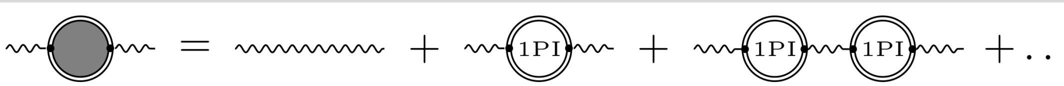
This is a part of SFQED-Loops script collection
developed for calculating loop processes in
Strong-Field Quantum Electrodynamics.
The scripts are available on https://github.com/ArsenyMironov/SFQED-Loops

If you use this script in your research, please, consider
citing our papers:
- A. A. Mironov, S. Meuren, and A. M. Fedotov, PRD 102, 053005 (2020),
  https://doi.org/10.1103/PhysRevD.102.053005
- A. A. Mironov, A. M. Fedotov, arXiv:2109.00634 (2021)

If you have any questions, please, don’t hesitate to contact:
mironov.hep@gmail.com




```
NotebookEvaluate[NotebookOpen[NotebookDirectory[] <> "definitions.nb"]]
```

FeynCalc 9.2.0. For help, use the documentation center, check out the wiki or write to the mailing list.

See also the supplied examples. If you use FeynCalc in your research, please cite

- V. Shtabovenko, R. Mertig and F. Orellana,
  Comput. Phys. Commun., 207C, 432–444, 2016, arXiv:1601.01167
- R. Mertig, M. Böhm, and A. Denner, Comput. Phys. Commun., 64, 345–359, 1991.

Exact photon propagator in momentum representation

$D^{c}{}_{\mu\nu}(l) =$

$$D_0\left(l^2,\chi_l\right) g_{\mu\nu} + D_1\left(l^2,\chi_l\right) \epsilon^{(1)}_{\mu}(l)\, \epsilon^{(1)}_{\nu}(l) + D_2\left(l^2,\chi_l\right) \epsilon^{(2)}_{\mu}(l)\, \epsilon^{(2)}_{\nu}(l);$$

l^{μ} – the photon propagator 4 – momentum;

$$\chi_l = \frac{e}{m^3}\sqrt{-\left(F_{\mu\nu}\, l^{\nu}\right)^2};$$

$$\epsilon^{(1)}_{\mu}(l) = \frac{e F_{\mu\nu}\, l^{\nu}}{m^3\, \chi_l};$$

$$\epsilon^{(2)}_{\mu}(l) = \frac{e F^{*}{}_{\mu\nu}\, l^{\nu}}{m^3\, \chi_l}; \qquad F^{*\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\delta\lambda} F_{\delta\lambda};$$

$$\left(\epsilon^{(i)}(l)\right)^2 = -1;$$

$$D_0\left(l^2,\chi_l\right) = \frac{-i}{l^2 - l^2\,\hat{\Pi}}, \quad D_{1,2}\left(l^2,\chi_l\right) = \frac{i\,\Pi_{1,2}}{\left(l^2 - l^2\,\hat{\Pi}\right)\left(l^2 - l^2\,\hat{\Pi} - \Pi_{1,2}\right)};$$

$$l^2\,\hat{\Pi} = l^2\,\hat{\Pi}\left(l^2,\chi_l\right),$$

$\Pi_{1,2} = \Pi_{1,2}\left(l^2,\chi_l\right)$ – polarization operator eigenfunctions;

Our goal : exact photon propagator in coordinate representation

$$D^{c}{}_{\mu\nu}(x) = \frac{\Lambda^{4-D}}{(2\pi)^D}\int d^D l\, D^{c}{}_{\mu\nu}(l)\, e^{-i l x};$$

Let’s define photon momentum and the coordinate variables

```
NewMomentum["l"]
NewCoordinate["x"]
```

$$\left\{l^{\alpha},\ l^2,\ k\cdot l,\ \mathrm{Fl}^{\alpha},\ \mathrm{FFl}^{\alpha},\ \mathrm{FDl}^{\alpha},\ a\cdot l,\ 0,\ 0,\ 0,\ -a^2\,(k\cdot l),\ 0,\ 0,\ -\frac{m^6\,\chi l^2}{e^2},\ -\frac{m^6\,\chi l^2}{e^2},\ \frac{m^6\,\chi l^2}{e^2},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\right\}$$

$$\left\{x^{\alpha},\ x^2,\ k\cdot x,\ a\cdot x,\ \mathrm{Fx}^{\alpha},\ \mathrm{FFx}^{\alpha},\ \mathrm{FDx}^{\alpha},\ k\cdot x,\ 0,\ 0,\ 0,\ -a^2\,(k\cdot x),\ 0,\ 0,\ -\frac{m^2\,\xi^2\,(k\cdot x)^2}{e^2},\ -\frac{m^2\,\xi^2\,(k\cdot x)^2}{e^2},\ \frac{m^2\,\xi^2\,(k\cdot x)^2}{e^2},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\right\}$$

Eigenvectors and tensor structures

We intentionally leave tensors and vectors uncontracted

```
ϵ1[μ_, ν_] = e Ft[μ, ν] lv[ν] / m^3 / χl
ϵ2[μ_, ν_] = e FDt[μ, ν] lv[ν] / m^3 / χl
T0[μ_, ν_] = MTD[μ, ν]
T1[μ_, ν_] = ϵ1[μ, α] ϵ1[ν, β]
T2[μ_, ν_] = ϵ2[μ, α] ϵ2[ν, β]
Contract[Contract[T1[μ, μ]] /. FieldSubstitutions]
Contract[Contract[T2[μ, μ]] /. FieldSubstitutions]
```

$$\frac{e\, l^{\nu}\, F(\mu,\nu)}{m^3\, \chi\mathrm{l}}$$

$$\frac{e\, l^{\nu}\, \mathrm{FD}(\mu,\nu)}{m^3\, \chi\mathrm{l}}$$

$$g^{\mu\nu}$$

$$\frac{e^2\, l^{\alpha}\, l^{\beta}\, F(\alpha,\mu)\, F(\beta,\nu)}{m^6\, \chi\mathrm{l}^2}$$

$$\frac{e^2\, l^{\alpha}\, l^{\beta}\, \mathrm{FD}(\alpha,\mu)\, \mathrm{FD}(\beta,\nu)}{m^6\, \chi\mathrm{l}^2}$$

-1

-1

We write $D^{c}{}_{\mu\nu}$ in the following form

$$\int d^D l\ \left[\mathrm{Coeff} * \mathrm{Matrix} * \mathrm{Exp}\left(i\ \mathrm{Phase}\right)\right],$$

where
Coeff – is a general multiplier for all terms,
Matrix – tensor part,
Phase – total phase of the expression,

We assume
$\mathrm{Dk} = \mathrm{Dk}\left(l^2, \chi_l\right)$

```
Coeff = Λ^(4 - D) / (2 π)^D
Matrix = D0 * T0[μ, ν] + D1 * T1[μ, ν] + D2 * T2[μ, ν]
Phase = -Contract[xv[α] lv[α]]
```

$(2\,\pi)^{-D}\,\Lambda^{4-D}$

$$\frac{\mathrm{D1}\,e^2\,l^\alpha\,l^\beta\,F(\alpha,\mu)\,F(\beta,\nu)}{m^6\,\chi\mathrm{l}^2}+\frac{\mathrm{D2}\,e^2\,l^\alpha\,l^\beta\,\mathrm{FD}(\alpha,\mu)\,\mathrm{FD}(\beta,\nu)}{m^6\,\chi\mathrm{l}^2}+\mathrm{D0}\,g^{\mu\nu}$$

$-(l\cdot x)$

We need to calculate the integrals of two types

$$\int d^D l\, D_0\left(l^2,\chi_l\right) e^{-i l x};$$

and

$$\int d^D l\, l_\alpha\, l_\beta\, D_{1,2}\left(l^2,\chi_l\right) e^{-i l x} = i\frac{\partial}{\partial x^\alpha}\, i\frac{\partial}{\partial x^\beta}\int d^D l\; D_{1,2}\left(l^2,\chi_l\right) e^{-i l x};$$

Symbol $d_{\alpha,\beta}$ means that we need to differentiate the expression later

Let us now change the variables

$lm = l_- = l_0 - l_3;$

$lp = l_+ = \frac{l_0 + l_3}{2};$

$lt = l_\perp$ - transverse components of l (in $D = 4$ $l_\perp = (l_1, l_2)$);

$l2 = l^2$

$l^2 = 2\, l_-\, l_+ - l_\perp^2;$

Proper time

$s = x_- / 2\, l_- = kx / 2\, kl;$

$l_- = x_- / 2\, s;$

$l_+ = (l^2 + l_\perp{}^2) / 2\, l_- = s\, (l^2 + l_\perp{}^2) / x_-$ -- expressed via l^2;

Hereinafter

$xm = x_- = x_0 - x_3;$

$xp = x_+ = \frac{x_0 + x_3}{2};$

$xt = x_\perp$

Change of variables

$l^\mu \to \{l_-, l_+, l_\perp\} = \left\{\frac{x_-}{2\, s}, \frac{s}{x_-} (l^2 + l_\perp{}^2), l_\perp\right\} \to \{s, l^2, l_\perp\}$

New integration measure

$$\mathrm{d}^D l \ldots = \frac{\mathrm{d}s}{2\,|s|}\, \mathrm{d}l^2\, \mathrm{d}^{D-2} l_\perp$$

```
(*Checking Jacobian*)
D[{xm/2/s, s (lv2 + lt2)/xm}, {{s, lv2}}]
Jac = Abs[Det[%]]
```

$$\begin{pmatrix} -\frac{xm}{2\, s^2} & 0 \\ \frac{lt2 + l^2}{xm} & \frac{s}{xm} \end{pmatrix}$$

$$\frac{1}{2\,|s|}$$

```
Coeff1 = Coeff * Jac
Matrix1 = Matrix /. {lv[α_] → dα}
Phase1 = Phase;
```

$$\frac{2^{-D-1}\,\pi^{-D}\,\Lambda^{4-D}}{|s|}$$

$$\frac{\mathrm{D1}\,e^2\,d_\alpha\,d_\beta\,F(\alpha,\mu)\,F(\beta,\nu)}{m^6\,\chi l^2}+\frac{\mathrm{D2}\,e^2\,d_\alpha\,d_\beta\,\mathrm{FD}(\alpha,\mu)\,\mathrm{FD}(\beta,\nu)}{m^6\,\chi l^2}+\mathrm{D0}\,g^{\mu\nu}$$

Integration over

$\int d l^{D-2}\, l_\perp \ldots$

$$I_0=\int d^{D-2}\,l_\perp\,\mathrm{Exp}\left[-I\,A\,l_\perp^2+I\,(J.l_\perp)\right]=$$

$$=\mathrm{Exp}\left[-I\,\frac{\pi}{2}\,\frac{D-2}{2}\right]\pi^{\frac{D-2}{2}}(\det A)^{-\frac{1}{2}}\,\mathrm{Exp}\left[I\,\frac{1}{4}\,J.A^{-1}.J\right]$$

where

$A = s$,

$J = x_\perp$,

$\det A = s^{D-2}$,

$A^{-1} = 1/s$

In effect,

integration results into multiplication of Coeff by I_0 and changing Phase

```
Coeff2 = Coeff1;
Matrix2 = Matrix1;
Phase2 =
 Expand[Phase1 /. {Pair[Momentum[x, D], Momentum[l, D]] → xm lp + xp lm - xt lt} /.
   {lm → xm / 2 / s, lp → s (lv2 + lt^2) / xm} /. {xp → (xv2 + xt^2) / 2 / xm}]
```

$$-l^2\,s+\mathrm{lt}^2\,(-s)+\mathrm{lt}\,\mathrm{xt}-\frac{x^2}{4\,s}-\frac{\mathrm{xt}^2}{4\,s}$$

```
Amatr = -Coefficient[Phase2, lt^2]
J = Coefficient[Phase2, lt]
CI0 = Exp[-I Pi / 2 (D / 2 - 1)] Pi^(D / 2 - 1) / Amatr^(D / 2 - 1)
```

s

xt

$$e^{-\frac{1}{2}i\pi\left(\frac{D}{2}-1\right)}\,\pi^{\frac{D}{2}-1}\,s^{1-\frac{D}{2}}$$

```
Coeff3 = Coeff2 * CI0
Matrix3 = Matrix2;
Phase3 = Expand[((Phase2 /. {lt → 0}) + 1 / 4 J^2 / Amatr)]
```

$$\frac{2^{-D-1} e^{-\frac{1}{2} i \pi \left(\frac{D}{2}-1\right)} \pi^{-\frac{D}{2}-1} \Lambda^{4-D} s^{1-\frac{D}{2}}}{|s|}$$

$$l^2 (-s) - \frac{x^2}{4 s}$$

The integrals

$$\int_{-\infty}^{\infty} \frac{ds}{2|s|} e^{-i \frac{x^2}{4 s}} \int_{-\infty}^{\infty} dl^2 D_k \left(l^2, \chi_l\right) e^{-i l^2 s}$$

will remain

Let us intoduce

$Jk = J_k (s) =$

$-i \int_{-\infty}^{\infty} dl^2 D_k \left(l^2, \chi_l\right) e^{-i l^2 s}$ – note that this integral is dimensionless

It can be shown that $J_k (s <= 0) = 0$

```
Coeff4 = Simplify[Coeff3 * I, Assumptions → {s > 0}]
Matrix4 = Matrix3 /. {D0 → J0, D1 → J1, D2 → J2}
Phase4 = Phase3 /. {lv2 → 0}
```

$$-2^{-D-1} e^{-\frac{1}{4} i \pi D} \pi^{-\frac{D}{2}-1} \Lambda^{4-D} s^{-D/2}$$

$$\frac{e^2 \, \mathrm{J1} \, d_\alpha \, d_\beta \, F(\alpha, \mu) \, F(\beta, \nu)}{m^6 \, \chi l^2} + \frac{e^2 \, \mathrm{J2} \, d_\alpha \, d_\beta \, \mathrm{FD}(\alpha, \mu) \, \mathrm{FD}(\beta, \nu)}{m^6 \, \chi l^2} + \mathrm{J0} \, g^{\mu \nu}$$

$$-\frac{x^2}{4 s}$$

Calculation of the tensor structure

Let us perform the remaining differention
and contract the resulting vectors and tensors

```
Expand[Simplify[
  I FourDivergence[I FourDivergence[Exp[I Phase4], xv[α]], xv[β]] Exp[-I Phase4]]]
Matrix4 /. {dα dβ → %}
Matrix5 = Collect[
  Contract[Contract[%] /. FieldSubstitutions /. FieldSubstitutions], {J0, J1, J2}]
Coeff5 = Coeff4
Phase5 = Phase4
```

$$\frac{x^\alpha x^\beta}{4 s^2} + \frac{i\, g^{\alpha\beta}}{2 s}$$

$$\frac{e^2 \,\mathrm{J1}\, F(\alpha,\mu)\, F(\beta,\nu) \left(\frac{x^\alpha x^\beta}{4 s^2} + \frac{i\, g^{\alpha\beta}}{2 s}\right)}{m^6 \chi \mathrm{l}^2} + \frac{e^2 \,\mathrm{J2}\, \mathrm{FD}(\alpha,\mu)\, \mathrm{FD}(\beta,\nu) \left(\frac{x^\alpha x^\beta}{4 s^2} + \frac{i\, g^{\alpha\beta}}{2 s}\right)}{m^6 \chi \mathrm{l}^2} + \mathrm{J0}\, g^{\mu\nu}$$

$$\mathrm{J2}\left(\frac{e^2 \,\mathrm{FDx}^\mu \,\mathrm{FDx}^\nu}{4 m^6 s^2 \chi \mathrm{l}^2} - \frac{i\, e^2 \,\mathrm{FF}(\mu,\nu)}{2 m^6 s \chi \mathrm{l}^2}\right) + \mathrm{J1}\left(\frac{e^2 \,\mathrm{Fx}^\mu \,\mathrm{Fx}^\nu}{4 m^6 s^2 \chi \mathrm{l}^2} - \frac{i\, e^2 \,\mathrm{FF}(\mu,\nu)}{2 m^6 s \chi \mathrm{l}^2}\right) + \mathrm{J0}\, g^{\mu\nu}$$

$$-2^{-D-1} e^{-\frac{1}{4} i \pi D} \pi^{-\frac{D}{2}-1} \Lambda^{4-D} s^{-D/2}$$

$$-\frac{x^2}{4 s}$$

Change to dimensionless proper time

```
Coeff6 = Simplify[Coeff5/m^2 /. {s → t/m^2}, Assumptions → {m > 0}]
Matrix6 = Matrix5 /. {χl → ξ kl/m^2} /. {kl → kx/2/s} /. {kx → ϕ} /.
    {FFt[μ, ν] → -av2 kv[μ] kv[ν]} /. {av2 → -m^2 ξ^2/e^2} /. {s → t/m^2}
Phase6 = Phase5 /. {s → t/m^2}
```

$$-2^{-D-1} e^{-\frac{1}{4} i \pi D} \pi^{-\frac{D}{2}-1} \Lambda^{4-D} m^{D-2} t^{-D/2}$$

$$\mathrm{J2}\left(\frac{e^2 \,\mathrm{FDx}^\mu \,\mathrm{FDx}^\nu}{m^2 \xi^2 \phi^2} - \frac{2\, i\, t\, k^\mu k^\nu}{m^2 \phi^2}\right) + \mathrm{J1}\left(\frac{e^2 \,\mathrm{Fx}^\mu \,\mathrm{Fx}^\nu}{m^2 \xi^2 \phi^2} - \frac{2\, i\, t\, k^\mu k^\nu}{m^2 \phi^2}\right) + \mathrm{J0}\, g^{\mu\nu}$$

$$-\frac{m^2 x^2}{4 t}$$

Final result

$$D^c{}_{\mu\nu}(x) = \frac{\Lambda^{4-D}}{(2\pi)^D}\int d^D l\, D^c{}_{\mu\nu}(l)\, e^{-ilx} =$$

$$= \text{Exp}\left[-i\,\frac{\pi}{2}\left(\frac{D}{2}-2\right)\right]\frac{\Lambda^{4-D}}{2^{D+1}\,\pi^{D/2+1}}$$

$$\int_0^\infty \frac{ds}{s^{D/2}}\, e^{-i\,\frac{x^2}{4s}}\left\{g_{\mu\nu}\, J_0(s) - i\,\frac{1}{2\,s\,m^6\,\chi_l{}^2}\left(e^2\left(F^2\right)_{\mu\nu} + i\,\frac{1}{2\,s}\,e^2\, Fx_\mu\, Fx_\nu\right) J_1(s)\right.$$

$$\left. - i\,\frac{1}{2\,s\,m^6\,\chi_l{}^2}\left(e^2\left(F^2\right)_{\mu\nu} + i\,\frac{1}{2\,s}\,e^2\, FDx_\mu\, FDx_\nu\right) J_2(s)\right\} =$$

$$= \text{Exp}\left[-i\,\frac{\pi}{2}\left(\frac{D}{2}-2\right)\right]\frac{1}{2^{D+1}\,\pi^{D/2+1}}\,\frac{\Lambda^{4-D}}{m^{2-D}}$$

$$\int_0^\infty \frac{dt}{t^{D/2}}\, e^{-i\,\frac{m^2\,x^2}{4\,t}}\left\{g_{\mu\nu}\, J_0\left(m^{-2}\,t\right) + \left(-2\,i\,t\,\frac{k_\mu\,k_\nu}{m^2\,\phi^2} + \frac{e^2\, Fx_\mu\, Fx_\nu}{m^2\,\xi^2\,\phi^2}\right) J_1\left(m^{-2}\,t\right)\right.$$

$$\left. + \left(-2\,i\,t\,\frac{k_\mu\,k_\nu}{m^2\,\phi^2} + \frac{e^2\, FDx_\mu\, FDx_\nu}{m^2\,\xi^2\,\phi^2}\right) J_2\left(m^{-2}\,t\right)\right\};$$

$$J_k(s, \chi_l) = -i\int_{-\infty}^{\infty} dl^2\, D_k\left(l^2, \chi_l\right) e^{-i l^2 s};$$

$$\phi = kx;$$

$$\chi_l = \xi\, kl \big/ m^2 = \xi\,\phi \,/\, 2\,m^2\,s = \xi\,\phi\,/\,2\,t;$$

$$\xi^2 = -e^2\,a^2 \big/ m^2;$$

$$Fx_\mu = F_{\mu\nu}\, x^\nu;$$

$$FDx_\mu = F^*{}_{\mu\nu}\, x^\nu;$$