Polinomis de Legendre

Els definim a partir de

$$\begin{cases} P_0(x) = 1 \\ P_n(x) = \dfrac{1}{2^n n!} \cdot \dfrac{d^n}{dx^n}\left[(x^2-1)^n\right], & n \geq 1 \end{cases}$$

Observem que

1) $P_n(x)$ té grau n $\left(P_1(x) = \frac{1}{2} \frac{d}{dx}(x^2-1) = x \right.$;

$$P_2(x) = \frac{1}{4.2} \frac{d^2}{dx^2}\left[\underbrace{(x^2-1)^2}_{x^4-2x^2+1}\right] = \frac{1}{8}(12x^2-4) = \left.\frac{3}{2}x^2 - \frac{1}{2}\right)$$

2) El coeficient de grau màxim de $P_n(x)$ és

$$\frac{1}{2^n n!}(2n)(2n-1)\cdots\underbrace{(2n-(n-1))}_{n+1} = \frac{(2n)!}{2^n (n!)^2}$$

3) $P_n(-x) = (-1)^n P_n(x)$ $\left(\text{En } \frac{d^n}{d(-x)^n} : \text{cada } \frac{d}{d(-x)} \text{ s'treu } (-1) \text{ multiplicant} \right)$

<u>Proposició</u>: La família $\{P_n(x)\}_{n\geq 0}$ és ortogonal en $[-1,1]$ respecte de la funció pes $w(x) = 1$, i.e.

$$(P_n, P_j) = \int_{-1}^{1} P_n(x) P_j(x) = \begin{cases} 0 & \text{si } n \neq j \\ \dfrac{2}{2n+1} & \text{si } n = j \end{cases} \qquad \left(\text{i.e. } (P_j, P_j) = \frac{2}{2j+1} \right)$$

<u>Prova</u>. No és restrictiu suposar que $j \leq n$. Definim $f(x) = (x^2-1)^j$, $g(x) = (x^2-1)^n$

$$2^{n+j} n! j! (P_n, P_j) = \int_{-1}^{1} (g(x))^{(n)} (f(x))^{(j)} dx =$$

per parts

$u = (f(x))^{(j)}$

$dv = (g(x))^{(n)} dx$

$$= \cancelto{0}{\left(f(x)\right)^{(j)} \left(g(x)\right)^{(n-1)} \Big|_{-1}^{1}} - \int_{-1}^{1} \left(f(x)\right)^{(j+1)} \left(g(x)\right)^{(n-1)} dx = \dots =$$

successivament per parts

$$= (-1)^j \int_{-1}^{1} \underbrace{\left(f(x)\right)^{(2j)}}_{(2j)!} \left(g(x)\right)^{(n-j)} =$$

$$= (-1)^j (2j)! \int_{-1}^{1} \left[(x^2-1)^n\right]^{(n-j)} dx = (*)$$

– si $j<n$ $\quad (*) = \text{ctant} \displaystyle\int_{-1}^{1} \left\{\left[(x^2-1)^n\right]^{(n-j-1)}\right\}' dx =$

$$= \text{ctant} \left[(x^2-1)^n\right]^{(n-j-1)} \Big|_{-1}^{1} = 0 \Longrightarrow \boxed{(P_n, P_j) = 0}$$

$$\cancelto{1}{(-1)^{2n}}(2n)!\, 2 \int_0^1 (1-x^2)^n dx$$

– si $j=n$ $\quad (*) = (-1)^n (2n)! \displaystyle\int_{-1}^{1} (x^2-1)^n dx = \left| \begin{matrix} x = \cos t \\ dx = -\sin t \, dt \end{matrix} \right| =$

$$= 2(2n)! \int_{\frac{\pi}{2}}^{0} (\sin^2 t)^n (-\sin t) dt = 2(2n)! \int_0^{\frac{\pi}{2}} \sin^{2n+1} t \, dt =$$

$$\underset{\substack{(**)\\ \text{ex.}}}{=} 2 \frac{(2n)!!}{(2n+1)!!} (2n)! \Longrightarrow (P_n, P_j) = \frac{2(2n)!!\,(2n)!}{(2n+1)!!\, 2^{2n} (n!)^2} \underset{\text{ex}}{=} \frac{2}{2n+1}$$

(x-1) Vegiem

A-20

Usem la fórmula

$$\int \sin^m x\,dx = \frac{-\sin^{m-1} x \cos x}{m} + \frac{m-1}{m}\int \sin^{m-2} x\,dx$$

Ara:

$$(2n)!\int_0^{\pi/2} \sin^{2n+1} t\,dt = (2n)!\left[\cancel{-\frac{\sin^{2n} t \cos t}{2n+1}}\Big|_0^{\pi/2}\right] + \frac{(2n)!\,2n}{2n+1}\int_0^{\pi/2} \sin^{2n-1} t\,dt =$$

$$0$$

$$= \frac{(2n)!\,2n}{2n+1}\,\frac{2n-2}{2n-1}\int_0^{\pi/2} \sin^{2n-3} t\,dt =$$

$$= \frac{(2n)!\;2n\;(2n-2)}{(2n+1)\,(2n-1)}\,\frac{(2n-4)}{(2n-3)}\int_0^{\pi/2} \sin^{2n-5} t\,dt =$$

$$= \cdots = \frac{(2n)!\,(2n)(2n-2)(2n-4)\cdots 2}{(2n+1)(2n-1)\cdots 3}\underbrace{\int_0^{\pi/2} \sin t\,dt}_{1}$$

$$= \frac{(2n)!\;(2n)!!}{(2n+1)!!}$$

Ara calculem la recurrència: (on $\psi_j(x) = P_j(x)$)

$$\psi_{j+1}(x) = \alpha_j (x-\beta_j)\psi_j(x) - \gamma_j \psi_{j-1}(x), \quad j \geq 0$$

amb $\alpha_j = \dfrac{A_{j+1}}{A_j}$; $\beta_j = \dfrac{(\psi_j, x\psi_j)}{(\psi_j, \psi_j)}$; $\gamma_j = \dfrac{\alpha_j (\psi_j, \psi_j)}{\alpha_{j-1}(\psi_{j-1}, \psi_{j-1})}, \; j \geq 1$

$$\boxed{\alpha_j =} \frac{\dfrac{\overbrace{(2j+2)(2j+1)(2j)\cdots(2j+2-j)}^{j+1}}{2^{j+1}(j+1)!}}{\dfrac{2j(2j-1)\cdots(j+1)}{2^j\, j!}} = \frac{\cancel{(2j+2)}(2j+1)}{\cancel{2(j+1)}(j+1)} = \boxed{\frac{2j+1}{j+1}}$$

$$\boxed{\beta_j = 0}$$

$$(\psi_j, x\psi_j) = \int_{-1}^{1} x \underbrace{\frac{1}{(2^n n!)^2}\left\{\frac{d^n}{dx^n}\left[(x^2-1)^n\right]\right\}^2}_{(\psi_j(x))^2} dx = 0$$

$\psi_j(x) = (-1)^n \psi_j(-x)$

$(\psi_j(x))^2 = (\psi_j(-x))^2$

$$\boxed{\gamma_j} = \frac{\dfrac{2j+1}{j+1}}{\dfrac{\cancel{2j-1}}{j}} \cdot \frac{\dfrac{\cancel{2}}{\cancel{2j+1}}}{\dfrac{\cancel{2}}{\cancel{2j-1}}} = \frac{j}{j+1}$$

Tenim finalment

$$\boxed{\begin{array}{l} P_0(x) = 1, \quad P_1(x) = x \\ P_{j+1}(x) = \dfrac{2j+1}{j+1}\, x\, P_j(x) - \dfrac{j}{j+1} P_{j-1}(x), \quad j \geq 1 \end{array}}$$

Exemple Prenem $f(x)=e^x$ a $[-1,1]$ i l'aproximem per polinomis de Legendre (sabem que els polinomis són ortogonals respecte de la funció pes $w(x)=1$, prenent el producte escalar

$$(f,g)=\int_{-1}^{1} f(x)\,g(x)\,dx$$

i sabem que

$$(P_k,P_j)=\begin{cases} 0 & \text{si } k\neq j \\ \dfrac{2}{2j+1} & \text{si } k=j \end{cases}$$

Llavors l'aproximació mínim-quadràtica serà:

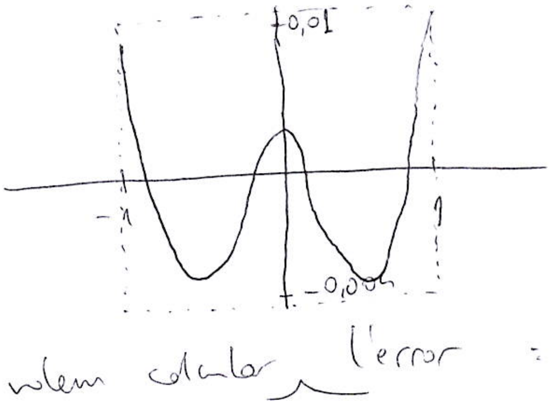
$$P_n^*(x)=\sum_{j=0}^{n} c_j^*\,P_j(x) \quad \text{on} \quad c_j^*=\frac{2j+1}{2}\int_{-1}^{1} e^x P_j(x)\,dx$$

– Si prenem per exemple $n=3$ obtenim

$c_0^*=1{,}17520119$ $\quad c_1^*=1{,}10363832$

$c_2^*=0{,}35781435$ $\quad c_3^*=0{,}07045563$

La funció error $e^x-P_3^*(x)$ té la gràfica (aproximadament)



– Si volem calcular $\underbrace{\text{l'error}}_{\text{la norma de}}$: $\|e^x-P_n^*(x)\|_2^2=\|e^x\|_2^2-\|P_n^*(x)\|_2^2$

vist a pàg. A-5

Exercici: 1) Fer els càlculs 2) Generar els plots mostrats a classe

NOTA Si $I=[a,b]$, $w(x)=1$ els polinomis de Legendre $\varphi_j(x)$ s'obtenen dels polinomis de Legendre a $[-1,1]$ a través d'un canvi de variable

$$\frac{x-a}{b-a} = \frac{t-(-1)}{2} \iff x = a + \frac{b-a}{2}(t+1) \iff t = \frac{2}{b-a}\left(x-\frac{a+b}{2}\right)$$

$x\in[a,b]$, $t\in[-1,1]$

és a dir $\varphi_j(x) = P_j\left(\frac{2}{b-a}\left(x-\frac{a+b}{2}\right)\right)$, $j\geq 0$

Se satisfà la recurrència següent en $t\in[-1,1]$

$$P_{j+1}(t) = \frac{2j+1}{j+1}\, t\, P_j(t) - \frac{j}{j+1} P_{j-1}(t)$$

$$\underbrace{P_{j+1}\left(\frac{2}{b-a}\left(x-\frac{a+b}{2}\right)\right)}_{\varphi_{j+1}(x)} = \frac{2j+1}{j+1}\,\frac{2}{b-a}\left(x-\frac{a+b}{2}\right) \underbrace{P_j\left(\frac{2}{b-a}\left(x-\frac{a+b}{2}\right)\right)}_{\varphi_j(x)} -$$

$$- \frac{j}{j+1}\underbrace{P_{j-1}\left(\frac{2}{b-a}\left(x-\frac{a+b}{2}\right)\right)}_{\varphi_{j-1}(x)}$$

és a dir

$$\boxed{\varphi_{j+1}(x) = \frac{2j+1}{j+1}\,\frac{2}{b-a}\left(x-\frac{a+b}{2}\right)\varphi_j(x) - \frac{j}{j+1}\varphi_{j-1}(x), \quad j\geq 1}$$

Altres exemples de polinomis ortogonals:

- Polinomis de Txebyxev: amb funció pes $w(x) = \frac{1}{\sqrt{1-x^2}}$ a $[-1,1]$

i.e. el producte escalar és

$$(g,h) = \int_{-1}^{1} \frac{g(x)h(x)}{\sqrt{1-x^2}}\,dx$$

(veure llista de problemes)

- Polinomis d'Hermite: $w(x) = e^{-x^2}$ a $(-\infty,\infty)$

$$(g,h) = \int_{-\infty}^{\infty} g(x)h(x)e^{-x^2}\,dx$$

que satisfan la recurrència

$$H_{j+1}(x) = 2xH_j(x) - 2jH_{j-1}(x), \quad j \geq 0$$

- Polinomis generalitzats de Laguerre: $L_j^{\alpha}(x)$; $w(x) = x^{\alpha}e^{-x}$

a $(0,\infty)$

$$(g,h) = \int_0^{\infty} g(x)h(x)x^{\alpha}e^{-x}\,dx$$

Els coeficients per a la relació de recurrència són:

$$\alpha_j = -\frac{1}{j+1}, \quad \beta_j = 2j+\alpha+1, \quad \gamma_j = \frac{j+\alpha}{j+1}.$$

Aproximació trigonomètrica

Suposem que la funció que volem aproximar expressa un fenomen 2π-periòdic. En aquest cas serà natural prendre com a funcions bàsiques independents:

$$\varphi_0(x) = \frac{1}{2}, \quad \varphi_1(x) = \cos x, \quad \varphi_2(x) = \sin x, \quad \varphi_3(x) = \cos 2x, \quad \varphi_4(x) = \sin 2x, \ldots$$

$$\varphi_{2n-1}(x) = \cos nx, \quad \varphi_{2n}(x) = \sin nx$$

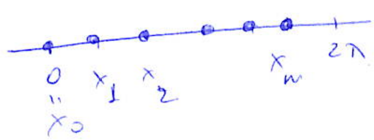
Propietats

Les funcions $\{\varphi_j(x)\}_{j\geq 0}$ són ortogonals tant en el cas continu com el discret, i es té

(i) Al cas continu.

$$(\varphi_j, \varphi_l) = \int_0^{2\pi} \varphi_j(x)\varphi_l(x)\,dx = \begin{cases} 0 & \text{si } j \neq l,\ j,l \geq 0 \\ \pi/2 & \text{si } j = l = 0 \\ \pi & \text{si } j = l > 0 \end{cases}$$

(ii) Al cas discret: prenem $x_k = \dfrac{2\pi k}{m+1}$, $k = 0, \ldots, m$, $2n \leq m$



$$(\varphi_j, \varphi_l)_m = \sum_{k=0}^{m} \varphi_j(x_k)\varphi_l(x_k) =$$

$$= \begin{cases} 0 & \text{si } j \neq l,\ j,l = 0 \div 2n \\ \dfrac{m+1}{4} & \text{si } j = l = 0 \\ \dfrac{m+1}{2} & \text{si } j = l = 1 \div 2n \end{cases}$$

Prova – a) cas continu.

(i) Usem $\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$$

Prenem, per exemple, $j \neq l$, $\varphi_j(x) = \cos jx$, $\varphi_l(x) = \cos lx$

- $$(\varphi_j, \varphi_l) = (\cos jx, \cos lx) = \int_0^{2\pi} \cos jx \cos lx \, dx = \frac{1}{2}\int_0^{2\pi} \left[\cos(j-l)x + \cos(j+l)x\right] dx =$$

$$= \frac{1}{2}\left[\frac{1}{j-l} \sin(j-l)x \Big|_0^{2\pi} + \frac{1}{j+l} \sin(j+l)x \Big|_0^{2\pi}\right] = 0 \checkmark$$

- Si $j = l = 0$ $\quad \int_0^{2\pi} \frac{1}{4\pi} \, dx = \frac{\pi}{2}$ $\checkmark$

- Si $j = l > 0$ $\quad \int_0^{2\pi} \cos^2 jx \, dx = \int_0^{2\pi} \left(\frac{1}{2} + \frac{\cos 2jx}{2}\right) dx = \pi$

(Anàleg amb $\sin jx$)

- Anàleg amb la resta de casos

(ii) Al cas discret:

• Si $j \neq l$, prenem per exemple $\varphi_j(x) = \cos jx$, $\varphi_l(x) = \cos lx$

$$(\varphi_j, \varphi_l)_m = \sum_{k=0}^{m} \cos jx_k \cos lx_k = \frac{1}{2}\sum_{k=0}^{m}\underbrace{\left[\cos(j-l)x_k + \cos(j+l)x_k\right]}_{(E)}$$

Abans de seguir, utilitzem:

$$\boxed{\sum_{k=0}^{m} \cos kA} + i\sum_{k=0}^{m} \sin kA = \sum_{k=0}^{m} e^{ikA} = \frac{1-e^{i(m+1)A}}{1-e^{iA}} =$$

$$= \frac{\overbrace{\cos(m+1)A}^{\tilde{c}} - 1 + i\overbrace{\sin(m+1)A}^{\tilde{s}}}{\cos A - 1 + i\sin A} \cdot \frac{\cos A - 1 - i\sin A}{\cos A - 1 - i\sin A} =$$

$$= \frac{(\tilde{c}-1)(\cos A - 1) - \tilde{s}\sin A + i\left[(-\tilde{c}+1)\sin A + \tilde{s}(\cos A - 1)\right]}{(\cos A - 1)^2 + \sin^2 A} =$$

$$(\cos A - 1)^2 + \sin^2 A = 2(1-\cos A)$$

$$= \boxed{\frac{\left(\cos(m+1)A - 1\right)\left(\cos A - 1\right) - \sin(m+1)A\sin A}{2(1-\cos A)}} + i\,\frac{\sin A\left(1-\cos(m+1)A\right) + (\cos A - 1)\sin(m+1)A}{2(1-\cos A)}$$

Ara tornem a l'expressió (E):

$$\sum_{k=0}^{m} \cos\left(\underbrace{\frac{(j\mp l)2\pi}{m+1}}_{A}\right)k = \frac{1}{2(1-\cos A)}\Big[\big(\underbrace{\cos((j\mp l)2\pi) - 1}_{=0}\big)\big)(\cos A - 1) -$$

$$(m+1)A = (j\mp l)2\pi$$

$$- \underbrace{\sin(j\mp l)2\pi}_{=0} \cdot \sin A = 0 \checkmark$$

Així obtenim que (E) = 0

- Si $j = l = 0$

$$(\varphi_0, \varphi_0)_m = \sum_{k=0}^{m} \frac{1}{4} = \frac{m+1}{4} \checkmark$$

- Si $j = l > 0$

$$(\varphi_j, \varphi_l)_m = \sum_{k=0}^{m} \cos^2 j x_k = \sum_{k=0}^{m}\left(\frac{1}{2} + \frac{1}{2}\cos 2j\frac{2\pi k}{m+1}\right) = \sum_{k=0}^{m}\frac{1}{2} = \frac{m+1}{2} \checkmark$$

$$\cos^2\theta = \frac{1}{2} + \frac{\cos 2\theta}{2}$$

Com abans,

$$\sum_{k=0}^{m} \cos 2j x_k = 0$$

- Anàleg amb la resta de cossos

Sigui ara

$$F_n = \left\{ t_n(x) = \frac{a_0}{2} + \sum_{j=1}^{n} \left(a_j \cos(jx) + b_j \sin(jx) \right), \ \forall a_0, a_j, b_j \in \mathbb{R} \right\}$$

El problema d'aproximació trigonomètrica per mínims quadrats consisteix en:

- Al cas continu: donada $f : [0, 2\pi] \to \mathbb{R}$ contínua, $f(0) = f(2\pi)$

volem $t_n^* \in F_n$ tq

$$\| f - t_n^* \|_2 = \min_{t_n \in F_n} \| f - t_n \|_2$$

- Al cas discret: donat $m \geq 2n$ i $f : I_m \to \mathbb{R}$,

on $I_m = \{ x_k = kh$, amb $h = \frac{2\pi}{m+1}$, $k = 0 \div m \}$

volem $t_n^* \in F_n$ tq

$$\| f - t_n^* \|_2 = \min_{t_n \in \tilde{F}_n} \| f - t_n \|_2$$

Com que les funcions bàsiques són ortogonals, el sistema d'equacions normals serà diagonal i per tant la solució t_n^* ve donada per

$$t_n^*(x) = \frac{a_0^*}{2} + \sum_{j=1}^{n} \left(a_j^* \cos jx + b_j^* \sin jx \right)$$

on a_0^*, a_j^*, b_j^* venen donats per:

Cas continu	Cas discret
$a_0^* = \dfrac{(\varphi_0, f)}{(\varphi_0,\varphi_0)} = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)dx,$	$\dfrac{2}{m+1}\displaystyle\sum_{k=0}^{m} f(x_k)$
$a_j^* = \dfrac{(\cos jx, f)}{(\cos jx, \cos jx)} = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\cos jx\,dx,$	$\dfrac{2}{m+1}\displaystyle\sum_{k=0}^{m} f(x_k)\cos jx_k$
$b_j^* = \dfrac{(\sin jx, f)}{(\sin jx, \sin jx)} = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\sin jx\,dx,$	$\dfrac{2}{m+1}\displaystyle\sum_{k=0}^{m} f(x_k)\sin jx_k$

NOTA — 1. Si prenem $f:\mathbb{R}\longrightarrow\mathbb{R}$ 2π-periòdica contínua, podem substituir qualsevol interval de la forma $[a, a+2\pi]$ per $[0, 2\pi]$ sense alterar l'aproximació.

2. Calculem l'error

$$\|f-t_n^*\|_2^2 \underset{\text{usl}}{=} \|f\|_2^2 - \|t_n^*\|_2^2 = \sum_{k=0}^{m} f_k^2 - \frac{m+1}{2}\left[\frac{a_0^{*2}}{2} + \sum_{j=1}^{n}\left(a_j^{*2}+b_j^{*2}\right)\right]$$

3. Usant

$$e^{ix} = \cos x + i\sin x, \quad \cos x = \frac{e^{ix}+e^{-ix}}{2}, \quad \sin x = \frac{e^{ix}-e^{-ix}}{2i}.$$

$$\cos jx + i\sin jx = \left(e^{ix}\right)^j = \left(\cos x + i\sin \cancel{j}x\right)^j$$

podem expressar

$$t_n(x) = \frac{a_0}{2} + \sum_{j=1}^{n} \left(a_j \cos jx + b_j \sin jx \right)$$

per la representació de Fourier complexa:

$$t_n(x) = \sum_{j=-n}^{n} c_j e^{ijx}$$

on $c_0 = \frac{a_0}{2}$, $c_j = \frac{1}{2}(a_j - ib_j)$, $c_{-j} = \overline{c_j} = \frac{1}{2}(a + ib_j)$, $j = 1 \div n$

o a l'inrevés

$$\left. \begin{aligned} a_0 = 2c_0, \quad a_j &= c_j + c_{-j} = c_j + \overline{c_j} = 2\,\mathrm{Re}(c_j) \\ b_j &= i(c_j - c_{-j}) = i(c_j - \overline{c_j}) = -2\,\mathrm{Im}(c_j) \end{aligned} \right\} j = 1 \div n$$

anomenats coeficients de Fourier reals o complexos.

En efecte,

$c_0 = \frac{a_0}{2}$ ✓

De $t_n(x) = \sum_{j=-n}^{n} c_j e^{ijx}$

tenim

$$\underbrace{c_j e^{ijx}}_{\text{terme } j\text{-èssim}} + \underbrace{c_{-j} e^{-ijx}}_{\text{terme } -j\text{-èssim}} = c_j(\cos jx + i \sin jx) + c_{-j}(\cos jx - i \sin jx) =$$

$$= (c_j + c_{-j}) \cos jx + i(c_j - c_{-j}) \sin jx =$$

$$= a_j \cos jx + b_j \sin jx$$

Ara bé $\underset{\mathbb{R}}{\underset{\cap}{a_j}} = c_j + c_{-j} = c_j + \overline{c_j} = 2\,\mathrm{Re}(c_j)$

$b_j = i(c_j - c_{-j}) = i(c_j - \overline{c_j}) = -2\,\mathrm{Im}(c_j)$

(Nota.- $\left.\begin{array}{l} c_j + c_{-j} \in \mathbb{R} \\ i(c_j - c_{-j}) \in \mathbb{R} \end{array}\right\} \Rightarrow c_{-j} = \overline{c_j}$:

en efecte

$\left.\begin{array}{lll} A+iB + E+iF \in \mathbb{R} & \Rightarrow B+F=0 & \Rightarrow F=-B \\ i(A+iB - E-iF) \in \mathbb{R} & \Rightarrow A-E=0 & \Rightarrow A=E \end{array}\right\} \Rightarrow c_j = \overline{c_j}$ ✓)

4. Canvi d'escala: En el cas d'una funció T-periòdica, $f(t)$, el podem reduir al cas 2π-periòdic definint la transformació:

$$t \in [0,T] \longleftrightarrow x \in [0,2\pi] \qquad x = \frac{2\pi}{T}t$$

$$f(t) = f\left(\frac{T}{2\pi}x\right) = \tilde{f}(x)$$

llavors $\tilde{f}(x) \approx \frac{a_0}{2} + \sum_{j=1}^{n} (a_j \cos jx + b_j \sin jx)$ i o dir

$$\left[f(t) \approx \frac{a_0}{2} + \sum_{j=1}^{n} \left(a_j \cos j\frac{2\pi}{T}t + b_j \sin j\frac{2\pi}{T}t\right) \right]$$

- Exemple $m=8$

x_i	0	$2\pi/9$	$4\pi/9$	$2\pi/3$	$8\pi/9$	$10\pi/9$	$4\pi/3$	$14\pi/9$	$16\pi/9$
f_i	3,0004	5,7203	3,1993	-1,0981	-0,8679	2,9890	4,0985	1,1477	-0,1882

Calculem

$$t_n^* = \frac{a_0^*}{2} + \sum_{j=1}^{2} \left(a_j^* \cos jx + b_j^* \sin jx \right)$$

Obtenim $a_0^* = 4,00022$, $a_1^* = 0,99998$, $b_1^* = 0,00029$

$a_2^* = 0,00011$, $b_2^* = 2,99997$

Error $= 0,00031$

(De fet els valors de f_i presos són una pertorbació de

$f(x) = 2 + \cos x + 3 \sin 2x$)