1 Statistical Signal Modelling

1.2: Modelling of memoryless sources

1.2.b: Quantization

Statistical Signal Modelling

1.2

1. Introduction to IPA and Random variable
2. **Modelling of memoryless processes**
   - Sample-wise operators
   - Uniform and non-uniform quantization
   - Examples: Sample-wise video processing
3. **Discrete Stochastic Processes**
   - Definition
   - Autocorrelation: Deterministic signals and processes
   - Stationarity and Ergodicity
   - Power Spectral Density (PSD)
   - Stochastic processes filtering
   - Examples

Unit Structure

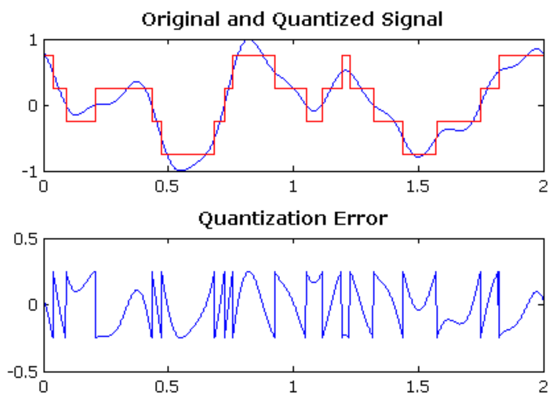
1.2

1. **Introduction**
   - Need of quantization, Loss of information, Distortion
2. **Uniform quantization**
   - Mid-tread and Mid-rise quantizers
   - Quality measures
3. **Non-uniform quantization**
   - Compander
   - A-law and μ-law in speech
4. **Optimal quantization**
   - Max-Lloyd algorithm
   - Max-Lloyd initialization

Introduction: Quantization

1.2

- A very common and useful sample-wise operation is quantization. **Quantization** is involved in nearly all digital signal processing.
- **Scalar quantization** maps values from a **scalar continuous signal**, or a discrete one with very high resolution, onto values from a **set with a finite number of elements**:
  - **Truncation** and **rounding** are simple cases of quantization
  - It can be extended to **vector quantization**: very useful tool
  - The difference between an input value and its quantized value is referred to as **quantization error**



https://commons.wikimedia.org/windex.php?curid=3606377

Introduction: Quantization

1.2

- **Storage** implies **quantization**. All kinds of data are to be stored: original data, transformed data, computed descriptors, …
- **Quantization** implies **loss of information and quality**: It is a non-reversible operation.

JPEG degradation test (just for illustrative purposes)
https://www.youtube.com/watch?v=EPEkrLvzUHI

- It is necessary to define **measures of quality** to assess the performance of the quantization. Commonly, **distortion measures** are defined:
  - MSE, SNR, Perceptual measures: $d(x, q(x))$
  - The assessment will be done in **statistical terms**: $E\{d(X, q(X))\}$

Distortion / Quality Measures

1.2

$$e[n] = x[n] - q(x[n])$$

- **Mean Square Error (MSE):**
  - Estimation of expectation

$$\sigma^2_{MSE} = \sigma^2_e = E\left\{\left|e[n]\right|^2\right\} = \frac{1}{N}\sum_{i=1}^{N}\left|e[n]\right|^2$$

- **Mean Absolute Difference (MAD):**
  - Faster computation
  - Less sensitive to outliers

$$C_{MAD} = E\left\{\left|e[n]\right|\right\} = \frac{1}{N}\sum_{i=1}^{N}\left|e[n]\right|$$

- **Signal to Noise Ratio (SNR):**
  - Comparison of estimated powers

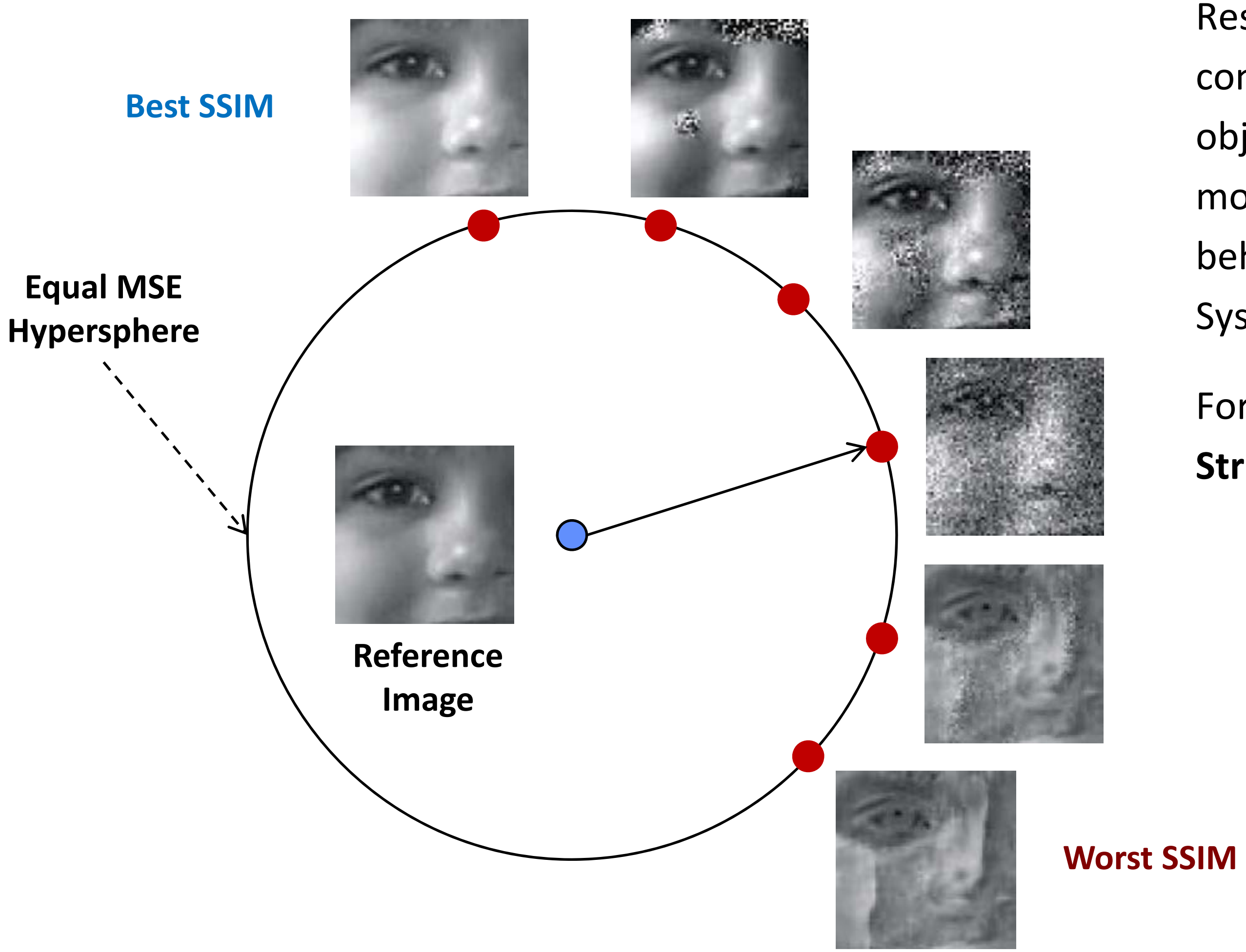
$$SNR(dB) = 10\log\frac{\sigma^2_x}{\sigma^2_e}$$

- **Peak Signal to Noise Ratio (PSNR):**
  - Very used in image processing
  - M = Maximum peak-to-peak value of the representation

$$PSNR(dB) = 10\log\frac{M^2}{\sigma^2_e}$$

Distortion / Quality Measures

1.2



Researchers are continuously looking for objective measures that model better the subjective behavior of the Human Visual System.

For instance, the so-called **SSIM**: **Structural Similarity Index**.

Z. Wang and A.C. Bovik, " Mean Square Error: Love it or Leave it?"
IEEE Signal Processing Magazine, pp. 98 - 117, January 2009.

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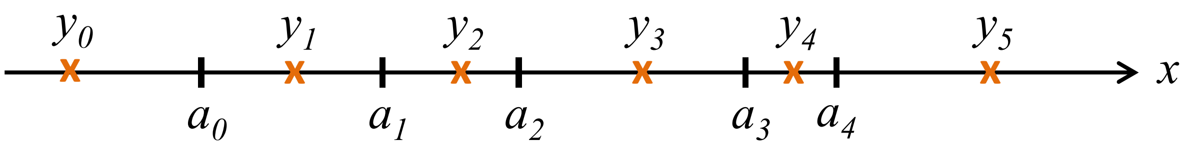
Quantization

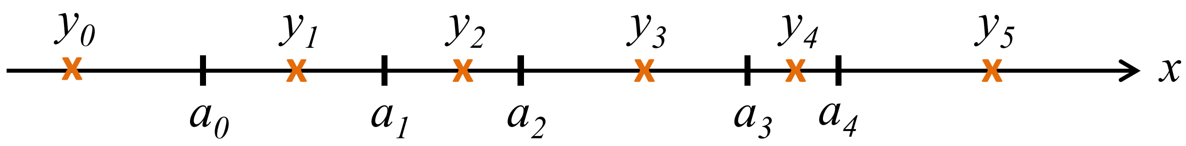
1.2



- The input (x) is a **scalar analog value** or a digital one with very high resolution
- The output (y) is **one from N possible values**
- A **quantizer** ($q(.)$) performs a mapping: q: $R \rightarrow C$
  - The **encoder** can be seen as **selection of a cell in a partition**:
    - **Classification** $i = \alpha(x)$ where the index links to a set of disjoint cells $S = \{S_i,\ i = 1, \ldots, \mathrm{N}\}$ that forms a **partition**.
  - The **decoder** can be seen as **selection of a codeword** (cell representative)
    - **Representation** $y_i = \beta(i)$ where a specific codeword is assigned $C = \{y_1, y_2, \ldots, y_N\} \subset R$.
- Therefore, the **quantizer** is defined as:

$$\boxed{\begin{aligned} q(x) &= \beta(\alpha(x)) \\ q &\equiv \{S, C\} = \{\alpha, \beta\} \end{aligned}}$$

Quantization



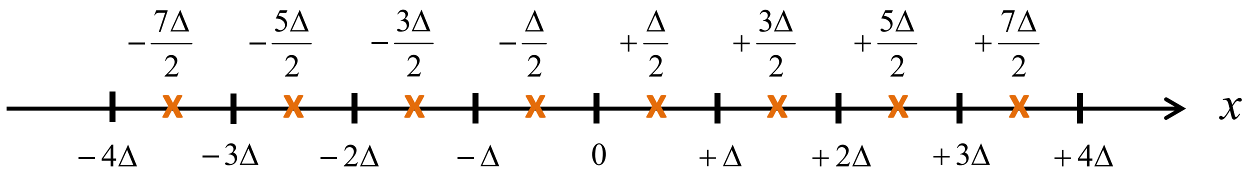
- The quantization step **introduces losses** in the signal representation that have to be assessed. These losses are defined as the quantization error:
  - The **quantization error**: $e_x = x - q(x)$

- **Fixed length encoding**: The **entropic coding** of the source created by the quantized signal assuming fixed length codes leads to:

$$R = \lceil \log_2 N \rceil = \lceil \log_2 |C| \rceil \text{ binary digits/symbol}$$

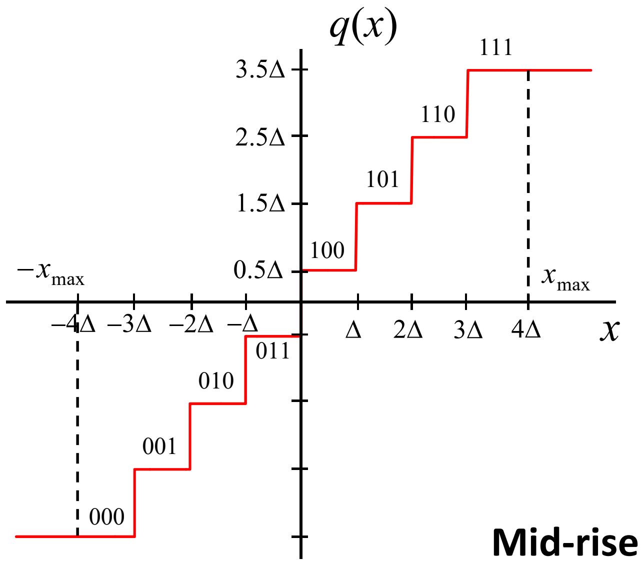
Uniform Quantization: Mid-rise

1.2



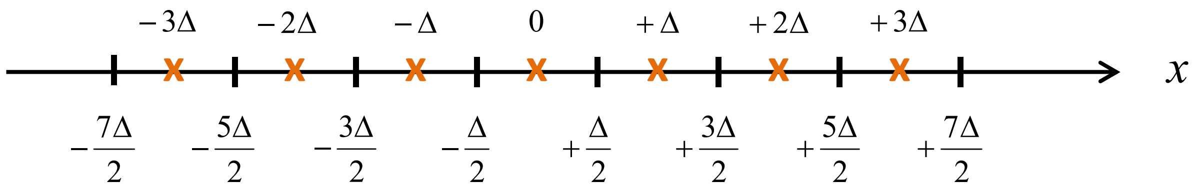
An example with ***L* = 8 levels of quantization** (3 bits assuming fixed length entropy coding):

- Levels y_i are equispaced (Δ)
- Thresholds a_i are midway between levels.



Uniform Quantization: Mid-tread

1.2

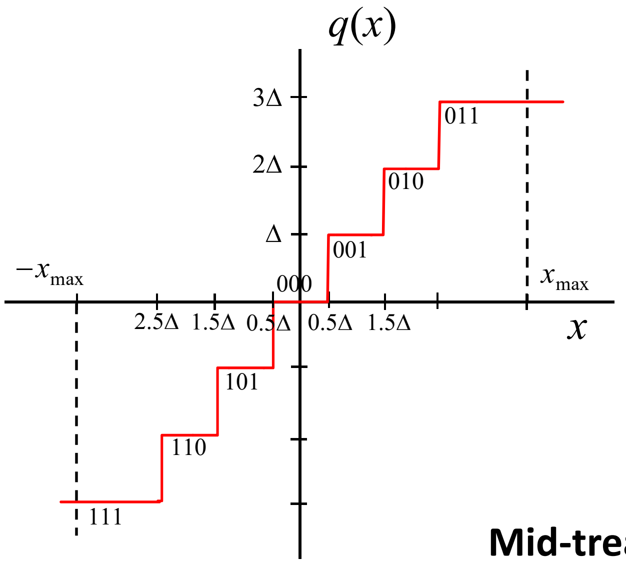


An example with L **= 7 levels of quantization** (3 bits assuming fixed length entropy coding):

- Levels y_i are equispaced (Δ)
- Thresholds a_i are midway between levels.

Symmetric use around 0:

- Loss of **one quantization level**: negligible when L is large.

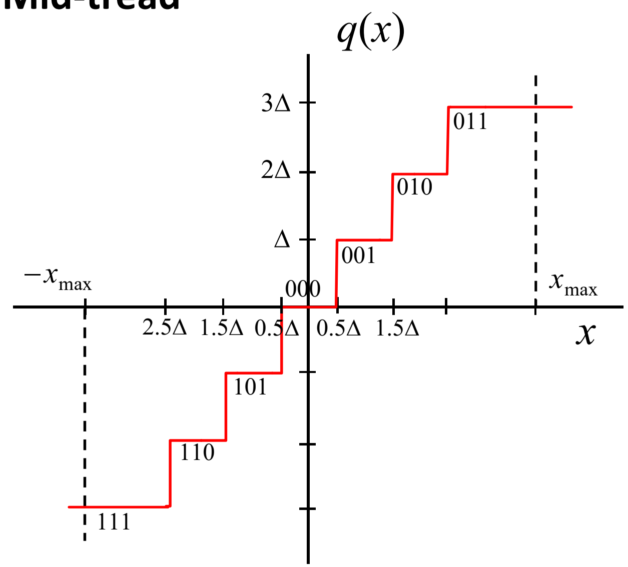


Mid-tread

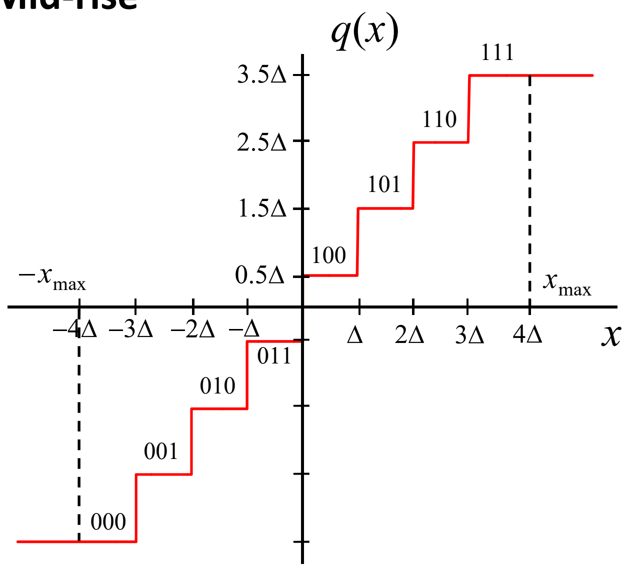
Mid-tread versus Mid-rise

1.2

Mid-tread



Mid-rise



Mid-tread:

- There is a zero output level
- There are $L = 2^R$-1 levels

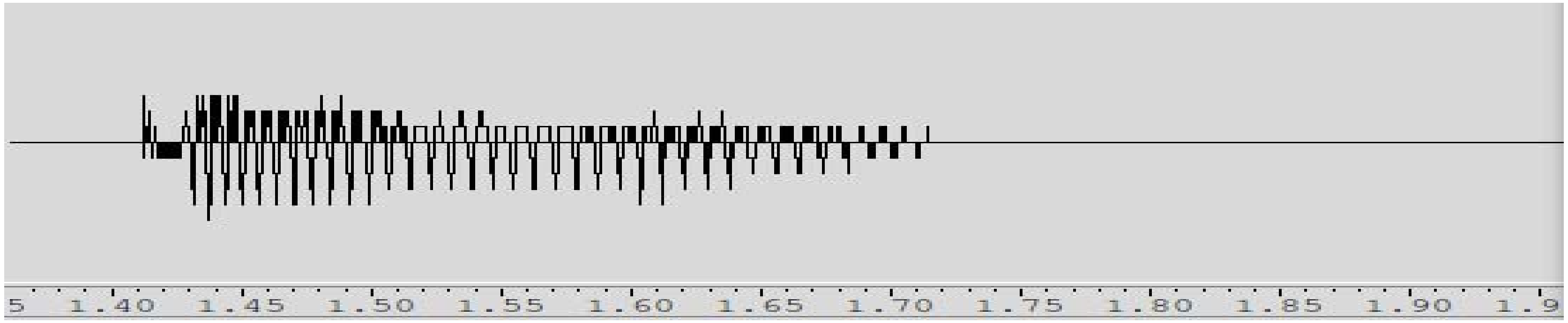
Mid-rise:

- There is no zero output level
- There are $L = 2^R$ levels

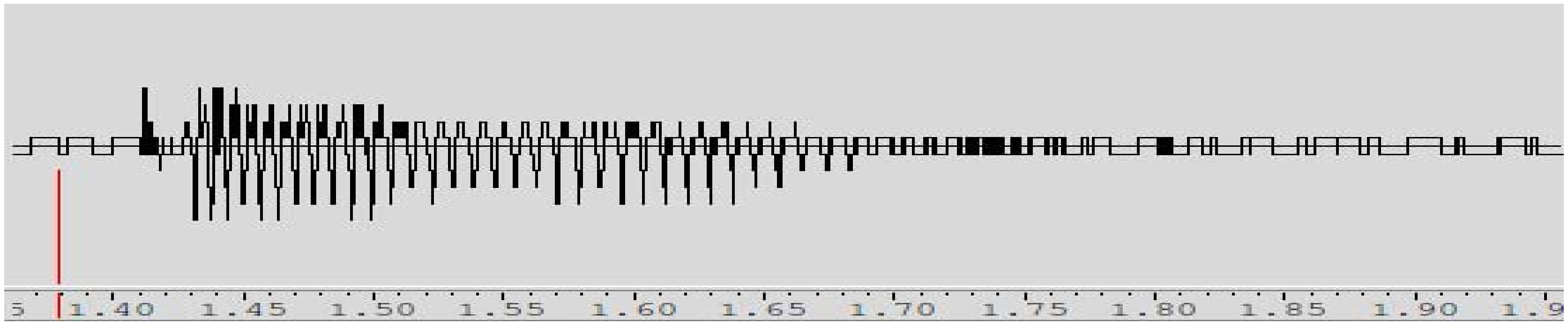
Which one would you use in voice quantization?

Mid-tread versus Mid-rise

1.2



Silent signal quantized with a mid-tread quantizer (5 bits)



Silent signal quantized with a mid-rise quantizer (5 bits)

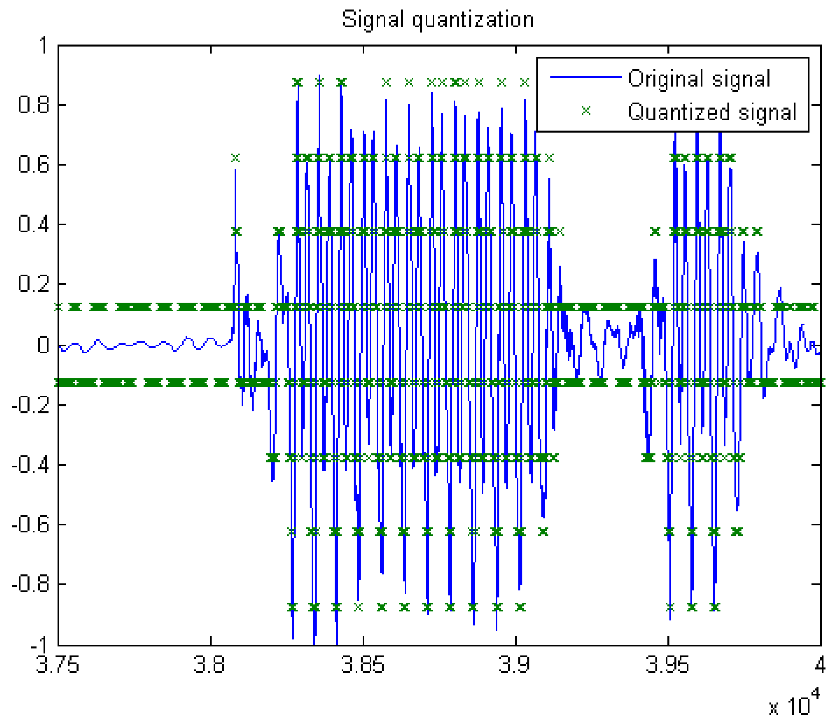
Mid-tread versus Mid-rise

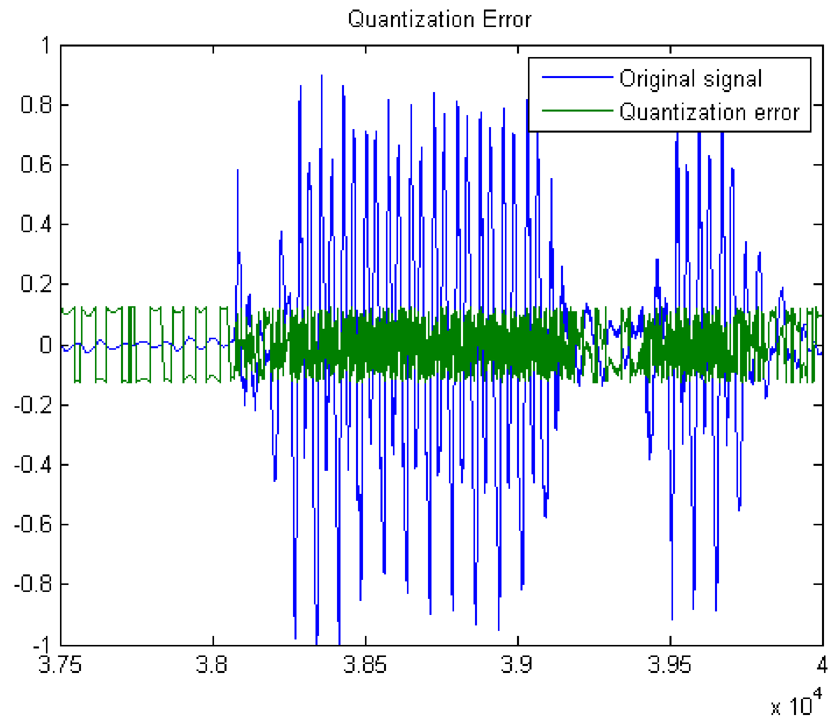
1.2

- Comparison of mid-tread and mid-rise quantization at different levels of quality: “***La pluja ja no m’estima***”
  - Original track
  - Mid-rise quantization 6 bits
  - Mid-tread quantization 6 bits
  - Mid-rise quantization 8 bits
  - Mid-tread quantization 8 bits
  - Mid-rise quantization 12 bits
  - Mid-tread quantization 12 bits

Mid-tread versus Mid-rise

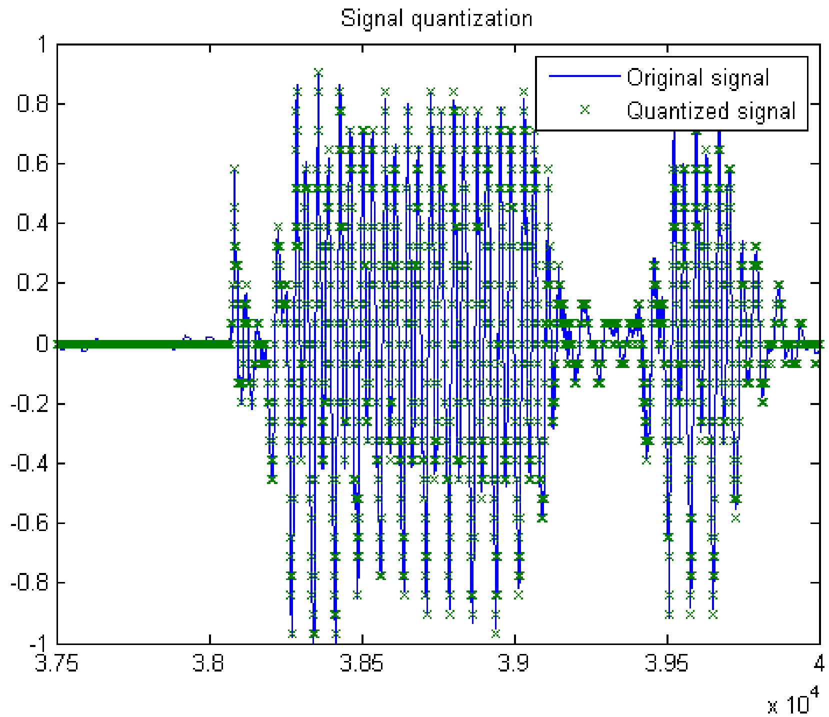
Example: Wave-form after **mid-rise quantization with 3 bits**:

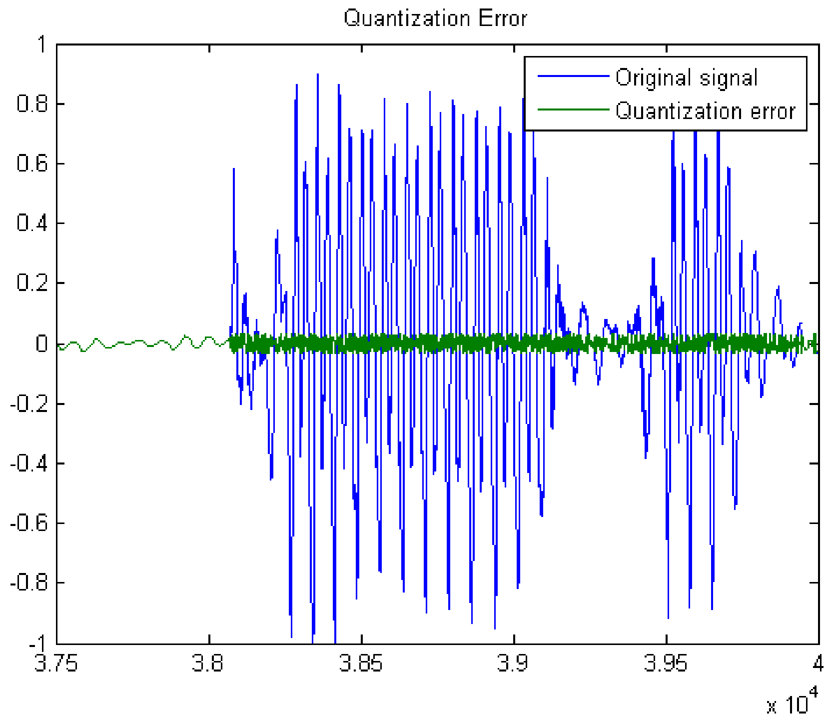




Mid-tread versus Mid-rise

Example: Wave-form after **mid-tread quantization with 5 bits**:

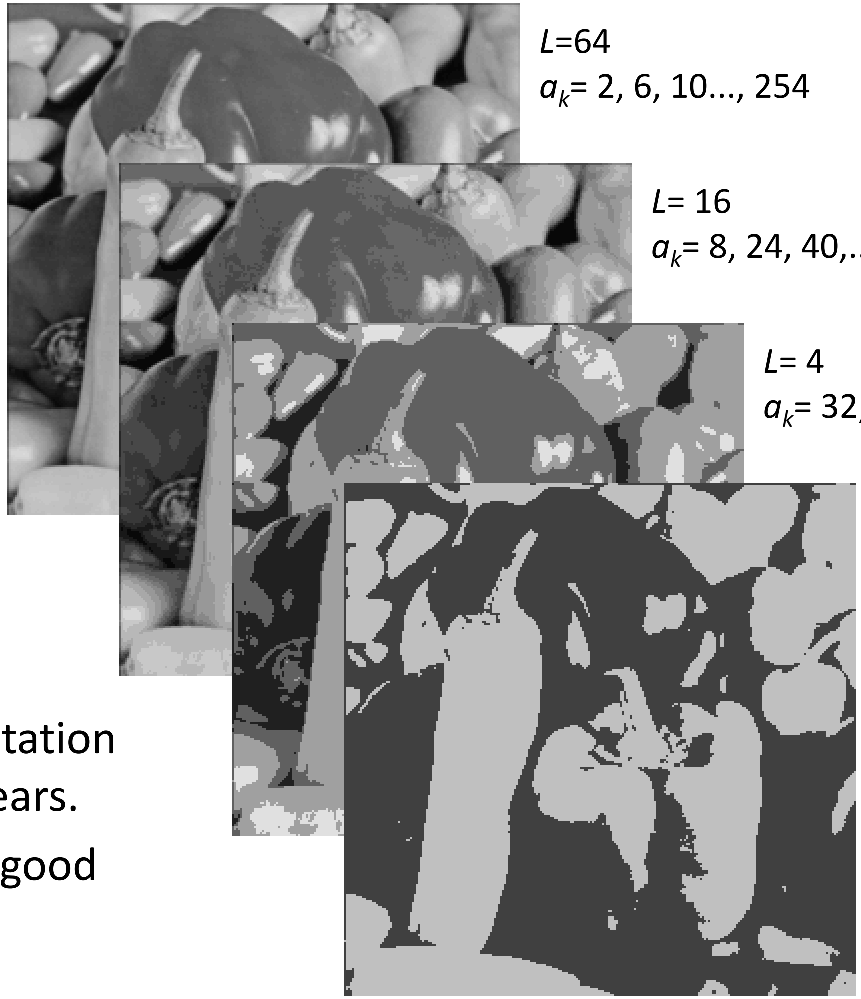




Uniform Quantization in Images

1.2

Original, 256 levels



L=64
a_k= 2, 6, 10..., 254

L= 16
a_k= 8, 24, 40,... 248

L= 4
a_k= 32, 96, 160, 224

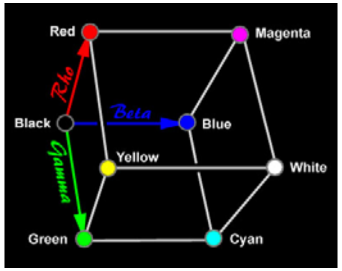
L= 2
a_k= 64, 192

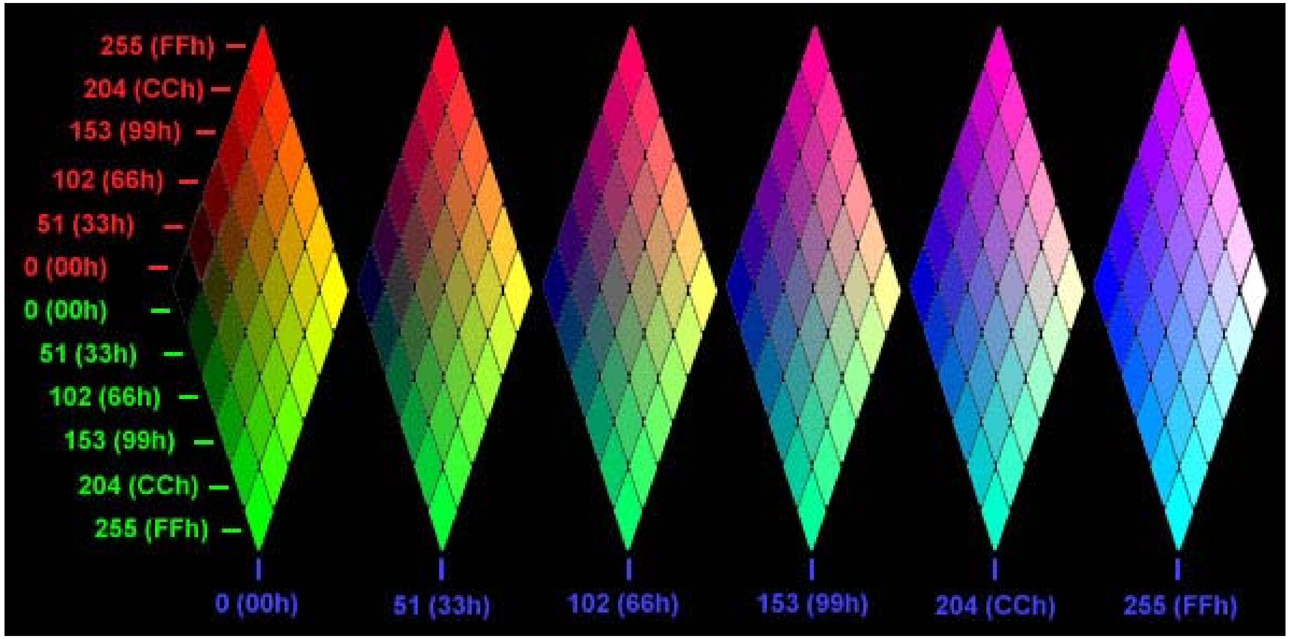
- When there are too few representation levels, a **false contour effect** appears.
- Commonly, **7 bits** are enough for good quality grey level images.

Uniform Quantization in Images

1.2

- Every color space component can be **uniformly** quantized:
  - Uniform quantization in ***L* = 6 levels** of every component in the RGB color space: **6x6x6 = 216 color palette**.





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1.2

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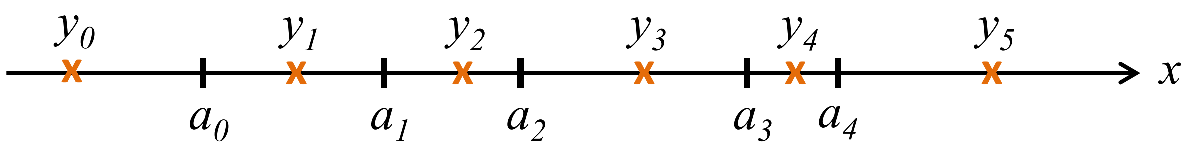
Quality measure: Distortion

1.2



- The **distortion** is based on a measure $d(x, y)$ that quantifies the cost of substituting the actual value x by its representative y:
  - Example: The **squared error** $d(x, y) = (x - y)^2$

- **Distortion**: It is actually computed as the **mean** of the measure $d(x, y)$ over the input X:

$$D(q) = E\{d(X, q(X))\} = \int d(x, q(x)) f(x) dx = \sum_{i=1}^{L} P(y_i) E\{d(X, y_i) / X \in S_i\}$$

Quality measure: Distortion

1.2

❑ **An example**:

- Uniform quantizer with L levels
- X presents a uniform probability density function
- The distortion measure is the squared error

$$D(q) = E\{d(X, q(X))\} = \int d(x, q(x)) f(x) dx = \sum_{i=1}^{L} P(y_i) E\{d(X, y_i) / X \in S_i\} = \frac{\Delta^2}{12}$$

This result is a good approximation for **smooth probability density functions**

Quality measure: Distortion

1.2

- **An example**:

  Show that a signal that is quantized with a uniform, scalar quantizer increases its quality (SNR) in **6 dBs with every additional bit** used in the quantizer.

Assumptions:

- The signal $x[n]$ is always within the interval $[-A_x, A_x]$,
- A *mid-rise* uniform quantizer of B bits is used,
- The signal $x[n]$ is uniformly distributed within the quantization step Δ, and
- The signal power can be approximated by $\sigma_x^2 = kA_x^2$ where k is a constant value that depends on the kind of signal.

Hint: $\log_{10}[2] \approx 0{,}3$

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Non-uniform Quantization: Motivation

1.2

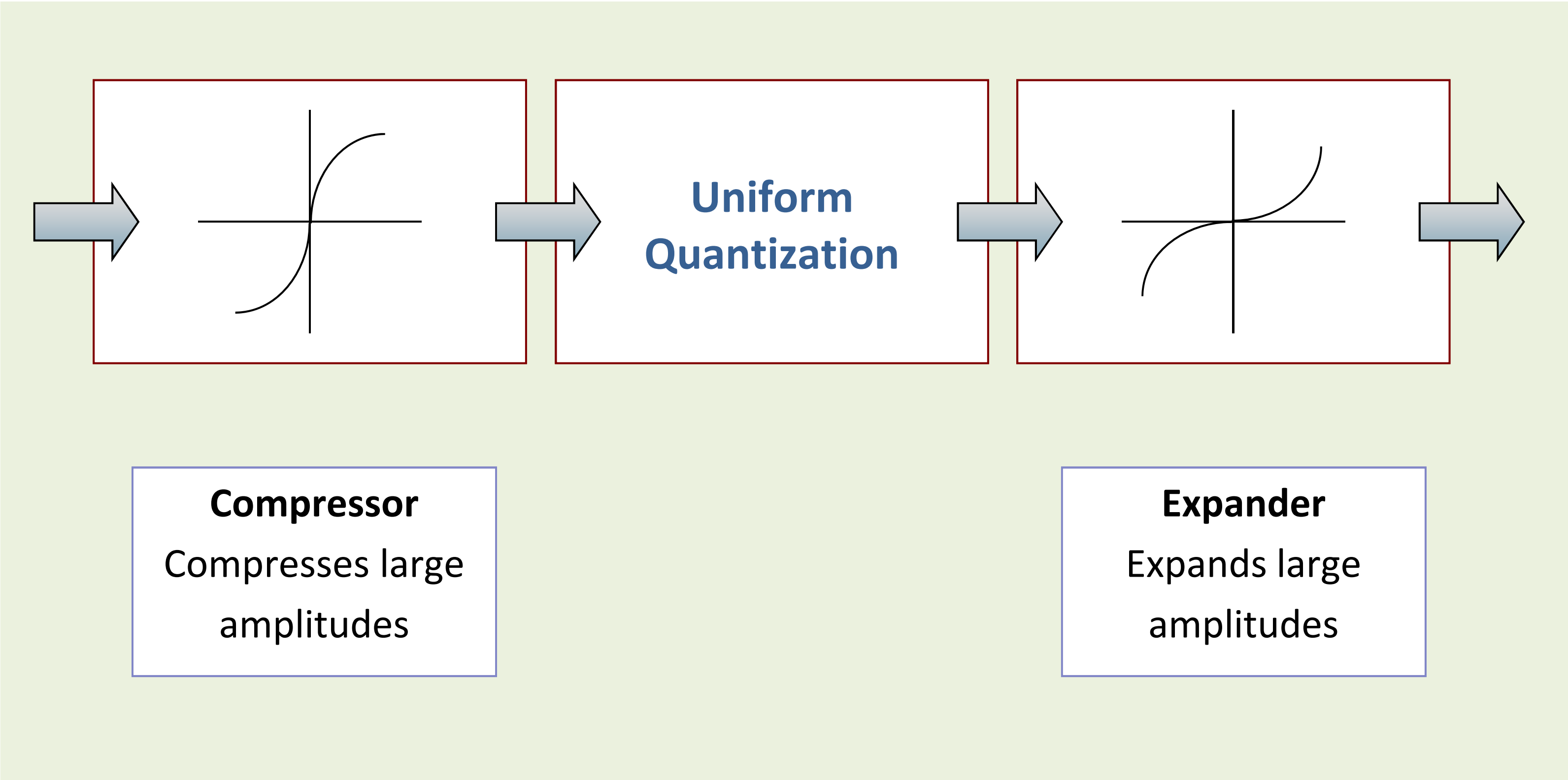
- When using uniform quantizers, the quantization error is **independent** of the signal level:
  - The **SNR for low level samples** is smaller that for high level samples
  - For some kinds of sources, this characteristic is **not desirable**

- Example of **voice signal**:
  - Low amplitude samples are **perceptually** important :
    - Logarithm compressor – expandor (**compansor**)
  - Its **range varies** through time:
    - **Adaptive quantizer**.
  - Its probability density function is not uniform (aprox. **Laplacian**) :
    - **Non-uniform quantizer**

Compander: Compressor + Expander

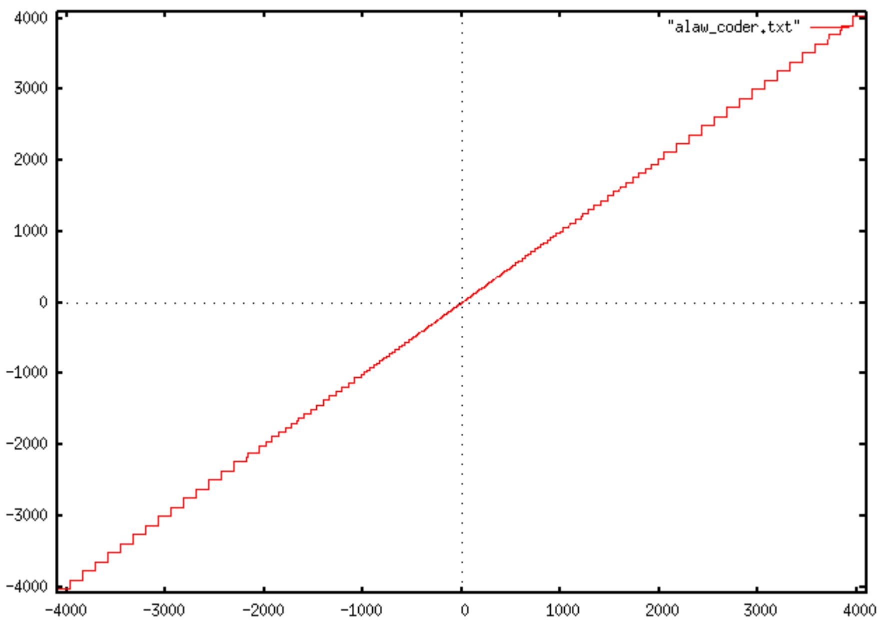
1.2

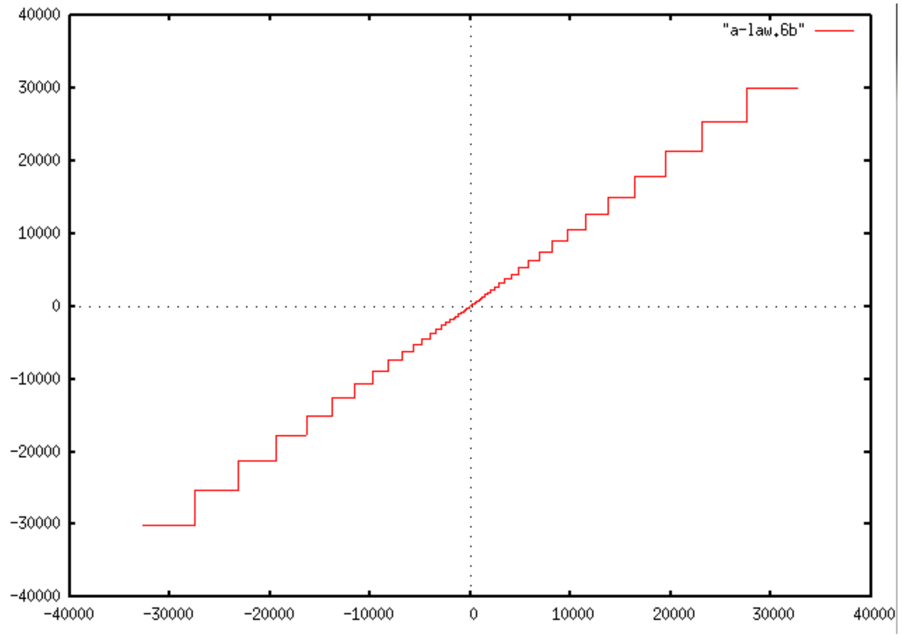
Non-uniform Q: Compander modeling of quantizers



Compander

- Signals with small amplitude: **small quantization** step
- Signals with large amplitude: **large quantization** step
  - As a result, the **SNR adapts**, becoming robust to the input signal level





Effect of the A-Law compandor on the quantization steps using 7+1 bits (left) and 5+1 bits (right)

A-law and μ-law

1.2

There exist different **implementations**:

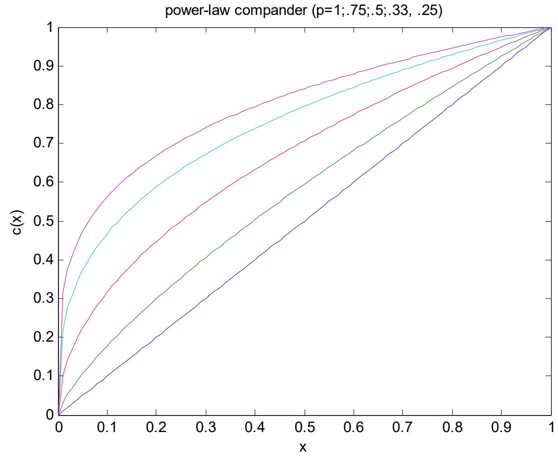
Power-law companding:

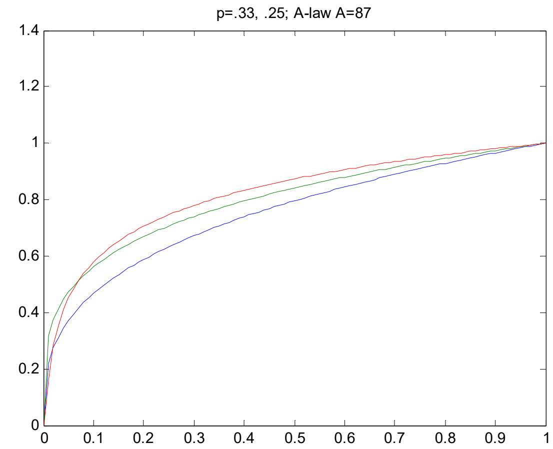
- $c(|x|) = |x|^p \ \ (0 < p < 1 \ ; p \approx 0.75)$
- Useful in Audio coding for transformed coefficients

Logarithm law:

- It keeps the ratio y_i / Δ_i constant.
- Speech telephony: A-law (Europe) and μ-law (USA) on speech samples

$$G_A(x) = \begin{cases} \dfrac{A|x|}{1+\ln A}\,\mathrm{sgn}(x) & 0 \le |x| \le V/A \\ \dfrac{V\left(1+\ln\left(A|x|/V\right)\right)}{1+\ln A}\,\mathrm{sgn}(x) & V/A \le |x| \le V \end{cases}$$





A-law examples

1.2

Mid-rise versus Mid-rise A-Law

- Comparison of mid-rise and mide-rise Law-A quantization at different levels of quality: “*La pluja ja no m’estima*”
  - Original track:
  - Mid-rise quantization 6 bits:
  - Mid-rise A-Law quantization 6 bits:
  - Mid-rise quantization 8 bits:
  - Mid-rise A-Law quantization 8 bits:

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Optimal Quantization: Motivation

1.2

- **Color image motivation (Vector Quantization)**: Given an image and a maximum number of colors to represent it, a **non-uniform quantization** can be obtained, leading to the **optimum palette**.
    - The optimum palette represents the image **minimizing a distortion measure**

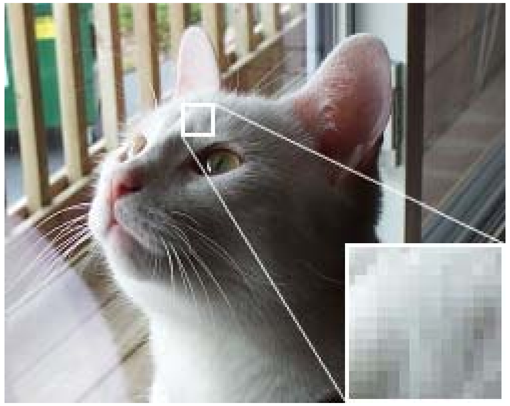
Original: 256x256x256 levels

Quantized: 16 levels

Optimal Quantization: Motivation

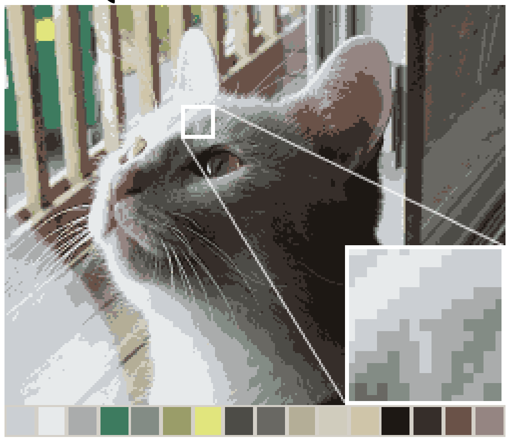
1.2

Original: 256x256x256 levels



Original: 256x256x256 levels

Quantized: 16 levels



What is this?

Quantized: 16 levels

https://en.wikipedia.org/wiki/Color_quantization

https://hbfs.wordpress.com/2013/12/31/dithering/

Optimal Quantization

1.2

- Given a source X, characterized by its probability density function, **define a quantizer** $q(x)$ with:
  - A given number of **levels L**
  - Leading to the **smallest distortion** $D(q, x)$
- There is **no close solution** to that problem:
  - Many theoretical results
  - Some **interesting algorithms**

- **Necessary conditions** (Lloyd)
  - The encoder (α, S_i) must be optimal given the decoder (β, y_i)
  - The decoder (β, y_i) must be optimal given the encoder (α, S_i)

- These conditions suggest an **iterative algorithm**.

Optimal Quantization

- **Necessary conditions** (Lloyd)
  - The encoder (α, S_i) must be optimal given the decoder (β, y_i)
  - The decoder (β, y_i) must be optimal given the encoder (α, S_i)

- It is easy to find the optimal α given β and the optimal β given α.
- An **iterative algorithm** can be find so that
  - $\alpha_0 \rightarrow \beta_1 \rightarrow \alpha_1 \rightarrow \beta_2 \rightarrow \alpha_2 \rightarrow \ldots$

- At each step, $D(q_i)$ decreases. As $D(q_i) > 0$, the algorithm converges.

- Necessary conditions ... but **not sufficient**. Therefore, the algorithm can get trapped in a local minimum of $D(q)$. Some results exist for particular pdf's

Max-Lloyd Algorithm

1.2

- Decision levels (a_k) and representation values (y_k) can be chosen to minimize a given criterion; for instance, the **mean square error** (MSE).
- Given a random variable x, **with known pdf** $p_x(x)$, we look for the decision levels (a_k) and representation values (y_k) that minimize the MSE for a given number of levels L:

$$\varepsilon = E\left\{(x-q(x))^2\right\} = \int_{a_1}^{a_L} (x-q(x))^2 f_x(x)dx = \sum_{k=1}^{L-1}\int_{a_k}^{a_{k+1}} (x-y_k)^2 f_x(x)dx$$

❑ **Optimization**: Computing the **derivatives** of this expression with respect to the variables (decision levels (a_k) and representation values (y_k)) and **equalizing them to zero**, the following results are obtained:

$$a_k = \frac{y_k + y_{k+1}}{2} \longleftarrow \textbf{Find and interpret} \longrightarrow y_k = \frac{\int_{a_k}^{a_{k+1}} x f_x(x)dx}{\int_{a_k}^{a_{k+1}} f_x(x)dx}$$

Max-Lloyd Algorithm

The previous results lead to the following **interpretation**:

- **Decision values** (**thresholds**) are the mean value of consecutives reconstruction values (nearest neighbor - NN- condition):

$$a_k = \frac{y_k + y_{k+1}}{2}$$

- **Reconstruction values** are the centroids of the area under $p_u(u)$ within its corresponding interval (centroid condition):

$$y_k = \frac{\int_{a_k}^{a_{k+1}} x f_x(x)dx}{\int_{a_k}^{a_{k+1}} f_x(x)dx}$$

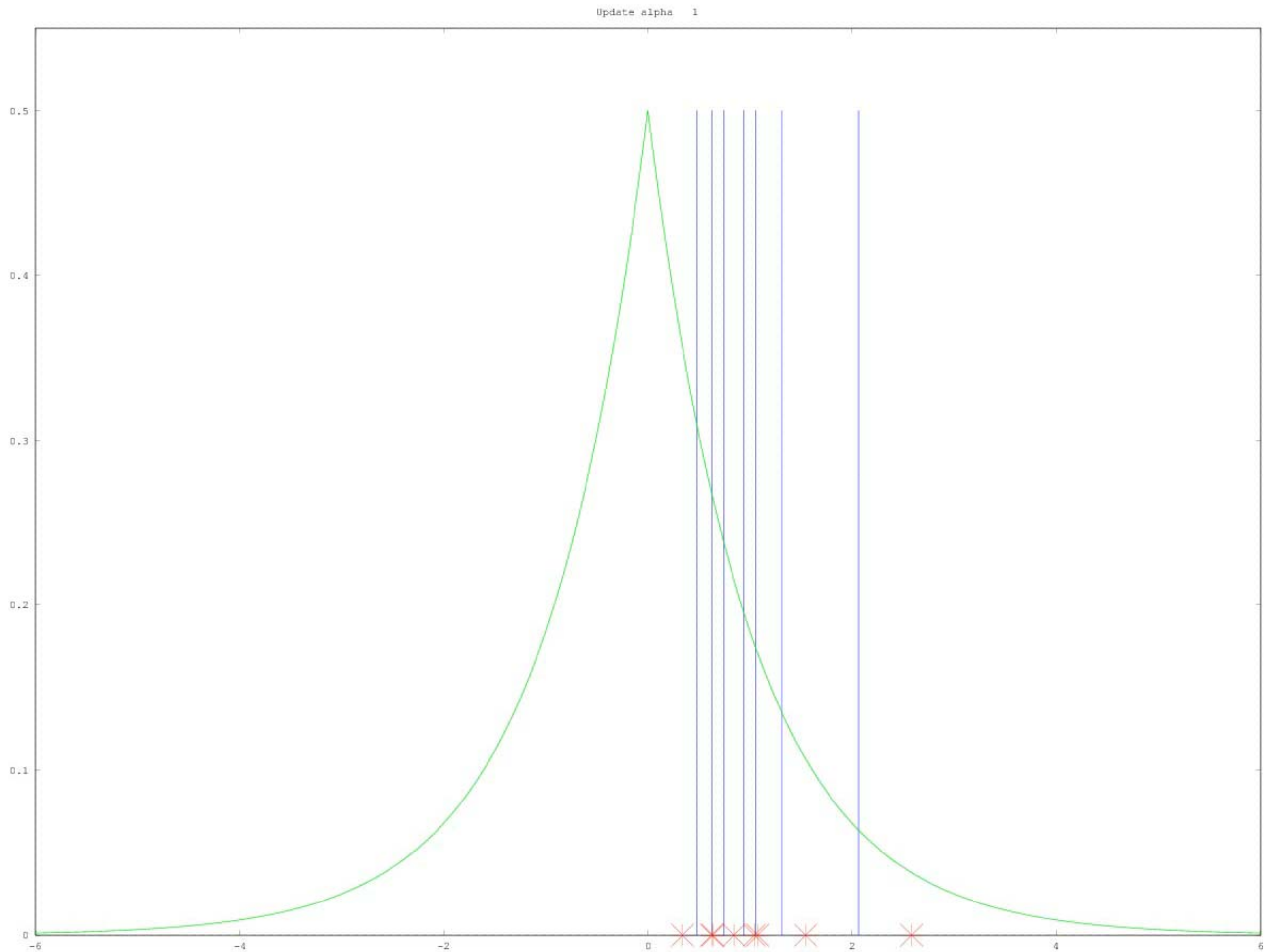
Commonly, the solution is obtained by an **iterative algorithm** (it does not ensure reaching the optimum):

Given a known distribution $p_x(x)$ and a set of initial reconstruction values $\{y_k\}$

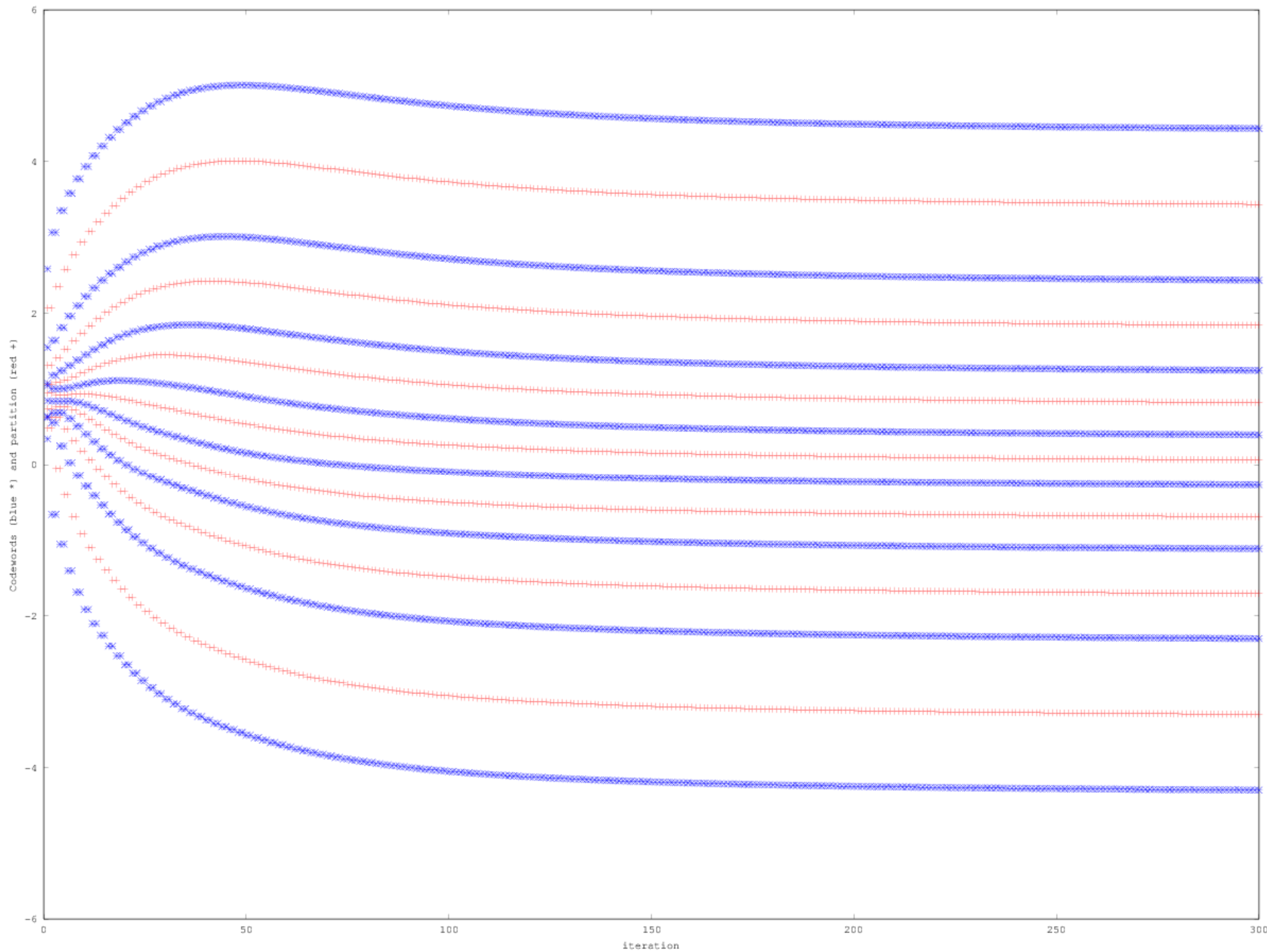
1. Compute the decision values $\{a_k\}$ with the NN condition
2. Compute the reconstruction values $\{y_k\}$ with the centroid condition
3. If the MSE is larger than a given value, repeat 1 and 2.

Max-Lloyd Algorithm: Example

1.2



Max-Lloyd Algorithm: Example



Unit Structure

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   - Definition, Types, Performance
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Max-Lloyd Initialization

1.2

- Since the Max-Lloyd algorithm **does not ensure reaching the optimum**, the initialization step is very important:
  - The algorithm may get trapped in the **local minimum** closer to the initial solution.

- Several **strategies** have been proposed:
  - **Random selection**: N elements from the initial training data.
  - **Regular lattice selection**: Product of uniform scalar quantizers
  - **Product codes**: Product derived from optimal scalar quantizers
  - **Splitting (or LBG algorithm)**: Sequential optimizations
    - Start from a quantization with one/two (K) representative(s)
    - Split each representative and optimize to obtain $2K$
    - Iterate until reaching N elements

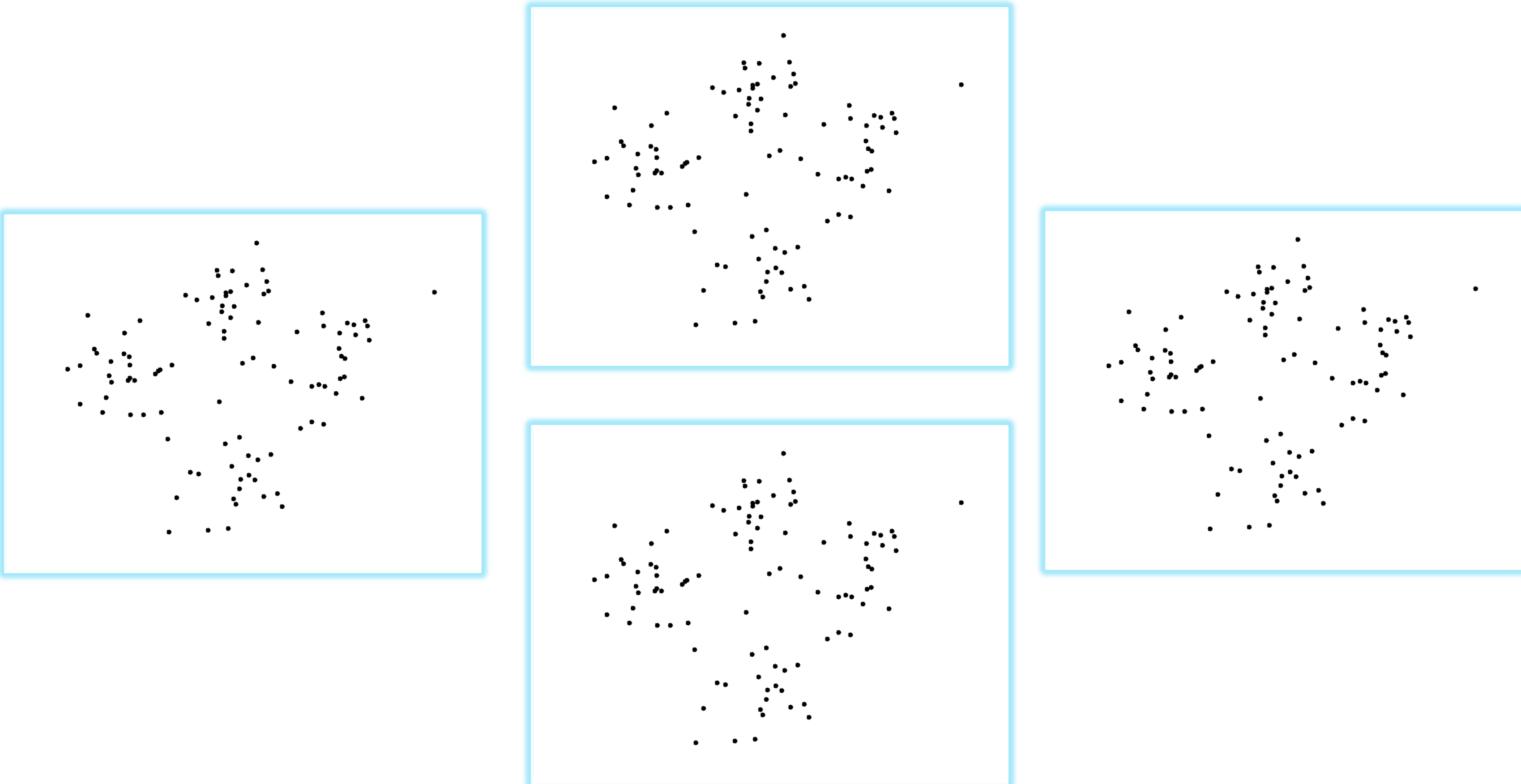
Max-Lloyd Initialization

1.2

The final result depends on the correct initialization of the algorithm.
Example with 4 representatives (random selection):

Max-Lloyd Initialization

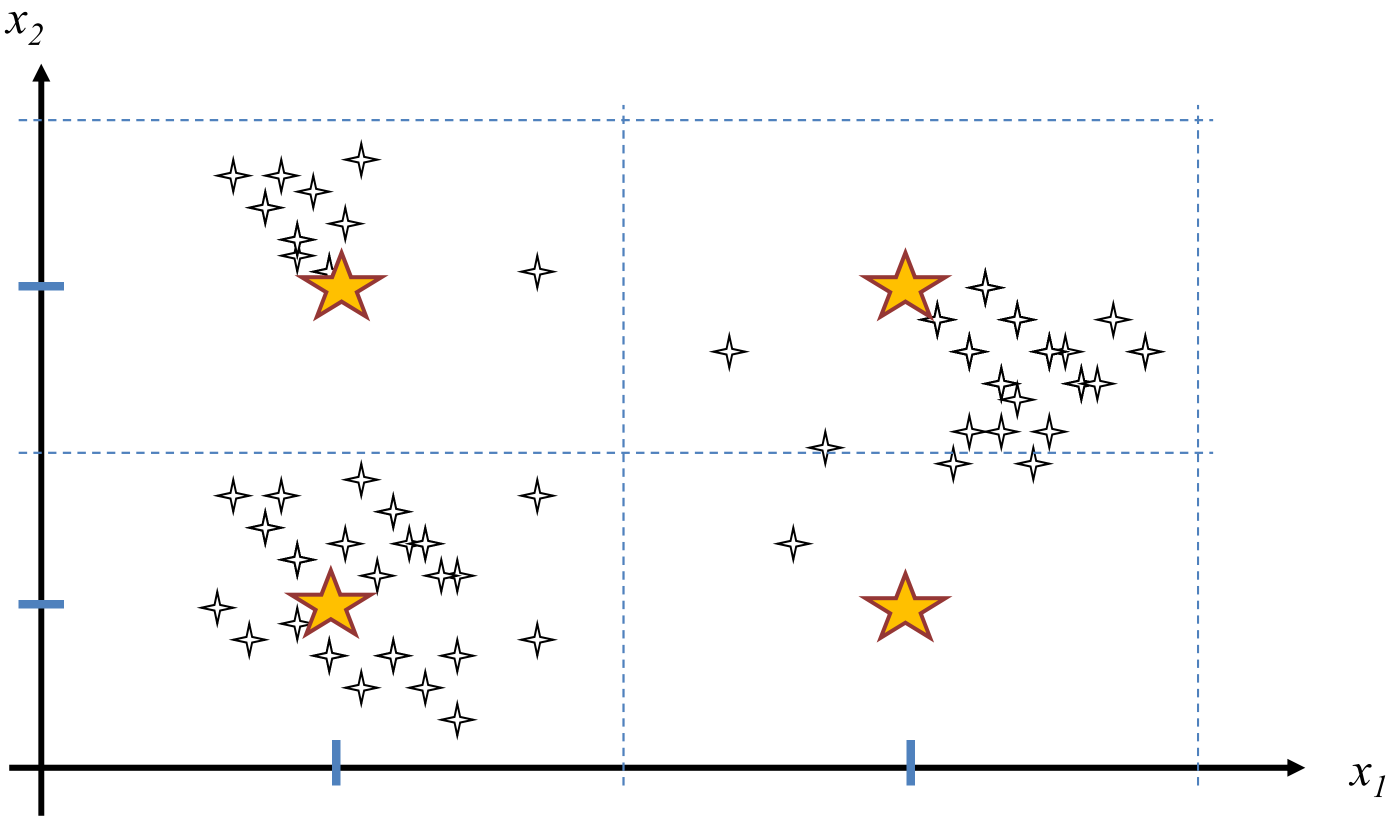
Four different initializations of the same problem (**random selection**):



Regular Lattice Selection

1.2

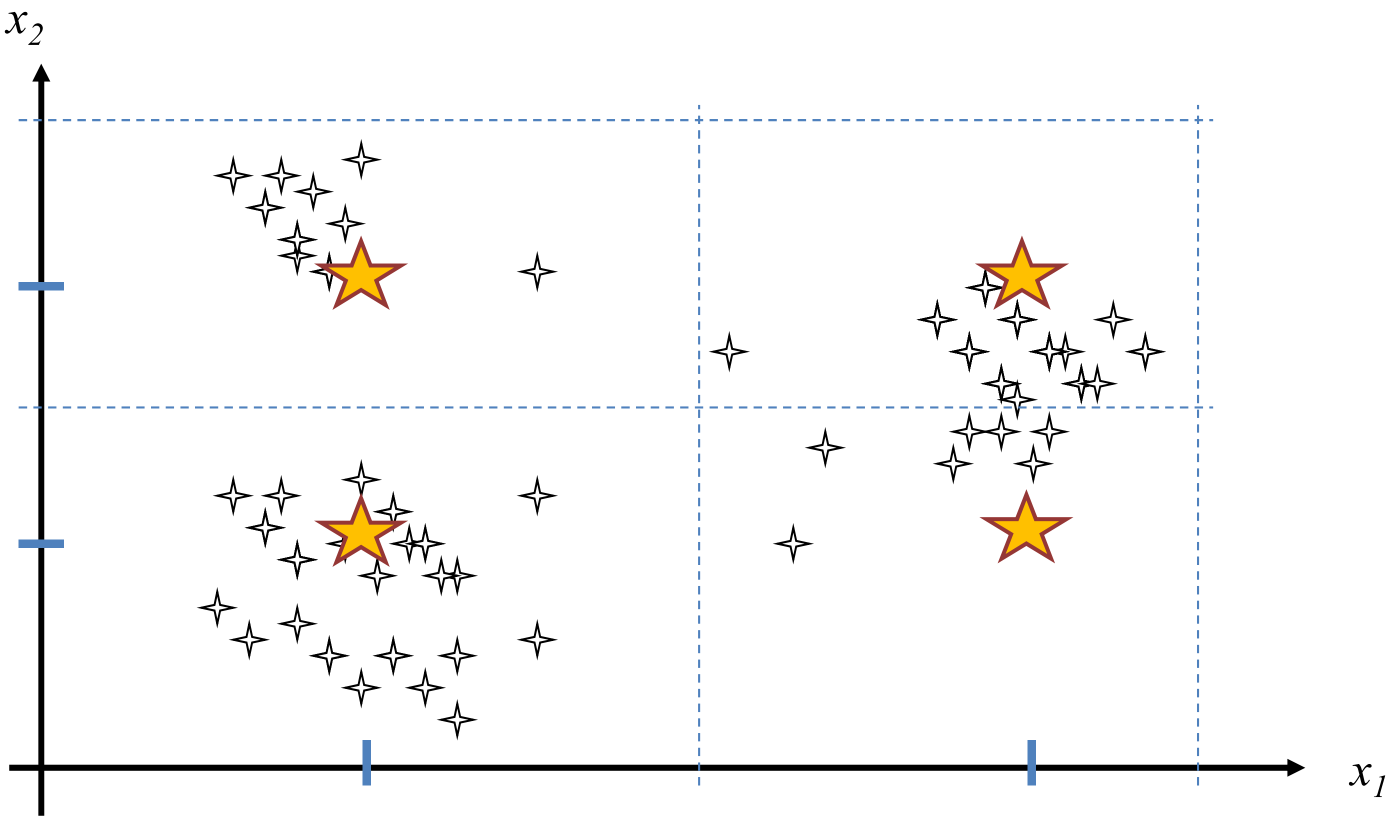
- **Product of uniform scalar quantizers**:



Product codes

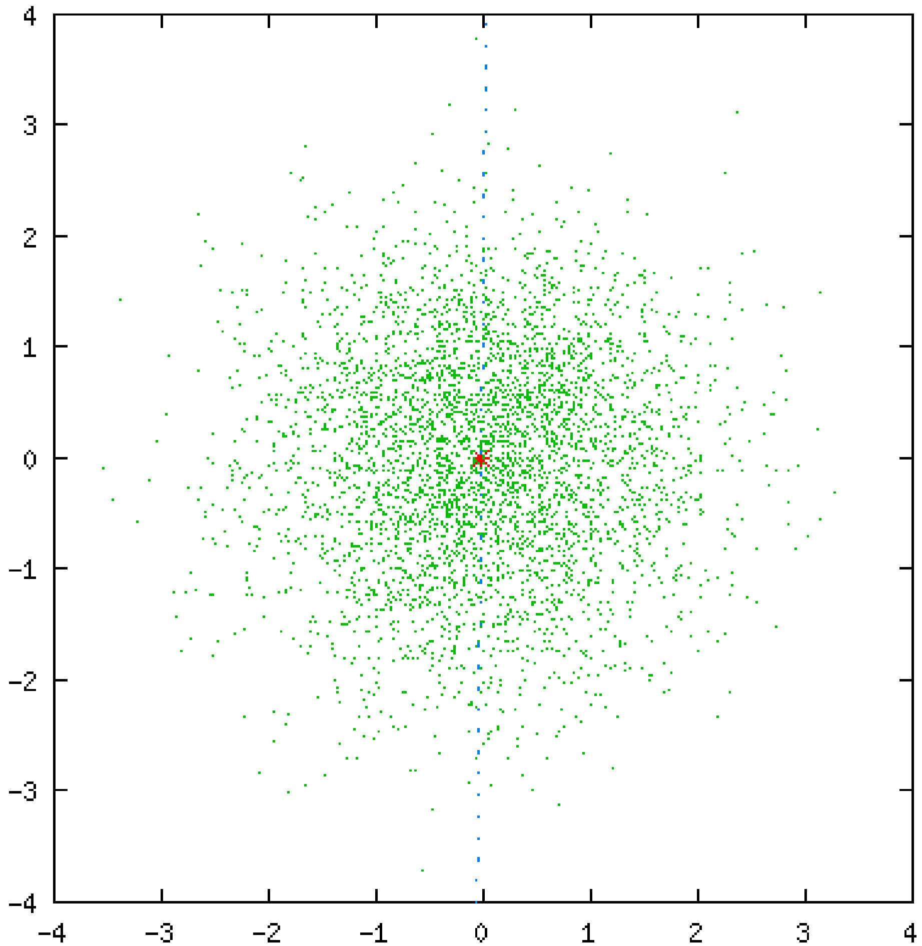
1.2

- **Product derived from optimal scalar quantizers**



Splitting Example

1.2



http://www.data-compression.com/vqanim.shtml

Example Data:

- The experiment is run with a white Gaussian source with zero-mean and unit variance.
- Green dots are training vectors: there are 4096 of them.
- The algorithm guarantees a locally optimal solution.
- The size of the training sequence should be sufficiently large. It is recommended that the number of training vectors should be 1000 times the number of representatives.

Conclusions

1.2

Memoryless processes assume that every sample of the process is **independent** of its neighbor samples.

Processing of memoryless processes: Only take into account the sample values, but neither their index (time instant/position) nor their neighbor sample values.

Quantization is involved in nearly all digital signal processing. **Storage** implies quantization. Quantization implies **loss of information and quality**: It is a non-reversible operation.

We have studied **uniform quantization** (no information available about the source), **non-uniform quantization** (non statistical information available) and **optimal quantization** (statistical optimization)

RM Gray and DL Neuhoff, *Quantization*, IEEE Trans. on Information Theory, 44 (6), 2325-2383