DERIVATIVES

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1. Introduction. In recent years there has been a considerable amount of research devoted to questions involving the derivative of a function of one real variable and its generalizations. This activity is due, in part, to the fundamental role played by the derivative in mathematics, and, also, to the difficulty of some of the interesting unsolved problems related to derivatives. It seems appropriate that some of the results of this activity, along with some of the interesting but not-so-well-known earlier results, be brought together and examined in one place. This is one of the purposes of the present expository article.

In deciding which topics to include in this article, we have given preference to ones which can be discussed without first having to develop a great deal of machinery. In addition, we have leaned toward topics in which recent work has been done and for which unsolved problems can be stated.

From the long list of references given at the end of this article, we single out the reference [**192**]. Many of the recent works on derivatives have their origin in this penetrating study by Zahorski.

2. Preliminaries. In this section we present a few of the definitions and concepts which appear in the sequel. To avoid having this discussion become prohibitively long, we restrict ourselves to those notions which appear prominently later on. For other real variable concepts which appear in this article (for example: approximate continuity, F_σ sets, big and little o notation, density conditions) the reader is referred to the texts [**50, 51, 56, 63, 64, 71, 132, 133, 173, 180**].

Throughout this article we shall be concerned with real valued functions of a real variable, usually defined on an interval $[a, b]$. Such a function belongs to Baire class 1 if it is the limit of a sequence of continuous functions. We define the other Baire classes inductively: if α is a countable ordinal, then f is in Baire class α provided it is the limit of a sequence of functions each of which is in a Baire class whose index is less than α. Detailed studies of the Baire classes can be found in [**3, 51, 56, 83, 133, 180**]. We note that our definition is an inclusive one: if f is in Baire class β and $\alpha > \beta$ then f is also in Baire class α. In some studies it is more convenient to have the Baire classes pairwise disjoint. This is the case, for example, in the recent text [**133**]. It is clear that every derivative is a function in Baire class 1.

A property possessed by every derivative of a continuous function is the Darboux property. The function f satisfies the Darboux condition (or f is a Darboux function) on $[a, b]$ provided the set $f[I]$ is connected for every interval $I \subset [a, b]$. This property is often called the *intermediate value property*, because a function has the Darboux property if and only if whenever x_1 and x_2 are points of $[a, b]$ and y is a number between $f(x_1)$ and $f(x_2)$, there is an x_3 between x_1 and x_2 such that $f(x_3) = y$.

Although there are many articles in research journals which deal with Darboux functions, a systematic study of such functions has escaped the standard real variable texts. It is worth noting that the Darboux property is far weaker than the property of continuity. Thus, there exist functions which take on every real value on every perfect set [**60**]. Such a function obviously satisfies the Darboux condition but is nowhere continuous. Another indication of the size of the class of Darboux functions is the fact that *every* function is the limit of a sequence of Darboux functions [**37**]. One more remark: the definition requires that the image (not the graph!) of a connected set be connected. There are Darboux-Baire functions whose graphs are not connected [**84**: p. 82]. However, it is shown in [**85**] that if f is in Baire class 1, then f satisfies the Darboux condition if and only if the graph of f is connected. (See also [**24**; **62**: pp. 289, 290; **84**: p. 81].) (It follows that every derivative has a connected graph.) For further remarks on this subject see [**107**]. The reader interested in Darboux functions is referred to the survey article [**13**].

We end this section by stating for reference the definition of two rather complicated density conditions due to Zahorski [**192**]. We refer to these important conditions in Sections 4, 12, and 14.

DEFINITION. *A nonempty set $E\subset[a, b]$ is said to be an M_4 set provided E is of type F_σ and there exists a sequence $\{F_n\}$ of closed sets and a sequence $\{\eta_n\}$ of numbers, $0<\eta_n<1$, such that $E=\bigcup_{n=1}^{\infty} F_n$ and for each $x\in F_n$ and every $c>0$ there is a number $\epsilon(x, c)>0$ enjoying the following property: for any numbers h and h_1 such that $hh_1>0$, $h/h_1<c$, $|h+h_1|<\epsilon(x, c)$ we have*

$$\frac{m(E\cap(x+h, x+h+h_1))}{|h_1|}>\eta_n.$$

We note that in this definition we require that the numbers η_n can be chosen to be strictly positive. If we relax this requirement to allowing some or all of the η_n to be zero, we arrive at the definition of an M_3 set. The definition of M_3 set can be stated in a different and perhaps simpler form: the set E is an M_3 set provided that if $x\in E$ and $\{I_n\}$ is a sequence of intervals not containing x such that $\{I_n\}\to x$ and $m(I_n\cap E)=0$ for all n, then

$$\lim_{n\to\infty}\frac{mI_n}{\text{dist}(x, I_n)}=0.$$

(The condition M_4 cannot be given in an analogous manner, see Lipiński [**91**].)

3. Continuity of the derivative. The student who has completed a first course in calculus is often not aware of the fact that the derivative of a differentiable function need not be continuous. In a later course he learns that the function f_1 given by $f_1(x)=x^2\sin(1/x)$, $f_1(0)=0$ is differentiable, but f_1' fails to be continuous at the origin. He might never learn, however, just how badly discontinuous a derivative can be. In this section we consider some questions concerning the continuity of derivatives. Our discussion points out, in addition, some of the pathological behavior possible of a derivative.

To show that not every bounded derivative is Riemann integrable, Volterra [**182**] gave an example of a function f_2 whose derivative is bounded but discontinuous on a set of positive (Lebesgue) measure. To construct such a function, he considered a nowhere dense perfect set $P \subset [0, 1]$ of positive measure, and constructed a function which on each interval contiguous to P behaves, roughly, as the function f_1 (above) behaves on $[0, 1]$. This function, f_2, was put together in such a way as to be differentiable on $[0, 1]$ and to produce on all of P the singularity f_1 exhibits at the origin. More precisely, $f_2' = 0$ on P, but f_2' oscillates between -1 and 1 in every neighborhood of an arbitrary point of P. It follows that f_2' must be discontinuous on P. For a precise formulation of such a function see Goffman [**51**: p. 210], Hobson [**63**: pp. 490, 491], or Thielman [**173**: p. 165]. A construction of the type referred to is possible relative to any nowhere dense perfect subset P of an interval I. Such subsets can have measure arbitrarily close to the measure of I, but since $I \sim P$ contains a dense open set, the set P cannot have full measure. It is natural to ask just how large the set of discontinuities of a derivative can be. Does there exist, for example, a derivative which is *everywhere* discontinuous? To see that this question must be answered in the negative, we need only observe that a derivative f' is of Baire class 1 from which it follows that f' must be continuous on a dense set [**133**: p. 143]. We weaken our requirement: is it possible for a derivative to be discontinuous except on a denumerable set? Again the answer is "no." To see this we recall first that the set of points of continuity of *any* function must be a G_δ. Since this set must also be dense, as was seen above, it cannot be denumerable, for a dense G_δ must be nondenumerable. (This fact follows readily from the Baire category theorem.)

We next ask whether or not it is possible for a derivative to be discontinuous on a dense set. To see that this question has an affirmative answer, we use the following approach. We seek a differentiable function whose derivative vanishes on one dense set but is different from zero on another. Such a derivative must, of course, be discontinuous at every point at which it does not vanish. The problem of constructing such derivatives is quite old. In 1887 Köpcke [**77**] claimed to have given an example of a function f possessing the following properties: (a) f has a bounded derivative f' on $[0, 1]$; (b) the set on which f' is positive is dense in $[0, 1]$; and (c) the set on which f' is negative is also dense in $[0, 1]$. Köpcke's original paper had a flaw which he corrected subsequently [**78, 79**]. There followed a sequence of articles on the subject, culminating in 1915, with a lengthy and penetrating study by Denjoy [**31**]. In this study the author provided a detailed discussion of differentiable functions whose derivatives take on both signs in every interval. He comments on some of the previous works on the subject, including Köpcke's, and gives several methods of constructing such functions. Zalcwasser [**196**] investigated the relative maxima and minima of such functions, obtaining results such as the following: *Let A and B be arbitrary* *nonoverlapping denumerable subsets of* $[a, b]$. *Then there exists a differentiable function* f, *having a bounded derivative, such that* A *is the set of strict local maxima of* f *and* B *is the set of strict local minima of* f.

Functions of the Köpcke type are too complicated to discuss here. However, a related kind of function, the so-called function of Pompeiu [**151**], is easier to understand and still exhibits the property that its derivative vanishes on one dense set but is different from zero on another. Again, the derivative of such a function is discontinuous at every point at which it does not vanish. Let us observe that if d is any real number then the function $(x-d)^{1/3}$ has a finite derivative except at d, at which point the derivative is infinite. Let $\{A_n\}$ be any sequence of positive numbers such that $\sum A_n < \infty$, and let $\{d_i\}$ be any denumerable dense subset of $[0, 1]$. Then the series $\sum A_n(x-d_n)^{1/3}$ defines a strictly increasing function f. It can be shown that f has a finite positive derivative at all points for which the differentiated series $\sum \frac{1}{3} A_n(x-d_n)^{-2/3}$ converges, and an infinite derivative otherwise. It can further be shown that the inverse function f^{-1} is a strictly increasing differentiable function and its derivative vanishes on a dense set. A lengthy and detailed study of such functions can be found in Marcus [**106, 113**]. See also [**11**] and [**90**] for answers to some questions raised in [**113**].

Differentiable functions whose derivatives are discontinuous on a preassigned denumerable set $\{d_n\}$ (which may be dense) can be constructed by considering any uniformly convergent series of derivatives, $\sum f_n$, such that the function f_n is discontinuous only at d_n. For example, the function f given by $f(x) = \sum_{n=1}^{\infty} n^{-2} \cos (x-d_n)^{-1}$ is a derivative with discontinuities on the set $\{d_n\}$. (See Halperin [**61**].)

We have seen that the set of discontinuities of a derivative can be dense, but that the set of points of continuity must also be dense and must be nondenumerable. Finally, we ask: what are necessary and sufficient conditions on a set E that it be *the* set of discontinuities of a derivative? It is easy to verify that such a set must be an F_σ of the first category. Conversely, if E is *any* first category F_σ, $E \subset [a, b]$, then E can be expressed as the union of an expanding sequence of nowhere dense closed sets E_n. With each E_n we associate a "Volterra type" function f_n with the property that if $x_0 \in E_n$ then f_n' oscillates between -1 and 1 in each neighborhood of x_0. It is not hard to verify that the function f defined by $f(x) = \sum f_n(x)/3^n$ is differentiable on $[a, b]$ and its derivative is continuous at each point of $\sim E$, but discontinuous at each point of E. Thus we have

THEOREM. *A necessary and sufficient condition that a set* $E \subset [a, b]$ *be the set of discontinuities of a derivative, is that* E *be an* F_σ *of the first category.* (Although we imagine that this theorem is known, we have been unable to find a reference.)

In particular, there are derivatives which are discontinuous a.e. on $[a, b]$. For if to each positive integer n we make correspond a nowhere dense closed subset E_n of $[a, b]$ having measure greater than $b-a-1/n$, then the set E which is the union of the E_n's is a first category F_σ of measure $b-a$. The result follows from the theorem above.

4. An unsolved problem. Many classes of functions can be characterized in terms of what the inverse mapping of a function in the class does to certain open sets. The chart below summarizes some of these characterizations. Let α be any real number and let

$$E_\alpha(f) = \{x: f(x) > \alpha\}, \qquad E^\alpha(f) = \{x: f(x) < \alpha\}.$$

Then f is	*if and only if for all real* α, β
continuous	$E_\alpha(f)$ and $E^\alpha(f)$ are open
Baire class 1	$E_\alpha(f)$ and $E^\alpha(f)$ are sets of type F_σ
Baire class ξ (ξ a countable ordinal)	$E_\alpha(f)$ and $E^\alpha(f)$ are additive Borel class ξ if ξ finite, $\xi+1$ if ξ infinite
in some Baire class	$E_\alpha(f)$ and $E^\alpha(f)$ are Borel sets
upper semi-continuous	$E^\alpha(f)$ is open
lower semi-continuous	$E_\alpha(f)$ is open
measurable	$E^\alpha(f)$ and $E_\alpha(f)$ are measurable
approximately continuous	each $x \in E^\alpha(f) \cap E_\beta(f)$ is a point of density of that set, and that set is an F_σ.

It is natural to ask what the corresponding characterizations are for various classes of derivatives. This question has been studied by Zahorski [**192**]. In this work, he found necessary conditions and also sufficient conditions in terms of the sets $E_\alpha(f)$ and $E^\alpha(f)$ for a function to be a bounded derivative, a finite derivative, or a derivative, possibly infinite, but he was unable to find *characterizations* of these classes of derivatives. (See Section 14 for a more detailed discussion of Zahorski's results.) The question of characterization of these classes is still open.

5. Derivatives a.e. and universal generalized antiderivatives. As we saw in Section 4, the problem of characterizing derivatives in terms of the sets $E^\alpha(f)$ and/or $E_\alpha(f)$ has not yet been resolved. We turn now to the problem of finding such a characterization for the class of functions which have the property of being *almost everywhere* the derivative of a continuous function. As we shall see, this requirement imposes very little restriction on a function.

We first observe that every Lebesgue summable function is almost everywhere the derivative of its integral. The same is true of any function integrable in the sense of Denjoy-Perron. (See Section 10.) There are, however, measurable functions not integrable in either of the above senses. In 1915, Lusin [**97, 99**] published the following theorem, which completely solves the problem of characterizing those functions which are derivatives a.e. of continuous functions: *Every measurable function f (finite a.e.) is almost everywhere the derivative of a continuous function F.* The condition of measurability of f as well as that of a.e. finiteness is obviously necessary as well as sufficient. Thus, *the a.e. finite function f is a.e. the derivative of a continuous function F if and only if f is measurable, or equivalently, if and only if each set of the form $\{x: f(x) < \alpha\}$ or of the form $\{x: f(x) > \alpha\}$ is measurable.* Of course the function f is not the only one which is a.e. the derivative of F, but any other such function is equivalent to f (i.e., agrees with f a.e.).

Let us see what happens if we weaken the requirement of being a.e. a derivative still further. Let f be an arbitrary function on $[a, b]$ and suppose there exists a sequence $\{h_n\}$ of numbers with $h_n \downarrow 0$ and a continuous function F such that

$$f(x) = \lim_{n\to\infty} \frac{F(x + h_n) - F(x)}{h_n}$$

a.e. on $I = [a, b]$. Then F may be called a *generalized antiderivative* of f. It is clear that such an F may be a generalized antiderivative of many functions not equivalent to f. How many? Marcinkiewicz [**101**] has proved the following remarkable theorem: *There exists a continuous function F which is a generalized antiderivative for every a.e. finite measurable function.* (That is, F is a universal generalized antiderivative.) It was also shown in [**101**] that *most* functions are universal generalized antiderivatives, in that the class of continuous functions which are *not* universal generalized antiderivatives form a set of the first category in $C[a, b]$. A proof of the theorem of Lusin and mention of the theorem of Marcinkiewicz can be found in [**157**: pp. 215–218].

Other results of the above type have been obtained by Sierpiński [**163**] and Eilenberg and Saks [**35**]. One such result [**35**] deals with the generalized antiderivative of arbitrary (not necessarily measurable) functions: *Let f be any function defined on an interval I and let H be any denumerable set of real numbers. Then there exists a continuous function F such that*

$$f(x) = \lim_{n\to\infty} F[(x + h_n) - F(x)]/h_n$$

for every null sequence $\{h_n\}$ from H. (There is no exceptional set here; the result holds for *all* x.)

6. Dini Derivatives. A function defined on an interval $I = [a, b]$ has defined at each point of I four Dini derivatives (except at a and b, at which only two of the Dini derivatives are defined). For example, the upper right Dini derivative of f, $D^+ f$, is defined by

$$D^+f(x) = \limsup_{h\to 0+} \frac{f(x + h) - f(x)}{h},$$

and the other three are defined in an analogous manner. An elementary result is that if one of the Dini derivatives of f is continuous at a point x_0, then f is differentiable at x_0.

In 1915 Denjoy [**30**] proved a theorem relating the four Dini derivatives for continuous functions. This was generalized to measurable functions by Young [**189**] in 1916 and to arbitrary functions by Saks [**156**] in 1924.

THEOREM. *Let f be finite on $[a, b]$. Then, with the possible exception of a null set, $[a, b]$ can be decomposed into four sets:*

A_1, on which f has a finite (ordinary) derivative,

A_2, on which $D^+f = D_-f$ (finite), $D^-f = \infty$, $D_+f = -\infty$,

A_3, on which $D^+f = \infty$, $D_-f = -\infty$, $D^-f = D_+f$ (finite), and

A_4, on which $D^+f = D^-f = \infty$, $D_+f = D_-f = -\infty$.

The theorem is valid if one replaces $[a, b]$ by any set A (not necessarily measurable).

Some immediate consequences of this theorem are the following:

(1) An increasing function is differentiable a.e. (for the sets A_2, A_3, and A_4 are empty in this case);

(2) A function of bounded variation is differentiable a.e. (for such a function is the difference of two increasing functions);

(3) If f is finite on $[a, b]$, then the set on which f' is infinite is a null set. (It is interesting to observe, by way of contrast, that there exist functions which have $D^+f \equiv \infty$, even though f is right-continuous. Here, right-continuity cannot be replaced by continuity. See [**7**: pp. 125, 126; **124**; **167**].)

The Denjoy-Young-Saks theorem has been extended by Garg [**42**], who showed that the exceptional null set also has an image of measure zero. Garg has also considered the set at which the Dini derivatives vanish. Some applications of this extension may be found in [**43, 44, 45**]. For results on Dini derivatives of nowhere monotone functions, the reader is referred to Garg [**46, 47, 48**].

For continuous functions the sets $\{D^+f \neq D^-f\}$ and $\{D_+f \neq D_-f\}$ are small in the sense of category. The following result is due to Neugebauer [**140**]: *If f is continuous then the sets $\{D^+f \neq D^-f\}$ and $\{D_+f \neq D_-f\}$ are of first category. If in addition f is of bounded variation on every closed interval, then these sets are of measure zero as well.* The characteristic function of the rationals shows that continuity cannot be dropped from the first statement; nor can the hypothesis of bounded variation be dropped from the second statement, as is shown by Example III of Denjoy [**30**]. Neugebauer's theorem, as well as certain related results, is a consequence of a result found in [**203**].

Although a Dini derivative does not, in general, satisfy the Darboux condition, some interesting results about the intermediate values taken on by Dini derivatives have been advanced by Morse [**130**]. One such result is the following: *If f is continuous, $-\infty < \lambda < \infty$, if the set $\{x: D^+f(x) \geqq \lambda\}$ is dense and the set $\{x: D^+f(x) < \lambda\}$ is nonempty, then the set $\{x: D^+f(x) = \lambda\}$ has the power of the continuum.*

A derivative (finite or infinite) of a real valued function is always in Baire class 1. The corresponding statement for Dini derivatives is not valid, even for continuous functions. However, if f is in Baire class α, then the four Dini derivatives are in Baire class $\alpha+2$ [**162**] and if f is measurable, then so are its Dini derivatives [**4**]. If f is not measurable, then the same may be true of its Dini derivatives. However, Hájek [**58**] has advanced the surprising result that for any finite function (measurable or not) the extreme *bilateral* derivatives must be of Baire class 2. (The upper bilateral derivative of a function f is defined by

$$\bar{f}(x) = \limsup_{h \to 0} \frac{f(x + h) - f(x)}{h} \cdot$$

The lower bilateral derivative is defined analogously.) This result is the best possible, for there exists a function f satisfying a Lipschitz condition, with $\bar{f}$ not in Baire class 1 [**168**].

7. Approximate derivatives. In certain instances a function fails to have a derivative at a point x_0, yet the restriction of the function to a set whose complement is very "thin" near x_0 has a derivative at x_0. If one properly interprets "thin" in terms of density, then one arrives at the notion of an approximate derivative.

DEFINITION. *Let f be defined on $[a, b]$, and let $x_0 \in (a, b)$. If there exists a set E such that* (1) *$x_0 \in E$,* (2) *x_0 is a point of zero density with respect to $\sim E$, and* (3)

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists, for x restricted to E, then this limit is called the approximate derivative of f at x_0 and is written $f'_{ap}(x_0)$. (The obvious modifications are made if $x_0 = a$ or $x_0 = b$.)

The notion of approximate derivative was introduced by Denjoy [**29**] and plays an important role in the theory of the Denjoy-Khintchine integral (see Section 10).

The approximate derivative also arises in connection with certain questions involving the approximation of functions of several real variables. Thus, let f be defined on, say, the unit square S in Euclidean two-space. According to Lusin's Theorem f is measurable if and only if for every $\epsilon > 0$ there exists a function g continuous on S such that $g = f$ except on some set of measure less than ϵ. Suppose we wish to approximate f in this sense by a function which is not only continuous, but also has a continuous total differential. This is possible if and only if f has an approximate total differential almost everywhere [**216**]. That is, in order that f have the property that for every $\epsilon > 0$ there exists a continuously differentiable function g such that $g = f$ except on a set of measure less than ϵ, it is necessary and sufficient that f be approximately differentiable a.e. This last condition is equivalent to the condition that the partial approximate derivatives of f exist a.e. [**213**], see also [**157**: p. 300]. (We note that the corresponding statement for ordinary partial derivatives is false. There is a continuous function of two variables whose partial derivatives exist a.e. but whose total differential exists *nowhere* [**213**: p. 515].)

Approximate derivatives possess some of the properties of ordinary derivatives. Thus, if f is approximately differentiable on $[a, b]$, then f'_{ap} is of Baire class 1 and possesses the Darboux property (see Khintchine [**73**, **74**] and Tolstoff [**174**]). For a unified development of these results, among others, the reader is referred to Goffman and Neugebauer [**53**].

If one allows the approximate derivative to be infinite, then the situation is a little different. Zahorski [**191**] has given an example of a function f which has at every point an approximate derivative (finite or infinite), but f'_{ap} is not of Baire class 1, nor does it satisfy the Darboux condition. In the same article, however, he shows that if f has at each point an approximate derivative (finite or infinite), then f'_{ap} is of Baire class 2. In addition, f must be of Baire class 2 as well. In case f is approximately continuous and has at each point a finite or infinite approximate derivative, then f'_{ap} (as well as f) must be of Baire class 1. (See Tolstoff, [**174**].) In addition, f'_{ap} must satisfy the Darboux condition in this case [**82**]. Some additional results involving the Baire class of approximate derivatives can be found in Krzyzewski [**81**] and Matysiak [**119**].

It is of interest to note that while the set of discontinuities of a function having everywhere a derivative (possibly infinite) must be denumerable, Lipiński [**92**] has shown that the set of points of approximate discontinuity of a function having everywhere an approximate derivative (possibly infinite) can be nondenumerable, although the set must have zero measure and be of the first category.

Under certain conditions a point of approximate differentiability is actually a point of differentiability. Thus if f is monotonic, f is differentiable wherever f is approximately differentiable [**73**, **74**]. Khintchine has also shown that if an approximate derivative (possibly infinite) is dominated by an ordinary derivative, then this approximate derivative is in fact an ordinary derivative. This result has been used by Tolstoff to prove that if f is approximately continuous and has at each point a finite or infinite approximate derivative f'_{ap}, then except possibly on a nowhere dense set, f'_{ap} is the ordinary derivative of f. ([**175**]; see also [**53**].)

In 1916 Denjoy [**33**] noted the following property of a (finite) derivative: *If $\alpha<\beta$ then the set $E_{\alpha\beta}=\{x:\alpha<f'(x)<\beta\}$ is either empty or has positive measure.* This result was extended by Clarkson [**26**] to derivatives (which might be infinite) of continuous functions. A more detailed description of the sets $E_{\alpha\beta}$ was advanced by Hsiang [**65**]. Finally, in 1962, Marcus [**108**] showed that the corresponding results are valid if one replaces "derivative" by "approximate derivative" in the hypothesis and conclusion of Clarkson's theorem. In case one allows the approximate derivative to be infinite at some points, additional assumptions are necessary. As mentioned in Section 4, Zahorski [**192**] has obtained results concerning the structure of sets of the form $\{x:f'(x)<\beta\}$. Corresponding results for approximate derivatives and Peano derivatives have been advanced by Weil [**185**] and Kulbacka [**82**].

Whether or not a function is approximately differentiable, it always has four extreme unilateral approximate derivates. For a detailed study of these, the reader is referred to Jeffery [**71**]. We mention the interesting fact, which may be found on pages 198–199 of [**71**], that the theorem of Denjoy-Saks-Young for Dini derivatives (see Section 6 above) has a virtually identical analogue. If f happens to be measurable, then the sets which correspond to A_2 and A_3 (of Section 6) are null sets. For results on nonmeasurable functions, see Chow [**25**].

It is of interest to note that for functions of *several* variables the extreme unilateral partial approximate derivatives of a function f reflect the measurability properties of f, whereas the partial Dini derivatives do not. Thus if f is a Lebesgue (Borel) measurable real valued function of several variables, the same is true of its extreme unilateral partial approximate derivatives. On the other hand, there are Lebesgue (Borel) measurable functions of two variables whose partial Dini derivatives are not Lebesgue (resp. Borel) measurable. It is true, however, that if f is continuous (Borel measurable), then its partial Dini derivatives are Borel measurable (resp. Lebesgue measurable). (For functions of one real variable, the Dini derivatives as well as the extreme unilateral approximate derivatives inherit the Lebesgue or Borel measurability of the primitive function.) For results of this sort see [**83**: p. 421], [**157**: pp. 113, 171, 299], and [**209**].

We conclude by mentioning that the word "thin" mentioned in the introductory paragraph can be interpreted in other ways, giving rise to different sorts of derivatives. Thus, for example, S. Marcus [**111, 112**] has interpreted "thin" in terms of category (rather than measure) and arrived at the notion of a qualitative derivative. The notion of "preponderant" derivative, due to Denjoy [**29**], is related to the notion of approximate derivative but the complements need not be quite so "thin" for a preponderant derivative to exist as for an approximate derivative to exist. For a fuller discussion of this matter in a slightly broader context, consult Section 8.

8. Other generalizations of the derivative. It is scarcely surprising that such a fundamental concept as the derivative has received generalization in a number of different directions for various special purposes. Many generalizations are arrived at by weakening the sense in which the limit of the difference quotient $[f(x+h)-f(x)]/h$ is obtained, although other avenues of definition are sometimes used. Usually the existence of the generalized derivative together with some regularity condition implies the existence of the ordinary derivative, and the restrictiveness of this regularity condition can be used as an index of the degree of generalization obtained. Where the ordinary derivative exists, it is equal to the generalized derivative. We shall give the definitions of various generalizations and discuss briefly some of the more important among them.

The Dini derivates, discussed in Section 6, represent the first generalization of the ordinary derivative, in that we do not restrict ourselves to the limit of the difference quotient, which limit may fail to exist, but rather consider the one-sided limit inferior and limit superior. In this way we are assured of the existence of these derivatives at each two-sided limit point of the domain of the function. We need only be given the continuity of one Dini derivate at a point to conclude the existence of the ordinary derivative at that point.

The approximate derivative, discussed in Section 7, is a natural generalization of the ordinary derivative in which the limit of the difference quotient is taken in the metric sense of the approximate limit. As pointed out in Section 7, an approximately continuous function with an approximate derivative everywhere in an interval $[a, b]$ possesses an ordinary derivative on a set of intervals which is dense in $[a, b]$ (see [**174**]).

Weakening the density requirements for the existence of a limit in the definition of approximate derivative (see Section 7) to *the set E has mean density greater than $1/2$ on all sufficiently small intervals including* x_0, we obtain the *preponderant* derivates and derivative of Denjoy [**29**]. Replacing metric considerations with the concept of category, one arrives at the *approximate qualitative* derivative defined by S. Marcus [**111, 112**], where the upper qualitative limit of f at x_0 is defined as inf $\{y: \{x: f(x) > y\}$ is first category at $x_0\}$, the lower qualitative limit is similarly defined, and these limiting operations are applied to the difference quotient to yield approximate qualitative derivates, which share many of the properties of the Dini derivates.

In taking the limit of the difference quotient, we may restrict ourselves to considering only values of $x_0 + h$ which belong to a given set E, which set has x_0 as a limit point. This will give us the *derivative of f relative to the set E.* Many of the theorems found in Saks [**157**] hold for this form of the derivative. Similar in concept is the *congruent derivative* of Sindalovskiĭ [**164, 165, 212**], in which the values of h used in forming the difference quotient are restricted to belong to a set Q, which has 0 as a limit point, but where this difference quotient is defined for every $x \in [a, b]$, always using the same set Q. The idea of passing to the limit while neglecting values obtained on "negligible" sets belonging to a particular family has been advanced by Császár [**27**].

Changing the form of the difference quotient gives rise to many generalizations of the derivative. The most common is the symmetric derivative (also called the Riemann derivative), defined by

$$f^{[']}(x) = \lim_{h \to 0} \frac{1}{2h} [f(x + h) - f(x - h)].$$

This derivative has the virtue of not involving the behavior of f at the point x itself. It is widely used in the theory of trigonometric series. (See, for example, [**198, 199**].) The existence of the symmetric derivative at all points of a set E implies the existence of the ordinary derivative a.e. in E [**73**]. The symmetric derivative is nicely arrived at through decomposing the function f at x_0 into its *even* and *odd* parts:

$$\phi_{x_0}(t) = \tfrac{1}{2}[f(x_0 + t) + f(x_0 - t)],$$
$$\psi_{x_0}(t) = \tfrac{1}{2}[f(x_0 + t) - f(x_0 - t)].$$

The differentiability of the odd part at $t = 0$ is then equivalent to the existence of the symmetric derivative, while the differentiability of the even part is equivalent to the property of *smoothness* [**197**, **139**] (see Section 9). The higher Riemann derivatives, given by

$$f^{[n]}(x_0) = \lim_{h\to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k f\left(x + h\left(\frac{n}{2} - k\right)\right),$$

extend the symmetric derivative [**18**, **20**, **72**, **199**]. The second Riemann derivative is often called the Schwarz derivative.

The difference quotient may be varied in other directions. The simplest variant is

$$f^*(x_0) = \lim_{\substack{x_1, x_2 \to x_0 \\ x_1 \neq x_2}} \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

This definition was considered by Peano [**146**], who felt that it portrayed the concept of the derivative used in the physical sciences more closely than does the usual definition, since f^* is always continuous, and coincides with f' whenever f' is continuous. This definition has been recently reconsidered by Esser and Shisha [**36**]. Another variation is Sindalovskiĭ's derivative [**166**],

$$\lim_{h\to 0} \frac{f(x - \phi(h)) - f(x - \phi(h) - h)}{h},$$

where ϕ is an arbitrary function, defined in a neighborhood of the origin, which approaches 0 with h. Murav'ev [**131**] dealt with the Gateaux derivative,

$$\lim_{h\to 0} \frac{f(x + h\alpha(x)) - f(x)}{h},$$

where f is differentiable (in the ordinary sense) on $[a, b]$, and $\alpha(x)$ is any bounded function defined on $[a, b]$. The linear function in the denominator of the difference quotient may be exchanged for an arbitrary function $g(x)$, yielding the *derivative with respect to* g given by

$$\lim_{x\to x_0} \frac{f(x_0) - f(x)}{g(x_0) - g(x)} \qquad [\mathbf{88}].$$

The physical sciences, in particular thermodynamics, led Borel to define a *mean derivative* [**10**],

$$f_B'(x) = \lim_{h\to 0} \lim_{\epsilon\to 0} \frac{1}{h} \int_\epsilon^h \frac{f(x+t) - f(x)}{t}\, dt.$$

Sargent [**158**] extended this definition to parallel the Dini derivates, and Marcinkiewicz and Zygmund [**103**] extended his results to a "smooth Borel derivative."

In a masterly paper, Khintchine [**73**] considered various candidates for a "generalized derivative," establishing their properties and constructing examples to illustrate a hierarchy of generality. The symmetric derivative was discarded since its existence, save on a null set, implies the existence of the ordinary derivative a.e. The Borel derivative generalizes the symmetric derivative and lacks this "flaw," but is itself generalized by the approximate derivative. The approximate derivative, however, is generalized by *la dérivée générale* of f, which is a function f_g', defined only a.e., such that for any $\epsilon > 0$,

$$m\left\{x: \left|\frac{f(x+h)-f(x)}{h} - f_g'(x)\right| > \epsilon\right\}$$

tends to 0 with h. The existence of f_{ap}' a.e. on an interval implies the existence of f_g' on the interval, and $f_{ap}' = f_g'$ a.e., while Khintchine constructed a function f such that f_g' exists on $[0, 1]$, and f_{ap}' only on a null set. However, f_g' bows to *la dérivée généralisée*, f_G', defined even less uniquely, which is any function with the property that, for some sequence $\{h_n\}$ decreasing to 0, we have $[f(x+h_n) - f(x)]/h_n \to f_G'$ a.e. This treatment reflects the generalized antiderivative of Section 5. Khintchine closed by constructing a continuous function which fails to have even a *dérivée généralisée*.

A completely different approach, which yields a generalization of ordinary derivatives of order greater than one, is given by polynomial approximation to a function. If $f(x_0+h)$ can be expressed as

$$f(x_0 + h) = f(x_0) + hf_1(x_0) + \frac{h^2}{2!}f_2(x_0) + \cdots + \frac{h^n}{n!}f_n(x_0) + o(h^n),$$

then the f_i's are referred to as the ith Peano derivatives [**145**] (referred to by Denjoy [**32**] as differential coefficients, and sometimes called de la Vallée Poussin derivatives [**181**], although this latter term is also used otherwise [**199**]). The nth Peano derivative f_n always equals the nth ordinary derivative when the latter exists. Oliver [**143**] studied the exact nth Peano derivative, which is one which exists at every point of an interval, and showed that it is of Baire class 1, enjoys the Darboux property and the Denjoy property of $E_{\alpha\beta}$, and coincides with the ordinary nth derivative on a dense open set. A side condition for the existence everywhere of the nth ordinary derivative is that the nth Peano derivative be bounded either above or below. If the nth Riemann derivative exists everywhere on a set E of positive measure, then the nth Peano derivative exists a.e. on E [**103**].

Generalization of the Peano derivative leads to the L^p derivative: if $f \in L^p$, $1 \leqq p \leqq \infty$, in some neighborhood of x_0, and if a polynomial

$$P(t) = \sum_{i=0}^{n} \frac{f_i}{i!} t^i$$

exists such that

$$\left[\frac{1}{2h}\int_{-h}^{h} |f(x_0 + t) - P(t)|^p \, dt\right]^{1/p} = o(h^n),$$

then f is said to be differentiable of order n at x_0 in L^p, and f_i is the ith L^p derivative. This generalization was introduced by Calderón and Zygmund [**21, 22**] because the property of differentiability in L^p at a point is preserved under various integral transformations, and was applied to solving partial differential equations. The L^p derivative has been used recently to establish the differentiability a.e. of functions [**136, 138, 169**] (see Section 9). Weiss [**186**] considered the symmetric kth derivative in L^p, and generalized the result that the existence of the kth symmetric derivative implies the existence of the kth (Peano) derivative a.e. Higher dimensions are considered in [**215**].

The nth Taylor derivative also arises out of considerations of polynomial approximation. It is defined [**19, 20**] as

$$f^{\{n\}}(x) = \lim_{h\to 0} \frac{n!}{h^n}\left[f(x+h) - \sum_{k=0}^{n-1} \frac{h^k}{k!} f^{(k)}(x)\right],$$

where $f^{(k)}$ is the ordinary kth derivative. Butzer [**19**] and Görlich and Nessel [**55**] studied the relationships between the Riemann, Peano, Taylor, and ordinary nth derivatives (listed in order of decreasing generality), where convergence was considered in both the usual metric and in the L^p norm.

For those interested in further generalizations of the derivative, we mention the fluents [**126**] and multiderivative [**125, 127**] of Menger, the Hölder and Cesaro derivatives of order n [**152**], the fractional derivative of Kuttner [**86**], Minetti's right oscillatory derivative [**128**], O'Neill's work on generalized derivates [**144**], and Shukla's on nonsymmetric differentiability [**160, 161**].

The extension of the concept of derivative to spaces other than the real line gives rise to a vast literature, which we will not explore. Introductions to this field are provided by Bögel's paper on higher-dimensional differentiation [**9**], Fréchet's study [**39**] of various definitions of differentiability in the plane, and the article of Rinehart and Wilson on differentiation in algebras [**153**].

9. Points of differentiability. Since the existence of the derivative of a function f at a point implies a certain degree of good behavior of f at that point, it is most natural to study the set of such points of differentiability. It is especially desirable to find under what conditions this set is large, and it is also of interest to know something about the size of the image of this set under f. We discuss these questions in this section.

If f' is defined and finite at x then x is a point of differentiability of f; if f' is defined (possibly infinite) at x, then we call x a point of extended differentiability. We let D represent the set of points of differentiability of f, and D^* the points of extended differentiability, and also let $N = \sim D$, $N^* = \sim D^*$.

The first result encountered is that a function which is monotonic on an interval has a finite derivative almost everywhere on that interval. The conclusion carries over easily to functions of bounded variation (Lebesgue's Theorem [**157**: p. 223]), and to functions which are VBG_*. Indeed, in these cases, not only is $mN=0$, but $mf[N^*]=0$. ([**157**: p. 230], due originally to Denjoy and Lusin). A condition, phrased in terms of a more restrictive notion of absolute continuity, which is both necessary and sufficient for f to be differentiable almost everywhere, under the restriction that f is continuous, has been presented by Pettineo [**149**, **150**].

The property of smoothness (see Section 8) and the related property Λ are of interest in investigating the differentiability of a function [**197**, **138**, **170**]. The function f is smooth at x_0 if

$$\Delta_x(h) = f(x+h) + f(x-h) - 2f(x) = o(h);$$

f satisfies condition Λ if this $\Delta_x(h) = O(h)$.

A continuous smooth function f on (a, b) has the property that the set D of its points of differentiability has the power of the continuum in every sub-interval of (a, b) [**197**]. Furthermore, f' satisfies the Darboux condition on D. (See [**197**].) If the continuity of f is replaced by measurability, the result on differentiability still holds, but in order to conclude the Darboux property of f' we need to know that D is "small" in the sense that $m(D \cap I) < m(I)$ for each interval $I \subset (a, b)$ [**139**].

If f satisfies condition Λ, then we may state a necessary and sufficient condition for the differentiability of f a.e.: *A measurable function f satisfying condition Λ at each point of a measurable set E is differentiable* a.e. *on E if and only if for almost every $x \in E$ there is an η_x such that $h^{-1}[\Delta_x(h)]^2$ is summable over* $(0, \eta_x)$. The necessity is due to Marcinkiewicz [**102**], and the sufficiency to Stein and Zygmund [**169**]. The summability of $h^{-1}[\Delta_x(h)]^2$ a.e. is equivalent to the existence of the L^2 derivative a.e. [**169**].

The condition Λ is dispensed with in a similar result due to Neugebauer [**136**]: *A measurable function f is equivalent to a function differentiable a.e. on a measurable set E if and only if for almost every $x \in E$ there is an $\eta_x > 0$ such that $[\Delta_x(h)]^2/h^3\phi(h^{-1}\Delta_x(h))$ is summable over* $(0, \eta_x)$, where the function ϕ is given by $\phi(x) = 1 - |x|$ in $(-1, 1)$, $\phi(x) = 0$ elsewhere.

Properties which are relevant to a discussion of differentiability are Banach's conditions (T_1) and (T_2) and Lusin's condition (N). For f defined on $I = [a, b]$ and $y \in f[I]$, we call the set $\{x : f(x) = y\}$ a level set of f. We say that f satisfies condition (T_1) if, for almost all $y \in f[I]$, the level sets are finite, and (T_2) if the level sets are at most denumerable. The condition (N) is satisfied if $B \subset I$ and $mB = 0$ imply that $mf[B] = 0$ also.

Marchaud [**100**] showed that if each level set of a continuous function f is finite, then f is differentiable almost everywhere. Iosifescu [**68**] has given a direct demonstration of this theorem, and has extended the result to discontinuous functions for which the set of points of nonmonotonicity (i.e., points having no neighborhood on which the function is monotonic) has measure zero. If the finiteness of the level sets is extended to denumerability, the result is almost totally lost, for Iosifescu has given a construction of such a function f_ϵ, for which $mD<\epsilon$, where ϵ is any arbitrarily preassigned positive number [**67**].

If f is continuous and satisfies property (N), then a theorem of Banach's [**157**: p. 286] assures us that D has positive measure. Weakening the hypothesis by replacing (N) with (T_1), we can conclude only that N^* has an image of measure zero (a property that characterizes continuous functions which are (T_1) [**157**: p. 278]). Still further weakening the hypothesis to (T_2), we can conclude only that D^* is nondenumerable, but can say nothing about its measure.

Finally, we know that it is possible for a continuous function to be so badly behaved as to be nowhere differentiable. It may be that D is empty but D^* is not empty, as in Cellerier's example [**23**], or we may even have D^* empty, as in Weierstrass' function [**184**], which does, however, admit one-sided derivatives on a dense set. Even this last remnant of good behavior can be removed, as in Besicovitch's example [**6, 147**], which at no point has even a unilateral derivative (even infinite). These are all discussed in Jeffery [**71**]. Functions such as Besicovitch's are "much rarer" than those of Weierstrass' example, in the sense that the former constitute a first category set in $C[a, b]$ while the latter form the complement of a first category set [**5, 123, 155**].

We pass from the study of the size of D and N to considerations of the structure of these sets. Through use of the concept of convergence classes [**62**: p. 309] one can prove that D is an $F_{\sigma\delta}$. The same is true of the set of points of left-differentiability or points of right-differentiability. It is not the case, however, that each $G_{\delta\sigma}$ is the set N for some function f. Zahorski [**190, 193**] showed for continuous functions that the set N is the union of a G_δ with a null $G_{\delta\sigma}$ and that any set of this form is the set of points of nondifferentiability for some continuous function. Exactly the same statement is true of N^*. For a function of bounded variation, the G_δ is dropped from the theorem. Brudno [**17**] extended the results to arbitrary functions, with exactly the same conditions holding. Zahorski's proof has been simplified by Piranian [**211**].

Since the distinction between having a derivative (possibly infinite) and having a finite derivative is so often critical (see, for example, Section 10 on inversion of derivatives), it is of interest to know just where a derivative may take on infinite values. We know from our discussion of Denjoy's theorem on Dini derivatives that the set $\{x : f'(x) \text{ is infinite}\}$ has measure zero. Conversely, for any set E of measure zero, there is a simple construction of a continuous, increasing function f_E with $f_E' = +\infty$ on E (see [**132**: p. 214]). Jarník [**70**] gave a construction of a continuous function with an infinite derivative on an arbitrarily given null G_δ and with finite Dini derivatives elsewhere, and Zahorski [**195**] improved this result to present an everywhere differentiable (in the extended sense) function with this property. For other results of this nature see Bojarski [**200**], Lipiński [**208**], Marcus [**116**], and Piranian [**210**].

The most complete result in this direction is due to Tzodiks [**177, 178, 179**]: *For a finite function f, necessary and sufficient conditions for the sets E_1 and E_2 to be sets where $f' = +\infty$ and $f' = -\infty$ respectively are*: (1) *E_1 and E_2 be $F_{\sigma\delta}$'s with* *measure zero, and* (2) *there exist disjoint* F_σ*'s* H_1 *and* H_2, *with* $E_1 \subset H_1$, $E_2 \subset H_2$. Other results concerning infinite derivatives are given in Filipczak [**38, 204**], Garg [**49**], Kronrod [**80**], Landis [**87**], Marcus [**109, 116**], and Marczewski [**118**].

10. Inversion of derivatives. We shall be concerned in this section with that half of the fundamental theorem of calculus which, roughly, recaptures a function from its derivative. The form which this theorem usually takes in elementary calculus is: *Let f be continuously differentiable on* $[a, b]$. *Then*

$$f(b) - f(a) = \int_a^b f'(x)\,dx, \tag{*}$$

the integral being taken in the sense of Riemann. The requirement that f' be continuous is usually weakened in a course in advanced calculus to the requirement that f' be Riemann integrable. The example of Volterra cited in Section 3 shows that this latter restriction cannot be weakened to insisting merely that f' be bounded. Now a desirable property of an integral is that the fundamental equation (*) hold for *any* derivative f', irrespective of whether or not f' is continuous or bounded. The above consideration shows that the Riemann integral does not have this property, even for bounded derivatives. The Lebesgue integral does a little better. The relevant theorem for Lebesgue integrals asserts that (*) holds whenever f' is summable. In particular, (*) holds for Lebesgue integrals whenever f' is bounded. If f' is not bounded, then f' might fail to be summable. The function $f(x) = x^2 \sin x^{-2}$, $f(0) = 0$ furnishes an example of a differentiable function on $[0, 1]$ whose derivative is not summable over any interval containing the origin. The difficulty, of course, lies in the fact that $\int |f'| = \infty$ over any such interval. It is of interest to note that even derivatives which are "tied down" by vanishing on a dense set of points (see Section 3) can be so large elsewhere that f' fails to be summable. (See [**11**] and [**90**].) We have seen that equation (*) is not valid for Lebesgue integrals in general.

Perron [**148**] and Denjoy [**28, 34**] independently defined integrals, both more general than the integral of Lebesgue's, which completely solved the problem of recapturing a function from its (finite) derivative; more precisely, of integrating arbitrary derivatives so that (*) holds. Although the methods of Denjoy and Perron were entirely different in approach, Hake [**59**], Alexandroff [**1, 2**], and Looman [**96**] proved that these two integrals were entirely equivalent; i.e., if a function is integrable in one of the two senses, it is integrable in the other, and the two integrals are equal. Thus, this integral is usually called the Denjoy-Perron integral.

In 1916 Khintchine [**75, 76**] modified the Denjoy construction to give rise to a more general integral, now referred to as the Denjoy-Khintchine integral, which integrated arbitrary approximate derivatives of continuous functions. *Descriptive* definitions of the Denjoy-Perron and Denjoy-Khintchine integrals were advanced by Lusin [**98**]. For a development of the Perron integral and both the constructive and descriptive definitions of the Denjoy-Perron and Denjoy-Khintchine integrals, see Saks [**157**]. An elegant development of the Perron and the Denjoy integrals, along with a proof of their equivalence, can be found in Natanson [**133**; Chap. 16]. A detailed discussion of how a function may be recaptured from its derivative in a countable number of steps is given in Jeffery [**71**].

For purposes of comparison we state the descriptive definitions of the Lebesgue, Denjoy-Perron, and Denjoy-Khintchine integrals. We begin with the definitions of four generalizations of the notion of absolute continuity of a function defined on an interval $[a, b]$: Let F be continuous on $[a, b]$ and let $E \subset [a, b]$. Then F is called $AC(AC_*)$ on E provided that for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $\{[a_k, b_k]\}$ is any sequence of nonoverlapping intervals with endpoints in E and with $\sum(b_k - a_k) < \delta$, then $\sum|f(b_k) - f(a_k)| < \epsilon$ ($\sum\omega_k < \epsilon$, where ω_k denotes the oscillation of f on $[a_k, b_k]$). If $[a, b] = \bigcup E_k$, such that F is $AC(AC_*)$ on each set E_k, then F is called $ACG(ACG_*)$ on $[a, b]$.

Descriptive definition of the Lebesgue integral: The function F is called the Lebesgue integral of a function f provided that:

(a) F is absolutely continuous on $[a, b]$, and

(b) $F' = f$ a.e.

Descriptive definition of the Denjoy-Perron integral: The function F is called the Denjoy-Perron integral of a function f provided that:

(a) F is ACG_* on $[a, b]$, and

(b) $F' = f$ a.e.

Descriptive definition of the Denjoy-Khintchine integral: The function F is called the Denjoy-Khintchine integral of a function f provided:

(a) F is ACG on $[a, b]$, and

(b) $F'_{ap} = f$ a.e.

For a detailed development of the relevant concepts, the reader is referred to Saks [**157**].

We conclude by observing that the derivatives considered in this section are taken to be finite. This requirement cannot be entirely deleted, for there exist two continuous functions F and G, such that $F' \equiv G'$, yet $F - G$ is not constant. These derivatives are equal to $+\infty$ on the Cantor set, and finite elsewhere. The difference $F - G$ is the Cantor function. The first to notice the existence of two such functions was Hahn [**57**]. See, also, Ruziewicz [**154**], and Saks [**157**: pp. 205, 206].

11. Stationary sets and determining sets. A standard theorem of elementary calculus asserts that if the derivative of a differentiable function vanishes on an interval, then the function is constant. One might ask the question: "On how large a set must the derivative be known to vanish, before it is known to vanish identically?" This leads us to the notion of a stationary set for a class of functions.

DEFINITION. *Let $\mathcal{C}$ be a collection of functions defined on* $[a, b]$. *A subset E of* $[a, b]$ *with the property that whenever $f \in \mathcal{C}$ is constant on E, then f must be constant on* $[a, b]$, *is said to be a stationary set for $\mathcal{C}$.*

For example, if $\mathcal{C}$ consists of the continuous functions on $[a, b]$, then the stationary sets for $\mathcal{C}$ are the dense sets, while if $\mathcal{C}$ consists of the analytic functions, then the stationary sets for $\mathcal{C}$ are those which contain at least one limit point.

If the collection of functions $\mathcal{C}$ is closed under the operation of subtraction, then every stationary set for $\mathcal{C}$ is also a *determining set* for $\mathcal{C}$. That is, two members of $\mathcal{C}$ which agree on this set must agree on all of $[a, b]$.

In recent years, the stationary sets and the determining sets for various classes of derivatives, as well as certain related classes of functions, have been characterized. See Boboc and Marcus [**8**], Bruckner [**12**], Bruckner and Leonard [**16**], Goffman and Neugebauer [**52**], Marcus [**104, 105, 110, 114, 115, 117**], Neugebauer [**134**], Sunyer Balaguer [**171, 172**]. The results of those investigations which bear directly on our subject are tabulated in the chart below, which lists the characterizations of stationary sets and determining sets for various classes of functions defined on $[a, b]$. If A is a set then $m_i(A)$ denotes its Lebesgue inner measure and card (A) its cardinality.

12. Intervals of constancy. An interesting function encountered by students of a course in real variables is the Cantor function f. This function is defined on $[0, 1]$ with $f(0)=0$, $f(1)=1$. It has the property that although it is continuous and nondecreasing on $[0, 1]$, it is constant on every interval contiguous to the Cantor set P. Thus $f'=0$ except on P. It can be shown, however, that f fails to have a derivative, finite or infinite, on a non-denumerable set. A natural question to ask is whether one can in some way "smoothe" the Cantor function to arrive at a differentiable function f such that $f(0)=0$, $f(1)=1$, and $f'=0$ on $\sim P$. The answer is in the negative, in view of the following result due to Zahorski [**192**: p. 21]: *If a continuous nonconstant function f of bounded variation has almost everywhere a vanishing derivative, then f fails to be differentiable on an uncountable set.*

This theorem does not eliminate the possibility that a nonconstant function be differentiable on an interval, yet constant on each interval of a set of intervals whose union is dense in $[0, 1]$. Such functions have actually been constructed; see Zahorski [**194**]. In fact, the following statement is valid [**192**; p. 43]: *A necessary and sufficient condition that E be the set of zeroes of a bounded derivative is that $\sim E$ be an M_4 set.* Using this theorem one can prove the following [**15**]: *Let G be an open dense subset of $[a, b]$ and let $P=\sim G$. A necessary and sufficient condition that there be a differentiable function f defined on $[a, b]$ such that f is constant on each component interval of G, but not constant on any open interval containing points of P, is that the intersection of P with any arbitrary open interval is either empty or has positive measure.*

CLASS OF FUNCTIONS	E IS A STATIONARY SET IF AND ONLY IF	E IS A DETERMINING SET IF AND ONLY IF
I. *Derivatives*		
Derivatives (possibly infinite)	$E = [a, b]$	$E = [a, b]$
Derivatives (possibly infinite) of continuous functions	$m_i(\sim E) = 0$	$m_i(\sim E) = 0$
Finite derivatives	$m_i(\sim E) = 0$	$m_i(\sim E) = 0$
Bounded derivatives	$m_i(\sim E) = 0$	$m_i(\sim E) = 0$
Riemann integrable derivatives	E is dense	E is dense
Bounded semicontinuous derivatives	$m_i(\sim E) = 0$	$m_i(\sim E) = 0$
II. *Approximate derivatives*		
Approximate derivatives (possibly infinite)	$E = [a, b]$	$E = [a, b]$
Approximate derivatives (possibly infinite) of Darboux functions	(See Note Below)	$E = [a, b]$
Approximate derivatives (possibly infinite) of approximately continuous functions	$m_i(\sim E) = 0$	$m_i(\sim E) = 0$
Approximate derivatives (possibly infinite) of continuous functions	$m_i(\sim E) = 0$	$m_i(\sim E) = 0$
Finite approximate derivatives	$m_i(\sim E) = 0$	$m_i(\sim E) = 0$
III. *Dini derivatives*		
Dini derivatives of Darboux Baire functions	E meets every perfect set	$E = [a, b]$
Dini derivatives of continuous functions	E meets every perfect set	E meets every perfect set
Finite Dini derivatives of continuous functions	E meets every perfect set	E meets every perfect set
IV. *Darboux functions*		
Darboux functions	card $(\sim E) < c$	$E = [a, b]$
Measurable Darboux functions	E meets every uncountable measurable set	$E = [a, b]$
Darboux Baire functions	E meets every perfect set	$E = [a, b]$
Darboux Baire class 1 functions	E meets every perfect set	$E = [a, b]$
Lower semicontinuous Darboux functions	E meets every perfect set	E meets every perfect set
Approximately continuous lower semicontinuous functions	$m_i(\sim E) = 0$	$m_i(\sim E) = 0$

Note. A necessary condition for E to be a stationary set for the class of approximate derivatives (possibly infinite) of Darboux functions is that $m_i(\sim E)=0$; a sufficient condition is that E meet every perfect set [12].

We also note that the stationary sets and the determining sets for both the class of approximately derivable functions and the uniform closure of this class are the sets which are dense in the interval $[a, b]$ [134].

13. Monotonicity. According to a theorem of elementary calculus, a differentiable function f whose derivative is nonnegative on an interval I must be nondecreasing on that interval. This theorem has been generalized in many ways. For example, the differentiability of f has been replaced by a weaker regularity condition, the derivative has been replaced by various types of generalized derivative, and the set on which the derivative is assumed to exist, as well as the set on which it is assumed to be nonnegative, has been assumed to be less than all of the interval I. For example, the standard monotonicity theorem which appears in the theory of Lebesgue integration asserts that a function which is absolutely continuous and has a nonnegative derivative a.e. must be nondecreasing. A similar theorem involving the approximate derivative appears in connection with the integral of Denjoy-Khintchine.

In this section we consider several theorems whose conclusions are that a function is nondecreasing. We begin with a theorem of Goldowski [**54**] and Tonelli [**176**] (see also [**157**; p. 206]).

THEOREM. *Let f be a function satisfying the following conditions on the interval I:*

(i) *f is continuous,*
(ii) *f' exists (finite or infinite), except perhaps on a denumerable set,*
(iii) *$f' \geqq 0$ a.e.*

Then f is nondecreasing on I.

We note that condition (ii) cannot be weakened to the condition that the derivative exists except perhaps on a null set. This can be seen by considering the negative of the Cantor function.

In 1939 Tolstoff [**175**] obtained an improvement of the theorem of Goldowski-Tonelli.

THEOREM. *Let f be a function satisfying the following conditions on an interval I:*

(i) *f is approximately continuous,*
(ii) *f'_{ap} exists (finite or infinite) except perhaps on a denumerable set,*
(iii) *$f'_{ap} \geqq 0$ a.e.*

Then f is continuous and nondecreasing on I.

Another generalization of the Goldowski-Tonelli theorem was obtained by Zahorski [**192**: p. 19] in 1950.

THEOREM. *Let f be a function satisfying the following conditions on an interval I:*

(i) *f is a Darboux function,*
(ii) *f' exists (finite or infinite) except perhaps on a denumerable set,*
(iii) *$f' \geqq 0$ a.e.*

Then f is continuous and nondecreasing on I.

We note that Zahorski's Theorem is stronger than Tolstoff's in so far as Zahorski assumed only Darboux continuity instead of approximate continuity of f. On the other hand, his theorem is weaker in so far as conditions (ii) and (iii) involve the ordinary derivative instead of the approximate derivative. We would like a theorem which implies both Tolstoff's Theorem and Zahorski's Theorem. An obvious candidate for such a theorem is obtained by considering the weaker of the corresponding conditions of the two theorems. That is, must a Darboux function satisfying conditions (ii) and (iii) of Tolstoff's Theorem be nondecreasing? This question is answered in the negative, as can be seen by considering the example below. This example is a slight modification of an example found in [**191**: pp. 321, 322].

Example. Let f be a function satisfying the following conditions on the interval (0, 1).

(i) If (a, b) is an interval contiguous to the Cantor set then $f(a)=0, f(b)=1$ and f is continuous and nondecreasing on $[a, b]$.

(ii) If x is a two sided limit point of the Cantor set then $f(x)=1$.

(iii) Every two sided limit point of the Cantor set is a point of density of the set $\{x: f(x)=1\}$.

It is not difficult to verify that this function has the required properties.

So our first attempt to obtain a simultaneous generalization of the two theorems fails. What next? We note that hypothesis (i) of Tolstoff's Theorem implies that f be in Baire class 1. The same is true of hypothesis (ii) in Zahorski's Theorem. (The function in our example is in Baire class 2, but not in Baire class 1.) What happens if we add the requirement that f be in Baire class 1? That is, if f is a Darboux function in Baire class 1 and satisfies conditions (ii) and (iii) of Tolstoff's Theorem, must f be nondecreasing? This question was asked by Zahorski [**192**: p. 8]. It turns out that this question has an affirmative answer [**201, 202, 214**], thus providing a theorem which includes both the theorem of Tolstoff and the theorem of Zahorski. In fact, the following more general theorem is valid [**201, 202**].

THEOREM. *Let* $\mathcal{P}$ *be a function-theoretic property sufficiently strong to imply*

(a) *Any Darboux function in Baire class* 1 *which satisfies property* $\mathcal{P}$ *on an interval* I *is VBG on* I [**157**: p. 221].

(b) *Any continuous function of bounded variation which satisfies property* $\mathcal{P}$ *on* I *is nondecreasing on* I.

Then any Darboux Baire 1 *function which satisfies property* $\mathcal{P}$ *on* I *is continuous and nondecreasing on* I.

To see that this theorem provides an affirmative answer to the question raised by Zahorski, we let $\mathcal{P}$ be the property of having, except perhaps on a denumerable set, an approximate derivative (finite or infinite) which is nonnegative a.e. Condition (a) follows from 10.8 of [**157**: p. 237] and condition (b) is a consequence of Tolstoff's Theorem.

Roughly speaking, the theorem states that if one wishes to show that a condition is strong enough to guarantee that every Darboux Baire 1 function satisfying the condition is nondecreasing, one need only show that every continuous function of bounded variation which satisfies the condition is nondecreasing. (Condition (a) is likely to be satisfied if the condition is at all "reasonable.") For example, one can use the theorem to show that Tolstoff's Theorem remains valid if approximate continuity is replaced by preponderant continuity and the approximate derivative is replaced by the preponderant derivative [**207**].

We conclude this section with two theorems concerning Dini derivatives.

Let f be a function defined on $[a, b]$ *which satisfies*

(a) $\limsup_{\xi\to x-} f(\xi) \leqq f(x) \leqq \limsup_{\xi\to x+} f(\xi)$ *for all* $x \in [a, b]$,

(b) $D^+f \geqq 0$ a.e. *on* $[a, b]$, *and*

(c) $D^+f > -\infty$ *except possibly on a denumerable set.*

Then f is nondecreasing.

This theorem is due to Gál [**41**]. Once more, none of the hypotheses of the theorem can be deleted with the conclusion still valid. Other theorems of this type, dealing with continuous functions, have been advanced by Garg [**43**] and Ważewski [**183**]. We state one such [**43**]: *Let f be a continuous function fulfilling Banach's condition* (T_2) (see Section 9). *Let* $Q = \{x: D^+f(x) < 0\}$. *If* $mf(Q) = 0$, *then f is nondecreasing.*

14. Derivatives and Darboux functions of Baire class 1. It was mentioned in Section 2 that every derivative belongs to Baire class 1 and possesses the Darboux property. The converse is not valid. Thus the requirement that a function f be a derivative is more stringent than the requirement that f be a Darboux Baire class 1 function. How much more stringent? On the one hand, Maximoff [**120**] has shown that the derivatives and the functions in Darboux Baire class 1 are topologically equivalent in the sense that any Darboux Baire class 1 function defined on $[a, b]$ can be transformed into a derivative by suitably transforming $[a, b]$ onto itself topologically. (See also [**24, 113**].) On the other hand, the two classes of functions exhibit quite different properties. We turn now to a consideration of some of these differences.

For α a real number and f any function defined on $[a, b]$ let $E_\alpha(f) = \{x: f(x) > \alpha\}$ and $E^\alpha(f) = \{x: f(x) < \alpha\}$. We mentioned in Section 4 that Zahorski considered such sets in trying to characterize derivatives. He was able to show that a necessary and sufficient condition that f be a Darboux Baire class 1 function is that each such set be an F_σ set with the property that each point of the set be a bilateral point of condensation of the set (he called this condition M_1). On the other hand, a *necessary* condition for a function f to be a derivative (possibly infinite) of a continuous function is that for every α, $E_\alpha(f)$ and $E^\alpha(f)$ be sets satisfying the condition which he called M_2. (A set E satisfies M_2 if E is an F_σ, and every one-sided neighborhood of each point in E intersects E is a set of positive measure.) This condition is not sufficient. Thus, each set of the type $E_\alpha(f)$ and $E^\alpha(f)$ must be considerably "more dense" near its members in order for f

to be a derivative (possibly infinite) than in order for f to be merely a Darboux Baire class 1 function. By requiring even more density of the sets $E_\alpha(f)$ and $E^\alpha(f)$, Zahorski managed to find necessary conditions for a function to be a *finite* derivative (condition M_3). The analogous necessary condition for f to be a *bounded* derivative is the still more stringent density condition that the sets $E_\alpha(f)$ and $E^\alpha(f)$ all be M_4 sets (see Section 4 for a definition of M_3 and M_4 sets). A *sufficient* density condition that the bounded function f be a derivative is that for every α, every point of $E_\alpha(f)(E^\alpha(f))$ be a point of (unit) density of $E_\alpha(f)$ (resp. $E^\alpha(f)$). This amounts to saying that the function is approximately continuous. The converse is, of course, not true; there exist bounded derivatives which are not approximately continuous. However, a partial converse, which characterizes approximately continuous functions, has been given by Lipiński [**94**]: *The function f is approximately continuous if and only if for every a and b the function $f_{ab}(x) = \max\{a, \min[b, f(x)]\}$ is a derivative.*

We mentioned in Section 4 that the question of *characterizing* the class of derivatives in terms of the set E_α and E^α has not yet been resolved. In this connection it should be mentioned that for the case of bounded derivatives no characterization solely in terms of the structure of the individual sets E^α and E_α is possible. This can be seen in the following way. Zahorski showed [**192**: pp. 45–47] that there are functions f, which are *not* bounded derivatives, but such that for all α the sets $E^\alpha(f)$ and $E_\alpha(f)$ are M_4 sets. Thus, if there were a condition of the type desired, it would have to be *more* stringent than the condition M_4. On the other hand, on page 35 one finds the result that for every M_4 set E there exists a bounded derivative f and a number α such that $E = E^\alpha(f)$. Thus the desired condition *cannot* be more stringent than M_4.

Some additional results concerning the sets $E^\alpha(f)$ and $E_\alpha(f)$ and M_k sets, $k=2, 3, 4$, can be found in Lipiński [**91**, **93**, **95**]. The results of Zahorski concerning the sets $E_\alpha(f)$ and $E^\alpha(f)$ for Darboux Baire 1 functions have been extended to more general spaces by Mišik [**129**].

Another kind of comparison involving convergent interval functions was advanced by Neugebauer [**135**]. In this article the author gives characterizations of each of the two classes presently under consideration. This is done in such a way as to allow an interesting comparison between the two classes. Two conditions are stated. The first one is necessary and sufficient for a function to be a Darboux function of Baire class 1, whereas the two conditions together are necessary and sufficient for a function to be a derivative. Thus, it is precisely the second condition which shows how much more stringent a requirement it is for a function f to be a derivative than it is for f to be a Baire class 1 Darboux function. A precise formulation of the relevent theorems would require more space than is appropriate here, so we omit the details.

As we saw in Section 11, a necessary and sufficient condition that a set E be stationary for the class of derivatives is that $\sim E$ have inner measure zero [**117**], whereas a necessary and sufficient condition that E be stationary for the class of Darboux Baire class 1 functions is that $\sim E$ be totally imperfect [**16**], that is, that $\sim E$ contain no nonempty perfect subset. For purposes of comparison, we mention that every totally imperfect set must have zero inner measure, but the converse statement is false. It is possible, however, for a totally imperfect set to have positive outer measure. In fact, the interval $[a, b]$ can be decomposed into two non-overlapping totally imperfect sets [**83**; p. 422]. It is clear that each of these sets must have outer measure equal to $b-a$. Similarly, the determining sets for the class of derivatives are those whose complements have zero inner measure, whereas the only determining set for the Darboux Baire class 1 functions is the interval $[a, b]$.

The remaining comparisons involve the algebraic and topological structures of the two classes. We begin by observing that the sum of two derivatives is again a derivative. The corresponding statement is not valid for the Darboux functions of Baire class 1. Thus let $f(x) = \sin(1/x)$, $f(0)=1$ and let $g(x) = -\sin(1/x)$, $g(0)=1$. Then $(f+g)(x)=0$, $(f+g)(0)=2$. The functions f and g are in the required class, but their sum is not.

On the other hand, if f is a Baire class 1 Darboux function, then so is f^2. This follows from the facts that a continuous function of a Baire function preserves the Baire class and a continuous function of a Darboux function is again Darboux. But the corresponding statement is not valid for derivatives. In fact if f is a square summable derivative, then f^2 is a derivative if and only if

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)|^2 \, dt = 0 \quad \text{for all } x.$$

(See Iosifescu [**66**].) Some interesting related results may be found in Hruška [**205**], Iosifescu [**206**], Iosifescu and Marcus [**69**], Neugebauer [**137**], Sélivanoff [**149**], Wilkosz [**187**] and Wolff [**188**].

For bounded functions we can state a simpler result: *If f and f^2 are bounded on $[a, b]$, then both functions are derivatives if and only if f is approximately continuous* [**187**]. These results show that the square of a derivative (even a bounded derivative) need not be a derivative. We also see that even though a continuous function of a Darboux Baire class 1 function is still in that class, the corresponding result for derivatives is not valid. A sufficient condition for such a composition can be found in Choquet [**24**; p. 89]: *If g is a lower semicontinuous derivative and f is a continuous function of bounded variation on $(-\infty, \infty)$, then the function $f \circ g$ is a derivative.* It is true that this theorem puts considerable restrictions on both f and g. It would be of interest to know just how much these restrictions can be weakened. To give some slight indication of the difficulties that arise if we put no restrictions (other than that of being a derivative) on g, we state the following result found in [**24**; p. 89]: *If f is nondecreasing and continuous, and such that for every bounded derivative g the function $f \circ g$ is still a derivative, then f must be linear.*

A certain other comparison has, to the best of our knowledge, not yet been resolved. Let $\{f_n\}$ be a sequence of continuous functions converging pointwise to a limit function f. We know that if each f_n is continuous and the convergence is *uniform*, then f is also continuous. Uniform convergence is, of course, not *necessary* for the limit function to be continuous. It has been known for a long time that a necessary and sufficient condition for the limit function to be continuous is that the convergence be quasi-uniform (see Hahn [**56**] for the relevant definition and theorem). One might ask for the corresponding types of convergence for the class of derivatives and for the class of Darboux Baire 1 functions: If $\{f_n\}$ is a sequence of derivatives (respectively Darboux Baire 1 functions) converging pointwise to a limit function f, then what additional restriction on the convergence is necessary and sufficient to guarantee that f also be a derivative (respectively Darboux Baire 1 function)? In this connection we mention that uniform convergence is sufficient in each case, but not necessary. The proof for derivatives is straightforward, and the proof for Darboux Baire 1 functions can be found in [**14**]. The relevant kind of convergence for functions in Baire class α (for fixed α) has been obtained by Gagaeff [**40**]. The results of Oeconomidis [**141, 142**] are relevant to this question for derivatives.

We conclude with a precise statement of Maximoff's deep theorem which we mentioned at the beginning of this section. (See also [**24**; p. 90].)

THEOREM. *Let f be a finite Darboux function of Baire class* 1 *on the interval* $[0, 1]$. *Then there exists a strictly increasing continuous function g such that* $g(0)=0$, $g(1)=1$ *and $f \circ g$ is a derivative.* (See [**120, 122**].)

Marcus [**113**] and Lipiński [**89**] consider some consequences of this theorem.

In [**121**], Maximoff showed that the word "derivative" can be replaced by the words "approximately continuous function" in the conclusion of the theorem.

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