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To cite this article:
R. H. Byrd, A. J. Goldman, Miriam Heller, (1987) Technical Note—Recognizing Unbounded Integer Programs. Operations Research 35(1):140-142. http://dx.doi.org/10.1287/opre.35.1.140

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TECHNICAL NOTE

RECOGNIZING UNBOUNDED INTEGER PROGRAMS

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(Received July 1982; revisions received August 1985, February 1986; accepted September 1986)

If an integer program (IP) has an unbounded continuous relaxation, is the IP also unbounded? We find the answer to be "no" in general but "yes" when the IP is feasible and has rational data. We also discuss related geometrical and algorithmic considerations.

Consider an integer linear program, denoted IP and written in the familiar form

$$\text{maximize} \quad cx \tag{1}$$

$$\text{subject to} \quad Ax = b, \tag{2}$$

$$x \geqslant 0, \tag{3}$$

$$x \text{ integral}, \tag{4}$$

where A is an $m \times n$ real matrix. Standard methods for treating IP begin by attempting to solve its continuous relaxation, denoted LP and obtained from IP by dropping constraint (4). That attempt may fail, however. If LP is infeasible, then so is IP. But what if LP is found to be unbounded? This natural question, which the writers found addressed in only a single optimization text (Papadimitriou and Steiglitz 1982, p. 321), prompted the present note.

For a first pertinent example, take $n = 4$ and $m = 2$, and let the constraints (2) consist of

$$x_3 - \sqrt{2}\,(x_1 - x_2) = 0, \quad x_2 + x_4 = 1. \tag{2'}$$

Then the constraint set defined by (2′) and (3) is unbounded, containing in particular all $(\alpha, 0, \alpha\sqrt{2}, 1)'$ with $\alpha \geqslant 0$, and so the objective function x_1 is unbounded on this set. But any *integral* x in this constraint set must have $x_1 = x_2$ and $x_3 = 0$ because of the first condition in (2′), and must have $x_2 \in \{0, 1\}$ by the second condition together with (3); thus the only solutions to (2′), (3) and (4) are $(0, 0, 0, 1)'$ and $(1, 1, 0, 0)'$, so that IP is bounded.

This example shows that the unboundedness of IP cannot in general be inferred from that of LP. However, the example includes irrational problem data. Such a situation is unlikely to arise in applications, where data tend to be either small integers known exactly, or rational numbers representing inexact measurements. And it could not arise in presenting IP to a digital computer program for solution. Fortunately, for problems whose data are rational, there is a positive result:

Theorem 1. *If LP has rational entries in matrix A and its constraint set (objective function) is unbounded, then IP is either infeasible or else also has an unbounded constraint set (objective function).*

Proof. (a) Since the convex polyhedral set $S = \{x \in R^n\colon Ax = b,\ x \geqslant 0\}$ is unbounded, it must contain a closed half-line, i.e., a set of the form $H = \{x^0 + ty\colon t \geqslant 0\}$ for some $x^0 \in R^n$ and nonzero $y \in R^n$. (This conclusion follows, for example, from the representation of S as the vector sum of a bounded convex polyhedron and a polyhedral convex cone; see Theorem 1 of Goldman 1956.) The inclusion of H in

Subject classification: 639 inferring integer program unboundedness from continuous relaxation

0030-364X/87/3501-0140 $01.25
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S requires that y obey

$$Ay = 0, \quad y \geqslant 0. \tag{5}$$

Since this system has a real nonzero solution y, and has rational data, it must have a *rational* nonzero solution y^*: for example, as constructed by applying the simplex method—which employs only rational operations—to the (unbounded) maximization of $y_1 + \ldots + y_n$ subject to (5). Since (5) is homogeneous, we can multiply y^* by a suitable positive integer ("clear of denominators") to obtain an *integral* solution of (5), which will also be denoted y^*.

If IP is assumed feasible, then there is an x^* satisfying (2), (3) and (4). The "integral half-line"

$$H^* = \{x^* + ty^*: t \geqslant 0 \text{ and integral}\}$$

is then readily seen to lie in IP's constraint set, which is therefore unbounded.

(b) Now suppose, further, that LP has an unbounded objective function. Then the half-line H of (a) can be chosen so that y satisfies not only (5) but alsy $cy > 0$, and furthermore is rational. (This conclusion is valid since the rational operations of the simplex method, applied to LP, in fact identify the "unbounded optimum" case by calculating from the entries of A a vector y satisfying $cy > 0$ as well as (5).) As in (a), we can clear y of denominators to obtain an integral y^*, satisfying (5) and $cy^* > 0$, which in turn yields an "integral half-line" H* of feasible points for IP along which cx is unbounded.

The possibility that IP is infeasible cannot be omitted from the statement of the theorem, as can be seen by taking $n = 2$, $m = 1$, $c = (1.0)'$, and (2) to consist of

$$x_1 - x_2 = 1/2;$$

here the problem-data are rational and LP is unbounded, but IP is infeasible.

Although Theorem 1 is apparently not well known, it is not new: Meyer (1974; Section 5) deduces it from deeper results (in a mixed-integer setting, with reference to Rubin 1970 for the pure-integer case). We found ourselves (Byrd, Goldman and Heller 1982) anticipated, in the preceding more direct proof, by Papadimitriou (1981, Lemma 3).

We can, however, provide some new information about the situation illustrated by (2′), in which LP was unbounded but IP bounded. (By Theorem 1, this situation can occur only when some problem data are irrational.) For this case, we will show that the continuous relaxation LP of IP is a very "loose" relaxation, in that it exaggerates the (linear) dimension of the feasible set.

Theorem 2. *If* $S = \{x \in R^n: Ax = b,\ x \geqslant 0\}$ *is unbounded, yet contains only a finite (but nonempty) set of integer points, then this set lies in a proper submanifold of the smallest linear manifold, L, that contains* S.

Proof. Suppose, to the contrary, that S contains integer points $\{u_k\}_{k=1}^K$ that do not lie in any proper submanifold of L. Then, by a standard result in n-dimensional analytic geometry, the points of L can be represented parametrically as $x = \sum_{k=1}^K \lambda_k u_k$ with $\sum_{k=1}^K \lambda_k = 1$, or equivalently, as $x = U\lambda$ with $e\lambda = 1$, where U is the matrix with u_k as its kth column and e is the $1 \times K$ vector of ones. This representation suggests forming in R^{n+2K} the convex polyhedral set

$$S^* = \{(x, \lambda^+, \lambda^-) \in R^{n+2K}: x - U(\lambda^+ - \lambda^-) = 0,$$
$$e(\lambda^+ - \lambda^-) = 1,\ x \geqslant 0,\ \lambda^+ \geqslant 0,\ \lambda^- \geqslant 0\},$$

which is nonempty because S is. (We abbreviate $(x', (\lambda^+)', (\lambda^-)')'$ as $(x, \lambda^+, \lambda^-)$, with similar abbreviations elsewhere.)

Since S is unbounded, at least one variable x_j is unbounded over S*. This variable defines the objective function of a linear program that is unbounded over S*. Since this program's data are integer, the preceding theorem applies to it, and so the corresponding program in *integer* variables $(x, \lambda^+, \lambda^-)$ is either unbounded or infeasible. But unboundedness is ruled out by the boundedness of the set of integer points in S, while infeasibility is ruled out by the presence in S* of the K points $(u_k, e_k, 0)$ where e_k is the kth unit K-dimensional row vector. So the desired contradiction has been obtained.

Finally, we revert to the situation of Theorem 1, in which the problem data are rational and LP has been found to be unbounded. Presumably, one would want to know whether IP was infeasible or unbounded—the only two alternatives, according to the theorem. We conclude with a brief discussion of that question.

First, since the data of (2)–(4) are rational (this is our first appeal to the rationality of b), we can clear denominators in (2) and obtain an equivalent set of constraints in which the (new) A and b are *integral.* Now (2), (3), (4) call for nonnegative solutions of a system of linear Diophantine equations. Determining the feasibility of such a problem would admit a polynomial algorithm (see Kannan and Bachem 1979, and Chou and Collins 1982) if the nonnegativity con-

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straints (3) were absent. But with them present, the question is known in general to be NP-hard (Sahni 1974, p. 277). This conclusion does not settle the matter, however, since in the situation at hand we would have the further information that the constraint set of LP is unbounded, and in fact would know (from the simplex method applied to LP) a specific half-line lying in that constraint set. Might not this additional knowledge somehow ameliorate the computational complexity of the feasibility question?

To rule out this possibility rigorously, we define a problem P that is polynomially reducible to another problem P′. An instance of P requires determining the feasibility in integers of $Cx = d$, $x \geqslant 0$ with $Cx = d$ a system of linear Diophantine equations, while an instance of P′ has the same form *except* that a half-line l feasible for the continuous relaxation of $\{Cx = d, x \geqslant 0\}$ is "given" at the outset. Since as noted previously P is known to be NP-hard, this reducibility implies that P′ is also NP-hard.

To establish the reducibility, consider any instance I of P, with data (C, d) and with, say, n variables. We will form from I, with only polynomial effort, an instance I' of P′ with data (C', d, l), for which it is obvious that I is feasible if and only if I' is. Adjoin to $Cx = d$ a new $(n+1)$st variable with zero coefficient in every equation, thus obtaining a Diophantine system $C'u = d$. The direction of half-line l is given by the $(n+1)$st unit vector in R^{n+1}; it remains only to specify the base point of l. A point in R^{n+1} is a suitable base point if and only if it has the form $(\bar{x}, \theta)$ for some $\bar{x} \in R^n$ feasible for the continuous relaxation of I and for some arbitrary $\theta \geqslant 0$. To obtain such an $\bar{x}$ and thus complete the specification of I', we apply a polynomial effort algorithm (e.g., the ellipsoid method) to the continuous relaxation of I; if the outcome is "infeasible," then I itself is infeasible. So polynomial reducibility is proven.

(We concur with a referee's observation that the preceding argument, though correct, is a "cheap shot." It would be desirable to investigate whether some contextually "natural" restrictions, on instances I of P or on how instances of I' of P′ could be formed from them, might possibly overturn the conclusion of NP-hardness.)

Given the unsatisfying outcome of the prior discussion, one may hope that, in applications for which unboundedness (hence, an infinity of feasible solutions) of an IP is at issue, the existence of at least one feasible solution would normally be apparent.

Acknowledgments

The writers are grateful to J. S. Provan, L. E. Trotter and the referees for helpful suggestions. Participation by Professors Byrd and Goldman was supported under National Science Foundation grants MCS-81-15475 and ECS-81-10035, respectively.

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