Trees

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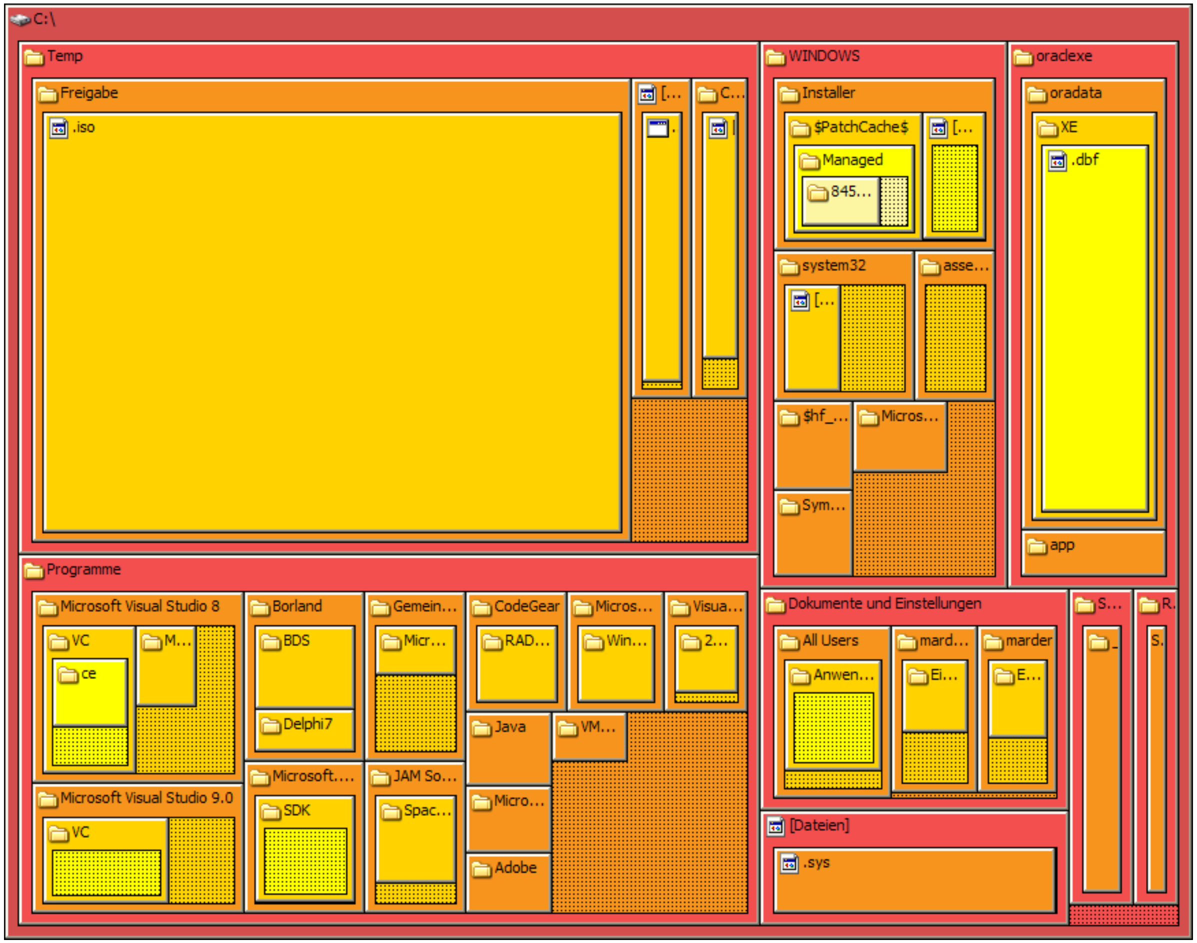
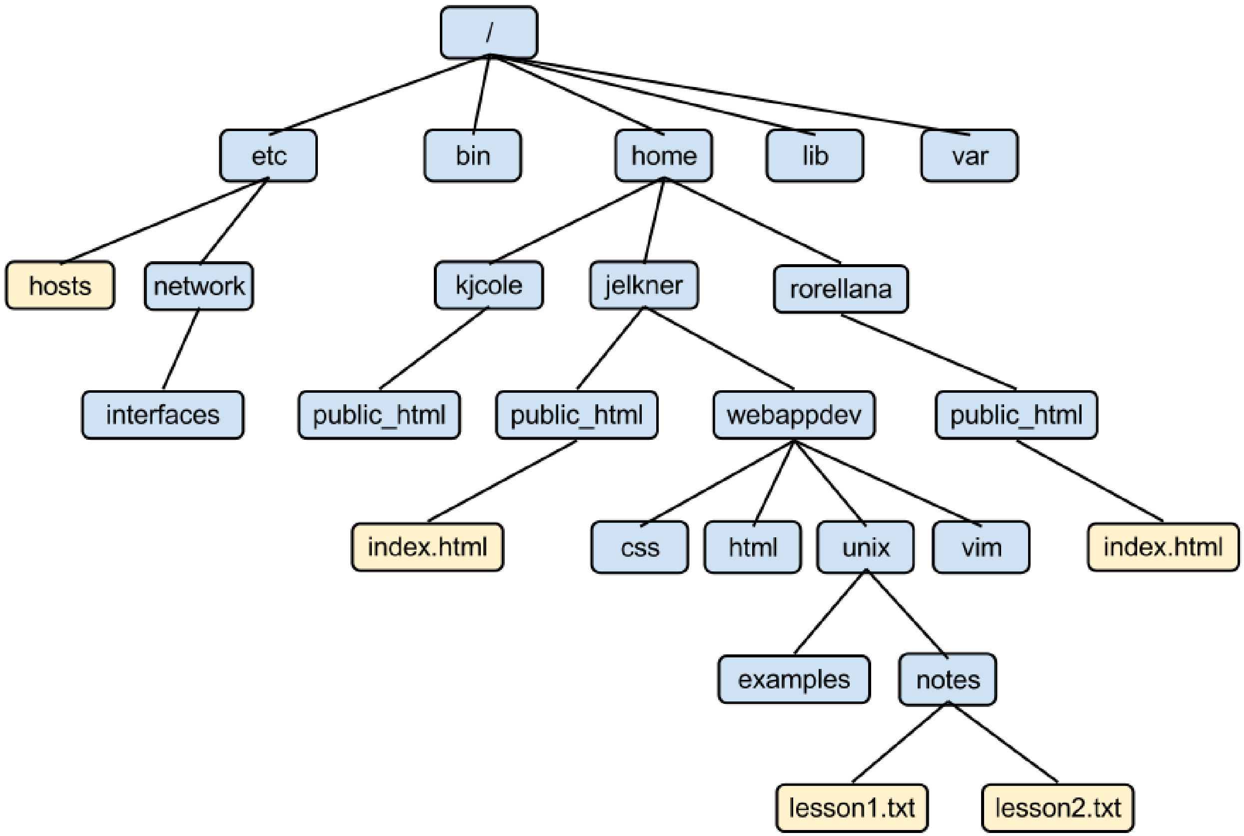
Department of Computer Science

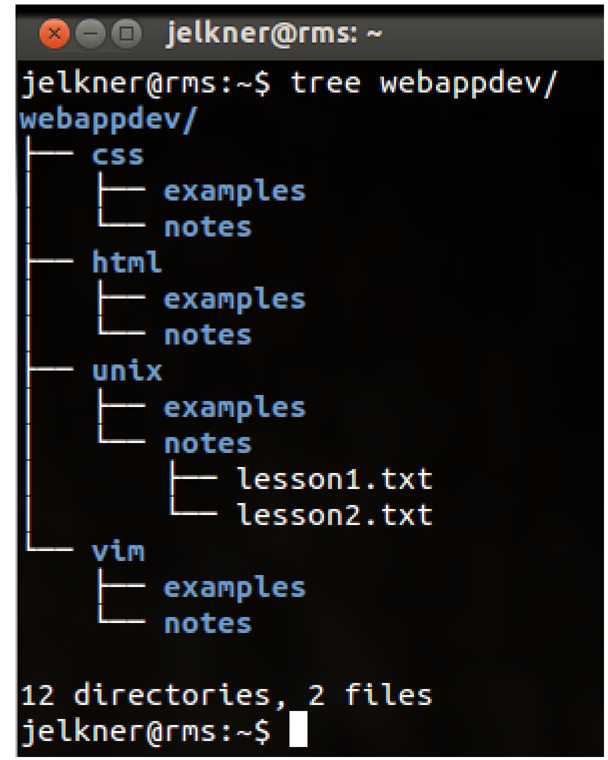
Data are often organized hierarchically



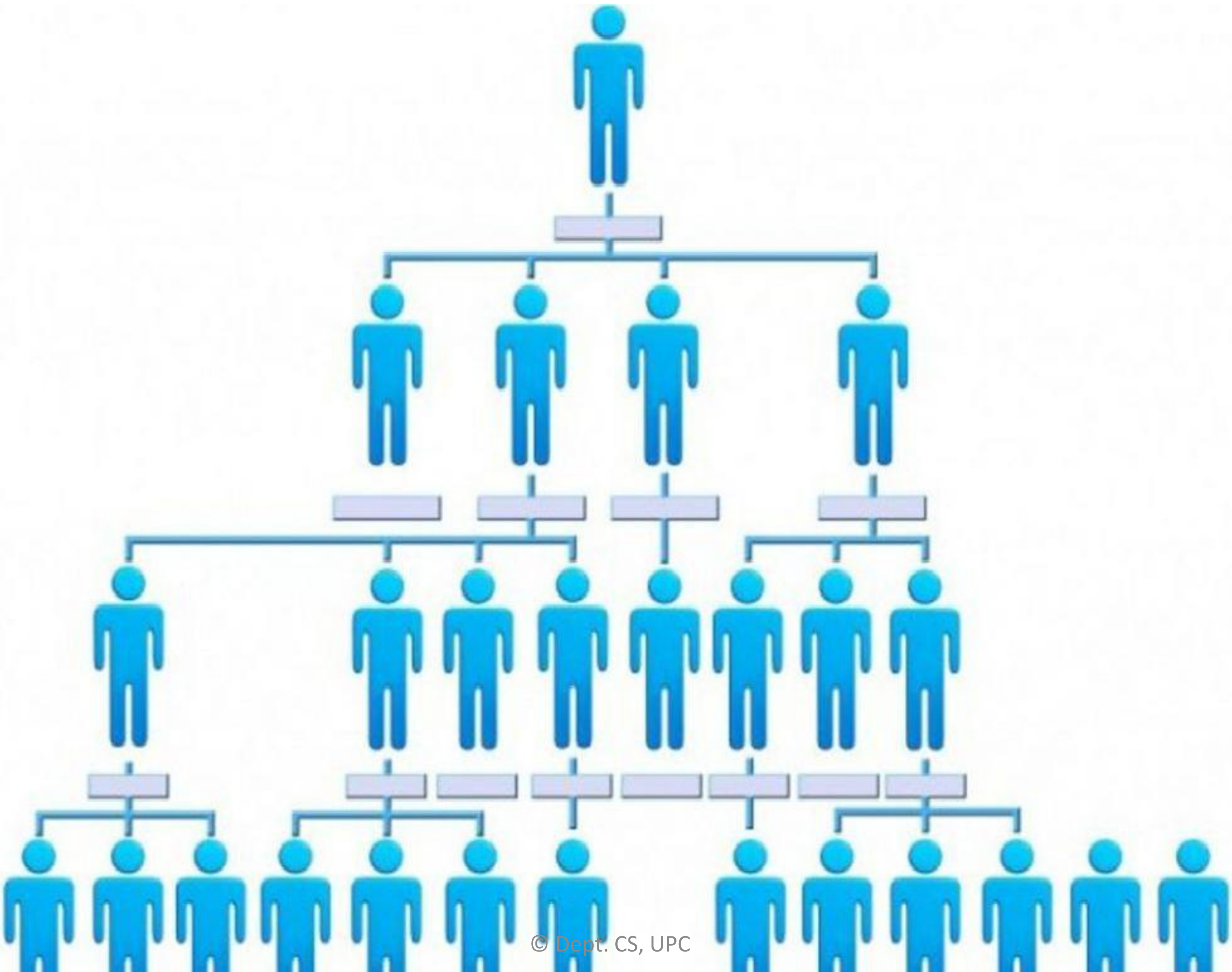
source: https://en.wikipedia.org/wiki/Tree_structure

Filesystems

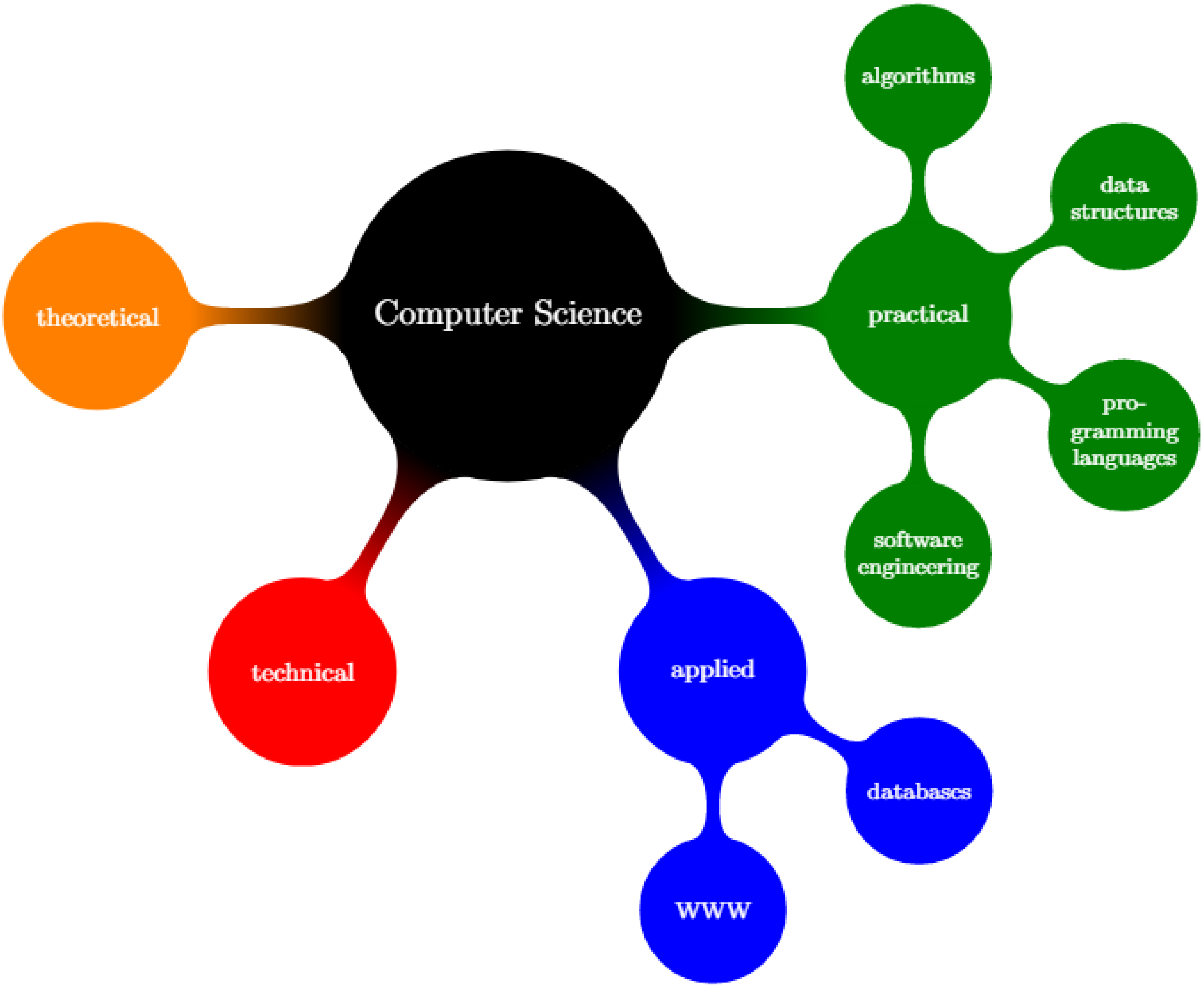




Company structure



Mind maps



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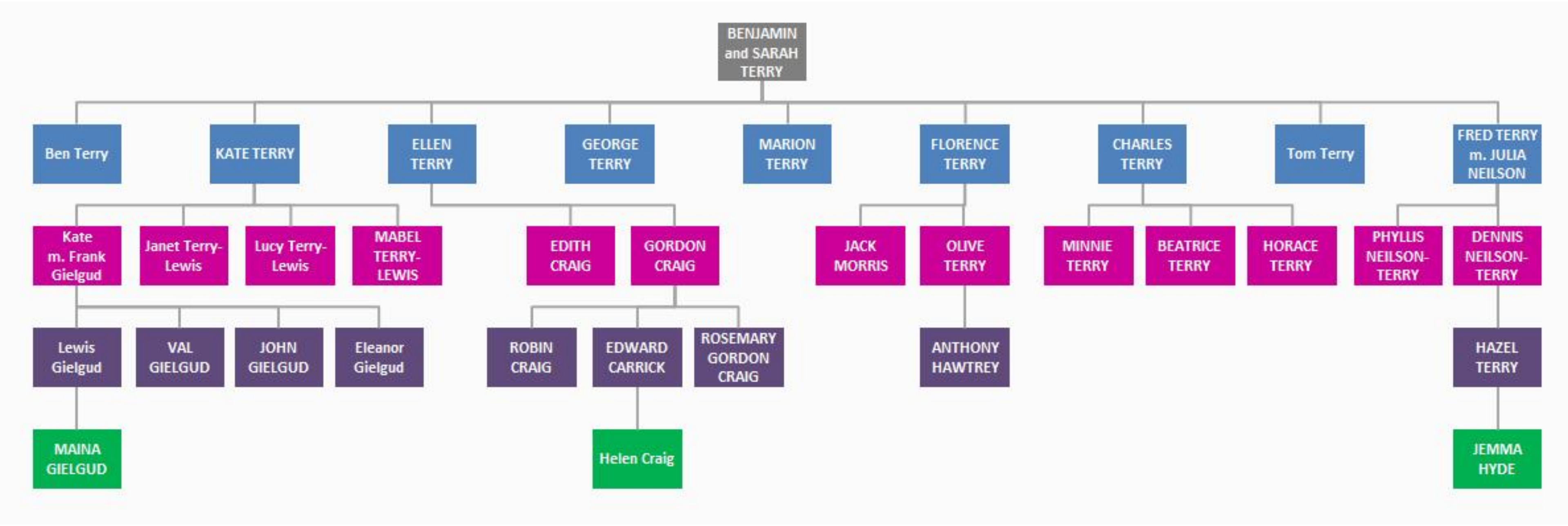
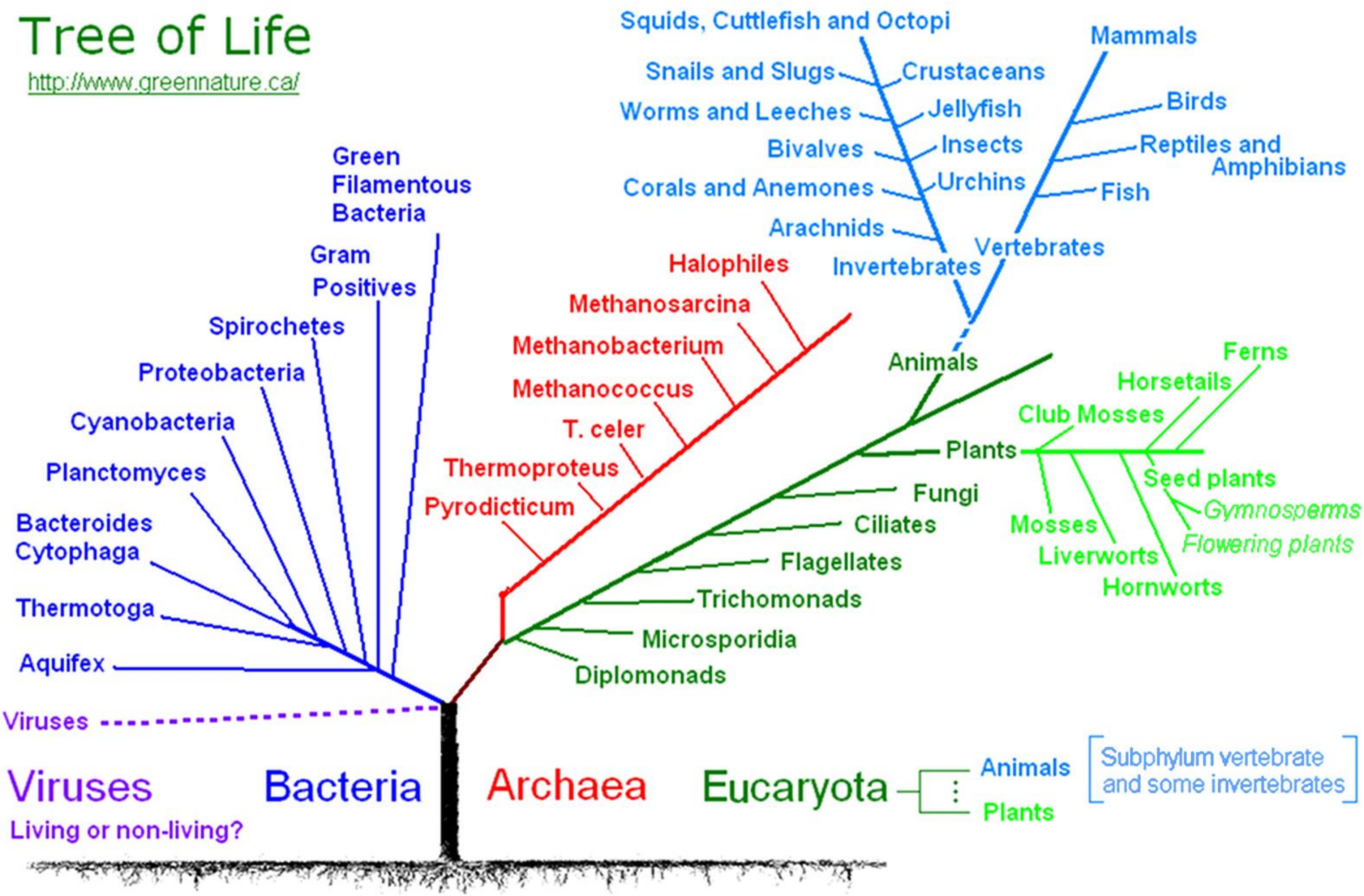
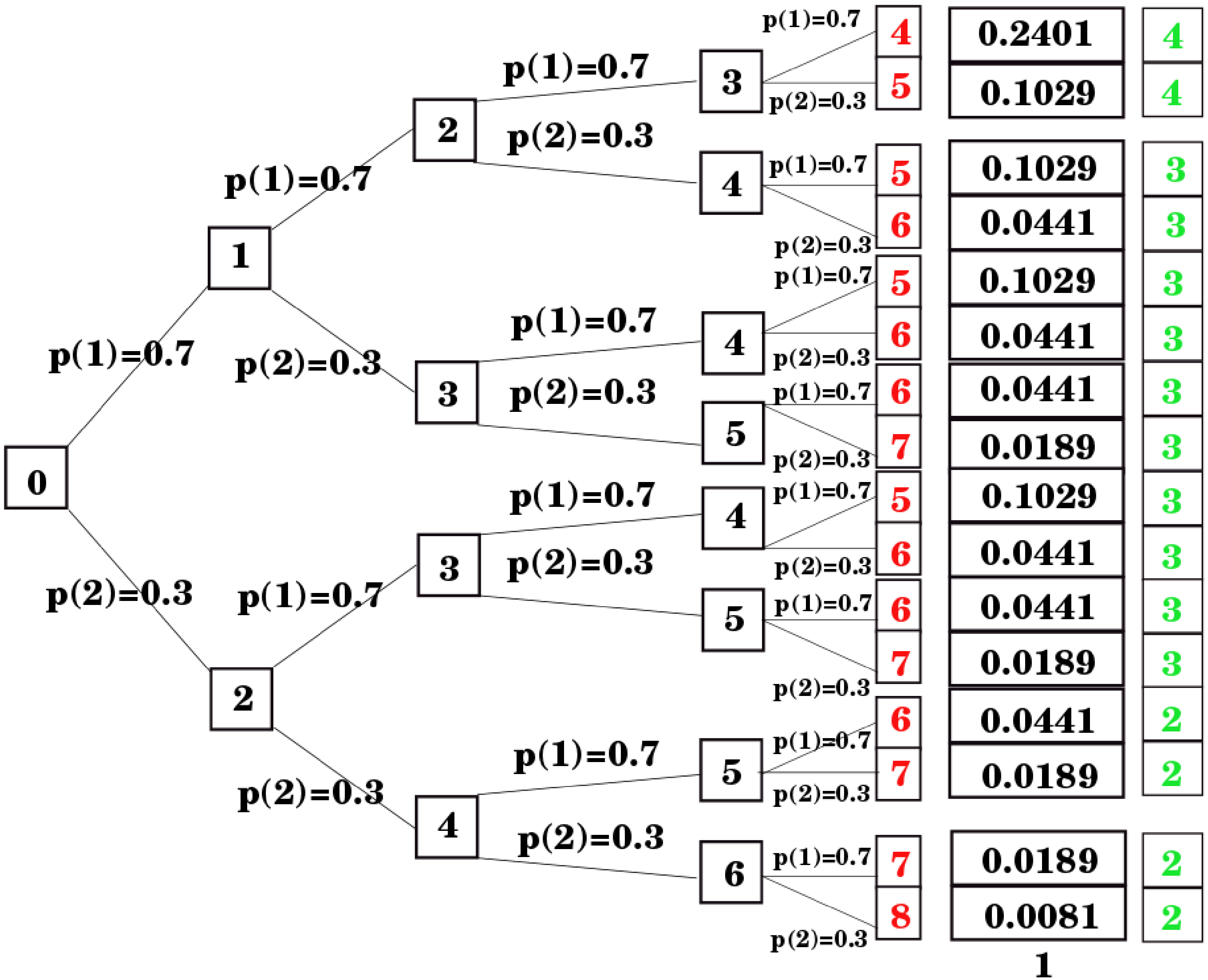
Genealogical trees



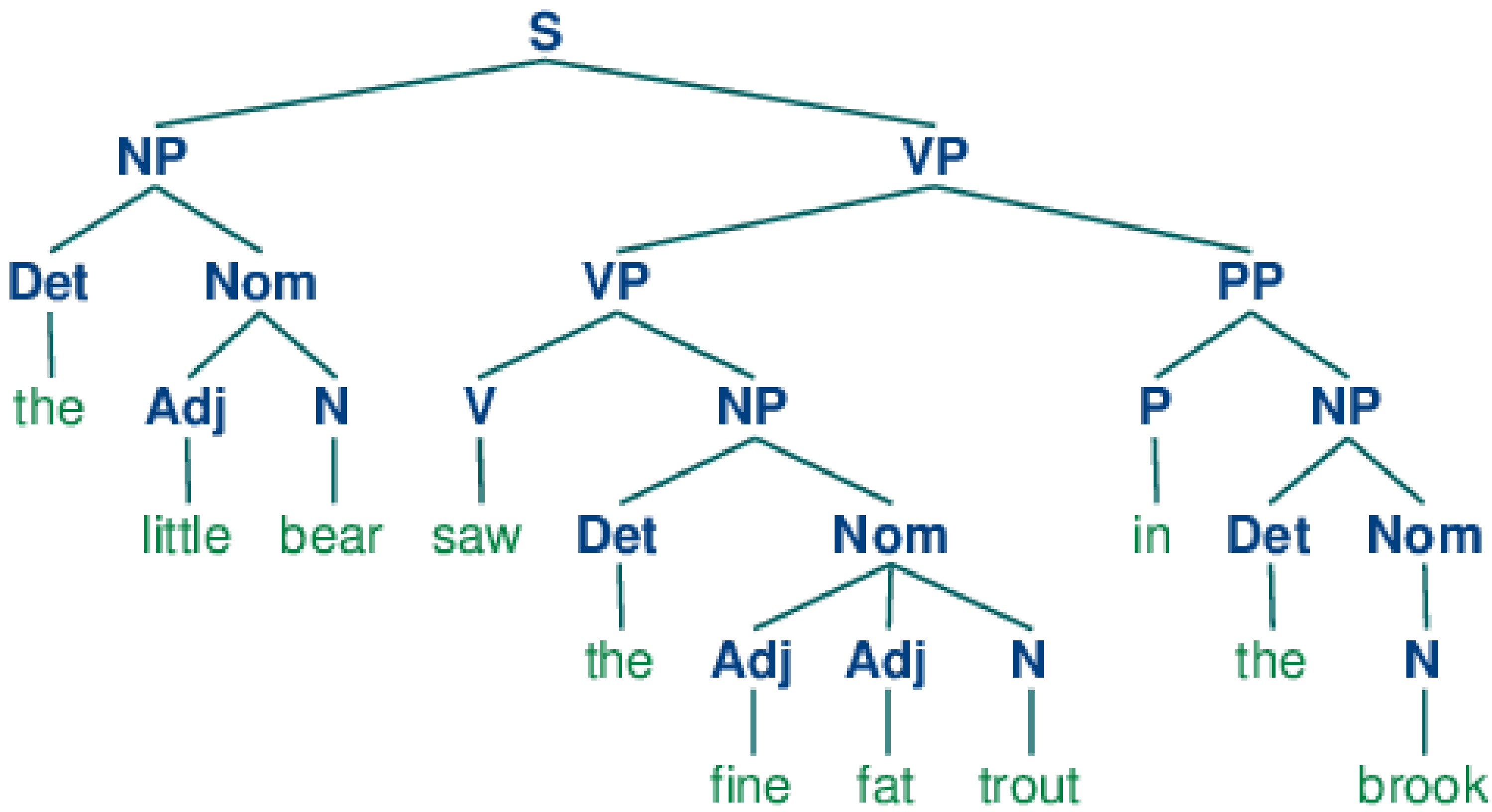


Tree of Life
http://www.greennature.ca/
Squids, Cuttlefish and Octopi
Snails and Slugs
Crustaceans
Worms and Leeches
Jellyfish
Bivalves
Insects
Corals and Anemones
Urchins
Arachnids
Invertebrates
Vertebrates
Mammals
Birds
Reptiles and Amphibians
Fish
Green Filamentous Bacteria
Gram Positives
Spirochetes
Proteobacteria
Cyanobacteria
Planctomyces
Bacteroides Cytophaga
Thermotoga
Aquifex
Halophiles
Methanosarcina
Methanobacterium
Methanococcus
T. celer
Thermoproteus
Pyrodicticum
Animals
Plants
Fungi
Ciliates
Flagellates
Trichomonads
Microsporidia
Diplomonads
Ferns
Horsetails
Club Mosses
Seed plants
Gymnosperms
Flowering plants
Mosses
Liverworts
Hornworts
Viruses
Viruses
Living or non-living?
Bacteria
Archaea
Eucaryota
Animals
Plants
Subphylum vertebrate and some invertebrates

Probability trees



Parse trees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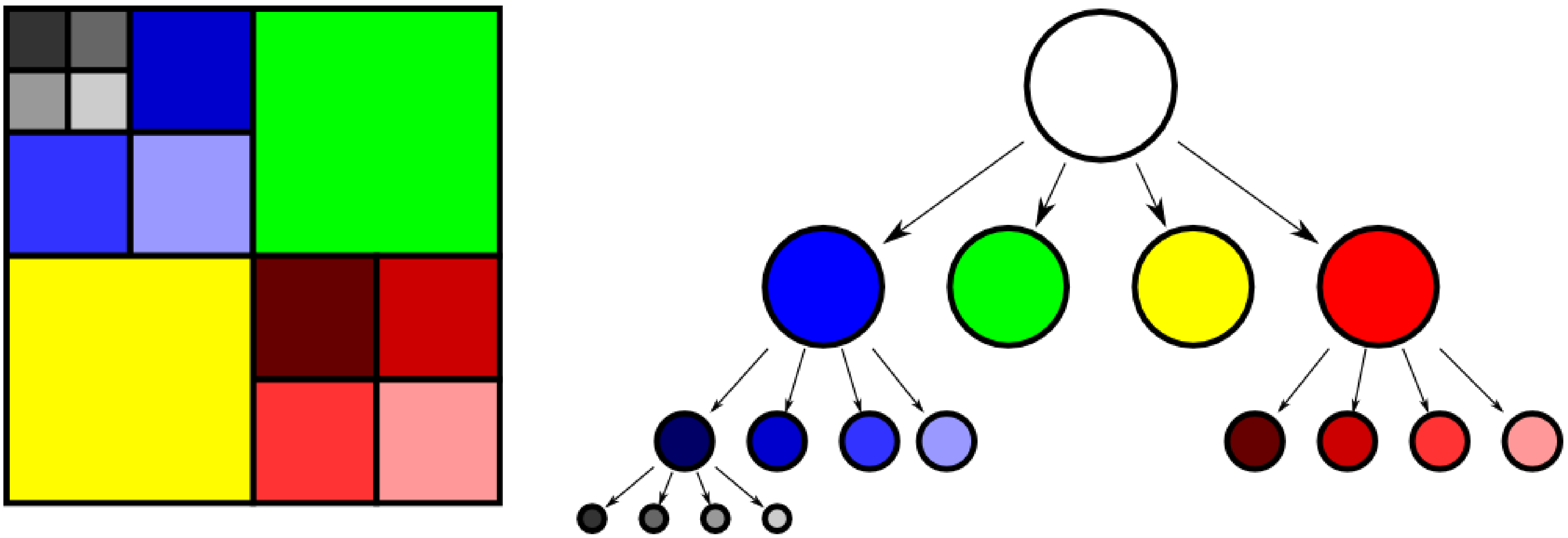
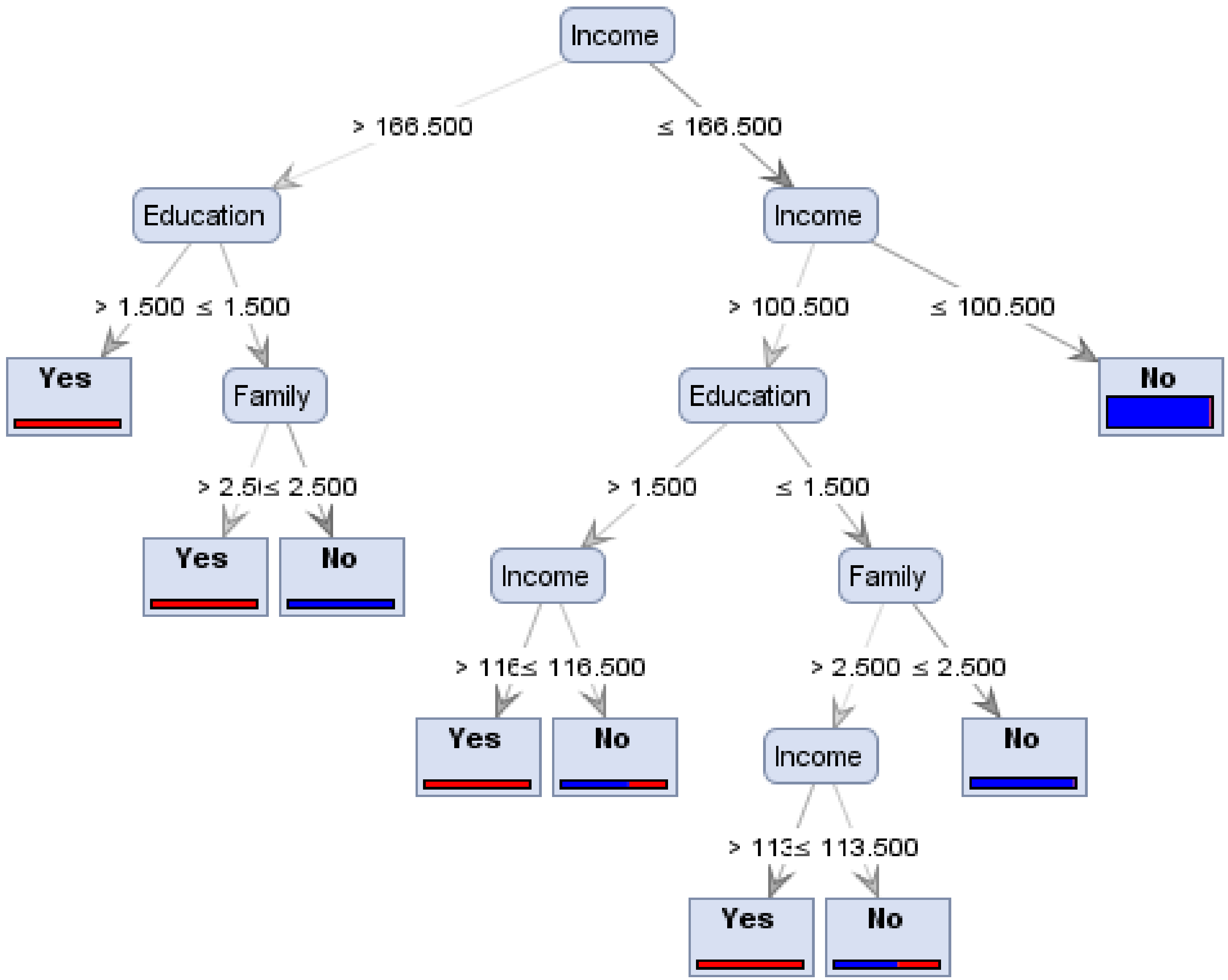
brook

Image representation (quad-trees)



Decision trees

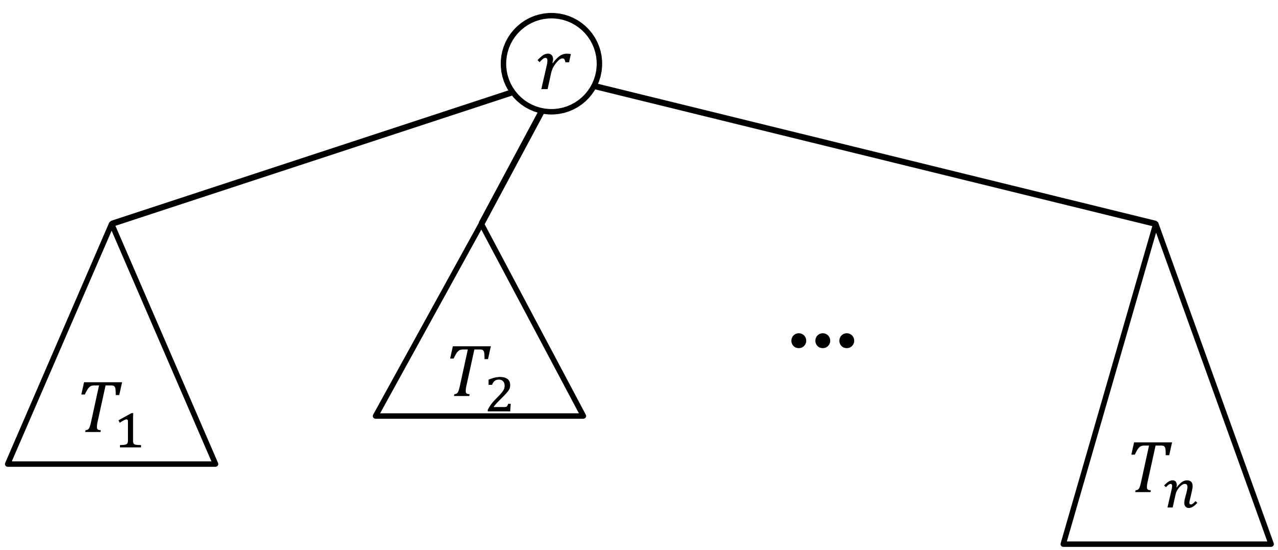


source: http://www.simafore.com/blog/bid/94454/A-simple-explanation-of-how-entropy-fuels-a-decision-tree-model

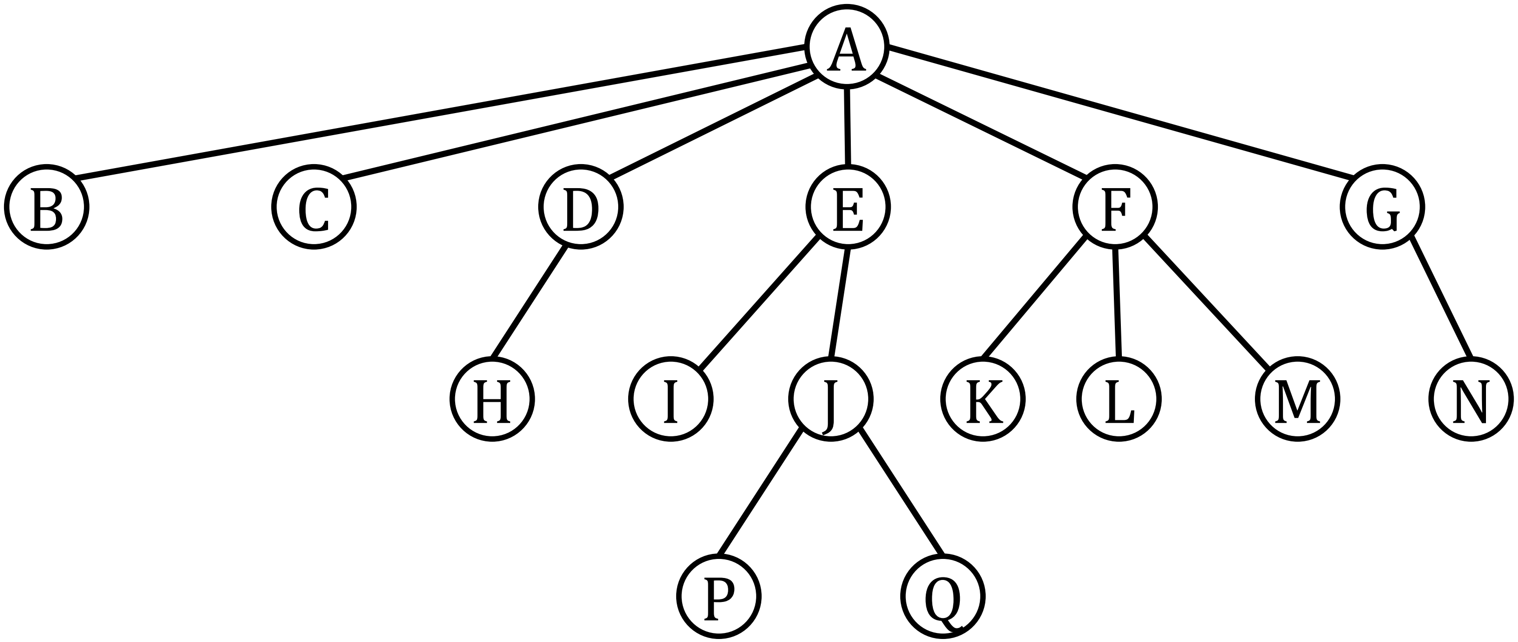
Tree: definition

- Graph theory: a tree is an undirected graph in which any two vertices are connected by exactly one path.

- Recursive definition (CS). A non-empty tree T consists of:
  - a root node r
  - a list of trees $T_1, T_2, \dots, T_n$ that hierarchically depend on r.

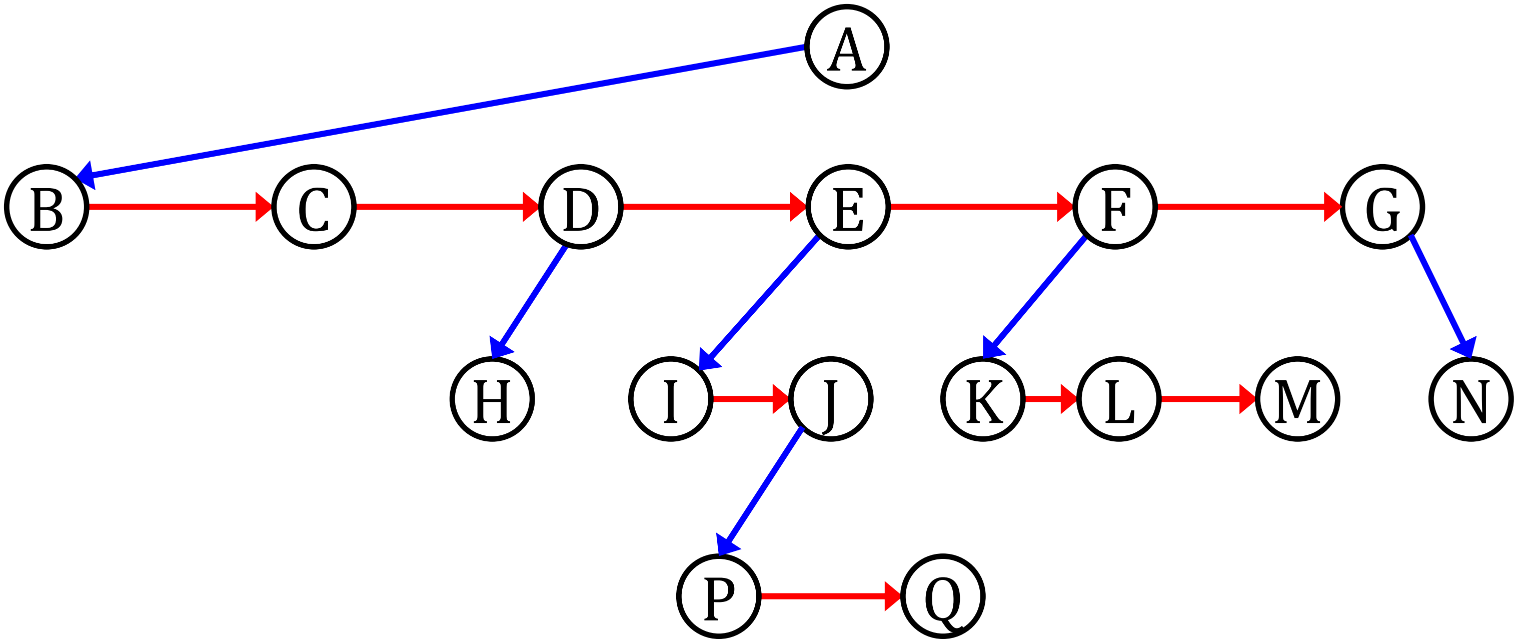


Tree: nomenclature



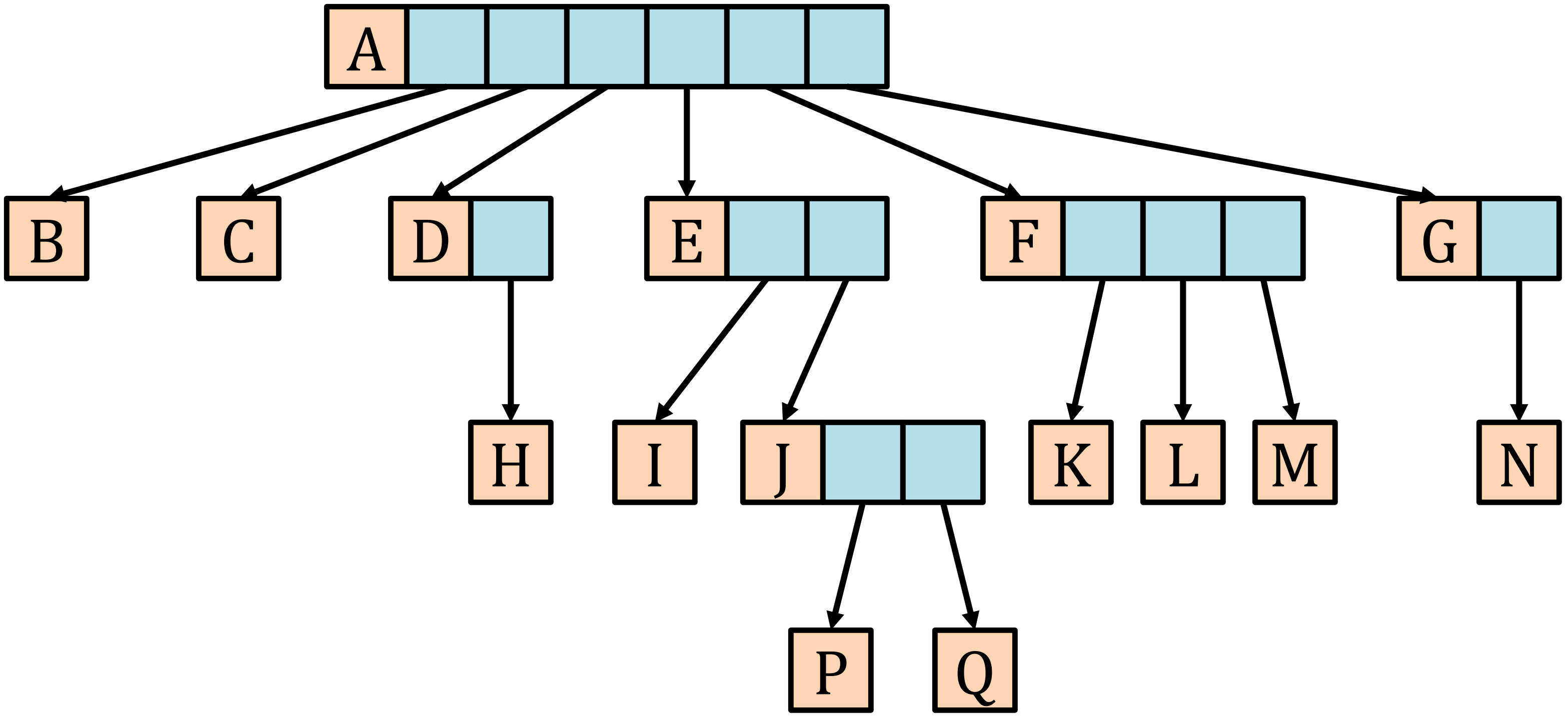
- A is the **root** node.
- Nodes with no children are **leaves** (e.g., B and P).
- Nodes with the same parent are **siblings** (e.g., K, L and M).
- The **depth** of a node is the length of the path from the root to the node. Examples: depth(A)=0, depth(L)=2, depth(Q)=3.

Tree: representation with linked lists



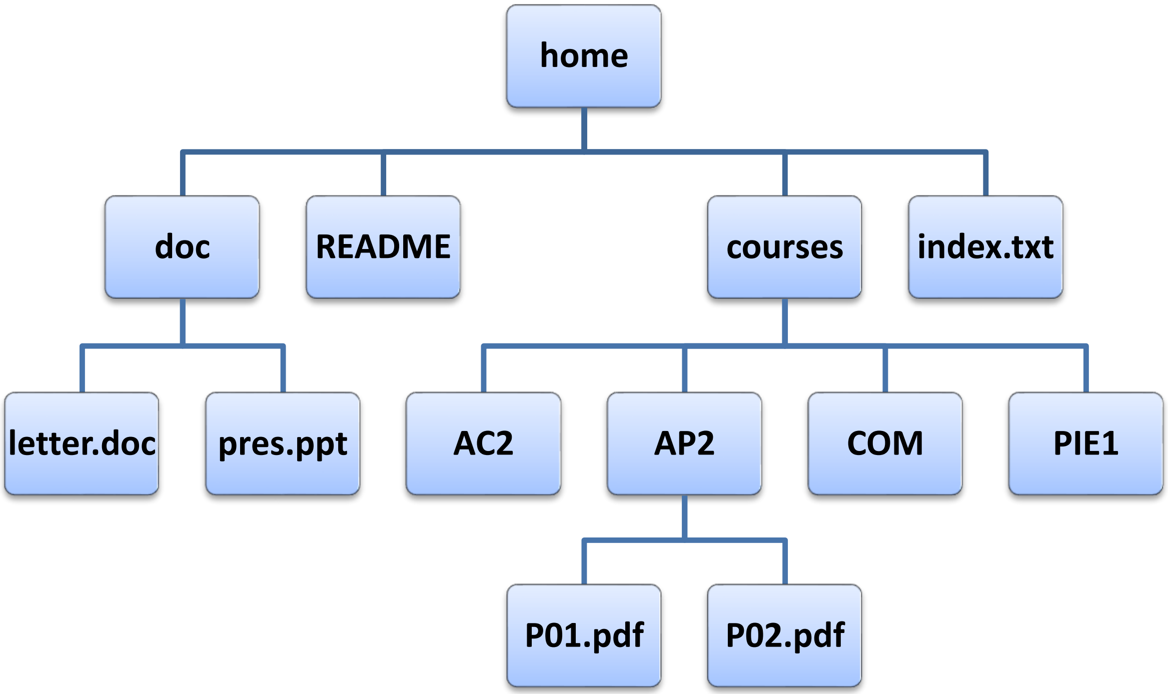

```
struct TreeNode {
  Type element;
  list<TreeNode> children; // Linked list of children
};
```

Tree: representation with vectors




```
struct TreeNode {
  Type element;
  vector<TreeNode> children; // Vector of children
};
```

Print a tree




```
home
  doc
    letter.doc
    pres.ppt
  README
  courses
    AC2
    AP2
      P01.pdf
      P02.pdf
    COM
    PIE1
  index.txt
```

```
struct Tree {
  string name;
  vector<Tree> children;
};
```

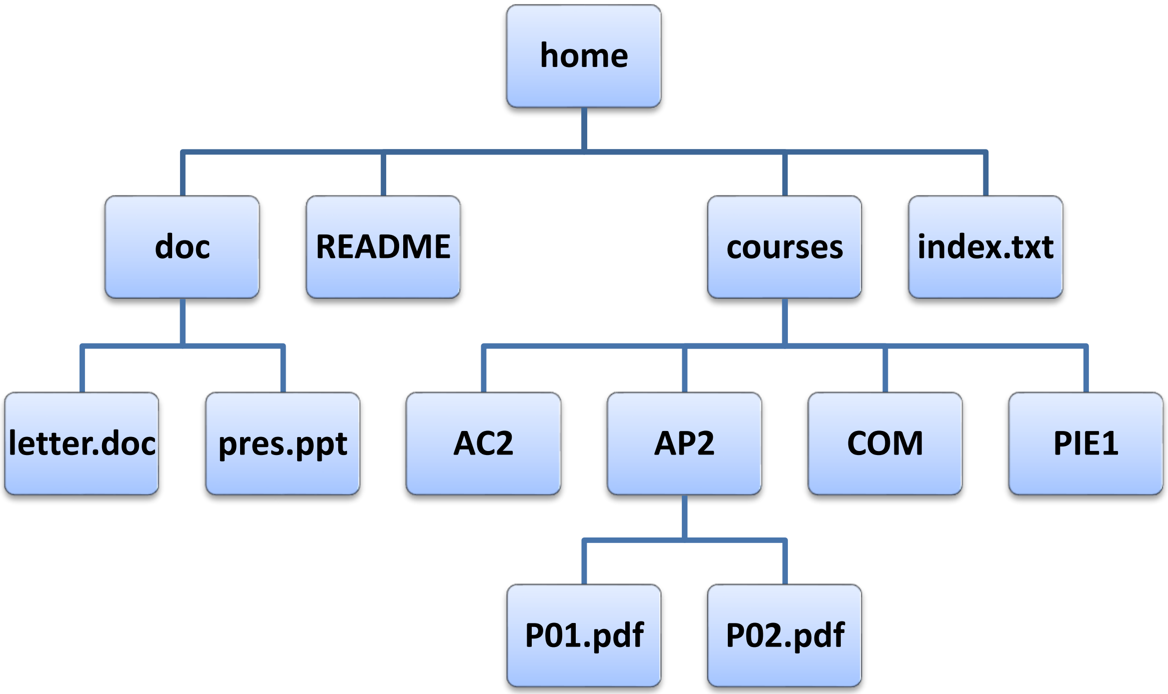
```
print(const Tree& T, int depth=0);
```

Print a tree

```
/** Prints a tree indented according to depth.
  * Pre: The tree is not empty. */
void print(const Tree& T, int depth) {

  // Print the root indented by 2*depth
  cout << string(2*depth, ' ') << T.name << endl;

  // Print the children with depth + 1
  for (const Tree& child: T.children)
      print(child, depth + 1);
}
```

This function executes a ***preorder*** traversal of the tree: each node is processed *before* the children.

Print a tree (postorder traversal)




```
    letter.doc
    pres.ppt
  doc
  README
    AC2
      P01.pdf
      P02.pdf
    AP2
    COM
    PIE1
  courses
  index.txt
home
```

Postorder traversal: each node is processed after the children.

Print a tree (postorder traversal)

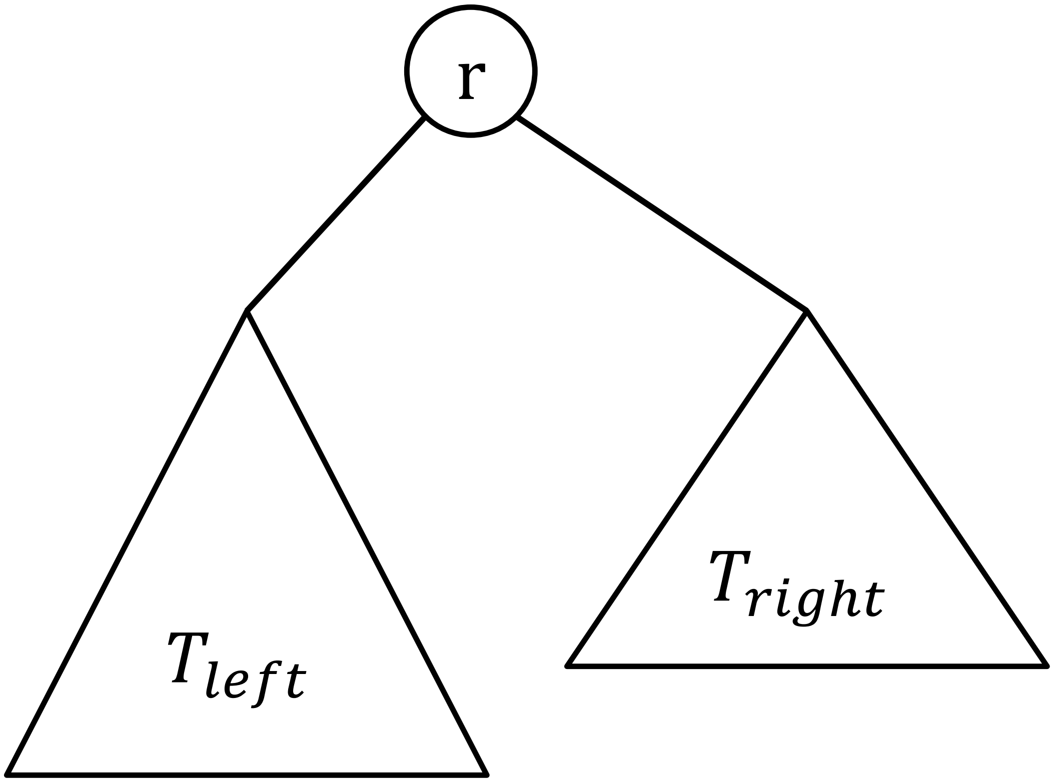
```
/** Prints a tree (in postorder) indented according to depth.
  * Pre: The tree is not empty. */
void printPostOrder(const Tree& T, int depth) {

  // Print the children with depth + 1
  for (const Tree& child: T.children)
      printPostOrder(child, depth + 1);

  // Print the root indented by 2*depth
  cout << string(2*depth, ' ') << T.name << endl;
}
```

This function executes a ***postorder*** traversal of the tree: each node is processed *after* the children.

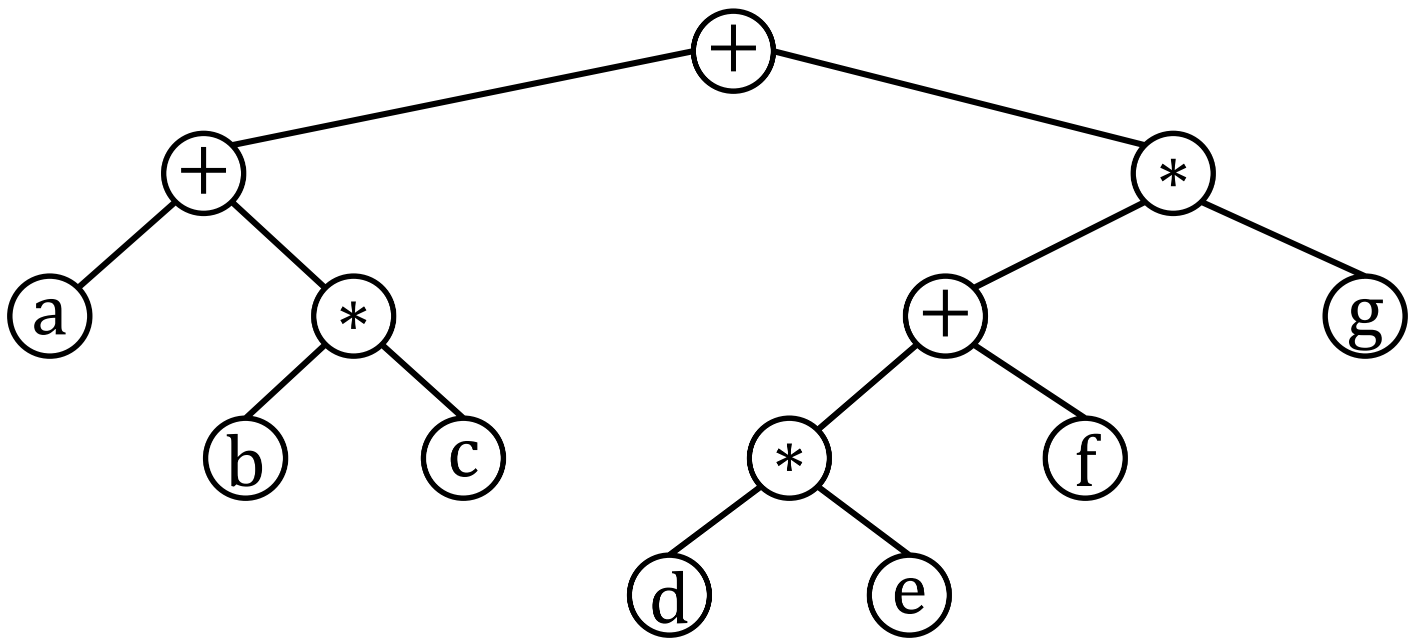
Binary trees

Nodes with at most two children.

```
struct BinTree {
  Type element;
  BinTree* left;
  BinTree* right;
};
```



Example: expression trees



Expression tree for: **a + b∗c + (d∗e + f) ∗ g**

Postfix representation: **a b c ∗ + d e ∗ f + g ∗ +**

How can the postfix representation be obtained?

Example: expression trees

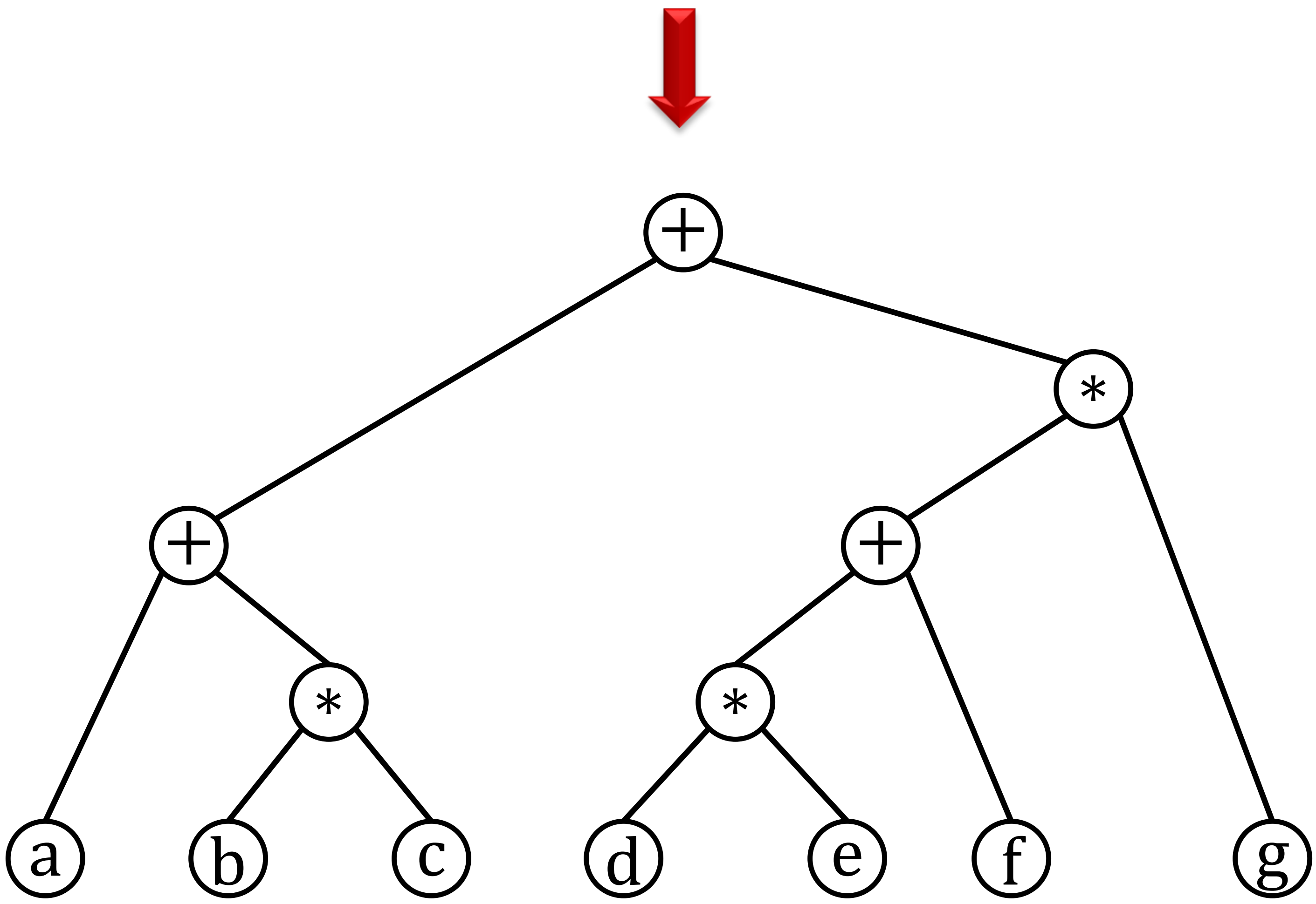
```
struct ExprTree {
  char op; // operand or operator
  ExprTree* left;
  ExprTree* right;
};

using Expr = ExprTree*;
```

Expressions are represented by strings in postfix notation in which the characters ‘a’…‘z’ represent operands and the characters ‘+’ and ‘∗’ represent operators.

```
/** Builds an expression tree from a correct
  * expression represented in postfix notation. */
Expr buildExpr(const string& expr);

/** Generates a string with the expression in
  * infix notation. */
string infixExpr(const Expr T);

/** Evaluates an expression taking V as the value of the
  * variables (e.g., V[‘a’] contains the value of a). */
int evalExpr(const Expr T, const map<char,int>& V);
```

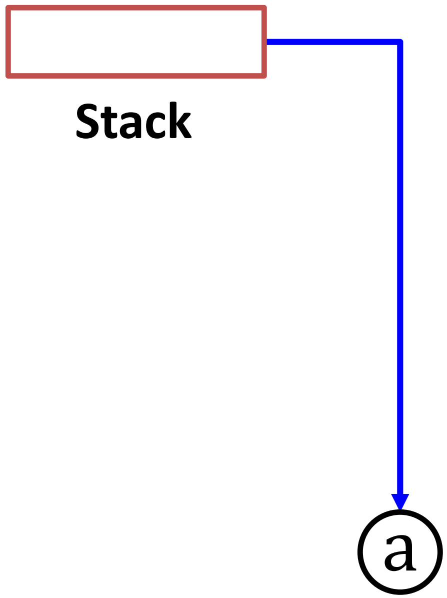
How to build an expression tree

a b c * + d e * f + g * +



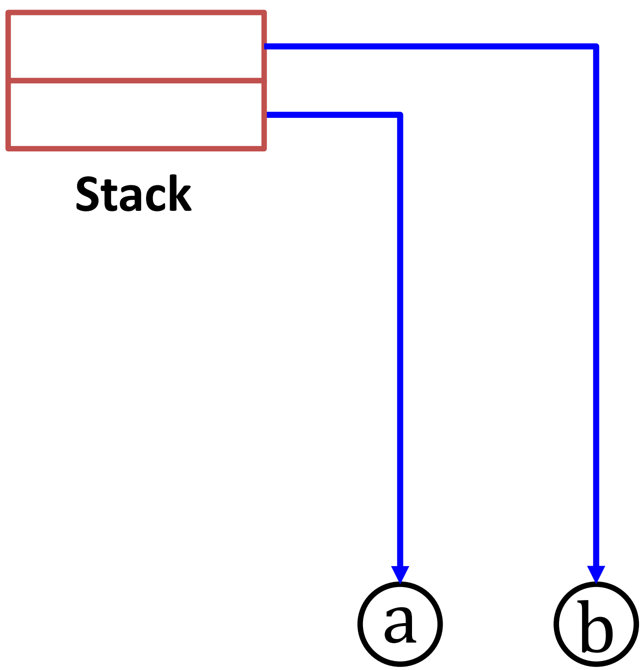
How to build an expression tree

a b c * + d e * f + g * +



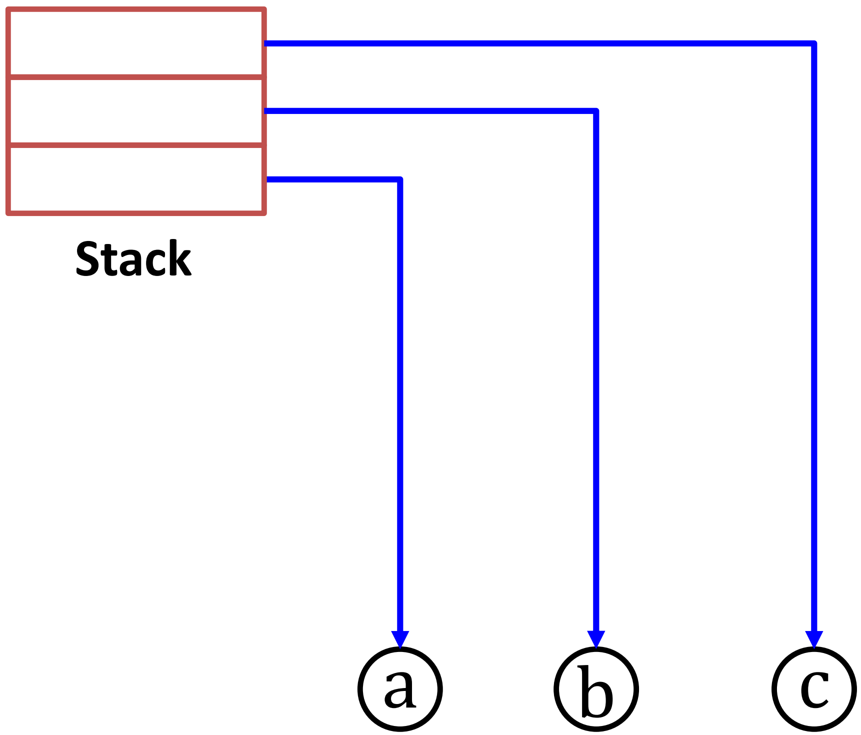
How to build an expression tree

a b c * + d e * f + g * +



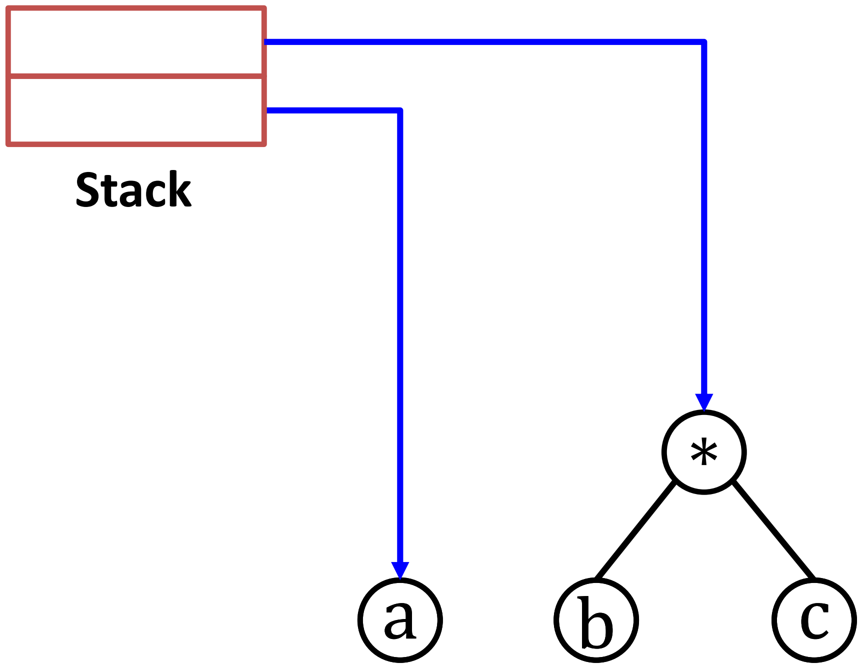
How to build an expression tree

a b c * + d e * f + g * +



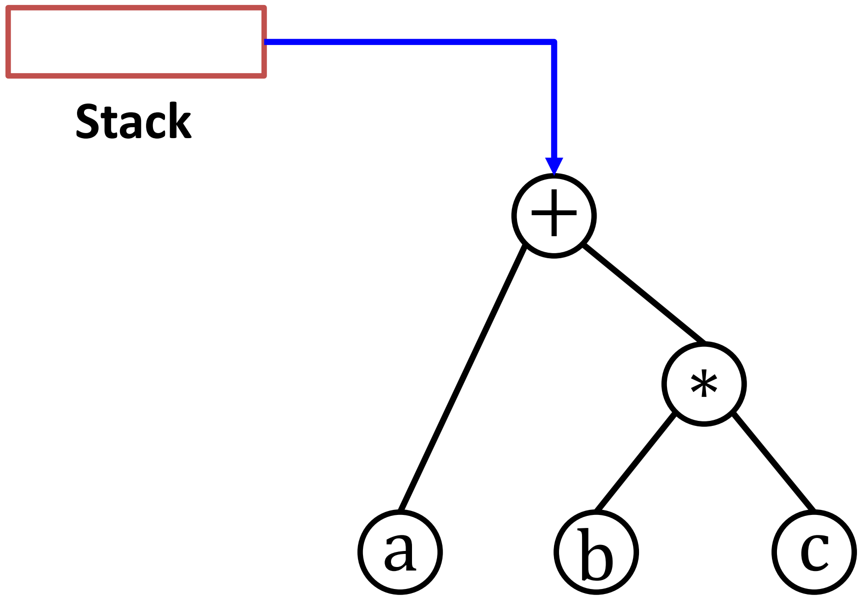
How to build an expression tree

a b c * **+ d e * f + g * +**



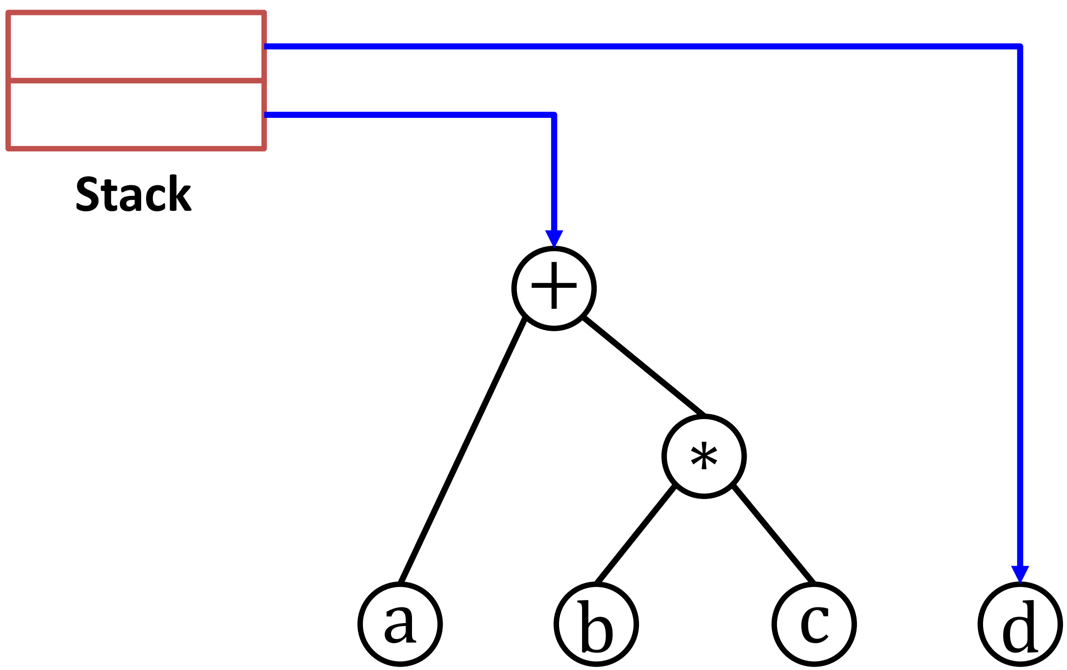
How to build an expression tree

a b c * + **d e * f + g * +**



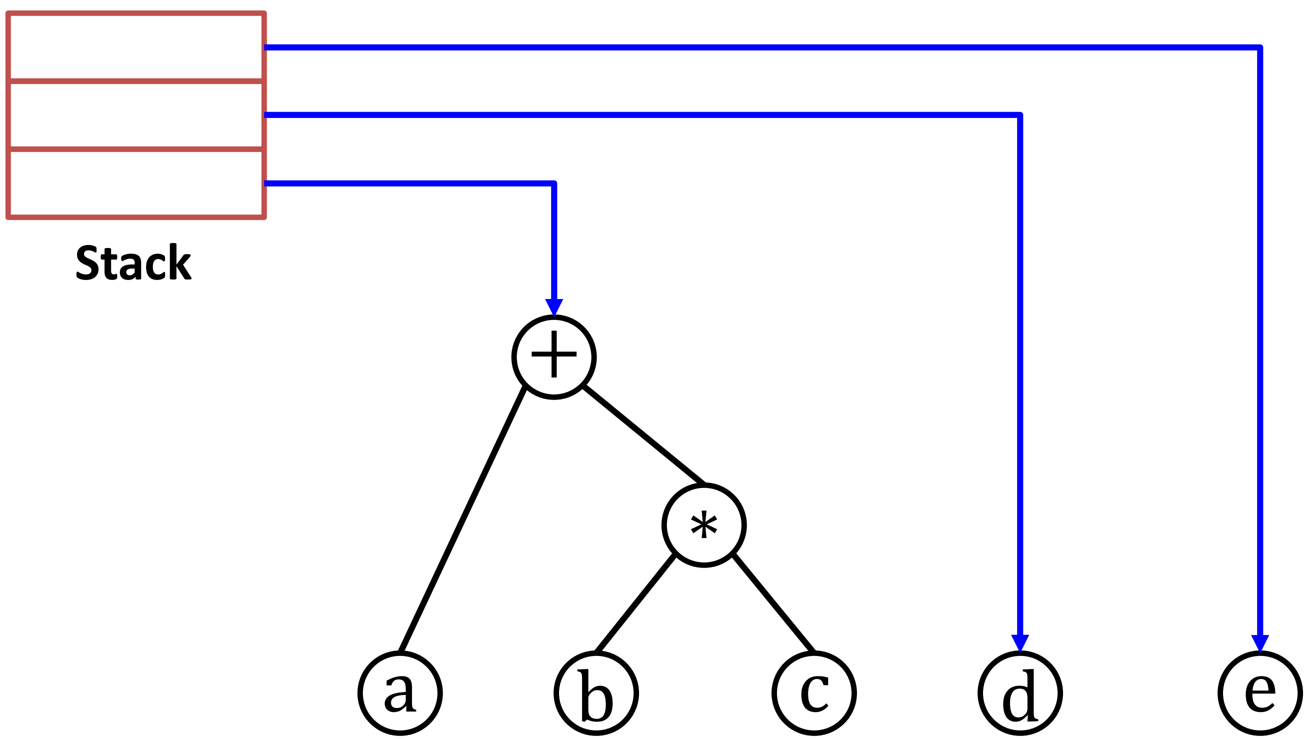
How to build an expression tree

a b c * + d **e * f + g * +**



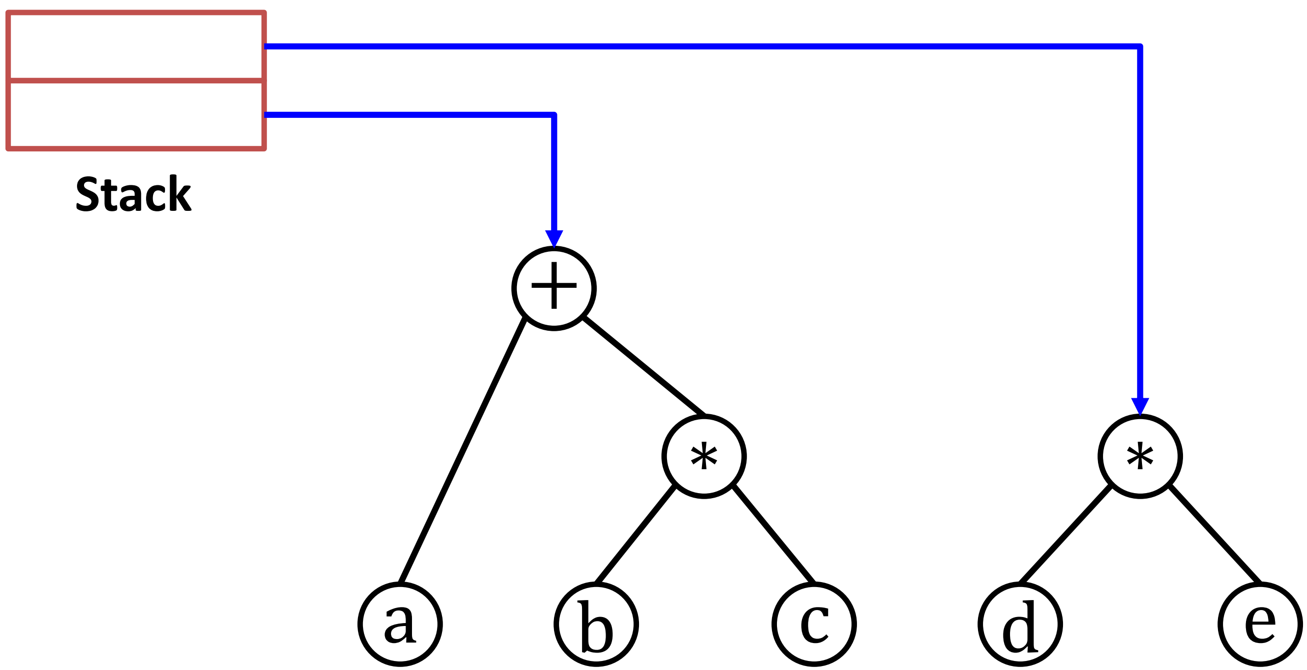
How to build an expression tree

a b c * + d e *** f + g * +**



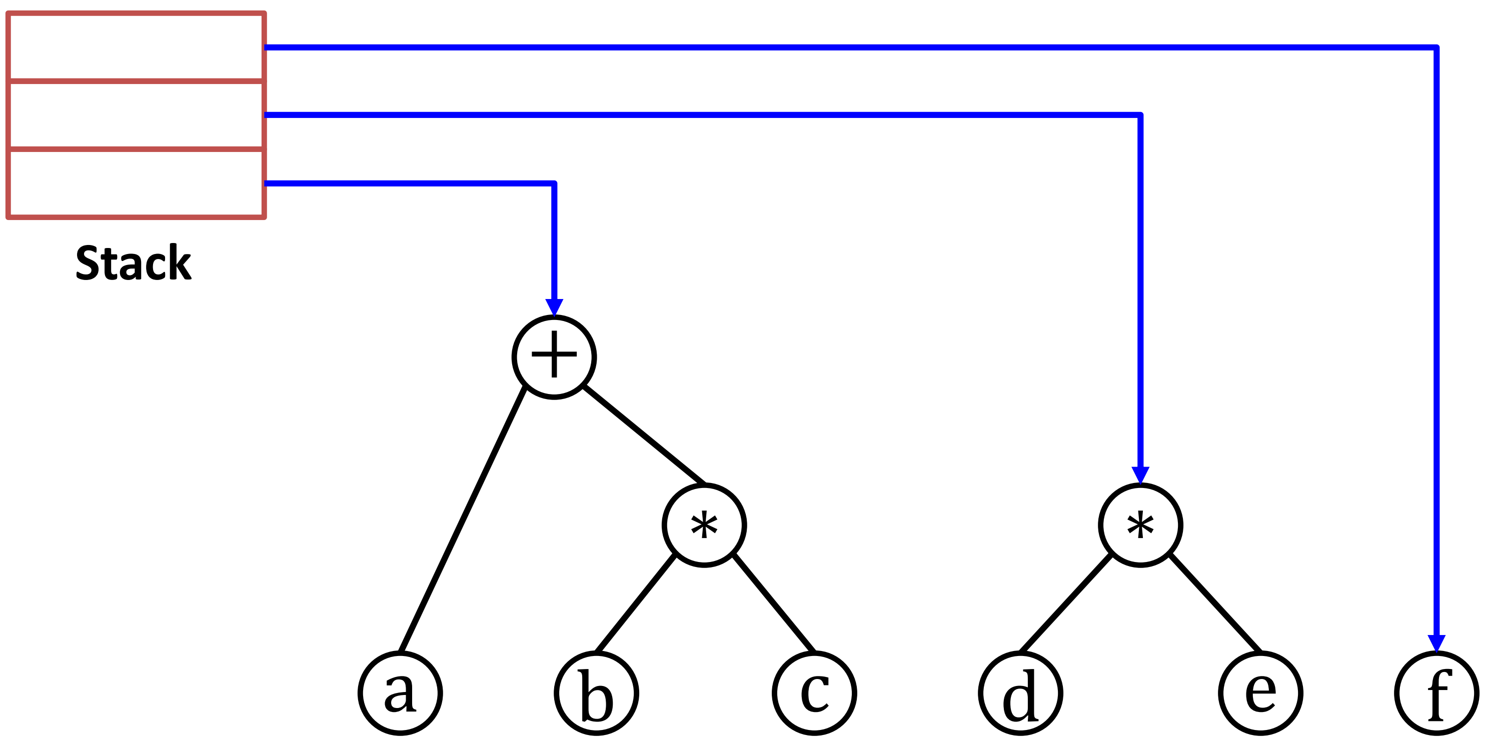
How to build an expression tree

a b c * + d e * **f + g * +**



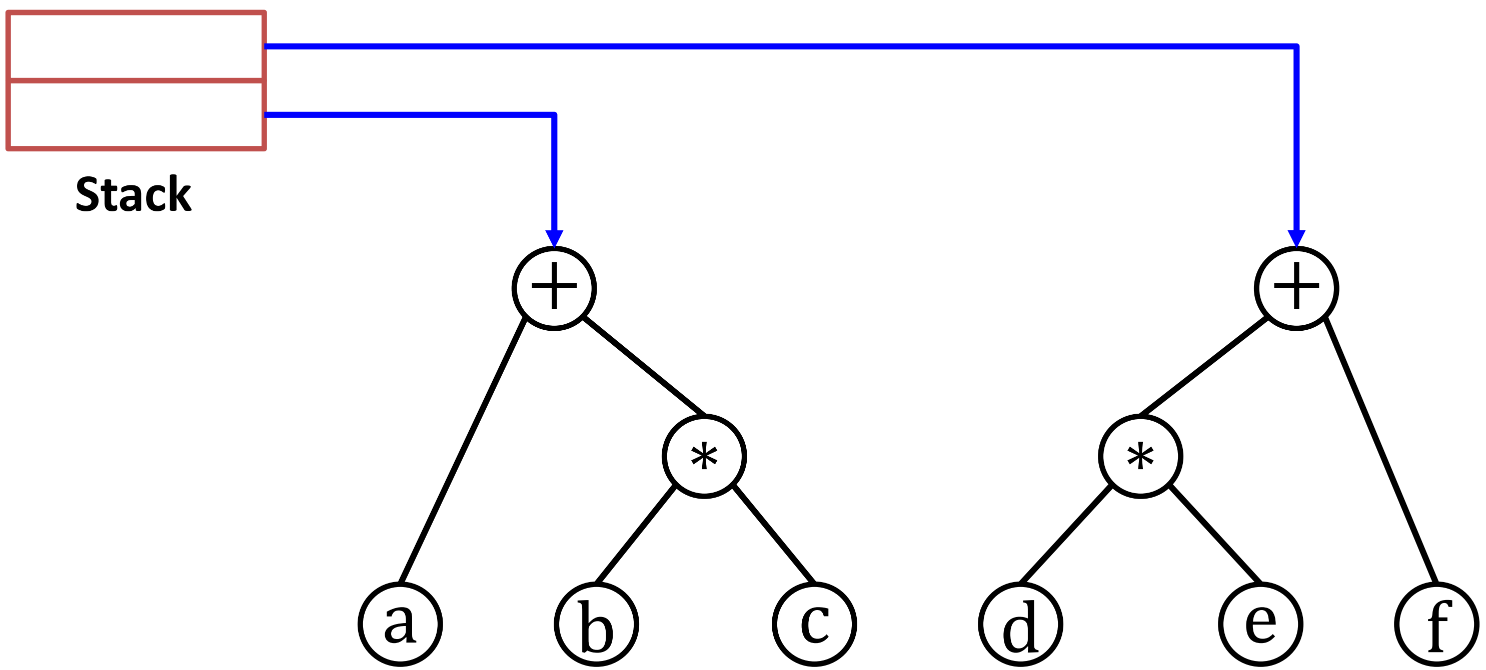
How to build an expression tree

a b c * + d e * f **+ g * +**



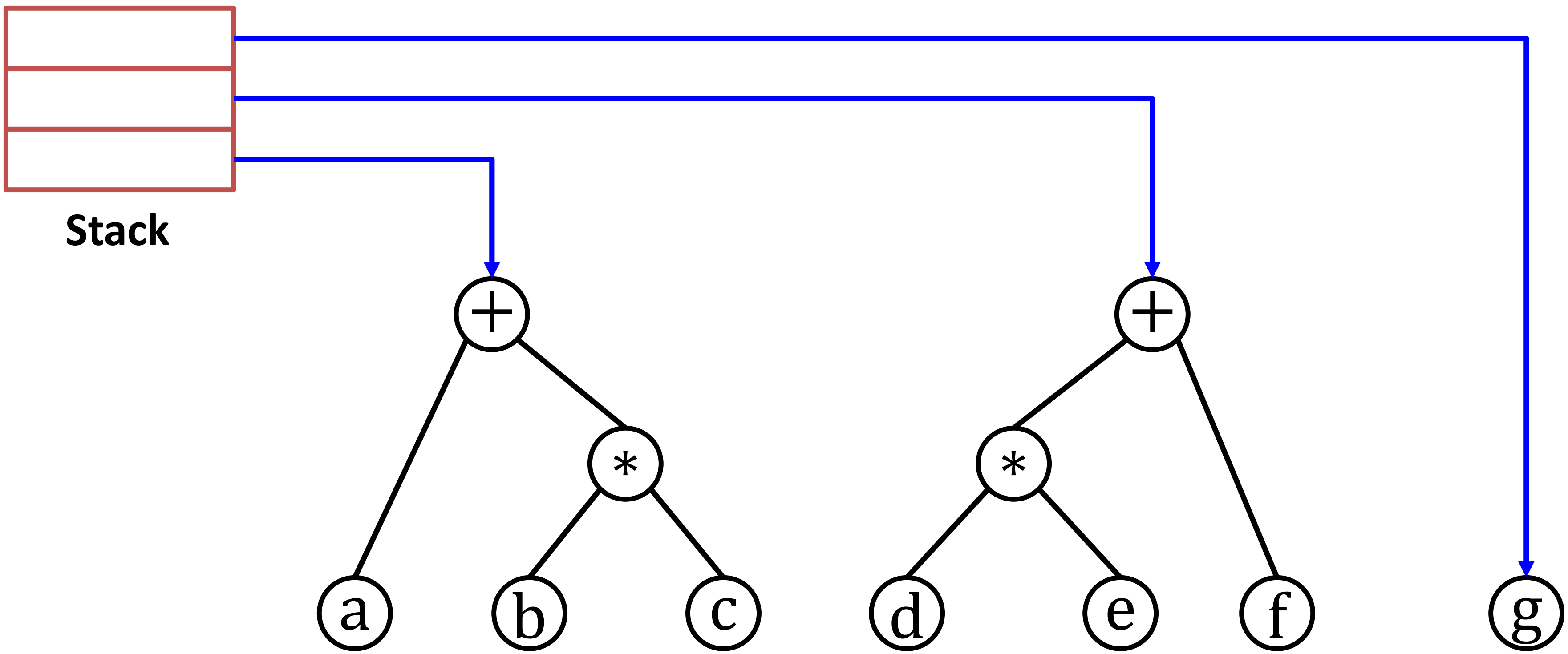
How to build an expression tree

a b c * + d e * f + **g * +**



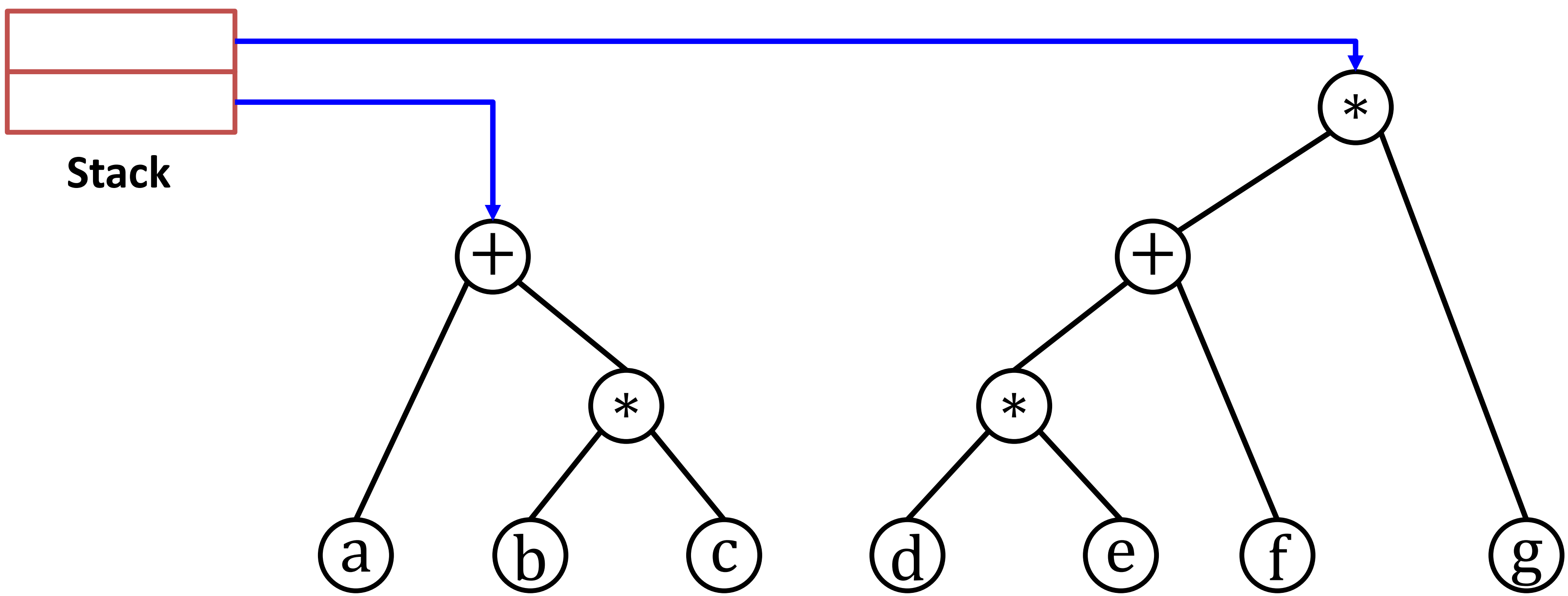
How to build an expression tree

a b c * + d e * f + g *** +**



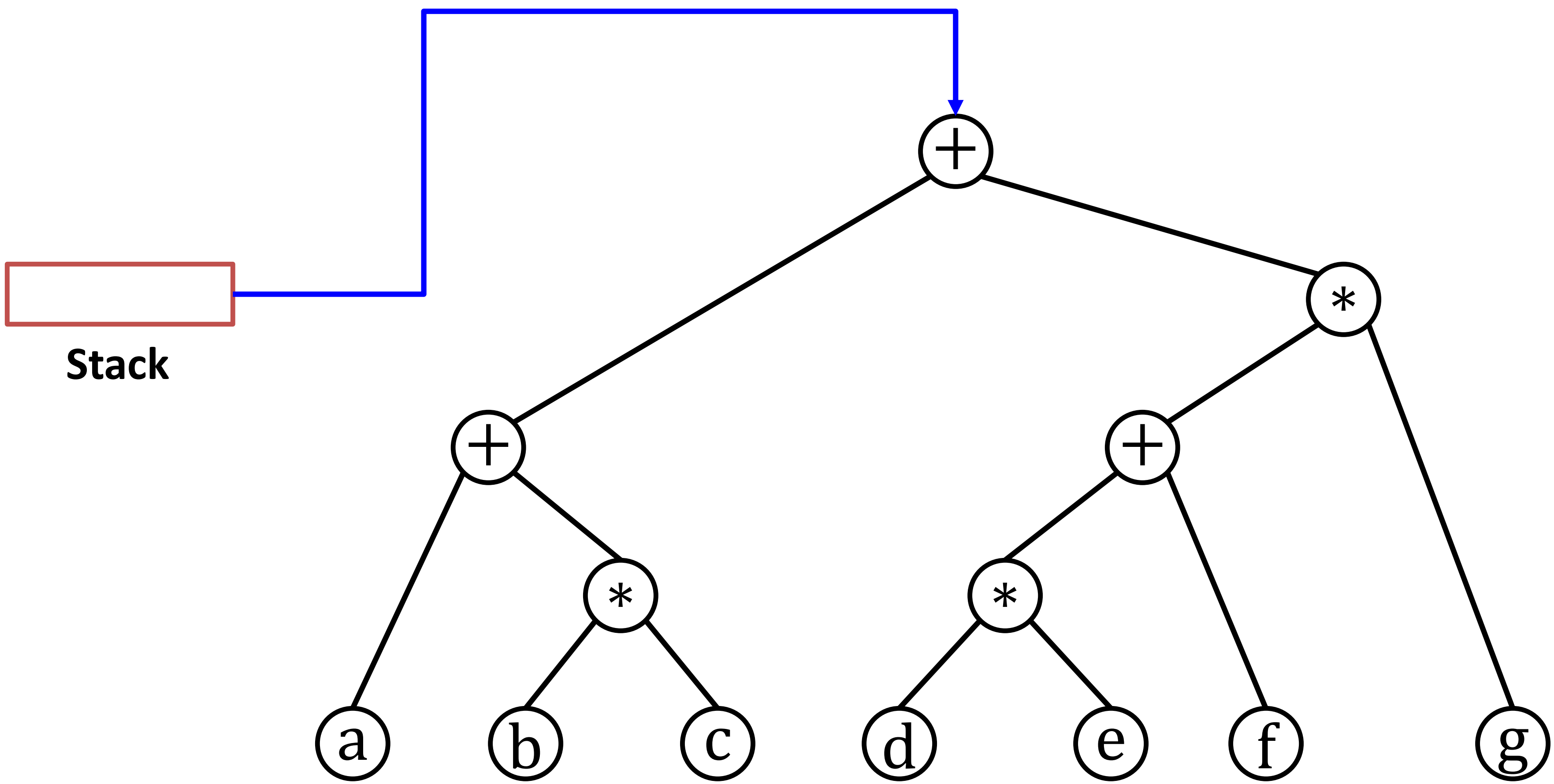
How to build an expression tree

a b c * + d e * f + g * **+**



How to build an expression tree

a b c * + d e * f + g * +



Example: expression trees

```
Expr buildExpr(const string& expr) {
  stack<Expr> S;
  // Visit the chars of the string sequentially
  for (char c: expr) {
    if (c >= ‘a’ and c <= ‘z’) {
      // We have an operand in {‘a’…‘z’}. Create a leaf node.
      S.push(new ExprTree{c, nullptr, nullptr});
    } else {
      // c is an operator (‘+’ or ‘*’)
      Expr right = S.top();
      S.pop();
      Expr left = S.top();
      S.pop();
      S.push(new ExprTree{c, left, right});
    }
  }
  // The stack has only one element and is freed after return
  return S.top();
}
```

Remember: `using Expr = ExprTree*;`

Example: expression trees

```
/** Returns a string with an infix representation of T. */
string infixExpr(const Expr T) {

  // Let us first check the base case (an operand)
  if (T->left == nullptr) return string(1, T->op);

  // We have an operator. Return ( T->left T->op T->right )
  return "(" +
            infixExpr(T->left) +
            T->op +
            infixExpr(T->right) +
         ")";
}
```

Inorder traversal: node is visited *between* the left and right children.

Example: expression trees

```
/** Evaluates an expression taking V as the value of the
  * variables (e.g., V[‘a’] contains the value of a). */
int evalExpr(const Expr T, const map<char,int>& V) {
  if (T->left == nullptr) return V.at(T->op);
  int l = evalExpr(T->left, V);
  int r = evalExpr(T->right, V);
  return T->op == ‘+’ ? l+r : l*r;
}

/** Example of usage of ExprTree. */
int main() {
  Expr T = buildExpr(“abc*+de*f+g*+”);
  cout << infixExpr(T) << endl;
  cout << “Eval = ”
       << evalExpr(T, {{‘a’,3}, {‘b’,1}, {‘c’,0}, {‘d’,5},
                       {‘e’,2}, {‘f’,1}, {‘g’,6}})
       << endl;
  freeExpr(T); // Not implemented yet
}
```

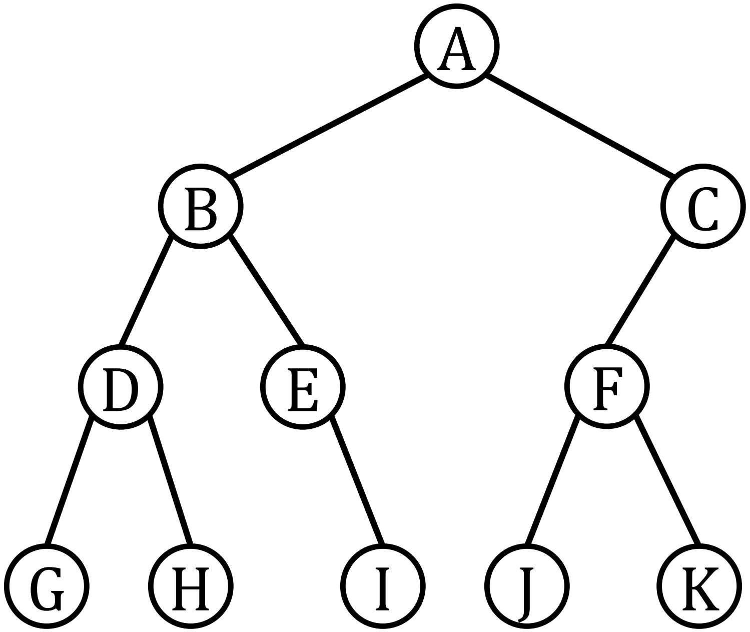
Exercises

- Design the function `freeExpr`.

- Modify `infixExpr` for a nicer printing:
  - Minimize number of parenthesis.
  - Add spaces around + (but not around *).

- Extend the functions to support other operands, including the unary – (e.g., –a/b).

Tree traversals



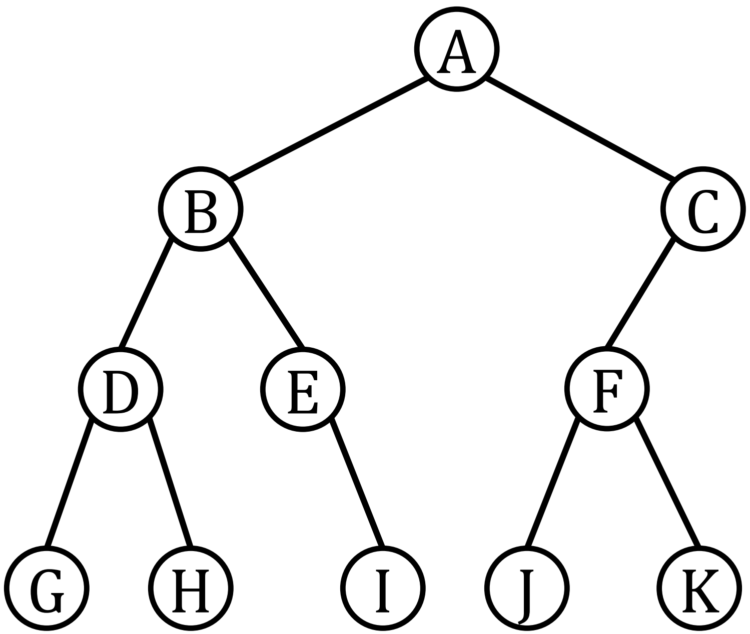
Traversal: algorithm to visit the nodes of a tree in some specific order.

The actions performed when visiting each node can be a parameter of the traversal algorithm.

```
struct TreeNode {
  Tinfo info;
  TreeNode* left;
  TreeNode* right;
};

using Tree = TreeNode*;
```

```
using visitor = void (int &);

// This function matches the type visitor
void print(int& i) {
  cout << i << endl;
}

void traversal(Tree T, visitor v);
```

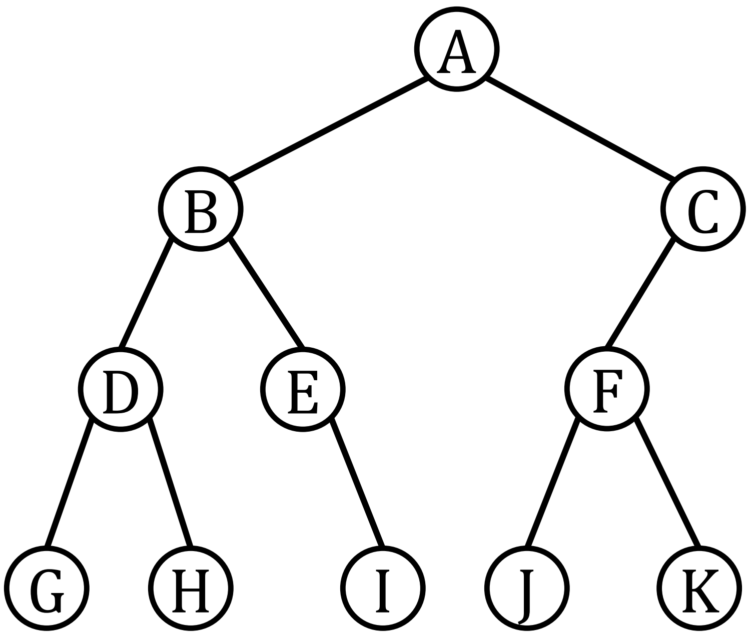
Tree traversals



Preorder: A B D G H E I C F J K

```
void preorder(Tree T, visitor v) {
  if (T != nullptr) {
    v(T->elem);
    preorder(T->left, v);
    preorder(T->right, v);
  }
}
```

Tree traversals

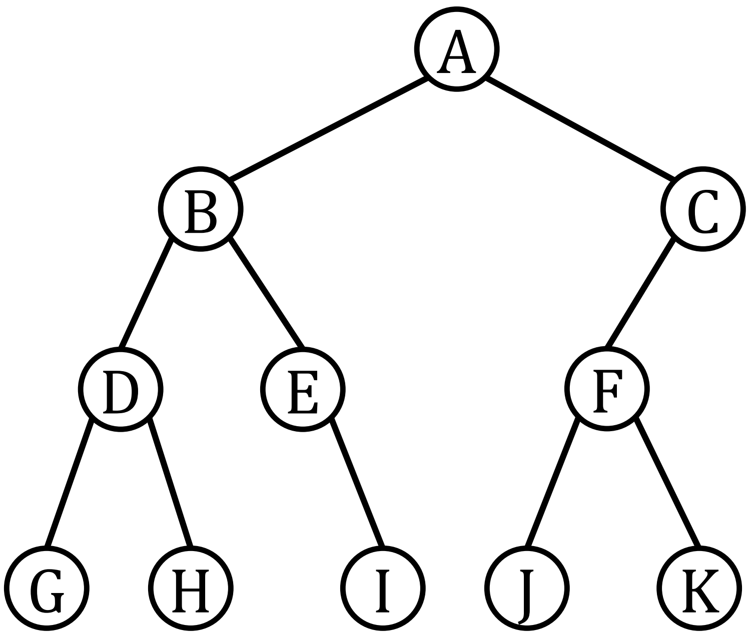


Preorder: A B D G H E I C F J K

Postorder: G H D I E B J K F C A

```
void postorder(Tree T, visitor v) {
  if (T != nullptr) {
    postorder(T->left, v);
    postorder(T->right, v);
    v(T->elem);
  }
}
```

Tree traversals



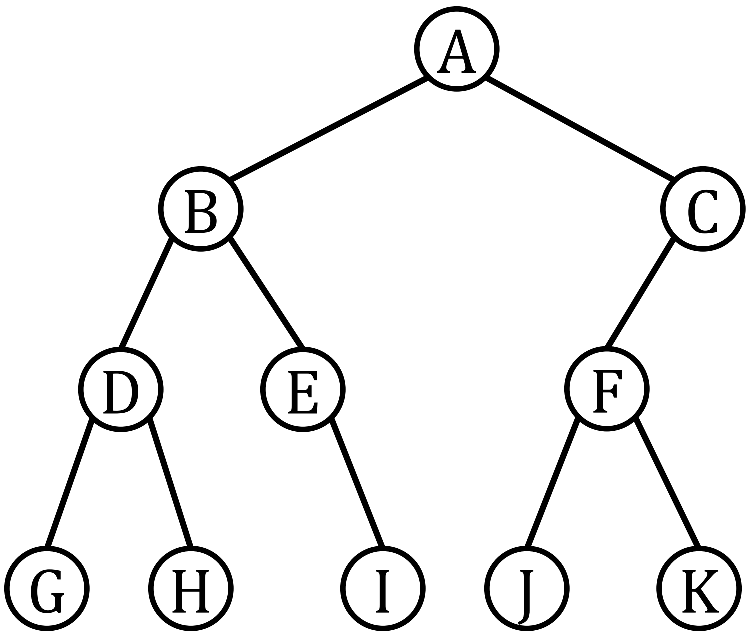
Preorder: A B D G H E I C F J K

Postorder: G H D I E B J K F C A

Inorder: G D H B E I A J F K C

```
void inorder(Tree T, visitor v) {
  if (T != nullptr) {
    inorder(T->left, v);
    v(T->elem);
    inorder(T->right, v);
  }
}
```

Tree traversals



Preorder: A B D G H E I C F J K

Postorder: G H D I E B J K F C A

Inorder: G D H B E I A J F K C

By levels: A B C D E F G H I J K

```
void byLevels(Tree T, visitor v) {
  queue<Tree> Q; Q.push(T);
  while (not Q.empty()) {
    T = Q.front(); Q.pop();
    if (T != nullptr) {
      v(T->elem);
      Q.push(T->left); Q.push(T->right);
    }
  }
}
```

EXERCISES

Traversals: Full Binary Trees

- A Full Binary Tree is a binary tree where each node has 0 or 2 children.
- Draw the full binary trees corresponding to the following tree traversals:
  - Preorder: 2 7 3 6 1 4 5; Postorder: 3 6 7 4 5 1 2
  - Preorder: 3 1 7 4 9 5 2 6 8; Postorder: 1 9 5 4 6 8 2 7 3
- Given the pre- and post-order traversals of a binary tree (not necessarily full), can we uniquely determine the tree?
  - If yes, prove it.
  - If not, show a counterexample.

Traversals: Binary Trees

- Draw the binary trees corresponding the following traversals:
  - Preorder: 3 6 1 8 5 2 4 7 9; Inorder: 1 6 3 5 2 8 7 4 9
  - Level-order: 4 8 3 1 2 7 5 6 9; Inorder: 1 8 5 2 4 6 7 9 3
  - Postorder: 4 3 2 5 9 6 8 7 1; Inorder: 4 3 9 2 5 1 7 8 6
- Describe an algorithm that builds a binary tree from the preorder and inorder traversals.

Drawing binary trees

We want to draw the skeleton of a binary tree as it is shown in the figure. For that, we need to assign (x, y) coordinates to each tree node. The layout must fit in a pre-defined bounding box of size $W \times H$, with the origin located in the top-left corner.

Design the function

```
void draw(Tree T, double W, double H)
```

to assign values to the attributes `x` and `y` of all nodes of the tree in such a way that the lines that connect the nodes do not cross.

Suggestion: calculate the coordinates in two steps. First assign (x, y) coordinates using some arbitrary unit. Next, shift/scale the coordinates to exactly fit in the bounding box.