Response of discrete-time LTI systems

U3

- Response to a single frequency of a LTI system:

$$x[n] = e^{j2\pi Fn} \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

eigenfunction

eigenvalue

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j2\pi F(n-k)} = e^{j2\pi Fn} \boxed{\sum_{k=-\infty}^{\infty} h[k] e^{-j2\pi Fk}}$$

Frequency response

$$H(F) = TF\{h[n]\}$$

- $x[n] = A\cos(2\pi F_0 n + \varphi) \overset{LTI}{\Longrightarrow} y[n] = A|H(F_0)|\cos\big(2\pi F_0 n + \varphi + \sphericalangle H(F_0)\big)$

Amplitude gain   Phase shift

$$and$$

$$real\ system\ (h[n] \in \mathbb{R} \Longrightarrow H(-F) = H^*(F))$$

Response of discrete-time LTI systems

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- In general, we may represent a signal as a weighted sum of complex exponentials. Then, if the system is LTI:

$$x[n] = \int_{\langle 1 \rangle} X(F)\, e^{j2\pi Fn}\, dF \quad \overset{LTI}{\Longrightarrow} y[n] = \int_{\langle 1 \rangle} X(F)\, H(F) e^{j2\pi Fn}\, dF$$

$$x[n],\ X(F) \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n],\quad Y(F) = X(F)H(F)$$

The convolution in the temporal domain is a product in the frequency domain

N samples average

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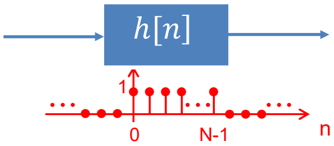
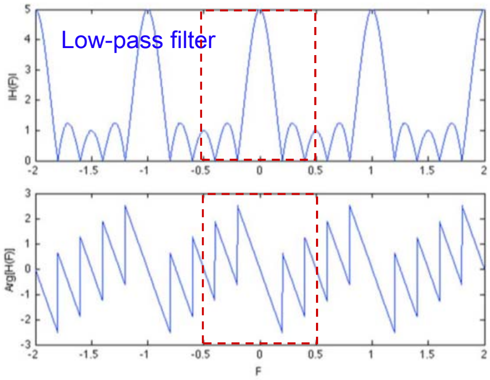


$$h[n] = p_N[n] = \begin{cases} 1, & 0 \leq n < N \\ 0, & n < 0,\ n \geq \mathrm{N} \end{cases} \quad \leftrightarrow \quad H(F) = \frac{\sin(\pi F N)}{\sin(\pi F)} e^{-j\pi F(N-1)}$$

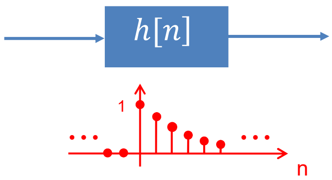


- $H(0) = N$
- Zeros of the numerator: $\pi F N = \pm k\pi \Leftrightarrow F = \pm \frac{k}{N}$
- Zeros of the denominator: $\pi F = \pm k\pi \Leftrightarrow F = \pm k$

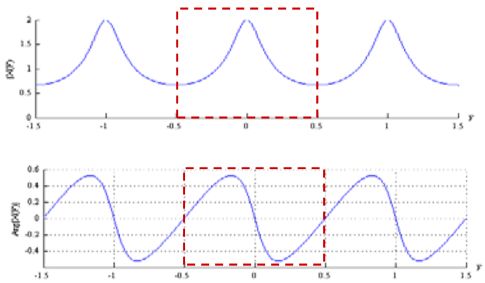
Plots are for N=5

Exponential average

U3



$$h[n] = a^n u[n], \qquad |a| < 1 \qquad \leftrightarrow \quad H(F) = \frac{1}{1 - ae^{-j2\pi Fn}}$$



- $|H(F)|^2 = \frac{1}{1 - 2a\cos(2\pi F) + a^2}$
- $\angle H(F) = -artg\,\frac{a\sin(2\pi F)}{1 - a\cos(2\pi F)}$

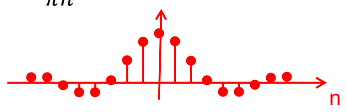
Plots are done for 0<a<1

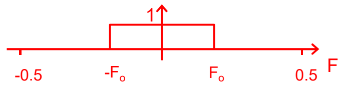
Ideal filters

U3

- Low-pass ideal filter

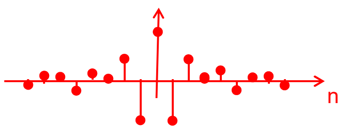
$$\frac{\sin(2\pi F_0 n)}{\pi n} = 2F_0 \mathrm{sinc}(2\pi F_0 n) \quad \leftrightarrow \quad H(F) = \sum_{k=-\infty}^{\infty} \Pi\left(\frac{F-k}{2F_0}\right)$$

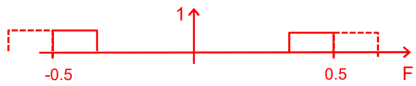




- High-pass ideal filter

$$(-1)^n 2F_0 \mathrm{sinc}(2\pi F_0 n) \quad \leftrightarrow \quad H(F) = \sum_{k=-\infty}^{\infty} \Pi\left(\frac{F-0.5-k}{2F_0}\right)$$





In both cases, $h[n]$ non-causal and non-stable

Frequency response of EDF filters

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$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \leftrightarrow \sum_{k=0}^{N} a_k Y(F) e^{-j2\pi Fk} = \sum_{k=0}^{M} b_k X(F) e^{-j2\pi Fk}$$

$$H(F) = \frac{Y(F)}{X(F)} = \frac{\sum_{k=0}^{M} b_k e^{-j2\pi Fk}}{\sum_{k=0}^{N} a_k e^{-j2\pi Fk}}$$ Frequency response

Example

$$y[n] = x[n] + ay[n-1] \Rightarrow H(F) = \frac{1}{1 - ae^{-j2\pi F}}$$

$$h[n] = a^n u[n]$$

Take away messages

U3

- LTI systems
  - cannot create new frequencies
  - can only scale magnitude and shift phase of existing components
- Therefore, LTI systems can be interpreted as filters: they can be designed to selectivily pass certain frequency bands

Thinking about signals by their frequency content and systems as filters has a large number of practical applications.

Filtering example

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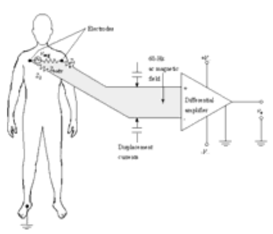
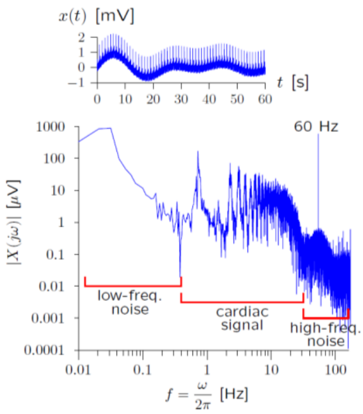
An electrocardiogram is a record of electrical potentials that are generated by the heart and measured on the surface of the chest





Filtering

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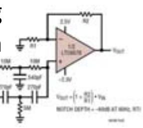
We wish to design a system to “clean” the signal:

x(t)

Electrical signal

Analog system



y(t)

Electrical signal

x(t) → A/D → x[n] → Digital system → y[n] → D/A → y(t)

- An ECG signal occupies the frequency range from 0.01 to 250 Hz (diagnostic-quality ECG)
- A/D conversion:
  - anti-aliasing filter (analog low pass filter with cutoff frequency 250 Hz)
  - sampling with sampling frequency 500 Hz

Digital filtering

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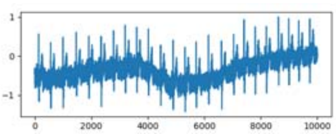
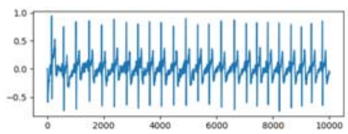
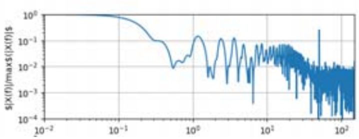
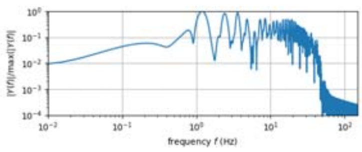
Sampling frequency fs=500Hz



→ LTI system →

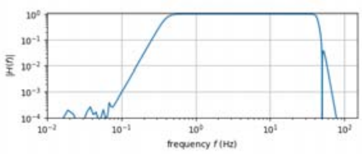






f=F·fs (Hz)

$2\cdot10^{-3}$ $2\cdot10^{-2}$ 0.2 F (cycles/sample)



A combination of notch filter, low pass filter, high pass filter