

ALGEBRA

2. VECTOR SPACES

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We consider the set of n -tuples of real numbers:

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R}\}$$

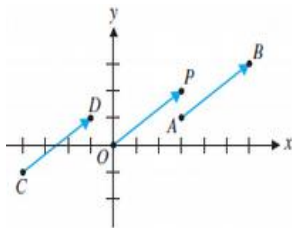
and we call its elements **vectors**.

Notation: When we talk about $v \in \mathbb{R}^n$ we usually think of v as a column vector,

$$v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

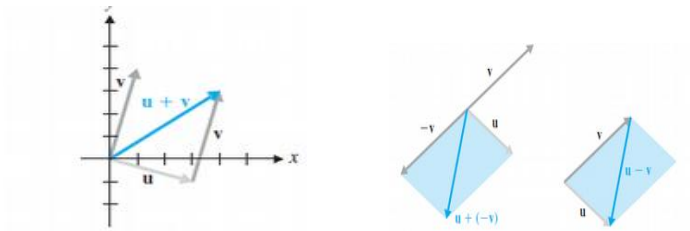
\mathbb{R}^2 : PHYSICAL INTERPRETATION

- $(x, y) \in \mathbb{R}^2$ can be thought either as a point or as a directed line segment between two points A and B , $(x, y) = \overrightarrow{AB}$. We think of them in this last sense (*vectors*).
- The vector \overrightarrow{AB} represents the displacement needed to get from A to B : x units along the x -axis and y -along the y -axis.
- Two vectors are equal if they represent the same displacement (equivalently, if they have the same length direction and sense).
- We can always think (x, y) as a vector of initial point $(0, 0)$ and end point (x, y) .

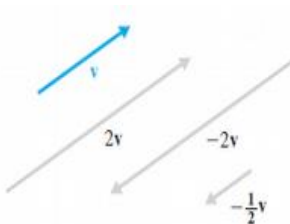


OPERATIONS IN \mathbb{R}^2

We can sum or subtract vectors



and we can multiply a vector by a constant (scalar)



- Vectors in \mathbb{R}^3 have a similar physical interpretation
- We can also sum two vectors and multiply a vector by a scalar. These operations can be done in coordinates: if $u = (x_1, x_2, x_3)$ and $v = (y_1, y_2, y_3)$, then

$$u + v = (x_1 + y_1, x_2 + y_2, x_3 + y_3),$$

$$c \cdot u = (cx_1, cx_2, cx_3) \text{ for any } c \in \mathbb{R}.$$

In \mathbb{R}^n we define the following operations:

sum: if $u = (x_1, x_2, \dots, x_n)$, $v = (y_1, y_2, \dots, y_n)$, then

$$u + v = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \in \mathbb{R}^n.$$

scalar multiplication: if $u = (x_1, x_2, \dots, x_n)$, $c \in \mathbb{R}$, then

$$c \cdot u = (c x_1, c x_2, \dots, c x_n) \in \mathbb{R}^n.$$

PROPOSITION

These operations in \mathbb{R}^n satisfy the following properties:

- ① $u + v = v + u$. *Commutativity*
- ② $(u + v) + w = u + (v + w)$. *Associativity*
- ③ \exists an element $\mathbf{0} \in \mathbb{R}^n$, called the zero vector, such that $u + \mathbf{0} = u$.
- ④ For each $u \in \mathbb{R}^n$, \exists an element $-u \in \mathbb{R}^n$ such that $u + (-u) = \mathbf{0}$.
- ⑤ $c \cdot (u + v) = c \cdot u + c \cdot v$. *Distributivity*
- ⑥ $(c + d) \cdot u = c \cdot u + d \cdot u$. *Distributivity*
- ⑦ $c \cdot (d \cdot u) = (cd) \cdot u$.
- ⑧ $1 \cdot u = u$.

VECTOR SPACE OVER \mathbb{K}

Let \mathbb{K} be \mathbb{Q} , \mathbb{R} , \mathbb{C} or any other (commutative) field (“cos”).

A **vector space over \mathbb{K}** (\mathbb{K} -e.v.) is a set E with two operations $+$ and \cdot ,

- + given $u, v \in E$, it assigns another element $u + v$ of E .
- given $u \in E$ and a scalar $c \in \mathbb{K}$, it assigns an element $c \cdot u \in E$

that satisfy the previous properties, i.e,

- $+$ is commutative, associative, has a neutral element (denoted $\mathbf{0}$ or $\vec{0}$) and every $u \in E$ has an inverse with respect to $+$ (denoted $-u$),
- \cdot and $+$ satisfy:

$$c \cdot (u + v) = c \cdot u + c \cdot v, \quad (c + d) \cdot u = c \cdot u + d \cdot u, \quad c \cdot (d \cdot u) = (cd) \cdot u, \quad 1 \cdot u = u$$

for any $u, v \in E$ and $c, d \in \mathbb{K}$.

The elements of a \mathbb{K} -e.v. are called **vectors**.

EXAMPLES OF VECTOR SPACES

- $\mathbb{K}^n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{K}\}$ is a \mathbb{K} -e.v. with the natural sum and product inherited by \mathbb{K} .
- $\mathcal{M}_{m \times n}(\mathbb{K}) = m \times n$ matrices with entries in \mathbb{K} and the natural operations of sum of matrices and multiplication by scalars is a \mathbb{K} -e.v.
- The set of polynomials of degree $\leq d$,
 $\mathbb{R}_d[x] = \{p(x) = a_0 + a_1x + \dots + a_dx^d \mid a_i \in \mathbb{R}\}$, is a vector space with the usual sum of polynomials and multiplication by a scalar.
- $\mathbb{R}[x] = \{\text{polynomials in one variable } x \text{ and coefficients in } \mathbb{R}\}$ is a \mathbb{R} -e.v.
- The set $\mathcal{F}(\mathbb{R}, \mathbb{R})$ of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a \mathbb{R} -e.v. with the usual sum of functions ($f + g$ is the function $(f + g)(x) = f(x) + g(x)$) and product by a scalar ($c \cdot f$ is the function $(c \cdot f)(x) = cf(x)$).

If E is a \mathbb{K} -e.v., then

(A) $0 \cdot u = \mathbf{0} = c \cdot \mathbf{0},$

(B) $(-1) \cdot u = -u,$

(C) $(-c) \cdot u = c \cdot (-u) = -(c \cdot u)$ (so we denote it by $-cu$),

(D) $c \cdot u = \mathbf{0} \Leftrightarrow c = 0 \text{ or } u = 0.$

Note: Usually we avoid the notation \cdot

DEFINITION

A vector u is a **linear combination** of vectors u_1, \dots, u_k if there are scalars c_1, \dots, c_k such that $u = c_1 u_1 + \dots + c_k u_k$ (the scalars c_i are the **coefficients** of the linear combination).

Finding out whether a vector in \mathbb{K}^n is a linear combination of a collection of given vectors is equivalent to solving a linear system of equations:

PROPOSITION

A system $Ax = b$ is consistent if and only if b is a linear combination of the columns of A .

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DEFINITION

Let E be a \mathbb{K} -e.v. Then a subset $V \neq \emptyset$ of E is a **vector subspace** if V is itself a vector space (with $+$ and \cdot of E). This is equivalent to:

- 1 If u and v are in V , then $u + v$ is in V .
- 2 If u is in V and c is a scalar, then $c \cdot u$ is in V .

Ex:

- $V = \mathbb{K}^n$ is a vector subspace of \mathbb{K}^n .
- $V = \{\mathbf{0}\}$ is a vector subspace (of any E).
- $V = \{(x, y, z) \in \mathbb{R}^3 \mid x - y = 0, 3z = 0\}$ is a vector subspace of \mathbb{R}^3 .
- $F = \{(a + 2b, 0, b) \in \mathbb{R}^3 \mid a, b \in \mathbb{R}\}$ is a vector subspace of \mathbb{R}^3 .

- Every subspace contains the zero vector.
- Properties 1 and 2 can be combined:
 $V \neq \emptyset$ is a subspace \Leftrightarrow for any u_1, \dots, u_k in V and c_1, \dots, c_k in \mathbb{K} , the linear combination

$$c_1 u_1 + \dots + c_k u_k$$

is also in V .

That is, vector subspaces are closed under linear combinations.

PROPOSITION

Let $Ax = 0$ be a linear system, where $A \in \mathcal{M}_{m,n}(\mathbb{K})$. Then, the set of solutions $V = \{v \in \mathbb{K}^n \mid Av = 0\}$ is a vector subspace of \mathbb{K}^n .

Let v_1, v_2, \dots, v_k be vectors in E .

DEFINITION

The set of all linear combinations of v_1, v_2, \dots, v_k ,

$$\{c_1 v_1 + \dots + c_k v_k \mid c_1, \dots, c_k \in \mathbb{K}\}$$

is called the **span of v_1, v_2, \dots, v_k** and is denoted as $[v_1, v_2, \dots, v_k]$.

PROPOSITION

$V = [v_1, v_2, \dots, v_k]$ is a vector subspace and is the smallest subspace containing $\{v_1, \dots, v_k\}$.

We say that $\{v_1, v_2, \dots, v_k\}$ is a **system of generators of V** , and also that V is **spanned by v_1, v_2, \dots, v_k** .

Examples:

- $\mathbb{R}^n = [(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)]$.
- $V = \{(x, y, z) \in \mathbb{R}^3 \mid x - y = 0, 3z = 0\} \Rightarrow V = [(1, 1, 0)]$.
- $V = \{(x, y, z) \in \mathbb{R}^3 \mid x - y = 0\} \Rightarrow V = [(1, 1, 0), (0, 0, 1)]$.
- $F = \{(a + 2b, 0, b) \in \mathbb{R}^3 \mid a, b \in \mathbb{R}\} = [(1, 0, 0), (2, 0, 1)]$.

A vector space E is **finitely generated** (f.g.) if it can be spanned by a finite collection of vectors.

- \mathbb{K}^n is f.g.
- $\mathbb{K}[x]$ is not f.g.

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DEFINITION

The vectors $v_1, v_2, \dots, v_k \in E$ are **linearly dependent** (l.d.) if there are scalars c_1, c_2, \dots, c_k , at least one $\neq 0$, such that $c_1 v_1 + \dots + c_k v_k = \mathbf{0}$. Otherwise, we say that v_1, v_2, \dots, v_k are **linearly independent**, l.i..

v_1, v_2, \dots, v_k are linearly independent if any linear combination of them that equals $\mathbf{0}$, $c_1 v_1 + \dots + c_k v_k = \mathbf{0}$, implies $c_1 = c_2 = \dots = c_k = 0$.

Remarks:

- 1 Any set of vectors containing $\mathbf{0}$ is linearly dependent.
- 2 Two vectors v_1, v_2 are linearly dependent \Leftrightarrow one is multiple of the other.
- 3 v_1, v_2, \dots, v_k in E are l.d. \Leftrightarrow at least one of the vectors can be expressed as a linear combination of the others.
- 4 If v_1, \dots, v_k are l.i. and $u \notin [v_1, \dots, v_k] \Rightarrow v_1, \dots, v_k, u$ are l.i.

DEFINITION

Let $V \subset E$ be a vector subspace. A collection of vectors $\{v_1, \dots, v_k\}$ is a **basis of V** if

- 1 $V = [v_1, \dots, v_k]$ and
- 2 $\{v_1, \dots, v_k\}$ is linearly independent.

Ex: $(1, 1, 0), (0, 0, 1)$ is a basis of $V = \{(x, y, z) \in \mathbb{R}^3 \mid x - y = 0\}$.

There are some **standard** (or *natural*, *canonical*) bases of certain vector spaces:

- \mathbb{K}^n : $\{e_1, e_2, \dots, e_n\}$ where $e_i = (0, \dots, \overset{i}{1}, \dots, 0)$ for $i = 1, 2, \dots, n$.
- $\mathcal{M}_{m \times n}$: $\{E_{i,j}\}_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$ where $E_{i,j} \in \mathcal{M}_{m \times n}$ has 1 at entry (i,j) and 0's elsewhere.
- $\mathbb{R}_d[x]$: $\{1, x, \dots, x^d\}$

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- $\mathcal{M}_{m \times n}$: $\{E_{i,j}\}_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$ where $E_{i,j} \in \mathcal{M}_{m \times n}$ has 1 at entry (i, j) and 0's elsewhere.
- $\mathbb{R}_d[x]$: $\{1, x, \dots, x^d\}$

THEOREM (STEINITZ SUBSTITUTION LEMMA)

Let E be an f.g. \mathbb{K} -e.v. Let v_1, \dots, v_m be generators of E and $u_1, \dots, u_n \in E$ be l.i. Then, $n \leq m$ and one can substitute n vectors of $\{v_1, \dots, v_m\}$ by u_1, \dots, u_n such that the new collection of vectors is still a system of generators for E .

COROLLARY (THE BASIS THEOREM)

Any two bases of a f.g. vector space have the same number of elements .

This number is called the **dimension** of the vector space and noted as $\dim(E)$. By convention, $\dim(\{\vec{0}\}) = 0$.

PROPOSITION

Let E be a vector space of dimension n . Then:

- ① *Any system of generators for E contains $\geq n$ vectors. Moreover, it contains a basis of E .*
- ② *Any linearly independent set in E contains $\leq n$ vectors. Moreover, it can be extended to a basis of E (by choosing vectors of a given basis of E conveniently).*
- ③ *Any linearly independent set of n vectors in E is a basis for E .*
- ④ *Any system of generators for E consisting of n vectors is a basis for E .*

Hence,

$$\begin{aligned} n &= \text{minimum number of elements in a system of generators of } E \\ &= \text{maximum number of l.i. vectors in } E. \end{aligned}$$

THEOREM

Let $V_1 \subset V_2$ be subspaces of E and $\dim(E) = n$. Then:

- ① $\dim V_1 \leq \dim V_2 \leq n$.
- ② $\dim V_1 = \dim V_2$ if and only if $V_1 = V_2$.

RANK (REVISITED)

THEOREM

Given $v_1, v_2, \dots, v_k \in \mathbb{K}^n$, write $A = (v_1, \dots, v_k) \in \mathcal{M}_{n,k}(\mathbb{K})$. Then,

- A) v_1, v_2, \dots, v_k are l.d. \Leftrightarrow the homogeneous system $Ax = 0$ has a nontrivial solution (indeterminate system).
- B) v_1, v_2, \dots, v_k are l.i. $\Leftrightarrow \text{rank}(A) = k$.
- C) v_1, v_2, \dots, v_k are a system of generators of $\mathbb{K}^n \Leftrightarrow \text{rank}(A) = n$.
- D) v_1, v_2, \dots, v_k is a basis of $\mathbb{K}^n \Leftrightarrow k = n$ and $\text{rank}(A) = n$.

PROPOSITION

The **rank** of a matrix A equals:

- the dimension of the row span of A (number of linearly independent rows) and
- the dimension of the column span of A (number of linearly independent columns).

FINDING A BASIS FROM GENERATORS

If $V = [v_1, v_2, \dots, v_k] \subset \mathbb{K}^n$, then a basis of V can be obtained by applying one the following methods:

- 1 Write the vectors v_1, \dots, v_k as the rows of a matrix A , and reduce A to row echelon form \bar{A} (Gaussian elimination). The nonzero rows of \bar{A} are a basis of V .
- 2 Write the vectors v_1, \dots, v_k as the columns of a matrix B . Then, reduce B to row echelon form \bar{B} (Gaussian elimination). The columns of \bar{B} with pivots indicate which vectors v_1, \dots, v_k to choose to obtain a basis of V .

EXTENDING TO A BASIS OF \mathbb{K}^n

If u_1, \dots, u_k are linearly independent vectors of \mathbb{K}^n , then they can be extended to a basis of \mathbb{K}^n :

- Write the vectors u_1, \dots, u_k as the columns of a matrix B , and take $M = (B \mid Id)$.
- Then, reduce M to row echelon form $\bar{M} = (\bar{B} \mid \bar{Id})$ (Gaussian elimination).
- Collect the columns of \bar{Id} with a pivot and choose the corresponding vectors of the standard basis (columns of Id) of \mathbb{K}^n .
- u_1, \dots, u_k together with these last vectors form a basis of \mathbb{K}^n .

The same can be done if u_1, \dots, u_k are linearly independent vectors of a vector subspace V :

instead of Id , take a matrix formed by a basis v_1, \dots, v_d of V and do the same process as above for $M = (u_1, \dots, u_k \mid v_1, \dots, v_d)$.

From “generators” to “equations”:

If $V = [v_1, \dots, v_k] \subset \mathbb{K}^n$:

Write $M = (v_1, \dots, v_k)$, and form an augmented matrix $(M|x)$ with $x =$ column with entries x_1, x_2, \dots, x_n .

Then $x \in [v_1, \dots, v_k]$ if and only if $\text{rank}(M|x) = \text{rank}(M)$.

There are 2 options:

- Reduce M to echelon form $(\bar{M}|\bar{x})$ by Gaussian elimination \Rightarrow a linear system of equations for V is obtained by writing the equations that correspond to zero rows of \bar{M} .
- If $\text{rank}(M) = k$, the equations are formed by the vanishing of the $(k+1) \times (k+1)$ minors of $(M|x)$ that contain a chosen non-zero $k \times k$ minor of M .

From “equations” to “generators”:

If $V = \{u \in \mathbb{K}^n \mid Au = 0\}$ (solutions to a homogeneous system):

- It is enough to solve the system to obtain a system of generators of V .
- Moreover, if we give values 0's and 1's to the free variables, these generators form a basis and $\dim(V) = n - \text{rank}(A)$.

We have proved:

COROLLARY

A subset V of \mathbb{K}^n is a subspace \Leftrightarrow it is the set of solutions to a homogeneous system.

VECTOR SPACES NOT FINITELY GENERATED

If we consider any collection of vectors $\{v_i\}_{i \in I}$ of a \mathbb{K} -e.v E (where I is any set of indices), then one has to redefine linear combinations:

- $v \in E$ is a **linear combination of** $\{v_i\}_{i \in I}$ if v is a linear combination of a finite collection of them,

$$v = x_1 v_{i_1} + x_2 v_{i_2} + \dots + x_k v_{i_k}$$

for certain $x_1, \dots, x_k \in \mathbb{K}$ and $i_1, \dots, i_k \in I$.

Therefore,

- $\{v_i\}_{i \in I}$ are linearly dependent if $\mathbf{0}$ is a linear combination of (a finite number of) them.
- $\{v_i\}_{i \in I}$ are linearly independent if any finite subfamily is linearly independent.

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THEOREM

Let $B = \{v_1, \dots, v_n\}$ be a basis of a \mathbb{K} -e.v. E . Then, for every vector $v \in E$, there is exactly one way to write v as a linear combination of the vectors in B , that is, there exist $c_1, \dots, c_n \in \mathbb{K}$ such that $v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$ and moreover, these c_1, \dots, c_n are unique.

DEFINITION

The c_1, c_2, \dots, c_n are called the **coordinates of v with respect to B** .

We will use the notation

$$v_B = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

Coordinates preserve linear combinations:

If v_1, \dots, v_k are in E and $B = \{v_1, \dots, v_n\}$ is a basis of E then

$$(x_1 u_1 + \dots + x_k u_k)_B = x_1 (u_1)_B + \dots + x_k (u_k)_B.$$

In particular,

- u_1, \dots, u_k are l.i. $\Leftrightarrow (u_1)_B, \dots, (u_k)_B$ are l.i. in \mathbb{K}^n .

CHANGE OF BASIS

Let $B = \{u_1, \dots, u_n\}$ and $C = \{v_1, \dots, v_n\}$ be bases of E .

Denote by $A_{B \rightarrow C}$ the $n \times n$ matrix whose columns are the coordinate vectors $((u_1)_C, \dots, (u_n)_C)$ of B with respect to C . This is the **change-of-basis matrix from B to C** :

$$A_{B \rightarrow C} = \begin{pmatrix} (u_1)_C & \dots & (u_n)_C \end{pmatrix}.$$

PROPOSITION

- ❶ $A_{B \rightarrow C} w_B = w_C$ for all $w \in E$.
- ❷ $A_{B \rightarrow C}$ is invertible, and $(A_{B \rightarrow C})^{-1} = A_{C \rightarrow B}$.
- ❸ If D is another basis of E , then $A_{C \rightarrow D} A_{B \rightarrow C} = A_{B \rightarrow D}$.

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INTERSECTION & SUM OF SUBSPACES

Given V_1, V_2 vector subspaces of E , define

- ① **Intersection of V_1 and V_2** is $V_1 \cap V_2 = \{v \in E \mid v \in V_1, v \in V_2\}$.
- ② **Sum of V_1 and V_2** is $V_1 + V_2 = \{v_1 + v_2 \in E \mid v_1 \in V_1, v_2 \in V_2\}$.
Computation: If $V_1 = [u_1, \dots, u_r]$ and $V_2 = [v_1, \dots, v_s]$, then
 $V_1 + V_2 = [u_1, \dots, u_r, v_1, \dots, v_s]$.

THEOREM

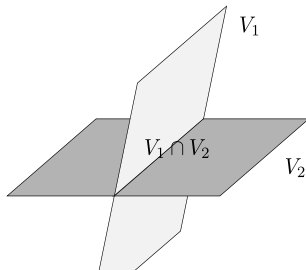
- ① $V_1 \cap V_2$ and $V_1 + V_2$ are vector subspaces of E .
- ② *Grassmann formula*
 $\dim(V_1 \cap V_2) + \dim(V_1 + V_2) = \dim(V_1) + \dim(V_2)$.

THEOREM

- 1 $V_1 \cap V_2$ and $V_1 + V_2$ are vector subspaces of E .
- 2 Grassmann formula: if $\dim(E) < \infty \Rightarrow$

$$\dim(V_1 \cap V_2) + \dim(V_1 + V_2) = \dim(V_1) + \dim(V_2)$$

$$\begin{array}{ll} V_1 = [(1, 0, 1), (0, 2, 3)] & V_1 \cap V_2 = [(1, 0, 1)] \\ V_2 = [(0, 1, 0), (1, 1, 1)] & V_1 + V_2 = \mathbb{R}^3 \end{array}$$



DEFINITION

E is a **direct sum** of subspaces F_1, F_2, \dots, F_m if any $w \in E$ can be written in a **unique way** as $w = v_1 + v_2 + \dots + v_m$ with $v_i \in F_i$, $i = 1, \dots, m$.

In this case we use the notation $E = F_1 \oplus F_2 \oplus \dots \oplus F_m$.

PROPOSITION

Let F_1, F_2 be two subspaces of E . Then $E = F_1 \oplus F_2$ if and only if the following two conditions hold:

$$E = F_1 + F_2,$$

$$F_1 \cap F_2 = \{0\}.$$

If $E = F_1 \oplus F_2$, we say that F_2 is a **complementary subspace** to F_1 (and vice-versa).

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QUOTIENT SPACE

Let E be a \mathbb{K} -e.v. and $F \subset E$ be a subspace.

DEFINITION

If $v \in E$, we define the **class of v modulo F** as

$$\bar{v} = \{u \in E \mid u - v \in F\};$$

it can be also denoted as $v + F$.

The set $E/F := \{\bar{v} \mid v \in E\}$ is called the **quotient** of E by F .

E/F is the quotient set of E by the equivalence relation

$$u \sim v \Leftrightarrow u - v \in F$$

and hence, $\bar{u} = \bar{v} \Leftrightarrow u - v \in F$.

Any element of \bar{v} is called a **representative** of the class.

AS A VECTOR SPACE

In E/F we can define two operations:

- $+ \quad \overline{u} + \overline{v} := \overline{u + v}, u, v \in E$
- $\cdot \quad c \cdot \overline{v} := \overline{c \cdot v}, v \in E, c \in \mathbb{K}.$

With these operations E/F is a **vector space**.

THEOREM

Let E be a \mathbb{K} -e.v. and $F \subset E$ be a subspace. Given $u_1, \dots, u_k \in E$ the following are equivalent:

- $\bullet \quad \overline{u_1}, \dots, \overline{u_k}$ is a basis of E/F
- $\bullet \quad u_1, \dots, u_k$ is a basis of a complementary subspace of F in E .

In particular, if E has finite dimension, then

$$\dim(E/F) = \dim(E) - \dim(F)$$

OUTLINE

- 1 \mathbb{R}^n AND OTHER VECTOR SPACES
- 2 VECTOR SUBSPACES
- 3 LINEAR DEPENDENCY, BASIS AND DIMENSION
- 4 COORDINATES AND CHANGE OF BASIS
- 5 INTERSECTION AND SUM
- 6 QUOTIENT SPACE
- 7 BIBLIOGRAPHY

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Additional

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