Unit 4. Z-Transform and Filter Design

1. The Z-transform (22/11/2019)
2. Transfer function and frequency response (26/11/2019)
3. Filter design (29/11/2019)

Lab 3 (Part 1: 11/12/2019, Part 2: 18/12/2019)

2019-2020

Signals and Systems (DSE)

The Z-Transform

U4

Another one?!

Why do we need another transform?

The Z-transform is widely used:

- To describe/analyze systems defined with a finite difference equation in a compact/easy way
- To design digital filters
- Actually, the FT of a sequence is a particular case of the Z-transform

Z-Transform definition

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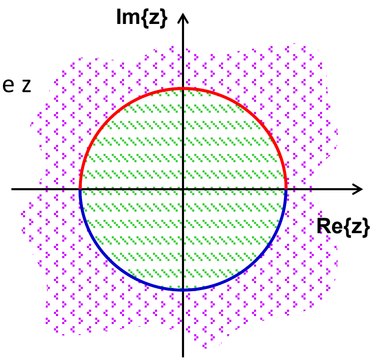
- Definition of Z-transform of a discrete sequence x[n]:

$$\boxed{X(z) = Z\left[x[n]\right] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}}$$

It is a complex function of complex variable z



ROC (Region of Convergence) definition:

$$z \in ROC \Leftrightarrow \sum_{n=-\infty}^{\infty} \left| x[n] z^{-n} \right| < +\infty$$

Z-transform and Fourier Transform

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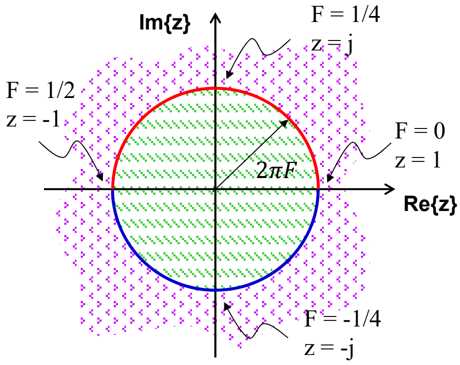
- The Fourier Transform is a particular case of the Z-transform:

$$X(F) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi F n} \quad = X(z)|_{z=e^{j2\pi F}}$$

Unit-circle of z-plane:

$|z| = 1, \qquad z = e^{j2\pi F}$

Note the 1-periodicity over F



Z-transform: examples (1)

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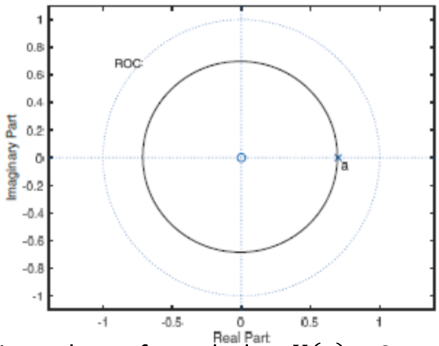
$$x[n] = a^n u[n] \leftrightarrow X(z) = \sum_{n=-\infty}^{\infty} a^n u[n]\, z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

Si $\left|\frac{a}{z}\right| < 1 \Rightarrow X(z) = \frac{1}{1-az^{-1}}$

The ROC is the outside region of the circle of radius |a|: ROC={$|z| > |a|$}



- 'o' zeros: Roots of the numerator, i.e. values of z such that $X(z)$ =0
- 'x' poles: Roots of the denominator, i.e. values of z such that $|X(z)| \rightarrow \infty$

Z-transform: examples (2)

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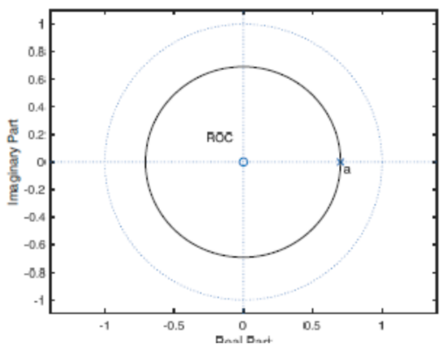
$$x[n] = -a^n u[-n-1] \leftrightarrow X(z) = -\sum_{n=-\infty}^{-1} (az^{-1})^n = -\sum_{m=1}^{\infty} \left(\frac{z}{a}\right)^m$$

Si $\left|\frac{z}{a}\right| < 1 \Rightarrow X(z) = \frac{1}{1-az^{-1}}$

The ROC is the inner region of the circle of radius |a|: ROC={$|z| < |a|$}

This sequence and the previous one have the same Z-transform: But the ROC is different!



ROC for causal sequences

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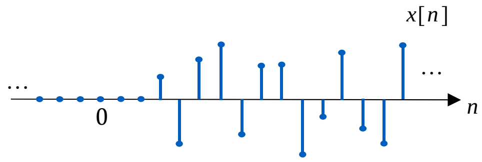
For signals such that *such that* $x[n] = 0,\ n < 0$



the ROC is either the outside of a circle or the complete complex plane:

$$z \in ROC \quad \Leftrightarrow \quad |z| > r \quad or \quad \forall z$$

Z-transform: examples (3)

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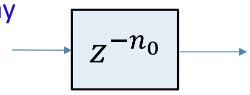
$$x[n] = \delta[n] \leftrightarrow X(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = 1$$

The ROC is the whole complex plane

$$x[n] = \delta[n-n_0] \leftrightarrow X(z) = \sum_{n=-\infty}^{\infty} \delta[n-n_0] z^{-n} = z^{-n_0}$$

For n_0>0, the ROC is the whole complex excepte z=0

Remember that a delay system is usually represented as



Signal	Waveform $x[n]$	Transform $X(z)$	ROC
Impulse	$\delta[n]$	1	$\forall z$
Delayed impulse	$\delta[n-n_0]$	z^{-n_0}	$\lvert z\rvert > 0,\ n_0 > 0$ $\forall z,\ n_0 < 0$
Rectangular pulse	$p_N[n] = \begin{cases} 1, & 0 \le n \le L-1 \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-z^{-L}}{1-z^{-1}}$	$\lvert z\rvert > 0$
Step function	$u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
Causal real potential function	$a^n u[n]$	$\dfrac{1}{1-a\cdot z^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
Causal cosinus	$\cos(2\pi F_0 n)u[n]$	$\dfrac{1-\cos(2\pi F_0)z^{-1}}{1-2\cos(2\pi F_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
Causal sinus	$\sin(2\pi F_0 n)u[n]$	$\dfrac{\sin(2\pi F_0)z^{-1}}{1-2\cos(2\pi F_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
Causal damped cosinus	$a^n\cos(2\pi F_0 n)u[n]$	$\dfrac{1-a\cos(2\pi F_0)z^{-1}}{1-2a\cos(2\pi F)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Z-transform: properties

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We will consider the the following ones:

- Linearity: $\alpha \cdot x_1[n] + \beta \cdot x_2[n] \quad \leftrightarrow \quad \alpha \cdot X_1(z) + \beta \cdot X_2(z)$
- Temporal shift (delay): $x[n-n_0] \quad \leftrightarrow \quad z^{-n_0}X(z)$
- Convolution: $x_1[n] * x_2[n] \quad \leftrightarrow \quad X_1(z) \cdot X_2(z)$

We already knew that the convolution of 2 sequences was a product of polynomials:

$$[\underline{1},2,3,4] * [\underline{5},6,7] \leftrightarrow (\underline{1} + 2z^{-1} + 3z^{-2} + 4z^{-3}) \cdot (\underline{5} + 6z^{-1} + 7z^{-2})$$

Transfer function of LTI systems

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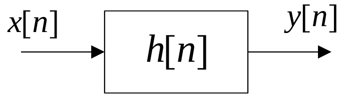

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Z-domain:

$$Y(z) = Z\left[y[n]\right] = Z\left[x[n] * h[n]\right] = X(z) \cdot H(z)$$

$$\boxed{H(z) = Z\left[h[n]\right] = \frac{Y(z)}{X(z)}, \quad X(z) \neq 0}$$

SYSTEM TRANSFER FUNCTION:
It does not depend on the input signal, but only on the system

Representation of LTI systems (a summary)

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Unit impulse response

$$h[n] = 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

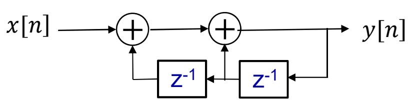
Output calculated through convolution

Finite difference equation:

$$y[n] = x[n] + y[n-1] + y[n-2]$$

- mathematically compact
- useful for step by step analysis

Block diagrams:



- also useful for step by step analysis
- illustrate signal flow paths
- different block diagrams for the same finite difference equation

Operator representations:

$$Y = X + Yz^{-1} + Yz^{-2} \quad \Rightarrow$$

$$H(z) = \frac{Y}{X} = \frac{1}{1 - z^{-1} - z^{-2}}$$

- analyze systems as polynomials
- transfer function (or system function): $H(z)$

Inverse Z-transform

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- General expression of the inverse Z-transform

$$h[n] = Z^{-1}[H(z)] = \frac{1}{2\pi j}\oint_C H(z)z^{n-1}dz$$

C: closed anti-clockwise circular path within the ROC and centered at the origin (z=0)

- Alternatives, when H(z) is the division of two polynomials:
  - Direct division of the polynomials, or
  - Identification of terms, or
  - Partial fraction expansion

Inverse Z-transform

U4

- Direct division of the polynomials:

$$H(z) = \frac{1}{1 - z^{-1} - z^{-2}} = 1 + \mathrm{z}^{-1} + 2\mathrm{z}^{-2} + 3\mathrm{z}^{-3} + 5\mathrm{z}^{-4} + \ldots$$

- Partial fraction expansion

  If degree(numerator) < degree(denominator) as power of z^{-1}

$$H(z) = \frac{1}{1-z^{-1}-z^{-2}} = \frac{\mathrm{A}}{1-az^{-1}} + \frac{\mathrm{B}}{1-bz^{-1}} \quad \text{with } a = \frac{1+\sqrt{5}}{2};\ b = \frac{1-\sqrt{5}}{2}$$

$$A = \frac{1}{1-z^{-1}-z^{-2}}(1 - az^{-1})\Big|_{z=a} = \frac{1}{1-bz^{-1}}\Big|_{z=a} = \frac{\mathrm{a}}{\mathrm{a}-b}$$

$$B = \frac{1}{1-z^{-1}-z^{-2}}(1 - bz^{-1})\Big|_{z=b} = \frac{1}{1-az^{-1}}\Big|_{z=b} = \frac{b}{b-a}$$

$$\boxed{h[n] = Aa^n u[n] + Bb^n u[n]}$$

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Systems defined with a finite difference equation

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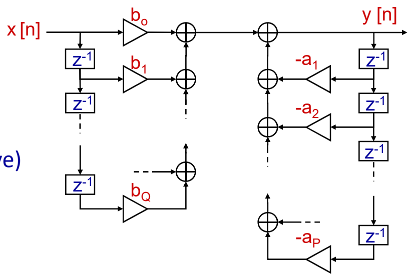
- Time-discrete systems of practical interest are usually defined with a finite difference equation:

$$y[n] = \underbrace{\sum_{k=0}^{Q} b_k x[n-k]}_{\text{forward (non-recursive) component}} - \underbrace{\sum_{k=1}^{P} a_k y[n-k]}_{\text{backward (recursive) component}}$$



when the initial conditions are zero the system is LTI

- These systems can be described/analyzed in a compact/easy way using the Z- transform.

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Transfer function

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$$y[n] = \sum_{k=0}^{Q} b_k x[n-k] - \sum_{k=1}^{P} a_k y[n-k] \leftrightarrow Y(z) = \sum_{k=0}^{Q} b_k X(z) z^{-k} - \sum_{k=1}^{P} a_k Y(z) z^{-k}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_Q z^{-Q}}{1 + a_1 z^{-1} + \cdots + a_P z^{-P}}$$

Order of $H(z) \equiv$ max(P,Q)

- $H(z)$ is the ratio of two polynomials:
  - 'o' zeros: Roots of the numerator, i.e. values of z such that $H(z)$ =0
  - 'x' poles: Roots of the denominator, i.e. values of z such that $|H(z)| \to \infty$
- If $\{a_k\}, \{b_k\}$ are real, then the zeros and poles are real or they appear in complex conjugate pairs.

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Real, causal and stable systems

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- For a system to be real $\{a_k\}, \{b_k\}$ must be real, then zeros and poles are real or they appear in complex conjugate pairs
- For a system to be causal (h[n]<0 for n<0): ROC is the outer region of a circle excluding poles (i.e. circle outside the out-most pole)
- For a system to be stable $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

  This is equivalent to $\sum_{n=-\infty}^{\infty} |h[n]| |z|^{-n} \, |_{|z|=1} < \infty$

  i.e. the ROC must contain the unit circle

  (in this case, the FT of h[n] converges uniformly and H(F), and all its derivatives, are continuous functions of F)

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Transfer function: FIR case

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- Finite difference equation for the FIR case:

$$y[n] = \sum_{k=0}^{Q} b_k x[n-k] = b_0 x[n] + b_1 x[n-1] + \cdots + b_Q x[n-Q]$$

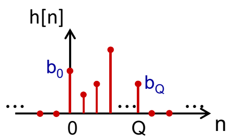
- Transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + \cdots + b_Q z^{-Q} = \frac{b_0 z^Q + b_1 z^{Q-1} + \cdots + b_Q}{z^Q}$$

All the poles at the origin!!

- Impulse response:

$$y[n] = b_0 \delta[n] + b_1 \delta[n-1] + \cdots + b_Q \delta[n-Q]$$



Transfer function: IIR case

U4

- Finite difference equation for the IIR case:

$$y[n] = \sum_{k=0}^{Q} b_k x[n-k] - \sum_{k=1}^{P} a_k y[n-k]$$

- Transfer function:

Poles or zeros at the origin

Q zeros out from the origin

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_Q z^{-Q}}{1 + a_1 z^{-1} + \cdots + a_P z^{-P}} = z^{P-Q} \frac{b_0 z^Q + b_1 z^{Q-1} \ldots + b_Q}{z^P + a_1 z^{P-1} + \cdots + a_P}$$

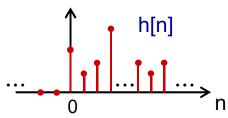
P poles out from the origin

$$= \frac{b_0 + b_1 z^{-1} + \cdots + b_Q z^{-Q}}{(1 - p_1 z^{-1})(1 - p_2 z^{-1}) \cdots (1 - p_P z^{-1})} = \frac{K_1}{1 - p_1 z^{-1}} + \cdots + \frac{K_P}{1 - p_P z^{-1}}$$

- Impulse response: P>Q simple poles

$$h[n] = K_1 p_1^n u[n] + K_2 p_2^n u[n] + \ldots + K_P p_P^n u[n]$$

Each exponential sequence has infinite length !



Finite difference eq. representation (summary)

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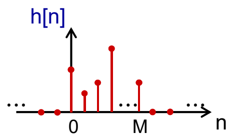
$$y[n]=\underbrace{\sum_{j=0}^{M} b_j x[n-j]}_{\text{forward non-recursive component}}-\underbrace{\sum_{i=1}^{N} a_i y[n-i]}_{\text{backward recursive component}}$$

- Finite Impulse Response (FIR) system: there is only a forward component, h[n] has finite length, the system is always stable

$$y[n]=\sum_{j=0}^{M} b_j x[n-j] \quad \Rightarrow \quad h[n]=\sum_{j=0}^{M} b_j \delta[n-j]$$



- Infinite Impulse Response (IIR) system: there is a recursive component, h[n] has infinite length, the system may be unstable (but not necessarily)

Z-transform versus frequency response

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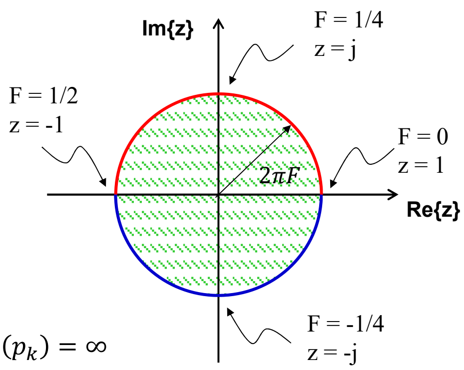
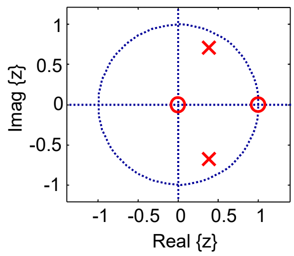
$$H(F)=\sum_{n=-\infty}^{\infty} h[n]e^{-j2\pi Fn} \quad = H(z)|_{z=e^{j2\pi F}}$$

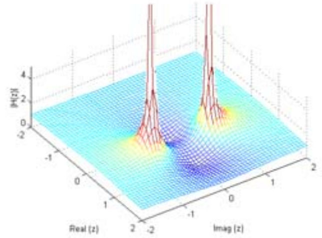


- **Poles and zeros**
  - Poles are represented with X $H(p_k)=\infty$
  - Zeros are represented with ○ $H(z_k)=0$

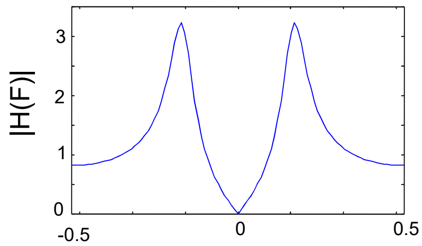
Z-transform versus frequency response

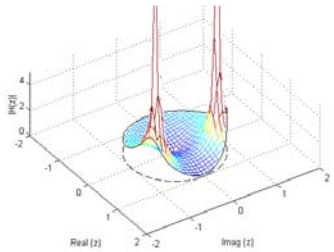
U4





$H(F)=H(z)|_{z=e^{j2\pi F}}$





Frequency response examples

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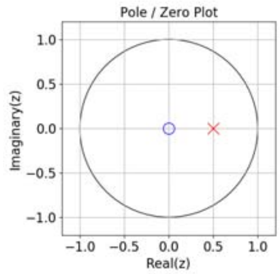
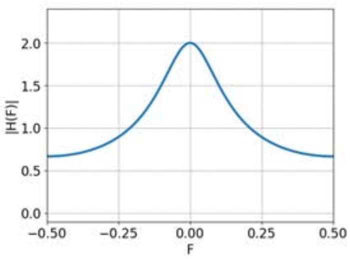
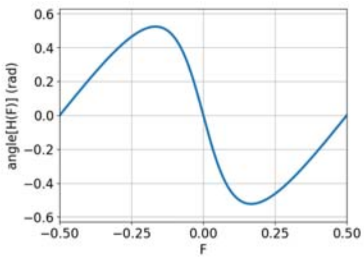
$$\boxed{H(z)=\frac{1}{1-0.5z^{-1}}} \qquad H(F)=\frac{1}{1-0.5e^{-j2\pi F}}$$

$$H(F)\Big|_{\substack{F=0\\ z=1}}=\frac{1}{1-0.5}=2$$

$$H(F)\Big|_{\substack{F=0.5\\ z=-1}}=\frac{1}{1+0.5}=0.66$$







Frequency response examples

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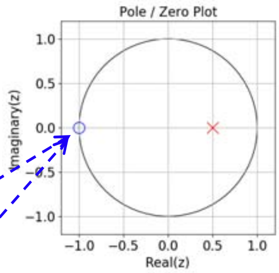
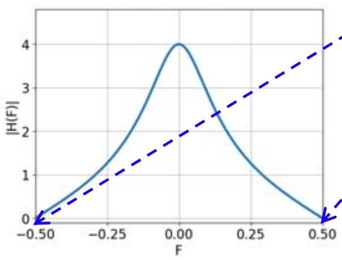
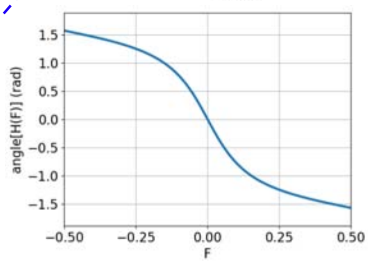
$$H(z) = \frac{1 + z^{-1}}{1 - 0.5z^{-1}}$$

Low-pass filter
(1^{st} order)

Transmission zeros:
zeros over the unit-circle generate zeros in the frequency response







Frequency response examples

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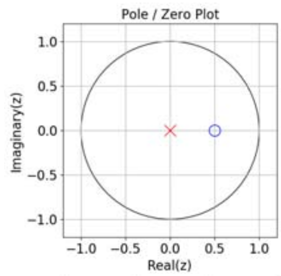
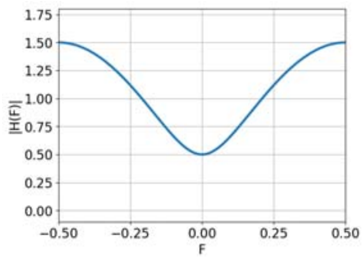
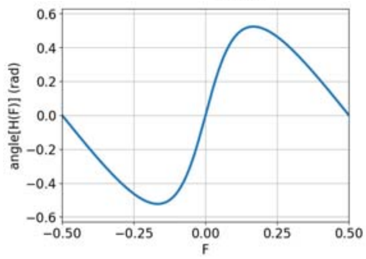
$$H(z) = 1 - 0.5z^{-1}$$

FIR filter: all poles at z=0







Frequency response examples

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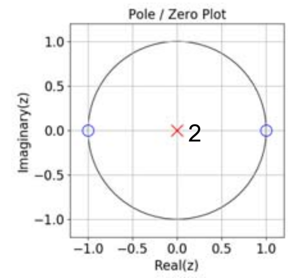
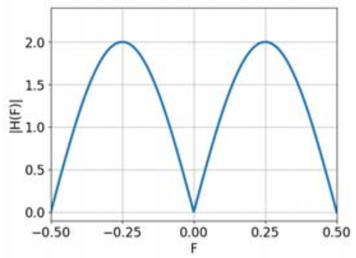
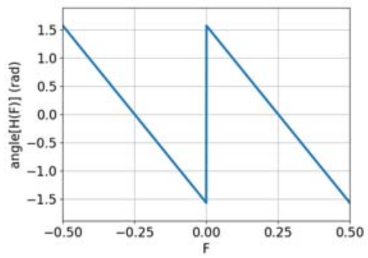
$$\boxed{H(z) = 1 - z^{-2}}$$

FIR filter: all poles (two) at z=0

Band-pass filter (2nd order)







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Frequency response examples

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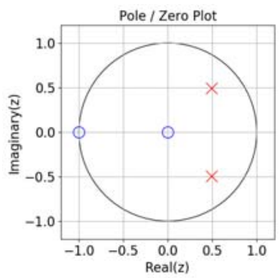
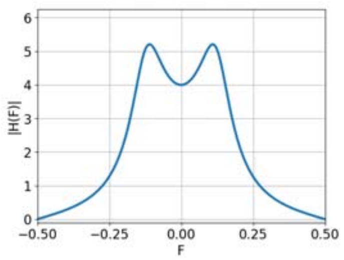
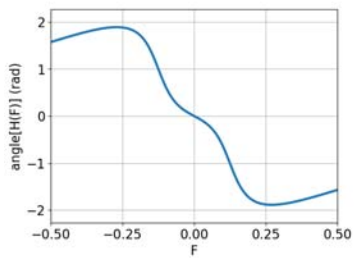
$$\boxed{H(z) = \frac{1 + z^{-1}}{1 - 2 \cdot 0.7\cos\left(\frac{\pi}{4}\right) z^{-1} + 0.7^2}}$$

$$p = 0.7e^{j\frac{\pi}{4}},\ p^* = 0.7e^{-j\frac{\pi}{4}}$$

Low-pass filter (2nd order)







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Frequency response examples

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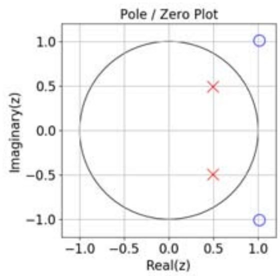
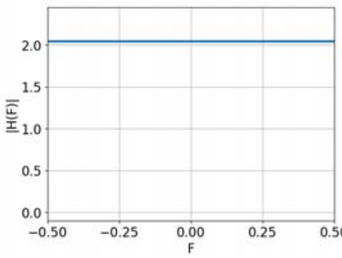
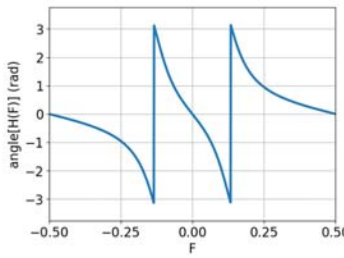
$$H(z) = \frac{1 - 2\cdot\frac{1}{0.7}\cos\left(\frac{\pi}{4}\right)z^{-1} + \left(\frac{1}{0.7}\right)^2}{1 - 2\cdot 0.7\cos\left(\frac{\pi}{4}\right)z^{-1} + 0.7^2}$$

$p = 0.7e^{j\frac{\pi}{4}}$, $p^* = 0.7e^{-j\frac{\pi}{4}}$

All-pass filter
(2nd order)







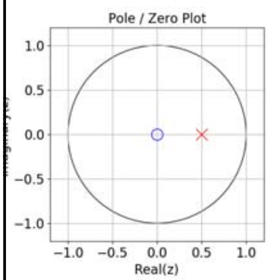
Impulse responses (P2 unit 4)

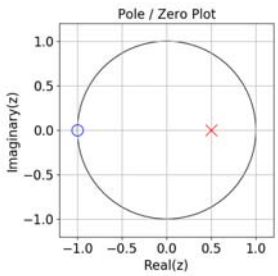
U4

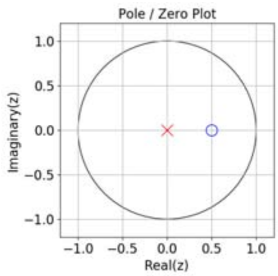
$H(z) = \dfrac{1}{1 - 0.5z^{-1}}$

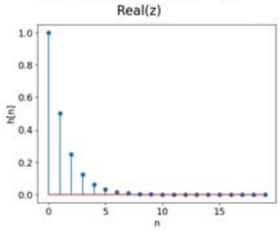
$H(z) = \dfrac{1 + z^{-1}}{1 - 0.5z^{-1}}$

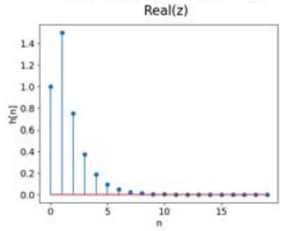
$H(z) = 1 - 0.5z^{-1}$

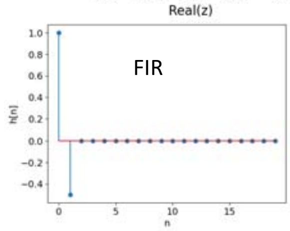












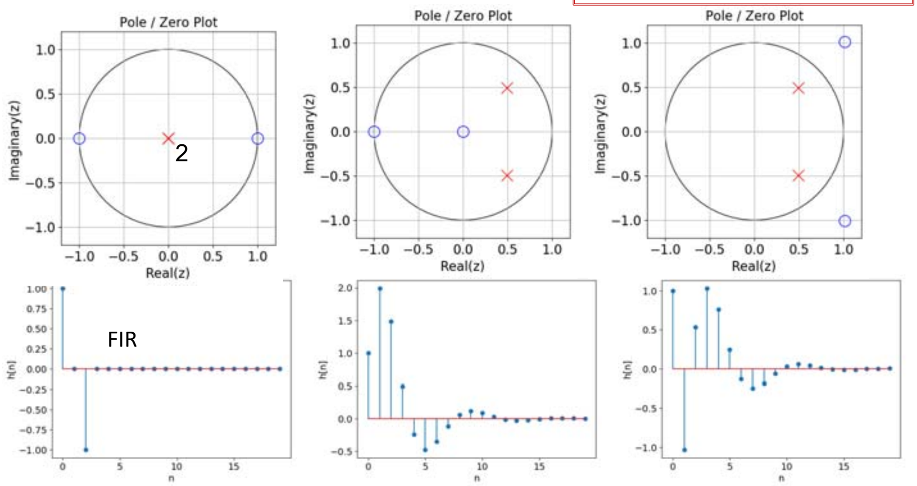
Impulse responses (P2 unit 4)

U4

$$H(z) = 1 - z^{-2}$$

$$H(z) = \frac{1 + z^{-1}}{1 - 2 \cdot 0.7\cos\left(\frac{\pi}{4}\right) z^{-1} + 0.7^2}$$

$$H(z) = \frac{1 - 2 \cdot \frac{1}{0.7}\cos\left(\frac{\pi}{4}\right) z^{-1} + \left(\frac{1}{0.7}\right)^2}{1 - 2 \cdot 0.7\cos\left(\frac{\pi}{4}\right) z^{-1} + 0.7^2}$$



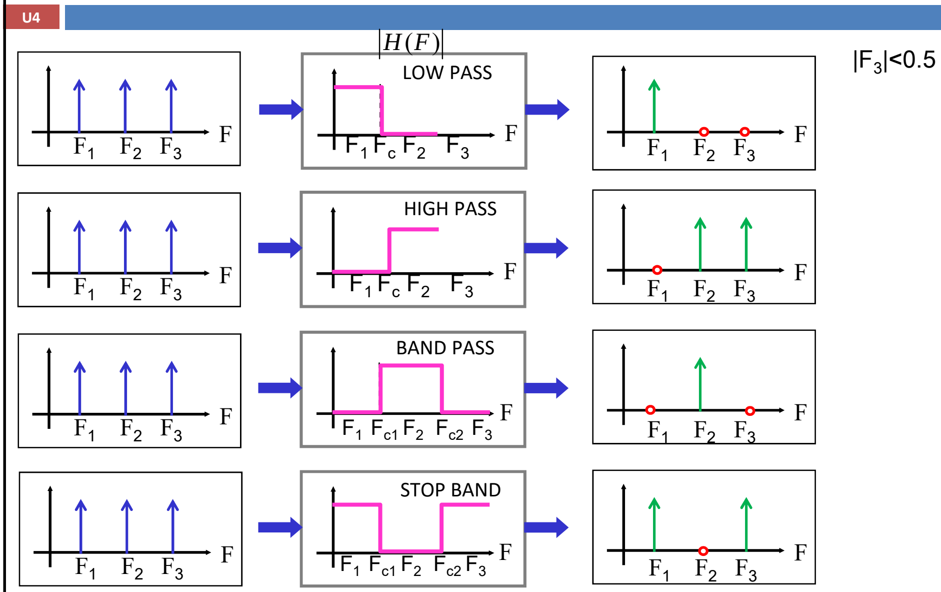
Unit 4. Z-Transform and Filter Design

1. The Z-transform (22/11/2019)
2. Transfer function and frequency response (26/11/2019)
3. Filter design (29/11/2019)

Lab 3 (Part 1: 11/12/2019, Part 2: 18/12/2019)

Ideal filters

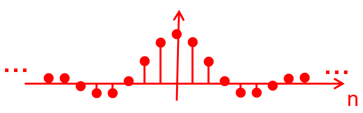
U4

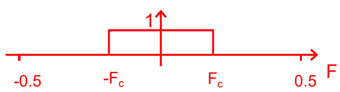


Ideal low pass filter

U4

$$h[n] = \frac{\sin(2\pi F_c n)}{\pi n} = 2F_c \mathrm{sinc}(2F_c n) \quad \leftrightarrow \quad H(F) = \sum_{r=-\infty}^{\infty} \Pi\left(\frac{F-r}{2F_c}\right)$$





The filter is non-stable ($\sum_{k=-\infty}^{\infty} |h[n]| = \infty$) and non-causal ($h[n] \neq 0 \; for \; n < 0$)

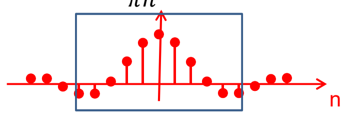
select a segment of the impulse response (windowing)

delay

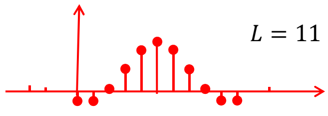
A real filter

U4

$$h[n] = \frac{\sin(2\pi F_c n)}{\pi n} = 2F_c \text{sinc}(2F_c n)$$

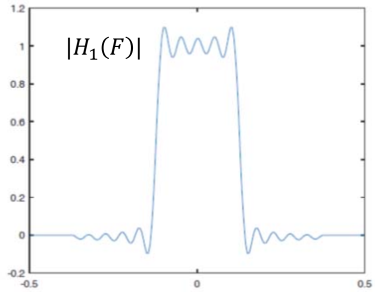


$$h_1[n] = h\left[n - \frac{L-1}{2}\right] p_L[n]$$



$$H_1(F) = H(F)e^{-j2\pi F\frac{L-1}{2}} \otimes P_L(F)$$

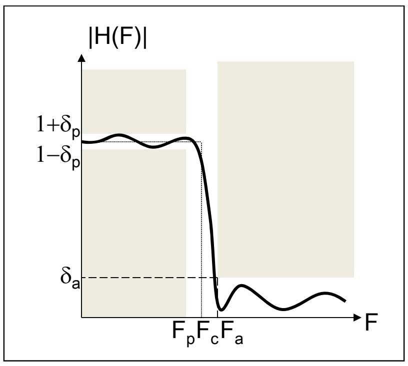
- $|H_1(F)|$ is not flat in the pass and stop bands: this is due to the secondary lobes of the window!
- Transition band: its duration decreases if we increase L!



Filter specifications

U4

We need to relax the requirements of the filter



- Pass band:
  $1 - \delta_p \leq |H(F)| \leq 1 + \delta_p$, for $|F| \leq F_p$
- Stop band:
  $|H(F)| \leq \delta_a$, for $|F| \geq F_a$
- Transition band:
  $|F_p| \leq |F| \leq |F_a|$

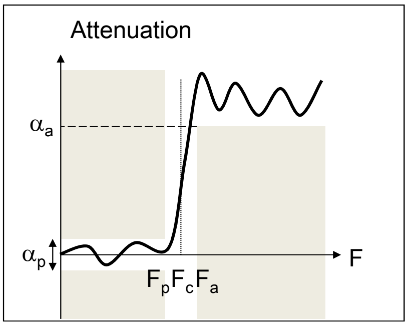
The specifications for other types of filters (besides low pass) are analogous

Filter attenuation

U4

Filter specifications are often expressed in terms of attenuation (in dB)

$$\alpha(F) = 10\log_{10}\frac{|H|^2_{max}}{|H(F)|^2}$$



- Pass band:

  $\alpha(F) \le \alpha_p(F) = 20\log_{10}\frac{1+\delta_p}{1-\delta_p}$, for $|F| \le F_p$

- Stop band:

  $\alpha(F) \ge \alpha_a(F) = 20\log_{10}\frac{1+\delta_p}{\delta_a}$, for $|F| \ge F_a$

- Transition band:

  $|F_p| \le |F| \le |F_a|$

- Transmission zeros: frequencies at which |H(F)|=0 (infinite attenuation)
- Attenuation zeros: frequencies at which |H(F)| is maximum

Approximating the ideal frequency response

U4

What do we want in a filter?

- Short transitions bands (a more selective filter)
- High difference in |H(F)| between pass and stop bands (a more discriminant filter)

Improving any of these features (or both) requires increasing the order of the filter

- To fulfill a set of specifications FIR filters requires higher order (compared to IIR filters), but FIR filters are always stable
- IIR filters require lower order for the same specifications, but they may become unstable (due to numerical precision errors)

Usual methods for FIR filters design

U4

- Window method: the ideal h[n] is windowed and delayed
  - ✓ The discrimination and selectivity of the filter depend on the window

    scipy.signal.firwin(numtaps, cutoff, window = 'hamming',...)

- Sampling of the ideal frequency response: $h[n] = IDFT_N\ \{H_{ideal}(F)|_{F=\frac{k}{N}}\}$
  - ✓ Exact desired values for some frequencies
  - ✓ There is no control over the frequency response at other frequencies

    scipy.signal.firwin2(numtaps, freq, gain, ...)

- Remez algorithm: FIR with constant ripple (optimum)

  Given the number of coefficients and Fp and Fa, the Remez algorithm computes the coefficients that minimize the maximum absolute error w.r.t. the ideal frequency response (weighting factors are possible)

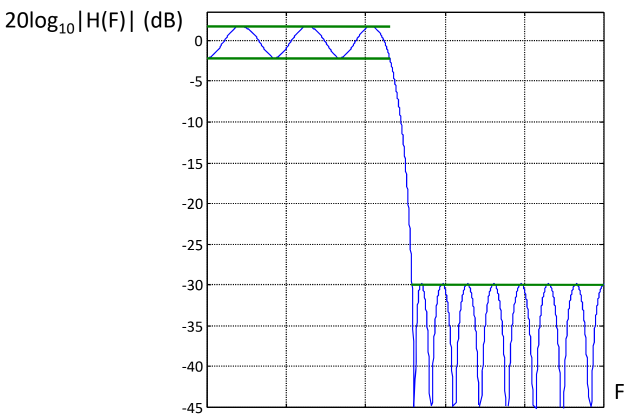
    scipy.signal.remez(numtaps, bands, desired, weight = None, ...)

Unit 4: Z-Transform and filter design 39

Remez algorithm is optimum in terms of order

U4

- Given the selectivity and discrimination constraints, the design with the lowest order has a constant amplitude ripple behaviour

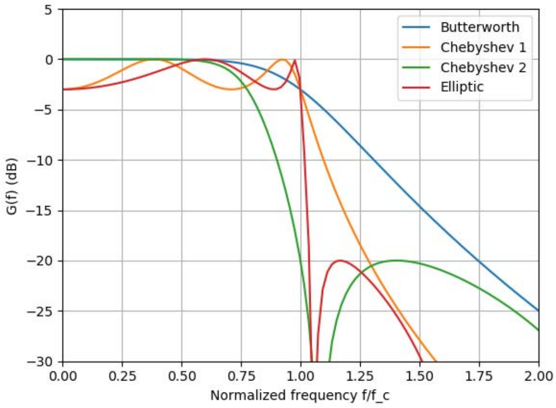


Unit 4: Z-Transform and filter design

IIR filters design

U4

- The design of digital IIR filters are based on a mathematical transformation of the classical approximations for analog filters



scipy.signal.iirdesign (wp, ws, gapss, gstop, ftype='ellip'...)

FIR / IIR: which one to choose?

U4

	FIR	IIR
Linear phase (it requires h[n] to have odd/even symmetry)	Yes (possible)	Only possible for non-causal
Stability	Always	Not always (due to numerical precision errors)
Order	High	Low

Phase distorsion

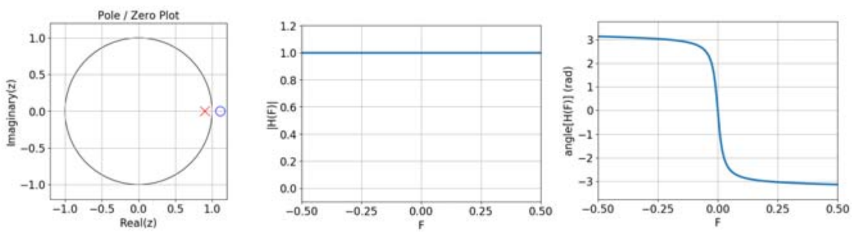
U4

When the filter is not ideal: we have linear distorsion

- Amplitude distorsion: if |H(F)|is not constant in the pass band
- Phase distorsion: if H(F) has non linear phase in the pass band

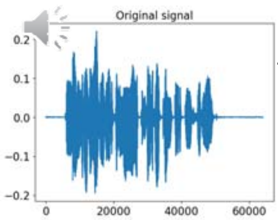
What is the effect of phase distorsion in audio signals?

$$H(F) = \frac{1 - az}{z - a}$$



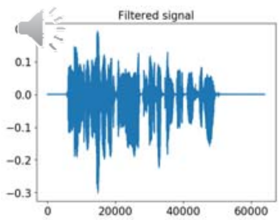
Effect of phase distortion in audio signals (1)

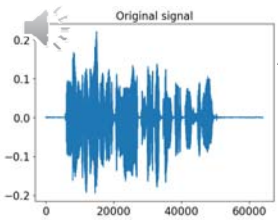
U4



$x[n]$ → $H(z) = \dfrac{1 - az}{z - a}$ → $y[n]$

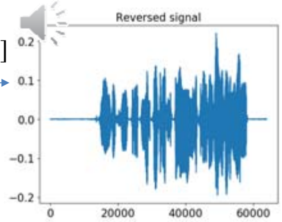
```
a = 0.9
num = -a*np.array([1,-1/a])
den = np.array([1,-a])
y = lfilter(num, den, x)
```





$x[n]$ → Reflection → $x[-n]$ → z^{-N} → $s[n]$

```
s = x[::-1]
```



Effect of phase distortion in audio signals (2)

U4

How are the phases of X(F) and X(-F) related?

$$x[n] \leftrightarrow X(F) = |X(F)|e^{jarg\{X(F)\}}$$

$$x[-n] \leftrightarrow X(-F) = X^*(F) = |X(F)|e^{-jarg\{X(F)\}}$$

x[n] is real valued

- Phase shifts are time-shifts of the basic signal components (i.e. sinusoids)
- Roughly speaking, the ear acts as a set of band-pass filters: we will 'hear phase shifts' if
  - ✓ the components fall close to each other,
  - ✓ and there are relative big shifts in the phase

The phase in images (1)

U4

- Position of the elements

image | magnitude DFT | phase DFT

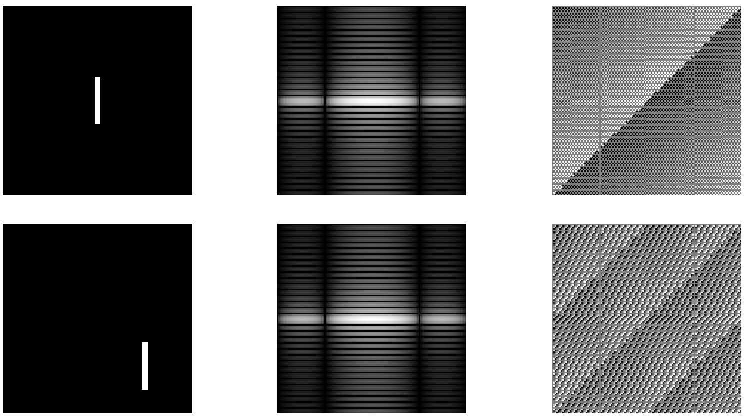


image | magnitude DFT | phase DFT

The phase in images (2)

U4

- Need of phase 0 in the processing

$x[m,n]$

$$DFT^{-1}\left\{DFT\{x[m,n]\}\cdot\ e^{-j2\pi\frac{30}{M}k}\cdot e^{-j2\pi\frac{20}{M}l}\right\}$$

The phase in images (3)

U4