1 Statistical Signal Modelling

1.1.b: Random Variable

Statistical Signal Modelling

1.1

1. **Introduction to IPA and Random variable**
2. **Modelling of memoryless processes**
   - Sample-wise operators
   - Uniform and non-uniform quantization
3. **Discrete Stochastic Processes**
   - Definition
   - Autocorrelation: Deterministic signals and processes
   - Stationarity and Ergodicity
   - Power Spectral Density (PSD)
   - Stochastic processes filtering
   - Examples

Random Variable

1.1

1. **Random variables**
    - Definition of random variable
    - Moments of a random variable
    - Random variable models
    - Processing random variables
    - Comparison of random variables
2. **Multivariate Random Variables**
    - Definition of multivariate random variable
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    - Multivariate random variable models
3. **Summary and Conclusions**

Random Variable

1.1

Random variable: Assignment to a variable of the result of an experiment performed multiple (infinite) times.

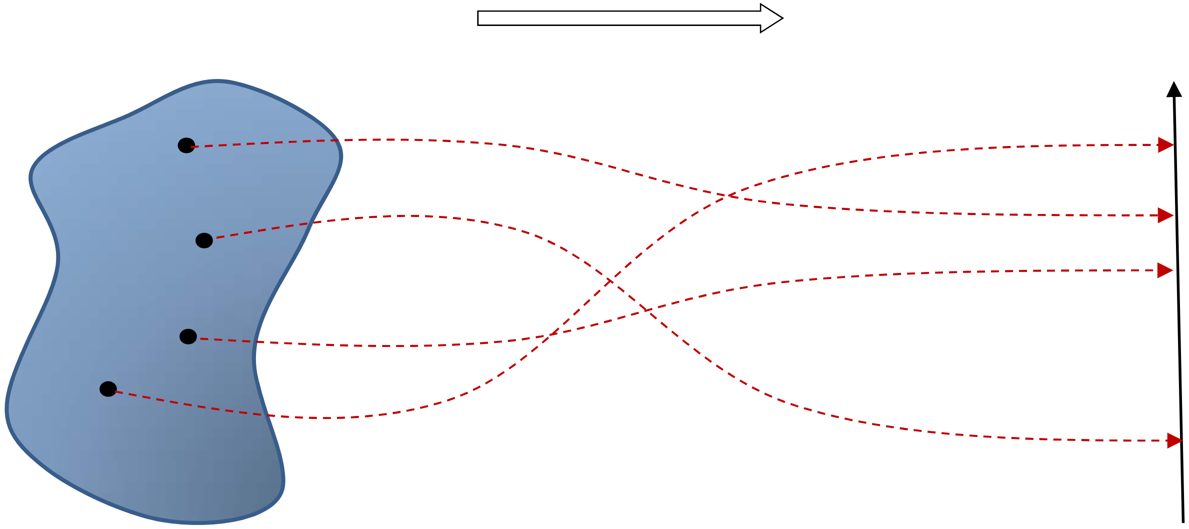
In the context of Introduction to Audiovisual Processing, values will be real numbers.

Ω: Set of possible outcomes of a given experiment

Random Variable

$X: \Omega \rightarrow E$

E: Measurable space. Typically real numbers



Random Variable

1.1

Random variable: Assignment to a variable of the result of an experiment performed multiple (infinite) times

Cumulative distribution function (**CDF**) determines the probability that a given random variable (X) takes a value less than or equal to a given value (x).

- It is a positive, bounded function:

$$F_X(x) = P(X \leq x)$$

$$0 \leq F_X(x) \leq 1$$

Probability density function (**PDF**) provides a *relative likelihood* that a result is given in an experiment:

- It is a positive function which fulfills:

$$f_X(x) = \frac{dF_X(x)}{dx}$$

$$f_X(x) \geq 0 \qquad \int_{-\infty}^{\infty} f_X(x)dx = 1 \qquad P(a \leq X \leq b) = \int_a^b f_X(x)dx$$

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Moments of a Random Variable

1.1

Often, the information conveyed by a random variable is represented by a **small set of parameters**, which have a practical interpretation; typically, its **moments**:

- **Expected value (first order moment):** measures the mean of the variable.

$$m_X = E\{X\} = \int_{-\infty}^{\infty} x f_X(x) dx$$

- **Quadratic mean** (**second order moment**): measures the dispersion of the variable around the origin. Related to **power**.

$$E\{X^2\} = \int_{-\infty}^{\infty} x^2 f_X(x) dx$$

- **Variance:** measures the dispersion of the variable around its mean:

$$\sigma_X^2 = var(X) = E\{[X - E\{X\}]^2\} = E\{[X - m_X]^2\} = \int_{-\infty}^{\infty} [x - m_X]^2 f_X(x) dx$$

➢ Demonstrate: $\sigma_X^2 = var(X) = E\{[X - E\{X\}]^2\} = E\{X^2\} - E^2\{X\}$.

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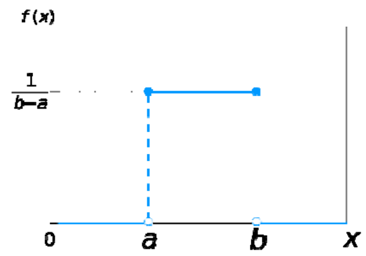
Common Probability Distributions

1.1

Commonly, we will assume that the behavior of the random variable that is being analyzed can be characterized by a given **probability distribution**.

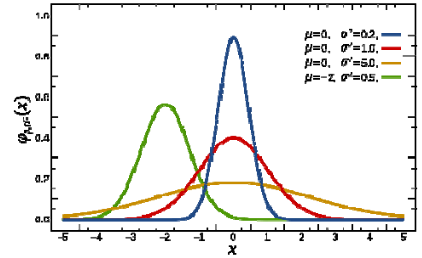
- **Uniform:**

$$f_X(x;a,b) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \leq x \leq b \\ \\ 0 & \text{otherwise} \end{cases}$$



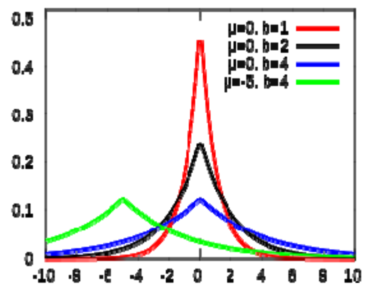
- **Gaussian:**

$$f_X(x;m_X,\sigma_X^2) = \frac{1}{\sqrt{2\pi\sigma_X^2}} exp\left[-\frac{[x-m_X]^2}{2\sigma_X^2}\right]$$



- **Laplacian:**

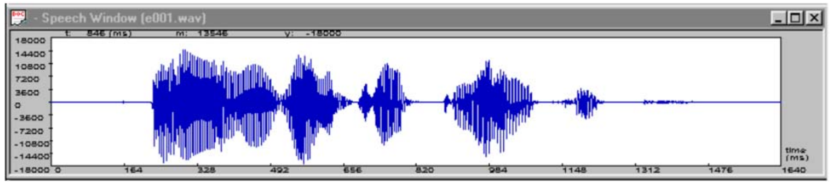
$$f_X(x;m_X,b) = \frac{1}{2b} exp\left[-\frac{|x-m_X|}{b}\right]$$

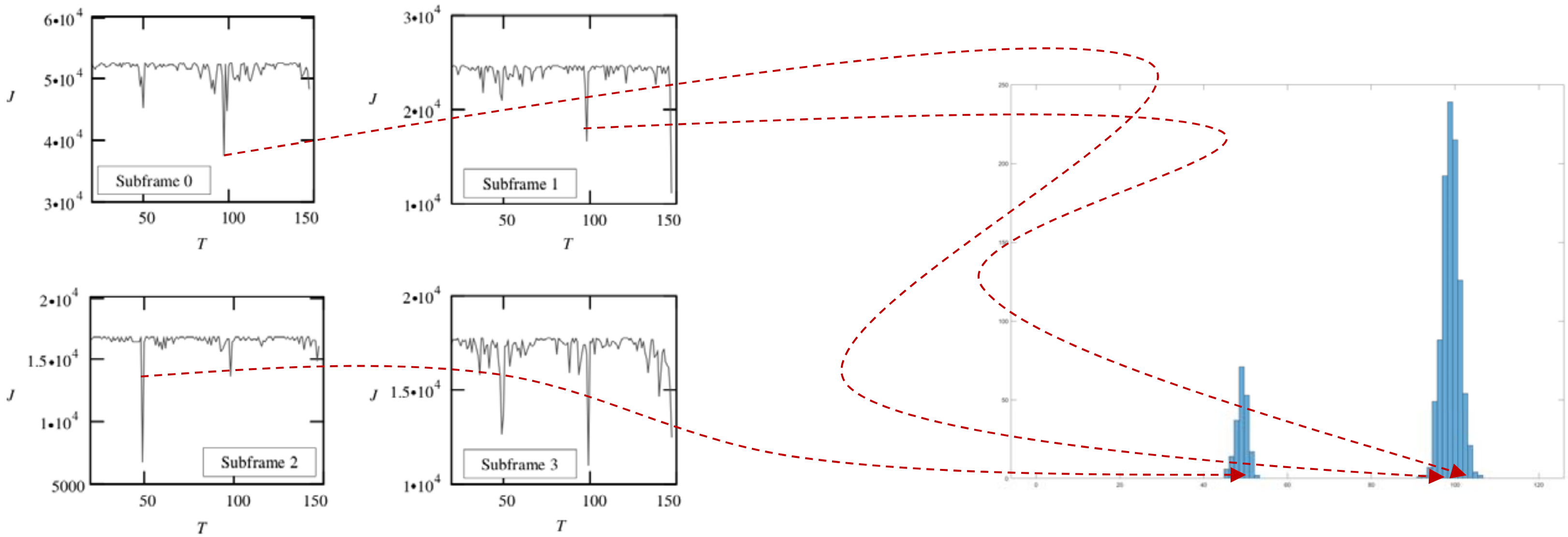


Random Variable Models

In some cases, there will not be a simple mathematical model that correctly fits the random variable behavior and we will use an **empirical model**.

Example: Estimation of the pitch of a given speaker



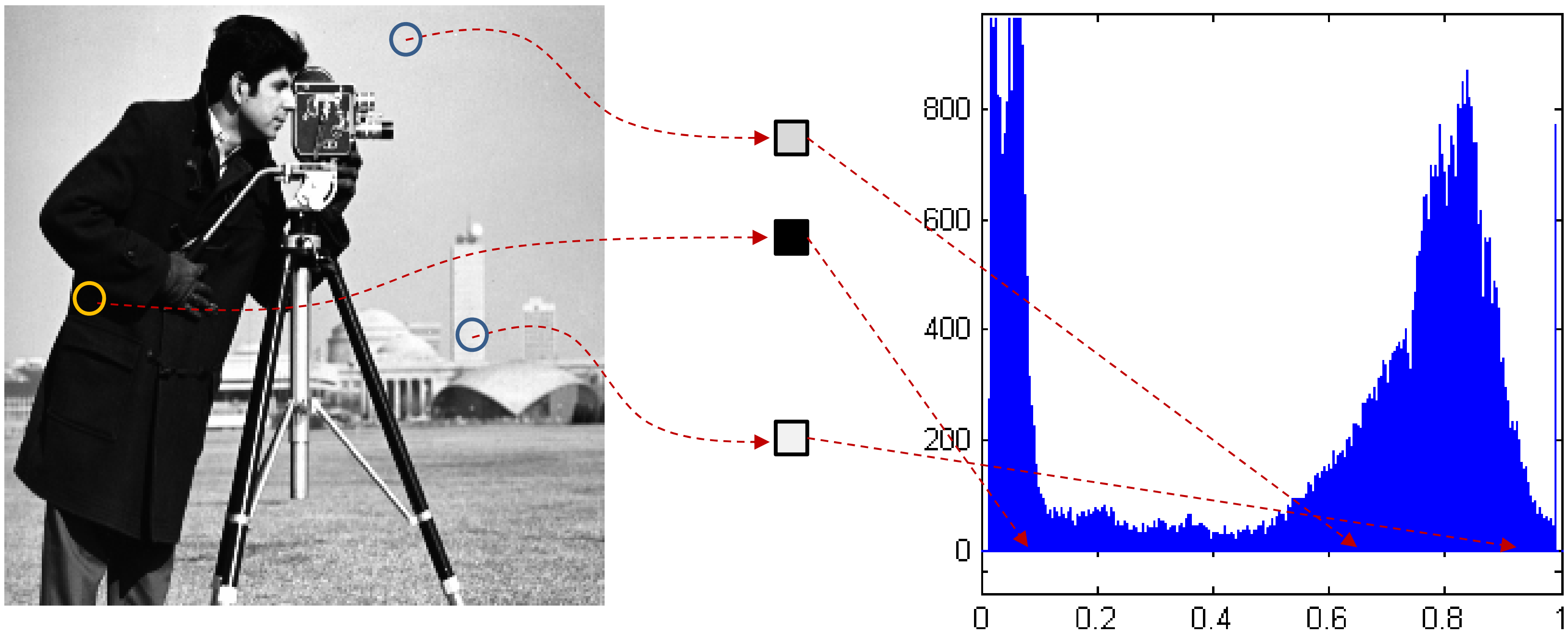


Random Variable Models

1.1

In some cases, there will not be a simple mathematical model that correctly fits the random variable behavior and we will use an **empirical model**.

Example: Estimation of the luminance of a given point in a scene



Random Variable

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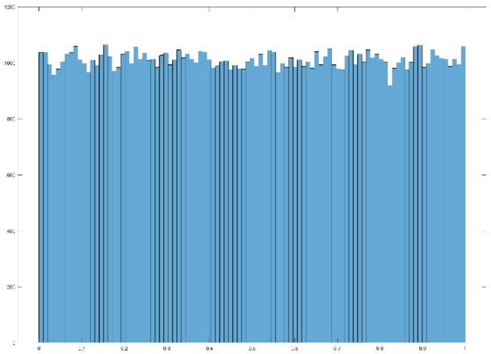
Processing of Random Variables

1.1

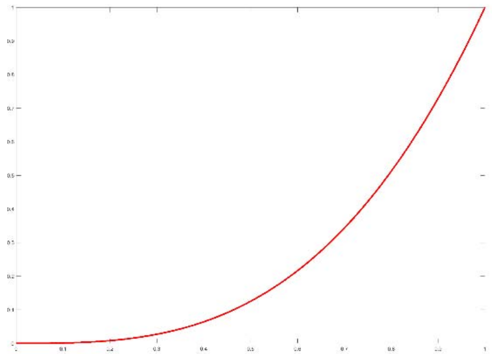
If X is a continuous random variable and $y = g(x)$ is a strictly monotonic function, in the interval where $f_X(x)$ is defined, with inverse function $x = g^{-1}(y)$, then the pdf of $Y = g(X)$ is given by:

$$f_Y(y) = \left|\frac{dg(x)}{dx}\right|^{-1} \cdot f_X(x) = \left|\frac{dg^{-1}(y)}{dy}\right| \cdot f_X(\, g^{-1}(y))$$

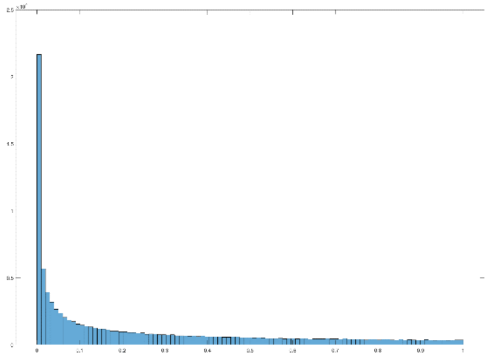
Example: $Y = X^3;\ X \in [0,1]$



Uniform distribution: pdf estimated through 10000 samples



$y = x^3$



Resulting distribution

Processing of Random Variables

Example:

$Y = X^2;\ X \in [0, \infty)$

$$f_Y(y) = \left|\frac{d\, g^{-1}(y)}{dy}\right| \cdot f_X(\,g^{-1}(y)) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y})$$

- Given a random variable $X \in [0,1]$ with uniform pdf $f_X(x)$, determine the pdf $f_Y(y)$ of $Y = X^2$

$$f_X(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \dfrac{1}{2\sqrt{y}} & \text{for } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

- Its **mean and variance** are:
  (Note that their values depend on the original pdf $f_X(x)$)

$$m_Y = E\{Y\} = \int_{-\infty}^{\infty} y f_Y(y) dy = 1/3$$

$$\sigma_Y^2 = E\{[Y - m_Y]^2\} = E\{Y^2\} - E^2\{Y\} = 4/45$$

Processing of Random Variables

1.1

Actually, to determine the mean (and other parameters) of $y = g(x)$, it is not necessary to obtain $f_Y(y)$ since:

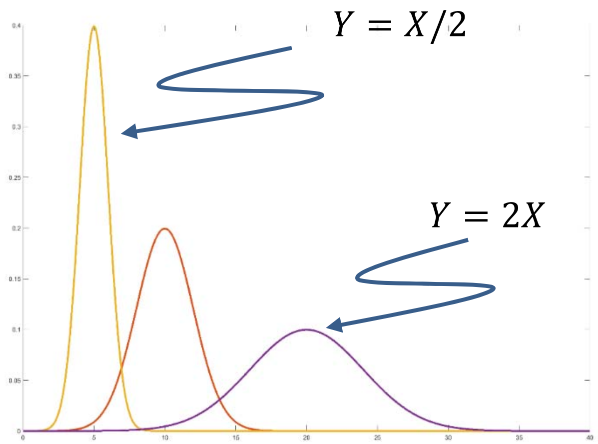
$$\boxed{E\{Y\} = E\{g(X)\} = \int_{-\infty}^{\infty} g(x) f_X(x) dx}$$

Example: $Y = kX$

➢ Its **mean and variance** are:

$$m_Y = E\{Y\} = k m_X$$

$$\sigma_Y^2 = E\{[Y - m_Y]^2\} = k^2 \sigma_X^2$$



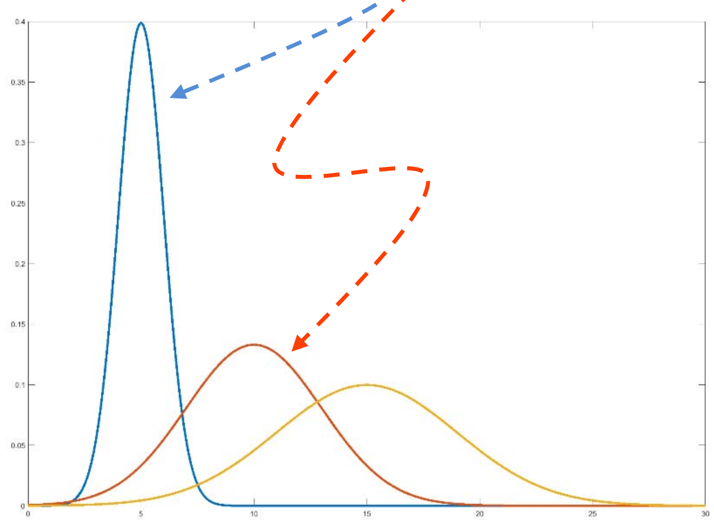
Adding Independent Random Variables

1.1

Given **two independent random variables** X and Y with probability density functions $f_x(x)$ and $f_Y(y)$ respectively, the pdf of its sum $Z = X + Y$ is the **convolution** of their pdf's:

$$f_Z(z) = f_X(x) * f_Y(y)$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(\omega) \cdot f_Y(z-\omega)d\omega$$



- Compute the **pdf of the sum** between a random variable X and a constant value a: $Z = X + a$.
- Compute its **mean** and **variance** (relate them with those of X)

Adding Independent Random Variables

1.1

- The **expected value of the sum** of a set of (~~independent~~) **random variables** is the sum of their expected values:

$$Z = \sum_{i=1}^{N} X_i \quad \rightarrow \quad m_Z = E\{Z\} = \sum_{i=1}^{N} m_{X_i}$$

- The **variance of the sum** of a set of **independent random variables** is the sum of their variances:

$$Z = \sum_{i=1}^{N} X_i \quad \rightarrow \quad \sigma_Z^2 = var(Z) = E\{[Z - E\{Z\}]^2\} = \sum_{i=1}^{N} \sigma_{X_i}^2 = \sum_{i=1}^{N} var(X_i)$$

- Compute the **mean and the variance** of the **difference** between two independent random variables: $Z = X - Y$

Random Variable Models: Notation

1.1

Models depend on **parameters** (θ) which may be variable and unknown (random variables (Θ)), deterministic but unknown (parameters) or deterministic and known (given values):

- Joint probability density distribution of the random variable X and the random variable Θ that parametrizes the pdf.

$$f_{X,\Theta}(x,\theta)$$

$$f_{X,\Theta}(x,\theta) = f_X(x|\Theta = \theta) f_\Theta(\theta)$$

- Conditional pdf of X given the occurrence of the value θ of Θ. Typically used in optimization processes over θ, when $f_\Theta(\theta)$ is known (MAP estimation).

$$f_X(x|\Theta = \theta) = \frac{f_{X,\Theta}(x,\theta)}{f_\Theta(\theta)} \rightarrow f_X(x|\theta)$$

- Probability density function of X given the value θ. Typically used in optimization processes over θ, assuming that θ is deterministic but unknown (ML estimation).

$$f_X(x;\theta)$$

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2. Multivariate Random Variables

- Definition of multivariate random variable
- Moments of a multivariate random variable
- Multivariate random variable models

3. Summary and Conclusions

Comparison of Random Variables

1.1

In some cases, it is interesting **to compare random variables** to understand **how they are related**. Typical comparison measures are extensions of the previous moments:

Covariance ($c_{X,Y}$): Given two random variables (X, Y), it measures its joint variability:

- Its **sign** shows the tendency in the linear relationship between the variables.
- Its **magnitude** depends on the magnitudes of the variable: not direct interpretation.
  - The **correlation coefficient** ($\rho_{X,Y}$) is a normalized version.

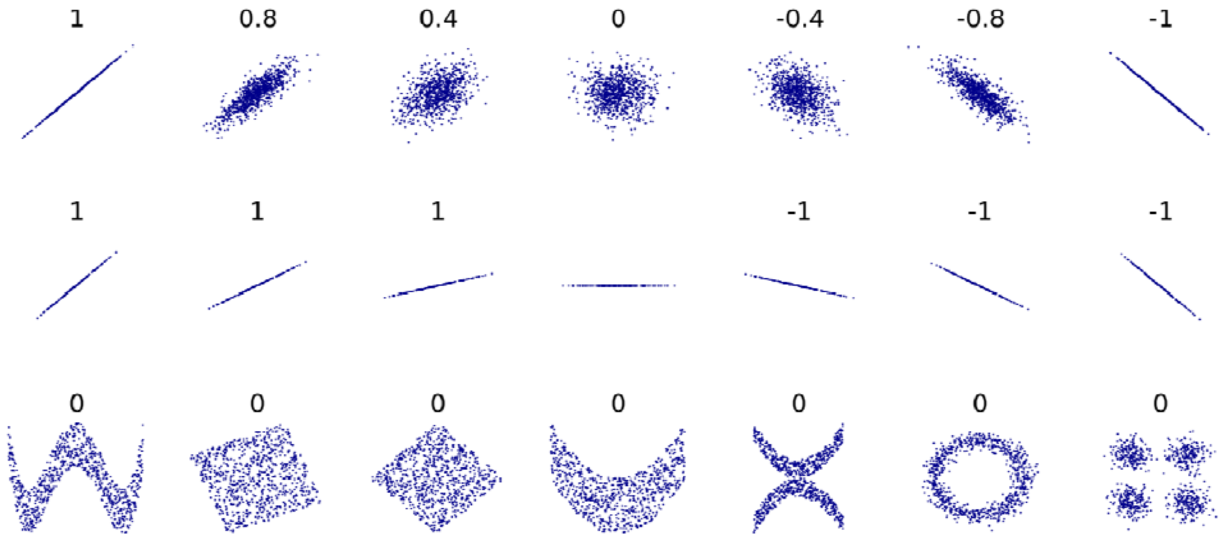
$$c_{X,Y} = E\{[X - m_X][Y - m_Y]\} = \iint_{-\infty}^{\infty} [x - m_X][y - m_Y] f_{X,Y}(x, y) dx dy$$

Comparison of Random Variables

1.1

The **correlation coefficient** is a normalized version of the covariance measure:

$$\rho_{X,Y} = \frac{c_{X,Y}}{\sigma_X \sigma_Y} = \frac{E\{[x - m_X][y - m_Y]\}}{\sigma_X \sigma_Y}$$



Several sets of (x, y) points, with the correlation coefficient of X and Y for each set. Note that the correlation reflects the noisiness and direction of a linear relationship (top row), but not the slope of that relationship (middle), nor many aspects of nonlinear relationships (bottom). [Wikimedia Commons]

Comparison of Random Variables

1.1

As an adaptation of the second order moment, we define as well a measure that does not depend of the random variable expected values (m_X, m_Y):

Correlation: Measures the **joint variability of two random variables** (X, Y) regardless their expected values:

- Same comments with respect to sign and magnitude as before.

$$r_{X,Y} = E\{XY\} = \iint_{-\infty}^{\infty} xy f_{X,Y}(x,y)dxdy$$

$$c_{X,Y} = r_{X,Y} - m_X m_Y$$

If the two random variables (x, y) are **independent**:

$$r_{X,Y} = E\{XY\} = E\{X\}E\{Y\} = m_X m_Y \qquad c_{X,Y} = 0$$

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Multivariate Random Variable

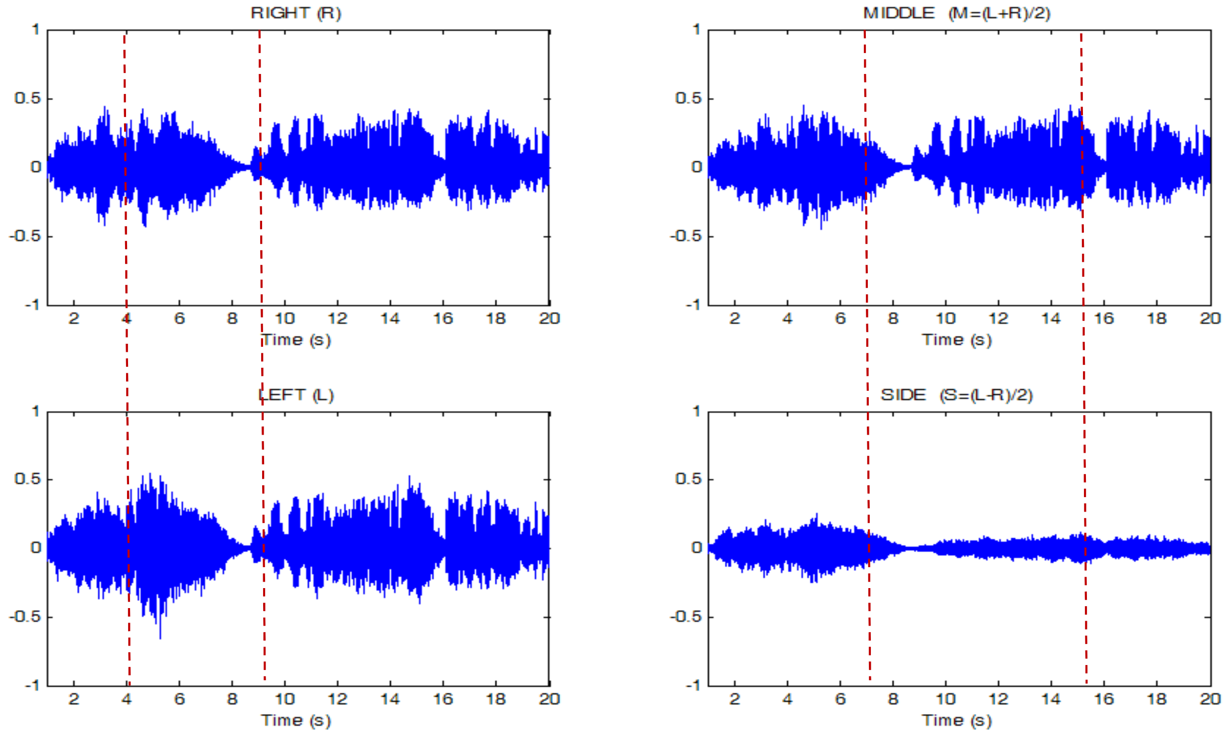
1.1

Another way to study sets of random variables is assuming that they create a **multivariate random variable**. Given N random variables, $X_1, X_2, \ldots, X_N$, we define a vector:

$$\underline{X} = [X_1, X_2, \ldots, X_N]^T$$

Example 1:
Stereo audio (N = 2)

$$\underline{X} = \begin{bmatrix} X_r[n] \\ X_l[n] \end{bmatrix}$$



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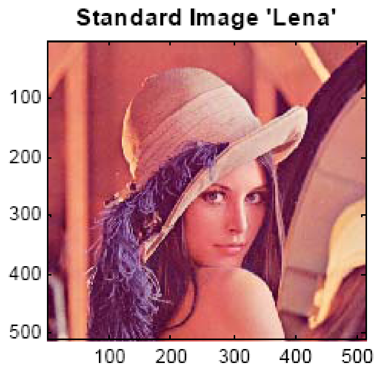
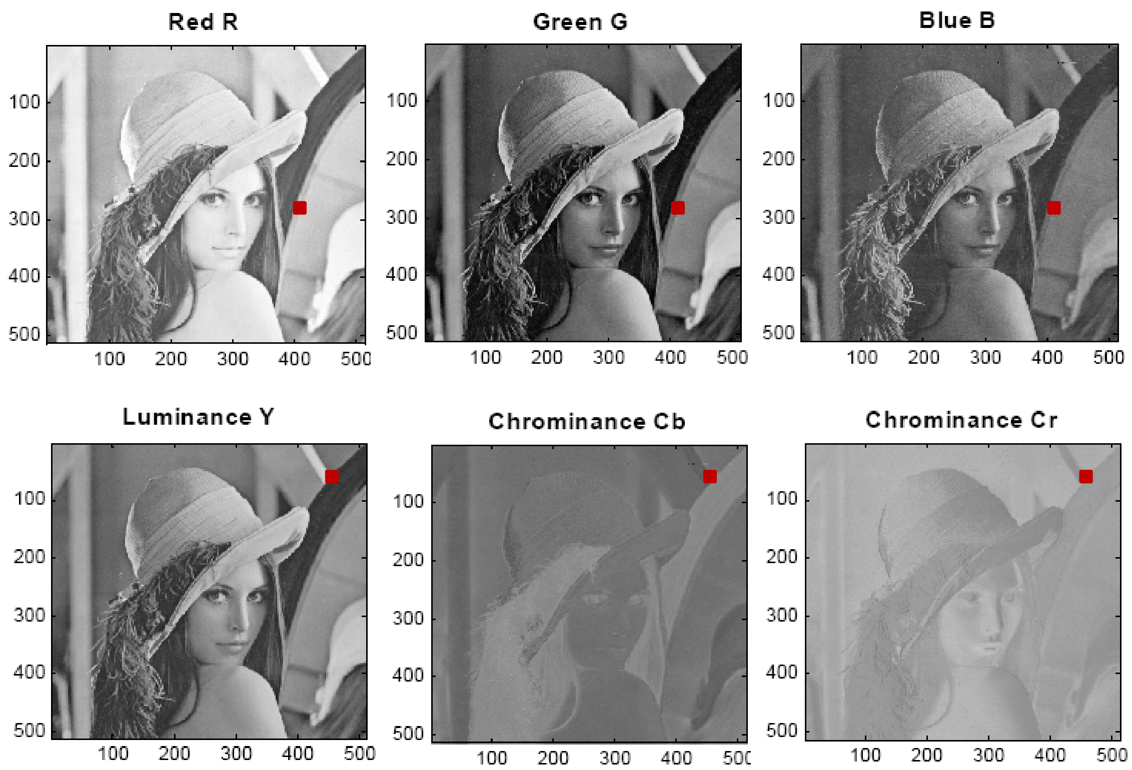
Example 2:
Color images (N = 3)





$$\underline{X} = \begin{bmatrix} X_R[m,n] \\ X_G[m,n] \\ X_B[m,n] \end{bmatrix}$$

Multivariate Random Variable

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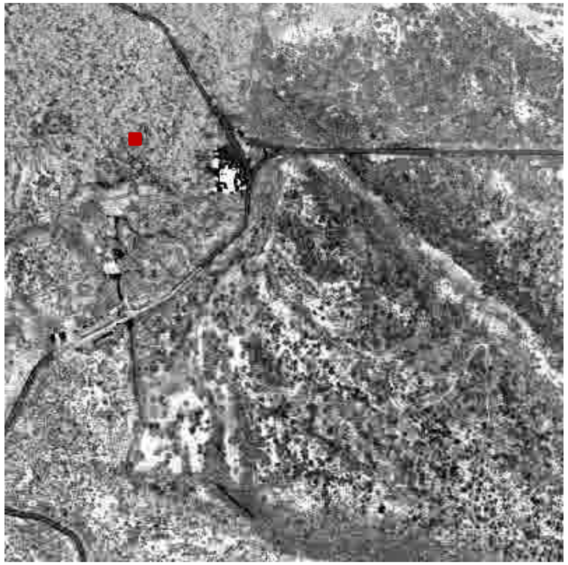
$$\underline{X} = [X_1, X_2, \ldots, X_N]^T$$

Example 3:
Hyperspectral images (N > 70)

$$\underline{X} = \begin{bmatrix} X_{B_1}[m,n] \\ X_{B_2}[m,n] \\ \cdots \\ X_{B_k}[m,n] \\ \cdots \\ X_{B_N}[m,n] \end{bmatrix}$$



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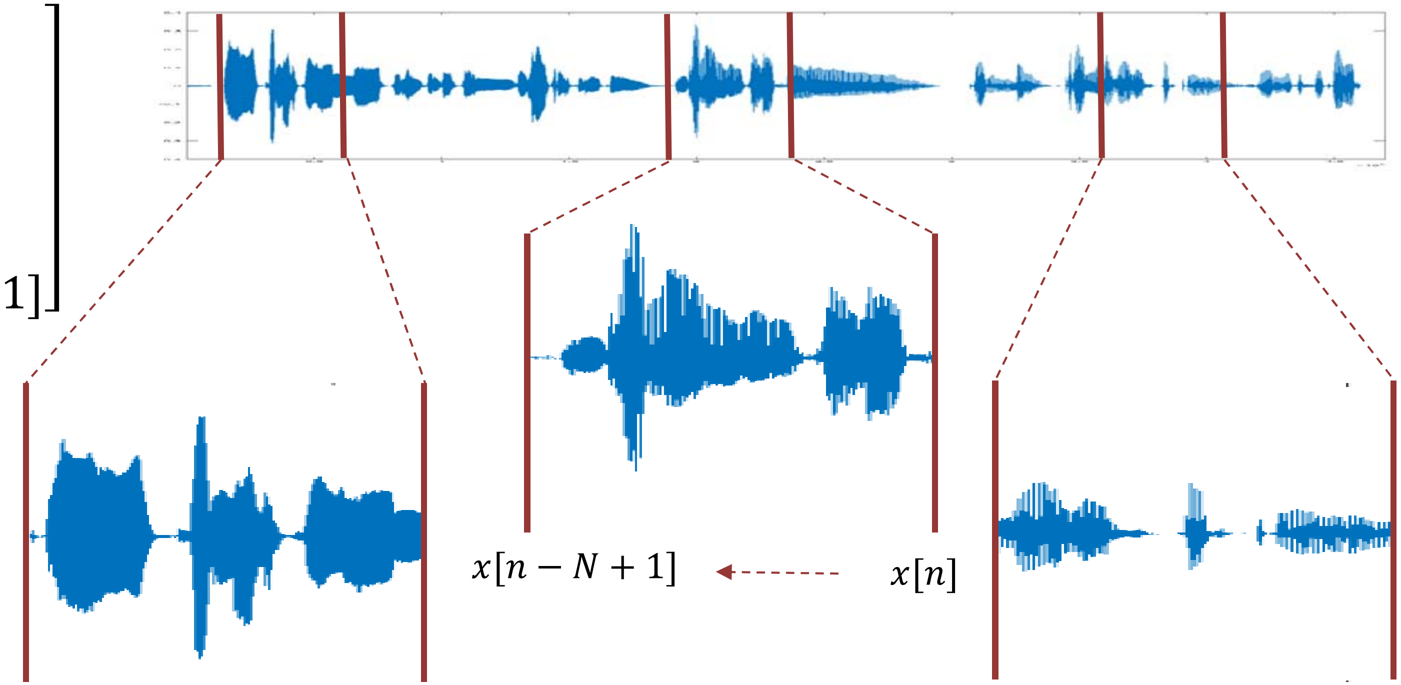
$$\underline{X} = [X_1, X_2, \ldots, X_N]^T$$

Example 3:
Frames of audio signals (N > 70)

$$\underline{X}[n] = \begin{bmatrix} X[n] \\ X[n-1] \\ \cdots \\ X[n-k] \\ \cdots \\ X[n-N+1] \end{bmatrix}$$



Multivariate Random Variable

1.1

As for scalar random variables, the (**joint**) **cumulative distribution** and **probability density functions** are defined:

- **Joint cumulative distribution function** determines the probability that every component of a random variable ($\underline{X} = [X_1, X_2, \ldots, X_N]^T$) takes a value less than or equal to the associated components of a given vector ($\underline{x}$):

$$F_{\underline{X}}(\underline{x}) = F_{\underline{X}}(x_1, x_2, \ldots, x_N) = P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_N \leq x_N)$$

- **Joint probability density function** provides a *relative likelihood* that a result is given in an experiment:

$$f_{\underline{X}}(\underline{x}) = f_{\underline{X}}(x_1, x_2, \ldots, x_N) = \frac{\partial^N F_{\underline{X}}(x_1, x_2, \ldots, x_N)}{\partial x_1 \partial x_2 \ldots \partial x_N}$$

$$f_{\underline{X}}(\underline{x}) \geq 0$$

$$\int_{-\infty}^{\infty} f_{\underline{X}}(\underline{x}) d\underline{x} = 1$$

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Moments of a Multivariate RV

The previous **moment definitions** can be extended to the case of multivariate random variables

Expected value (first order moment): It is a measure of the mean of the multivariate random variable.

$$\underline{m}_{\underline{X}} = E\{\underline{X}\} = E\{[X_1, X_2, \dots, X_N]^T\} = [E\{X_1\}, E\{X_2\}, \dots, E\{X_N\}]^T = \int_{-\infty}^{\infty} \underline{x}\, f_{\underline{X}}(\underline{x}) d\underline{x}$$

Covariance: It measures the dispersion of the multivariate random variable around its expected value:

$$\boxed{\underline{\underline{\mathbf{C}}}_{\underline{X}} = covar(\underline{X}) = E\left\{[\underline{X} - E\{\underline{X}\}][\underline{X} - E\{\underline{X}\}]^T\right\}}$$

- Example of **covariance matrix for $N = 2$**

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Commonly, we will assume that the behavior of the random variable that is being analyzed can be characterized by a given **probability distribution**.

A typical case is the **Gaussian model**:

$$f_{\underline{X}}(\underline{x}) = \frac{1}{\sqrt{(2\pi)^N |\underline{\underline{\mathbf{C}}}_{\underline{X}}|}} exp\left[-\frac{[\underline{x}-\underline{m}_{\underline{X}}]^T \underline{\underline{\mathbf{C}}}_{\underline{X}}^{-1} [\underline{x}-\underline{m}_{\underline{X}}]}{2}\right]$$

Real random variable:
Usual case in this course

$$f_{\underline{X}}(\underline{x}) = \frac{1}{\pi^N |\underline{\underline{\mathbf{C}}}_{\underline{X}}|} exp\left[-[\underline{x}-\underline{m}_{\underline{X}}]^H \underline{\underline{\mathbf{C}}}_{\underline{X}}^{-1} [\underline{x}-\underline{m}_{\underline{X}}]\right]$$

Complex random variable:
Transform domain

where $\underline{x}^H = (\underline{x}^*)^T$ denotes **Hermitian**; that is, conjugate transpose.

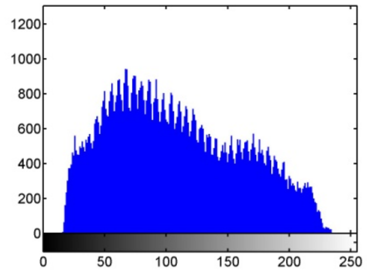
Random Variable Models

As in the scalar case, in some cases, there will not be a simple mathematical model that correctly fits the random variable behavior and we will use an **empirical model**:

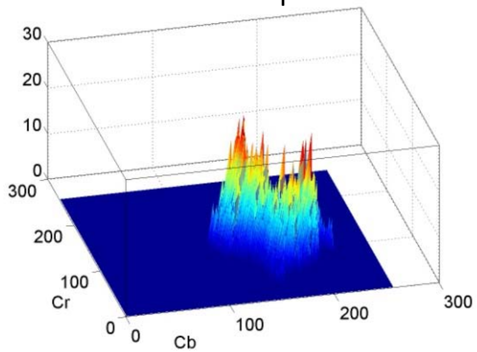
Y

C_b

C_r



1D Histogram (Y)



2D Histogram (C_bC_r)

The 2D histogram of the Cb, Cr components of an image as its joint pdf estimate

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