Graphs: Shortest paths

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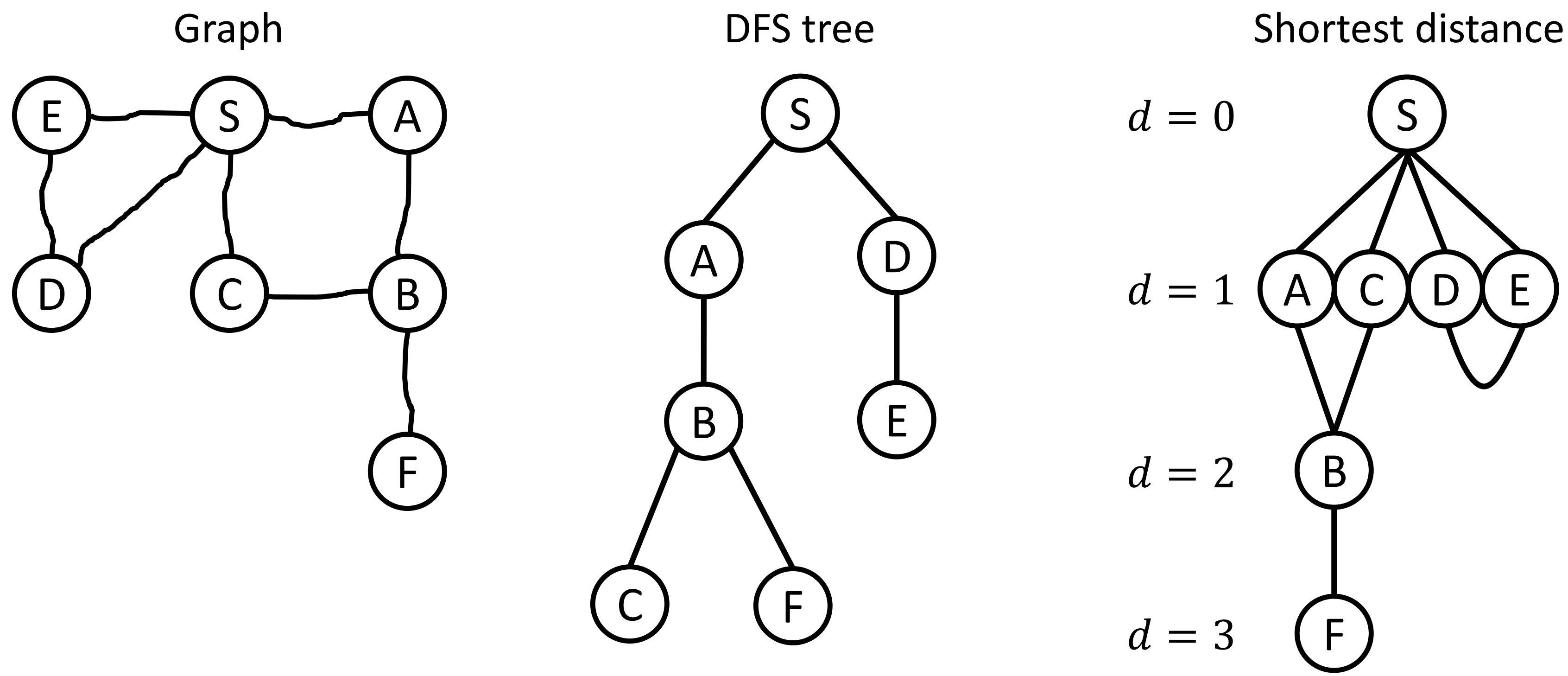
Distance in a graph

Depth-first search finds vertices reachable from another given vertex. The paths are not the shortest ones.

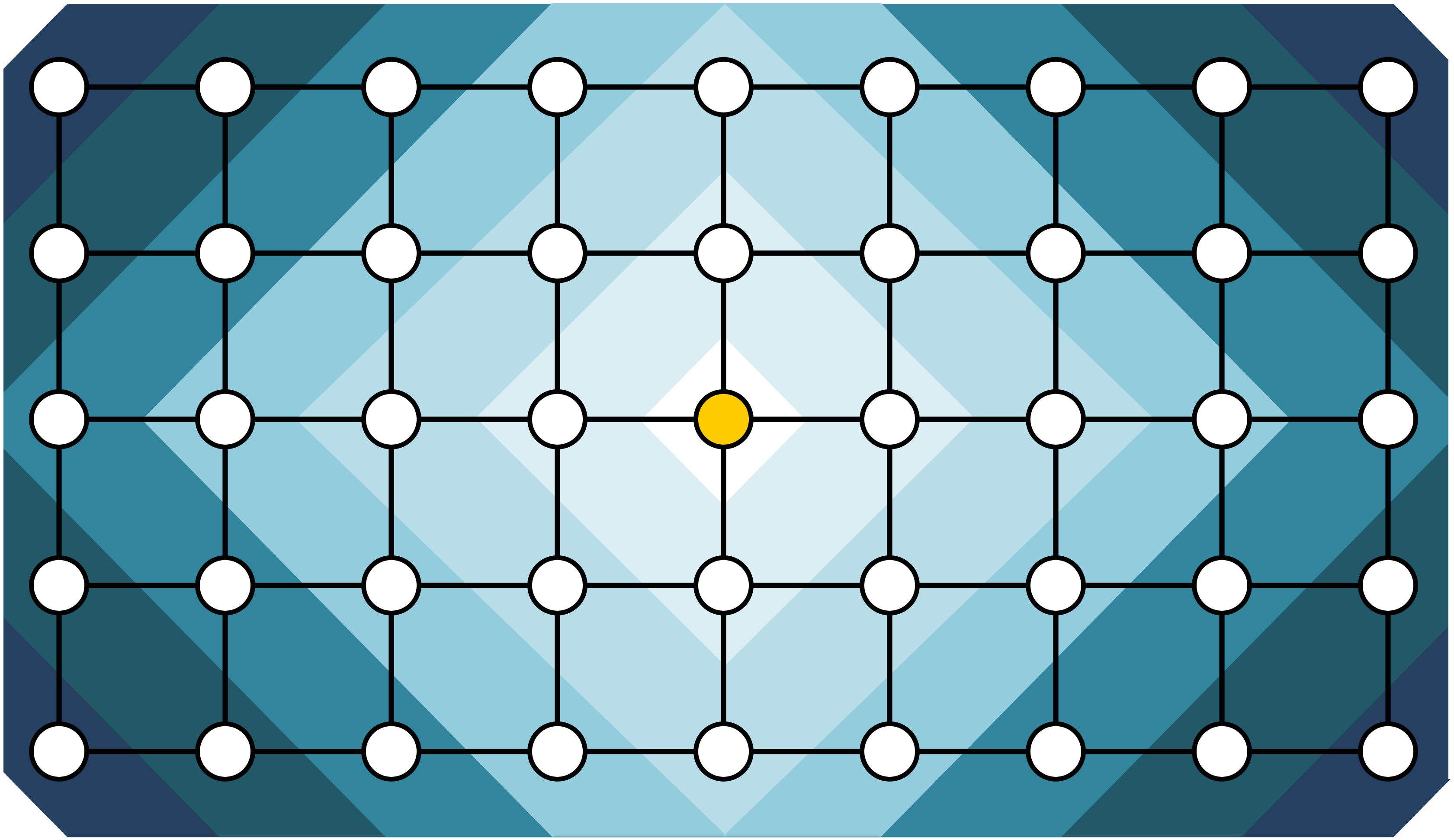


Distance between two nodes: length of the shortest path between them

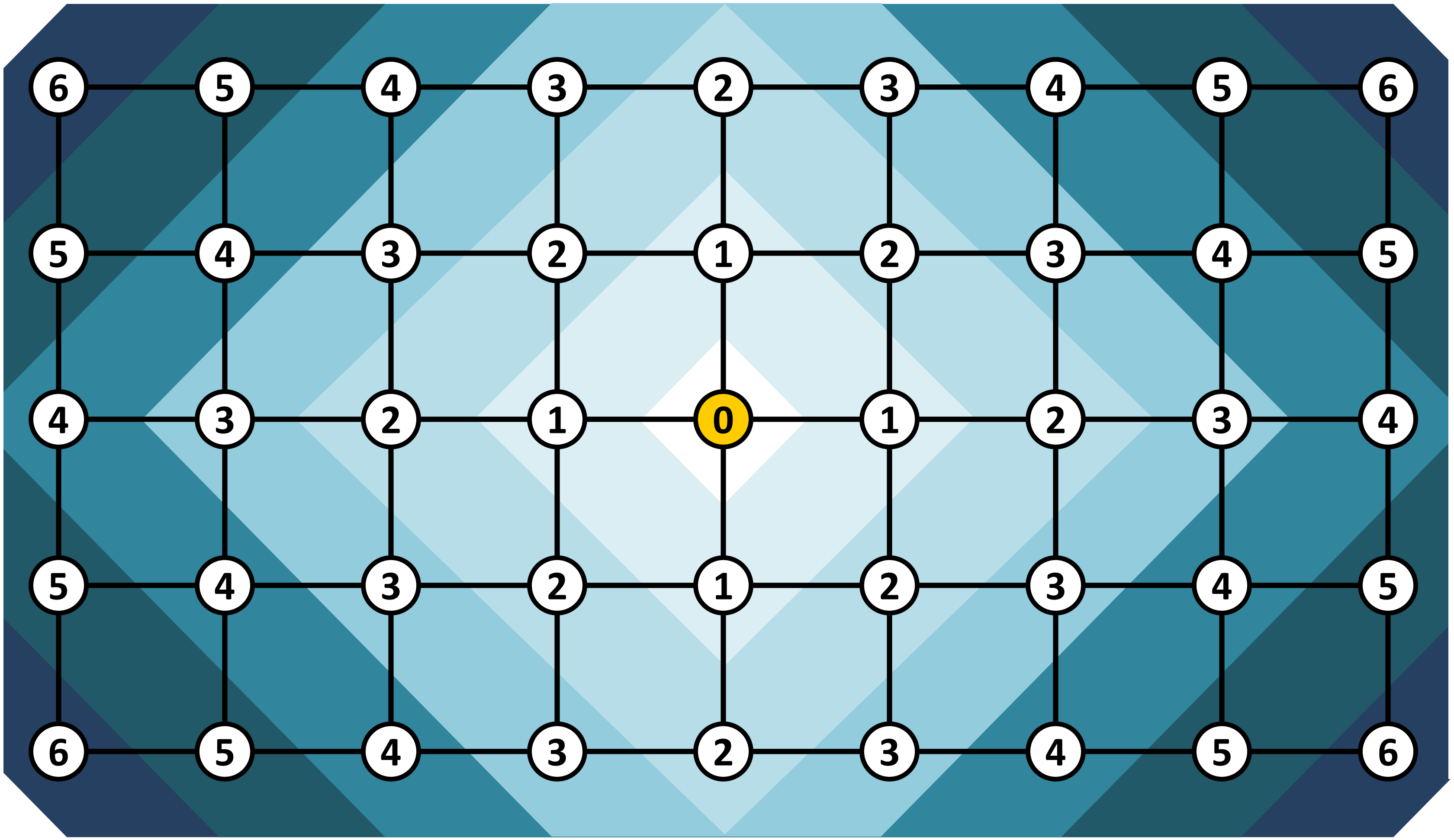
Breadth-first search

Similar to a wave propagation

Breadth-first search



Breadth-first search



BFS algorithm

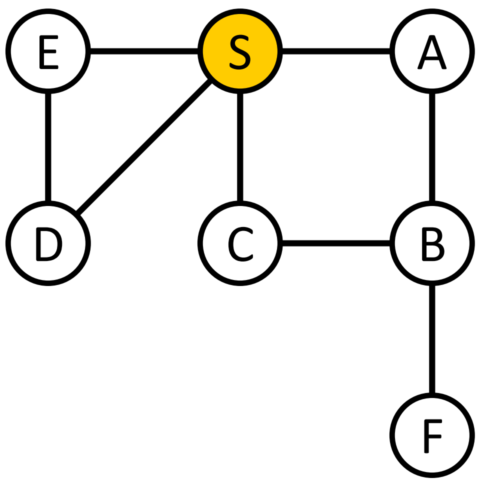
- BFS visits vertices layer by layer: $0,1,2, \ldots, d$.

- Once the vertices at layer d have been visited, start visiting vertices at layer $d+1$.

- Algorithm with two active layers:
  - Vertices at layer d (currently being visited).
  - Vertices at layer $d+1$ (to be visited next).

- Central data structure: a queue.

BFS algorithm



	Queue	S	A	B	C	D	E	F
	S_0	0	∞	∞	∞	∞	∞	∞
S_0	$A_1\ C_1\ D_1\ E_1$	0	1	∞	1	1	1	∞
A_1	$C_1\ D_1\ E_1\ B_2$	0	1	2	1	1	1	∞
C_1	$D_1\ E_1\ B_2$	0	1	2	1	1	1	∞
D_1	$E_1\ B_2$	0	1	2	1	1	1	∞
E_1	B_2	0	1	2	1	1	1	∞
B_2	F_3	0	1	2	1	1	1	3
F_3		0	1	2	1	1	1	3

BFS algorithm

```
function BFS(G, s)
// Input: Graph G(V,E), source vertex s.
// Output: For each vertex u, dist[u] is
//         the distance from s to u.

  for all u ∈ V: dist[u]=∞

  dist[s] = 0
  Q = {s} // Queue containing just s
  while not Q.empty():
    u = Q.pop_front()
    for all (u,v) ∈ E:
      if dist[v] = ∞:
        dist[v] = dist[u] + 1
        Q.push_back(v)
```

Runtime $O(|V| + |E|)$: Each vertex is visited once, each edge is visited once (for directed graphs) or twice (for undirected graphs).

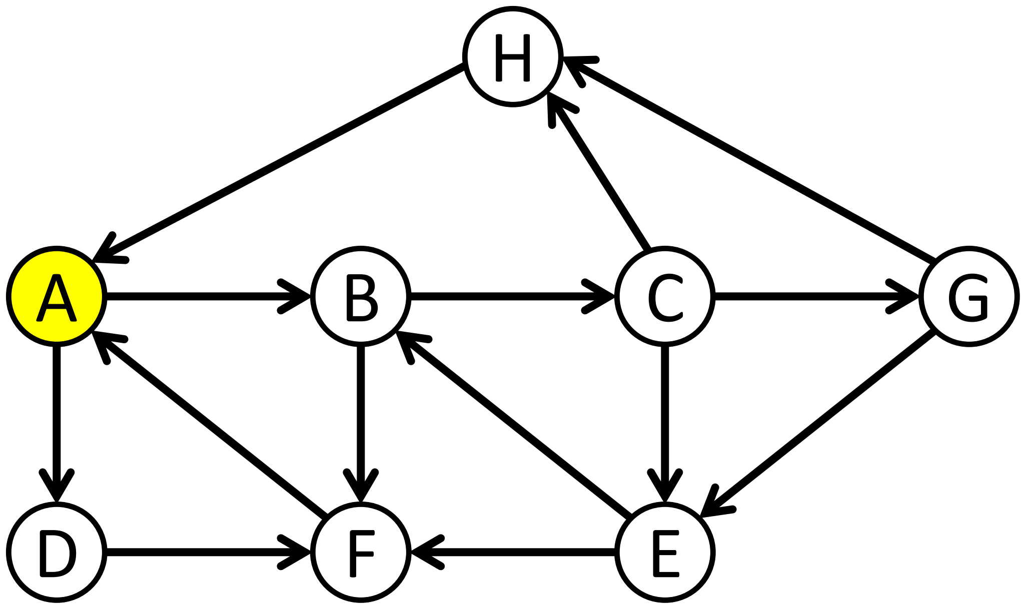
Reachability: BFS vs. DFS

Input: A graph G and a source node s.
Output: $\forall u \in V$: reached$[u] \Leftrightarrow u$ is reachable from s.

```
function BFS(G, s)

  for all u ∈ V:
    reached[u] = false

  Q = [ ]     // Empty queue
  Q.push_back(s)
  reached[s] = true
  while not Q.empty():
    u = Q.pop_front()
    for all (u,v) ∈ E:
      if not reached[v]:
        reached[v] = true
        Q.push_back(v)
```

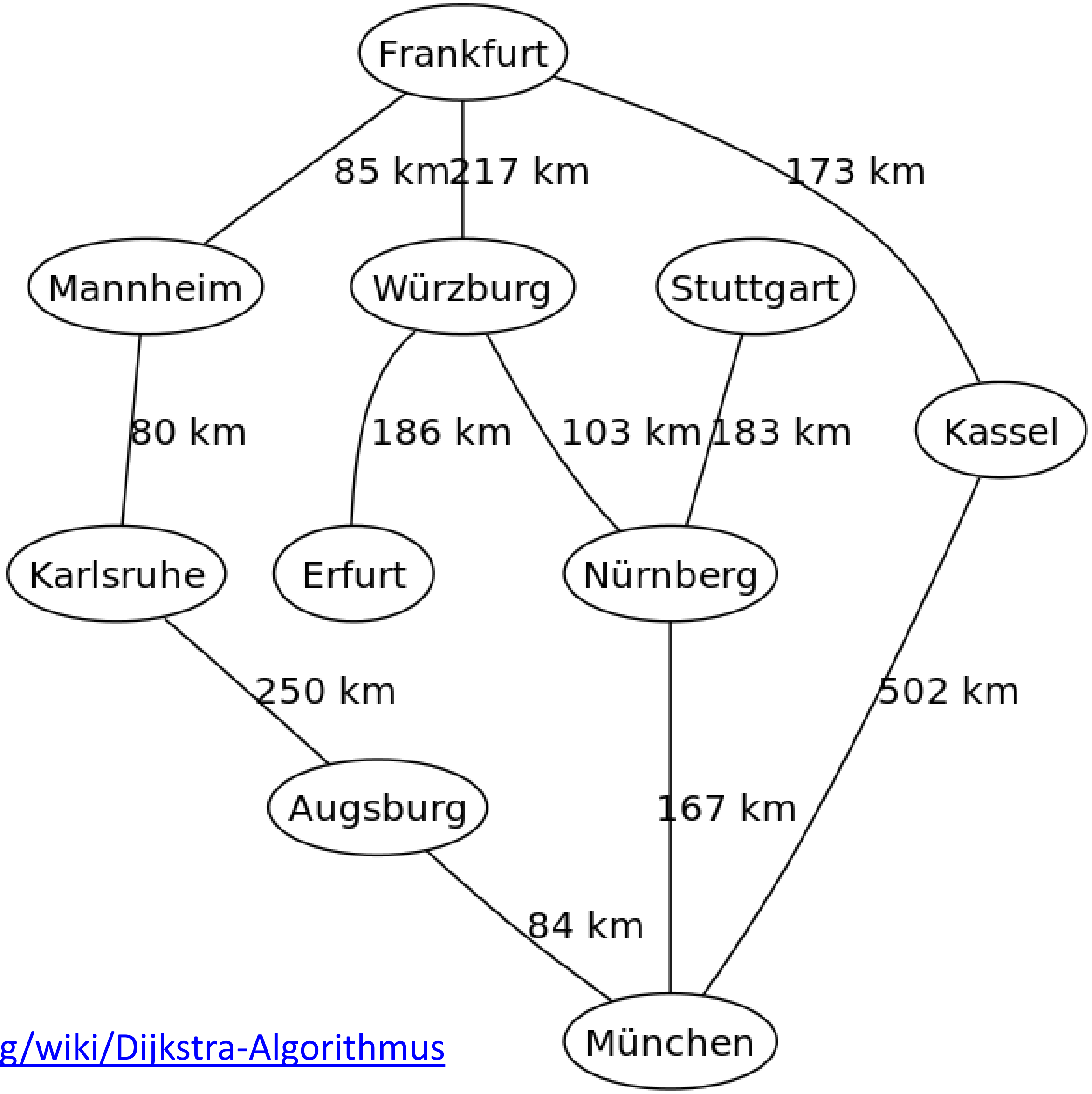
```
function DFS(G, s)

  for all u ∈ V:
    reached[u] = false

  S = [ ]     // Empty stack
  S.push(s)
  while not S.empty():
    u = S.pop()
    if not reached[u]:
      reached[u] = true
      for all (u,v) ∈ E:
        if not reached[v]:
          S.push(v)
```

Reachability: BFS vs. DFS



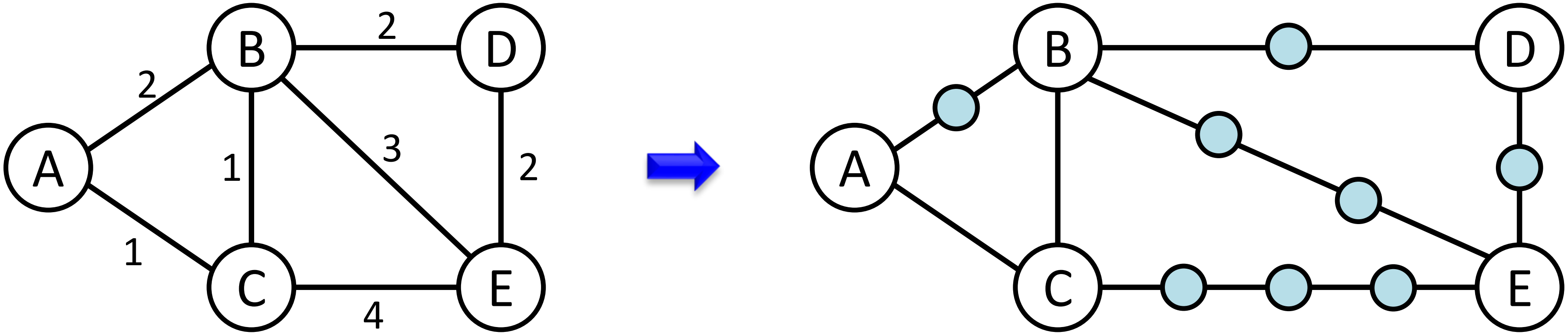
DFS order:	A	B	C	E	F	G	H	D
BFS order:	A	B	D	C	F	E	G	H
Distance:	0	1	1	2	2	3	3	3

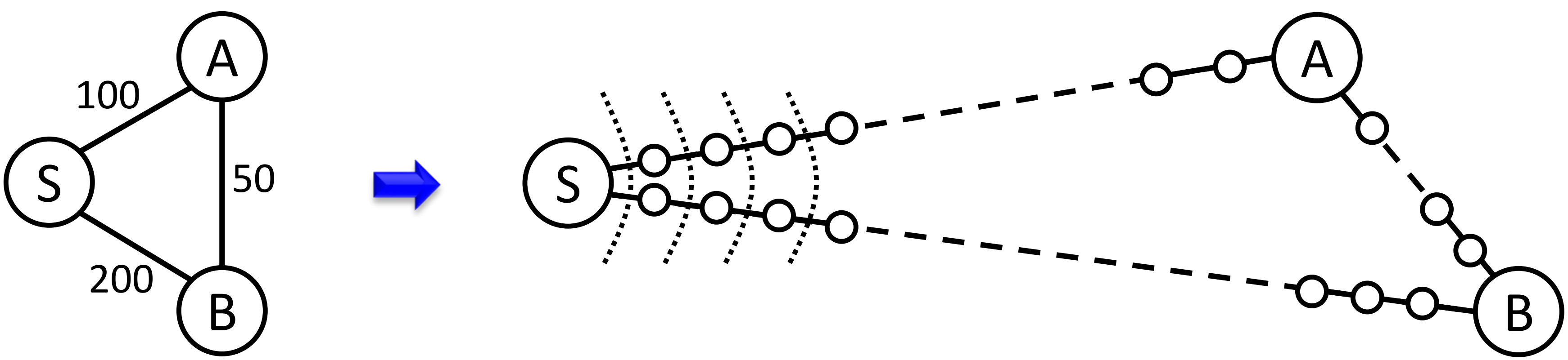
Distances on edges



https://de.wikipedia.org/wiki/Dijkstra-Algorithmus

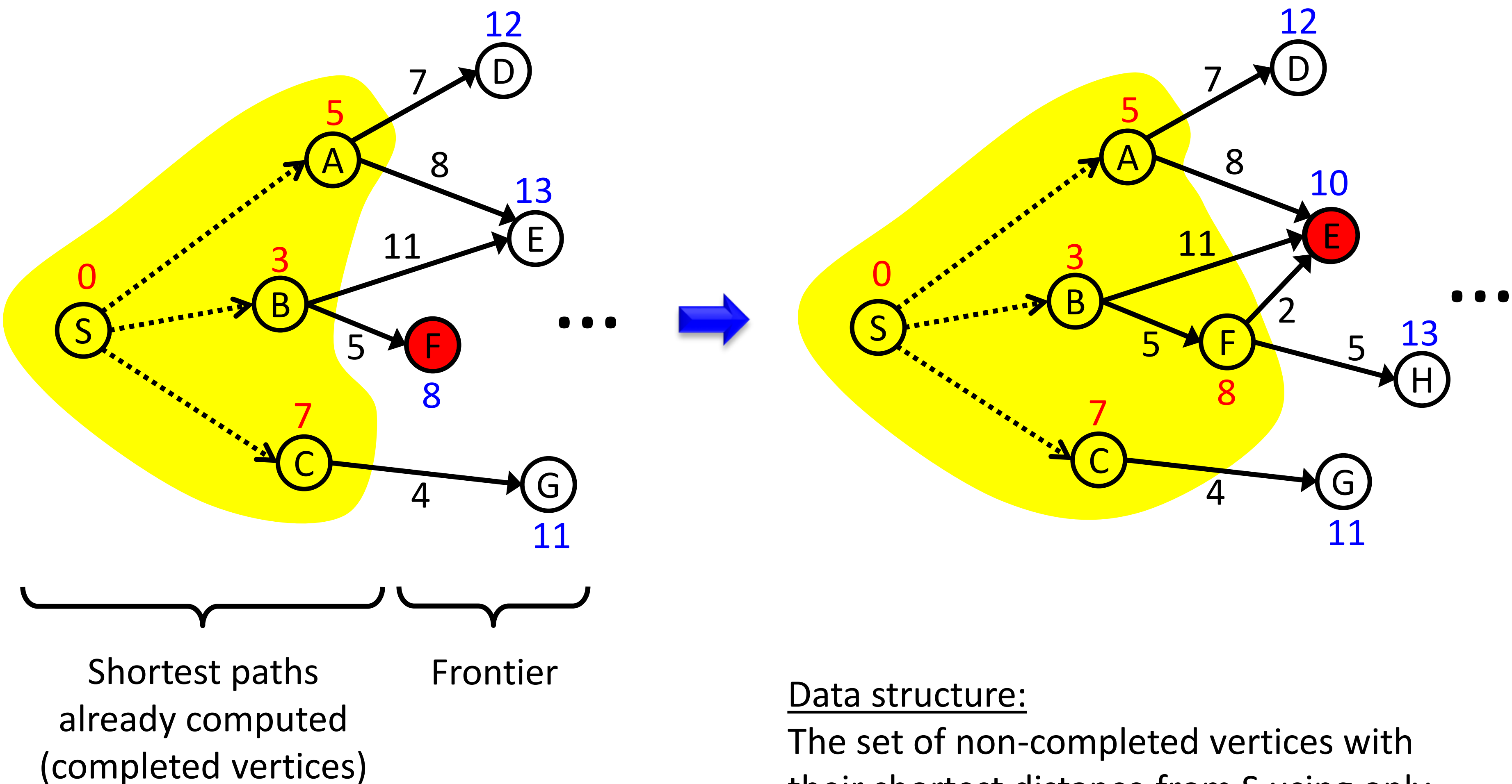
Reusing BFS





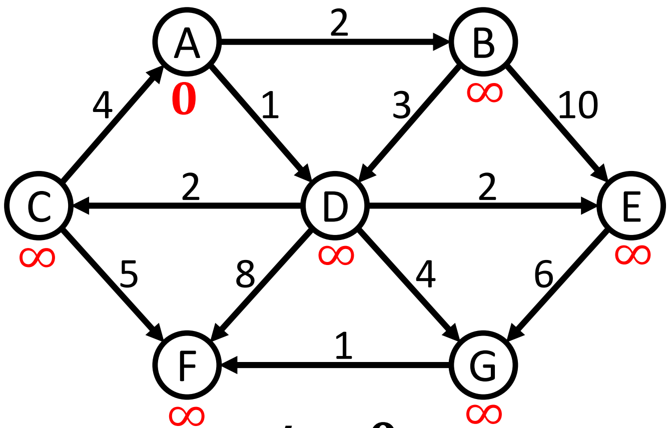
Inefficient: many cycles without any interesting progress. How about real numbers?

Dijkstra's algorithm: invariant

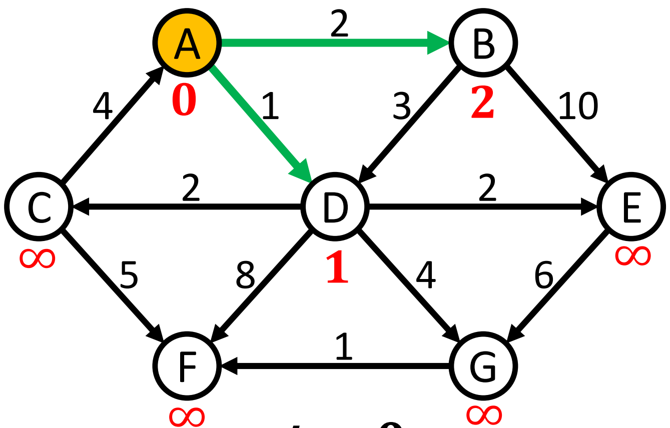


<u>Data structure:</u>
The set of non-completed vertices with their shortest distance from S using only the completed vertices.

Example



$t = 0$

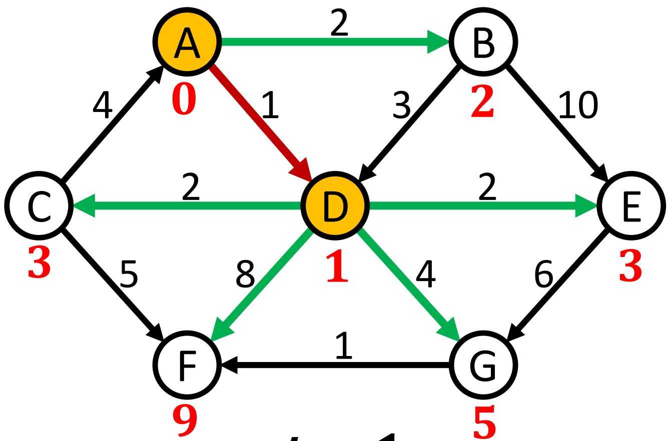


$t = 0$

Done	Queue
	A:0
	B:∞
	E:∞
	D:∞
	C:∞
	F:∞
	G:∞

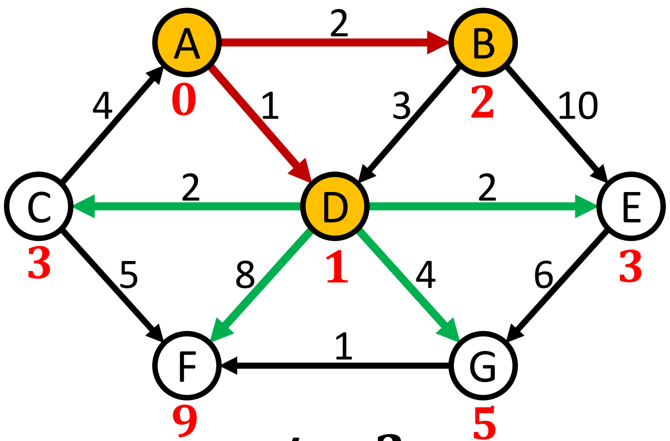
Done	Queue
A:0	D:1
	B:2
	E:∞
	C:∞
	F:∞
	G:∞

Example



$t = 1$

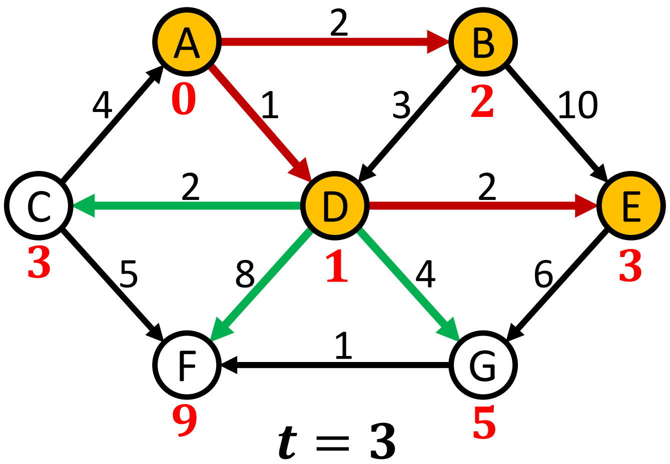
Done	Queue
A:0	B:2
D:1	E:3
	C:3
	G:5
	F:9

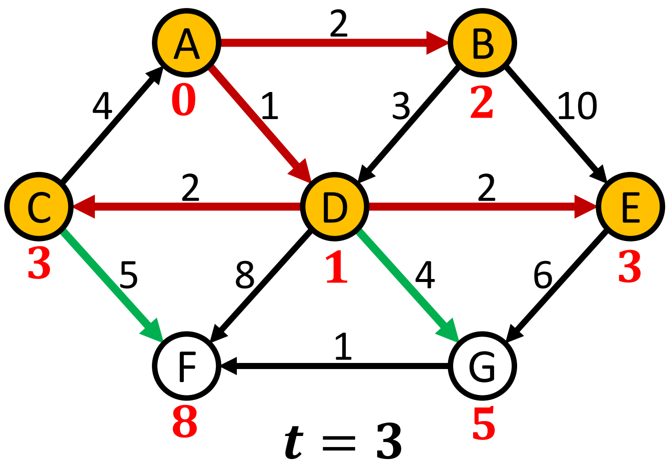


$t = 2$

Done	Queue
A:0	E:3
D:1	C:3
B:2	G:5
	F:9

Example

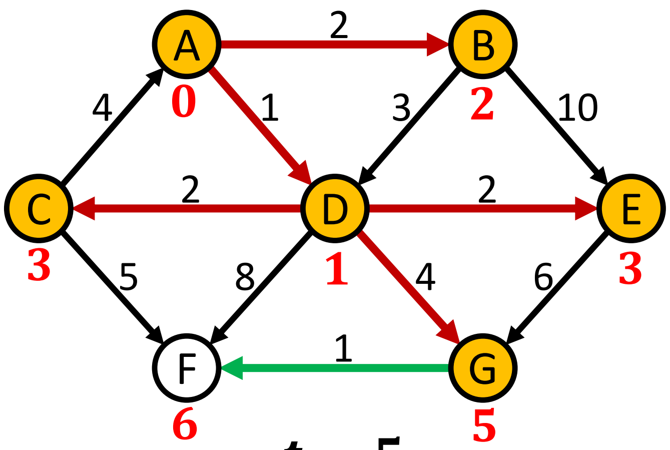




Done	Queue
A:0	**C:3**
D:1	**G:5**
B:2	**F:9**
E:3	

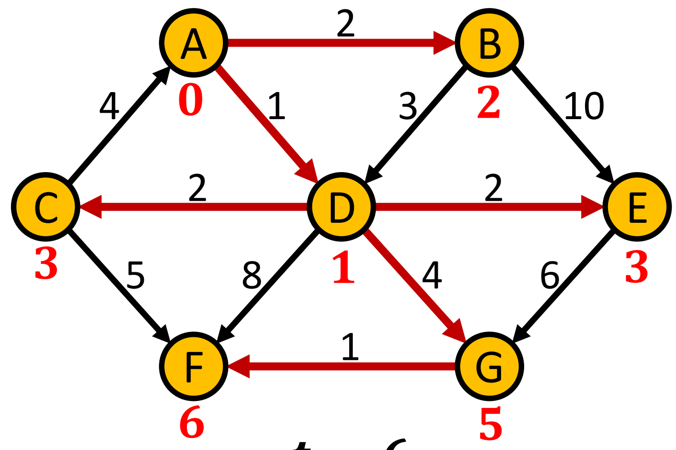
Done	Queue
A:0	**G:5**
D:1	**F:8**
B:2	
E:3	
C:3	

Example



$t = 5$

Done	Queue
A:0	F:6
D:1	
B:2	
E:3	
C:3	
G:5	

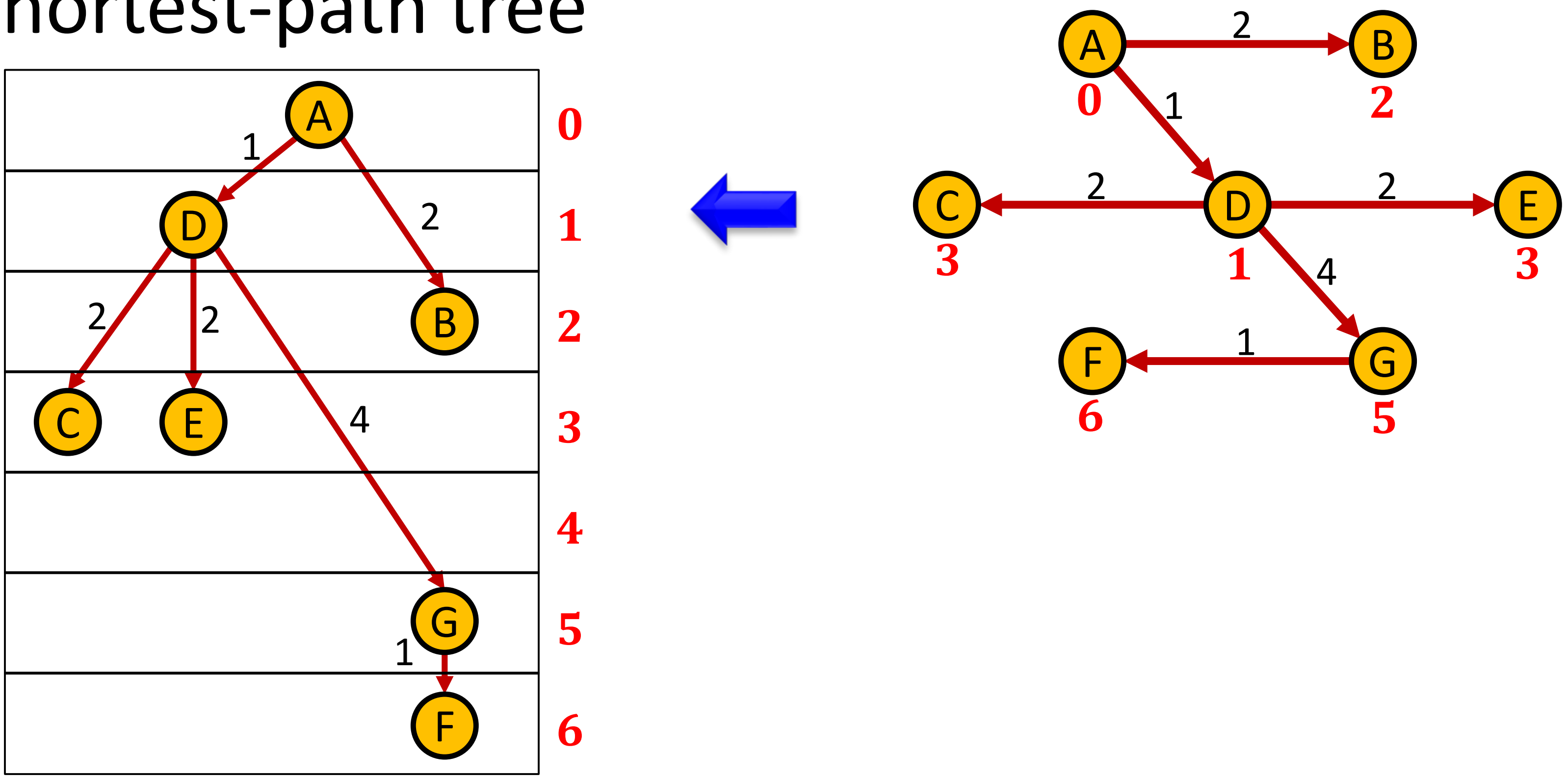


$t = 6$

Done	Queue
A:0	
D:1	
B:2	
E:3	
C:3	
G:5	
F:6	

Example

Shortest-path tree



We need to:

- keep a list non-completed vertices and their expected distances.
- select the non-completed vertex with shortest distance.
- update the distances of the neighbouring vertices.

Dijkstra’s algorithm for shortest paths

```
function ShortestPaths(G, l, s)

// Input: Graph G(V,E), source vertex s,
//        positive edge lengths {l_e: e ∈ E}
// Output: dist[u] has the distance from s,
//         prev[u] has the predecessor in the tree

  for all u ∈ V:
    dist[u] = ∞
    prev[u] = nil

  dist[s] = 0
  Q = makequeue(V)   // using dist as keys

  while not Q.empty():
    u = Q.deletemin()
    for all (u,v) ∈ E:
      if dist[v] > dist[u] + l(u,v):
        dist[v] = dist[u] + l(u,v)
        prev[v] = u
        Q.decreasekey(v)
```

Dijkstra's algorithm: complexity

```
Q = makequeue(V)
while not Q.empty():
   u = Q.deletemin()            ←  |V| times
   for all (u,v) ∈ E:
      if dist[v] > dist[u] + l(u,v):
         dist[v] = dist[u] + l(u,v)
         prev[v] = u
         Q.decreasekey(v)       ←  |E| times
```

- The skeleton of Dijkstra's algorithm is based on BFS, which is $O(|V|+|E|)$
- We need to account for the cost of:
  - **makequeue**: insert $|V|$ vertices to a list.
  - **deletemin**: find the vertex with min dist in the list ($|V|$ times)
  - **decreasekey**: update dist for a vertex ($|E|$ times)

- Let us consider two implementations for the list: **vector** and **binary heap**

Dijkstra's algorithm: complexity

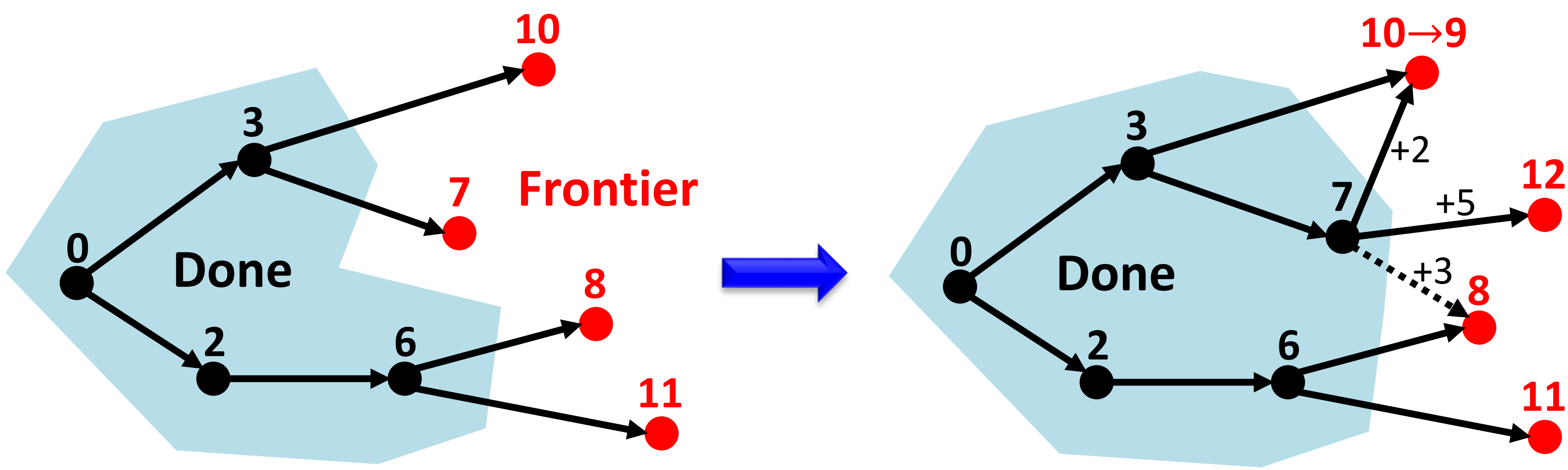
Implementation	deletemin	insert/ decreasekey	Dijkstra's complexity
Vector	$O(\lvert V \rvert)$	$O(1)$	$O(\lvert V \rvert^2)$
Binary heap	$O(\log \lvert V \rvert)$	$O(\log \lvert V \rvert)$	$O((\lvert V \rvert + \lvert E \rvert) \log \lvert V \rvert)$

Binary heap:

- The elements are stored in a complete (balanced) binary tree.
- **Insertion:** place element at the bottom and let it *bubble up* swapping the location with the parent (at most $\log_2 |V|$ levels).
- **Deletemin:** Remove element from the root, take the last node in the tree, place it at the root and let it *bubble down* (at most $\log_2 |V|$ levels).
- **Decreasekey:** decrease the key in the tree and let it *bubble up* (same as insertion). A data structure might be required to known the location of each vertex in the heap (table of pointers).

For connected graphs: $O((|V| + |E|) \log |V|) = O(|E| \log |V|)$

Why Dijkstra's works

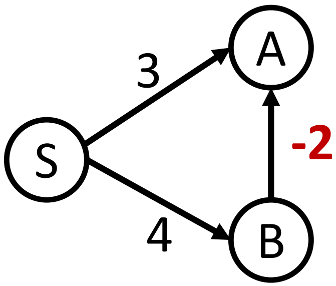


- A tree of open paths with distances is maintained at each iteration.
- The shortest paths for the internal nodes have already been calculated.
- The node in the frontier with shortest distance is "frozen" and expanded. Why? Because no other shorter path can reach the node.

Disclaimer: this is only true if the **distances are non-negative!**

Graphs with negative edges

- Dijkstra's algorithm does not work:



Dijkstra would say that the shortest path S→A has length=3.

- Dijkstra is based on a safe update each time an edge (u, v) is treated:

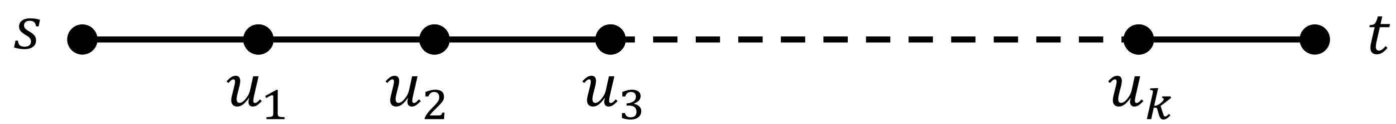
$$\text{dist}(v) = \min\{\text{dist}(v), \text{dist}(u) + l(u, v)\}$$

- Problem: shortest paths are consolidated too early.

- Possible solution: add a constant weight to all edges, make them positive, and apply Dijkstra.
  - It does not work, prove it!

Graphs with negative edges

- The shortest path from s to t can have at most $|V| - 1$ edges:



- If the sequence of updates includes

$$(s, u_1), (u_1, u_2), (u_2, u_3), \dots, (u_k, t),$$

in that order, the shortest distance from s to t will be computed correctly (updates are always safe). Note that the sequence of updates does not need to be consecutive.

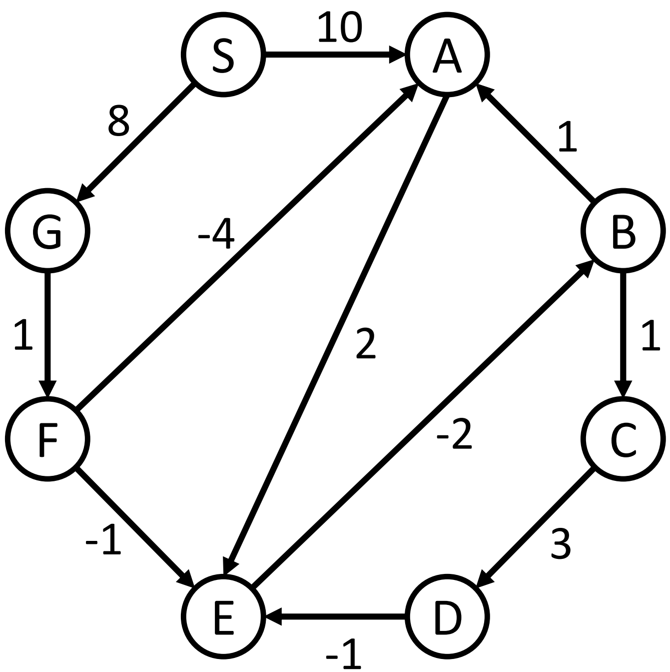
- Solution: update all edges $|V| - 1$ times !

- Complexity: $\mathrm{O}(|V| \cdot |E|)$.

Bellman-Ford algorithm

```
function ShortestPaths(G, l, s)

// Input: Graph G(V,E), source vertex s,
//        edge lengths {l_e: ∈ E}, no negative cycles.
// Output: dist[u] has the distance from s,
//         prev[u] has the predecessor in the tree

  for all u ∈ V:
    dist[u] = ∞
    prev[u] = nil

  dist[s] = 0
  repeat |V| − 1 times:
    for all (u,v) ∈ E:
      if dist[v] > dist[u] + l(u,v):
        dist[v] = dist[u] + l(u,v)
        prev[v] = u
```

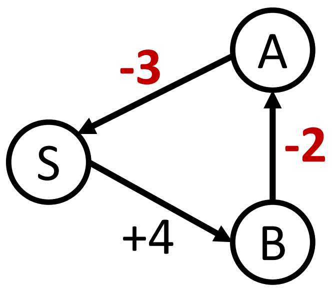
Bellman-Ford: example



	Iteration							
Node	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
S	**0**	0	0	0	0	0	0	0
A	∞	10	10	**5**	5	5	5	5
B	∞	∞	∞	10	6	**5**	5	5
C	∞	∞	∞	∞	11	7	**6**	6
D	∞	∞	∞	∞	∞	14	10	**9**
E	∞	∞	12	8	**7**	7	7	7
F	∞	∞	**9**	9	9	9	9	9
G	∞	**8**	8	8	8	8	8	8

Negative cycles

- What is the shortest distance between S and A?



Bellman-Ford does not work as it assumes that the shortest path will not have more than $|V| - 1$ edges.

- A negative cycle produces $-\infty$ distances by endlessly applying rounds to the cycle.

- How to detect negative cycles?
  - Apply Bellman-Ford (update edges $|V| - 1$ times)
  - Perform an extra round and check whether some distance decreases.

Shortest paths in DAGs

- DAG’s property:

  In any path of a DAG, the vertices appear in increasing topological order.

- Any sequence of updates that preserves the topological order will compute distances correctly.

- Only one round visiting the edges in topological order is sufficient: $O(|V| + |E|)$.

- How to calculate the longest paths?
  – Negate the edge lengths and compute the shortest paths.
  – Alternative: update with max (instead of min).

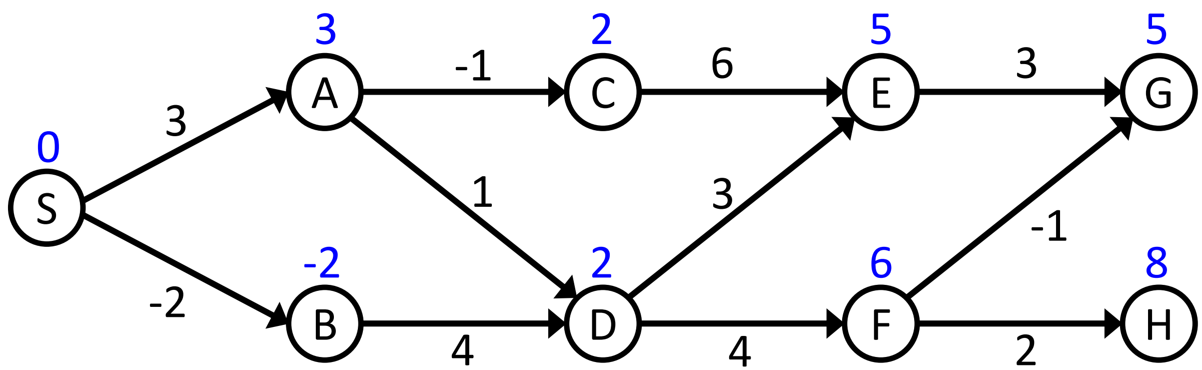
DAG shortest paths algorithm

```
function DagShortestPaths(G, l, s)

// Input: DAG G(V,E), source vertex s,
//        edge lengths {l_e:∈ E}.
// Output: dist[u] has the distance from s,
//         prev[u] has the predecessor in the tree

  for all u ∈ V:
    dist[u] = ∞
    prev[u] = nil

  dist[s] = 0
  Linearize G
  for all u ∈ V in linearized order:
    for all (u,v) ∈ E:
      if dist[v] > dist[u] + l(u,v):
        dist[v] = dist[u] + l(u,v)
        prev[v] = u
```

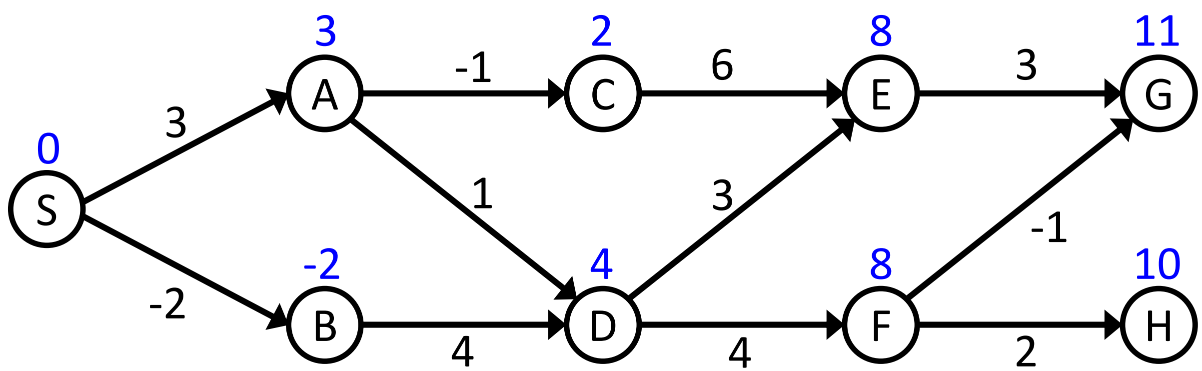
DAG shortest/longest paths: example

Linearization: S A B C D E F G H

Shortest paths



Longest paths



Shortest paths: summary

Single-source shortest paths

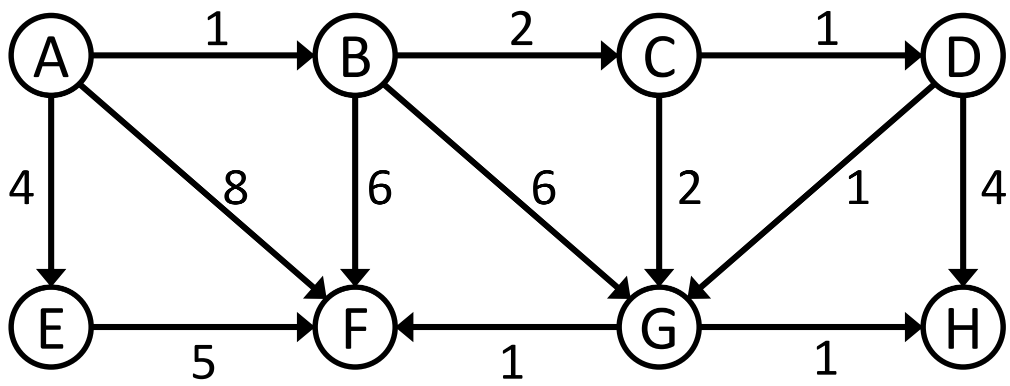
Graph	Algorithm	Complexity
Non-negative edges	Dijkstra	$\mathrm{O}\big((\lvert V\rvert + \lvert E\rvert)\log\lvert V\rvert\big)$
Negative edges	Bellman-Ford	$\mathrm{O}(\lvert V\rvert \cdot \lvert E\rvert)$
DAG	Topological sort	$\mathrm{O}(\lvert V\rvert + \lvert E\rvert)$

A related problem: All-pairs shortest paths

- Floyd-Warshall algorithm ($\mathrm{O}(\lvert V\rvert^3)$), based on dynamic programming.
- Other algorithms exist.

EXERCISES

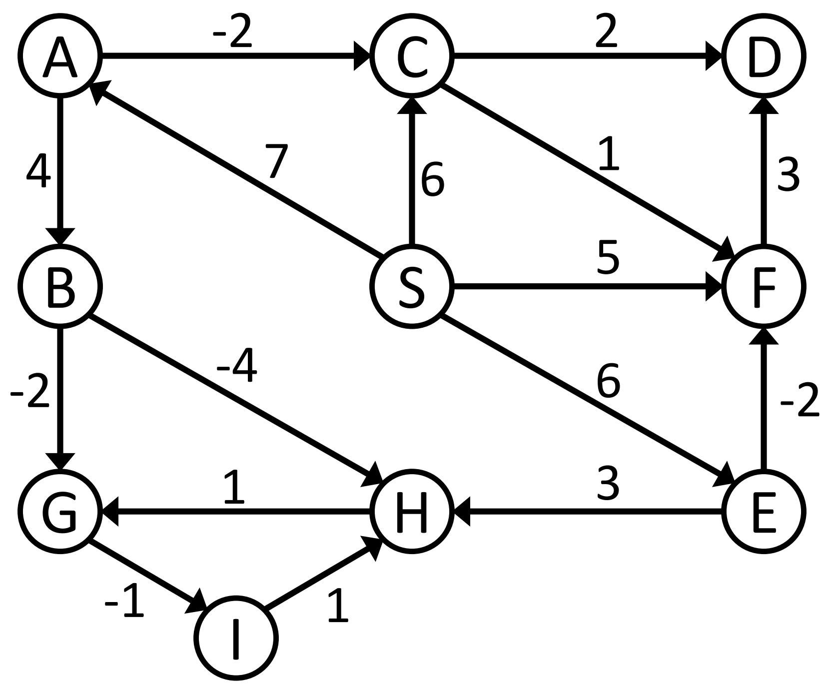
Dijkstra (from [DPV2008])



Run Dijkstra's algorithm starting at node A:

- Draw a table showing the intermediate distance values of all the nodes at each iteration
- Show the final shortest-path tree

Bellman-Ford (from [DPV2008])



Run Bellman-Ford algorithm starting at node S:

– Draw a table showing the intermediate distance values of all the nodes at each iteration
– Show the final shortest-path tree

New road (from [DPV2008])

There is a network of roads $G = (V, E)$ connecting a set of cities V. Each road in E has an associated length l_e. There is a proposal to add one new road to this network, and there is a list E' of pairs of cities between which the new road can be built. Each such potential road $e' \in E'$ has an associated length. As a designer for the public works department you are asked to determine the road $e' \in E'$ whose addition to the existing network G would result in the maximum decrease in the driving distance between two fixed cities s and t in the network. Give an efficient algorithm for solving this problem.

Nesting boxes

A d-dimensional box with dimensions $(x_1, x_2, \ldots, x_d)$ nests within another box with dimensions $(y_1, y_2, \ldots, y_d)$ if there exists a permutation π on $\{1,2, \ldots, d\}$ such that:

$$x_{\pi(1)} < y_1, x_{\pi(2)} < y_2, \ldots, x_{\pi(d)} < y_d.$$

a. Argue that the nesting relation is transitive.

b. Describe an efficient method to determine whether or not one d-dimensional box nests inside another.

c. Suppose that you are given a set of n d-dimensional boxes $\{B_1, B_2, \ldots, B_n\}$. Describe an efficient algorithm to determine the longest sequence $\langle B_{i_1}, B_{i_2}, \ldots, B_{i_k} \rangle$ of boxes such that B_{i_j} nests within $B_{i_{j+1}}$ for $j = 1,2, \ldots, k-1$. Express the running time of your algorithm in terms of n and d.

Source: Cormen, Leiserson and Rivest, Introduction to Algorithms, The MIT Press.