \frac{D \gamma \cdot x}{2 m s} + \frac{\gamma \cdot x}{m s} - \frac{i D e \gamma^\beta \cdot \bar{\gamma}^5 F D x^\beta}{2 m} + \frac{3 i e \gamma^\beta \cdot \bar{\gamma}^5 F D x^\beta}{m} - \frac{D e \Phi \gamma \cdot k(a \cdot x)}{2 m s(k \cdot p)} + \frac{e \Phi \gamma \cdot k(a \cdot x)}{m s(k \cdot p)} \Big) + \\
& \text{J2} \left(-\frac{D m \xi^2 \gamma \cdot k \phi^3}{16 s(k \cdot p)^2} + \frac{3 m \xi^2 \gamma \cdot k \phi^3}{16 s(k \cdot p)^2} + \frac{D m \xi^2 \gamma \cdot k \phi^2}{4(k \cdot p)} - \frac{3 m \xi^2 \gamma \cdot k \phi^2}{4(k \cdot p)} - \frac{1}{3} D m s \xi^2 \gamma \cdot k \phi + \right. \\
& \quad m s \xi^2 \gamma \cdot k \phi + \frac{i D e \sigma F \phi}{4(k \cdot p)} - \frac{3 i e \sigma F \phi}{4(k \cdot p)} - \frac{D e \Phi \gamma \cdot a \phi}{2 m s(k \cdot p)} + \frac{3 e \Phi \gamma \cdot a \phi}{2 m s(k \cdot p)} - \frac{i D e \gamma^\beta \cdot \bar{\gamma}^5 F D x^\beta \phi}{4 m s(k \cdot p)} + \\
& \quad \frac{3 i e \gamma^\beta \cdot \bar{\gamma}^5 F D x^\beta \phi}{4 m s(k \cdot p)} + \frac{D m \xi^2 \Phi^2 \gamma \cdot k \phi}{4 s(k \cdot p)^2} - \frac{3 m \xi^2 \Phi^2 \gamma \cdot k \phi}{4 s(k \cdot p)^2} - D - \frac{1}{2} i D e s \sigma F + \frac{3}{2} i e s \sigma F + \frac{D \gamma \cdot x}{2 m s} - \\
& \quad \left. \frac{3 \gamma \cdot x}{2 m s} + \frac{i D e \gamma^\beta \cdot \bar{\gamma}^5 F D x^\beta}{2 m} - \frac{3 i e \gamma^\beta \cdot \bar{\gamma}^5 F D x^\beta}{2 m} + \frac{D e \Phi \gamma \cdot k(a \cdot x)}{2 m s(k \cdot p)} - \frac{3 e \Phi \gamma \cdot k(a \cdot x)}{2 m s(k \cdot p)} + 3 - \frac{2 i t \gamma \cdot k}{m s \phi} \right)
\end{aligned}$$

Substituting the lightcone variables

We introduce the following notations

$$(xp, pp, qp, Gp) = (x, p, q, \gamma)_+,$$

$$xm = x_- = \phi / m,$$

$$pm = p_- = kp / m,$$

$$(xt, pt, Gt, at) = (x, p, \gamma, a)_\perp,$$

$$pp = p_+,$$

We introduce the following notations for $\gamma^5 (F^* x)_\beta$

$$GFDp = (\gamma^\beta \gamma^5 F^*_{\beta\mu})_+,$$

$$GFDt = (\gamma^\beta \gamma^5 F^*_{\beta\mu})_\perp,$$

$$\text{Note that } (\gamma^\beta \gamma^5 F^*_{\beta\mu})_- = 0.$$

We also will use the conservation law

$$qm = pm,$$

$$qt = pt;$$

Scalar products will take the form

$$x^2 = 2 x_+ x_- - x_\perp^2 = 2 x_p x_m - x_t^2,$$

$$(px) = p_+ x_- + p_- x_+ - p_\perp x_\perp = pp x_m + pm x_p - pt x_t,$$

$$(ax) = -a_\perp x_\perp = -at pt,$$

$$\gamma p = \gamma_- p_+ + \gamma_+ p_- - \gamma_\perp p_\perp = Gm \frac{s}{x_-} (p^2 + p_\perp^2) + Gp \frac{x_-}{2s} - Gt pt,$$

where we used that $p_+ =$

$$(p^2 + p_\perp^2) / 2 p_- \text{ and the definition of the proper time } s = x_- / 2 p_-;$$

$$\gamma k = \gamma_- k_+ = m Gm,$$

$$\gamma^\beta \gamma^5 (F^* x)_\beta = GFDp x_m - GFDt x_t.$$

The integration measure

$$dx_- = d\phi / m$$

$$dX_- = d\Phi / m$$

$$\int dX_- d^D x \dots = (m)^{-2} \int d\Phi d^{D-2} x_\perp d\phi dx_+ \dots$$

$$\text{remaining integrals : } \int d\Phi d^{D-2} x_\perp d\phi dx_+$$

$$\text{Coeff5} = \text{Coeff4} / (m)^2$$

$$\text{Phase5} = \text{Collect}[\text{Phase4} / .$$

$$\{xv2 \rightarrow 2 x_m x_p - x_t^2,$$

$$ax \rightarrow -at * xt,$$

$$\text{Pair}[\text{Momentum}[p, D], \text{Momentum}[x, D]] \rightarrow pp x_m + pm x_p - pt * xt,$$

$$\text{Pair}[\text{Momentum}[q, D], \text{Momentum}[x, D]] \rightarrow qp x_m + qm x_p - qt * xt,$$

$$\text{Pair}[\text{Momentum}[a, D], \text{Momentum}[p, D]] \rightarrow -at pt \} / .$$

$$\{qm \rightarrow pm, qt \rightarrow pt\} / . \{x_m \rightarrow \phi / m, p_m \rightarrow kp / m\}, \{x_p, \phi, x_t\}]$$

$$\text{Matrix5} =$$

$$\text{Collect}[\text{Expand}[\text{Matrix4} / . \{ \text{DiracGamma}[\text{Momentum}[x, D], D] \rightarrow Gp x_m + Gm * x_p - Gt * x_t,$$

$$\text{DiracGamma}[\text{LorentzIndex}[\beta, D], D] . \text{DiracGamma}[5]$$

$$\text{Pair}[\text{LorentzIndex}[\beta, D], \text{Momentum}[FDx, D]] \rightarrow GFDp x_m - GFDt * x_t,$$

$$\text{Pair}[\text{Momentum}[a, D], \text{Momentum}[x, D]] \rightarrow -at * xt \} / .$$

$$\{x_m \rightarrow \phi / m\}, \{J0, J1, J2, x_t, x_p\}]$$

$$- \frac{2^{-D-2} e^{-\frac{1}{2} i \pi D} e^2 s^{-D/2} t^{-D/2}}{\pi^2 m}$$

$$\begin{aligned}
& \phi \left(\frac{\text{at } e \text{pt } \Phi}{k \cdot p} + \frac{m^2 \xi^2 \Phi^2}{2(k \cdot p)} + \frac{\text{pp}}{2m} + \frac{\text{qp}}{2m} \right) + \text{xt}(-\text{at } e \Phi - \text{pt}) + \frac{m^2 \xi^2 \phi^3}{24(k \cdot p)} + \\
& \text{xp} \left(\frac{k \cdot p}{m} + \phi \left(-\frac{1}{2ms} - \frac{1}{2mt} \right) \right) - \frac{1}{12} m^2 \xi^2 s \phi^2 - m^2 s - \frac{\text{pp } \Phi}{m} + \frac{\text{qp } \Phi}{m} + \text{xt}^2 \left(\frac{1}{4s} + \frac{1}{4t} \right) \\
& \text{J1} \left(-\frac{m \xi^2 \gamma \cdot k \phi^3}{16s(k \cdot p)^2} - \frac{m \xi^2 \gamma \cdot k \phi^2}{4(k \cdot p)} - \frac{i e \text{GFDp } \phi^2}{4m^2 s(k \cdot p)} - \frac{1}{3} m s \xi^2 \gamma \cdot k \phi + \right. \\
& \quad \frac{\text{Gp } \phi}{2m^2 s} + \frac{i e \sigma \text{F } \phi}{4(k \cdot p)} - \frac{e \Phi \gamma \cdot a \phi}{2ms(k \cdot p)} - \frac{i e \text{GFDp } \phi}{2m^2} + \frac{m \xi^2 \Phi^2 \gamma \cdot k \phi}{4s(k \cdot p)^2} + \frac{\text{Gm xp}}{2ms} + \\
& \quad \left. \frac{1}{2} i e s \sigma \text{F} + \text{xt} \left(\frac{i e \text{GFDt}}{2m} + \frac{i e \phi \text{GFDt}}{4ms(k \cdot p)} - \frac{\text{Gt}}{2ms} - \frac{\text{at } e \Phi \gamma \cdot k}{2ms(k \cdot p)} \right) - 1 - \frac{2 i t \gamma \cdot k}{ms \phi} \right) + \\
& \text{J0} \left(\frac{D m \xi^2 \gamma \cdot k \phi^3}{16s(k \cdot p)^2} - \frac{m \xi^2 \gamma \cdot k \phi^3}{8s(k \cdot p)^2} - \frac{D m \xi^2 \gamma \cdot k \phi^2}{4(k \cdot p)} + \frac{3 m \xi^2 \gamma \cdot k \phi^2}{2(k \cdot p)} + \frac{i D e \text{GFDp } \phi^2}{4m^2 s(k \cdot p)} - \right. \\
& \quad \frac{i e \text{GFDp } \phi^2}{2m^2 s(k \cdot p)} + \frac{1}{3} D m s \xi^2 \gamma \cdot k \phi - \frac{2}{3} m s \xi^2 \gamma \cdot k \phi - \frac{D \text{Gp } \phi}{2m^2 s} + \frac{\text{Gp } \phi}{m^2 s} - \frac{i D e \sigma \text{F } \phi}{4(k \cdot p)} + \\
& \quad \frac{D e \Phi \gamma \cdot a \phi}{2ms(k \cdot p)} - \frac{e \Phi \gamma \cdot a \phi}{ms(k \cdot p)} - \frac{i D e \text{GFDp } \phi}{2m^2} + \frac{3 i e \text{GFDp } \phi}{m^2} - \frac{D m \xi^2 \Phi^2 \gamma \cdot k \phi}{4s(k \cdot p)^2} + \\
& \quad \frac{m \xi^2 \Phi^2 \gamma \cdot k \phi}{2s(k \cdot p)^2} + D + \left(\frac{\text{Gm}}{ms} - \frac{D \text{Gm}}{2ms} \right) \text{xp} + \frac{1}{2} i D e s \sigma \text{F} - 2 i e s \sigma \text{F} + \\
& \quad \left. \text{xt} \left(\frac{i D e \text{GFDt}}{2m} - \frac{3 i e \text{GFDt}}{m} - \frac{i D e \phi \text{GFDt}}{4ms(k \cdot p)} + \frac{i e \phi \text{GFDt}}{2ms(k \cdot p)} + \frac{D \text{Gt}}{2ms} - \frac{\text{Gt}}{ms} - \frac{\text{at } e \Phi \gamma \cdot k}{ms(k \cdot p)} + \frac{\text{at } D e \Phi \gamma \cdot k}{2ms(k \cdot p)} \right) \right) + \\
& \text{J2} \left(-\frac{D m \xi^2 \gamma \cdot k \phi^3}{16s(k \cdot p)^2} + \frac{3 m \xi^2 \gamma \cdot k \phi^3}{16s(k \cdot p)^2} + \frac{D m \xi^2 \gamma \cdot k \phi^2}{4(k \cdot p)} - \frac{3 m \xi^2 \gamma \cdot k \phi^2}{4(k \cdot p)} - \frac{i D e \text{GFDp } \phi^2}{4m^2 s(k \cdot p)} + \frac{3 i e \text{GFDp } \phi^2}{4m^2 s(k \cdot p)} - \right. \\
& \quad \frac{1}{3} D m s \xi^2 \gamma \cdot k \phi + m s \xi^2 \gamma \cdot k \phi + \frac{D \text{Gp } \phi}{2m^2 s} - \frac{3 \text{Gp } \phi}{2m^2 s} + \frac{i D e \sigma \text{F } \phi}{4(k \cdot p)} - \frac{3 i e \sigma \text{F } \phi}{4(k \cdot p)} - \frac{D e \Phi \gamma \cdot a \phi}{2ms(k \cdot p)} + \\
& \quad \frac{3 e \Phi \gamma \cdot a \phi}{2ms(k \cdot p)} + \frac{i D e \text{GFDp } \phi}{2m^2} - \frac{3 i e \text{GFDp } \phi}{2m^2} + \frac{D m \xi^2 \Phi^2 \gamma \cdot k \phi}{4s(k \cdot p)^2} - \frac{3 m \xi^2 \Phi^2 \gamma \cdot k \phi}{4s(k \cdot p)^2} - D + \\
& \quad \left(\frac{D \text{Gm}}{2ms} - \frac{3 \text{Gm}}{2ms} \right) \text{xp} - \frac{1}{2} i D e s \sigma \text{F} + \frac{3}{2} i e s \sigma \text{F} + \text{xt} \left(-\frac{i D e \text{GFDt}}{2m} + \frac{3 i e \text{GFDt}}{2m} + \frac{i D e \phi \text{GFDt}}{4ms(k \cdot p)} - \right. \\
& \quad \left. \frac{3 i e \phi \text{GFDt}}{4ms(k \cdot p)} - \frac{D \text{Gt}}{2ms} + \frac{3 \text{Gt}}{2ms} + \frac{3 \text{at } e \Phi \gamma \cdot k}{2ms(k \cdot p)} - \frac{\text{at } D e \Phi \gamma \cdot k}{2ms(k \cdot p)} \right) + 3 - \frac{2 i t \gamma \cdot k}{ms \phi} \Bigg)
\end{aligned}$$

Integration over **$\int dx_+$ and then $\int d\phi$...**

The phase is linear in x_+ ,
 the terms in the preexponent either do not depend or linear in x_+ ,
 therefore we can perform the integration using the equality

$$\int dx_+ \left(\frac{1}{x_+} \right) \text{Exp} [i x_+ P(\phi)] = 2\pi \left(\begin{array}{c} \delta(P(\phi)) \\ -i \delta'(P(\phi)) \end{array} \right)$$

To use it, we separate the terms linear in $x_p = x_+$.

Recall that J_k depends in ϕ , as

$$J_k = J_k[t, \chi_1 = \frac{\xi \phi}{2m^2 t}],$$

In the next steps we will use the shorthand notation

$$J_k = J_k[\phi].$$

```
Matrix52 = Collect[
  Coefficient[Expand[Matrix5], Gm xp] Gm xp /. {J0 -> J0[phi], J1 -> J1[phi], J2 -> J2[phi]},
  {Gm xp, J0[phi], J1[phi], J2[phi]}]
Matrix51 = Collect[
  (Expand[Matrix5] /. {J0 -> J0[phi], J1 -> J1[phi], J2 -> J2[phi]}) - Expand[Matrix52],
  {J0[phi], J1[phi], J2[phi], xt, xp}]
Gm xp (J0(phi) (1/m s - D/(2 m s)) + J2(phi) (D/(2 m s) - 3/(2 m s)) + J1(phi)/(2 m s))
```

$$\begin{aligned}
& J1(\phi) \left(-\frac{m\xi^2 \gamma \cdot k \phi^3}{16 s(k \cdot p)^2} - \frac{m\xi^2 \gamma \cdot k \phi^2}{4(k \cdot p)} - \frac{i e GFDp \phi^2}{4 m^2 s(k \cdot p)} - \right. \\
& \quad \frac{1}{3} m s \xi^2 \gamma \cdot k \phi + \frac{Gp \phi}{2 m^2 s} + \frac{i e \sigma F \phi}{4(k \cdot p)} - \frac{e \Phi \gamma \cdot a \phi}{2 m s(k \cdot p)} - \frac{i e GFDp \phi}{2 m^2} + \frac{m \xi^2 \Phi^2 \gamma \cdot k \phi}{4 s(k \cdot p)^2} + \\
& \quad \left. \frac{1}{2} i e s \sigma F + xt \left(\frac{i e GFDt}{2 m} + \frac{i e \phi GFDt}{4 m s(k \cdot p)} - \frac{Gt}{2 m s} - \frac{at e \Phi \gamma \cdot k}{2 m s(k \cdot p)} \right) - 1 - \frac{2 i t \gamma \cdot k}{m s \phi} \right) + \\
& J0(\phi) \left(\frac{D m \xi^2 \gamma \cdot k \phi^3}{16 s(k \cdot p)^2} - \frac{m \xi^2 \gamma \cdot k \phi^3}{8 s(k \cdot p)^2} - \frac{D m \xi^2 \gamma \cdot k \phi^2}{4(k \cdot p)} + \frac{3 m \xi^2 \gamma \cdot k \phi^2}{2(k \cdot p)} + \frac{i D e GFDp \phi^2}{4 m^2 s(k \cdot p)} - \frac{i e GFDp \phi^2}{2 m^2 s(k \cdot p)} + \right. \\
& \quad \frac{1}{3} D m s \xi^2 \gamma \cdot k \phi - \frac{2}{3} m s \xi^2 \gamma \cdot k \phi - \frac{D Gp \phi}{2 m^2 s} + \frac{Gp \phi}{m^2 s} - \frac{i D e \sigma F \phi}{4(k \cdot p)} + \frac{D e \Phi \gamma \cdot a \phi}{2 m s(k \cdot p)} - \frac{e \Phi \gamma \cdot a \phi}{m s(k \cdot p)} - \\
& \quad \frac{i D e GFDp \phi}{2 m^2} + \frac{3 i e GFDp \phi}{m^2} - \frac{D m \xi^2 \Phi^2 \gamma \cdot k \phi}{4 s(k \cdot p)^2} + \frac{m \xi^2 \Phi^2 \gamma \cdot k \phi}{2 s(k \cdot p)^2} + D + \frac{1}{2} i D e s \sigma F - 2 i e s \sigma F + \\
& \quad \left. xt \left(\frac{i D e GFDt}{2 m} - \frac{3 i e GFDt}{m} - \frac{i D e \phi GFDt}{4 m s(k \cdot p)} + \frac{i e \phi GFDt}{2 m s(k \cdot p)} + \frac{D Gt}{2 m s} - \frac{Gt}{m s} - \frac{at e \Phi \gamma \cdot k}{m s(k \cdot p)} + \frac{at D e \Phi \gamma \cdot k}{2 m s(k \cdot p)} \right) \right) + \\
& J2(\phi) \left(-\frac{D m \xi^2 \gamma \cdot k \phi^3}{16 s(k \cdot p)^2} + \frac{3 m \xi^2 \gamma \cdot k \phi^3}{16 s(k \cdot p)^2} + \frac{D m \xi^2 \gamma \cdot k \phi^2}{4(k \cdot p)} - \frac{3 m \xi^2 \gamma \cdot k \phi^2}{4(k \cdot p)} - \frac{i D e GFDp \phi^2}{4 m^2 s(k \cdot p)} + \right. \\
& \quad \frac{3 i e GFDp \phi^2}{4 m^2 s(k \cdot p)} - \frac{1}{3} D m s \xi^2 \gamma \cdot k \phi + m s \xi^2 \gamma \cdot k \phi + \frac{D Gp \phi}{2 m^2 s} - \frac{3 Gp \phi}{2 m^2 s} + \frac{i D e \sigma F \phi}{4(k \cdot p)} - \\
& \quad \frac{3 i e \sigma F \phi}{4(k \cdot p)} - \frac{D e \Phi \gamma \cdot a \phi}{2 m s(k \cdot p)} + \frac{3 e \Phi \gamma \cdot a \phi}{2 m s(k \cdot p)} + \frac{i D e GFDp \phi}{2 m^2} - \frac{3 i e GFDp \phi}{2 m^2} + \frac{D m \xi^2 \Phi^2 \gamma \cdot k \phi}{4 s(k \cdot p)^2} - \\
& \quad \frac{3 m \xi^2 \Phi^2 \gamma \cdot k \phi}{4 s(k \cdot p)^2} - D - \frac{1}{2} i D e s \sigma F + \frac{3}{2} i e s \sigma F + xt \left(-\frac{i D e GFDt}{2 m} + \frac{3 i e GFDt}{2 m} + \frac{i D e \phi GFDt}{4 m s(k \cdot p)} - \right. \\
& \quad \left. \frac{3 i e \phi GFDt}{4 m s(k \cdot p)} - \frac{D Gt}{2 m s} + \frac{3 Gt}{2 m s} + \frac{3 at e \Phi \gamma \cdot k}{2 m s(k \cdot p)} - \frac{at D e \Phi \gamma \cdot k}{2 m s(k \cdot p)} \right) + 3 - \frac{2 i t \gamma \cdot k}{m s \phi} \Big)
\end{aligned}$$

Phase5xp =

Collect[Simplify[Coefficient[Phase5, xp] /. {t → s ω / (s - ω)}], {kp}, Simplify]

φ0 = Collect[φ /. Solve[Phase5xp == 0, φ][[1]], {kp}, Simplify]

dP = Simplify[D[Phase5xp, φ]] /. {φ → φ0}

AbsdP = -dP

Phase5noxp = Collect[Phase5 /. {xp → 0} /. {t → s ω / (s - ω)}, {xp, φ, xt}, Simplify]

$$\frac{k \cdot p}{m} - \frac{\phi}{2 m \omega}$$

$$2 \omega(k \cdot p)$$

$$-\frac{1}{2 m \omega}$$

$$\frac{1}{2 m \omega}$$

$$\frac{1}{2} \phi \left(\frac{\Phi(2 at ept + m^2 \xi^2 \Phi)}{k \cdot p} + \frac{pp + qp}{m} \right) + xt(-at e \Phi - pt) + \frac{m^2 \xi^2 \phi^3}{24(k \cdot p)} - \frac{m^3 s + pp \Phi - qp \Phi}{m} - \frac{1}{12} m^2 \xi^2 s \phi^2 + \frac{xt^2}{4 \omega}$$

Now we can perform the integrations

$$\int dx_+ \left(\frac{1}{x_+} \right) \text{Exp} [i x_+ P(\phi)] =$$

$$2\pi \left(\frac{\delta(P(\phi))}{-i\delta'(P(\phi))} \right) = 2\pi \left(\frac{\delta(\phi - \phi_0) \frac{1}{|P'(\phi_0)|}}{-i\delta(\phi - \phi_0) \frac{1}{|P'(\phi_0)|} \left(-\frac{d}{d\phi} \frac{1}{P'(\phi)} \cdot \right)} \right)$$

Here $\left(-\frac{d}{d\phi} \frac{1}{P'(\phi)} \cdot \right)$ is an operator,
acting on a function in the place of ".". Then,

$$\int d\phi 2\pi \left(\frac{\delta(\phi - \phi_0) \frac{1}{|P'(\phi_0)|}}{-i\delta(\phi - \phi_0) \frac{1}{|P'(\phi_0)|} \left(-\frac{d}{d\phi} \frac{1}{P'(\phi)} \cdot \right)} \right) f(\phi) e^{ig(\phi)} =$$

$$\frac{2\pi}{|P'(\phi_0)|} e^{ig(\phi_0)} \left(i \frac{d}{d\phi} \left(f(\phi) \frac{1}{P'(\phi)} \right) \Big|_{\phi_0} - f(\phi_0) \frac{g'(\phi_0)}{P'(\phi_0)} \right)$$

$$P(\phi) = \frac{(k \cdot p)}{m} - \frac{\phi}{2m\omega},$$

$$P'(\phi) = -\frac{1}{2m\omega},$$

$$\omega^{-1} = s^{-1} + t^{-1}, \quad \omega = st / (s + t)$$

Note that we will take the derivative of $J_k[\phi]$ in ϕ too;

We introduce the shorthand

$$DJ_k \equiv \frac{d}{d\phi} J_k[\phi] \equiv \frac{d}{d\phi} J_k(t, \chi_1(\phi))$$

The remaining integrals : $\int d\Phi d^{D-2}x_\perp$

Coeff6 = Coeff5 * 2 π / AbsdP

Phase6 =

Collect[Expand[Simplify[Phase5 /. {xp → 0} /. {φ → φ0} /. {t → s ω / (s - ω)}]], {xt, Φ}]

$$-\frac{2^{-D} e^{-\frac{1}{2} i \pi D} e^2 \omega s^{-D/2} t^{-D/2}}{\pi}$$

$$\frac{1}{3} m^2 \xi^2 \omega^3 (k \cdot p)^2 - \frac{1}{3} m^2 \xi^2 s \omega^2 (k \cdot p)^2 + \Phi \left(2 \text{at } e p t \omega - \frac{p p}{m} + \frac{q p}{m} \right) +$$

$$x t (-\text{at } e \Phi - p t) + \frac{p p \omega (k \cdot p)}{m} + \frac{q p \omega (k \cdot p)}{m} + m^2 \xi^2 \Phi^2 \omega - m^2 s + \frac{x t^2}{4 \omega}$$

Note that Matrix52 produces terms that are proportional to J_k and $\frac{dJ_k}{ds}$.

We combine the ones that are proportional to J_k together in Matrix61, and leave the rest in Matrix62

We also again denote

$$J_k = J_k\left[t, \frac{\xi \phi_0}{2 m^2 t}\right]$$

Matrix61 =

```
Collect[Expand[Matrix51 - (Matrix52 /. {xp -> D[Phase5noxp, phi]/dP}]] /. {J0[phi] -> J0,
  J1[phi] -> J1, J2[phi] -> J2} /. {phi -> phi0}, {J0, J1, J2, xt, Gm, Gp, GFDp, pp}]
```

Matrix62 = I D[1/dP * Matrix52, phi] /. {xp -> 1} /. {phi -> phi0}

$$\begin{aligned}
& \text{J1} \left(-\frac{m \xi^2 \gamma \cdot k(k \cdot p) \omega^3}{2s} - m \xi^2 \gamma \cdot k(k \cdot p) \omega^2 + \frac{1}{2} i e \sigma F \omega - \right. \\
& \quad \frac{e \Phi \gamma \cdot a \omega}{ms} - \frac{2}{3} m s \xi^2 \gamma \cdot k(k \cdot p) \omega + \frac{Gp(k \cdot p) \omega}{m^2 s} + \frac{m \xi^2 \Phi^2 \gamma \cdot k \omega}{2s(k \cdot p)} + \frac{1}{2} i e s \sigma F + \\
& \quad \text{xt} \left(\frac{i e \omega GFDt}{2ms} + \frac{i e GFDt}{2m} - \frac{Gt}{2ms} - \frac{at e \Phi \gamma \cdot k}{2ms(k \cdot p)} \right) + GFDp \left(-\frac{i e(k \cdot p) \omega^2}{m^2 s} - \frac{i e(k \cdot p) \omega}{m^2} \right) + Gm \\
& \quad \left(\frac{m^2 \xi^2 (k \cdot p) \omega^3}{2s} - \frac{1}{3} m^2 \xi^2 (k \cdot p) \omega^2 + \frac{pp \omega}{2ms} + \frac{qp \omega}{2ms} + \frac{m^2 \xi^2 \Phi^2 \omega}{2s(k \cdot p)} + \frac{at e pt \Phi \omega}{s(k \cdot p)} \right) - 1 - \frac{i t \gamma \cdot k}{ms(k \cdot p) \omega} \Big) + \\
& \text{J0} \left(\frac{D m \xi^2 \gamma \cdot k(k \cdot p) \omega^3}{2s} - \frac{m \xi^2 \gamma \cdot k(k \cdot p) \omega^3}{s} - D m \xi^2 \gamma \cdot k(k \cdot p) \omega^2 + 6 m \xi^2 \gamma \cdot k(k \cdot p) \omega^2 - \right. \\
& \quad \frac{1}{2} i D e \sigma F \omega + \frac{D e \Phi \gamma \cdot a \omega}{ms} - \frac{2 e \Phi \gamma \cdot a \omega}{ms} + \frac{2}{3} D m s \xi^2 \gamma \cdot k(k \cdot p) \omega - \\
& \quad \frac{4}{3} m s \xi^2 \gamma \cdot k(k \cdot p) \omega - \frac{D m \xi^2 \Phi^2 \gamma \cdot k \omega}{2s(k \cdot p)} + \frac{m \xi^2 \Phi^2 \gamma \cdot k \omega}{s(k \cdot p)} + D + \frac{1}{2} i D e s \sigma F - 2 i e s \sigma F + \\
& \quad \text{xt} \left(-\frac{i D e \omega GFDt}{2ms} + \frac{i e \omega GFDt}{ms} + \frac{i D e GFDt}{2m} - \frac{3 i e GFDt}{m} + \frac{D Gt}{2ms} - \frac{Gt}{ms} - \frac{at e \Phi \gamma \cdot k}{ms(k \cdot p)} + \frac{at D e \Phi \gamma \cdot k}{2ms(k \cdot p)} \right) + \\
& \quad Gp \left(\frac{2 \omega(k \cdot p)}{m^2 s} - \frac{D \omega(k \cdot p)}{m^2 s} \right) + GFDp \left(\frac{i D e(k \cdot p) \omega^2}{m^2 s} - \frac{2 i e(k \cdot p) \omega^2}{m^2 s} - \frac{i D e(k \cdot p) \omega}{m^2} + \frac{6 i e(k \cdot p) \omega}{m^2} \right) + \\
& \quad Gm \left(-\frac{D m^2 \xi^2 (k \cdot p) \omega^3}{2s} + \frac{m^2 \xi^2 (k \cdot p) \omega^3}{s} + \frac{1}{3} D m^2 \xi^2 (k \cdot p) \omega^2 - \frac{2}{3} m^2 \xi^2 (k \cdot p) \omega^2 - \frac{D qp \omega}{2ms} + \right. \\
& \quad \left. \frac{qp \omega}{ms} - \frac{D m^2 \xi^2 \Phi^2 \omega}{2s(k \cdot p)} + \frac{m^2 \xi^2 \Phi^2 \omega}{s(k \cdot p)} + \frac{2 at e pt \Phi \omega}{s(k \cdot p)} - \frac{at D e pt \Phi \omega}{s(k \cdot p)} + pp \left(\frac{\omega}{ms} - \frac{D \omega}{2ms} \right) \right) \Big) + \\
& \text{J2} \left(-\frac{D m \xi^2 \gamma \cdot k(k \cdot p) \omega^3}{2s} + \frac{3 m \xi^2 \gamma \cdot k(k \cdot p) \omega^3}{2s} + D m \xi^2 \gamma \cdot k(k \cdot p) \omega^2 - 3 m \xi^2 \gamma \cdot k(k \cdot p) \omega^2 + \right. \\
& \quad \frac{1}{2} i D e \sigma F \omega - \frac{3}{2} i e \sigma F \omega - \frac{D e \Phi \gamma \cdot a \omega}{ms} + \frac{3 e \Phi \gamma \cdot a \omega}{ms} - \frac{2}{3} D m s \xi^2 \gamma \cdot k(k \cdot p) \omega + \\
& \quad 2 m s \xi^2 \gamma \cdot k(k \cdot p) \omega + \frac{D m \xi^2 \Phi^2 \gamma \cdot k \omega}{2s(k \cdot p)} - \frac{3 m \xi^2 \Phi^2 \gamma \cdot k \omega}{2s(k \cdot p)} - D - \frac{1}{2} i D e s \sigma F + \\
& \quad \frac{3}{2} i e s \sigma F + \text{xt} \left(\frac{i D e \omega GFDt}{2ms} - \frac{3 i e \omega GFDt}{2ms} - \frac{i D e GFDt}{2m} + \frac{3 i e GFDt}{2m} - \frac{D Gt}{2ms} + \right. \\
& \quad \left. \frac{3 Gt}{2ms} + \frac{3 at e \Phi \gamma \cdot k}{2ms(k \cdot p)} - \frac{at D e \Phi \gamma \cdot k}{2ms(k \cdot p)} \right) + Gp \left(\frac{D \omega(k \cdot p)}{m^2 s} - \frac{3 \omega(k \cdot p)}{m^2 s} \right) + \\
& \quad GFDp \left(-\frac{i D e(k \cdot p) \omega^2}{m^2 s} + \frac{3 i e(k \cdot p) \omega^2}{m^2 s} + \frac{i D e(k \cdot p) \omega}{m^2} - \frac{3 i e(k \cdot p) \omega}{m^2} \right) + Gm \left(\frac{D m^2 \xi^2 (k \cdot p) \omega^3}{2s} - \right. \\
& \quad \left. \frac{3 m^2 \xi^2 (k \cdot p) \omega^3}{2s} - \frac{1}{3} D m^2 \xi^2 (k \cdot p) \omega^2 + m^2 \xi^2 (k \cdot p) \omega^2 + \frac{D qp \omega}{2ms} - \frac{3 qp \omega}{2ms} + \frac{D m^2 \xi^2 \Phi^2 \omega}{2s(k \cdot p)} - \right. \\
& \quad \left. \frac{3 m^2 \xi^2 \Phi^2 \omega}{2s(k \cdot p)} - \frac{3 at e pt \Phi \omega}{s(k \cdot p)} + \frac{at D e pt \Phi \omega}{s(k \cdot p)} + pp \left(\frac{D \omega}{2ms} - \frac{3 \omega}{2ms} \right) \right) + 3 - \frac{i t \gamma \cdot k}{ms(k \cdot p) \omega} \Big) \\
& - 2 i Gm m \omega \left(\left(\frac{1}{ms} - \frac{D}{2ms} \right) J0'(2 \omega(k \cdot p)) + \left(\frac{D}{2ms} - \frac{3}{2ms} \right) J2'(2 \omega(k \cdot p)) + \frac{J1'(2 \omega(k \cdot p))}{2ms} \right)
\end{aligned}$$

Let us rewrite Matrix62

After the last integration

$$J_k = J_k \left(t, \frac{\xi \phi_0}{2 m^2 t} \right);$$

$$J_k'(\phi_0) = \frac{\partial J_k \left(t, \frac{\xi \phi}{2 m^2 t} \right)}{\partial \phi} \Big|_{\phi_0} = \frac{\partial}{\partial \chi_l} J_k \left(t, \chi_l(\phi_0) \right) * \frac{\xi}{2 m^2 t};$$

$$\chi_l(\phi_0) = \frac{\xi \phi_0}{2 m^2 t} = \frac{\xi \omega(k \cdot p)}{m^2 t};$$

In this step,

we use the initial assumption that J_0 does not depend on χ_l , therefore

$$\frac{\partial}{\partial \chi_l} J_0 = 0.$$

We denote

$$dJ_k d\chi_l = \frac{\partial}{\partial \chi_l} J_k \left(t, \chi_l(\phi_0) \right), \quad k = 1, 2$$

Coeff7 = Coeff6;

Phase7 = Phase6;

Matrix71 = Matrix61;

$$\chi_l \phi = \xi \phi / 2 / m^2 / t$$

$$\chi_l \phi_0 = \chi_l \phi /. \{\phi \rightarrow \phi_0\}$$

$$d\chi_l d\phi = D[\chi_l \phi, \phi] /. \{\phi \rightarrow \phi_0\}$$

Matrix72d =

$$\text{Collect}[\text{Matrix62} /. \{J_0'[2 \omega kp] \rightarrow dJ_0 d\chi_l * d\chi_l d\phi, J_1'[2 \omega kp] \rightarrow dJ_1 d\chi_l * d\chi_l d\phi, \\ J_2'[2 \omega kp] \rightarrow dJ_2 d\chi_l * d\chi_l d\phi\} /. \{dJ_0 d\chi_l \rightarrow 0\}, \{dJ_0 d\chi_l, dJ_1 d\chi_l, dJ_2 d\chi_l\}]$$

$$\frac{\xi \phi}{2 m^2 t}$$

$$\frac{\xi \omega(k \cdot p)}{m^2 t}$$

$$\frac{\xi}{2 m^2 t}$$

$$- \frac{i dJ_2 d\chi_l G m \xi \omega \left(\frac{D}{2 m s} - \frac{3}{2 m s} \right)}{m t} - \frac{i dJ_1 d\chi_l G m \xi \omega}{2 m^2 s t}$$

Matrix7 = Collect[Matrix71 + Matrix72d,

{J0, J1, J2, dJ0dχl, dJ1dχl, dJ2dχl, xt, Gm, Gp, GFDp, pp}]

$$\begin{aligned}
& - \frac{i \, dJ2d\chi l \, Gm \left(\frac{D}{2ms} - \frac{3}{2ms} \right) \xi \omega}{m t} - \frac{i \, dJ1d\chi l \, Gm \xi \omega}{2 m^2 s t} + \\
& J1 \left(- \frac{m \xi^2 \gamma \cdot k(k \cdot p) \omega^3}{2 s} - m \xi^2 \gamma \cdot k(k \cdot p) \omega^2 + \frac{1}{2} i e \sigma F \omega - \frac{e \Phi \gamma \cdot a \omega}{m s} - \right. \\
& \quad \frac{2}{3} m s \xi^2 \gamma \cdot k(k \cdot p) \omega + \frac{Gp(k \cdot p) \omega}{m^2 s} + \frac{m \xi^2 \Phi^2 \gamma \cdot k \omega}{2 s(k \cdot p)} + \frac{1}{2} i e s \sigma F + \\
& \quad \text{xt} \left(\frac{i e \omega GFDt}{2 m s} + \frac{i e GFDt}{2 m} - \frac{Gt}{2 m s} - \frac{at e \Phi \gamma \cdot k}{2 m s(k \cdot p)} \right) + GFDp \left(- \frac{i e(k \cdot p) \omega^2}{m^2 s} - \frac{i e(k \cdot p) \omega}{m^2} \right) + Gm \\
& \quad \left(\frac{m^2 \xi^2(k \cdot p) \omega^3}{2 s} - \frac{1}{3} m^2 \xi^2(k \cdot p) \omega^2 + \frac{pp \omega}{2 m s} + \frac{qp \omega}{2 m s} + \frac{m^2 \xi^2 \Phi^2 \omega}{2 s(k \cdot p)} + \frac{at e pt \Phi \omega}{s(k \cdot p)} \right) - 1 - \frac{i t \gamma \cdot k}{m s(k \cdot p) \omega} \Bigg) + \\
& J0 \left(\frac{D m \xi^2 \gamma \cdot k(k \cdot p) \omega^3}{2 s} - \frac{m \xi^2 \gamma \cdot k(k \cdot p) \omega^3}{s} - D m \xi^2 \gamma \cdot k(k \cdot p) \omega^2 + 6 m \xi^2 \gamma \cdot k(k \cdot p) \omega^2 - \right. \\
& \quad \frac{1}{2} i D e \sigma F \omega + \frac{D e \Phi \gamma \cdot a \omega}{m s} - \frac{2 e \Phi \gamma \cdot a \omega}{m s} + \frac{2}{3} D m s \xi^2 \gamma \cdot k(k \cdot p) \omega - \\
& \quad \frac{4}{3} m s \xi^2 \gamma \cdot k(k \cdot p) \omega - \frac{D m \xi^2 \Phi^2 \gamma \cdot k \omega}{2 s(k \cdot p)} + \frac{m \xi^2 \Phi^2 \gamma \cdot k \omega}{s(k \cdot p)} + D + \frac{1}{2} i D e s \sigma F - 2 i e s \sigma F + \\
& \quad \text{xt} \left(- \frac{i D e \omega GFDt}{2 m s} + \frac{i e \omega GFDt}{m s} + \frac{i D e GFDt}{2 m} - \frac{3 i e GFDt}{m} + \frac{D Gt}{2 m s} - \frac{Gt}{m s} - \frac{at e \Phi \gamma \cdot k}{m s(k \cdot p)} + \frac{at D e \Phi \gamma \cdot k}{2 m s(k \cdot p)} \right) + \\
& \quad Gp \left(\frac{2 \omega(k \cdot p)}{m^2 s} - \frac{D \omega(k \cdot p)}{m^2 s} \right) + GFDp \left(\frac{i D e(k \cdot p) \omega^2}{m^2 s} - \frac{2 i e(k \cdot p) \omega^2}{m^2 s} - \frac{i D e(k \cdot p) \omega}{m^2} + \frac{6 i e(k \cdot p) \omega}{m^2} \right) + \\
& \quad Gm \left(- \frac{D m^2 \xi^2(k \cdot p) \omega^3}{2 s} + \frac{m^2 \xi^2(k \cdot p) \omega^3}{s} + \frac{1}{3} D m^2 \xi^2(k \cdot p) \omega^2 - \frac{2}{3} m^2 \xi^2(k \cdot p) \omega^2 - \frac{D qp \omega}{2 m s} + \right. \\
& \quad \left. \frac{qp \omega}{m s} - \frac{D m^2 \xi^2 \Phi^2 \omega}{2 s(k \cdot p)} + \frac{m^2 \xi^2 \Phi^2 \omega}{s(k \cdot p)} + \frac{2 at e pt \Phi \omega}{s(k \cdot p)} - \frac{at D e pt \Phi \omega}{s(k \cdot p)} + pp \left(\frac{\omega}{m s} - \frac{D \omega}{2 m s} \right) \right) \Bigg) + \\
& J2 \left(- \frac{D m \xi^2 \gamma \cdot k(k \cdot p) \omega^3}{2 s} + \frac{3 m \xi^2 \gamma \cdot k(k \cdot p) \omega^3}{2 s} + D m \xi^2 \gamma \cdot k(k \cdot p) \omega^2 - 3 m \xi^2 \gamma \cdot k(k \cdot p) \omega^2 + \right. \\
& \quad \frac{1}{2} i D e \sigma F \omega - \frac{3}{2} i e \sigma F \omega - \frac{D e \Phi \gamma \cdot a \omega}{m s} + \frac{3 e \Phi \gamma \cdot a \omega}{m s} - \frac{2}{3} D m s \xi^2 \gamma \cdot k(k \cdot p) \omega + \\
& \quad 2 m s \xi^2 \gamma \cdot k(k \cdot p) \omega + \frac{D m \xi^2 \Phi^2 \gamma \cdot k \omega}{2 s(k \cdot p)} - \frac{3 m \xi^2 \Phi^2 \gamma \cdot k \omega}{2 s(k \cdot p)} - D - \frac{1}{2} i D e s \sigma F + \\
& \quad \frac{3}{2} i e s \sigma F + \text{xt} \left(\frac{i D e \omega GFDt}{2 m s} - \frac{3 i e \omega GFDt}{2 m s} - \frac{i D e GFDt}{2 m} + \frac{3 i e GFDt}{2 m} - \frac{D Gt}{2 m s} + \right. \\
& \quad \left. \frac{3 Gt}{2 m s} + \frac{3 at e \Phi \gamma \cdot k}{2 m s(k \cdot p)} - \frac{at D e \Phi \gamma \cdot k}{2 m s(k \cdot p)} \right) + Gp \left(\frac{D \omega(k \cdot p)}{m^2 s} - \frac{3 \omega(k \cdot p)}{m^2 s} \right) + \\
& \quad GFDp \left(- \frac{i D e(k \cdot p) \omega^2}{m^2 s} + \frac{3 i e(k \cdot p) \omega^2}{m^2 s} + \frac{i D e(k \cdot p) \omega}{m^2} - \frac{3 i e(k \cdot p) \omega}{m^2} \right) + Gm \left(\frac{D m^2 \xi^2(k \cdot p) \omega^3}{2 s} - \right. \\
& \quad \left. \frac{3 m^2 \xi^2(k \cdot p) \omega^3}{2 s} - \frac{1}{3} D m^2 \xi^2(k \cdot p) \omega^2 + m^2 \xi^2(k \cdot p) \omega^2 + \frac{D qp \omega}{2 m s} - \frac{3 qp \omega}{2 m s} + \frac{D m^2 \xi^2 \Phi^2 \omega}{2 s(k \cdot p)} - \right. \\
& \quad \left. \frac{3 m^2 \xi^2 \Phi^2 \omega}{2 s(k \cdot p)} - \frac{3 at e pt \Phi \omega}{s(k \cdot p)} + \frac{at D e pt \Phi \omega}{s(k \cdot p)} + pp \left(\frac{D \omega}{2 m s} - \frac{3 \omega}{2 m s} \right) \right) + 3 - \frac{i t \gamma \cdot k}{m s(k \cdot p) \omega} \Bigg)
\end{aligned}$$

Integration over

$$\int d^{D-2} x_{\perp} \dots :$$

The integral is gaussian

$$\begin{aligned} I_0 &= \int d^{D-2} x_{\perp} \text{Exp} \left[I A x_{\perp}^2 + I (J \cdot x_{\perp}) \right] = \\ &= \text{Exp} \left[I \frac{\pi}{2} \frac{D-2}{2} \right] \pi^{\frac{D-2}{2}} (\det A)^{-\frac{1}{2}} \text{Exp} \left[-I \frac{1}{4} J \cdot A^{-1} \cdot J \right] \end{aligned}$$

The preexponent contains terms linear in x_{\perp} , so we will also need

$$\begin{aligned} I_1 &= \int d^{D-2} x_{\perp} x_{\perp} \text{Exp} \left[I A x_{\perp}^2 + I (J \cdot x_{\perp}) \right] = \\ &= -\frac{1}{2} (A^{-1} \cdot J) I_0; \end{aligned}$$

$$\begin{aligned} I_2 &= \int d^{D-2} x_{\perp} x_{\perp}^2 \text{Exp} \left[I A x_{\perp}^2 + I (J \cdot x_{\perp}) \right] = \\ &= \left[I \frac{1}{2} \text{Tr} A^{-1} + \left(-\frac{1}{2} (A^{-1} \cdot J) \right)^2 \right] I_0 \end{aligned}$$

The remaining integrals : $\int d\Phi$

```
Amatr = Coefficient[Phase7, xt^2]
DetA = Amatr ^ (D - 2)
Jvec = Coefficient[Phase7, xt]
Jvec2 = Collect[Expand[Jvec^2], {#}]
CI0 = Exp[I Pi / 2 (D / 2 - 1)] Pi ^ (D / 2 - 1) (Amatr) ^ (- (D - 2) / 2)
PhaseI0 = -1 / 4 (1 / Amatr) Jvec2
xtQuadraticSubst = {xt^2 -> (I * 1 / 2 / Amatr * (D - 2) + (-1 / 2 Amatr ^ (-1) Jvec) ^ 2)}
xtLinearSubst = {xt -> -1 / 2 Amatr ^ (-1) Jvec}

1
4 ω

4^{2-D} \left( \frac{1}{\omega} \right)^{D-2}

-at e \Phi - pt

at^2 e^2 \Phi^2 + 2 at e pt \Phi + pt^2

2^{D-2} e^{\frac{1}{2} i \pi \left( \frac{D-1}{2} \right)} \pi^{\frac{D-1}{2}} \left( \frac{1}{\omega} \right)^{\frac{2-D}{2}}
```


$$\omega \left(-(\text{at}^2 e^2 \Phi^2 + 2 \text{at} e \text{pt} \Phi + \text{pt}^2) \right)$$

$$\{ \text{xt}^2 \rightarrow 4 \omega^2 (-\text{at} e \Phi - \text{pt})^2 + 2 i (D-2) \omega \}$$

$$\{ \text{xt} \rightarrow -2 \omega (-\text{at} e \Phi - \text{pt}) \}$$

We find that

$$A = \frac{1}{4 \omega},$$

$$J = -e a_{\perp} \Phi - p_{\perp},$$

$$\det A = \left(\frac{1}{4 \omega} \right)^{D-2},$$

$$A^{-1} = 4 \omega,$$

Also we use the following equalities for further simplifications :

$$a^{\mu} \gamma^{\beta} \gamma^5 F_{\beta\mu}^* = -a_{\perp} \left(\gamma^{\beta} \gamma^5 F_{\beta\mu}^* \right)_{\perp} = \text{at GFDt} = 0, \text{ as } a^{\mu} F_{\beta\mu}^* = 0,$$

$$\text{at Gt} = -(\gamma a),$$

$$\text{at pt} = (a p),$$

$$\text{at}^2 = -a^2 = \xi^2 m^2 / e^2$$

`Coeff8 = Simplify[Coeff7 * CI0, Assumptions → {ω > 0, m > 0, t > 0, s > 0}]`

`Phase8 = Collect[(Phase7 /. {xt → 0}) + PhaseI0, x] /. {at^2 → ξ^2 m^2 / e^2}`

`Matrix8 = Collect[`

`Expand[`

`Matrix7 /. xtQuadraticSubst /. xtLinearSubst`

`] /. {at GFDt → 0, at Gt → -DiracGamma[Momentum[a, D], D],`

`at^2 → ξ^2 m^2 / e^2, at pt → -Pair[Momentum[a, D], Momentum[p, D]]},`

`{J0, J1, J2, dJ0dxl, dJ1dxl, dJ2dxl, x, Gm, Gp, GFDp, pp}]`

$$\frac{1}{4} i e^{-\frac{1}{4} i \pi D} \pi^{\frac{D}{2}-2} e^2 \left(\frac{s t}{\omega} \right)^{-D/2}$$

$$\frac{1}{3} m^2 \xi^2 \omega^3 (k \cdot p)^2 - \frac{1}{3} m^2 \xi^2 s \omega^2 (k \cdot p)^2 + \frac{p p \omega (k \cdot p)}{m} + \frac{q p \omega (k \cdot p)}{m} - m^2 s + \Phi \left(\frac{q p}{m} - \frac{p p}{m} \right) - p t^2 \omega$$

$$- \frac{i dJ1dxl Gm \xi \omega}{2 m^2 s t} + dJ2dxl Gm \left(\frac{3 i \xi \omega}{2 m^2 s t} - \frac{i D \xi \omega}{2 m^2 s t} \right) +$$

$$J1 \left(-\frac{m \xi^2 \gamma \cdot k (k \cdot p) \omega^3}{2 s} - m \xi^2 \gamma \cdot k (k \cdot p) \omega^2 + \frac{i e GFDt p t \omega^2}{m s} + \frac{i e GFDt p t \omega}{m} + \frac{1}{2} i e \sigma F \omega - \right.$$

$$\left. \frac{2}{3} m s \xi^2 \gamma \cdot k (k \cdot p) \omega + \frac{Gp (k \cdot p) \omega}{m^2 s} - \frac{Gt p t \omega}{m s} + \frac{1}{2} i e s \sigma F + \Phi^2 \left(\frac{Gm m^2 \xi^2 \omega}{2 s (k \cdot p)} - \frac{m \xi^2 \omega \gamma \cdot k}{2 s (k \cdot p)} \right) + \right.$$

$$\left. \Phi \left(\frac{e \omega \gamma \cdot k (a \cdot p)}{m s (k \cdot p)} - \frac{e Gm \omega (a \cdot p)}{s (k \cdot p)} \right) + GFDp \left(-\frac{i e (k \cdot p) \omega^2}{m^2 s} - \frac{i e (k \cdot p) \omega}{m^2} \right) + \right.$$

$$\begin{aligned}
& \text{Gm} \left(\frac{m^2 \xi^2 (k \cdot p) \omega^3}{2s} - \frac{1}{3} m^2 \xi^2 (k \cdot p) \omega^2 + \frac{\text{pp} \omega}{2ms} + \frac{\text{qp} \omega}{2ms} \right) - 1 - \frac{i t \gamma \cdot k}{ms(k \cdot p) \omega} \Big) + \\
& \text{J0} \left(\frac{D m \xi^2 \gamma \cdot k (k \cdot p) \omega^3}{2s} - \frac{m \xi^2 \gamma \cdot k (k \cdot p) \omega^3}{s} - D m \xi^2 \gamma \cdot k (k \cdot p) \omega^2 + 6 m \xi^2 \gamma \cdot k (k \cdot p) \omega^2 - \right. \\
& \quad \frac{i D e \text{GFDt pt } \omega^2}{ms} + \frac{2 i e \text{GFDt pt } \omega^2}{ms} + \frac{i D e \text{GFDt pt } \omega}{m} - \frac{6 i e \text{GFDt pt } \omega}{m} - \frac{1}{2} i D e \sigma \text{F} \omega + \\
& \quad \frac{2}{3} D m s \xi^2 \gamma \cdot k (k \cdot p) \omega - \frac{4}{3} m s \xi^2 \gamma \cdot k (k \cdot p) \omega + \frac{D \text{Gt pt } \omega}{ms} - \frac{2 \text{Gt pt } \omega}{ms} + D + \\
& \quad \left. \frac{1}{2} i D e s \sigma \text{F} - 2 i e s \sigma \text{F} + \Phi^2 \left(\frac{D m \omega \gamma \cdot k \xi^2}{2s(k \cdot p)} - \frac{m \omega \gamma \cdot k \xi^2}{s(k \cdot p)} + \text{Gm} \left(\frac{m^2 \xi^2 \omega}{s(k \cdot p)} - \frac{D m^2 \xi^2 \omega}{2s(k \cdot p)} \right) \right) \right) + \\
& \quad \Phi \left(-\frac{D e \omega \gamma \cdot k (a \cdot p)}{ms(k \cdot p)} + \frac{2 e \omega \gamma \cdot k (a \cdot p)}{ms(k \cdot p)} + \text{Gm} \left(\frac{D e \omega (a \cdot p)}{s(k \cdot p)} - \frac{2 e \omega (a \cdot p)}{s(k \cdot p)} \right) \right) + \\
& \quad \text{Gp} \left(\frac{2 \omega (k \cdot p)}{m^2 s} - \frac{D \omega (k \cdot p)}{m^2 s} \right) + \text{GFDp} \left(\frac{i D e (k \cdot p) \omega^2}{m^2 s} - \frac{2 i e (k \cdot p) \omega^2}{m^2 s} - \frac{i D e (k \cdot p) \omega}{m^2} + \frac{6 i e (k \cdot p) \omega}{m^2} \right) + \\
& \quad \text{Gm} \left(-\frac{D m^2 \xi^2 (k \cdot p) \omega^3}{2s} + \frac{m^2 \xi^2 (k \cdot p) \omega^3}{s} + \frac{1}{3} D m^2 \xi^2 (k \cdot p) \omega^2 - \right. \\
& \quad \left. \frac{2}{3} m^2 \xi^2 (k \cdot p) \omega^2 - \frac{D \text{qp} \omega}{2ms} + \frac{\text{qp} \omega}{ms} + \text{pp} \left(\frac{\omega}{ms} - \frac{D \omega}{2ms} \right) \right) \Big) + \\
& \text{J2} \left(-\frac{D m \xi^2 \gamma \cdot k (k \cdot p) \omega^3}{2s} + \frac{3 m \xi^2 \gamma \cdot k (k \cdot p) \omega^3}{2s} + D m \xi^2 \gamma \cdot k (k \cdot p) \omega^2 - 3 m \xi^2 \gamma \cdot k (k \cdot p) \omega^2 + \right. \\
& \quad \frac{i D e \text{GFDt pt } \omega^2}{ms} - \frac{3 i e \text{GFDt pt } \omega^2}{ms} - \frac{i D e \text{GFDt pt } \omega}{m} + \frac{3 i e \text{GFDt pt } \omega}{m} + \frac{1}{2} i D e \sigma \text{F} \omega - \\
& \quad \frac{3}{2} i e \sigma \text{F} \omega - \frac{2}{3} D m s \xi^2 \gamma \cdot k (k \cdot p) \omega + 2 m s \xi^2 \gamma \cdot k (k \cdot p) \omega - \frac{D \text{Gt pt } \omega}{ms} + \frac{3 \text{Gt pt } \omega}{ms} - D - \\
& \quad \left. \frac{1}{2} i D e s \sigma \text{F} + \frac{3}{2} i e s \sigma \text{F} + \Phi^2 \left(-\frac{D m \omega \gamma \cdot k \xi^2}{2s(k \cdot p)} + \frac{3 m \omega \gamma \cdot k \xi^2}{2s(k \cdot p)} + \text{Gm} \left(\frac{D m^2 \xi^2 \omega}{2s(k \cdot p)} - \frac{3 m^2 \xi^2 \omega}{2s(k \cdot p)} \right) \right) \right) + \\
& \quad \Phi \left(\frac{D e \omega \gamma \cdot k (a \cdot p)}{ms(k \cdot p)} - \frac{3 e \omega \gamma \cdot k (a \cdot p)}{ms(k \cdot p)} + \text{Gm} \left(\frac{3 e \omega (a \cdot p)}{s(k \cdot p)} - \frac{D e \omega (a \cdot p)}{s(k \cdot p)} \right) \right) + \text{Gp} \\
& \quad \left(\frac{D \omega (k \cdot p)}{m^2 s} - \frac{3 \omega (k \cdot p)}{m^2 s} \right) + \text{GFDp} \left(-\frac{i D e (k \cdot p) \omega^2}{m^2 s} + \frac{3 i e (k \cdot p) \omega^2}{m^2 s} + \frac{i D e (k \cdot p) \omega}{m^2} - \frac{3 i e (k \cdot p) \omega}{m^2} \right) + \\
& \quad \text{Gm} \left(\frac{D m^2 \xi^2 (k \cdot p) \omega^3}{2s} - \frac{3 m^2 \xi^2 (k \cdot p) \omega^3}{2s} - \frac{1}{3} D m^2 \xi^2 (k \cdot p) \omega^2 + m^2 \xi^2 (k \cdot p) \omega^2 + \right. \\
& \quad \left. \frac{D \text{qp} \omega}{2ms} - \frac{3 \text{qp} \omega}{2ms} + \text{pp} \left(\frac{D \omega}{2ms} - \frac{3 \omega}{2ms} \right) \right) + 3 - \frac{i t \gamma \cdot k}{ms(k \cdot p) \omega} \Big)
\end{aligned}$$

Next, we recollect all scalar products into covariant notations

After this step the dependence on \oplus in the preexponent should vanish

$$\text{GFDp pm} = \gamma^\beta \gamma^5 (F^* p)_\beta + \text{GFDt pt}$$

$$\text{Gp pm} = -\text{Gm pp} + (\gamma p) + \text{GFDt pt}$$

$$\text{Gm} = (\gamma k) / m = (\gamma F^2 p) / m [-a^2 (kp)]$$

Note that after this step the preexponential does not depend on \oplus

SubstitutionStep91 = {DiracGamma[Momentum[k, D], D] (kp) →

DiracGamma[Momentum[FFp, D], D] e^2 / m^2 / ξ^2,

GFDp → m / kp (GFDt pt + DiracGamma[LorentzIndex[β, D], D].DiracGamma[5]
Pair[LorentzIndex[β, D], Momentum[FDp, D]]),

Gp → m / kp (DiracGamma[Momentum[p, D], D] - Gm pp + Gt pt) }

SubstitutionStep92 = {Gm → DiracGamma[Momentum[k, D], D] / m}

SubstitutionStep93 = {DiracGamma[Momentum[k, D], D] →

DiracGamma[Momentum[FFp, D], D] e^2 / m^2 / ξ^2 / (kp) }

$$\left\{ \gamma \cdot k (k \cdot p) \rightarrow \frac{e^2 \gamma \cdot \text{FFp}}{m^2 \xi^2}, \text{GFDp} \rightarrow \frac{m (\text{FDp}^\beta \gamma^\beta \cdot \bar{\gamma}^5 + \text{GFDt pt})}{k \cdot p}, \text{Gp} \rightarrow \frac{m (-\text{Gm pp} + \text{Gt pt} + \gamma \cdot p)}{k \cdot p} \right\}$$

$$\left\{ \text{Gm} \rightarrow \frac{\gamma \cdot k}{m} \right\}$$

$$\left\{ \gamma \cdot k \rightarrow \frac{e^2 \gamma \cdot \text{FFp}}{m^2 \xi^2 (k \cdot p)} \right\}$$

Coeff9 = Coeff8

Phase9 = Phase8

Matrix9 =

Collect[

Expand[

Simplify[

Expand[

Matrix8 /. SubstitutionStep91 /. SubstitutionStep92 /.

SubstitutionStep93 /. {ω → s t / (s + t) }

]

]

],

{J0, J1, J2, dJ0dχl, dJ1dχl, dJ2dχl, ⊕, e^2 DiracGamma[Momentum[FFp, D], D],

DiracGamma[LorentzIndex[β, D], D].DiracGamma[5]

Pair[LorentzIndex[β, D], Momentum[FDp, D]],

DiracGamma[Momentum[p, D], D], Pair[Momentum[a, D], Momentum[p, D]]}]

$$\frac{1}{4} i e^{-\frac{1}{4} i \pi D} \pi^{\frac{D}{2}-2} e^2 \left(\frac{s t}{\omega} \right)^{-D/2}$$

$$\frac{1}{3} m^2 \xi^2 \omega^3 (k \cdot p)^2 - \frac{1}{3} m^2 \xi^2 s \omega^2 (k \cdot p)^2 + \frac{p p \cdot \omega (k \cdot p)}{m} + \frac{q p \cdot \omega (k \cdot p)}{m} - m^2 s + \Phi \left(\frac{q p}{m} - \frac{p p}{m} \right) - p t^2 \omega$$

$$\begin{aligned}
& \text{dJ1d}\chi^1\gamma\cdot\text{FFp}\left(-\frac{is}{2m^5(s+t)^2\xi(k\cdot p)}-\frac{it}{2m^5(s+t)^2\xi(k\cdot p)}\right)e^2+ \\
& \text{dJ2d}\chi^1\gamma\cdot\text{FFp}\left(-\frac{iDs}{2m^5(s+t)^2\xi(k\cdot p)}+\frac{3is}{2m^5(s+t)^2\xi(k\cdot p)}-\frac{iDt}{2m^5(s+t)^2\xi(k\cdot p)}+\frac{3it}{2m^5(s+t)^2\xi(k\cdot p)}\right)e^2+ \\
& \text{J1}\left(\frac{ie\sigma F s^3}{2(s+t)^2}+\frac{3iet\sigma F s^2}{2(s+t)^2}-\frac{s^2}{(s+t)^2}+\frac{iet^2\sigma F s}{(s+t)^2}-\frac{2ts}{(s+t)^2}+\left(\frac{t^2}{m(s+t)^2}+\frac{st}{m(s+t)^2}\right)\gamma\cdot p+ \right. \\
& \quad \left(-\frac{iet s^2}{m(s+t)^2}-\frac{2iet^2 s}{m(s+t)^2}\right)\gamma^\beta\cdot\bar{\gamma}^5\text{FDp}^\beta+e^2\gamma\cdot\text{FFp}\left(-\frac{2ts^3}{3m(s+t)^2}-\frac{2t^2s^2}{m(s+t)^2}-\frac{\text{pp}ts}{2m^4(s+t)^2\xi^2(k\cdot p)}+ \right. \\
& \quad \frac{\text{qp}ts}{2m^4(s+t)^2\xi^2(k\cdot p)}-\frac{is}{m^3(s+t)^2\xi^2(k\cdot p)^2}-\frac{\text{pp}t^2}{2m^4(s+t)^2\xi^2(k\cdot p)}+\frac{\text{qp}t^2}{2m^4(s+t)^2\xi^2(k\cdot p)}- \\
& \quad \left.\frac{3it}{m^3(s+t)^2\xi^2(k\cdot p)^2}-\frac{3it^2}{m^3(s+t)^2\xi^2(k\cdot p)^2s}-\frac{it^3}{m^3(s+t)^2\xi^2(k\cdot p)^2s^2}\right)-\frac{t^2}{(s+t)^2}\Bigg)+ \\
& \text{J0}\left(\frac{iDe\sigma F s^3}{2(s+t)^2}-\frac{2ie\sigma F s^3}{(s+t)^2}+\frac{iDet\sigma F s^2}{2(s+t)^2}-\frac{4iet\sigma F s^2}{(s+t)^2}+\frac{Ds^2}{(s+t)^2}-\frac{2iet^2\sigma F s}{(s+t)^2}+ \right. \\
& \quad \frac{2Dts}{(s+t)^2}+\left(-\frac{Dt^2}{m(s+t)^2}+\frac{2t^2}{m(s+t)^2}-\frac{Dst}{m(s+t)^2}+\frac{2st}{m(s+t)^2}\right)\gamma\cdot p+ \\
& \quad \left(-\frac{iDet s^2}{m(s+t)^2}+\frac{6iet s^2}{m(s+t)^2}+\frac{4iet^2 s}{m(s+t)^2}\right)\gamma^\beta\cdot\bar{\gamma}^5\text{FDp}^\beta+ \\
& \quad e^2\gamma\cdot\text{FFp}\left(\frac{2Dts^3}{3m(s+t)^2}-\frac{4ts^3}{3m(s+t)^2}+\frac{4t^2s^2}{m(s+t)^2}+\frac{D\text{pp}ts}{2m^4(s+t)^2\xi^2(k\cdot p)}-\frac{\text{pp}ts}{m^4(s+t)^2\xi^2(k\cdot p)}- \right. \\
& \quad \frac{D\text{qp}ts}{2m^4(s+t)^2\xi^2(k\cdot p)}+\frac{\text{qp}ts}{m^4(s+t)^2\xi^2(k\cdot p)}+\frac{D\text{pp}t^2}{2m^4(s+t)^2\xi^2(k\cdot p)}- \\
& \quad \left.\frac{\text{pp}t^2}{m^4(s+t)^2\xi^2(k\cdot p)}-\frac{D\text{qp}t^2}{2m^4(s+t)^2\xi^2(k\cdot p)}+\frac{\text{qp}t^2}{m^4(s+t)^2\xi^2(k\cdot p)}\right)+\frac{Dt^2}{(s+t)^2}\Bigg)+ \\
& \text{J2}\left(-\frac{iDe\sigma F s^3}{2(s+t)^2}+\frac{3ie\sigma F s^3}{2(s+t)^2}-\frac{iDet\sigma F s^2}{2(s+t)^2}+\frac{3iet\sigma F s^2}{2(s+t)^2}-\frac{Ds^2}{(s+t)^2}+\frac{3s^2}{(s+t)^2}-\frac{2Dts}{(s+t)^2}+\frac{6ts}{(s+t)^2}+ \right. \\
& \quad \left(\frac{Dt^2}{m(s+t)^2}-\frac{3t^2}{m(s+t)^2}+\frac{Dst}{m(s+t)^2}-\frac{3st}{m(s+t)^2}\right)\gamma\cdot p+\left(\frac{iDes^2t}{m(s+t)^2}-\frac{3ies^2t}{m(s+t)^2}\right)\gamma^\beta\cdot\bar{\gamma}^5\text{FDp}^\beta+ \\
& \quad e^2\gamma\cdot\text{FFp}\left(-\frac{2Dts^3}{3m(s+t)^2}+\frac{2ts^3}{m(s+t)^2}-\frac{D\text{pp}ts}{2m^4(s+t)^2\xi^2(k\cdot p)}+\frac{3\text{pp}ts}{2m^4(s+t)^2\xi^2(k\cdot p)}+ \right. \\
& \quad \frac{D\text{qp}ts}{2m^4(s+t)^2\xi^2(k\cdot p)}-\frac{3\text{qp}ts}{2m^4(s+t)^2\xi^2(k\cdot p)}-\frac{is}{m^3(s+t)^2\xi^2(k\cdot p)^2}-\frac{D\text{pp}t^2}{2m^4(s+t)^2\xi^2(k\cdot p)}+ \\
& \quad \frac{3\text{pp}t^2}{2m^4(s+t)^2\xi^2(k\cdot p)}+\frac{D\text{qp}t^2}{2m^4(s+t)^2\xi^2(k\cdot p)}-\frac{3\text{qp}t^2}{2m^4(s+t)^2\xi^2(k\cdot p)}-\frac{3it}{m^3(s+t)^2\xi^2(k\cdot p)^2}- \\
& \quad \left.\frac{3it^2}{m^3(s+t)^2\xi^2(k\cdot p)^2s}-\frac{it^3}{m^3(s+t)^2\xi^2(k\cdot p)^2s^2}\right)-\frac{Dt^2}{(s+t)^2}+\frac{3t^2}{(s+t)^2}\Bigg)
\end{aligned}$$

```

Coefficient[Matrix9,  $\Phi$ ]
Coefficient[Matrix9,  $\Phi^2$ ]
0
0

```

The integration over

$$\int_{-\infty}^{\infty} d\Phi \dots$$

is now trivial,
as the phase is linear in Φ and the preexponent does not depend on Φ

$$\int d\Phi \exp\left[i\Phi \frac{q_+ - p_+}{m}\right] = 2\pi \delta\left(\frac{q_+ - p_+}{m}\right) = 2\pi m \delta(q_+ - p_+)$$

After this step we have collected the full delta - function, so that
 $M(q, p) = \Lambda^{D-4} (2\pi)^D \delta^{(D)}(q - p) M(p)$

In what follows we will consider $M(p)$

We substitute

$$\begin{aligned}
 qp &= q_+ \rightarrow pp = p_+, \\
 pp &= \frac{1}{2p_-} (p^2 + pt^2) = \frac{m}{2kp} (p^2 + pt^2)
 \end{aligned}$$

and also introduce

$$\chi = \chi_p = \frac{\varepsilon(kp)}{m^2}$$

After this step all the feasible integrations are done.

We are left with two integrals over the proper times

$$\int_0^\infty ds \int_0^\infty dt \dots$$

and the implicit integration

$$\int_{-\infty}^\infty dl^2 \quad \text{in } J_k(t, \chi_l)$$

$$\text{Coeff10} = \text{Coeff9} * 2 \pi * m / (2 \pi)^{\wedge D} / \Lambda^{\wedge (D - 4)}$$

$$\text{Phase10} = \text{Collect}[$$

$$\text{Expand}[$$

$$\text{Phase9} /. \{qp \rightarrow pp\} /. \{pp \rightarrow (pv2 + pt^{\wedge 2}) m / (2 kp)\} /. \{kp \rightarrow m^{\wedge 2} \chi / \xi\}$$

$$],$$

$$\chi, \text{Simplify}]$$

$$\text{Matrix10} = \text{Collect}[\text{Expand}[\text{Matrix9} /. \{qp \rightarrow pp\} /. \{pp \rightarrow (pv2 + pt^{\wedge 2}) m / (2 kp)\}]] /.$$

$$\{\text{DiracGamma}[\text{Momentum}[k, D], D] \rightarrow$$

$$\text{DiracGamma}[\text{Momentum}[\text{FFp}, D], D] e^{\wedge 2} / m^{\wedge 2} / \xi^{\wedge 2} / (kp)\} /. \{kp \rightarrow m^{\wedge 2} \chi / \xi\},$$

$$\{J0, J1, J2, dJ0d\chi l, dJ1d\chi l, dJ2d\chi l, e^{\wedge 2} \text{DiracGamma}[\text{Momentum}[\text{FFp}, D], D],$$

$$e \sigma F, \text{DiracGamma}[\text{LorentzIndex}[\beta, D], D] . \text{DiracGamma}[5]$$

$$\text{Pair}[\text{LorentzIndex}[\beta, D], \text{Momentum}[\text{FDp}, D]], \text{DiracGamma}[\text{Momentum}[p, D], D],$$

$$\text{Pair}[\text{Momentum}[a, D], \text{Momentum}[p, D]], \chi\}, \text{Simplify}]$$

$$i 2^{-D-1} e^{-\frac{1}{4} i \pi D} \pi^{-\frac{D}{2}-1} e^2 m \Lambda^{4-D} \left(\frac{s t}{\omega} \right)^{-D/2}$$

$$\frac{1}{3} m^6 \chi^2 \omega^2 (\omega - s) - m^2 s + p^2 \omega$$

$$J0 \left(-\frac{i e s t \text{FDp}^{\beta} \gamma^{\beta} \cdot \bar{\gamma}^5 ((D-6) s - 4 t)}{m (s+t)^2} + \frac{2 e^2 s^2 t ((D-2) s + 6 t) \gamma \cdot \text{FFp}}{3 m (s+t)^2} + \frac{i e s \sigma F ((D-4) s - 4 t)}{2 (s+t)} - \frac{(D-2) t \gamma \cdot p}{m (s+t)} + D \right) +$$

$$J2 \left(\frac{i (D-3) e s^2 t \text{FDp}^{\beta} \gamma^{\beta} \cdot \bar{\gamma}^5}{m (s+t)^2} + e^2 \gamma \cdot \text{FFp} \left(-\frac{2 (D-3) s^3 t}{3 m (s+t)^2} - \frac{i (s+t)}{m^7 s^2 \chi^2} \right) - \frac{i (D-3) e s^2 \sigma F}{2 (s+t)} + \frac{(D-3) t \gamma \cdot p}{m (s+t)} - D + 3 \right) +$$

$$J1 \left(-\frac{i e s t \text{FDp}^{\beta} (s+2 t) \gamma^{\beta} \cdot \bar{\gamma}^5}{m (s+t)^2} + e^2 \gamma \cdot \text{FFp} \left(-\frac{2 s^2 t (s+3 t)}{3 m (s+t)^2} - \frac{i (s+t)}{m^7 s^2 \chi^2} \right) + \frac{i e s \sigma F (s+2 t)}{2 (s+t)} + \frac{t \gamma \cdot p}{m s + m t} - 1 \right) -$$

$$\frac{i (D-3) dJ2d\chi l e^2 \gamma \cdot \text{FFp}}{2 m^7 \chi (s+t)} - \frac{i dJ1d\chi l e^2 \gamma \cdot \text{FFp}}{2 m^7 \chi (s+t)}$$

Coeff10 /. {D → 4}

Phase10 /. {D → 4}

Matrix10 /. {D → 4}

$$-\frac{i e^2 m \omega^2}{32 \pi^3 s^2 t^2}$$

$$\omega \vec{p}^2 + \frac{1}{3} m^6 \chi^2 \omega^2 (\omega - s) - m^2 s$$

$$-\frac{i dJ1 d\chi1 e^2 \bar{\gamma} \cdot \overline{FFp}}{2 m^7 \chi (s+t)} - \frac{i dJ2 d\chi1 e^2 \bar{\gamma} \cdot \overline{FFp}}{2 m^7 \chi (s+t)} +$$

$$J0 \left(\frac{2 e^2 s^2 t (2 s + 6 t) \bar{\gamma} \cdot \overline{FFp}}{3 m (s+t)^2} - \frac{i e s t (-2 s - 4 t) \bar{\gamma}^\beta \cdot \bar{\gamma}^5 \overline{FDp}^\beta}{m (s+t)^2} - \frac{2 t \bar{\gamma} \cdot \vec{p}}{m (s+t)} - \frac{2 i e s \sigma F t}{s+t} + 4 \right) +$$

$$J1 \left(e^2 \bar{\gamma} \cdot \overline{FFp} \left(-\frac{2 s^2 t (s+3 t)}{3 m (s+t)^2} - \frac{i (s+t)}{m^7 s^2 \chi^2} \right) - \frac{i e s t (s+2 t) \bar{\gamma}^\beta \cdot \bar{\gamma}^5 \overline{FDp}^\beta}{m (s+t)^2} + \frac{t \bar{\gamma} \cdot \vec{p}}{m s + m t} + \frac{i e s \sigma F (s+2 t)}{2 (s+t)} - 1 \right) +$$

$$J2 \left(e^2 \bar{\gamma} \cdot \overline{FFp} \left(-\frac{2 s^3 t}{3 m (s+t)^2} - \frac{i (s+t)}{m^7 s^2 \chi^2} \right) + \frac{i e s^2 t \bar{\gamma}^\beta \cdot \bar{\gamma}^5 \overline{FDp}^\beta}{m (s+t)^2} + \frac{t \bar{\gamma} \cdot \vec{p}}{m (s+t)} - \frac{i e s^2 \sigma F}{2 (s+t)} - 1 \right)$$

Let us change the variables :

$$(s, t) \rightarrow (u, \sigma),$$

where

$$s = \frac{1}{m^2} \frac{1+u}{\chi^{2/3} u^{1/3}} \sigma$$

$$t = \frac{1}{m^2} \frac{1+u}{\chi^{2/3} u^{4/3}} \sigma$$

then

$$\omega = \frac{1}{m^2} \frac{1}{\chi^{2/3} u^{1/3}} \sigma,$$

$$\int_0^\infty dt \int_0^\infty ds \dots = \int_0^\infty du \int_0^\infty d\sigma \mid J^{-1} \mid \dots$$

$$\mid J^{-1} \mid = \frac{\sigma (u+1)^2}{m^4 u^{8/3} \chi^{4/3}}$$


```
$Assumptions = {χ > 0, u > 0, σ > 0};
```

```
suσ = m^(-2) (1 + u) / u^(1/3) / χ^(2/3) σ
```

```
tuσ = m^(-2) (1 + u) / u^(4/3) / χ^(2/3) σ
```

```
Jac = Simplify[D[suσ, u] D[tuσ, σ] - D[suσ, σ] D[tuσ, u]]
```

```
Simplify[(1 / suσ + 1 / tuσ) ^ (-1)]
```

$$\frac{\sigma(u+1)}{m^2 \sqrt[3]{u} \chi^{2/3}}$$

$$\frac{\sigma(u+1)}{m^2 u^{4/3} \chi^{2/3}}$$

$$\frac{\sigma(u+1)^2}{m^4 u^{8/3} \chi^{4/3}}$$

$$\frac{\sigma}{m^2 \sqrt[3]{u} \chi^2}$$

```
Coeff11 = Simplify[Coeff10 * Jac /. {ω → s t / (s + t)} /. {s → suσ, t → tuσ},  
Assumptions → {χ > 0, u > 0, σ > 0}]
```

```
Phase11 = Collect[  
Expand[  
Simplify[  
(Phase10) /. {ω → s t / (s + t)} /. {s → suσ, t → tuσ},  
Assumptions → {χ > 0, u > 0, σ > 0}  
]  
, {pv2, σ},  
Simplify]
```

```
Matrix11 =
```

```
Collect[  
Expand[  
Simplify[  
Matrix10 /. {kp → χ / ξ m^2} /. {ω → s t / (s + t)} /. {s → suσ, t → tuσ},  
Assumptions → {u > 0, σ > 0, χ > 0}]  
],
```

```
{J0, J1, J2, dJ0dχl, dJ1dχl, dJ2dχl, e^2 DiracGamma[Momentum[FFp, D], D],  
e σF, DiracGamma[LorentzIndex[β, D], D].DiracGamma[5]  
Pair[LorentzIndex[β, D], Momentum[FDp, D]],  
DiracGamma[Momentum[p, D], D], pv2, σ, χ}, Simplify]
```

$$\frac{i 2^{-D-1} e^{-\frac{1}{4} i \pi D} \pi^{-\frac{D}{2}-1} e^2 m \Lambda^{4-D} \left(\frac{\sigma(u+1)^2}{m^2 u^{4/3} \chi^{2/3}} \right)^{2-\frac{D}{2}}}{\sigma(u+1)^2}$$

$$\frac{p^2 \sigma}{m^2 \sqrt[3]{u} \chi^2} - \frac{\sigma^3}{3} - \frac{\sigma(u+1)}{\sqrt[3]{u} \chi^2}$$

$$\begin{aligned}
& J0 \left(\sigma \text{FDp}^\beta \gamma^\beta \cdot \bar{\gamma}^5 \left(\frac{4 i e}{m^3 \sqrt[3]{u} (u+1)^2 \chi^{2/3}} - \frac{i e u^{2/3} (D u + D - 6 u - 10)}{m^3 (u+1)^2 \chi^{2/3}} \right) + \frac{2 e^2 \sigma^2 ((D-2) u + 6) \gamma \cdot \text{FFp}}{3 m^5 u^{2/3} \chi^{4/3}} + \right. \\
& \quad \left. e \sigma \sigma \text{F} \left(\frac{i u^{2/3} (D(u+1)^2 - 4(u^2 + 3 u + 3))}{2 m^2 (u+1)^2 \chi^{2/3}} - \frac{2 i}{m^2 \sqrt[3]{u} (u+1)^2 \chi^{2/3}} \right) + \frac{(2-D) \gamma \cdot p}{m u + m} + D \right) + \\
& J2 \left(\frac{i (D-3) e \sigma u^{2/3} \text{FDp}^\beta \gamma^\beta \cdot \bar{\gamma}^5}{m^3 (u+1) \chi^{2/3}} + e^2 \gamma \cdot \text{FFp} \left(-\frac{2 (D-3) \sigma^2 \sqrt[3]{u}}{3 m^5 \chi^{4/3}} - \frac{i}{m^5 \sigma u^{2/3} \chi^{4/3}} \right) - \right. \\
& \quad \left. \frac{i (D-3) e \sigma \sigma \text{F} u^{2/3}}{2 m^2 \chi^{2/3}} + \frac{(D-3) \gamma \cdot p}{m (u+1)} - D + 3 \right) + \\
& J1 \left(\sigma \text{FDp}^\beta \gamma^\beta \cdot \bar{\gamma}^5 \left(-\frac{i e u^{2/3} (u+3)}{m^3 (u+1)^2 \chi^{2/3}} - \frac{2 i e}{m^3 \sqrt[3]{u} (u+1)^2 \chi^{2/3}} \right) + e^2 \gamma \cdot \text{FFp} \left(-\frac{2 \sigma^2 (u+3)}{3 m^5 u^{2/3} \chi^{4/3}} - \frac{i}{m^5 \sigma u^{2/3} \chi^{4/3}} \right) + \right. \\
& \quad \left. e \sigma \sigma \text{F} \left(\frac{i u^{2/3} (u^2 + 4 u + 5)}{2 m^2 (u+1)^2 \chi^{2/3}} + \frac{i}{m^2 \sqrt[3]{u} (u+1)^2 \chi^{2/3}} \right) + \frac{\gamma \cdot p}{m u + m} - 1 \right) - \\
& \frac{i (D-3) \text{dJ}2 \text{d}\chi l e^2 u^{4/3} \gamma \cdot \text{FFp}}{2 m^5 \sigma (u+1)^2 \sqrt[3]{\chi}} - \frac{i \text{dJ}1 \text{d}\chi l e^2 u^{4/3} \gamma \cdot \text{FFp}}{2 m^5 \sigma (u+1)^2 \sqrt[3]{\chi}}
\end{aligned}$$

In the next step we rewrite

$$dJ_k d\chi_l = \frac{\partial}{\partial \chi_l} J_k(t, \chi_l),$$

where $\chi_l = \chi_l(u) = \frac{u}{1+u} \chi$ and $t = t(u) = \frac{1}{m^2} \frac{1+u}{\chi^{2/3} u^{4/3}} \sigma$ as defined above

using integration by parts

First, we may use the equality

$$\frac{\partial}{\partial \chi_l} J_k(t, \chi_l) = (\chi_l'(u))^{-1} \left[\frac{d}{du} J_k(t(u), \chi_l(u)) - \frac{\partial}{\partial t} J_k(t, \chi_l) t'(u) \right].$$

Then, we may integrate the first term by parts, i.e.

$$\int_0^\infty du f(u) \frac{d}{du} J_k(t, \chi_l) = - \int_0^\infty du J_k(t, \chi_l) \frac{d}{du} f(t),$$

where we assumed that $J_k(t, 0) = f(\infty) = 0$.

As for second term, we write

$$\begin{aligned} \frac{\partial}{\partial t} J_k(t, \chi_l) &= \frac{\partial}{\partial t} (-i) \int_{-\infty}^\infty dl^2 D_k(l^2, \chi_l) e^{-il^2 t} = \\ &(-i)^2 \int_{-\infty}^\infty dl^2 l^2 D_k(l^2, \chi_l) e^{-il^2 t} = -i m^2 \tilde{J}_k(t, \chi_l). \end{aligned}$$

We denote

$$J_k t = \tilde{J}_k$$

```

χlu = Simplify[χlϕ0 /. {ω → s t / (s + t)} /. {s → suσ, t → tuσ} /. {kp → χ / ξ m^2}] /.
  {kp → χ / ξ m^2}
dχldu = Simplify[D[χlu, u]]
dtdu = Simplify[D[tuσ, u]]
Coefficient[Matrix11, dJ0dχl] * dJ0dχl +
  Coefficient[Matrix11, dJ1dχl] * dJ1dχl + Coefficient[Matrix11, dJ2dχl] * dJ2dχl
Matrix12t = % /. {dJ0dχl → 1/dχldu (-(-I m^2 J0t) dtdu),
  dJ1dχl → 1/dχldu (-(-I m^2 J1t) dtdu), dJ2dχl → 1/dχldu (-(-I m^2 J2t) dtdu)}
Matrix12d = %% /. {dJ0dχl → 1/dχldu dJ0du,
  dJ1dχl → 1/dχldu dJ1du, dJ2dχl → 1/dχldu dJ2du }
u χ
u + 1
χ
(u + 1)^2

```

$$\begin{aligned}
& -\frac{\sigma(u+4)}{3 m^2 u^{7/3} \chi^{2/3}} \\
& -\frac{i(D-3) dJ2d\chi l e^2 u^{4/3} \gamma \cdot \text{FFp}}{2 m^5 \sigma(u+1)^2 \sqrt[3]{\chi}} - \frac{i dJ1d\chi l e^2 u^{4/3} \gamma \cdot \text{FFp}}{2 m^5 \sigma(u+1)^2 \sqrt[3]{\chi}} \\
& -\frac{(D-3) e^2 J2t(u+4) \gamma \cdot \text{FFp}}{6 m^5 u \chi^2} - \frac{e^2 J1t(u+4) \gamma \cdot \text{FFp}}{6 m^5 u \chi^2} \\
& -\frac{i(D-3) dJ2du e^2 u^{4/3} \gamma \cdot \text{FFp}}{2 m^5 \sigma \chi^{4/3}} - \frac{i dJ1du e^2 u^{4/3} \gamma \cdot \text{FFp}}{2 m^5 \sigma \chi^{4/3}}
\end{aligned}$$

f = Coeff11 * Exp[I Phase11]

Matrix12dmod = Collect[

Expand[Simplify[-D[Matrix12d * f, u] / f] /. {dJ0du → J0, dJ1du → J1, dJ2du → J2}],
{J0, J1, J2}, Simplify]

$$\begin{aligned}
& \frac{1}{\sigma(u+1)^2} i 2^{-D-1} \pi^{-\frac{D}{2}-1} e^2 m \Lambda^{4-D} \left(\frac{\sigma(u+1)^2}{m^2 u^{4/3} \chi^{2/3}} \right)^{2-\frac{D}{2}} \exp \left(i \left(\frac{p^2 \sigma}{m^2 \sqrt[3]{u \chi^2}} - \frac{\sigma^3}{3} - \frac{\sigma(u+1)}{\sqrt[3]{u \chi^2}} \right) - \frac{i \pi D}{4} \right) \\
& \left(e^2 J1 \gamma \cdot \text{FFp} \left(p^2 \sigma(u+1) + m^2 \left(\sigma(2u^2 + u - 1) - i(D-2)(u-2) \sqrt[3]{u \chi^2} \right) \right) \right) / (6 m^7 \sigma(u+1) \chi^2) + \\
& \left((D-3) e^2 J2 \gamma \cdot \text{FFp} \left(p^2 \sigma(u+1) + m^2 \left(\sigma(2u^2 + u - 1) - i(D-2)(u-2) \sqrt[3]{u \chi^2} \right) \right) \right) / (6 m^7 \sigma(u+1) \chi^2)
\end{aligned}$$

Coeff12 = Coeff11

Phase12 = Phase11

Matrix12 = Collect[

(Matrix11 /. {dJ0dχl → 0, dJ1dχl → 0, dJ2dχl → 0}) + Matrix12t + Matrix12dmod,

{J0, J1, J2, J0t, J1t, J2t, e^2 DiracGamma[Momentum[FFp, D], D],

e σF, DiracGamma[LorentzIndex[β, D], D].DiracGamma[5]

Pair[LorentzIndex[β, D], Momentum[FDp, D]],

DiracGamma[Momentum[p, D], D], pv2, lv2, σ, χ}, Simplify]

$$\frac{i 2^{-D-1} e^{-\frac{1}{4} i \pi D} \pi^{-\frac{D-1}{2}} e^2 m \Lambda^{4-D} \left(\frac{\sigma (u+1)^2}{m^2 u^{4/3} \chi^{2/3}} \right)^{2-\frac{D}{2}}}{\sigma (u+1)^2}$$

$$\frac{p^2 \sigma}{m^2 \sqrt[3]{u} \chi^2} - \frac{\sigma^3}{3} - \frac{\sigma (u+1)}{\sqrt[3]{u} \chi^2}$$

$$J2 \left(\frac{i (D-3) e \sigma u^{2/3} \text{FDp}^\beta \gamma^\beta \bar{\gamma}^5}{m^3 (u+1) \chi^{2/3}} + \right.$$

$$e^2 \gamma \cdot \text{FFp} \left(\frac{-\frac{i (D^2-5 D+6) (u-2) \sqrt[3]{u} \chi^2}{6 m^5 (u+1) \chi^2} - \frac{i}{m^5 u^{2/3} \chi^{4/3}}}{\sigma} + \frac{(D-3) p^2}{6 m^7 \chi^2} - \frac{2 (D-3) \sigma^2 \sqrt[3]{u}}{3 m^5 \chi^{4/3}} + \frac{(D-3) (2 u-1)}{6 m^5 \chi^2} \right) -$$

$$\left. \frac{i (D-3) e \sigma \sigma F u^{2/3}}{2 m^2 \chi^{2/3}} + \frac{(D-3) \gamma \cdot p}{m (u+1)} - D + 3 \right) +$$

$$J0 \left(-\frac{i e \sigma ((D-6) u-4) \text{FDp}^\beta \gamma^\beta \bar{\gamma}^5}{m^3 \sqrt[3]{u} (u+1) \chi^{2/3}} + \frac{2 e^2 \sigma^2 ((D-2) u+6) \gamma \cdot \text{FFp}}{3 m^5 u^{2/3} \chi^{4/3}} + \right.$$

$$\left. \frac{i e \sigma \sigma F ((D-4) u-4)}{2 m^2 \sqrt[3]{u} \chi^{2/3}} + \frac{(2-D) \gamma \cdot p}{m u+m} + D \right) + J1$$

$$\left(-\frac{i e \sigma (u+2) \text{FDp}^\beta \gamma^\beta \bar{\gamma}^5}{m^3 \sqrt[3]{u} (u+1) \chi^{2/3}} + e^2 \gamma \cdot \text{FFp} \left(\frac{-\frac{i (D-2) (u-2) \sqrt[3]{u} \chi^2}{6 m^5 (u+1) \chi^2} - \frac{i}{m^5 u^{2/3} \chi^{4/3}}}{\sigma} + \frac{p^2}{6 m^7 \chi^2} - \frac{2 \sigma^2 (u+3)}{3 m^5 u^{2/3} \chi^{4/3}} + \frac{2 u-1}{6 m^5 \chi^2} \right) + \right.$$

$$\left. \frac{i e \sigma \sigma F (u+2)}{2 m^2 \sqrt[3]{u} \chi^{2/3}} + \frac{\gamma \cdot p}{m u+m} - 1 \right) - \frac{(D-3) e^2 J2t (u+4) \gamma \cdot \text{FFp}}{6 m^5 u \chi^2} - \frac{e^2 J1t (u+4) \gamma \cdot \text{FFp}}{6 m^5 u \chi^2}$$

Let us rewrite the result using the following notation

$$\begin{aligned}\text{MFFp} &= \frac{e^2 \gamma^\mu \text{FFp}_\mu}{m^5 \chi^2} \left(\frac{\chi}{u} \right)^{2/3}; \\ \text{MFDp} &= \frac{e \gamma^\mu \gamma^5 \text{FDp}_\mu}{m^3 \chi} \left(\frac{\chi}{u} \right)^{1/3}, \quad \text{FDp}_\mu = (F^* p)_\mu; \\ \text{M}\sigma\text{F} &= \frac{e \sigma^{\mu\nu} F_{\mu\nu}}{m^2 \chi} \left(\frac{\chi}{u} \right)^{1/3};\end{aligned}$$

Coeff13 = Coeff12

Phase13 = Phase12

Matrix13 =

Collect[

Expand[

Matrix12 /. {DiracGamma[Momentum[FFp, D], D] → MFFp / e^2 * m^5 χ^ (4 / 3) u^ (2 / 3),

DiracGamma[LorentzIndex[β, D], D].DiracGamma[5]

Pair[LorentzIndex[β, D], Momentum[FDp, D]] →

MFDp / e * m^3 χ^ (2 / 3) u^ (1 / 3), σF → MσF / e * m^2 χ^ (2 / 3) u^ (1 / 3) }

],

{J0, J1, J2, dJ0dχl, dJ1dχl, dJ2dχl, MFFp, MFDp, MσF,

DiracGamma[Momentum[p, D], D], Pair[Momentum[a, D], Momentum[p, D]], σ}, Simplify]

$$\frac{i 2^{-D-1} e^{-\frac{1}{4} i \pi D} \pi^{-\frac{D}{2}-1} e^2 m \Lambda^{4-D} \left(\frac{\sigma(u+1)^2}{m^2 u^{4/3} \chi^{2/3}} \right)^{2-\frac{D}{2}}}{\sigma(u+1)^2}$$

$$\frac{p^2 \sigma}{m^2 \sqrt[3]{u \chi^2}} - \frac{\sigma^3}{3} - \frac{\sigma(u+1)}{\sqrt[3]{u \chi^2}}$$

$$J2 \left(\text{MFFp} \left(-\frac{i((D^2 - 5D + 6)u^2 - 2(D^2 - 5D + 3)u + 6)}{6\sigma(u+1)} + \frac{(D-3)\left(\frac{u}{\chi}\right)^{2/3}(m^2(2u-1) + p^2)}{6m^2} - \frac{2}{3}(D-3)\sigma^2 u \right) + \right.$$

$$\left. \frac{(D-3)\gamma \cdot p}{m(u+1)} + \frac{i \text{MFDp} \sigma u}{u+1} - \frac{1}{2} i (D-3) \text{M}\sigma\text{F} \sigma u - D + 3 \right) +$$

$$J0 \left(\frac{(2-D)\gamma \cdot p}{m u + m} - \frac{i \text{MFDp} \sigma ((D-6)u - 4)}{u+1} + \text{MFFp} \sigma^2 \left(\frac{2}{3}(D-2)u + 4 \right) + \frac{1}{2} i \text{M}\sigma\text{F} \sigma ((D-4)u - 4) + D \right) +$$

$$J1 \left(\text{MFFp} \left(-\frac{i((D-2)u^2 - 2(D-5)u + 6)}{6\sigma(u+1)} + \frac{\left(\frac{u}{\chi}\right)^{2/3}(m^2(2u-1) + p^2)}{6m^2} - \frac{2}{3}\sigma^2(u+3) \right) + \right.$$

$$\left. \frac{\gamma \cdot p}{m u + m} - \frac{i \text{MFDp} \sigma (u+2)}{u+1} + \frac{1}{2} i \text{M}\sigma\text{F} \sigma (u+2) - 1 \right) - \frac{\text{MFFp} (u+4) ((D-3)J2t + J1t)}{6 \sqrt[3]{u \chi^2}}$$

In order to remove the terms in

Matrix13 that are proportional to $(\text{MFFp } \sigma^{-1})$,

we integrate over σ by parts.

Let us consider the integral

$$\int_0^\infty d\sigma g(\sigma) \sigma^{3-D/2} \text{Exp}[-i\sigma^3 - iz\sigma], \quad 4-D=\epsilon > 0,$$

where $g(\sigma) = J_{1,2}(t(u, \sigma), \chi_1)$, and

where $g(\sigma)$ satisfies the condition :

$$g(\sigma) \sigma^{1-D/2} \rightarrow 0, \quad \sigma \rightarrow 0; \quad g(\infty) = 0.$$

Then, we may write

$$\begin{aligned} \int_0^\infty d\sigma g(\sigma) \sigma^{3-D/2} \text{Exp}[-i\sigma^3 - iz\sigma] &= \\ &= i \int_0^\infty d(\text{Exp}[-i\sigma^3]) g(\sigma) \sigma^{1-D/2} \text{Exp}[-i\sigma^3 - iz\sigma] = \\ &= -i \int_0^\infty d\sigma \sigma^{1-D/2} \text{Exp}[-i\sigma^3 - iz\sigma] \times \left[g'(\sigma) + \left(-\frac{D/2-1}{\sigma} - iz \right) g(\sigma) \right], \end{aligned}$$

therefore

$$\begin{aligned} \int_0^\infty d\sigma \sigma^{1-D/2} \text{Exp}[-i\sigma^3 - iz\sigma] \frac{1}{\sigma} g(\sigma) &= \\ &= \int_0^\infty d\sigma \sigma^{1-D/2} \left[\text{Exp}[-i\sigma^3 - iz\sigma] \times \frac{1}{D/2-1} \left[g'(\sigma) - i(\sigma^2 + z) g(\sigma) \right] \right]. \end{aligned}$$

For $g'(\sigma)$ we have

$$\begin{aligned} g'(\sigma) &= \frac{d}{d\sigma} J_{1,2}(t(u, \sigma), \chi_1) = \\ &= \frac{d}{d\sigma} (-i) \int_{-\infty}^\infty dl^2 D_k(l^2, \chi_1) \text{Exp}\left[-i\left(\frac{u}{\chi}\right)^{2/3} \frac{1+u}{u^2} \frac{l^2}{m^2} \sigma\right] = \\ &= (-i) \int_{-\infty}^\infty dl^2 \left(-i\left(\frac{u}{\chi}\right)^{2/3} \frac{1+u}{u^2} \frac{l^2}{m^2}\right) D_k(l^2, \chi_1) \text{Exp}\left[-it(u, \sigma) l^2\right] \\ &= -i \left(\frac{u}{\chi}\right)^{2/3} \frac{1+u}{u^2} \tilde{J}_k. \end{aligned}$$

At this step we substitute J_k and \tilde{J}_k explicitly :

$$\begin{aligned} J_k &= -i \int_{-\infty}^\infty dl^2 D_k(l^2, \chi_1) \text{Exp}\left[-it(u, \sigma) l^2\right], \\ \tilde{J}_k &= -i \int_{-\infty}^\infty dl^2 \frac{l^2}{m^2} D_k(l^2, \chi_1) \text{Exp}\left[-it(u, \sigma) l^2\right] \end{aligned}$$

and introduce notations

$$\begin{aligned} D_0 &= D_0(l^2), \\ D_k &= D_k(l^2, \chi_1), \quad k = 1, 2. \end{aligned}$$

Outer integration : $\int_0^\infty ds \int_0^\infty dt \int_{-\infty}^\infty dl^2 \dots$

The nonrenormalized diagonal part of the mass operator in D dimensions :

Coeff14 = Coeff13

Phase14 = Phase13

Matrix14 = Collect[Expand[Simplify[Matrix13 - Coefficient[Matrix13, 1/σ]/σ +
(Coefficient[Matrix13, 1/σ] * I (-σ - z/σ) * σ/(D/2 - 1) /.
{z → (u/χ)^(2/3) (1 - (pv2 - m^2)/m^2/u + (1+u)/u^2 lv2/m^2)}]] /. {lv2 J1 → m^2 J1t, lv2 J2 → m^2 J2t}, {J0, J1, J2, J1t, J2t, MFFp,
MFDp, MσF, DiracGamma[Momentum[p, D], D], pv2, u/χ, σ}, Simplify]

$$\frac{i 2^{-D-1} e^{-\frac{1}{4} i \pi D} \pi^{-\frac{D}{2}-1} e^2 m \Lambda^{4-D} \left(\frac{\sigma (u+1)^2}{m^2 u^{4/3} \chi^{2/3}} \right)^{2-\frac{D}{2}}}{\sigma (u+1)^2}$$

$$\frac{p^2 \sigma}{m^2 \sqrt[3]{u \chi^2}} - \frac{\sigma^3}{3} - \frac{\sigma (u+1)}{\sqrt[3]{u \chi^2}}$$

$$J2 \left(MFFp \left(\frac{p^2 ((D^2 - 5D + 6) u^2 - (D^2 - 5D + 2) u + 4)}{2(D-2) m^2 \sqrt[3]{u} (u+1) \chi^{2/3}} - \frac{\sigma^2 ((D^2 - 5D + 6) u^2 + 2u + 2)}{(D-2)(u+1)} + \frac{(D^2 - 5D + 2) u - 4}{2(D-2) \sqrt[3]{u} \chi^{2/3}} \right) + \right. \\ \left. \frac{(D-3) \gamma \cdot p}{m(u+1)} + \frac{i(D-3) MFDp \sigma u}{u+1} - \frac{1}{2} i(D-3) M\sigma F \sigma u - D + 3 \right) -$$

$$\frac{J2t MFFp ((D^2 - 5D + 6) u^2 + 4u + 4)}{2(D-2) u^{4/3} \chi^{2/3}} +$$

$$J0 \left(\frac{(2-D) \gamma \cdot p}{m u + m} - \frac{i MFDp \sigma ((D-6) u - 4)}{u+1} + MFFp \sigma^2 \left(\frac{2}{3} (D-2) u + 4 \right) + \frac{1}{2} i M\sigma F \sigma ((D-4) u - 4) + D \right) +$$

$$J1 \left(MFFp \left(\frac{p^2 ((D-2) u^2 - (D-6) u + 4)}{2(D-2) m^2 \sqrt[3]{u} (u+1) \chi^{2/3}} + \frac{\sigma^2 (2(u^2 + u + 1) - D(u^2 + 2u + 2))}{(D-2)(u+1)} + \frac{(D-6) u - 4}{2(D-2) \sqrt[3]{u} \chi^{2/3}} \right) + \right.$$

$$\left. \frac{\gamma \cdot p}{m u + m} - \frac{i MFDp \sigma (u+2)}{u+1} + \frac{1}{2} i M\sigma F \sigma (u+2) - 1 \right) - \frac{J1t MFFp ((D-2) u^2 + 4u + 4)}{2(D-2) u^{4/3} \chi^{2/3}}$$

(*Coeff14=Coeff13/(I)

Phase14=Phase13-lv2 tuσ

Expand[Matrix13]/.

{J0→ D0,J1→ D1,J2→ D2,J0t→ lv2/m^2 D0,J1t→ lv2/m^2 D1,J2t→ lv2/m^2 D2};

Matrix14=Collect[Expand[Simplify[%-Coefficient[%,1/σ]/σ+

(Coefficient[%,1/σ]*I(-σ-z/σ)*σ/(D/2-1) /.

{z→(u/χ)^(2/3) (1-(pv2-m^2)/m^2/u+(1+u)/u^2 lv2/m^2)}]]],

{D0,D1,D2,MFFp,MFDp, MσF, DiracGamma[Momentum[p,D],D],pv2,lv2,χ,σ},

Simplify]*)

$$z = (u/\chi)^{2/3}$$

$$\text{Collect}[\text{Expand}[-\text{Coefficient}[\text{Phase14}, \sigma] * (\chi/u)^{2/3}], \{pv2, lv2\}, \text{Simplify}]$$

$$z = \left(\frac{u}{\chi}\right)^{2/3} \left(-\frac{p^2}{m^2 u} + \frac{1}{u} + 1\right)$$

The nonrenormalized diagonal part of the mass operator in D = 4 :

$$\text{Coeff14D4} = \text{Coeff14} * \sigma /. \{D \rightarrow 4\}$$

$$\text{Phase14D4} = \text{Phase14} /. \{D \rightarrow 4\}$$

$$\text{Matrix14D4} = \text{Collect}[\text{Matrix14} / \sigma /. \{D \rightarrow 4\},$$

$$\{J0, J1, J2, J1t, J2t, \text{MFFp}, \text{MFDp}, \text{M}\sigma\text{F}, \text{DiracGamma}[\text{Momentum}[p, D], D] /. \{D \rightarrow 4\},$$

$$pv2 /. \{D \rightarrow 4\}, lv2 /. \{D \rightarrow 4\}, \chi, \sigma\}, \text{Simplify}]$$

$$-\frac{i e^2 m}{32 \pi^3 (u+1)^2}$$

$$\frac{\sigma \vec{p}^2}{m^2 \sqrt[3]{u} \chi^2} - \frac{\sigma^3}{3} - \frac{\sigma(u+1)}{\sqrt[3]{u} \chi^2}$$

$$J0 \left(-\frac{2 \bar{\gamma} \cdot \vec{p}}{\sigma(m u + m)} + \frac{2 i \text{MFDp}(u+2)}{u+1} + \frac{4}{3} \text{MFFp} \sigma(u+3) - 2 i \text{M}\sigma\text{F} + \frac{4}{\sigma} \right) +$$

$$J1 \left(\text{MFFp} \left(\frac{(u^2 + u + 2) \vec{p}^2}{2 m^2 \sigma \sqrt[3]{u} (u+1) \chi^{2/3}} - \frac{\sigma(u^2 + 3 u + 3)}{u+1} + \frac{-u-2}{2 \sigma \sqrt[3]{u} \chi^{2/3}} \right) + \right.$$

$$\left. \frac{\bar{\gamma} \cdot \vec{p}}{\sigma(m u + m)} - \frac{i \text{MFDp}(u+2)}{u+1} + \frac{1}{2} i \text{M}\sigma\text{F}(u+2) - \frac{1}{\sigma} \right) +$$

$$J2 \left(\text{MFFp} \left(\frac{(u^2 + u + 2) \vec{p}^2}{2 m^2 \sigma \sqrt[3]{u} (u+1) \chi^{2/3}} - \frac{\sigma(u^2 + u + 1)}{u+1} + \frac{-u-2}{2 \sigma \sqrt[3]{u} \chi^{2/3}} \right) + \frac{\bar{\gamma} \cdot \vec{p}}{\sigma(m u + m)} + \right.$$

$$\left. \frac{i \text{MFDp} u}{u+1} - \frac{1}{2} i \text{M}\sigma\text{F} u - \frac{1}{\sigma} \right) - \frac{J1t \text{MFFp}(u^2 + 2 u + 2)}{2 \sigma u^{4/3} \chi^{2/3}} - \frac{J2t \text{MFFp}(u^2 + 2 u + 2)}{2 \sigma u^{4/3} \chi^{2/3}}$$

Renormalization

$$M = M_0 + \delta M$$

where we attribute the term incorporating J_0 to M_0 ,
and which coincides with the 1 - loop order unrenormalized mass operator,
and the rest to δM ,
which is associated with the polarization loop insertions.

M_0 should be renormalized, δM is finite. Therefore,
we take the limit $D \rightarrow 4$ in δM .

We renormalize M_0 as follows

$$M_0(p, F) \rightarrow [M_0(p, 0)]_{\text{ren}} + [M_0(p, F) - M_0(p, 0)]$$

The second term gives a regular field dependent
part. $[M_0(p, 0)]_{\text{ren}}$ is the renormalized field - free mass operator.

In what follows, we write only the field – dependent part of M

Let us introduce the Ritus functions

$$f(z) = i \int_0^\infty d\sigma \exp\left(-i \frac{\sigma^3}{3} - iz\sigma\right),$$

$$f'(z) = \int_0^\infty d\sigma \sigma \exp\left(-i \frac{\sigma^3}{3} - iz\sigma\right),$$

$$f_1(z) = \int_0^\infty \frac{d\sigma}{\sigma} e^{-iz\sigma} \left[\exp\left(-i \frac{\sigma^3}{3}\right) - 1 \right]$$

$$z = \left(\frac{u}{\chi}\right)^{2/3} \left(1 - \frac{1}{u} \frac{p^2 - m^2}{m^2} + \frac{1+u}{u^2} \frac{l^2}{m^2}\right)$$

In effect,

in the limit $D \rightarrow 4$ the renormalization of M_0 is reduced to the substitution

$$\frac{1}{\sigma} \exp\left(-i \frac{\sigma^3}{3} - iz\sigma\right) \rightarrow \frac{1}{\sigma} e^{-iz\sigma} \left[\exp\left(-i \frac{\sigma^3}{3}\right) - 1 \right]$$

or

$$\int_0^\infty \frac{d\sigma}{\sigma} \exp\left(-i \frac{\sigma^3}{3} - iz\sigma\right) \rightarrow f_1(z)$$

We also substitute

$$\int_0^\infty d\sigma \sigma \exp\left(-i \frac{\sigma^3}{3} - iz\sigma\right) \rightarrow f'(z)$$

$$\int_0^\infty d\sigma \exp\left(-i \frac{\sigma^3}{3} - iz\sigma\right) \rightarrow -if(z)$$

and use

$$J_0 = 2\pi i \theta(t)$$

Matrix15M0 now contains the phase factor inside the Ritus f – functions.

```
pv2D4 = pv2 /. {D -> 4};
lv2D4 = lv2 /. {D -> 4};

zarg = Collect[Phase14D4 /. {σ^3 -> 0} /. {σ -> -1} /. {1/√[3]{u χ^2} -> (u/χ)^(2/3)/u,
  1/(u^(4/3) χ^(2/3)) -> (u/χ)^(2/3)/u^2}, {(u/χ)^(2/3)}, Simplify]
Clear[f];
Coeff15M0 = Coeff14D4 * 2 π I * 2 /. {e^2 -> α 4 π}
Matrix15M0 =
  Collect[Expand[Coefficient[Matrix14D4, J0]/2] /. {1/σ -> f1[z], σ -> f'[z],
    MFDp -> -I MFDp f[z], MσF -> -I MσF f[z]}, {f[z], f'[z], f1[z]}]
```

$$\left(\frac{u}{\chi}\right)^{2/3} \left(-\frac{\vec{p}^2}{m^2 u} + \frac{1}{u} + 1\right)$$

$$\frac{\alpha m}{2\pi(u+1)^2}$$

$$f_1(z) \left(2 - \frac{\bar{\gamma} \cdot \bar{p}}{m u + m} \right) + \left(\frac{2 \text{MFFp } u}{3} + 2 \text{MFFp} \right) f'(z) + f(z) \left(\frac{\text{MFDp } u}{u+1} + \frac{2 \text{MFDp}}{u+1} - \text{M}\sigma\text{F} \right)$$

The nontrivial part of δM

$$\text{Coeff15}\delta M = \text{Coeff14D4} /. \{e^2 \rightarrow \alpha^4 \pi\}$$

$$\text{Phase15} = \text{Phase14D4}$$

$$\text{Matrix15}\delta M = \text{Matrix14D4} - \text{Coefficient}[\text{Matrix14D4}, J0] J0$$

$$-\frac{i \alpha m}{8\pi^2(u+1)^2}$$

$$\frac{\sigma \bar{p}^2}{m^2 \sqrt[3]{u} \chi^2} - \frac{\sigma^3}{3} - \frac{\sigma(u+1)}{\sqrt[3]{u} \chi^2}$$

$$J1 \left(\text{MFFp} \left(\frac{(u^2 + u + 2) \bar{p}^2}{2 m^2 \sigma \sqrt[3]{u} (u+1) \chi^{2/3}} - \frac{\sigma(u^2 + 3u + 3)}{u+1} + \frac{-u-2}{2 \sigma \sqrt[3]{u} \chi^{2/3}} \right) + \right.$$

$$\left. \frac{\bar{\gamma} \cdot \bar{p}}{\sigma(m u + m)} - \frac{i \text{MFDp}(u+2)}{u+1} + \frac{1}{2} i \text{M}\sigma\text{F}(u+2) - \frac{1}{\sigma} \right) +$$

$$J2 \left(\text{MFFp} \left(\frac{(u^2 + u + 2) \bar{p}^2}{2 m^2 \sigma \sqrt[3]{u} (u+1) \chi^{2/3}} - \frac{\sigma(u^2 + u + 1)}{u+1} + \frac{-u-2}{2 \sigma \sqrt[3]{u} \chi^{2/3}} \right) + \frac{\bar{\gamma} \cdot \bar{p}}{\sigma(m u + m)} + \right.$$

$$\left. \frac{i \text{MFDp } u}{u+1} - \frac{1}{2} i \text{M}\sigma\text{F } u - \frac{1}{\sigma} \right) - \frac{J1t \text{MFFp}(u^2 + 2u + 2)}{2 \sigma u^{4/3} \chi^{2/3}} - \frac{J2t \text{MFFp}(u^2 + 2u + 2)}{2 \sigma u^{4/3} \chi^{2/3}}$$

Let us rewrite the answer in the following form

$$M(p, F) = \sum_{n=0}^2 \left[m S_n(p^2, \chi) + (\gamma p) V_n^{(1)}(p^2, \chi) + \frac{(\gamma F^2 p)}{m^4 \chi^2} V_n^{(2)}(p^2, \chi) + \frac{(\sigma F)}{m \chi} T_n(p^2, \chi) + \frac{(\gamma F^* p) \cdot \gamma^5}{m^2 \chi} A_n(p^2, \chi) \right]$$

$$\text{MFFpV2} = \text{MFFp} * m \left(\frac{\chi}{u} \right)^{-2/3} == \frac{e^2 \gamma^\mu \text{FFp}_\mu}{m^4 \chi^2};$$

$$\text{MFDpA} = \text{MFDp} * m \left(\frac{\chi}{u} \right)^{-1/3} == \frac{e \gamma^\mu \chi^5 \text{FDp}_\mu}{m^2 \chi}, \quad \text{FDp}_\mu == (F^* p)_\mu;$$

$$\text{M}\sigma\text{FT} = \text{M}\sigma\text{F} * m \left(\frac{\chi}{u} \right)^{-1/3} == \frac{e \sigma^{\mu\nu} F_{\mu\nu}}{m \chi};$$

```

Coeff15M01 = Collect[Matrix15M0 * Coeff15M0 /. {MFFp → MFFpV2 / m (χ / u) ^ (2 / 3),
  MFDp → MFDpA / m (χ / u) ^ (1 / 3), MσF → MσFT / m (χ / u) ^ (1 / 3)},
  {D0, f[z], f'[z], f1[z], DiracGamma[Momentum[p]], MFDpA, MσFT}, Simplify]

```

```

Coeff15δM1 = Collect[Expand[Matrix15δM * Coeff15δM] /.
  {MFFp → MFFpV2 / m (χ / u) ^ (2 / 3),
  MFDp → MFDpA / m (χ / u) ^ (1 / 3), MσF → MσFT / m (χ / u) ^ (1 / 3)},
  {MFFpV2, J1, J1t, J2, J2t, DiracGamma[Momentum[p]], MFDpA, MσFT, σ}, Simplify]

```

$$f1(z) \left(\frac{\alpha m}{\pi (u+1)^2} - \frac{\alpha \bar{\gamma} \cdot \bar{p}}{2 \pi (u+1)^3} \right) + \frac{\alpha \text{MFFpV2} (u+3) \left(\frac{\chi}{u} \right)^{2/3} f'(z)}{3 \pi (u+1)^2} + f(z) \left(\frac{\alpha \text{MFDpA} (u+2) \sqrt[3]{\frac{\chi}{u}}}{2 \pi (u+1)^3} - \frac{\alpha \text{M}\sigma\text{FT} \sqrt[3]{\frac{\chi}{u}}}{2 \pi (u+1)^2} \right)$$

$$\begin{aligned} & \text{MFFpV2} \left(J1 \left(\frac{i \alpha (m^2 (u^2 + 3 u + 2) - (u^2 + u + 2) \bar{p}^2)}{16 \pi^2 m^2 \sigma u (u+1)^3} + \frac{i \alpha \sigma (u^2 + 3 u + 3) \left(\frac{\chi}{u} \right)^{2/3}}{8 \pi^2 (u+1)^3} \right) + \right. \\ & J2 \left(\frac{i \alpha (m^2 (u^2 + 3 u + 2) - (u^2 + u + 2) \bar{p}^2)}{16 \pi^2 m^2 \sigma u (u+1)^3} + \frac{i \alpha \sigma (u^2 + u + 1) \left(\frac{\chi}{u} \right)^{2/3}}{8 \pi^2 (u+1)^3} \right) + \\ & \left. \frac{i \alpha J1t (u^2 + 2 u + 2)}{16 \pi^2 \sigma u^2 (u+1)^2} + \frac{i \alpha J2t (u^2 + 2 u + 2)}{16 \pi^2 \sigma u^2 (u+1)^2} \right) + \\ & J1 \left(- \frac{i \alpha \bar{\gamma} \cdot \bar{p}}{8 \pi^2 \sigma (u+1)^3} + \frac{i \alpha m}{8 \pi^2 \sigma (u+1)^2} - \frac{\alpha \text{MFDpA} (u+2) \sqrt[3]{\frac{\chi}{u}}}{8 \pi^2 (u+1)^3} + \frac{\alpha \text{M}\sigma\text{FT} (u+2) \sqrt[3]{\frac{\chi}{u}}}{16 \pi^2 (u+1)^2} \right) + \\ & J2 \left(- \frac{i \alpha \bar{\gamma} \cdot \bar{p}}{8 \pi^2 \sigma (u+1)^3} + \frac{i \alpha m}{8 \pi^2 \sigma (u+1)^2} + \frac{\alpha \text{MFDpA} \sqrt[3]{u^2 \chi}}{8 \pi^2 (u+1)^3} - \frac{\alpha \text{M}\sigma\text{FT} \sqrt[3]{u^2 \chi}}{16 \pi^2 (u+1)^2} \right) \end{aligned}$$

```

"S0 = " TraditionalForm[
  Coeff15M01 / m /. {MFFpV2 → 0, DiracGamma[Momentum[p]] → 0, MFDpA → 0, MσFT → 0} ]
"V0(1) = " TraditionalForm[ Coefficient[ Coeff15M01, DiracGamma[Momentum[p]] ] ]
"V0(2) = " TraditionalForm[ (- 8 π^2 / α (1 + u) ^ 2 / (-I) ^ 2) ^ (-1) Collect[Expand[
  Coefficient[ Coeff15M01 * (- 8 π^2 / α (1 + u) ^ 2 / (-I) ^ 2), MFFpV2]],
  {σ, J1, J1t, J2, J2t, (pv2 /. {D → 4})}], Simplify] ]
"T0 = " TraditionalForm[ (16 π^2 χ / α (χ / u) ^ (-1/3) (1 + u) ^ 2) ^ (-1) Collect[Expand[
  Coefficient[ Coeff15M01, MσFT] * (16 π^2 χ / α (χ / u) ^ (-1/3) (1 + u) ^ 2)],
  {J1, J1t, J2, J2t, σ, (pv2 /. {D → 4})}], Simplify]]
"A0 = " TraditionalForm[ (-8 π^2 χ / α (χ / u) ^ (-1/3) (1 + u) ^ 3) ^ (-1) Collect[
  Expand[Coefficient[ Coeff15M01, MFDpA] (-8 π^2 χ / α (χ / u) ^ (-1/3) (1 + u) ^ 3)],
  {σ, J1, J1t, J2, J2t, (pv2 /. {D → 4})}], Simplify]]

```

$$S_0 = \frac{\alpha f_1(z)}{\pi (u+1)^2}$$

$$V_0^{(1)} = \left(-\frac{\alpha f_1(z)}{2\pi (u+1)^3} \right)$$

$$V_0^{(2)} = \frac{(u+3)\alpha \left(\frac{\chi}{u}\right)^{2/3} f'(z)}{3\pi (u+1)^2}$$

$$T_0 = \left(-\frac{\alpha \sqrt[3]{\frac{\chi}{u}} f(z)}{2\pi (u+1)^2} \right)$$

$$A_0 = \frac{(u+2)\alpha \sqrt[3]{\frac{\chi}{u}} f(z)}{2\pi (u+1)^3}$$

```

"S1,2 = " TraditionalForm[
  Coeff15δM1 / m /. {MFFpV2 → 0, DiracGamma[Momentum[p]] → 0, MFDpA → 0, MσFT → 0} ]
"V1,2(1) = " TraditionalForm[ Coefficient[ Coeff15δM1, DiracGamma[Momentum[p]] ] ]
"V1,2(2) = " TraditionalForm[ (- 8 π^2 / α (1 + u) ^ 2 / (-I) 2) ^ (-1) Collect[Expand[
  Coefficient[ Coeff15δM1 * (- 8 π^2 / α (1 + u) ^ 2 / (-I) 2), MFFpV2]],
  {σ, J1, J1t, J2, J2t, (pv2 /. {D → 4})}], Simplify] ]
"T1,2 = " TraditionalForm[ (16 π^2 / α (χ / u) ^ (-1/3) (1 + u) ^ 2) ^ (-1) Collect[Expand[
  Coefficient[ Coeff15δM1, MσFT] * (16 π^2 / α (χ / u) ^ (-1/3) (1 + u) ^ 2)],
  {J1, J1t, J2, J2t, σ, (pv2 /. {D → 4})}], Simplify] ]
"A1,2 = " TraditionalForm[ (- 8 π^2 / α (χ / u) ^ (-1/3) (1 + u) ^ 3) ^ (-1) Collect[
  Expand[Coefficient[ Coeff15δM1, MFDpA] (- 8 π^2 / α (χ / u) ^ (-1/3) (1 + u) ^ 3)],
  {σ, J1, J1t, J2, J2t, (pv2 /. {D → 4})}], Simplify] ]

```

$$S_{1,2} = \frac{\frac{i J_1 m \alpha}{8 \pi^2 (u+1)^2 \sigma} + \frac{i J_2 m \alpha}{8 \pi^2 (u+1)^2 \sigma}}{m}$$

$$V_{1,2}^{(1)} = \left(-\frac{i J_1 \alpha}{8 \pi^2 (u+1)^3 \sigma} - \frac{i J_2 \alpha}{8 \pi^2 (u+1)^3 \sigma} \right)$$

$$V_{1,2}^{(2)} = \frac{1}{16 \pi^2 (u+1)^2} i \alpha \left(\sigma \left(\frac{2 J_2 \left(\frac{\chi}{u} \right)^{2/3} (u^2 + u + 1)}{u + 1} + \frac{2 J_1 (u^2 + 3 u + 3) \left(\frac{\chi}{u} \right)^{2/3}}{u + 1} \right) + \frac{1}{\sigma} \right. \\ \left. \left(\frac{J_1 t (u^2 + 2 u + 2)}{u^2} + \frac{J_2 t (u^2 + 2 u + 2)}{u^2} + J_1 \left(\frac{u + 2}{u} - \frac{(u^2 + u + 2) \vec{p}^2}{m^2 u (u + 1)} \right) + J_2 \left(\frac{u + 2}{u} - \frac{(u^2 + u + 2) \vec{p}^2}{m^2 u (u + 1)} \right) \right) \right)$$

$$T_{1,2} = \frac{(J_1 (u + 2) - J_2 u) \alpha \sqrt[3]{\frac{\chi}{u}}}{16 \pi^2 (u + 1)^2}$$

$$A_{1,2} = \left(-\frac{(J_1 (u + 2) - J_2 u) \alpha \sqrt[3]{\frac{\chi}{u}}}{8 \pi^2 (u + 1)^3} \right)$$

The elastic scattering amplitude $\mathcal{M} = \overline{u}_p M(p) u_p$

We assume that $\mathbf{p}^2 = 0$ and calculate the matrix element

$$\mathcal{M} = \bar{u}_p M(p) u_p,$$

where u_p is a free Dirac bispinor and $\bar{u}_p = \gamma^0 u_p^\dagger$.

To perform this calculation, we use the following relations

$$\bar{u}_p u_p = 2m,$$

$$\bar{u}_p (\gamma p) u_p = 2m^2,$$

$$\bar{u}_p e^2 (\gamma F^2 p) u_p = 2m^6 \chi^2,$$

$$\bar{u}_p e (\sigma_{\mu\nu} F^{\mu\nu}) u_p = \frac{2}{m} e (\bar{u}_p \gamma^\beta \gamma^5 u_p) (F^* p)_\beta = 4 s^\beta e (F^* p)_\beta,$$

where $s^\beta = \frac{1}{2m} \bar{u}_p \gamma^\beta \gamma^5 u_p$ is the electron spin 4 - vector.

Recall

$$MFFp = \frac{e^2 \gamma^\mu F F p_\mu}{m^5 \chi^2} \left(\frac{\chi}{u} \right)^{2/3},$$

$$MFDp = \frac{e \gamma^\mu \gamma^5 F D p_\mu}{m^3 \chi} \left(\frac{\chi}{u} \right)^{1/3}, \quad FDp_\mu = (F^* p)_\mu,$$

$$M\sigma F = \frac{e \sigma^{\mu\nu} F_{\mu\nu}}{m^2 \chi} \left(\frac{\chi}{u} \right)^{1/3},$$

in effect, we should perform the following substitutions

$$\bar{u}_p MFFp u_p = 2m \left(\frac{\chi}{u} \right)^{2/3},$$

$$\bar{u}_p MFDp u_p = \frac{2e}{m^2 \chi} s^\nu FDp_\nu \left(\frac{\chi}{u} \right)^{1/3},$$

$$\bar{u}_p M\sigma F u_p = \frac{4e}{m^2 \chi} s^\nu FDp_\nu \left(\frac{\chi}{u} \right)^{1/3}.$$

```
Simplify[S + DiracGamma[Momentum[p]] / m V1 +
  MFFp V2 (χ / u) ^ (-2 / 3) + MσF T (χ / u) ^ (-1 / 3) + MFDp A (χ / u) ^ (-1 / 3) /.
  {DiracGamma[Momentum[p]] → 2 m ^ 2, MFFp → 2 m χ ^ (2 / 3) / u ^ (2 / 3),
  MσF → e Contract[FVD[s, ν] FDpν[ν]] 4 / m ^ 2 / χ * (χ / u) ^ (1 / 3),
  MFDp → e Contract[FVD[s, ν] FDpν[ν]] 2 / m ^ 2 / χ * (χ / u) ^ (1 / 3), S → 2 m S}]
2 e (A + 2 T) (FDp · s)
m^2 χ + 2 m (S + V1 + V2)
```



```

Coeff16M0 = Coeff15M0 * 2 m
zargOS = zarg /. { (pv2 /. {D -> 4}) -> m^2 }
TermI0 = Expand[Simplify[
  Matrix15M0 /. {DiracGamma[Momentum[p]] -> 0, MFFp -> 0, MσF -> 0, MFDp -> 0}]]];
1/2/m Expand[Matrix15M0 - TermI0 + 2 m TermI0] /.
  {DiracGamma[Momentum[p]] -> 2 m^2, MFFp -> 2 m χ^(2/3) / u^(2/3),
   MσF -> e Contract[FVD[s, v] FDpv[v]] 4 / m^2 / χ * (χ / u)^(1/3),
   MFDp -> e Contract[FVD[s, v] FDpv[v]] 2 / m^2 / χ * (χ / u)^(1/3)} /. {pv2 ->
   m^2} /. {D -> 4};
Matrix16M0 = Collect[

```

```

  Expand[
    Simplify[% - Coefficient[%, e Pair[Momentum[s], Momentum[FDp]]]
      e Pair[Momentum[s], Momentum[FDp]]]]]
  + Simplify[Coefficient[%, e Pair[Momentum[s], Momentum[FDp]]]
    e Pair[Momentum[s], Momentum[FDp]]]
  ,
  {f[z], f'[z], f1[z]}, Simplify]

```

$$\frac{\alpha m^2}{\pi (u+1)^2}$$

$$\left(\frac{u}{\chi}\right)^{2/3}$$

$$-\frac{ef(z)\left(\frac{u}{\chi}\right)^{2/3}(\overline{\text{FDp}} \cdot \vec{s})}{m^3(u+1)} + \frac{2}{3}(u+3)\left(\frac{\chi}{u}\right)^{2/3}f'(z) + \frac{(2u+1)f1(z)}{u+1}$$

```

Coeff16δM = Coeff15δM * 2 m
Phase16 = Collect[Phase15 /. {pv2D4 → m^2}, σ, Simplify]
TermIδ = Expand[Simplify[
  Matrix15δM /. {DiracGamma[Momentum[p]] → 0, MFFp → 0, MσF → 0, MFDp → 0}]]];
1/2/m Expand[Matrix15δM - TermIδ + 2 m TermIδ] /.
  {DiracGamma[Momentum[p]] → 2 m^2, MFFp → 2 m χ^(2/3)/u^(2/3),
   MσF → e Contract[FVD[s, v] FDpv[v]] 4/m^2/χ * (χ/u)^(1/3),
   MFDp → e Contract[FVD[s, v] FDpv[v]] 2/m^2/χ * (χ/u)^(1/3)} /.
  {pv2D4 → m^2} /. {D → 4};
Matrix16δM = Collect[
  Expand[
    Simplify[% - Coefficient[%, e Pair[Momentum[s], Momentum[FDp]]]
      e Pair[Momentum[s], Momentum[FDp]]]]
  + Simplify[Coefficient[%, e Pair[Momentum[s], Momentum[FDp]]]
    e Pair[Momentum[s], Momentum[FDp]]]
  ,
  {J1, J2, J1t, J2t, MFFp, MFDp, MσF, DiracGamma[Momentum[p]], χ, σ}, Simplify]

```

$$\begin{aligned}
& -\frac{i \alpha m^2}{4 \pi^2 (u+1)^2} \\
& -\frac{\sigma^3}{3} - \sigma \left(\frac{u}{\chi} \right)^{2/3} \\
& J1 \left(\frac{i e (u+2) \left(\frac{u}{\chi} \right)^{2/3} (\overline{\text{FDp}} \cdot \vec{s})}{m^3 (u+1)} - \frac{1}{\sigma} + \frac{\sigma (-u^2 - 3 u - 3) \chi^{2/3}}{u^{2/3} (u+1)} \right) + \\
& J2 \left(-\frac{i e u \left(\frac{u}{\chi} \right)^{2/3} (\overline{\text{FDp}} \cdot \vec{s})}{m^3 (u+1)} - \frac{1}{\sigma} - \frac{\sigma \left(u^{4/3} + \frac{1}{u^{2/3}} + \sqrt[3]{u} \right) \chi^{2/3}}{u+1} \right) - \frac{J1t (u^2 + 2 u + 2)}{2 \sigma u^2} - \frac{J2t (u^2 + 2 u + 2)}{2 \sigma u^2}
\end{aligned}$$

Let us rewrite Matrix16M0

we perform an additional transformation of the D_0 term

Let us use the following integral equality (we will prove it below)

$$\int_0^\infty \frac{du}{(1+u)^2} \left[\frac{2(u-2)u}{3(u+1)} \left(\frac{x}{u}\right)^{2/3} f'(z_0) + -\frac{2u}{u+1} f_1(z_0) \right] = 0$$

We add this expression to Matrix16M0

After these transformation we arrive at the expression corresponding to Eq. (11) in [A.A.Mironov, S.Meuren, A.M.Fedotov PRD 102, 053005 (2020)] .

$$\begin{aligned} \text{Matrix16M01} = & \text{Collect}\left[\text{Matrix16M0} + \left(\frac{2(u-2)u}{3(u+1)} \left(\frac{x}{u}\right)^{2/3} + f_1[z] \left(-\frac{2u}{u+1}\right)\right), \right. \\ & \left. \{f[z], f'[z], f_1[z]\}, \text{Simplify}\right] \\ & -\frac{ef(z)\left(\frac{u}{x}\right)^{2/3}(\overline{\text{FDp}} \cdot \vec{s})}{m^3(u+1)} + \frac{2(2u^2+2u+3)\left(\frac{x}{u}\right)^{2/3}f'(z)}{3(u+1)} + \frac{f_1(z)}{u+1} \end{aligned}$$

Proof of the integral equality

$$I = \int_0^{\infty} \frac{du}{(1+u)^2} \left[\frac{2(u-2)u}{3(u+1)} \left(\frac{\chi}{u} \right)^{2/3} f'(z) + \left(\frac{12}{m^2} - \frac{2u}{u+1} \right) f_1(z) \right] = 0$$

Let us restore the explicit integral form of the Ritus f - functions and integrate the term with $f'(z)$ by parts

$$\begin{aligned} f'(z) &= \int_0^{\infty} d\sigma \sigma \exp\left(-i\frac{\sigma^3}{3} - iz\sigma\right) = i \int_0^{\infty} d\left[\exp\left(-i\frac{\sigma^3}{3}\right) - 1\right] \times \frac{1}{\sigma} e^{-iz\sigma} = \\ &= -i \int_0^{\infty} d\sigma \times \left[\exp\left(-i\frac{\sigma^3}{3}\right) - 1\right] \times \left[-\frac{1}{\sigma^2} - \frac{iz}{\sigma}\right] e^{-iz\sigma}, \end{aligned}$$

where we used the fact that

$$\left[\exp\left(-i\frac{\sigma^3}{3}\right) - 1\right] \times \frac{1}{\sigma} e^{-iz\sigma} = 0 \text{ for } \sigma = 0 \text{ and } \sigma = \infty.$$

Note that after rewriting $f'(z)$,

the factor $\left[\exp\left(-i\frac{\sigma^3}{3}\right) - 1\right]$ will be common in I ,

$$I = \left[\exp\left(-i\frac{\sigma^3}{3}\right) - 1\right] \int_0^{\infty} du e^{-iz(u)\sigma} [\dots]$$

so we will only write out the expression in $[\dots]$.

We also substitute

$$z(u) = \left(\frac{u}{\chi}\right)^{2/3} \left(\frac{(u+1)\tilde{t}^2}{m^2 u^2} + 1\right).$$

$$\text{zarg1} = \text{zarg0S} + \text{lv2D4} \, tu\sigma / \sigma$$

$$\text{Int} = \text{Collect}\left[1/(1+u)^2\right.$$

$$\left. \left(\frac{2(u-2)u}{3(1+u)} \left(\chi/u\right)^{2/3} f'[z] + \left(\text{lv2D4}/m^2 - 2u/(1+u)\right) f_1[z] \right) / .\right.$$

$$\left. \left\{ f'[z] \rightarrow -i(-1/\sigma^2 - iz/\sigma), f_1[z] \rightarrow 1/\sigma \right\} / . \{z \rightarrow \text{zarg1}\},\right.$$

$$\left. \{\sigma, \text{lv2D4}\}, \text{Simplify}\right]$$

$$\frac{(u+1)\tilde{t}^2}{m^2 u^{4/3} \chi^{2/3}} + \left(\frac{u}{\chi}\right)^{2/3}$$

$$\frac{\frac{(u+4)\tilde{t}^2}{3m^2 u(u+1)^2} - \frac{2u}{3(u+1)^2}}{\sigma} + \frac{2i\sqrt[3]{u(u-2)}\chi^{2/3}}{3\sigma^2(u+1)^3}$$

In the next step we prove that in fact this expression is a total derivative

$$I = \left[\text{Exp} \left(-i \frac{\sigma^3}{3} \right) - 1 \right] \int_0^\infty du \left[\dots \right] = \left[\text{Exp} \left(-i \frac{\sigma^3}{3} \right) - 1 \right] \int_0^\infty du \frac{d}{du} \left[P(u) e^{-iz(u)\sigma} \right],$$

and $P(0) = P(\infty) = 0$, which makes the statement evident.

To find $P(u)$, we expand the derivative $\frac{d}{du} [\dots]$ and equal the coefficient of l^2 to the corresponding coefficient in the expression for Int .

We find that

$$P(u) = -\frac{i u^{4/3} \chi^{2/3}}{\sigma^2 (u+1)^2}.$$

In the last two lines we check that the result is correct.

$$\begin{aligned} I &= \int_0^\infty \frac{du}{(1+u)^2} \left[\frac{2(u-2)u}{3(u+1)} \left(\frac{\chi}{u} \right)^{2/3} f'(z) + \left(\frac{l^2}{m^2} - \frac{2u}{u+1} \right) f_1(z) \right] = \\ &= -\frac{i}{\sigma^2} \left[\text{Exp} \left(-i \frac{\sigma^3}{3} \right) - 1 \right] \int_0^\infty du \frac{d}{du} \left[\frac{u^2}{(u+1)^2} \left(\frac{\chi}{u} \right)^{2/3} e^{-iz\sigma} \right] = 0. \end{aligned}$$

```
IntTest = Collect[
  Simplify[D[P[u] Exp[-I zarg1 σ], u] / Exp[-I zarg1 σ], {λ, ν, P'[u], σ}, Simplify]
```

$$P'(u) - \frac{i \sigma P(u) (2 m^2 u^2 - (u+4) l^2)}{3 m^2 u^{7/3} \chi^{2/3}}$$

```
Psol[u] =
```

```
P[u] /. Solve[Coefficient[Int, lv2D4] == Coefficient[IntTest, lv2D4], P[u]][[1]]
Collect[(IntTest /. {P[u] → Psol[u], P'[u] → D[Psol[u], u]}], {lv2D4, σ}, Simplify]
```

```
Simplify[Int - % /. {√[3]{u χ^2} → χ^(2/3) u^(1/3)}]
```

$$-\frac{i u^{4/3} \chi^{2/3}}{\sigma^2 (u+1)^2}$$

$$\frac{(u+4) l^2}{3 m^2 \sigma u (u+1)^2} + \frac{2 i \sqrt[3]{u} (u-2) \chi^{2/3}}{3 \sigma^2 (u+1)^3} - \frac{2 u}{3 \sigma (u+1)^2}$$

```
0
```