Advanced Algebra and Calculus
Degree in Data Science and Egineering

Unit 2: Fourier

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Outline

Review of norms and scalar product

Recall that in the vector space $\mathbb{R}^n$ the scalar (dot) product of two vectors $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ is defined as:

$$\boldsymbol{x} \cdot \boldsymbol{y} = \langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^{n} x_i y_i$$

and has the following properties: bilinear, symmetric and positive definite.

The euclidean norm is defined as:

$$\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle} = \sqrt{\sum x_i^2},$$

and must be interpreted as the size of the vector $\boldsymbol{x}$.

Using the norm one can define the euclidean distance

$$d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|,$$

and orthogonality and angle between two vectors

$$\boldsymbol{x} \perp \boldsymbol{y} \Leftrightarrow \langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0, \qquad \cos\left(\widehat{\boldsymbol{x}\boldsymbol{y}}\right) = \frac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}{\|\boldsymbol{x}\| \|\boldsymbol{y}\|}.$$

The euclidean norm has the following properties:

- $\|\boldsymbol{x}\| \geq 0, \quad \|\boldsymbol{x}\| = 0 \Leftrightarrow \boldsymbol{x} = \boldsymbol{0}$;
- $\|\lambda \boldsymbol{x}\| = |\lambda| \|\boldsymbol{x}\|$ for $\lambda \in \mathbb{R}$;
- $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$ (triangular inequality).

Other norms $\|\cdot\| \colon \mathbb{R}^n \to \mathbb{R}$ satisfy the same properties, for example:

- The maximum (or supremum) norm $\|\boldsymbol{x}\|_\infty = \max\big\{|x_i| : 1 \leq i \leq n\big\}$;
- The sum of components or taxicab norm $\|\boldsymbol{x}\|_1 = \sum_{i=1}^n |x_i|$.

They can be used to define other distances in $\mathbb{R}^n$:

$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|_\infty, \qquad\qquad d_1(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|_1.$$

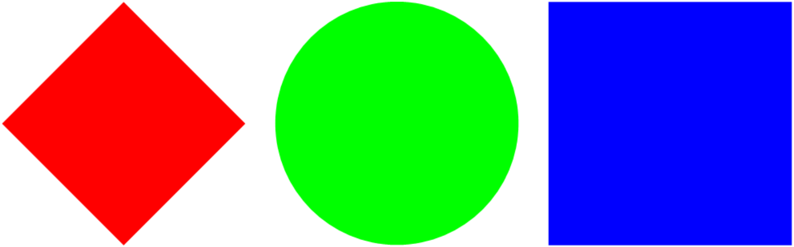
More generally, a distance on an arbitrary set X is a map $d \colon X \times X \to \mathbb{R}$ satisfying the properties:

- $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$ (positiveness);
- $d(y, x) = d(x, y)$ (symmetry);
- $d(x, z) \leq d(x, y) + d(y, z)$ (triangular inequality).

The balls centered at a point $\boldsymbol{x}_0 \in \mathbb{R}^2$ of given radius r

$$\{\boldsymbol{x} \in \mathbb{R}^2 : \|\boldsymbol{x} - \boldsymbol{x}_0\| < r\}$$

have the following shape for the norms $\|\boldsymbol{x}\|_1$, $\|\boldsymbol{x}\|_2$ (euclidean) and $\|\boldsymbol{x}\|_\infty$:



Spaces of functions

Let $I \subseteq \mathbb{R}$ be an interval, it can be open or closed at the endpoints, finite or infinite. Consider sets of functions $f : I \to \mathbb{R}$ satisfying certain property that are closed by sums and product by scalars. For example,

- all functions $I \to \mathbb{R}$;
- functions $I \to \mathbb{R}$ that are bounded;
- continuous functions $I \to \mathbb{R}$;
- functions of class $\mathcal{C}^1$: derivable with continuous derivative;
- functions of class $\mathcal{C}^\infty$ (smooth functions) that are infinitely derivable;
- (absolutely) integrable functions $L^1(I)$: functions with
$$\int_I \left| f(t) \right| dt < \infty;$$
- square integrable functions $L^2(I)$: functions with
$$\int_I \left| f(t) \right|^2 dt < \infty.$$

These sets are vector spaces over the field $\mathbb{R}$ of infinite dimension.

Norms in spaces of functions

In some of these spaces one can define norms analogous to those in $\mathbb{R}^n$ in order to measure the "size" of functions.
The general idea is to replace finite sums with integrals. For example:

- in the space of bounded functions the supremum norm:
$$\|f\|_\infty = \sup\left\{|f(x)| : x \in I\right\};$$
- in the space $L^1(I)$ the integral norm or L^1-norm:
$$\|f\|_1 = \int_I |f(x)|\,dx;$$
- in the space $L^2(I)$ the L^2-norm:
$$\|f\| = \|f\|_2 = \sqrt{\int_I |f(x)|^2\,dx},$$
analogous to the euclidean norm in the finite-dimensional space $\mathbb{R}^n$.

Dot product of functions

Moreover, in the space of square integrable functions $L^2(I)$ one can define the *dot (scalar) product*, also in analogy with the finite-dimensional case:

$$\langle f, g \rangle = \int_I f(x)g(x)\,dx, \qquad f, g \in L^2(I),$$

from which the L^2-norm is expressed as:

$$\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_I |f(x)|^2\,dx}.$$

These norms and scalar products can be used, as in the finite-dimensional case, to define distances between functions, orthogonality and angles.

Technical remark about L^1 and L^2-norms: they are called seminorms because some nonzero functions have integral (hence norm) equal to zero.

Due to that sometimes the spaces $L^1(I)$ and $L^2(I)$ are considered with the identification of functions whose difference has zero integral.

Convergence of sequences and series of functions

Many interesting numbers appear in natural ways as limits of sequences or sums of numerical series. For example,

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \sum_{n=0}^{\infty}\frac{1}{n!}.$$

$$\pi = \lim_{n\to\infty}\frac{4}{n^2}\sum_{k=1}^{n}\sqrt{n^2-k^2} = \sum_{n=0}^{\infty}\frac{2^{n+1}n!^2}{(2n+1)!}.$$

In the same way functions can be obtained as limits of sequences or sums of series of functions:

$$f = \lim_{n\to\infty} f_n, \qquad g = \sum_{n=0}^{\infty} g_n = \lim_{n\to\infty}\left(\sum_{k=0}^{n} g_k\right).$$

The limits of functions may be considered *pointwise* or with respect to some norm.

Pointwise limit means that the values of the f_n tend to the value of f at every point:

$$\lim_{n\to\infty} f_n(x) = f(x), \qquad \text{for every} \quad x \in I \subseteq \mathbb{R}.$$

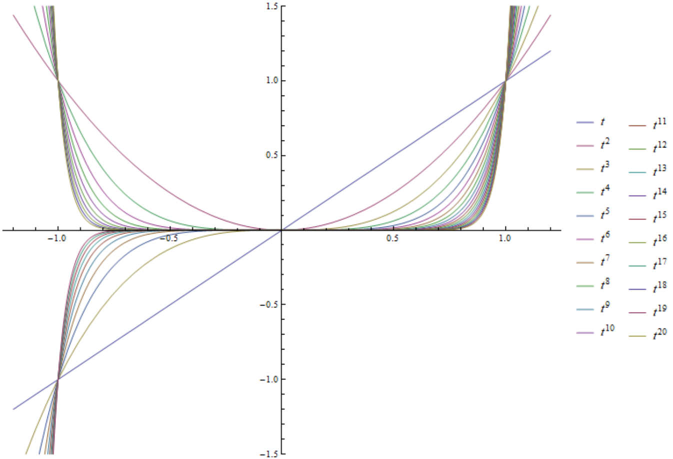
When the functions f_n and f belong to a certain space of functions in which a norm $\|\cdot\|$ is defined, the limit of (f_n) is f with respect to that norm when the functions f_n get close to f when n is large:

$$\lim_{n\to\infty} \|f_n - f\| = 0.$$

The limit with respect to the supremum norm $\|\cdot\|_\infty$ is called *uniform limit*. Uniform convergence implies pointwise convergence at every point.

It may happen that a sequence converges in the L^1-norm or in the L^2-norm but does not converge pointwise at some points.

Example: consider the functions f_n defined by $f_n(t) = t^n$ on some real interval $I \subseteq \mathbb{R}$.



- The (f_n) are not pointwise convergent at points t with $|t| > 1$ nor at $t = -1$; at $t = 1$ they converge to 1 and at $t \in (-1, 1)$ converge to 0.

- Consider the interval $I = [0, 1]$ and the two functions

$$f(t) = \begin{cases} 0, & t \neq 1, \\ 1, & t = 1, \end{cases} \qquad \text{and} \qquad g(t) = 0 \quad \forall t \in [0, 1]$$

  In the interval I the sequence $(f_n)_{n \geq 1}$

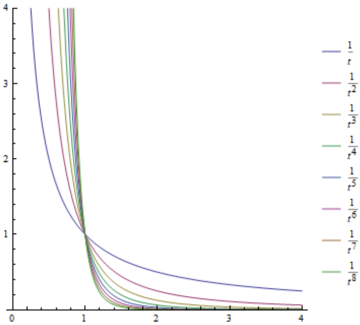
  - converges pointwise to the function f but not to the function g;
  - does not converge uniformly to f nor to g (the supremum norms $\|f_n - f\|_\infty$ and $\|f_n - g\|_\infty$ are always equal to 1);
  - converges in norm L^1 and norm L^2 to both f and g; these two functions are identified in the two spaces.

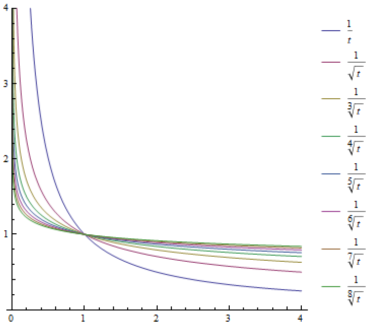
- In every interval $I = [0, b]$ for $0 < b < 1$ the sequence converges pointwise, uniformly and in norms L^1 and L^2 to the function f (which is equal to g in this interval).

Example: consider the negative power functions $g_u \colon [0, \infty) \to \mathbb{R}$ defined as

$$g_u(t) = \begin{cases} \frac{1}{t^u}, & \text{if} \quad t \neq 0, \\ 0, & \text{if} \quad t = 0, \end{cases}$$

for real (not only integer) exponents $u > 0$:





Integration over intervals $[a, b]$ gives:

$$\int_a^b t^{-u}\,dt = \begin{cases} \left[\frac{t^{-u+1}}{-u+1}\right]_a^b, & \text{if} \quad u \neq 1, \\ [\log(t)]_a^b, & \text{if} \quad u = 1. \end{cases}$$

Over the interval $I = [1, \infty)$ the integral is:

$$\int_1^\infty t^{-u}\,dt = \begin{cases} \frac{1}{1-u}, & \text{if} \quad u > 1, \\ \infty, & \text{if} \quad 0 < u \leq 1. \end{cases}$$

Since $g_u(t)^2 = \frac{1}{t^{2u}} = g_{2u}(t)$ it follows that:

- $g_u(t)$ belongs to $L^1\big([1, \infty)\big)$ if, and only if, $u > 1$;
- $g_u(t)$ belongs to $L^2\big([1, \infty)\big)$ if, and only if, $2u > 1 \Leftrightarrow u > \frac{1}{2}$.

In particular $f(t) = g_1(t) = \frac{1}{t}$ is not integrable but is square integrable.
The same is true for the functions $f_n(t) = \frac{1}{t^{1-1/n}}$ for all exponents $n \geq 3$.

The sequence (f_n) converges to f in L^2-norm but not in L^1-norm. Indeed,

- all distances $\|f_n - f\|_1$ are ∞: this means that the areas between f and the f_n are infinite; but
- the functions $(f_n - f)^2$ are integrable with integral

$$\begin{aligned}\int_1^\infty \left(\frac{1}{t^{1-1/n}} - \frac{1}{t}\right)^2 dt &= \int_1^\infty \left(\frac{1}{t^{2-2/n}} - \frac{2}{t^{2-1/n}} + \frac{1}{t^2}\right) dt \\ &= \left[-\frac{n}{(n-2)t^{1-2/n}} + \frac{2n}{(n-1)t^{1-1/n}} - \frac{1}{t}\right]_1^\infty \\ &= \frac{-n}{n-2} + \frac{2n}{n-1} - 1 = \frac{2}{n^2-3n+2}.\end{aligned}$$

and it follows that

$$\lim_{n\to\infty} \|f_n - f\|_2 = 0.$$

The sequence (f_n) converges to f also pointwise and uniformly:

$$\lim_{n\to\infty} f_n(t) = f(t) \quad \forall t \in [0,\infty), \qquad \text{and} \qquad \lim_{n\to\infty} \|f_n - f\|_\infty = 0$$

Something completely analogous happens in the interval $[0, 1]$.
The integral is:

$$\int_0^1 t^{-u}\, dt = \begin{cases} \frac{1}{u-1}, & \text{if} \quad 0 < u < 1, \\ \infty, & \text{if} \quad u \geq 1. \end{cases}$$

Recalling that $g_u(t)^2 = \frac{1}{t^{2u}} = g_{2u}(t)$, now one has:

- $g_u(t)$ belongs to $L^1\big([0, 1]\big)$ if, and only if, $u < 1$;
- $g_u(t)$ belongs to $L^2\big([0, 1]\big)$ if, and only if, $2u < 1 \Leftrightarrow u < \frac{1}{2}$.

The function $f(t) = g_{\frac{1}{2}}(t) = \frac{1}{\sqrt{t}}$ is integrable but is not square integrable over the interval $[0, 1]$ since $f(t)^2 = g_1(t)$.

The same is true for the functions $f_n(t) = \frac{1}{t^{1/2+1/n}}$ for all exponents $n \geq 3$.
The sequence (f_n) converges to f in L^1-norm but not in L^2-norm.

Series of functions

For many applications it is convenient to write functions of interest as sums of series of a class of very simple functions.

One fixes an infinite family of basic simple functions $\varphi_n(x)$ and tries to write a given function f as a sum of a series of the type

$$f(x) = \sum_{n=0}^{\infty} c_n \varphi_n(x), \quad c_n \in \mathbb{R}.$$

This expression can also be thought of as an infinite linear combination of the basic functions $\varphi_n(x)$, with coefficients c_n which are interpreted as the "amount of the basic function φ_n" needed to get the function f.

Once the basic functions φ_n are fixed, the function f can be given just by listing the coefficients $c_0, c_1, c_2, c_3, \ldots$
Sometimes the indices n spread over all integers:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \varphi_n(x) = \sum_{n \in \mathbb{Z}} c_n \varphi_n(x).$$

The choice of the basic functions φ_n depends on the application, and gives rise to several types of series and of mathematical theories: *power series*, *trigonometric series*, *Fourier series*, *wavelet series*, etc.

Finite partial sums

$$f_N(x) := \sum_{n=0}^{N} c_n \varphi_n(x) \quad \text{or} \quad \sum_{n=-N}^{N} c_n \varphi_n(x)$$

with sufficiently large N produce approximations to $f(x)$ with any desired accuracy, and in practice is enough to work with the f_N for large N. Hence, only a finite number of coefficients $c_0, c_1, c_2, \dots, c_N$ are used in practice to work with (an approximation of) the function f.

The type of convergence considered for the equality $f = \sum c_n \varphi_n$ depends on the context: pointwise, uniform (sup norm), in L^1 or L^2 norm, etc.

Example: power series

This is the simplest example. The powers $\varphi_n(x) = x^n$ are the basic functions, and give rise to the *power series* (or Taylor series):

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = \sum_{n=0}^{\infty} c_n x^n, \quad c_n = \frac{f^{(n)}(0)}{n!},$$

which can be seen just as polynomials of infinite degree.
They converge pointwise in an interval of the form $(-R, R)$ for some *radius of convergence* R.

Well known examples are the *geometric series* and the *logarithm*:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} 1 \cdot x^n,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n,$$

representing the functions in the interval $x \in (-1, 1)$. Namely, $R = 1$.

and also the exponential and trigonometric functions:

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}\, x^n, \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, x^{2n}, \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - \frac{x^9}{9!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1},
\end{aligned}
$$

that converge for every $x \in \mathbb{R}$ (i.e. $R = \infty$ for them).

For playing with power series see *GeoGebra* or *GeoGebra* or *GeoGebra*.

Outline

Signals

In science and engineering many phenomena related to the physical world are described in terms of magnitudes that vary depending on one or several parameters. For example,

- *Kinematics* studies the motion of bodies in space: position, velocity, acceleration, etc. depending on time;
- Thermodynamics studies how the temperature varies with time;
- Variations of atmospheric pressure are used for weather forecast;
- In economy and social sciences the evolution of many variables of interest is described by sequences of numbers. For example, prices in a stock market, population of a city or country, data about income, tax payments, health, etc.;
- Sound is the variation of air pressure with time;
- Image is the brightness of light incident on a surface.

The parameter is usually refereed as "time" because many important examples of signals represent the variation of magnitudes with time, but in other cases the parameters are spatial (for example an image) or may have other meanings and interpretations.

Signals are called continuous[1] when time varies continuously (for example in a sound or an image) or discrete if time takes only discrete values (for example in daily stock prices or yearly population).

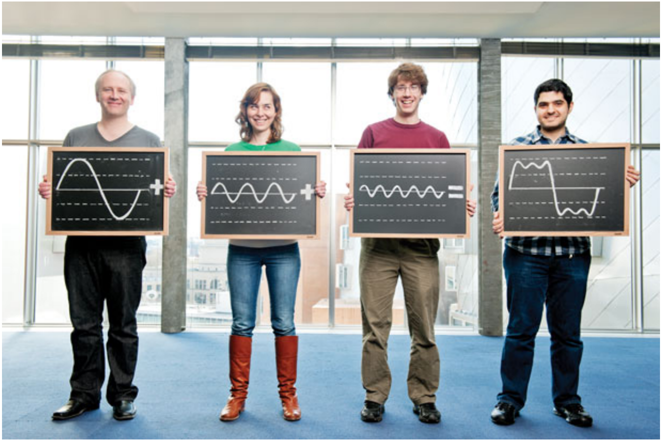
Continuous signals are modelized by functions $f\colon I \to \mathbb{R}$ that associate a number $f(t)$ to every $t \in I$ in some interval of the reals and also $f\colon D \subseteq \mathbb{R}^n \to \mathbb{R}$ depending on several variables ($n = 2$ for an image). Discrete signals are modelized by sequences $(a_n)_n$ of numbers $a_n \in \mathbb{R}$ indexed by integers n.

Here we will be mainly concerned by continuous signals. Mathematically this means that we want to study functions $f\colon I \to \mathbb{R}$, which we will refer to as signals to emphasize that our objective is to understand properties of functions that are of interest to scientists and engineers for understanding the phenomena they modelize.

[1] Do not confuse with continuity in the sense of calculus.

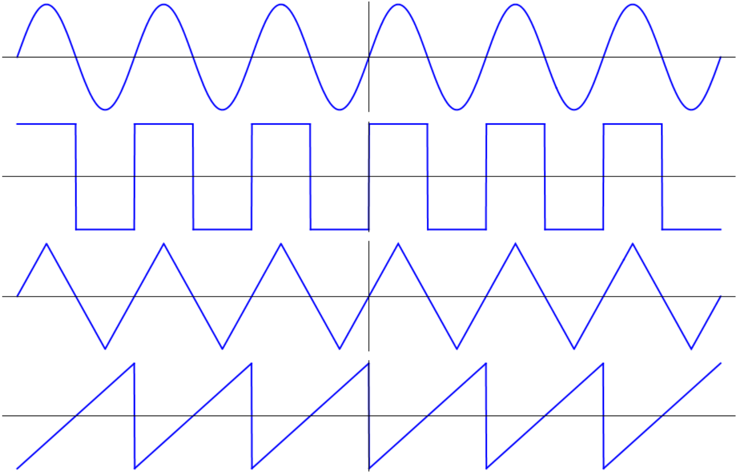
What is Fourier analysis?

Fourier analysis is the main tool for studying signals. What it does is to decompose a signal into their frequencies, producing a different way to look at the signal and to represent it.



Examples of signals: periodic

Sinusoid, square, triangle and sawtooth waves:



If a function f has period T then the integral over any interval of length T takes the same value independently of the initial point of the interval:

$$\forall t_0 \in \mathbb{R}, \quad \int_{t_0}^{t_0+T} f(t)\,dt = \int_0^T f(t)\,dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,dt.$$

Indeed, for every integer n one has $f(t+nT) = f(t)$ for every $t \in \mathbb{R}$ and the change of variables $u = t + nT$ gives

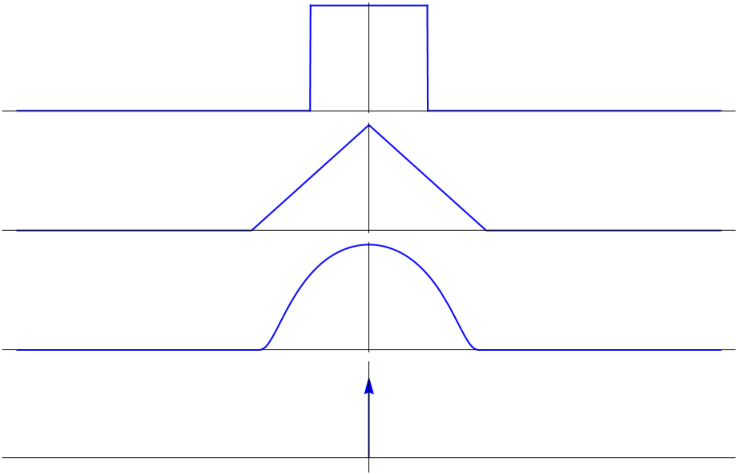
$$\int_a^b f(t)\,dt = \int_{a+nT}^{b+nT} f(t)\,dt.$$

Then, given a t_0, if n is the integer such that $t_0 \leq nT < t_0 + T$ one has

$$\begin{aligned}\int_{t_0}^{t_0+T} f(t)\,dt &= \int_{t_0}^{nT} f(t)\,dt + \int_{nT}^{t_0+T} f(t)\,dt \\ &= \int_{t_0}^{nT} f(t)\,dt + \int_{(n-1)T}^{t_0} f(t)\,dt = \int_{(n-1)T}^{nT} f(t)\,dt \\ &= \int_0^T f(t)\,dt.\end{aligned}$$

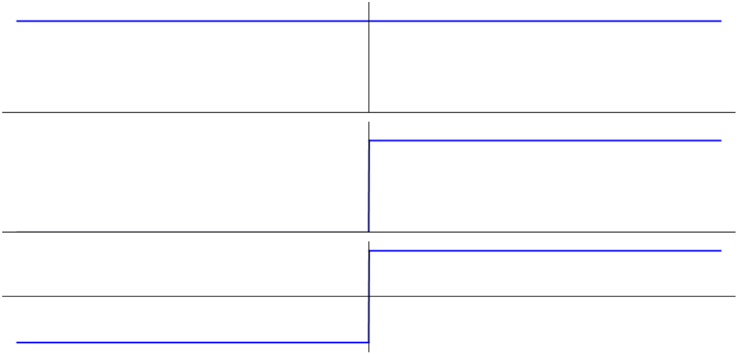
Examples of signals: compact support

Rectangular ant triangular pulses, *bump function*, Dirac delta:



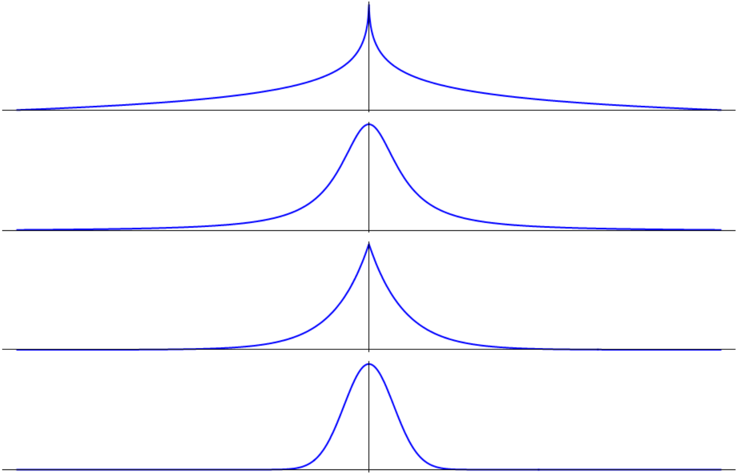
Examples of signals

Constant, Heaviside step function, sign:

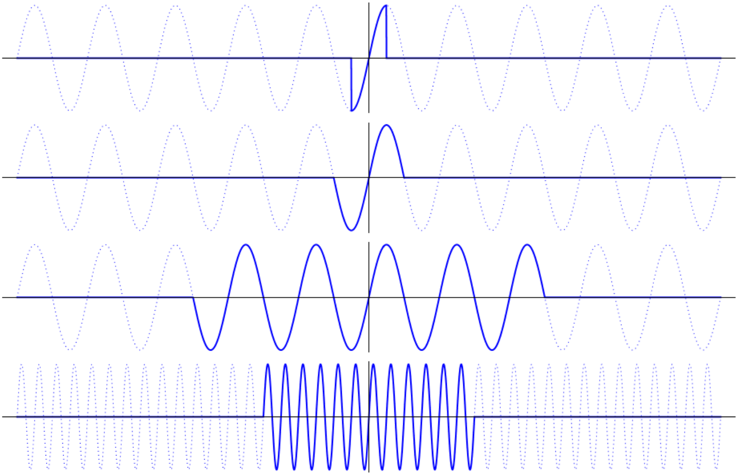


Examples of signals

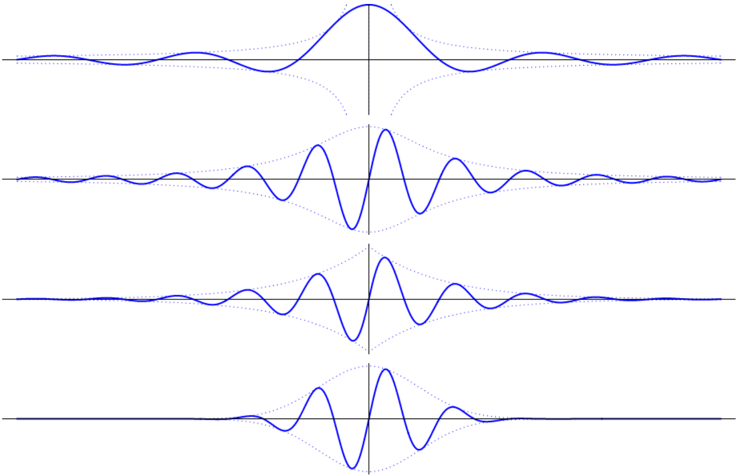
Slow ($\frac{1}{1+|t|}$), moderate ($\frac{1}{1+t^2}$) and rapid ($e^{-|t|}$ or e^{-t^2}) decrease:



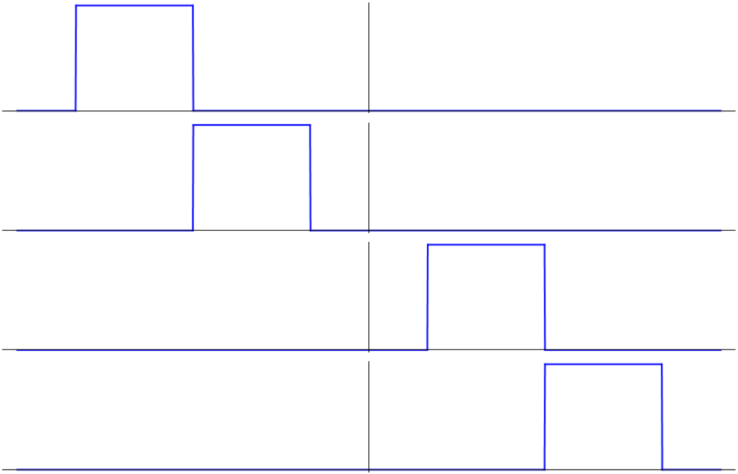
Examples of signals: zero-extension (windowing)



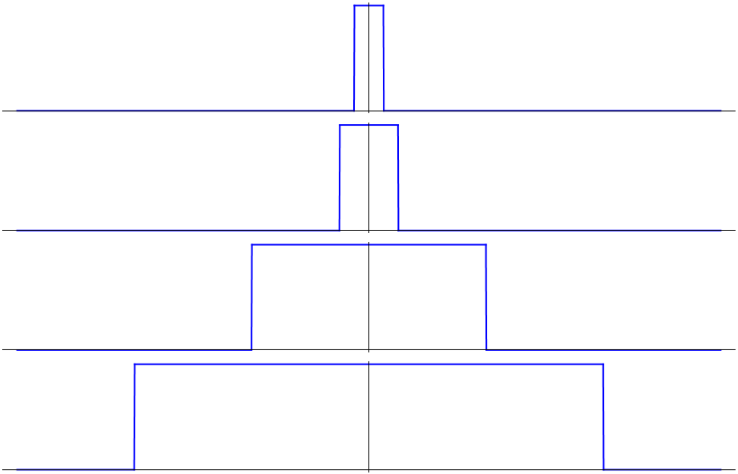
Examples of signals: enveloping



Examples of signals: shift



Examples of signals: scaling



A Fourier series or Fourier transform produces data that describe a given signal in terms of the frequencies present on it, and from which one is able to recover the original signal.

They transform the signal between the time domain (or spatial domain, or physical domain) into the frequency domain, and backwards.

By means of appropriate sensors signals are converted into electrical signals, where their processing can be easily performed. Moreover, in most cases the signals are digitized and the processing is done by digital processors.

For the study, transmission, storage and processing of signals, their versions in the frequency domain turn out to be most appropriate descriptions: signals are much better understood by looking at them in the frequency domain than in the time domain.

Sine waves

Sine waves or sinusoids are functions of the form

$$A\sin(2\pi nt+\phi), \qquad A>0, \quad \phi\in\mathbb{R}, \quad n\in Z.$$

They have the following characteristics:

- are *periodic* of period 1;
- oscillate at a *frequency* n (number of oscillations per unit of time);
- the *amplitude* (width) of the oscillations is the coefficient A;
- *phase* angle ϕ determines the position at the initial time $t=0$.

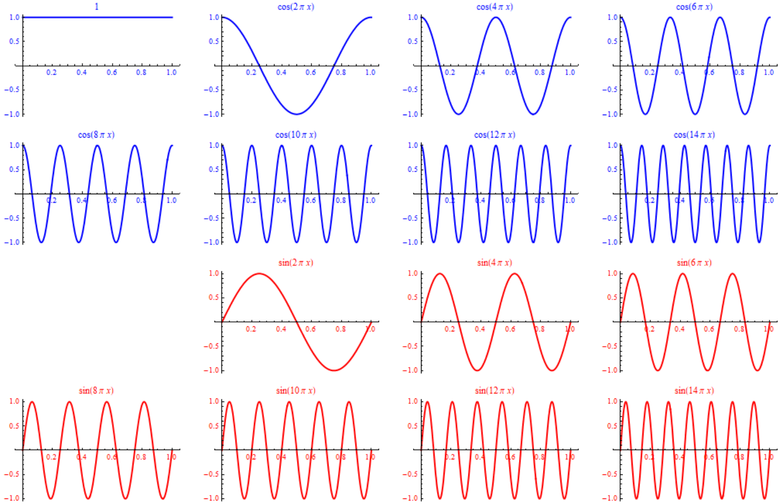
They can be written as sums of multiples of pure sine and cosine waves oscillating at frequency n:

$$A\sin(2\pi nt+\phi)=a\cos(2\pi nt)+b\sin(2\pi nt), \quad a,b\in\mathbb{R}.$$

Fourier series express a given signal f as an infinite sum of sinusoids at all frequencies $n\geq 0$.

Trigonometric Fourier basis

Functions $\cos(2\pi nt)$ for $n \geq 0$ and $\sin(2\pi nt)$ for $n \geq 1$:



Orthogonality

Considered as functions of the space $L^2([0,1])$ the trigonometric Fourier basis functions are "orthonormal":

$$\int_0^1 \cos(2\pi nt)\cos(2\pi mt)\,dt = \begin{cases} 1, & n = m = 0, \\ \frac{1}{2}, & n = m \neq 0, \\ 0, & n \neq m. \end{cases}$$

$$\int_0^1 \sin(2\pi nt)\sin(2\pi mt)\,dt = \begin{cases} \frac{1}{2}, & n = m, \\ 0, & n \neq m. \end{cases}$$

$$\int_0^1 \cos(2\pi nt)\sin(2\pi mt)\,dt = 0, \qquad \forall n, m.$$

They play the same role in the infinite-dimensional space $L^2([0,1])$ than any orthonormal basis in an euclidean finite-dimensional vector space.

The Fourier series of a function f is just the expression of this function as a linear combination of functions of this orthonormal basis.

Trigonometric Fourier series

Any reasonable function $f\colon [0,1] \to \mathbb{R}$ defined in $[0,1]$ can be represented by a trigonometric *Fourier series*:

$$\boxed{f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi n t) + b_n \sin(2\pi n t)}$$

The convergence of the series is understood in the L^1 norm.
Pointwise convergence may be tricky.

The Fourier coefficients a_n and b_n are obtained with the integrals:

$$a_n = 2\int_0^1 f(t)\cos(2\pi n t)\,dt, \quad b_n = 2\int_0^1 f(t)\sin(2\pi n t)\,dt,$$

which are the dot products $2\langle f(t), \cos(2\pi n t)\rangle$ and $2\langle f(t), \sin(2\pi n t)\rangle$ integrating the function f against the basic trigonometric functions.

Everything works also for any other interval $[t_0, t_0+1]$ of length 1.

The N-th. approximations

$$\frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(2\pi n t) + b_n \sin(2\pi n t)$$

are called trigonometric polynomials. Geometrically they can be interpreted as the orthogonal projection of the function f on the finite-dimensional space of functions generated by the basic functions for $n \leq N$.
Recall from euclidean geometry that this projection minimizes the distance.

Since the sine waves are periodic of period 1, a Fourier series is always a *periodic function*.

For an arbitrary function f defined only over a length one interval the Fourier series produces the periodic extension of f outside of that interval.

Example: square wave

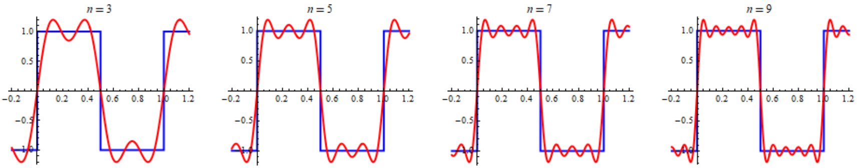
The *square wave* function of period 1 is the periodic extension of the function:

$$f(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2}, \\ -1, & \frac{1}{2} \le t < 1, \end{cases} \qquad t \in [0,1),$$

and has Fourier series:

$$f(t) = \sum_{n=1}^{\infty} \frac{4}{(2n+1)\pi} \sin(2\pi(2n+1)t).$$

It has only sine components at odd frequencies.



Example: sawtooth wave

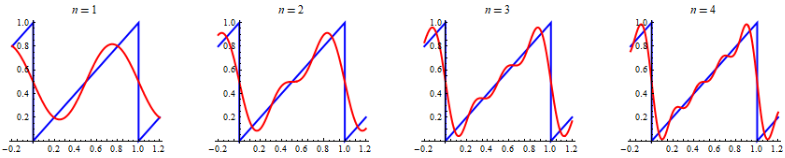
The *sawtooth wave* of period 1 is the periodic extension of the identity function in the interval $[0, 1)$

$$f(t) = t, \qquad t \in [0, 1),$$

and has Fourier series:

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-1}{2n\pi} \sin(2\pi n t).$$

It has only the constant (cosine) component and sine components.



Example: triangle wave

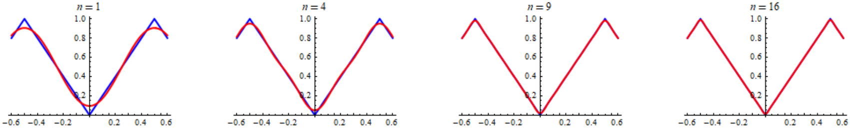
The *triangle wave* of period 1 is the periodic extension of the function:

$$f(t) = \begin{cases} 2t, & 0 \le t < \frac{1}{2}, \\ 2 - 2t, & \frac{1}{2} \le t < 1, \end{cases} \qquad t \in [0, 1),$$

and has Fourier series:

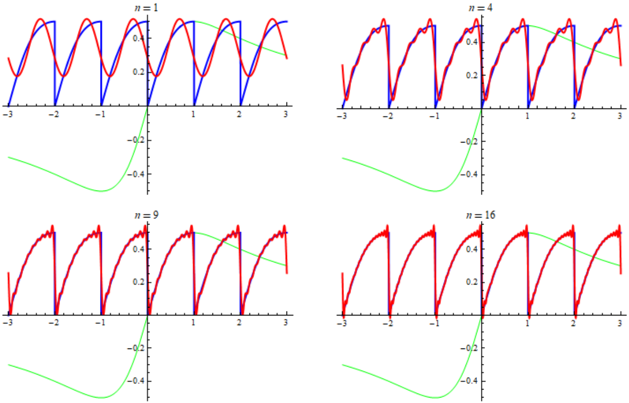
$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-4}{(2n+1)^2 \pi^2} \cos(2\pi(2n+1)t).$$

It has only cosine components.



Example: periodic extension

Fourier series of the (periodic extension of the) function $f(x) = \dfrac{x}{1+x^2}$ on the interval $[0,1)$:



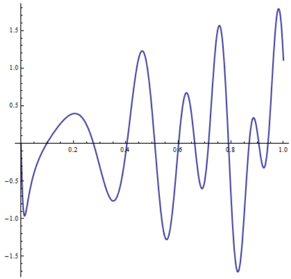
See *GeoGebra*, *GeoGebra*.

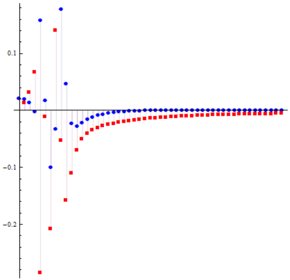
Frequency domain

A function $f(t)$ of a continuous variable t (usually interpreted as time) is completely determined in a length one interval by its Fourier coefficients: the amount of each frequency present in f.

The sequence of Fourier coefficients is called the representation of f in the *discrete frequency domain*. See *Applet*.

They are usually represented in the following way: a function f on the interval $[-\frac{1}{2}, \frac{1}{2}]$ and its trigonometric Fourier coefficients a_n and b_n:





Example: sound

The human *auditory system* perceives sound through frequencies: humans can only hear frequencies in the range 20-22000 Hz, being most sensitive at those in the range 2-5 kHz. See *Hearing range*. For this reason:

- For transmission of speech over telephony lines the bandwidth is often limited to $300 - 3500$ Hz, which is enough to reproduce speech with good accuracy.
- *CD-Digital Audio* contains the soundwave sampled at $44\,100$ samples per second ($> 22\,000 \times 2$, due to the *Nyquist-Shannon theorem*).
- *Audio compression* methods (*MP3*, *AAC*, *WMA*, etc.) transform sound into frequencies; some frequencies are deleted because they are not important or are *masked* by the presence of other frequencies, according to *psychoacoustic models*; the simplified frequency domain version of the sound is then encoded. The audio recovered by decoders is virtually indistinguishable from the original one by human ears.

Parity: cosine and sine series

Definition (Parity)

A function $f\colon \mathbb{R} \to \mathbb{R}$ is

- *even* if $f(-t) = f(t)$ (for example $\cos(2\pi n t)$);
- *odd* if $f(-t) = -f(t)$ (for example $\sin(2\pi n t)$).

With respect to the the product parity of functions behaves exactly as parity for the sum of two integers: a product of two even or odd functions is even and a product of an even and an odd function is odd.

Every function is the sum (in a unique way) of an even function and an odd function:

$$f(t) = f_{\mathrm{even}}(t) + f_{\mathrm{odd}}(t)$$

with

$$f_{\mathrm{even}}(t) = \frac{f(t) + f(-t)}{2}, \quad f_{\mathrm{odd}}(t) = \frac{f(t) - f(-t)}{2}.$$

When integrating even and odd functions over a symmetric interval $[-x, x]$ one has:

$$\int_{-x}^{x} f(t)\,dt = 0 \quad \text{if } f \text{ is odd,}$$

$$\int_{-x}^{x} f(t)\,dt = 2\int_{0}^{x} f(t)\,dt \quad \text{if } f \text{ is even.}$$

This has consequences on Fourier coefficients: a periodic function f is even if, and only if, all its Fourier coefficients b_n are zero.
In this case it can be written as a *cosine series*:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi n t).$$

It is odd if, and only if, all its Fourier coefficients a_n are zero, and in this case can be written as a sine series:

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(2\pi n t).$$

Cosine series are an important tool in image and audio compression.

Outline

Review of complex numbers

Complex numbers are expressions of the form

$$z = x + iy, \qquad x, y \in \mathbb{R},$$

with x and y real numbers, the *Cartesian coordinates* of z, and i the *imaginary unit*: a "number" with the property that $i^2 = -1$.

The set of complex numbers is denoted $\mathbb{C}$ and geometrically identified with the set of points (x, y) of the plane $\mathbb{R}^2$.
The number $\overline{z} = x - iy$ is the *complex conjugate* of z. Geometrically, complex conjugation is the symmetry with respect to the real axis.

The basic operations, with $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, are:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2), \quad z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2),$$

$$z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1), \quad \frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2}.$$

Complex conjugation respects the two basic operations:

$$\overline{z + w} = \overline{z} + \overline{w}, \qquad \overline{z \cdot w} = \overline{z} \cdot \overline{w}.$$

Polar coordinates

Every nonzero complex number $x = x + iy$ is determined by its *polar coordinates*:

- the *modulus* $r = |z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$ is the distance to the origin;
- the *argument* $\alpha = \arg(z) = \arctan \frac{y}{x} = \mathrm{atan2}(x, y)$ is the angle with respect to the positive x axis.

Cartesian and polar coordinates are related through the identity:

$$z = x + iy = r \cos \alpha + i\, r \sin \alpha = r(\cos \alpha + i \sin \alpha).$$

Multiplication is easily performed in polar coordinates:

Proposition

Given two nonzero complex numbers z and w one has

$$|zw| = |z||w|, \qquad \arg(zw) = \arg(z) + \arg(w).$$

It follows that $\arg(z^{-1}) = -\arg(z)$, $|z^n| = |z|^n$ and $\arg(z^n) = n \arg(z)$.

Exponential form

The fundamental *Euler formula* relates trigonometric and exponential functions over the complex numbers:

$$\boxed{e^{i\alpha} = \cos\alpha + i\sin\alpha \qquad \text{for} \quad \alpha \in \mathbb{R}}$$

It is obtained from the Taylor series for the involved functions.

The exponential form of a complex number is the expression

$$z = re^{i\alpha}, \qquad r = |z|, \quad \alpha = \arg(z),$$

that comes from Euler identity:

$$z = r\cos\alpha + ir\sin\alpha = r(\cos\alpha + i\sin\alpha) = re^{i\alpha}$$

This form behaves well for multiplication. If $z = re^{i\alpha}$, $w = se^{i\beta}$:

$$zw = (rs)e^{i(\alpha+\beta)}, \qquad \frac{z}{w} = \frac{r}{s}\,e^{i(\alpha-\beta)}, \qquad z^n = r^n e^{in\alpha}.$$

Notice that the complex numbers of modulus one are the $e^{i\alpha}$ for $\alpha \in \mathbb{R}$.

Since $z\overline{z} = r^2$ it follows that $\arg(z) + \arg(\overline{z}) = \arg(r^2) = 0$, and

$$\arg(\overline{z}) = -\arg(z).$$

From this identity, or also from the parity of the trigonometric functions, one obtains the conjugation of complex numbers in exponential form:

$$\begin{aligned}\overline{re^{i\alpha}} = \overline{r\cos\alpha + i\sin\alpha} &= r\cos\alpha - ir\sin\alpha \\ &= r\cos(-\alpha) + ir\sin(-\alpha) = re^{-i\alpha}.\end{aligned}$$

And the trigonometric functions are given in terms of complex exponentials:

$$\cos\alpha = \frac{1}{2}(e^{i\alpha} + e^{-i\alpha}), \qquad \sin\alpha = \frac{1}{2i}(e^{i\alpha} - e^{-i\alpha}).$$

Roots

Every nonzero complex number $z = re^{i\alpha}$ has exactly n different n-th. roots, that are most conveniently expressed in terms of polar coordinates: their common modulus is $\sqrt[n]{r}$ and their arguments are $\frac{\alpha}{n} + \frac{2\pi ik}{n}$ for integers $k = 0, 1, \dots, n-1$:

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\alpha + 2\pi k}{n}\right) + i\cos\left(\frac{\alpha + 2\pi k}{n}\right)\right], \quad k = 0, \dots, n-1,$$

or, in exponential form,

$$\sqrt[n]{re^{i\alpha}} = \sqrt[n]{r}\, e^{i\left(\frac{\alpha}{n} + \frac{2\pi k}{n}\right)}, \quad k = 0, \dots, n-1.$$

An important fact about complex numbers is the:

Theorem (Fundamental theorem of Algebra)

Every non-constant polynomial P with complex coefficients has a complex root. In fact, if $n = \deg P$, it has exactly n roots, counting multiplicities.

Roots of unity

The complex n-th *roots of unity* are the complex numbers z with $z^n = 1$: the n different roots of the polynomial $X^n - 1$. They are:

$$e^{2\pi ik/n} = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}, \quad k = 0, \ldots, n-1.$$

Geometrically, the n-th. roots of unity are the vertices of a n-sided *regular polygon* centered at the origin with *circumscribed circle* of radius one.

They can be obtained as powers of the generating root $\zeta_n = e^{2\pi i/n}$:

$$e^{2\pi ik/n} = \zeta_n^k.$$

Functions with complex values

Even though most important signals take real values in practice signals with complex values are considered because Fourier series can be written in a more elegant and uniform way as series of complex exponentials. Moreover, Fourier transforms take place in a natural way on functions with complex values.

For this reason from now on we will consider functions $f\colon I \to \mathbb{C}$ defined over an interval $I \subseteq \mathbb{R}$ whose values $f(t) \in \mathbb{C}$ are complex numbers.

Such a function can be written in terms of its real and imaginary (cartesian) components $x, y\colon I \to \mathbb{R}$:

$$f(t) = x(t) + i\,y(t), \qquad x(t) = \mathrm{Re}\,\big(f(t)\big), \quad y(t) = \mathrm{Im}\,\big(f(t)\big),$$

or also in terms of the polar coordinates modulus and argument:

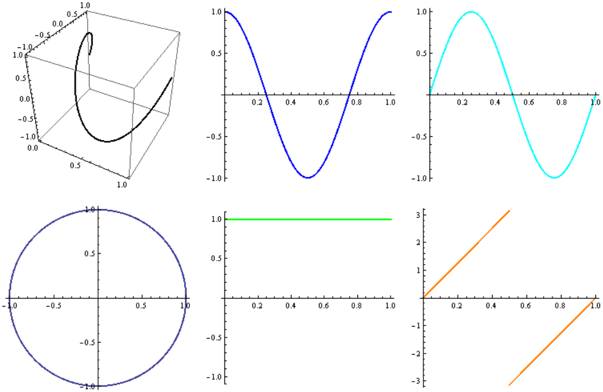
$$f(t) = r(t)e^{i\,\alpha(t)}, \qquad r(t) = |f(t)|, \quad \alpha(t) = \arg\big(f(t)\big),$$

Example: Exponential function

To geometrically represent a function $f\colon I \to \mathbb{C}$ one may plot the curve drawn by its values when the parameter travels over I, or the cartesian components $x(t)$ and $y(t)$, or the polar components $r(t)$ and $\alpha(t)$.

The most basic example is the complex exponential function

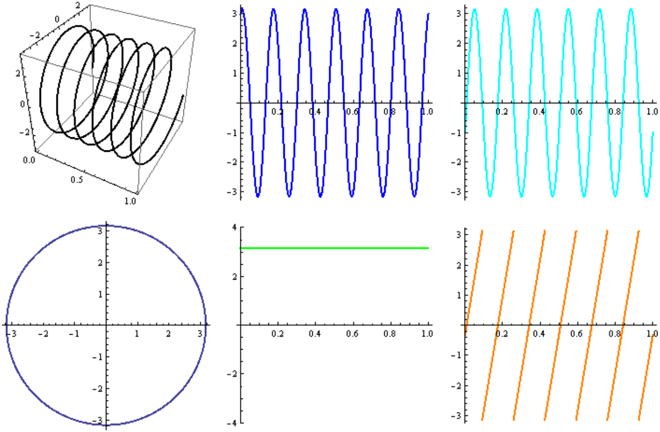
$$f(t) = e^{2\pi i t} = 1 e^{i\,2\pi t} = \cos 2\pi t + i \sin 2\pi t, \qquad t \in \mathbb{R}.$$



More generally, a complex oscillation function of the form

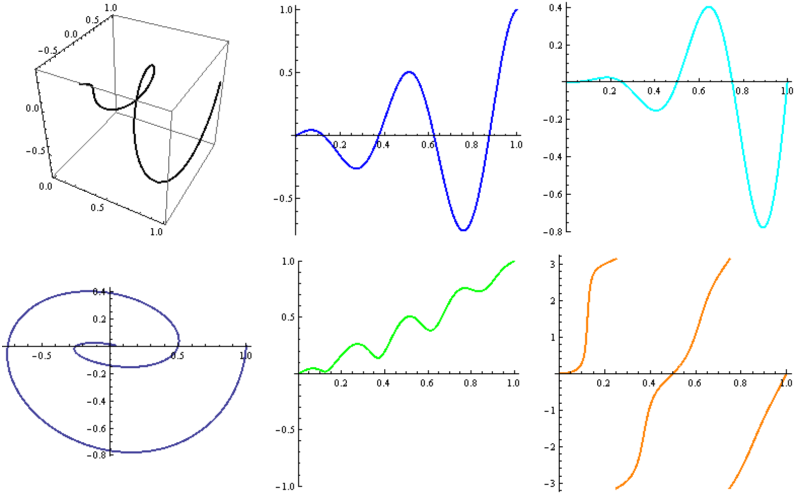
$$f(t) = ce^{2\pi int}, \qquad c = (a + bi) \in \mathbb{C}, \quad n \in \mathbb{Z}$$

has the following shape for the values $c = 3 - i$ and $n = 6$, with constant amplitude $r(t) = \sqrt{3^2 + 1^2} = \sqrt{10} \simeq 3.1623$



And the function

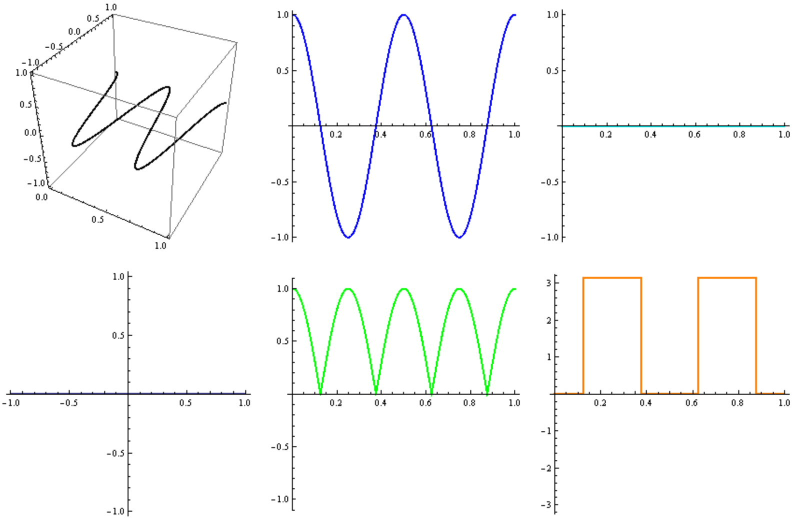
$$f(t) = t\cos 4\pi t + i\, t^2 \sin 4\pi t$$



Real valued functions can also be looked at as taking complex values:

$$f(t) = \cos 4\pi t = \color{blue}{\cos 4\pi t} + i\,\color{cyan}{0} = \color{green}{|\cos 4\pi t|}e^{i\,\color{orange}{\arg(\cos 4\pi t)}}$$

with argument $\color{orange}{\arg\left(f(t)\right)}$ taking only the values 0 or π, depending on whether $f(t)$ is positive or negative.



Derivation and integration

Derivation and integration (definite or indefinite) of complex-valued functions are performed on the components: if $f(t) = x(t) + iy(t)$,

$$f'(t) = x'(t) + iy'(t), \qquad \int f(t)\,dt = \int x(t)\,dt + i\int y(t)\,dt.$$

For example, if $f(t) = e^{it} = \cos t + i\sin t$, then

$$f'(t) = -\sin t + i\cos t = i(\cos t + i\sin t) = ie^{it}.$$

More generally, if $f(t) = ce^{2\pi i\xi t}$ with $c \in \mathbb{C}$ and $\xi \in \mathbb{R}$, then

$$f'(t) = 2\pi i\xi c e^{2\pi i\xi t}$$

and the function $F(t) = \frac{c}{2\pi i\xi}e^{2\pi i\xi t}$ is a primitive of f.

The usual formulas and techniques for computation of derivatives, primitives and definite integrals do work also for complex-valued functions.

Norms and dot products

In the finite dimensional complex vector space $\mathbb{C}^n$ there is *dot product* analogous to the scalar product in $\mathbb{R}^n$ called *Hermitian inner product*.

It is defined in the following way: given two vectors $\boldsymbol{z} = (z_1, \ldots, z_n)$ and $\boldsymbol{w} = (w_1, \ldots, w_n)$ with complex components $z_i, w_i \in \mathbb{C}$, one defines:

$$\langle \boldsymbol{z}, \boldsymbol{w} \rangle = \sum_{i=1}^{n} z_i \overline{w}_i \in \mathbb{C}.$$

In particular, thanks to the conjugation on the second vector, one has $\langle \boldsymbol{z}, \boldsymbol{z} \rangle = \sum_{i=1}^{n} |z_i|^2 \in \mathbb{R}$ and one can define a norm

$$\|\boldsymbol{z}\| = \sqrt{\langle \boldsymbol{z}, \boldsymbol{z} \rangle} = \sqrt{\sum |z_i|^2}$$

which has the same properties than the norms over $\mathbb{R}^n$ seen before, and can be used to measure sizes of vectors and distances between them.

Using the dot product one can define the concept of orthogonality.

Spaces of functions

As before one can consider spaces of functions defined over an interval $I \subseteq \mathbb{R}$ but now with complex values $f \colon I \to \mathbb{C}$,

- continuous functions or of class $\mathcal{C}^1$ (continuously differentiable);
- bounded functions, with the sup norm $\|f\|_\infty = \sup\left\{|f(t)| : t \in I\right\}$;
- space $L^1(I)$ of (absolutely) integrable functions: $\int_I |f(t)|\,dt < \infty$, with norm $\|f\|_1 = \int_I |f(t)|\,dt$;
- space $L^2(I)$ of square integrable functions: $\int_I |f(t)|^2\,dt < \infty$ with dot product and L^2-norm:

$$\langle f, g\rangle = \int_I f(t)\,\overline{g(t)}\,dt, \qquad \|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_I |f(t)|^2\,dt}.$$

More generally also spaces of functions $f \colon D \subseteq \mathbb{R}^n \to \mathbb{C}$ of more variables, representing signals that depend on many parameters, may be considered.

Exponential Fourier series

Using Euler's formula

$$e^{2\pi int} = \cos(2\pi nt) + i\sin(2\pi nt)$$

the Fourier series of a function $f\colon [0,1] \to \mathbb{R}$ can be written in compact form using complex numbers in the form of an exponential Fourier series:

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi int}, \qquad c_n = \int_0^1 f(t)\, e^{-2\pi int}\, dt \in \mathbb{C}}$$

The relation between the complex and real coefficients is:

$$c_n = \begin{cases} \frac{1}{2}(a_n - ib_n), & n > 0, \\ \frac{1}{2}a_0, & n = 0, \\ \frac{1}{2}(a_{-n} + ib_{-n}), & n < 0, \end{cases} \qquad \begin{aligned} a_n &= c_n + c_{-n}, \\ b_n &= i(c_n - c_{-n}). \end{aligned}$$

Indeed, given a trigonometric Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi nt) + b_n \sin(2\pi nt),$$

writing $a_n = c_n + c_{-n}$ and $-ib_n = c_n - c_{-n}$ for two complex numbers c_n and c_{-n} one computes:

$$\begin{aligned}
a_n \cos(2\pi nt) + b_n \sin(2\pi nt) &= a_n \cos(2\pi nt) + i(-ib_n) \sin(2\pi nt) \\
&= (c_n + c_{-n}) \cos(2\pi nt) + i(c_n - c_{-n}) \sin(2\pi nt) \\
&= c_n\big(\cos(2\pi nt) + i \sin(2\pi nt)\big) \\
&\qquad\qquad + c_{-n}\big(\cos(2\pi nt) - i \sin(2\pi nt)\big) \\
&= c_n e^{2\pi int} + c_{-n} e^{-2\pi int},
\end{aligned}$$

and the trigonometric Fourier series becomes an exponential series:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi int}.$$

This Fourier exponential series expansion can be applied more generally to any reasonable function $f\colon [0,1] \to \mathbb{C}$ with complex values.

Now the basis of functions is

$$e^{2\pi int} = \cos(2\pi nt) + i\sin(2\pi nt), \qquad n \in \mathbb{Z},$$

and they form an orthonormal basis of the space of functions because

$$\begin{aligned}\langle e^{2\pi int}, e^{2\pi imt}\rangle &= \int_0^1 e^{2\pi int}\overline{e^{2\pi imt}}\,dt = \int_0^1 e^{2\pi int}e^{-2\pi imt}\,dt \\ &= \int_0^1 e^{2\pi i(n-m)t}\,dt.\end{aligned}$$

For $n = m$ this integral is

$$= \int_0^1 e^{2\pi i0t}\,dt = \int_0^1 1\,dt = 1,$$

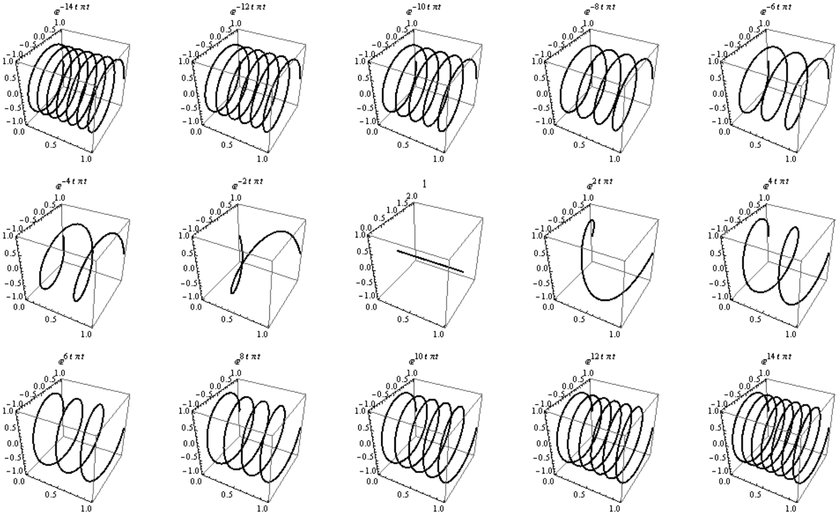
and for $n \neq m$ it is computed from a primitive obtaining

$$= \left[\frac{e^{2\pi i(n-m)t}}{2\pi i(n-m)}\right]_0^1 = \frac{e^{2\pi i(n-m)}}{2\pi i(n-m)} - \frac{e^0}{2\pi i(n-m)} = \frac{1-1}{2\pi i(n-m)} = 0.$$

The Fourier exponential basis

Plot of the functions of the Fourier exponential basis

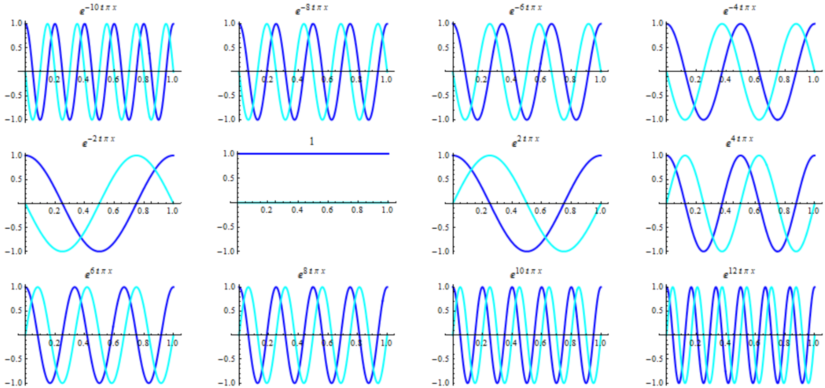
$$e^{2\pi i n t} = \cos(2\pi n t) + i\sin(2\pi n t)$$



plot of the components:

$$e^{2\pi int} = \cos(2\pi nt) + i\sin(2\pi nt)$$

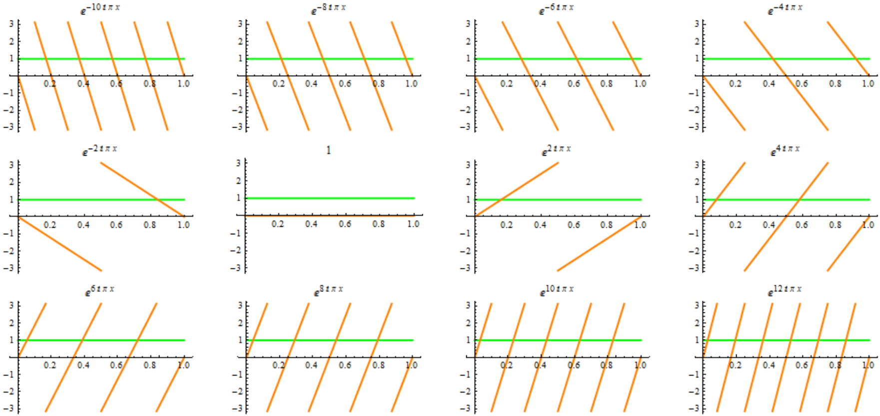
with real and imaginary parts:



plot of the polar coordinates

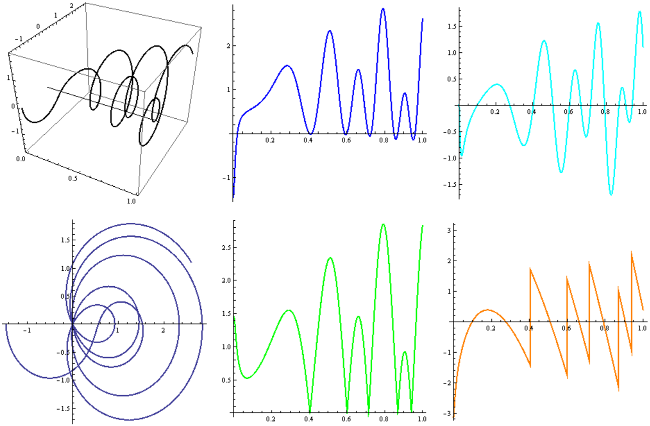
$$e^{2\pi int} = {\color{green}1}\, e^{i\,{\color{orange}2\pi nt}}$$

with modulus and argument:

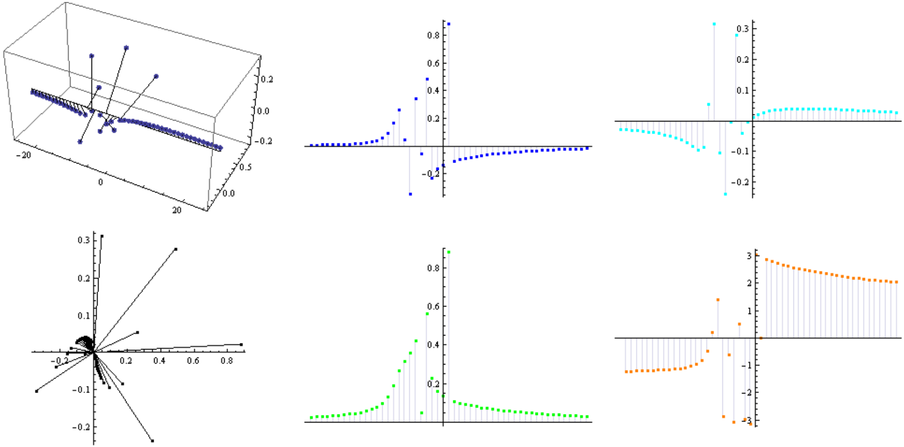


Time versus frequency

Again the exponential Fourier series transforms between time domain and frequency domain. Here is a complex-valued function in the time domain:



and this is the same function in the frequency domain represented from its Fourier coefficients:



Other intervals

Sine waves $\cos(2\pi nt)$ and $\sin(2\pi nt)$ are periodic of period one.
Fourier series as presented represent periodic functions of period 1, and produce periodic extensions of arbitrary functions on an interval of length 1.

Intervals of arbitrary length T and functions of arbitrary period T can be handled using the bases of sine waves or exponentials of period T:

$$\cos(2\pi nt/T), \qquad \sin(2\pi nt/T), \qquad e^{2\pi int/T}.$$

The corresponding trigonometric or exponential Fourier series produce the T-periodic extension of the function over the length T interval.
For example, given a function $f\colon [t_0, t_0+T] \to \mathbb{R}$, one has

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi int/T}, \qquad c_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t) e^{-2\pi int/T}\, dt.$$

Higher dimension

Everything works for functions of two (or more) variables.
For example, a function $f\colon [0,1] \times [0,1] \to \mathbb{R}$ on the unit square can be given as a Fourier series:

$$f(x,y) = \sum_{n,m \geq 0} a_{n,m} \operatorname{trig}(2\pi n x) \operatorname{trig}(2\pi m y),$$

as a combination of basic functions that are 2-dimensional sine waves with certain Fourier coefficients $a_{n,m}$. This same expression can also be written more elegantly using complex exponentials:

$$f(x,y) = \sum_{n,m=-\infty}^{\infty} c_{n,m}\, e^{2\pi i (nx+my)},$$

where the Fourier coefficients are given by the double integrals:

$$c_{n,m} = \int_0^1 \int_0^1 f(x,y)\, e^{-2\pi i (nx+my)}\, dx\, dy.$$

2-dimensional frequency domain

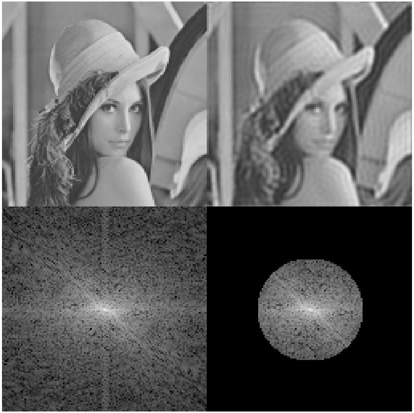
The Fourier coefficients of a 2-variable function are indexed by two integers, and in this case the graphic representation of the function in the frequency domain depends also on these two variables.

Example:

Lena (upper left);

magnitude of complex Fourier coefficients (down left);

and effect of a low-pass filter that eliminates large frequencies (right).



See *Applet*

Outline

Fourier series vs. Fourier transform

Fourier series give a dictionary between:

- Periodic functions $f(t)$ of a continuous variable $t \in \mathbb{R}$, completely determined by their values on a finite interval $[t_0, t_0 + T]$, and
- Infinite sequences c_n of complex numbers indexed by a discrete variable $n \in \mathbb{Z}$,

through the formulas

$$c_n = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) e^{-2\pi i n t/T} \, dt, \qquad f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t/T}.$$

The Fourier transform is defined by analogous formulas to achieve a more symmetric correspondence between non periodic functions defined over the entire real line $\mathbb{R}$ and other functions also defined over $\mathbb{R}$.

Even though most signals of interest take real values we will always consider functions $f : \mathbb{R} \to \mathbb{C}$ with complex values.

Fourier transform and its inverse

Definition

The *Fourier transform* of a function $f\colon \mathbb{R} \to \mathbb{C}$ is the function $\widehat{f}\colon \mathbb{R} \to \mathbb{C}$ defined (when it exists) as

$$\boxed{\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \xi t}\, dt, \qquad \xi \in \mathbb{R}}$$

Theorem (Inverse Fourier theorem)

Under reasonable conditions a function can be recovered from its Fourier transform with the following formula, called inverse Fourier transform:

$$\boxed{f(t) = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2\pi i t \xi}\, d\xi, \qquad t \in \mathbb{R}}$$

In this case the functions f and $\widehat{f}$ are known as a Fourier Pair, and denoted

$$f \leftrightarrow \widehat{f}$$

Since $|f(t)e^{-2\pi i\xi t}| = |f(t)|$ for every $t \in \mathbb{R}$ and every $\xi \in \mathbb{R}$, the existence of the integral $\int_{-\infty}^{\infty} |f(t)|\, dt$ implies that the integrals

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i\xi t}\, dt$$

are convergent for every ξ, and the Fourier transform does exist.
This means that all functions of the space $L^1(\mathbb{R})$ of absolutely integrable functions do have a Fourier transform.

For example, functions with $f(t) = O\left(\frac{1}{t^a}\right)$ for some $a > 1$ or $O(e^{-|t|})$ at $\pm\infty$. Typical examples are:

- Functions with finite *support*: $f(t) = 0$ except in a finite interval $[a, b]$. Every function defined in $[a, b]$ can be zero-extended to all $\mathbb{R}$.
- Functions of the type $\frac{1}{1+t^2}$, e^{-t^2}, and similar.
- For every bounded function f (in particular most periodic ones) the function $f(t)e^{-t^2}$ is the function f *enveloped* by a Gaussian.

For the same reason, functions of the space $L^1(\mathbb{R})$ have inverse Fourier transforms, but in general is no true that for every function $f \in L^1(\mathbb{R})$ one has $\widehat{f} \in L^1(\mathbb{R})$.

When the inverse Fourier theorem holds one has a Fourier pair $f \leftrightarrow \widehat{f}$. This establishes a correspondence between:

- Certain functions $t \mapsto f(t) \colon \mathbb{R} \to \mathbb{C}$ of the variable t (time), and
- Certain functions $\xi \mapsto \widehat{f}(\xi) \colon \mathbb{R} \to \mathbb{C}$ of the variable ξ (frequency).

In the inverse Fourier theorem one does not require that the inverse transform of $\widehat{f}$ equals f at every point $t \in \mathbb{R}$, but only that they coincide in the space L^1: the integral of the difference equals zero.

Analogy between transform and series

Comparison of the formulas for Fourier series and Fourier transform:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t/T}, \qquad c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-2\pi i \frac{n}{T} t}\, dt,$$

$$f(t) = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2\pi i \xi t}\, d\xi, \qquad \widehat{f}(\xi) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \xi t}\, dt,$$

shows the analogy between them.

In the Fourier transform the interval length T tends to infinity but instead of discrete coefficients c_n depending on integers $n \in \mathbb{Z}$ one computes $\widehat{f}(\xi)$ depending on a continuous parameter $\xi \in \mathbb{R}$.

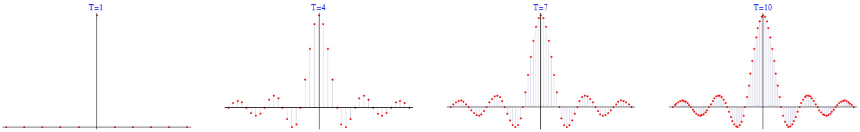
This analogy between the two theories justifies that $\widehat{f}(\xi)$ is interpreted as the "amount of frequency ξ" in the function f. The modulus $|\widehat{f}(\xi)|$ is also called the *amplitude* and the argument $\arg\left(\widehat{f}(\xi)\right)$ is called the *phase*.

For example, an approach to the transform of the zero-extension of a function $f\colon [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{C}$ can be obtained from the series computed over large intervals $[-\frac{T}{2}, \frac{T}{2}]$.

For the rectangular pulse function

$$f(t) = \begin{cases} 1, & \text{if } |t| \leq \frac{1}{2}, \\ 0, & \text{otherwise,} \end{cases}$$

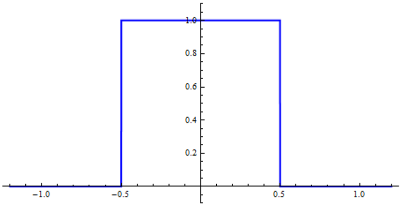
with several values of T and taking also a proportionally increasing number of coefficients, one obtains the following (real) coefficients:

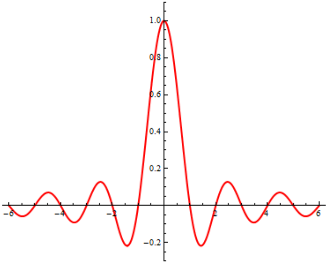


Example: rectangular function

The Fourier transform of the *rectangular function* is the normalized *cardinal sine function*:

$$f(t) = \mathrm{rect}(t) = \begin{cases} 1, & \text{if} \quad |t| < \frac{1}{2}, \\ 0, & \text{if} \quad |t| \geq \frac{1}{2}, \end{cases} \qquad \widehat{f}(\xi) = \mathrm{sinc}(\xi) = \frac{\sin \pi\xi}{\pi\xi}.$$





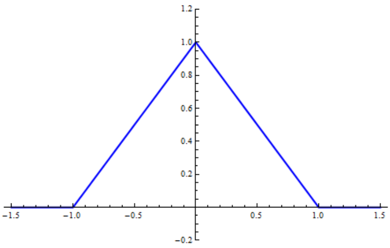
Indeed, recalling that $\sin\alpha = \frac{1}{2i}(e^{i\alpha} - e^{-i\alpha})$, one computes:

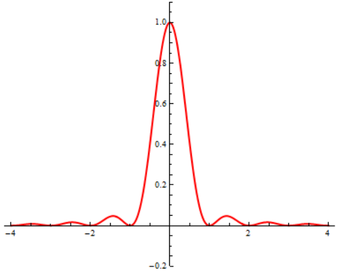
$$\begin{aligned}\widehat{\mathrm{rect}}(\xi) &= \int_{-\infty}^{\infty} \mathrm{rect}(t) e^{-2\pi i \xi t}\, dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i \xi t}\, dt \\ &= \left[\frac{e^{-2\pi i \xi t}}{-2\pi i \xi}\right]_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{e^{-\pi i \xi} - e^{\pi i \xi}}{-2\pi i \xi} = \frac{-2i \sin \pi\xi}{-2\pi i \xi} = \frac{\sin \pi\xi}{\pi\xi}.\end{aligned}$$

Example: triangular function

The Fourier transform of the *triangular function* is the square of the sinc function:

$$f(t) = \mathrm{tri}(t) = \begin{cases} 1 - |t|, & \text{if} \quad |t| < 1, \\ 0, & \text{if} \quad |t| \geq 1, \end{cases} \qquad \widehat{f}(\xi) = \mathrm{sinc}^2(\xi) = \frac{\sin^2 \pi\xi}{\pi^2\xi^2}.$$

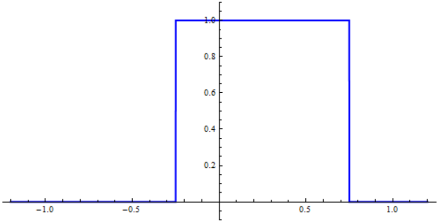


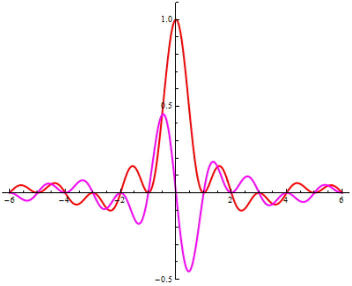


Example: rectangular function shifted

The Fourier transform of the shifted rectangular function $\mathrm{rect}(t - \frac{1}{4})$ is the complex valued function

$$\frac{e^{-\pi i\xi/2}\sin \pi\xi}{\pi\xi} = \frac{\cos\frac{\pi\xi}{2}\sin\pi\xi}{\pi\xi} + i\frac{-\sin\frac{\pi\xi}{2}\sin\pi\xi}{\pi\xi}$$

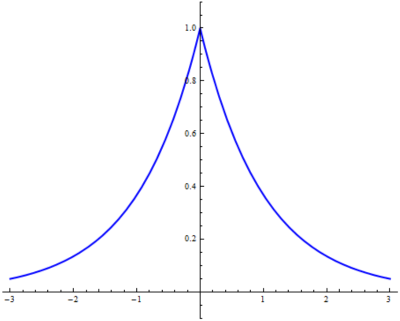


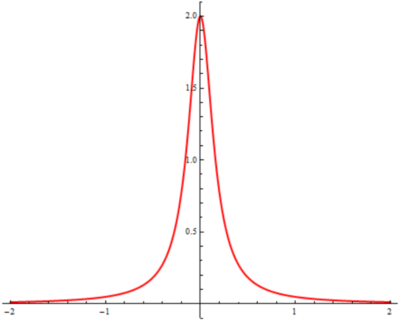


Notice that now the Fourier transform is not real valued.
The reason for that is because the function $f(t)$ is not even.

Example: exponential function

$$f(t) = e^{-|t|}, \qquad \widehat{f}(\xi) = \frac{2}{1 + 4\pi^2\xi^2}.$$

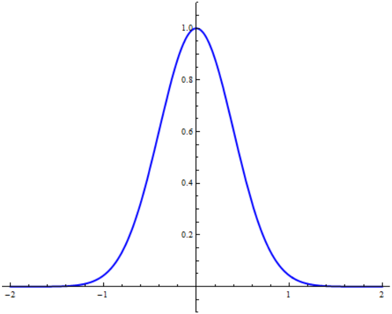


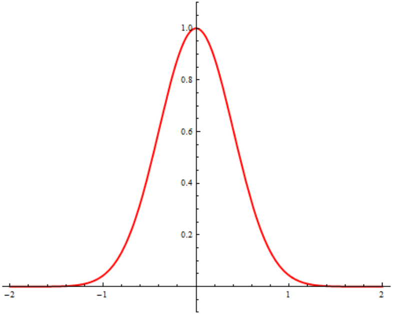


Example: Gaussian function

The Fourier transform of the *Gaussian function* is the same function:

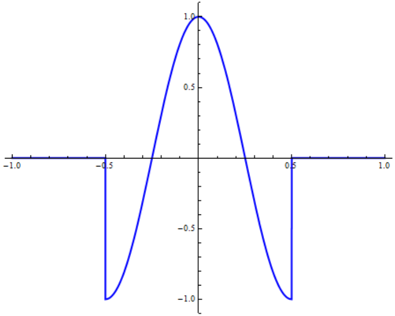
$$f(t) = e^{-\pi t^2}, \qquad \widehat{f}(\xi) = e^{-\pi \xi^2}.$$

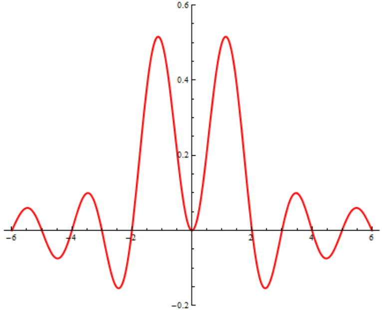




Example: zero extension of cosine

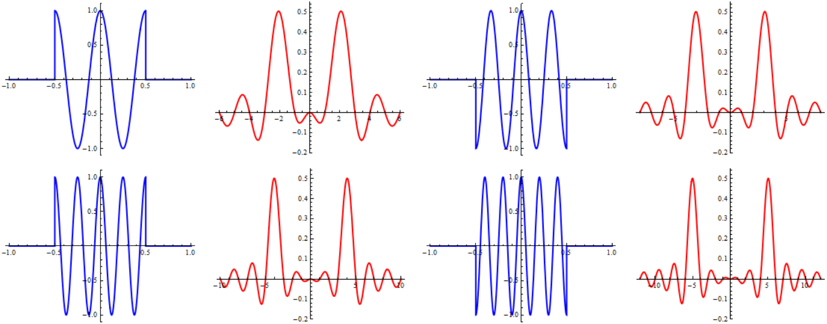
$$f(t) = \begin{cases} \cos(2\pi t), & \text{if} \quad |t| < \frac{1}{2}, \\ 0, & \text{if} \quad |t| \geq \frac{1}{2}, \end{cases} \qquad \widehat{f}(\xi) = \frac{\xi \sin \pi\xi}{\pi(\xi^2 - 1)}.$$





Zero extension of cosine $\cos(2\pi n t)$ at higher frequencies $n = 2, 3, 4, 5$:

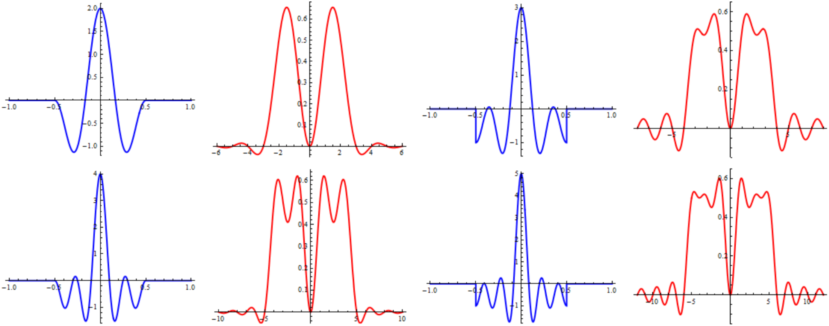
$$\widehat{f}(\xi) = \frac{\xi \sin \pi\xi}{\pi(\xi^2 - n^2)}.$$



Zero extension of sums of n first harmonics

$$f(t) = \sum_{k=1}^{n} \cos(2\pi n k t), \quad \widehat{f}(\xi) = \frac{\xi P_n(\xi)\sin \pi\xi}{\pi Q_n(\xi)}, \qquad n = 2, 3, 4, 5, \ldots$$

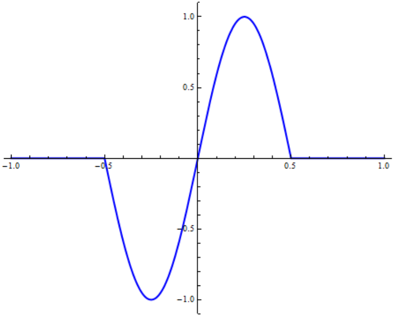
with P_n and Q_n polynomials.

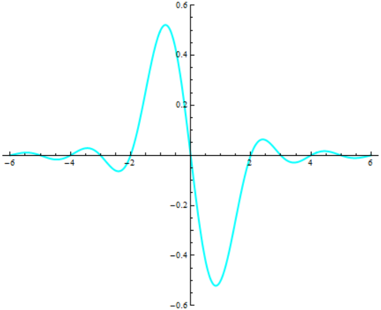


Example: zero extension of sine

The Fourier transform of the zero-extension of the sine function is purely imaginary and odd:

$$f(t) = \begin{cases} \sin(2\pi t), & \text{if} \quad |t| < \frac{1}{2}, \\ 0, & \text{if} \quad |t| \geq \frac{1}{2}, \end{cases} \qquad \widehat{f}(\xi) = 0 + i\,\frac{\sin \pi\xi}{\pi(\xi^2 - 1)}.$$

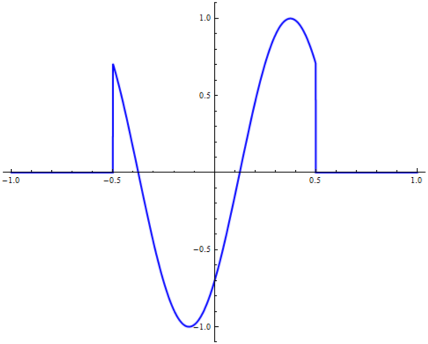


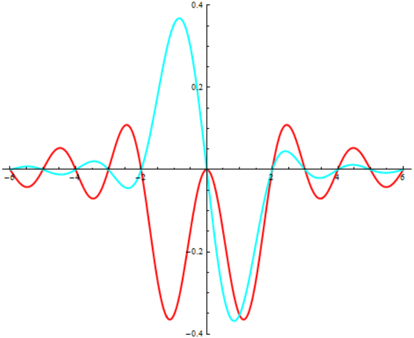


Example: shifted sine

The zero extension of the shifted sine function $\sin\left(2\pi(t-\frac{1}{8})\right)$ has complex valued Fourier transform with even real and odd imaginary components:

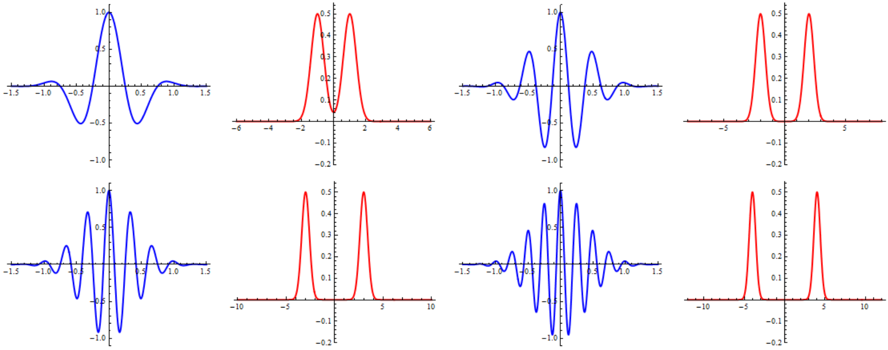
$$\widehat{f}(\xi)=\frac{\xi\sin\pi\xi}{\sqrt{2}\pi(\xi^2-1)}+i\,\frac{\sin\pi\xi}{\sqrt{2}\pi(\xi^2-1)}.$$





Example: cosine enveloped by Gaussian

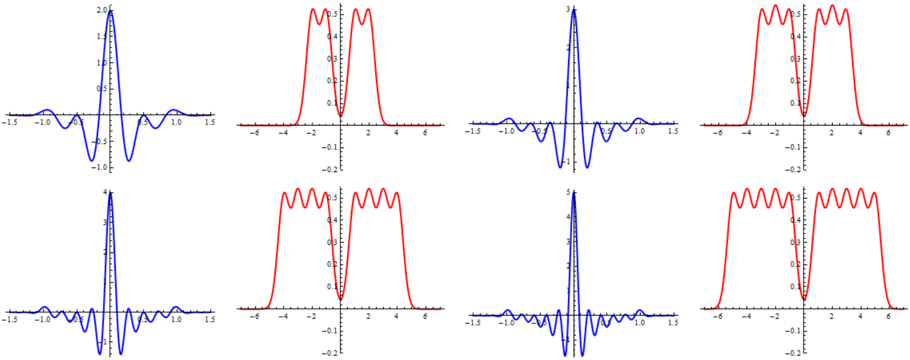
$$f(t) = \cos(2\pi n t)e^{-\pi t^2}, \quad \widehat{f}(\xi) = \frac{e^{-\pi(\xi+n)^2} + e^{-\pi(\xi-n)^2}}{2}, \qquad n = 1, 2, 3, 4.$$



And now with more frequencies:

$$f(t) = e^{-\pi t^2} \sum_{k=1}^{n} \cos(2\pi k t), \quad \widehat{f}(\xi) = \sum_{k=1}^{n} \frac{e^{-\pi(\xi+k)^2} + e^{-\pi(\xi-k)^2}}{2},$$

for $n = 2, 3, 4, 5$.



Properties of FT: linearity

From the linearity of the integral it immediately follows the linearity of the Fourier transform:

Proposition (Linearity)

The Fourier transform is linear with respect to functions:

$$\widehat{f+g}(\xi) = \widehat{f}(\xi) + \widehat{g}(\xi), \qquad \widehat{cf}(\xi) = c\widehat{f}(\xi), \quad c \in \mathbb{C}.$$

Symmetries: parity, reality and Hermiticity

Besides parity, already discussed, two new types of symmetry can be considered for complex-valued functions (signals) $f\colon \mathbb{R} \to \mathbb{C}$:

Definition (Real)

The function f is

- real if it takes real values: $\overline{f(t)} = f(t)$; and
- purely imaginary if it takes purely imaginary values: $\overline{f(t)} = -f(t)$.

Definition (Hermitian)

A function $f\colon \mathbb{R} \to \mathbb{C}$ is called

- *Hermitian* if $\overline{f(t)} = f(-t)$; and
- anti-Hermitian if $\overline{f(t)} = -f(-t)$.

As for parity, every function can be written, in a unique way, as the sum of a real and a purely imaginary function, and also as a sum of an Hermitian and an anti-Hermitian function.

There are several relations between the three symmetries. In each of the following situations, two of the symmetries implies the third:

- even, real and Hermitian;
- even, purely imaginary and anti-Hermitian;
- odd, real and anti-Hermitian;
- odd, purely imaginary and Hermitian.

Notice that the symmetries of the functions are related to symmetries of their components: if a complex valued function $f\colon \mathbb{R} \to \mathbb{C}$ has cartesian coordinate functions

$$f(t) = x(t) + i\,y(t) \quad x, y\colon \mathbb{R} \to \mathbb{R},$$

- $f(t)$ real $\Leftrightarrow y(t) = 0$;
- $f(t)$ purely imaginary $\Leftrightarrow x(t) = 0$;
- $f(t)$ even $\Leftrightarrow x(t)$ and $y(t)$ are both even;
- $f(t)$ odd $\Leftrightarrow x(t)$ and $y(t)$ are both odd;
- $f(t)$ Hermitian $\Leftrightarrow x(t)$ even and $y(t)$ odd;
- $f(t)$ anti-Hermitian $\Leftrightarrow x(t)$ odd and $y(t)$ even,

Similar relations can be found for the modulus and argument functions if

$$f(t) = A(t)e^{i\alpha(t)}, \qquad A(t) = |f(t)|, \quad \alpha(t) = \arg\big(f(t)\big),$$

with $A\colon \mathbb{R} \to (0, \infty)$ and $\alpha\colon \mathbb{R} \to (-\pi, \pi]$.

Properties of FT: symmetries

Fourier transform affects symmetry properties in the following way:

Proposition (Properties of symmetry)

Let $\widehat{f}$ be the Fourier transform of a function f. Then,

- *Time inversion: $\widehat{f}(-\xi)$ is the Fourier transform of $f(-t)$;*
- *f is even (resp. odd) if, and only if, $\widehat{f}$ is even (resp. odd);*
- *f is real (resp totally imaginary) if, and only if, $\widehat{f}$ is Hermitian (resp. anti-Hermitian);*
- *f is Hermitian (resp. anti-Hermitian) if, and only if, $\widehat{f}$ is real (resp. totally imaginary);*
- *Duality: $f(-\xi)$ is the Fourier transform of $\widehat{f}(t)$.*

One deduces symmetry properties of the amplitude and argument of $\widehat{f}$ from symmetry properties of the function f.

The proof of all these properties is obtained from the change of variables $t = -u$ in the integrals that define the Fourier transform.

For example, let $g(t) = f(-t)$. Its Fourier transform is

$$\begin{aligned}\widehat{g}(\xi) &= \int_{t=-\infty}^{t=\infty} g(t)e^{-2\pi i\xi t}\,dt = \int_{u=\infty}^{u=-\infty} g(-u)e^{2\pi i\xi u}\,(-du) \\ &= \int_{-\infty}^{\infty} f(u)e^{2\pi i\xi u}\,du = \int_{-\infty}^{\infty} f(u)e^{-2\pi i(-\xi)u}\,du = \widehat{f}(-\xi).\end{aligned}$$

Another example: assume that f is anti-Hermitian. Then,

$$\begin{aligned}\overline{\widehat{f}(\xi)} &= \int_{-\infty}^{\infty} \overline{f(t)e^{-2\pi i\xi t}}\,dt = \int_{-\infty}^{\infty} \overline{f(t)}e^{2\pi i\xi t}\,dt \\ &= \int_{-\infty}^{\infty} -f(-t)e^{2\pi i\xi t}\,dt = \int_{\infty}^{-\infty} -f(u)e^{-2\pi i\xi u}\,(-du) \\ &= -\int_{-\infty}^{\infty} f(u)e^{-2\pi i\xi u}\,du = -\widehat{f}(\xi),\end{aligned}$$

and the Fourier transform is purely imaginary.

Properties of FT: time shifting and scaling

The effect of the two most basic changes of variables: shift $t \mapsto t - t_0$ and scaling $t \mapsto at$ for nonzero $a \in \mathbb{R}$ can be easily determined by making simple changes of variables in the defining integrals:

Proposition (Shift and scaling)

Let $\widehat{f}$ be the Fourier transform of a function t. Then,

- *Time scale:* $\frac{1}{|a|}\widehat{f}\left(\frac{\xi}{a}\right)$ *is the transform of* $f(at)$;
- *Frequency scale:* $\widehat{f}(a\xi)$ *is the transform of* $\frac{1}{|a|}f\left(\frac{t}{a}\right)$;
- *Time shift:* $e^{-2\pi i\xi t_0}\widehat{f}(\xi)$ *is the transform of* $f(t-t_0)$;
- *Frequency shift:* $\widehat{f}(\xi-\xi_0)$ *is the transform of* $e^{2\pi i\xi_0 t}f(t)$.

For example, from the fact that $\widehat{\mathrm{rect}} = \mathrm{sinc}$ it follows that the Fourier transform of the shifted scaled multiple $c\,\mathrm{rect}(a(t-t_0))$ is the function:

$$\frac{c}{|a|}e^{-2\pi i\xi t_0}\,\mathrm{sinc}\left(\frac{\xi}{a}\right), \qquad c \in \mathbb{C}, \quad a \in \mathbb{R}, a \neq 0.$$

Proof of time scaling:

Let a be a positive real number and define $g(t) = f(at)$. The Fourier transform of g is computed from the change of variables $u = at$ (that preserves the integration limits)

$$\begin{aligned}\widehat{g}(\xi) &= \int_{-\infty}^{\infty} g(t) e^{-2\pi i \xi t}\, dt = \int_{-\infty}^{\infty} f(at) e^{-2\pi i \xi t}\, dt \\ &= \int_{-\infty}^{\infty} f(u) e^{-2\pi i \xi (u/a)}\, \frac{1}{a}\, du = \frac{1}{a} \int_{-\infty}^{\infty} f(u) e^{-2\pi i (\xi/a) u}\, du = \frac{1}{a} \widehat{f}\left(\frac{\xi}{a}\right).\end{aligned}$$

For negative values of the scaling factor a everything works exactly in the same way except that the change of variables $u = at$ interchanges the integration limits. Hence there appears a minus sign when putting the limits in the correct order and one obtains

$$\widehat{g}(\xi) = -\frac{1}{a} \widehat{f}\left(\frac{\xi}{a}\right) = \frac{1}{|a|} \widehat{f}\left(\frac{\xi}{a}\right),$$

and now this expression works for both signs of the scaling factor.

The frequency scaling property is proven in the same way.

Proof of time shift:
Consider the function $g(t) = f(t - t_0)$ for some real number $t_0 \in \mathbb{R}$. The Fourier transform of g is computed using the change of variables $u = t - t_0$:

$$\begin{aligned}\widehat{g}(\xi) &= \int_{-\infty}^{\infty} f(t - t_0) e^{-2\pi i \xi t}\, dt = \int_{-\infty}^{\infty} f(u) e^{-2\pi i \xi (u + t_0)}\, du \\ &= e^{-2\pi i \xi t_0} \int_{-\infty}^{\infty} f(u) e^{-2\pi i \xi u}\, du = e^{-2\pi i \xi t_0} \widehat{f}(\xi).\end{aligned}$$

Proof of frequency shift:
Consider the function $g(t) = e^{2\pi i \xi_0 t} f(t)$ for some real number $\xi_0 \in \mathbb{R}$. The Fourier transform of g is computed directly:

$$\begin{aligned}\widehat{g}(\xi) &= \int_{-\infty}^{\infty} g(t) e^{-2\pi i \xi t}\, dt = \int_{-\infty}^{\infty} e^{2\pi i \xi_0 t} f(t) e^{-2\pi i \xi t}\, dt \\ &= \int_{-\infty}^{\infty} f(t) e^{-2\pi i (\xi - \xi_0) t}\, dt = \widehat{f}(\xi - \xi_0).\end{aligned}$$

Convolution

The *convolution* is an operation between two functions with applications in probability, signal processing and many other fields.

Definition (Convolution)

The convolution of two functions $f, g\colon \mathbb{R} \to \mathbb{C}$ is the function $f * g\colon \mathbb{R} \to \mathbb{C}$ defined as:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(x)g(t - x)\, dx, \qquad t \in \mathbb{R}.$$

It can be thought of as a *cross-correlation* with one of the functions time-reversed. *see* and *see*.

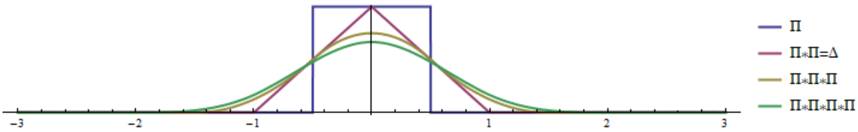
For example, the *density probability function of the sum* of two independent continuous random variables is the convolution of the corresponding density probability functions: $f_{X+Y} = f_X * f_Y$.

Example: convolution with the square pulse produces an "averaging" of the function:

$$(f * \mathrm{rect})(t) = \int_{-\infty}^{\infty} f(x)\,\mathrm{rect}(t-x)\,dx = \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} f(x)\,dx.$$

By convolving with the square pulse, the value of f at the point t is replaced by the average of f on the interval of length one centered at t.

This has the effect of smoothing the function: discontinuities of f may disappear and the function become more derivable. For example, the convolution of the square pulse $\mathrm{rect} = \Pi$ with itself produces the functions



In a digital image an analogous "discrete 2-dimensional convolution" replaces each pixel by the average of its neighbours.

Properties of FT: convolution

Under Fourier transform products and convolutions are converted one to each other:

Proposition (Convolution)

Under Fourier transform product corresponds to convolution:

$$\widehat{f * g}(\xi) = (\widehat{f} \cdot \widehat{g})(\xi), \qquad \widehat{f \cdot g}(\xi) = (\widehat{f} * \widehat{g})(\xi).$$

Proof of the convolution property

It is obtained as a consequence of Fubini's theorem on interchanging the order of integration in iterated integrals, together with a change of variables $u = t - x$ in the inner integral:

$$\begin{aligned}
\widehat{f * g}(\xi) &= \int_{-\infty}^{\infty} (f * g)(t) e^{-2\pi i \xi t}\, dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x) g(t - x)\, dx \right) e^{-2\pi i \xi t}\, dt \\
&= \int_{-\infty}^{\infty} f(x) \left(\int_{-\infty}^{\infty} g(t - x) e^{-2\pi i \xi t}\, dt \right) dx \\
&= \int_{-\infty}^{\infty} f(x) \left(\int_{-\infty}^{\infty} g(u) e^{-2\pi i \xi (u + x)}\, du \right) dx \\
&= \left(\int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x}\, dx \right) \left(\int_{-\infty}^{\infty} g(u) e^{-2\pi i \xi u}\, du \right) \quad = \widehat{f}(\xi) \widehat{g}(\xi).
\end{aligned}$$

Example

The convolution of the square pulse with itself is the triangle pulse:

$$\operatorname{rect} * \operatorname{rect}(t) = \int_{-\infty}^{\infty} \operatorname{rect}(x) \operatorname{rect}(t-x)\, dx$$

$$= \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \operatorname{rect}(x)\, dx = \int_{[t-\frac{1}{2}, t+\frac{1}{2}] \cap [-\frac{1}{2}, \frac{1}{2}]} 1\, dx$$

$=$ length of the interval $[t-\frac{1}{2}, t+\frac{1}{2}] \cap [-\frac{1}{2}, \frac{1}{2}]$, which is equal to $\operatorname{tri}(t)$:

- the two intervals do not overlap if $t < -1$ or $t > 1$;
- the intersection is $[-\frac{1}{2}, t+\frac{1}{2}]$ for $-1 \leq t \leq 0$, with length $1+t$;
- the intersection is $[t-\frac{1}{2}, \frac{1}{2}]$ for $0 \leq t \leq 1$, with length $1-t$.

Hence we recover the relation between their Fourier transforms:

$$\widehat{\operatorname{tri}}(\xi) = \widehat{\operatorname{rect} * \operatorname{rect}}(t) = \widehat{\operatorname{rect}}(\xi)\widehat{\operatorname{rect}}(\xi) = \operatorname{sinc}(\xi)^2$$

Properties of FT: conservation of energy

The *following identities* say that the Fourier transform preserves (Hermitian) dot products and norms of functions. As in Euclidean spaces, the two properties are equivalent: the first implies the second by definition and the second implies the first thanks to the polarization identity.

Theorem (Parseval theorem)

$$\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle \qquad \textit{i.e.} \qquad \int_{-\infty}^{\infty} f(t)\overline{g(t)}\,dt = \int_{-\infty}^{\infty} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,d\xi$$

Theorem (Plancherel theorem)

$$\|f\| = \|\widehat{f}\| \qquad \textit{i.e.} \qquad \int_{-\infty}^{\infty} |f(t)|^2\,dt = \int_{-\infty}^{\infty} \left|\widehat{f}(\xi)\right|^2 d\xi$$

This second identity is sometimes known as *conservation of energy* by the Fourier transform, because the *energy of a signal* $f : \mathbb{R} \to \mathbb{C}$ is defined as the square of the L^2-norm: $\|f\|^2 = \int_{-\infty}^{\infty} |f(t)|^2\,dt$.

Proof of the Parseval identity: using the formulas for the Fourier transform of g and the inverse transform of f, and interchanging the order of integration,

$$\begin{aligned}
\int_{-\infty}^{\infty} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\, d\xi &= \int_{-\infty}^{\infty} \widehat{f}(\xi) \left(\int_{-\infty}^{\infty} \overline{g(t)} e^{2\pi i \xi t}\, dt \right) d\xi \\
& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widehat{f}(\xi)\overline{g(t)} e^{2\pi i \xi t}\, dt\, d\xi \\
&= \int_{-\infty}^{\infty} \overline{g(t)} \left(\int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2\pi i t \xi}\, d\xi \right) dt = \int_{-\infty}^{\infty} \overline{g(t)} f(t)\, dt.
\end{aligned}$$

The Parseval identity says that the Fourier transform is a *unitary operator*: it preserves the Hermitian dot product.

Properties of FT: derivation

Using integration by parts with a given function f and the function $g(t) = e^{-2\pi i\xi t}$ one obtains Fourier transforms of derivatives:

Proposition (Derivation)

Let $\widehat{f}$ be the Fourier transform of a function f. Then,

- $2\pi i\xi\,\widehat{f}(\xi)$ *is the transform of* $f'(t)$*;*
- $\widehat{f}\,'(\xi)$ *is the transform of* $-2\pi it\,f(t)$*.*

Indeed,

$$\widehat{f'}\,(\xi) = \int_{-\infty}^{\infty} f'(t)e^{-2\pi i\xi t}\,dt = \left[f(t)e^{-2\pi i\xi t}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)(-2\pi i\xi)e^{-2\pi i\xi t}\,dt$$

$$= 0 - 0 + 2\pi i\xi\int_{-\infty}^{\infty} f(t)e^{-2\pi i\xi t}\,dt = 2\pi i\xi\widehat{f}(\xi).$$

The values are equal to zero at $\pm\infty$ because the existence of the Fourier transform of f implies that it must tend to zero at $\pm\infty$.

Outline

Generalized functions

In many applications of Fourier theory, specially in signal processing, it is common to work with objects that are not functions but in some sense behave as true functions would do in many situations. They are called *generalized functions*.

There are at several ways to approach them and do computations:

- Limits of sequences: generalized functions are seen as limits of sequences of functions which do not have a limit function in the ordinary sense. Here one works with the functions of the sequence.
- *Distributions*: by integrating a function g against other functions f one obtains a functional: a linear map sending functions f to numbers

$$f \mapsto \langle f, g \rangle = \int_{-\infty}^{\infty} f(t)g(t)\,dt.$$

  Generalized functions can also be seen as such linear maps sending functions to numbers that do not come from a function in this way.

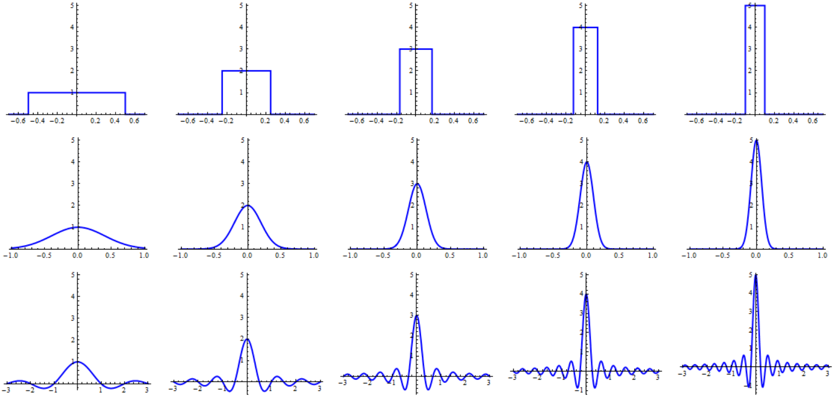
The Dirac delta function

The *Dirac delta function* δ is a generalized function that has the following properties, when looked at as if it was a function (which is not):

- Value at real numbers: $\delta(t) = \begin{cases} 0, & t \neq 0, \\ \infty, & t = 0. \end{cases}$.
- Integral: $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$.
- Functional: $\int_{-\infty}^{\infty} f(t)\delta(t)\, dt = f(0)$.
- Convolution: $\delta * f = f$ for every f continuous. Namely, δ is the neutral element of the convolution operation.
- Fourier transform: $\widehat{\delta} = \mathbf{1}$ and $\widehat{\mathbf{1}} = \delta$ with $\mathbf{1}$ the constant function $= 1$.
- Derivative as a functional: $\int_{-\infty}^{\infty} f(t)\delta'(t)\, dt = -f'(0)$.
- Primitive: $\delta = H'$ is the derivative of the Heaviside step function.

Dirac delta as a limit

The Dirac delta function can be introduced as the “limit” of the functions $n\,\mathrm{rect}(nt)$ or $ne^{-\pi(nt)^2}$, or even the functions $n\,\mathrm{sinc}(nt)$ when $n \to \infty$:



One can show the properties of δ working with the functions of the sequence. We take for example the sequence $f_n(t) = n\,\mathrm{rect}(nt)$.

The value at a point t is the limit of the values of the f_n:

$$\delta(t) = \lim_{n\to\infty} n\,\mathrm{rect}(nt) = \begin{cases} 0, & t \neq 0, \\ \infty, & t = 0. \end{cases}$$

The integral is the limit of the integrals of the f_n:

$$\int_{-\infty}^{\infty} \delta(t)\,dt = \lim_{n\to\infty} \int_{-\infty}^{\infty} n\,\mathrm{rect}(nt)\,dt = \lim_{n\to\infty} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n\,dt = \lim_{n\to\infty} 1 = 1.$$

For a function f continuous at 0:

$$\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = \lim_{n\to\infty} \int_{-\infty}^{\infty} f(t) n\,\mathrm{rect}(nt)\,dt = \lim_{n\to\infty} \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(t)\,dt = f(0).$$

Applying this property one obtains the convolution:

$$(\delta * f)(t) = \int_{-\infty}^{\infty} \delta(x) f(t-x)\,dx = f(t-0) = f(t) \quad \Rightarrow \quad \delta * f = f.$$

Since the Fourier transform of the functions $f_n(t) = n\,\mathrm{rect}(nt)$ is the function $\widehat{f_n}(\xi) = n\frac{1}{n}\,\mathrm{sinc}(\frac{\xi}{n}) = \mathrm{sinc}(\frac{\xi}{n})$, the Fourier transform of δ is:

$$\widehat{\delta}(\xi) = \lim_{n\to\infty} \widehat{f_n}(\xi) = \lim_{n\to\infty} \mathrm{sinc}\left(\frac{\xi}{n}\right) = \mathrm{sinc}(0) = 1 \quad \Rightarrow \quad \widehat{\delta} = \mathbf{1}.$$

Or also

$$\widehat{\delta}(\xi) = \int_{-\infty}^{\infty} \delta(t) e^{-2\pi i \xi t}\, dt = e^{-2\pi i \xi 0} = 1 \quad \forall \xi.$$

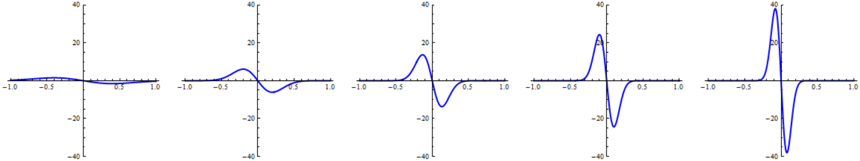
The function $\mathbf{1}$ does not have Fourier transform which is a function: the integrals in the definition diverge.
But it has a Fourier transform which is a generalized function: the δ function. To justify the identity $\widehat{\mathbf{1}} = \delta$ as generalized functions write $\mathbf{1}$ as the limit of the wide rectangular pulses $g_n(t) = \mathrm{rect}(\frac{t}{n})$.

Then $\widehat{g_n}(\xi) = n\,\mathrm{sinc}(n\xi)$ and these functions converge to δ for $n \to \infty$, hence

$$\widehat{\mathbf{1}} = \lim_{n\to\infty} \widehat{g_n} = \lim_{n\to\infty} n\,\mathrm{sinc}(n\xi) = \delta.$$

The derivative of δ is a strange "function": it is zero everywhere, and has integral $\int_a^b \delta'(t)\,dt = 0$ except $\int_a^0 \delta'(t) = \infty$ and $\int_0^b \delta'(t) = -\infty$.

To construct it we need to see δ as a limit of derivable functions. For example of the functions $f_n(t) = ne^{-\pi(nt)^2}$. Their derivatives are $f_n'(t) = -2\pi n^3 t e^{-\pi(nt)^2}$:



For a derivable function f, using properties of δ already proved, integration by parts gives

$$\int_{-\infty}^{\infty} f(t)\delta'(t)\,dt = [f(t)\delta(t)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(t)\delta(t)\,dt = 0 - 0 - f'(0) = -f'(0).$$

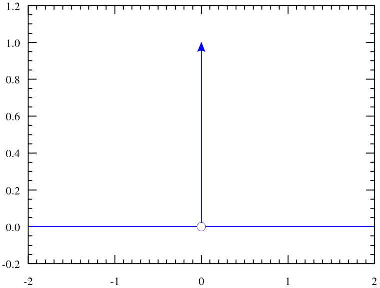
This could be also computed using the f_n and taking limits.

The primitive H of δ can be constructed as

$$H(x) = \lim_{n\to\infty} \int_{-\infty}^{x} n \operatorname{rect}(nt)\, dt = \lim_{n\to\infty} \int_{-\frac{1}{2n}}^{\min\{x,\frac{1}{2n}\}} n\, dt = \begin{cases} 0, & x < 0, \\ \frac{1}{2}, & x = 0, \\ 1, & x > 0. \end{cases}$$

Indeed, for $x = 0$ the integral is always equal to $\frac{1}{2}$, and for $x \neq 0$ it is equal to 0 or 1 depending on whether $x > 0$ or $x < 0$ for all n such that $\frac{1}{2n} < |x|$.

In signal theory the Dirac δ function is known as the unit impulse symbol and is graphically represented as an arrow pointing upwards placed on the origin of the real line:



Even though $\delta(0) = \infty$ the arrow in the plot has height one.
Arrows of other sizes are used to represent multiples of the function δ.

Distributions

Let $\mathcal{S}$ be some vector space of functions $f\colon \mathbb{R} \to \mathbb{C}$.
A given function $\phi\colon \mathbb{R} \to \mathbb{C}$ may produce, by integrating it against the functions of the space $\mathcal{S}$, a *functional* $\langle \cdot, \phi \rangle$: a $\mathbb{C}$-linear continuous map $\mathcal{S} \to \mathbb{C}$ (an element of the *dual space* of $\mathcal{S}$), defined in the following way:

$$f \mapsto \langle f, \phi \rangle = \int_{-\infty}^{\infty} f(t)\phi(t)\, dt.$$

For a given function ϕ one needs that the integrals[2] exist for all $f \in \mathcal{S}$.

In this way one can identify functions ϕ for which the integration paring is always defined with functionals $\mathcal{S} \to \mathbb{C}$. The linearity of the integral implies that this identification is linear: sums of functions produce the sum of the corresponding functionals and multiplication by scalars is also respected.

There are functionals that do not arise from any function in this way.
The theory of *distributions* enlarges the set of functions by allowing these new functionals to be considered together with the true functions.

[2]Warning! here there is no conjugation of the second function

The space $\mathcal{S}$ used to identify functions with functionals is called space of test functions. There are several choices giving rise to several theories. In Fourier theory the standard space is the *Schwarz space* of functions that are infinitely derivable and rapidly decreasing at infinity, in the sense that all derivatives tend to zero at infinity faster than any negative power:

$$\mathcal{S} = \left\{ f \colon \mathbb{R} \to \mathbb{C} : \lim_{x \to \pm\infty} |x|^n \left| f^{(k)}(x) \right| = 0 \quad \forall n, \forall k \right\}.$$

The corresponding distributions are called tempered distributions.

For example every $\mathcal{C}^\infty$ function with compact support belongs to $\mathcal{S}$. Another example of Schwarz function is e^{-x^2}.

Since these functions are so small at infinity there are many functions $\phi \colon \mathbb{R} \to \mathbb{C}$ that produce distributions because all integrals $\int_{-\infty}^{\infty} f(t)\phi(t)\, dt$ are convergent for all $f \in \mathcal{S}$.
For example, bounded functions with not too many discontinuities, polynomial functions, etc. give rise to distributions.

Dirac delta as a distribution

The map $f \mapsto f(0) \colon \mathcal{S} \to \mathbb{C}$ is a functional: it is a continuous linear map. But it is not of the form $\langle \cdot, \phi \rangle$ for some function ϕ. Hence it gives rise to a true new object: a distribution not coming from a function.
Sometimes one uses the notation $\phi(t)$ as if ϕ was a function.

The Dirac delta is defined as the distribution $\langle \cdot, \delta \rangle \colon \mathcal{S} \to \mathbb{C}$ corresponding to this linear map, by the following identity:

$$\langle f, \delta \rangle := f(0), \qquad \text{for every test function} \quad f \in \mathcal{S}.$$

In terms of integration of functions this corresponds to the identity:

$$\int_{-\infty}^{\infty} f(t)\delta(t)\, dt = f(0), \qquad \text{for every test function} \quad f \in \mathcal{S}.$$

In this interpretation δ is not a function of time in the usual way but a function that can be evaluated at functions: to every function $f \in \mathcal{S}$ it attaches a complex number. This number is sometimes denoted as

$$\delta[f] = \langle f, \delta \rangle = \int_{-\infty}^{\infty} f(t)\delta(t)\, dt = f(0).$$

Linearity: The sum and product by scalars $c_1, c_2 \in \mathbb{C}$ of distributions ϕ_1, ϕ_2 is defined by the image of test functions:

$$f \mapsto \langle f, c_1\phi_1 + c_2\phi_2 \rangle := c_1\langle f, \phi_1 \rangle + c_2\langle f, \phi_2 \rangle, \qquad \forall f \in \mathcal{S}.$$

Time shifting: For a function $g \in \mathcal{S}$, the functional corresponding to the time-shifted $g(t - t_0)$ is:

$$\begin{aligned} \langle f(t), g(t - t_0) \rangle &= \int_{-\infty}^{\infty} f(t)g(t - t_0)\, dt \\ &= \int_{-\infty}^{\infty} f(u + t_0), g(u)\, du = \langle f(t + t_0), g(t) \rangle. \end{aligned}$$

It sends a function $f(t)$ to the image by the functional $\langle \cdot, g \rangle$ of the function $f(t + t_0)$. Hence, for a distribution $\langle \cdot, \phi \rangle$ one defines the distribution $\phi(t - t_0)$ as the functional

$$\langle f(t), \phi(t - t_0) \rangle := \langle f(t + t_0), \phi(t) \rangle.$$

For the Dirac δ one obtains the shifted $\delta(t - t_0)$:

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)\, dt = \int_{-\infty}^{\infty} f(t + t_0)\delta(t)\, dt = f(0 + t_0) = f(t_0).$$

Time scaling: Also, for a nonzero real a, given a function $g(t) \in \mathcal{S}$, the functional corresponding to the function $g(at)$ is:

$$\langle f(t), g(at) \rangle = \int_{-\infty}^{\infty} f(t) g(at)\, dt$$
$$= \int_{-\infty}^{\infty} f\left(\frac{u}{a}\right) g(u) \frac{1}{|a|}\, du = \frac{1}{|a|} \left\langle f\left(\frac{t}{a}\right), g(t) \right\rangle.$$

It sends a function $f(t)$ to the image by the functional $\frac{1}{|a|}\langle \cdot, g \rangle$ of the function $f(t/a)$. Hence, for a distribution $\langle \cdot, \phi \rangle$ one defines the distribution $\phi(at)$ as the functional

$$\langle f(t), \phi(at) \rangle := \frac{1}{|a|} \left\langle f\left(\frac{t}{a}\right), \phi(t) \right\rangle.$$

For the Dirac δ one obtains the scaled $\delta(at)$:

$$\langle f(t), \delta(at) \rangle = \frac{1}{|a|} \left\langle f\left(\frac{t}{a}\right), \delta(t) \right\rangle = \frac{1}{|a|} f\left(\frac{0}{a}\right) = \frac{1}{|a|} f(0) = \frac{1}{|a|} \langle f(t), \delta(t) \rangle.$$

which can be stated as the identity $\delta(at) = \frac{\delta(t)}{|a|}$.

Fourier transform of distributions

Manipulation of distributions is performed looking at the effect on functions and translating it in terms of functionals.
Due to its rapid decreasing at infinity all test functions have Fourier transforms. If f and g are two test functions then one has the identity

$$\langle f, \widehat{g} \rangle = \langle \widehat{f}, g \rangle.$$

Indeed, interchanging the order of integration,

$$\begin{aligned}\langle f, \widehat{g} \rangle &= \int_{-\infty}^{\infty} f(t)\widehat{g}(t)\, dt = \int_{-\infty}^{\infty} f(t) \left(\int_{-\infty}^{\infty} g(x) e^{-2\pi i t x}\, dx \right) dt \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) e^{-2\pi i x t}\, dt \right) g(x)\, dx = \int_{-\infty}^{\infty} \widehat{f}(x) g(x)\, dx = \langle \widehat{f}, g \rangle.\end{aligned}$$

The identity $\langle f, \widehat{g} \rangle = \langle \widehat{f}, g \rangle$ suggests to define the Fourier transform of a distribution ϕ, corresponding to the functional $\langle \cdot, \phi \rangle$, as the distribution that sends $f \in \mathcal{S}$ to $\langle \widehat{f}, \phi \rangle$. Namely, $\langle \cdot, \widehat{\phi} \rangle$ is the functional defined by

$$\langle f, \widehat{\phi} \rangle := \langle \widehat{f}, \phi \rangle, \qquad \forall f \in \mathcal{S}.$$

Example: Fourier transform of δ

The Fourier transform of δ is the (distribution corresponding to the) constant $\mathbf{1}$ function:

$$\langle f, \widehat{\delta} \rangle = \langle \widehat{f}, \delta \rangle = \widehat{f}(0) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i t 0}\, dt = \int_{-\infty}^{\infty} \mathbf{1} f(t)\, dt = \langle f, \mathbf{1} \rangle.$$

This agrees with the direct computation of the values of $\widehat{\delta}$ from the definition of Fourier transform:

$$\widehat{\delta}(\xi) = \int_{-\infty}^{\infty} \delta(t) e^{-2\pi i \xi t}\, dt = e^{-2\pi i \xi 0} = 1 \qquad \forall \xi \in \mathbb{R}.$$

Now something very interesting happens. Even though the constant function $\mathbf{1}$ has no Fourier transform as a function since the integrals $\int_{-\infty}^{\infty} \mathbf{1} e^{-2\pi i \xi t}\, dt$ are not convergent at any single point $\xi \in \mathbb{R}$, it does have Fourier transform as a distribution. Namely, $\widehat{\mathbf{1}} = \delta$:

$$\langle f, \widehat{\mathbf{1}} \rangle = \langle \widehat{f}, \mathbf{1} \rangle = \int_{-\infty}^{\infty} \widehat{f}(t) \mathbf{1}\, dt = \int_{-\infty}^{\infty} \widehat{f}(t) e^{2\pi i 0 t}\, dt = f(0) = \langle f, \delta \rangle.$$

Example: $\delta(t-c) \leftrightarrow e^{-2\pi i c t}$

In the same way one checks that the Fourier transform of the time-shifted delta function $\delta_{t_0} = \delta(t - t_0)$ is the exponential function $\widehat{\delta}_{t_0} = e^{-2\pi i t_0 t}$ and, conversely, the transform of the exponential distribution [3] $e^{2\pi i t_0 t}$ is δ_{t_0}:

$$\langle f, \widehat{\delta}_{t_0} \rangle = \langle \widehat{f}, \delta_{t_0} \rangle = \widehat{f}(t_0) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i t_0 t}\, dt = \langle f, e^{-2\pi i t_0 t} \rangle,$$

$$\langle f, \widehat{e^{2\pi i t_0 t}} \rangle = \langle \widehat{f}, e^{2\pi i t_0 t} \rangle = \int_{-\infty}^{\infty} \widehat{f}(t) e^{2\pi i t_0 t}\, dt = f(t_0) = \langle f, \delta_{t_0} \rangle.$$

This correspondence may be interpreted as the fact that the waveform $e^{2\pi i t_0 t}$ contains only the single frequency $\xi = t_0$ and no other frequencies; hence its Fourier transform takes the value zero at all $\xi \neq t_0$ and the nonzero value ∞ at the frequency $\xi = t_0$.

The Fourier pair $\delta_{t_0} = \delta(t - t_0) \leftrightarrow e^{-2\pi i t_0 t}$ can also be obtained by applying the time-shifting property of Fourier transforms (which does also work for distributions: exercise!) to the Fourier pair $\delta \leftrightarrow \mathbf{1}$.

[3] It does not have Fourier transform as a function: integrals never converge!

Example: Fourier transforms of periodic functions

Using the δ function and its time-shifts one can enlarge the class of functions having Fourier transform and include all periodic functions, which have no Fourier transform as functions, but they do have as distributions.

Indeed, by linearity one gets:

$$\widehat{f}(\xi) = \sum_{n=-\infty}^{\infty} c_n \delta(\xi - n), \qquad \text{if} \quad f = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t}$$

for a periodic function f given as a Fourier series.

The Fourier pair $\sum c_n e^{2\pi i n t} \leftrightarrow \sum c_n \delta_n$ is interpreted in the sense that the signal given by the Fourier series contains only frequencies at the discrete values $\xi = n \in \mathbb{Z}$, the coefficient c_n being the amount of each frequency.

Derivation of distributions

Again one can look at the effect of derivation on the distributions defined by functions. All test functions are derivable and if $f, g \in \mathcal{S}$ then $fg \in \mathcal{S}$ too; in particular this product function is derivable and zero at $\pm\infty$. Applying integration by parts one obtains:

$$\langle f, g' \rangle = \int_{-\infty}^{\infty} f(t) g'(t)\, dt = \big[f(t) g(t) \big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(t) g(t)\, dt = -\langle f', g \rangle.$$

This identity suggests the definition of the derivative of a distribution ϕ: the derivative ϕ' is the distribution corresponding to the functional

$$f \mapsto \langle f, \phi' \rangle := -\langle f', \phi \rangle.$$

Namely, the value of the functional ϕ' at a test function f is defined as minus the value of the given functional ϕ at the test function f'.

Example: derivation and δ

The derivative of the delta function (distribution) is the distribution corresponding to the linear map

$$\delta' : \mathcal{S} \to \mathbb{C}, \qquad f \mapsto \langle f, \delta' \rangle = -\langle f', \delta \rangle = -f'(0).$$

Namely, the value of δ at a test function f is $-f'(0)$:

$$\int_{-\infty}^{\infty} f(t)\delta'(t)\, dt = -f'(0).$$

Moreover, δ is the (distributional) derivative of the *Heaviside step function*

$$H(t) = \begin{cases} 1, & \text{if} \quad x \geq 0, \\ 0 & \text{if} \quad x < 0. \end{cases}$$

Indeed, looking at H as a distribution its derivative H' satisfies

$$\begin{aligned} \langle f, H' \rangle = -\langle f', H \rangle = -\int_{-\infty}^{\infty} f'(t)H(t)\, dt &= -\int_{0}^{\infty} f'(t)\, dt \\ &= -\big[f(t)\big]_0^{\infty} = f(0) - f(\infty) = f(0) = \langle f, \delta \rangle. \end{aligned}$$

Distributional derivative

Notice that enlarging the set of functions with the addition of distributions some non-derivable functions can now be derived, with this new concept of distributional derivative. For example functions with step discontinuities become derivable.

Of course, the derivative is a (distribution coming from a) true function only in the "classical" case of functions derivable in the ordinary sense. The derivative of a discontinuous function is always a genuine non-function distribution, typically involving the δ function.

Multiplication and convolution

Even though distributions "behave like functions" in many respects, this is not always the case, and one needs to be careful in using them, going to the rigorous definitions in case of doubt.
For example, in general there is no such thing as the multiplication of two distributions. In particular the square δ^2 has no meaning.

What is always defined is the product of a distribution by a reasonable function. For functions f, g, h such that $f, h, fg, gh \in \mathcal{S}$ one has

$$\langle f, gh \rangle = \int_{-\infty}^{\infty} f(t)g(t)h(t)\,dt = \langle fg, h \rangle,$$

and this suggests to define the product $g\phi$ for a distribution ϕ from

$$\langle f, g\phi \rangle := \langle fg, \phi \rangle.$$

For example, the product $g\delta$ of a function g by the Dirac delta is the functional that sends every function $f \in \mathcal{S}$ to the number $g(0)f(0)$.

Something analogous happens with convolution: two distributions cannot be convolved in general, but, as for the other cases, looking at the effect of convolution on distributions determined by functions, it is natural to define the convolution $g * \phi$ of a function g and a distribution ϕ as the functional determined by the identity:

$$\langle f, g * \phi \rangle := \langle f * \widetilde{g}, \phi \rangle, \qquad \widetilde{g}(t) = g(-t).$$

From this definition one easily checks again that the δ function is the neutral element of the convolution:

$$\langle f, g * \delta \rangle = \langle f * \widetilde{g}, \delta \rangle = (f * \widetilde{g})(0) = \int_{-\infty}^{\infty} f(t)\widetilde{g}(0 - t)\, dt = \langle f, g \rangle.$$

Signals and distributions

There are many reasonable signals: constant, step, periodic, delta, train of deltas, etc. to which the definition of the Fourier transform cannot be applied because is not defined or the integrals are not convergent.

Since Fourier transform is a basic tool for the study of signals one needs a way to interpret the Fourier transforms of such signals. This is where the theory of distributions and generalized functions enters.

By enlarging the class of signals to be considered to include generalized functions one can play with these very basic signals and use Fourier transforms in a well-defined and precise way. Moreover, this new theory includes some surprises, like the derivability of some discontinuous functions.

Applying the standard rules of manipulation of functions and integrals one usually gets correct results, and one soon becomes used to look at δ and other distributions as if they where functions. Just be careful.

Remarks on notation

In different disciplines: mathematics, physics, signal theory, etc., and even in different specialities inside the same discipline, people uses different *notations* and *conventions* for the Fourier transform. For example:

- The Fourier transform of a function f is also denoted $\mathcal{F}(f)$ or just $\mathcal{F}f$, and the inverse Fourier transform of g is denoted $\mathcal{F}^{-1}(g)$ or $\mathcal{F}^{-1}g$.
- Sometimes the variable used in the transform is the *angular frequency* ω related to the frequency ξ by $\omega = 2\pi\xi$.

In signal processing the following notations are usual:

- the imaginary unit is denoted j instead of i;
- complex conjugation is denoted z^* instead of $\overline{z}$;
- signals, $f(t)$ in our notation, are denoted by lowercase letters $x(t)$, $y(t)$, $z(t)$, etc. depending on the variable t (time);
- their Fourier transforms, $\widehat{f}(\xi)$ in our notation, are denoted by the corresponding uppercase letters $X(f)$, $Y(f)$, $Z(f)$, etc. depending on the variable f (frequency, ξ in our notation).

The following table recapitulates definitions, properties and examples in signal processing notation: $x \leftrightarrow X$ and $y \leftrightarrow Y$ are Fourier pairs.

Properties	Signal	Transform
definition	$x(t) = \int_{-\infty}^{\infty} X(f) e^{2\pi j f t}\, df$	$X(f) = \int_{-\infty}^{\infty} x(t) e^{-2\pi j f t}\, dt$
linearity	$c_1 x(t) + c_2 y(t)$	$c_1 X(f) + c_2 Y(f)$
convolution	$x(t) * y(t)$	$X(f) \cdot Y(f)$
product	$x(t) \cdot y(t)$	$X(f) * Y(f)$
time derivative	$x'(t)$	$2\pi j f\, X(f)$
freq. derivative	$-2\pi j t\, x(t)$	$X'(f)$
Parseval	$\langle x, y \rangle =$	$= \langle X, Y \rangle$
energy	$\|x\|^2 =$	$= \|X\|^2$

Properties	Signal	Transform
time scale	$x(at)$	$X(f/a)/\|a\|$
frequency scale	$x(t/a)/\|a\|$	$X(af)$
time shift	$x(t-t_0)$	$e^{-2\pi jft_0}X(f)$
frequency shift	$e^{2\pi jf_0t}x(t)$	$X(f-f_0)$
parity	$x(-t)=\pm x(t)$	$X(-f)=\pm X(f)$
reality-Hermiticity	$x(t)^*=\pm x(t)$	$X(f)^*=\pm X(-f)$
Hermiticity-reality	$x(t)^*=\pm x(-t)$	$X(f)^*=\pm X(f)$
duality	$X(t)$	$x(-f)$

Examples	Signal	Transform
square pulse	$\mathrm{rect}(t) = \Pi(t)$	$\mathrm{sinc}(f)$
triangle pulse	$\mathrm{tri}(t) = \Delta(t)$	$\mathrm{sinc}^2(f)$
exponential	$e^{-\|t\|}$	$2/(1+4\pi^2 f^2)$
Gaussian	$e^{-\pi t^2}$	$e^{-\pi f^2}$
Dirac delta	$\delta(t)$	$\mathbf{1}$
shifted delta	$\delta(t - t_0)$	$e^{2\pi j t_0 f}$
single frequency	$e^{2\pi j f_0 t}$	$\delta(f - f_0)$
periodic Fourier series	$\sum c_n e^{-2\pi j n t}$	$\sum c_n \delta(f - n)$