ALGEBRA
2. VECTOR SPACES

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Outline

1. $\mathbb{R}^n$ and other vector spaces
2. Vector subspaces
3. Linear dependency, basis and dimension
4. Coordinates and change of basis
5. Intersection and sum
6. Quotient space
7. Bibliography

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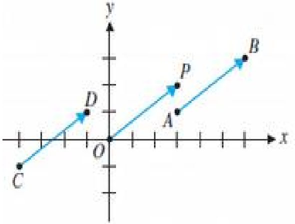
THE VECTOR SPACE $\mathbb{R}^n$

We consider the set of n-tuples of real numbers:

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R}\}$$

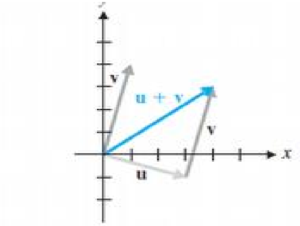
and we call its elements vectors.

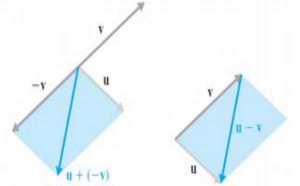
- $(x, y) \in \mathbb{R}^2$ can be thought either as a point or as a directed line segment between two points A and B, $(x, y) = \overrightarrow{AB}$. We think of them in this last sense (*vectors*).
- The vector $\overrightarrow{AB}$ represents the displacement needed to get from A to B: x units along the x-axis and y-along the y-axis.
- Two vectors are equal if they represent the same displacement (equivalently, if they have the same length direction and sense).
- We can always think (x, y) as a vector of initial point $(0, 0)$ and end point (x, y).



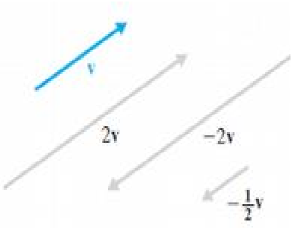
OPERATIONS IN $\mathbb{R}^2$

We can sum or substract vectors





and we can multiply a vector by a constant (scalar)



$\mathbb{R}^3$

- Vectors in $\mathbb{R}^3$ have a similar physical interpretation
- We can also sum two vectors and multiply a vector by a scalar. These operations can be done in coordinates: if $u = (x_1, x_2, x_3)$ and $v = (y_1, y_2, y_3)$, then

$$u + v = (x_1 + y_1, x_2 + y_2, x_3 + y_3),$$

$$c \cdot u = (cx_1, cx_2, cx_3) \text{ for any } c \in \mathbb{R}.$$

Operations in $\mathbb{R}^n$

In $\mathbb{R}^n$ we define the following operations:

sum: if $u = (x_1, x_2, \ldots, x_n), v = (y_1, y_2, \ldots, y_n)$, then

$$u + v = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \in \mathbb{R}^n.$$

scalar multiplication: if $u = (x_1, x_2, \ldots, x_n), c \in \mathbb{R}$, then

$$c \cdot u = (c\,x_1, c\,x_2, \ldots, c\,x_n) \in \mathbb{R}^n.$$

Proposition

These operations in $\mathbb{R}^n$ satisfy the following properties:

1. $u + v = v + u$. *Commutativity*
2. $(u + v) + w = u + (v + w)$. *Associativity*
3. $\exists$ *an element* $\boldsymbol{0} \in \mathbb{R}^n$, *called the zero vector, such that* $u + \boldsymbol{0} = u$.
4. *For each* $u \in \mathbb{R}^n$, $\exists$ *an element* $-u \in \mathbb{R}^n$ *such that* $u + (-u) = \boldsymbol{0}$.
5. $c \cdot (u + v) = c \cdot u + c \cdot v$. *Distributivity*
6. $(c + d) \cdot u = c \cdot u + d \cdot u$. *Distributivity*
7. $c \cdot (d \cdot u) = (cd) \cdot u$.
8. $1 \cdot u = u$.

Vector space over $\mathbb{K}$

Let $\mathbb{K}$ be $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ or any other (commutative) field (*"cos"*).

A vector space over $\mathbb{K}$ ($\mathbb{K}$-*e.v.*) is a set E with two operations $+$ and $\cdot$,

- $+$ given $u, v \in E$, it assigns another element $u + v$ of E.
- $\cdot$ given $u \in E$ and a scalar $c \in \mathbb{K}$, it assigns an element $c \cdot u \in E$

that satisfy the previous properties, i.e,

- $+$ is commutative, associative, has a neutral element (denoted **0** or $\vec{0}$) and every $u \in E$ has an inverse with respect to $+$ (denoted $-u$),
- $\cdot$ and $+$ satisfy:

$$c{\cdot}(u{+}v) = c{\cdot}u{+}c{\cdot}v,\ \ (c{+}d){\cdot}u = c{\cdot}u{+}d{\cdot}u,\ \ c{\cdot}(d{\cdot}u) = (cd){\cdot}u,\ \ 1{\cdot}u = u$$

for any $u, v \in E$ and $c, d \in \mathbb{K}$.

The elements of a $\mathbb{K}$-e.v. are called vectors.

- $\mathbb{K}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{K}\}$ is a $\mathbb{K}$-e.v. with the natural sum and product inherited by $\mathbb{K}$.
- $\mathcal{M}_{m\times n}(\mathbb{K})$=$m \times n$ matrices with entries in $\mathbb{K}$ and the natural operations of sum of matrices and multiplication by scalars is a $\mathbb{K}$-e.v.
- The set of polynomials of degree $\leq d$, $\mathbb{R}_d[x] = \{p(x) = a_0 + a_1 x + \ldots + a_d x^d \mid a_i \in \mathbb{R}\}$, is a vector space with the usual sum of polynomials and multplication by a scalar.
- $\mathbb{R}[x] = \{\text{polynomials in one variable x and coefficients in } \mathbb{R}\}$ is a $\mathbb{R}$-e.v.
- The set $\mathcal{F}(\mathbb{R}, \mathbb{R})$ of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a $\mathbb{R}$-e.v. with the usual sum of functions ($f + g$ is the function $(f + g)(x) = f(x) + g(x)$) and product by a scalar ($c \cdot f$ is the function $(c \cdot f)(x) = cf(x)$).

PROPERTIES

If E is a $\mathbb{K}$-e.v., then

(A) $0 \cdot u = \mathbf{0} = c \cdot \mathbf{0}$,

(B) $(-1) \cdot u = -u$,

(C) $(-c) \cdot u = c \cdot (-u) = -(c \cdot u)$ (so we denote it by $-cu$),

(D) $c \cdot u = \mathbf{0} \Leftrightarrow c = 0$ or $u = 0$.

Note: Usually we avoid the notation $\cdot$

DEFINITION

A vector u is a linear combination of vectors $u_1, \ldots, u_k$ if there are scalars $c_1, \ldots, c_k$ such that $u = c_1\, u_1 + \ldots + c_k\, u_k$ (the scalars c_i are the coefficients of the linear combination).

Finding out whether a vector in $\mathbb{K}^n$ is a linear combination of a collection of given vectors is equivalent to solving a linear system of equations:

PROPOSITION

A system $Ax = b$ is consistent if and only if b is a linear combination of the columns of A.

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3. LINEAR DEPENDENCY, BASIS AND DIMENSION
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Vector subspaces

Definition

Let E be a $\mathbb{K}$-e.v. Then a subset $V \subseteq E$ is a vector subspace of E if V is itself a vector space (with $+$ and $\cdot$ of E). If $V \neq \emptyset$, this is equivalent to:

1. If u and v are in V, then $u + v$ is in V.
2. If u is in V and c is a scalar, then $c \cdot u$ is in V.

Ex:

- $V = \mathbb{K}^n$ is a vector subspace of $\mathbb{K}^n$.
- $V = \{\mathbf{0}\}$ is a vector subspace (of any E).
- $V = \{(x, y, z) \in \mathbb{R}^3 \mid x - y = 0, 3z = 0\}$ is a vector subspace of $\mathbb{R}^3$.
- $F = \{(a + 2b, 0, b) \in \mathbb{R}^3 \mid a, b \in \mathbb{R}\}$ is a vector subspace of $\mathbb{R}^3$.

- Every subspace $V \neq \emptyset$ contains the zero vector.
- Properties 1 and 2 can be combined:
  $V \neq \emptyset$ is a subspace $\Leftrightarrow$ for any $u_1, \ldots, u_k$ in V and $c_1, \ldots, c_k$ in $\mathbb{K}$, the linear combination

$$c_1 u_1 + \ldots + c_k u_k$$

  is also in V.
  That is, vector subspaces are closed under linear combinations.

PROPOSITION

Let $Ax = 0$ be a linear system, where $A \in M_{m,n}(\mathbb{K})$. Then, the set of solutions $V = \{v \in \mathbb{K}^n \mid Av = 0\}$ is a vector subspace of $\mathbb{K}^n$.

Let $v_1, v_2, \ldots, v_k$ be vectors in E.

Definition

The set of all linear combinations of $v_1, v_2, \ldots, v_k$,

$$\{c_1 v_1 + \ldots + c_k v_k \mid c_1, \ldots, c_n \in \mathbb{K}\}$$

is the called the span of $v_1, v_2, \ldots, v_k$ and is denoted as $[v_1, v_2, \ldots, v_k]$.

Proposition

$V = [v_1, v_2, \ldots, v_k]$ is a vector subspace and is the smallest subspace containing $\{v_1, \ldots, v_k\}$.

We say that $\{v_1, v_2, \ldots, v_k\}$ is a system of generators of V, and also that V is spanned by $v_1, v_2, \ldots, v_k$.

Examples:

- $\mathbb{R}^n = [(1, 0, \ldots, 0), \ldots, (0, 1, 0, \ldots, 0)]$.
- $V = \{(x, y, z) \in \mathbb{R}^3 \mid x - y = 0, 3z = 0\} \Rightarrow V = [(1, 1, 0)]$.
- $V = \{(x, y, z) \in \mathbb{R}^3 \mid x - y = 0\} \Rightarrow V = [(1, 1, 0), (0, 0, 1)]$.
- $F = \{(a + 2b, 0, b) \in \mathbb{R}^3 \mid a, b \in \mathbb{R}\} = [(1, 0, 0), (2, 0, 1)]$.

A vector space E is finitely generated (f.g.) if it can be spanned by a finite collection of vectors.

- $\mathbb{K}^n$ is f.g.
- $\mathbb{K}[x]$ is not f.g.

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Linear dependency

Definition

The vectors $v_1, v_2, \ldots, v_k$ are linearly dependent (l.i.) if there are scalars $c_1, c_2, \ldots, c_k$, at least one $\neq 0$, such that $c_1\, v_1 + \ldots + c_k\, v_k = \mathbf{0}$. Otherwise, we say that $v_1, v_2, \ldots, v_k$ are linearly independent.

$v_1, v_2, \ldots, v_k$ are linearly independent if any linear combination of them that equals $\mathbf{0}$, $c_1\, v_1 + \ldots + c_k\, v_k = \mathbf{0}$, implies $c_1 = c_2 = \ldots = c_k = 0$.

Remarks:

1. Any set of vectors containing $\mathbf{0}$ is linearly dependent.
2. Two vectors v_1, v_2 are linearly dependent $\Leftrightarrow$ one is multiple of the other.
3. $v_1, v_2, \ldots, v_k$ in E are l.d. $\Leftrightarrow$ at least one of the vectors can be expressed as a linear combination of the others.
4. If $v_1, \ldots, v_k$ are l.i. and $u \notin [v_1, \ldots, v_k] \Rightarrow v_1, \ldots, v_k, u$ are l.i.

Basis of a vector subspace

Definition

Let $V \subset E$ be a vector subspace. A collection of vectors $\{v_1, \dots, v_k\}$ is a basis of V if

1. $V = [v_1, \dots, v_k]$ and
2. $\{v_1, \dots, v_k\}$ is linearly independent.

Example

If $e_i = (0, \dots, 1, \dots, 0)$ for $i = 1, 2, \dots, n$, then $\{e_1, e_2, \dots, e_n\}$ is a basis for $\mathbb{R}^n$, called the *standard basis* for $\mathbb{R}^n$.

Ex: $(1, 1, 0), (0, 0, 1)$ is a basis of $V = \{(x, y, z) \in \mathbb{R}^3 \mid x - y = 0\}$.

THEOREM (STEINITZ SUBSTITUTION LEMMA)

Let E be a $\mathbb{K}$-e.v. f.g. Let $v_1, \ldots, v_m$ be generators of E and $u_1, \ldots, u_n$ be linearly independent vectors. Then, $n \leq m$ and one can substitute n vectors of $\{v_1, \ldots, v_m\}$ by $u_1, \ldots, u_n$ such that the new collection of vectors is still a system of generators for E.

COROLLARY (THE BASIS THEOREM)

Any two bases of a f.g. vector space have the same number of elements .

This number is called the dimension of the vector space and noted as $\dim(E)$. By convention, $\dim(\{\vec{0}\}) = 0$.

Proposition

Let E be a vector space of dimension n. Then:

1. *Any system of generators for E contains $\geq n$ vectors. Moreover, it contains a basis of E.*
2. *Any linearly independent set in E contains $\leq n$ vectors. Moreover, it can be extended to a basis of E (by choosing conveniently vectors of a given basis of E).*
3. *Any linearly independent set of n vectors in E is a basis for E.*
4. *Any system of generators for E consisting of n vectors is a basis for E.*

Hence,

$$\begin{aligned} n &= \textit{minimum number of elements in a system of generators of } E \\ &= \textit{maximum number of l.i. vectors in } E. \end{aligned}$$

Theorem

Let $V_1 \subset V_2$ *be subspaces of* E *and* $\dim(E) = n$. *Then:*

1. $dim V_1 \leq dim V_2 \leq n$.
2. $dim V_1 = dim V_2$ *if and only if* $V_1 = V_2$.

Rank (revisited)

Theorem

Given $v_1, v_2, \dots, v_k \in \mathbb{K}^n$, write $A = (v_1, \dots, v_k) \in \mathcal{M}_{n,k}(\mathbb{K})$. Then,

A) *$v_1, v_2, \dots, v_k$ are l.d. $\Leftrightarrow$ the homogeneous system $Ax = 0$ has a nontrivial solution (indeterminate system).*

B) *$v_1, v_2, \dots, v_k$ are l.i. $\Leftrightarrow$ $\mathrm{rank}(A) = k$.*

C) *$v_1, v_2, \dots, v_k$ are a system of generators of $\mathbb{K}^n$ $\Leftrightarrow$ $\mathrm{rank}(A) = n$.*

D) *$v_1, v_2, \dots, v_k$ is a basis of $\mathbb{K}^n$ $\Leftrightarrow$ $k = n$ and $\mathrm{rank}(A) = n$.*

Proposition

*The **rank** of a matrix A equals:*

- *the dimension of the row span of A (number of linearly independent rows) and*
- *the dimension of the column span of A (number of linearly independent columns).*

If $V = [v_1, v_2, \ldots, v_k] \subset \mathbb{K}^n$, then a basis of V can be obtained by applying one the following methods:

1. Write the vectors $v_1, \ldots, v_k$ as the rows of a matrix A, and reduce A to row echelon form $\bar{A}$ (Gaussian elimination). The nonzero rows of $\bar{A}$ are a basis of V.
2. Write the vectors $v_1, \ldots, v_k$ as the columns of a matrix B. Then, reduce B to row echelon form $\bar{B}$ (Gaussian elimination). The columns of $\bar{B}$ with pivots indicate which vectors $v_1, \ldots, v_k$ to choose to obtain a basis of V.

Extending to a basis of $\mathbb{K}^n$

If $u_1, \ldots, u_k$ are linearly independent vectors of $\mathbb{K}^n$, then they can be extended to a basis of $\mathbb{K}^n$:

- Write the vectors $u_1, \ldots, u_k$ as the columns of a matrix B, and take $M = (B \mid Id)$.
- Then, reduce M to row echelon form $\bar{M} = (\bar{B} \mid \bar{Id})$ (Gaussian elimination).
- Collect the columns of $\bar{Id}$ with a pivot and choose the corresponding vectors of the standard basis (columns of Id) of $\mathbb{K}^n$.
- $u_1, \ldots, u_k$ together with these last vectors form a basis of $\mathbb{K}^n$.

The same can be done if $u_1, \ldots, u_k$ are linearly independent vectors of a vector subspace V:
instead of Id, take a matrix formed by a basis $v_1, \ldots, v_d$ of V and do the same process as above for $M = (u_1, \ldots, u_k | v_1, \ldots, v_d)$.

From “generators” to “equations”:
If $V = [v_1, \dots, v_k] \subset \mathbb{K}^n$:
Write $M = (v_1, \dots, v_k)$, and form an augmented matrix $(M|x)$ with $x =$ column with entries $x_1, x_2, \dots, x_n$.
Then $x \in [v_1, \dots, v_k]$ if and only if $rank(M|x) = rank(M)$.
There are 2 options:

- Reduce M to echelon form $(\bar{M}|\bar{x})$ by Gaussian elimination $\Rightarrow$ a linear system of equations for V is obtained by writing the equations that correspond to zero rows of $\bar{M}$.
- If $rank(M) = k$, the equations are formed by the vanishing of the $(k+1) \times (k+1)$ minors of $(M|x)$ that contain a chosen non-zero $k \times k$ minor of M.

From "equations" to "generators":

If $V = \{u \in \mathbb{K}^n \mid Au = 0\}$ (solutions to a homogeneous system):

- It is enough to solve the system to obtain a system of generators of V.
- Moreover, if we give values 0's and 1's to the free variables, these generators form a basis and $dim(V) = n - rank(A)$.

We have proved:

COROLLARY

A subset V of $\mathbb{K}^n$ is a subspace $\Leftrightarrow$ it is the set of solutions to a homogeneous system.

Vector spaces not finitely generated

If we consider any collection of vectors $\{v_i\}_{i\in I}$ of a $\mathbb{K}$-e.v E (where I is any set of indices), then one has to redefine linear combinations:

- $v \in E$ is a linear combination of $\{v_i\}_{i\in I}$ if v is a linear combination of a finite collection of them,

  $$v = x_1 v_{i_1} + x_2 v_{i_2} + \ldots + x_k v_{i_k}$$

  for certain $x_1, \ldots, x_k \in \mathbb{K}$ and $i_1, \ldots, i_k \in I$.
  Therefore,
- $\{v_i\}_{i\in I}$ are linearly dependent if $\mathbf{0}$ is a linear combination of (a finite number of) them.
- $\{v_i\}_{i\in I}$ are linearly independent if any finite subfamily is linearly independent.

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Coordinates

Theorem

Let $B = \{v_1, \dots, v_n\}$ be a basis of a $\mathbb{K}$-e.v. E. Then, for every vector $v \in E$, there is exactly one way to write v as a linear combination of the vectors in B, that is, there exist $c_1, \dots, c_n \in \mathbb{K}$ such that $v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$ and moreover, these $c_1, \dots, c_n$ are unique.

Definition

The $c_1, c_2, \dots, c_n$ are called the coordinates of v with respect to B.

We will use the notation

$$v_B = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

Coordinates: from E to $\mathbb{K}^n$

Coordinates preserve linear combinations:
If $v_1, \ldots, v_k$ are in E and $B = \{v_1, \ldots, v_n\}$ is a basis of E then

$$(x_1 u_1 + \ldots + x_k u_k)_B = x_1 (u_1)_B + \ldots + x_k (u_k)_B.$$

In particular,

- $u_1, \ldots, u_k$ are l.i. $\Leftrightarrow (u_1)_B, \ldots, (u_k)_B$ are l.i. in $\mathbb{K}^n$.

Let $B = \{u_1, \ldots, u_n\}$ and $C = \{v_1, \ldots, v_n\}$ be bases of E.
Denote by $A_{B \to C}$ the $n \times n$ matrix whose columns are the coordinate vectors $((u_1)_C, \ldots, (u_n)_C)$ of B with respect to C. This is the change-of-basis matrix from B to C:

$$A_{B \to C} = \begin{pmatrix} (u_1)_C & \ldots & (u_n)_C \end{pmatrix}.$$

PROPOSITION

1. *$A_{B \to C} w_B = w_C$ for all $w \in \mathbb{R}^n$.*
2. *$A_{B \to C}$ is invertible, and $(A_{B \to C})^{-1} = A_{C \to B}$.*
3. *If D is another basis of E, then $A_{C \to D} A_{B \to C} = A_{B \to D}$.*

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Given V_1, V_2 vector subspaces of E, define

1. Intersection of V_1 and V_2 is $V_1 \cap V_2 = \{v \in E \mid v \in V_1, v \in V_2\}$.
2. Sum of V_1 and V_2 is $V_1 + V_2 = \{v_1 + v_2 \in E \mid v_1 \in V_1, v_2 \in V_2\}$.
   Computation: If $V_1 = [u_1, \ldots, u_r]$ and $V_2 = [v_1, \ldots, v_s]$, then $V_1 + V_2 = [u_1, \ldots, u_r, v_1, \ldots, v_s]$.

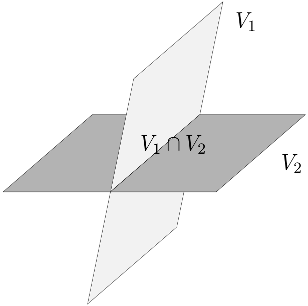
THEOREM

1. *$V_1 \cap V_2$ and $V_1 + V_2$ are vector subspaces of E.*
2. *Grassmann formula*
   $dim(V_1 \cap V_2) + dim(V_1 + V_2) = dim(V_1) + dim(V_2)$.

Theorem

1. *$V_1 \cap V_2$ and $V_1 + V_2$ are vector subspaces of E.*
2. *Grassmann formula*
   $dim(V_1 \cap V_2) + dim(V_1 + V_2) = dim(V_1) + dim(V_2)$.

$$V_1 = [(1,0,1),(0,2,3)] \qquad V_1 \cap V_2 = [(1,0,1)]$$
$$V_2 = [(0,1,0),(1,1,1)] \qquad V_1 + V_2 = \mathbb{R}^3$$



Direct sum

Definition

E is a direct sum of subspaces $F_1, F_2, \ldots, F_m$ if any $w \in E$ can be written in a **unique way** as $w = v_1 + v_2 + \cdots + v_m$ whith $v_i \in F_i$, $i = 1, \ldots, m$.

In this case we use the notation $E = F_1 \oplus F_2 \oplus \cdots \oplus F_m$.

Proposition

Let F_1, F_2 be two subspaces of E. Then $E = F_1 \oplus F_2$ if and only if the following two conditions hold:

$$E = F_1 + F_2,$$

$$F_1 \cap F_2 = \{0\}.$$

If $E = F_1 \oplus F_2$, we say that F_2 is a complementary subspace to F_1 (and vice-versa).

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Quotient space

Let E be a $\mathbb{K}$-e.v. and $F \subset E$ be a subspace.

Definition

If $v \in E$, we define the class of v modulo F as

$$\overline{v} = \{u \in E \mid u - v \in F\};$$

it can be also denoted as $v + F$.
The set $E/F := \{\overline{v} \mid v \in E\}$ is called the quotient of E by F.

E/F is the quotient set of E by the equivalence relation

$$u \sim v \Leftrightarrow u - v \in F$$

and hence, $\overline{u} = \overline{v} \Leftrightarrow u - v \in F$.
Any element of $\overline{v}$ is called a representative of the class.

As a vector space

In E/F we can define two operations:

+ $\overline{u} + \overline{v} := \overline{u+v}$, $u, v \in E$
. $c \cdot \overline{v} := \overline{c \cdot v}$, $v \in E$, $c \in \mathbb{K}$.

With these operations E/F is a **vector space.**

Theorem

Let E be a $\mathbb{K}$-e.v. and $F \subset E$ be a subspace. Given $u_1, \ldots, u_k \in E$ the following are equivalent:

- $\overline{u_1}, \ldots, \overline{u_k}$ *is a basis of* E/F
- $u_1, \ldots, u_k$ *is a basis of a complementary subspace of* F *in* E.

In particular, if E has finite dimension, then

$$\dim(E/F) = \dim(E) - \dim(F)$$

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