Chapter 4. Greedy Algorithms

Algorithmics and Programming III

FIB

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Example: Coin Exchange

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  - While there is money to give, use the largest possible coin
- What if we add a coin of 8 cents?
  - The previous *greedy* algorithm uses 4 coins: 50 + 10 + 2 + 1
  - However, there is a solution with only 3 coins: 50 + 8 + 5

What are Greedy Algorithms?

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- In the following we will:
  - Introduce and revisit greedy algorithms for well-known problems
  - Prove them correct using a variety of techniques

Interval Scheduling Problem

- Imagine we have a single lab room and N courses that request to use it
- We know the starting time $s(i)$ and the finishing time $f(i)$ of each course
- Requests of two overlapping courses cannot be both met
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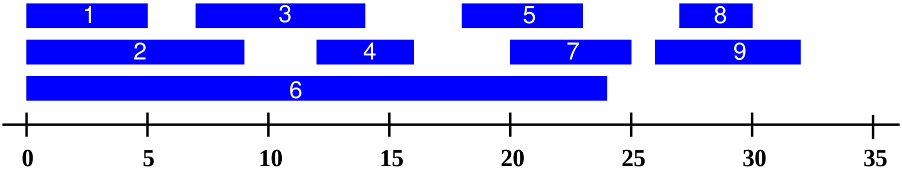

- $\{1,3,5,8\}$ or $\{2,4,7,9\}$ are feasible choices. Are they optimal?

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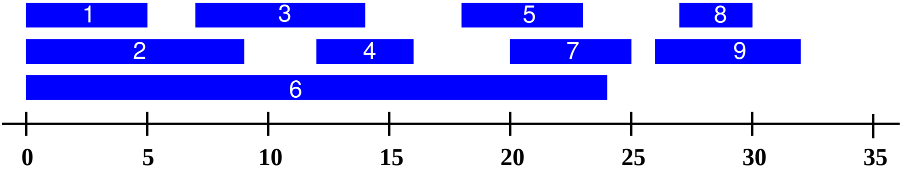

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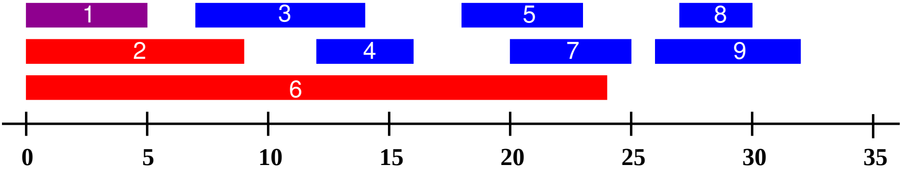

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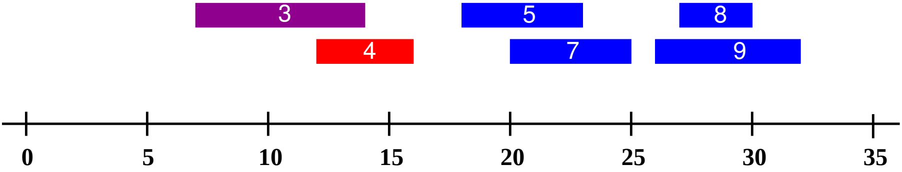

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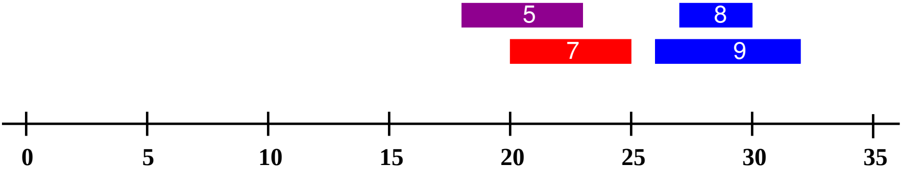

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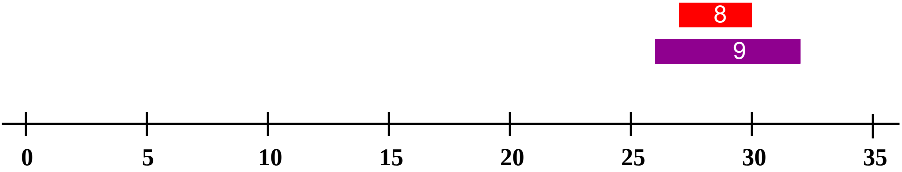

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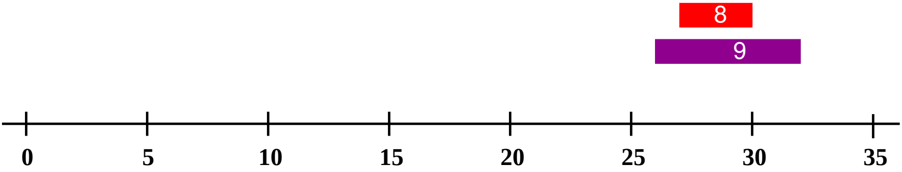

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- Is this algorithm correct? Can we cook up an example for which it fails?

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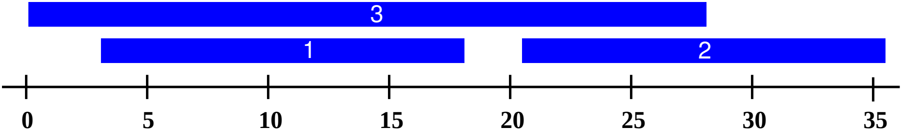

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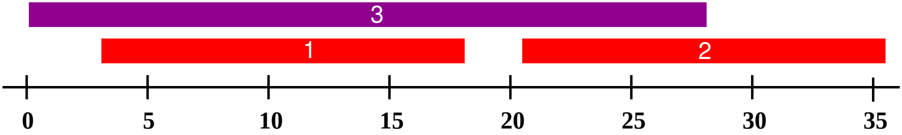

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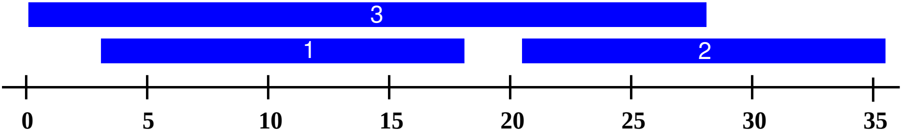

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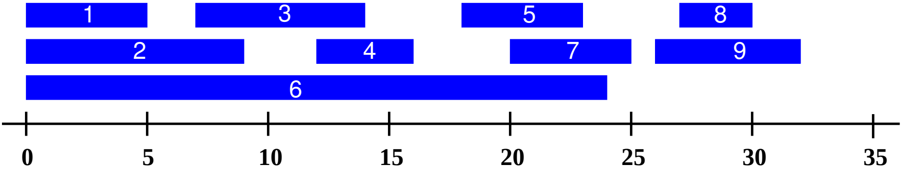

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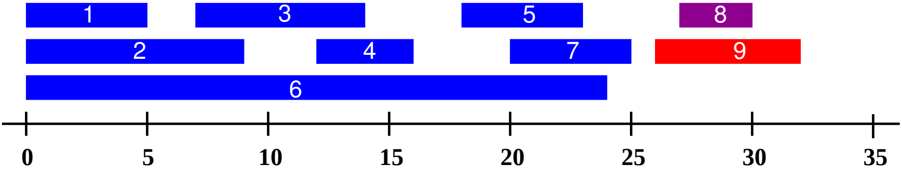

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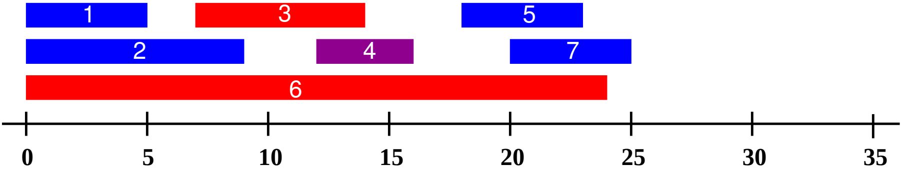

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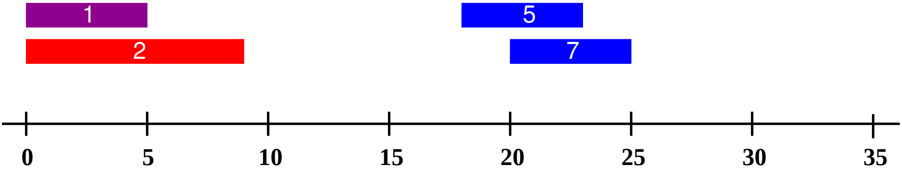

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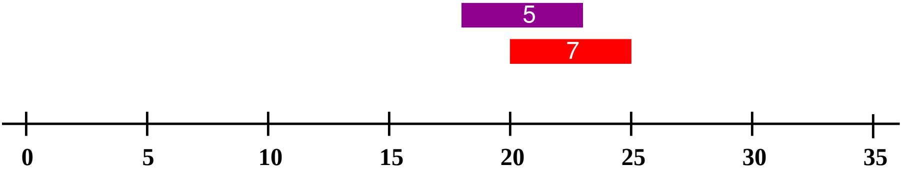

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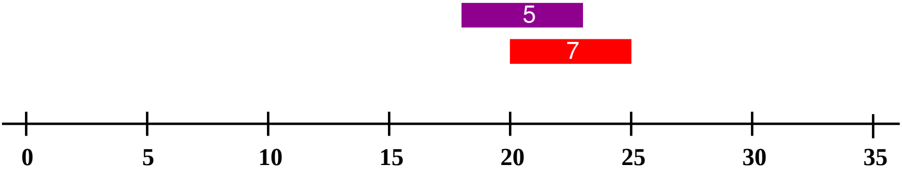

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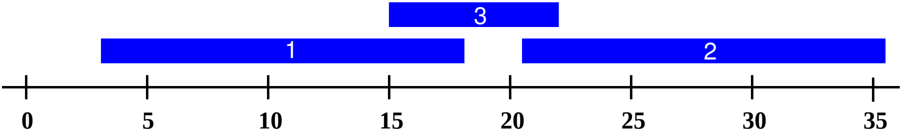

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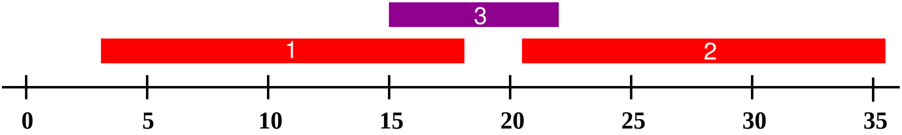

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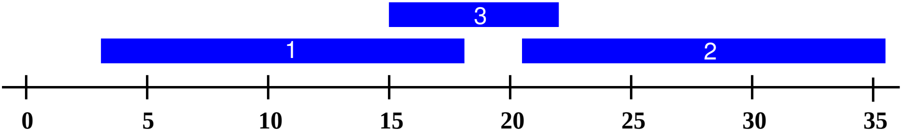

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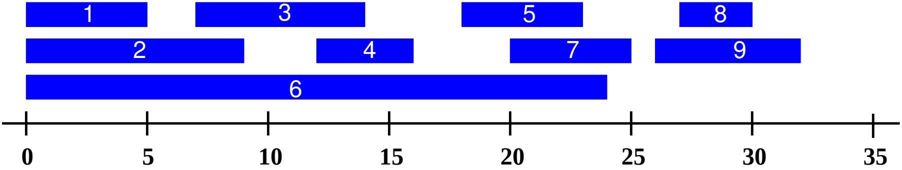

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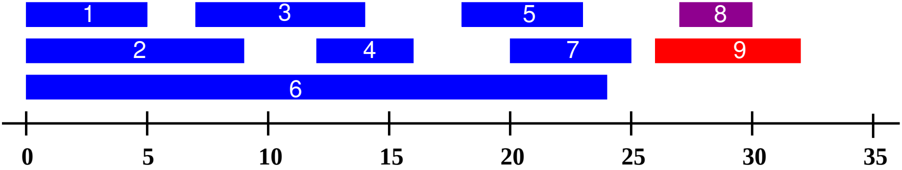

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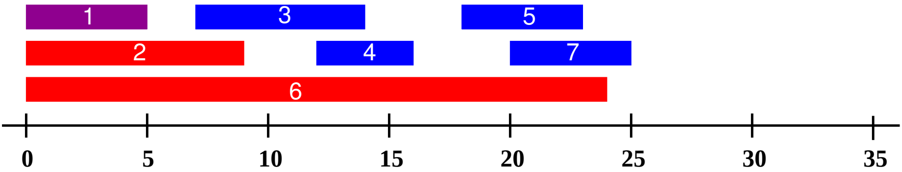

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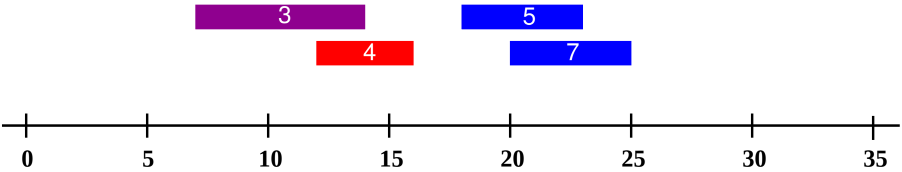

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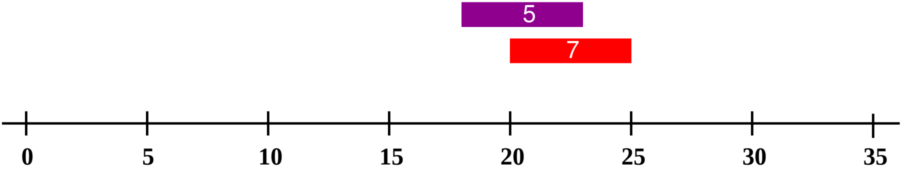

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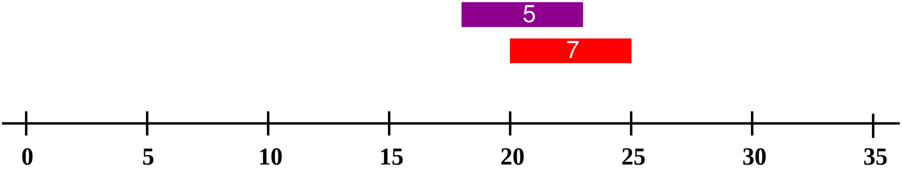

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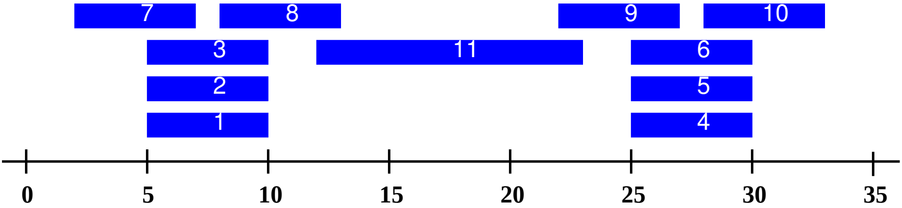

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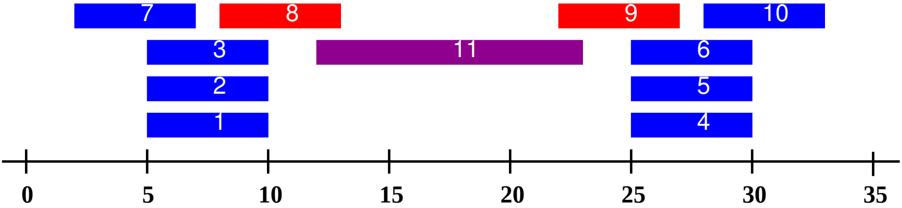

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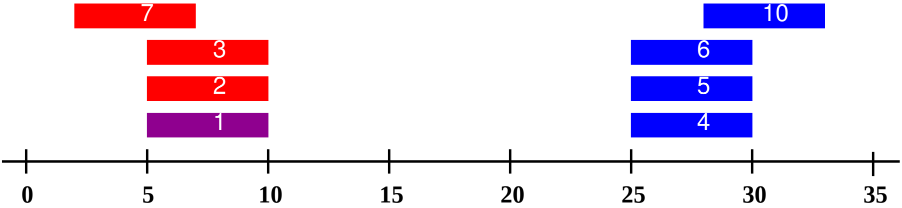

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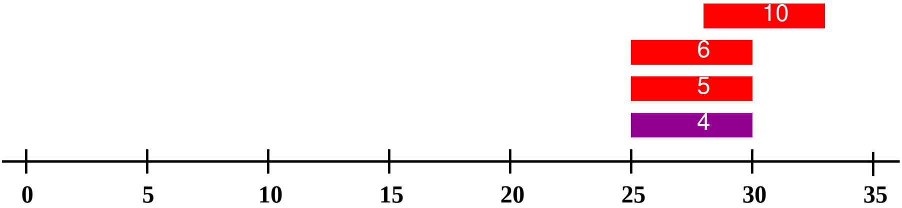

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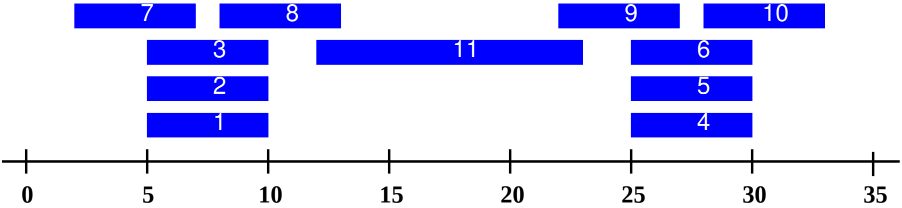

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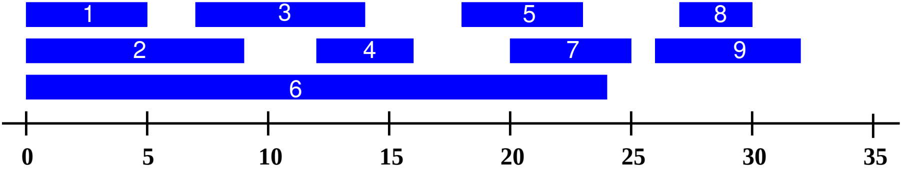

GREEDY ALGORITHM 4:

- Idea: choose tasks that finish early
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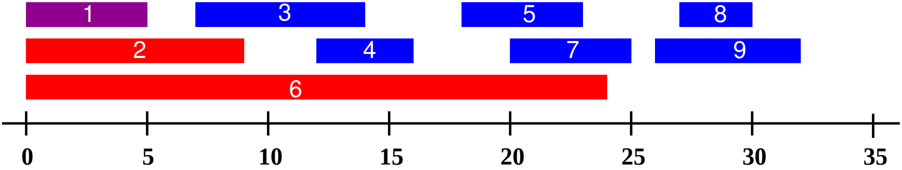

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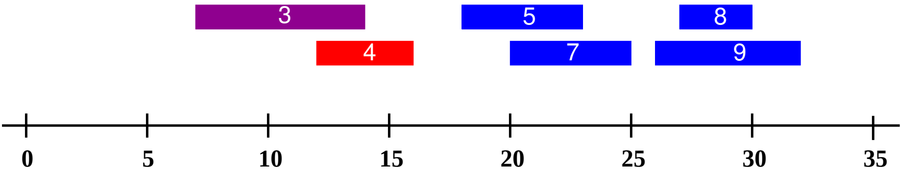

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- Algorithm: while there are available courses, select the available course i that has the lowest $f(i)$ among the available ones. If there is a tie, pick the one with the lowest index. Mark all courses that overlap with i as non-available.
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  1. Choose 1. Courses 2 and 6 are discarded due to overlapping.
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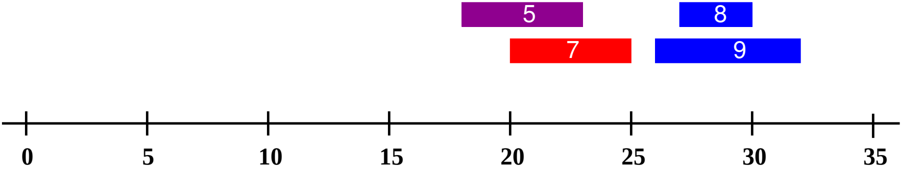

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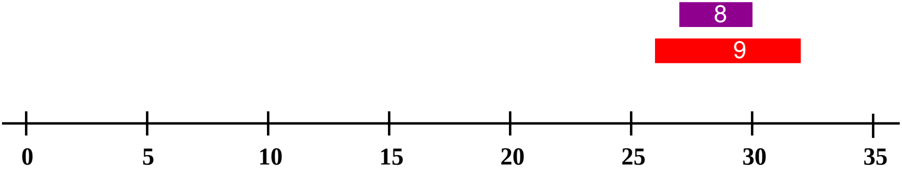

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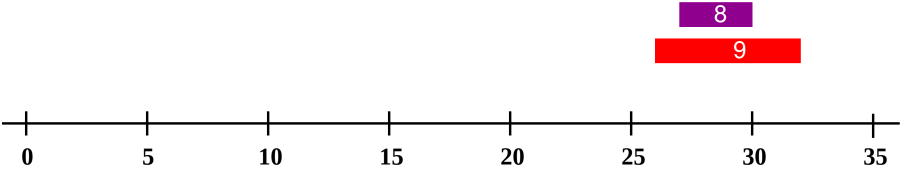

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- Is this algorithm correct? Can we cook up an example for which it fails?

Interval Scheduling Problem

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- The intuition is that at each step our algorithm has a better subproblem, as it works with courses in the interval $[f(a_r), \ldots]$ rather than in $[f(o_r), \ldots]$

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So when our algorithm selected a_r, one of the candidates to choose was o_r.
As it chooses the one with the smallest finishing time, we have $f(a_r) \leq f(o_r)$.
$\square$

Interval Scheduling Problem

Remember:

$\mathcal{A} = \{a_1, a_2, \ldots, a_k\}$ (the algorithm solution)

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Since $f(o_k) \leq s(o_{k+1})$, we have $f(a_k) \leq s(o_{k+1})$.

As $f(a_1) \leq \ldots \leq f(a_k)$, we have o_{k+1} does not overlap with any of $a_1, \ldots, a_k$.
So o_{k+1} was still available when our algorithm finished: contradiction!
(because it finishes when no available course exists).

$\square$

Interval Scheduling Problem

```
struct Task {
  int id;
  int starting;
  int finishing;
};

bool before (const Task& t1, const Task& t2) {
  return t1.finishing < t2.finishing;
}

int main(){
  vector<Task> tasks;
  int s, f, id = 1;
  while (cin >> s >> f) {
    tasks.push_back(Task{id,s,f});
    ++id;
  }
  sort(tasks.begin(),tasks.end(), before);
  vector<int> result = interval_scheduling(tasks);
  for (auto& x: result) cout << x << endl;
}
```

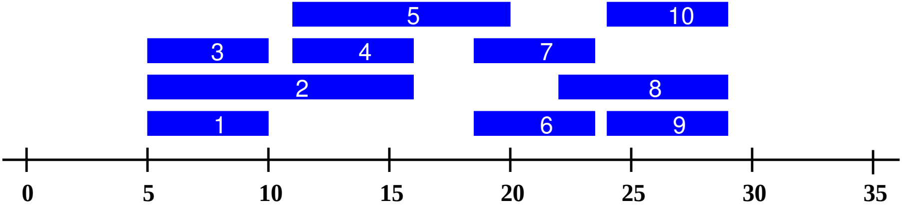
```
// PRE: tasks are sorted increasingly by finishing time
vector<int> interval_scheduling(const vector<Task>& tasks){
  int n = tasks.size();
  vector<int> res;         // Result
  int idx = 0;
  int last_f = INT_MIN;    // Acts as −∞

  while (idx < n) {
    // Look for non-overlapping task w/ smallest finishing
    while (tasks[idx].starting < last_f and idx < n) ++idx;
    if (idx < n) {  // Add tasks[idx] to solution
      res.push_back(tasks[idx].id);
      last_f = tasks[idx].finishing;
    }
  }
  return res;
}
```

Interval Partitioning Problem

- Imagine we have N courses that need to be allocated to some rooms
- Again, we know starting time $s(i)$ and finishing time $f(i)$ of each course
- The difference is that now:
  - we have to allocate all courses
  - we have an unlimited number of rooms
- We wish to find out the minimum number of rooms needed so that no two overlapping courses are assigned to the same room

Interval Partitioning Problem

- Scheduling using 4 rooms:



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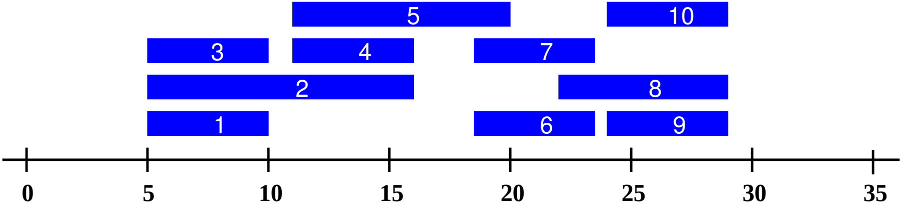

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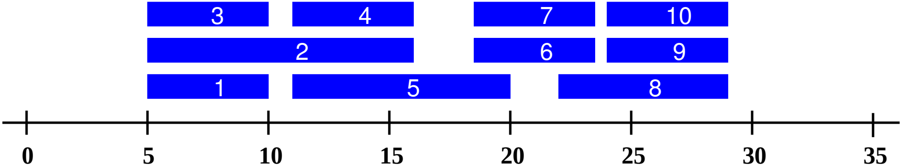

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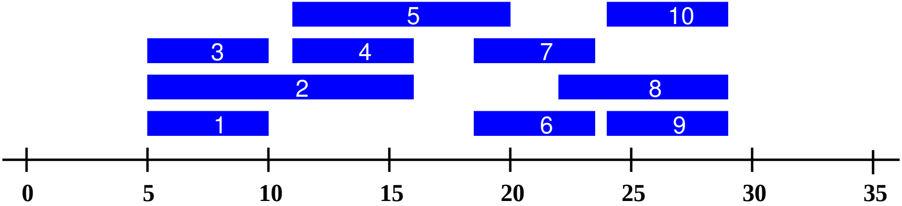

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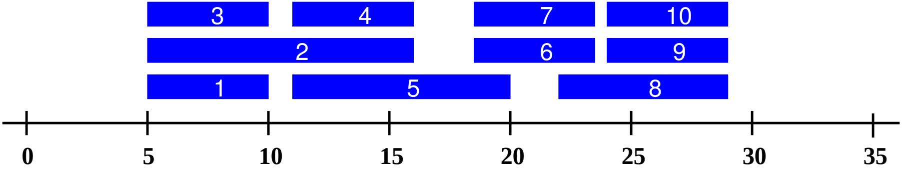

- Latter is optimal, as $\{1, 2, 3\}$ are overlapping and hence need 3 rooms.

- Let us define the depth of the problem as
  the maximum number of courses that overlap on any instant of time
- **CLAIM:** The number of rooms needed is $\geq$ the depth of the problem
- If d is the depth,
  can we find algorithm that schedules the courses with exactly d rooms?

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  1. Select d rooms (only these are available)
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  3. For each course c (in the order), select one room that is not used by the preceding courses (in the order) that overlap with c

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  3. For each course c (in the order), select one room that is not used by the preceding courses (in the order) that overlap with c
- We should prove:
  - In the second step there is always at least one free room
  - No overlapping courses get the same rooms (easy!)

Interval Partitioning Problem

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This means that at time $s(c)$,
all d rooms are occupied by previous courses that are active at that moment.

Those courses, together with c, form a set of $d + 1$ overlapping courses,
which is a contradiction with the definition of the depth d.

Interval Partitioning Problem

```
bool overlap(const Task& t1, const Task& t2) {
  return t1.finishing > t2.starting and
         t2.finishing > t1.starting;  }

bool before (const Task& t1, const Task& t2) {
  return t1.starting < t2.starting;  }

int main(){
  vector<Task> tasks;
  int s, f, id = 1;
  while (cin >> s >> f) {
    tasks.push_back(Task{id,s,f});
    ++id;
  }
  sort(tasks.begin(),tasks.end(), before);
  vector<int> labelling = interval_part(tasks);
  for (uint i = 0; i < tasks.size(); ++i)
  cout << "Task " <<  tasks[i].id << " gets room " <<
      labelling[i] << endl;
}
```

Interval Partitioning Problem

```
// PRE: tasks are sorted increasingly by starting time
vector<int> interval_part (const vector<Task>& tasks) {

  int n = tasks.size();
  vector<int> labelling(n,-1); // -1 for unassigned label

  for (int i = 0; i < n; ++i) {
    // Mark not available labels for the i-th task
    vector<int> available_label(n, true); // large UB
    for (int j = 0; j < i; ++j)
      if (overlap(tasks[i], tasks[j]))
         available_label[labelling[j]] = false;

    int l = 0;
    while (not available_label[l]) ++l;
    labelling[i] = l;
  }

  return labelling;
}
```

Lateness Minimization Problem

- Imagine we have a set of N tasks that request to use the same resource, of which we only have one
- For each task i we know its duration $du(i)$ and its deadline $d(i)$ (ideally, the task should be finished before this time)

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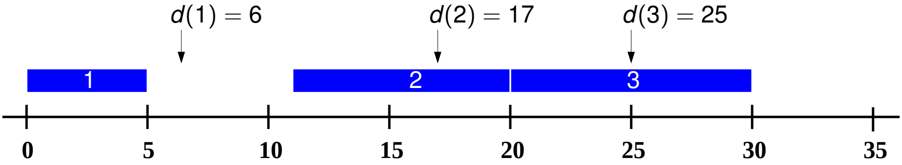
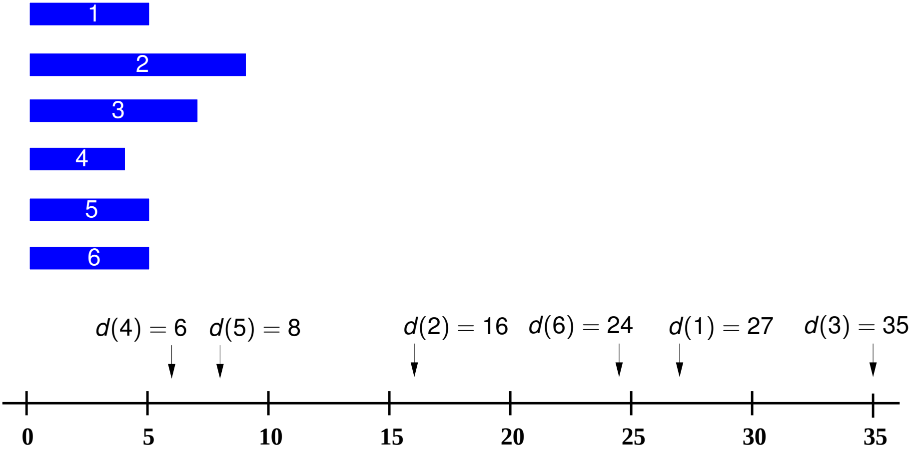
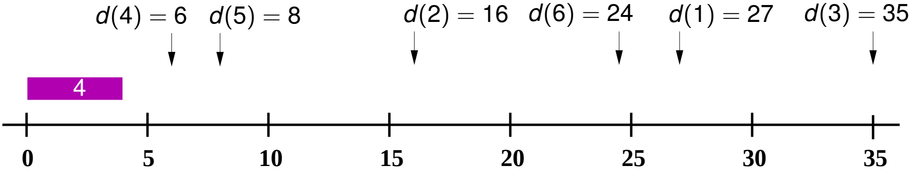
- Example:



- This non-optimal schedule has a maximum lateness of 5

Lateness Minimization Problem





GREEDY ALGORITHM

- Idea: schedule first the tasks with earliest deadline
- Algorithm: while there are tasks to schedule, pick the one with earliest deadline and place it right after the task scheduled in the previous step.
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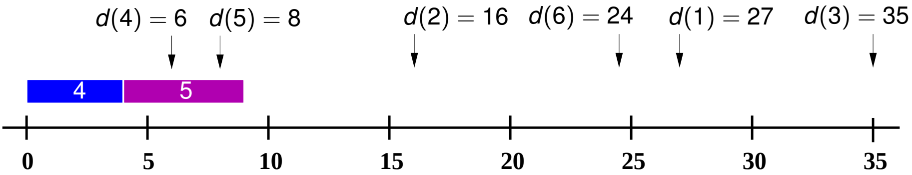
  1. Choose task 4. Starts at 0, finishes at 4. Lateness is 0.



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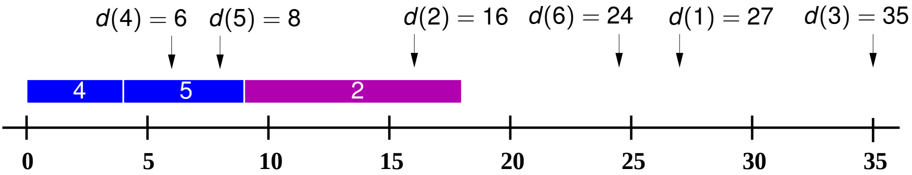
  2. Choose task 5. Starts at 4, finishes at 9. Lateness is 1.



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  3. Choose task 2. Starts at 9, finishes at 18. Lateness is 2.

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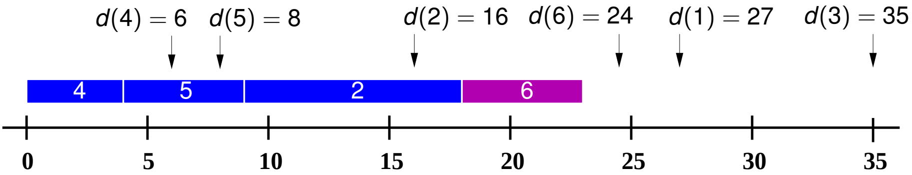

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  4. Choose task 6. Starts at 18, finishes at 23. Lateness is 0.

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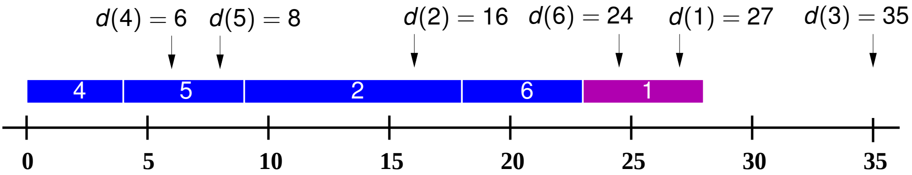

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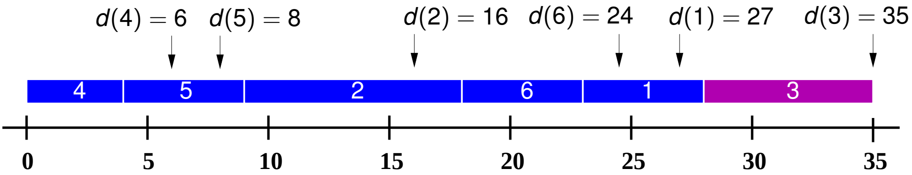
Lateness Minimization Problem



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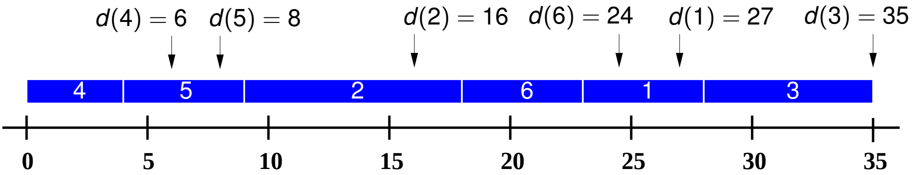
  3. Choose task 2. Starts at 9, finishes at 18. Lateness is 2.
  4. Choose task 6. Starts at 18, finishes at 23. Lateness is 0.
  5. Choose task 1. Starts at 23, finishes at 28. Lateness is 1.
  6. Choose task 3. Starts at 28, finishes at 35. Lateness is 0.



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  3. Choose task 2. Starts at 9, finishes at 18. Lateness is 2.
  4. Choose task 6. Starts at 18, finishes at 23. Lateness is 0.
  5. Choose task 1. Starts at 23, finishes at 28. Lateness is 1.
  6. Choose task 3. Starts at 28, finishes at 35. Lateness is 0.

     Maximum lateness is 2

Lateness Minimization Problem

We will prove that the previous algorithm produces optimal schedules.
Our proof strategy is the following:

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1. We prove all schedules with no inversions and no idle time have the same maximum lateness
2. We prove there's an optimal schedule with no inversions and no idle time

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Proposition

All schedules with no inversions and no idle time have the same max lateness

Proof: In a schedule with no inversions, if $d(i) > d(j)$ then $s(i) > s(j)$, i.e., task i necessarily follows task j.

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This happens for all deadlines d,
so the two schedules have the same maximum lateness.

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There is an optimal schedule with no inversions and no idle time.

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If it has no inversions, we are done.

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Let $a_0 = a, a_1, a_2, \ldots, a_{n-1}, a_n = b$ be the sequence of tasks between a and b.

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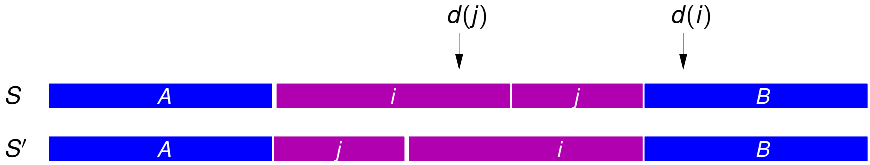
By repeating this swapping process,
we will end up having an optimal schedule with no inversions.

Proof (continued):



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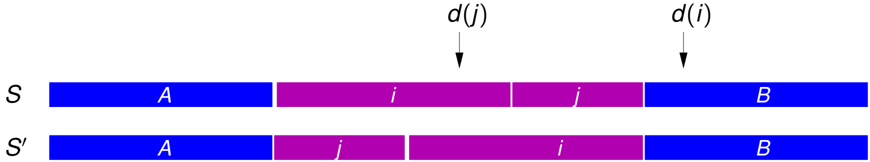

- S' has one less inversion than S:

  No new inversions among tasks in A or B are created.

  Inversions in S' of the form (k, j), (k, i) with k in A belong to S too.

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Lateness Minimization Problem

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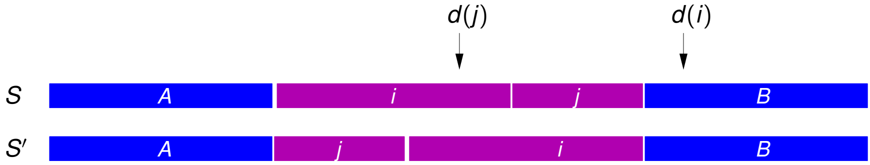

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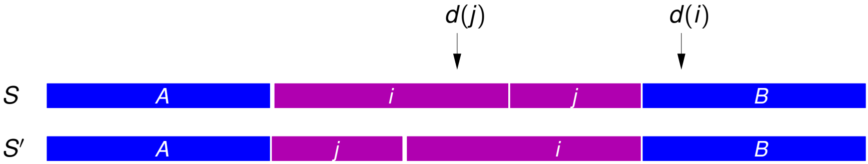

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  As the finishing time of j is smaller, only i may have a larger lateness

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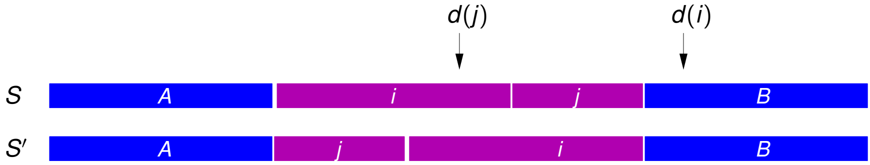

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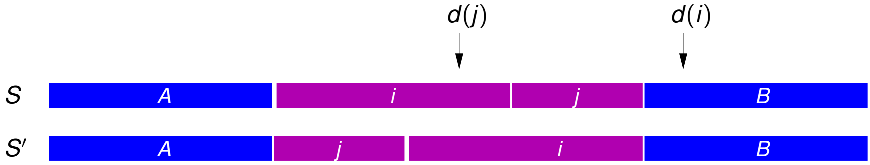

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  But since $d(i) > d(j)$, we have i cannot be later in S' than j in S:

$$\ell'(i) = s(j) + du(j) - d(i) < s(j) + du(j) - d(j) = \ell(j).$$

$\square$

```
struct Task {
  int id;
  int duration;
  int deadline;
  Task(int i, int du, int de):
  id(i), duration(du), deadline(de){}  };

bool before (const Task& t1, const Task& t2) {
  return t1.deadline < t2.deadline; }

int main(){
  vector<Task> tasks;
  int du, de, id = 1;
  while (cin >> du >> de) {
    tasks.push_back(Task(id,du,de));  ++id;}
  sort(tasks.begin(), tasks.end(), before);
  vector<int> starting_time = lateness(tasks);
  for (uint i = 0; i < tasks.size(); ++i)
     cout << "Task " <<  tasks[i].id << " starts at " <<
         starting_time[i] << endl;
}
```

Lateness Minimization Problem

```
// PRE: tasks is sorted increasingly by deadline
vector<int> lateness(const vector<Task>& tasks) {
  int n = tasks.size();
  vector<int> starting_time(n);

  int t = 0;
  for (int i = 0; i < n; ++i) {
    starting_time[i] = t;
    t += tasks[i].duration;
  }

  return starting_time;
}
```

Dijkstra's Algorithm

Dijkstra's algorithm for shortest paths

Input: Graph $G = (V, E)$, directed or not;
non-negative weights in the edges $\{w(u, v) \mid (u, v) \in E\}$;
initial vertex s;

Output: For all vertex u reachable from s,
$dist(u) =$ distance from s to u (denoted in the following by $\delta(s, u)$)

DIJKSTRA (G,w,s)
  for all vertex $u \in V$ **do** $dist(u) = \infty$
  $dist(s) = 0$
  H =create-priority-queue(V) (using *dist* as keys)
  while not empty(H)
    u =remove-min(H) $\leftarrow$ This is the **greedy** part
    for all edge $(u, v) \in E$ **do**
      if $dist(v) > dist(u) + w(u, v)$ (relax edge)
        $dist(v) = dist(u) + w(u, v)$
        decrement-key(H, v)

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Proposition

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We will do that by induction on the step in which the update was made.

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Consider now the k-th update.
Assume the property holds up to the $k-1$-th update.
The k-th update is done with the command

$$dist(v) = dist(u) + w(u, v)$$

Since $dist(u) < +\infty$ corresponds to the length of some path from s to u, and there is an edge $u \to v$ with weight $w(u, v)$, the property is true. □

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Corollary

For all vertices u, it always holds that $dist(u) \geq \delta(s,u)$.

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Let u be the first vertex that is removed from H with $dist(u) \neq \delta(s, u)$.

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Now let us consider the shortest path from s to u.

Before we removed u from H, this path went from $s \notin H$ to $u \in H$.

Let $x \to y$ be the first edge in this path from a vertex not in H to a vertex in H.
(Note that it could be $s = x$ or $y = u$).

This path is of the form $s \rightsquigarrow x \to y \rightsquigarrow u$.

Proof (continued):

Let us consider the state of the algorithm right after x was removed from H. When processing edge $x \to y$, two things may happen:

- If $dist(y) \leq dist(x) + w(x, y)$, nothing is done
- If $dist(y) > dist(x) + w(x, y)$, the edge is relaxed and $dist$ is updated so that $dist(y) = dist(x) + w(x, y)$.

So after processing edge (x, y) we have $dist(y) \leq dist(x) + w(x, y)$.

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This property is preserved from that moment on:

- $dist(x)$ is not changed ($dist(x) = \delta(s, x)$, so it cannot decrease)
- $dist(y)$ can only decrease

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But then the algorithm should have chosen y instead of u: contradiction! $\square$

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But at the end of the algorithm, H is empty.
As all vertices are eventually removed, all distances are duly computed $\square$

General Algorithm

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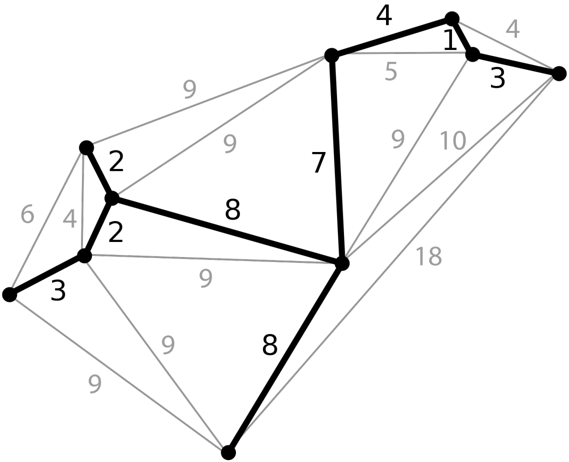
Definition

A minimum spanning tree (MST) of G is a spanning tree A of G such that its total weight

$$\omega(A) = \sum_{e \in A} \omega(e)$$

is minimum among all spanning trees of G



Source: `https://commons.wikimedia.org/wiki/File:Minimum_spanning_tree.svg`

General Algorithm

There are many different algorithms for computing MST's.

They typically follow the scheme:

```
A = ∅;
Cand = E;
while (|A| ≠ |V| − 1) {
  choose e ∈ Cand that does not close a cycle in A;
  A = A ∪ {e};
  Cand = Cand - {e}
}
```

General Algorithm

Definition

A set $A \subseteq E$ is promising if A is a subset of a MST of G

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A cut in a graph G is a partition of the set of vertices V, i.e.,
a pair (C, C') such that $C, C' \neq \emptyset$ and

- $C \cup C' = V$
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Definition

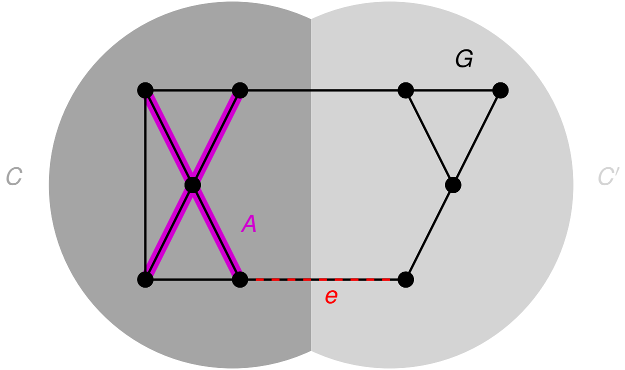
An edge e respects a cut (C, C') if the ends of e belong both to C or to C'; otherwise, we say that e crosses the cut.

General Algorithm

Theorem

Let A be a promising set of edges that respects a cut (C, C') of G.
Let e be an edge of minimum weight among those that cross the cut (C, C').
Then $A \cup \{e\}$ is promising.

General Algorithm



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1. start with an empty set of edges A
2. define a cut of G which is respected by A
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4. add e to A

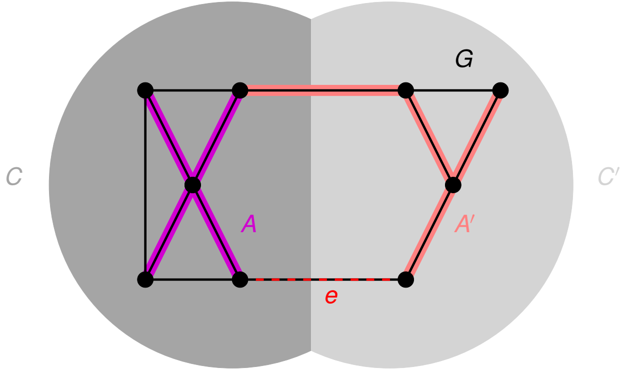
Any algorithm that follows this scheme is automatically correct.

Note that step 3 chooses the edge greedily
(the one with minimum weight among those that cross the cut)

Proof of the theorem

Let A' be an MST such that $A \subseteq A'$ (which exists since A is promising).

General Algorithm



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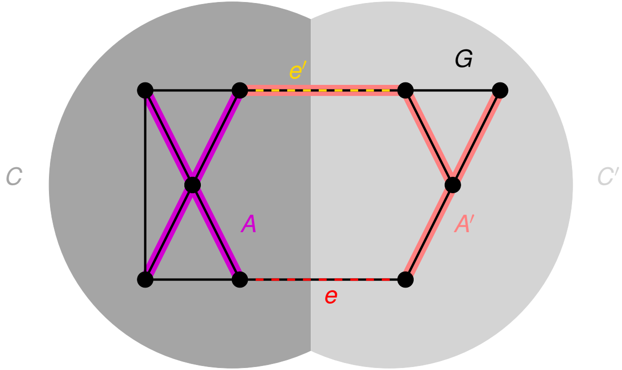
As A respects the cut, there is an edge $e' \in A' - A$ that crosses the cut (otherwise, A' would not be connected).

General Algorithm



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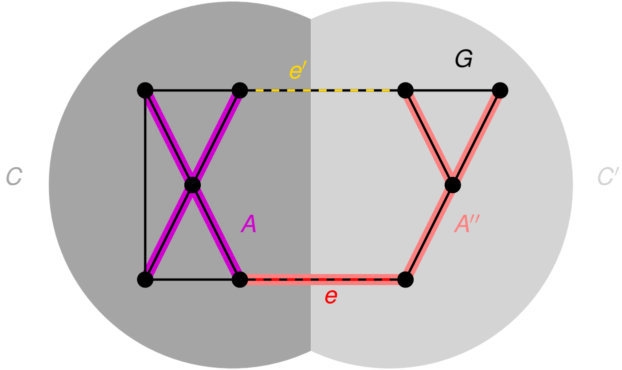

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Now let us assume that $e \notin A'$.

As A respects the cut, there is an edge $e' \in A' - A$ that crosses the cut (otherwise, A' would not be connected).

The subgraph $A'' = A' - \{e'\} \cup \{e\}$ is a spanning tree and

$$\omega(A') \leq \omega(A'') = \omega(A') - \omega(e') + \omega(e)$$

Hence $\omega(e') \leq \omega(e)$.

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But by definition of e, we have $\omega(e') \geq \omega(e)$, and therefore $\omega(A') = \omega(A'')$.

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Hence $\omega(e') \leq \omega(e)$.

But by definition of e, we have $\omega(e') \geq \omega(e)$, and therefore $\omega(A') = \omega(A'')$.

Since A'' is a MST, $A \cup \{e\}$ is promising.

Prim's Algorithm

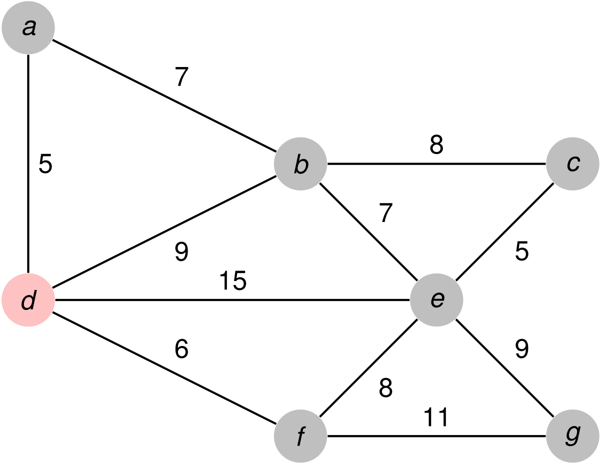
In Prim's Algorithm (also known as Prim-Jarník's Algorithm),
we keep a subset of visited vertices.

The set of vertices is thus partitioned into visited and non-visited vertices.

Each iteration of the algorithm chooses an edge of minimum weight among those that join a visited vertex and a non-visited one.

By the theorem, the algorithm is correct.

Prim's Algorithm



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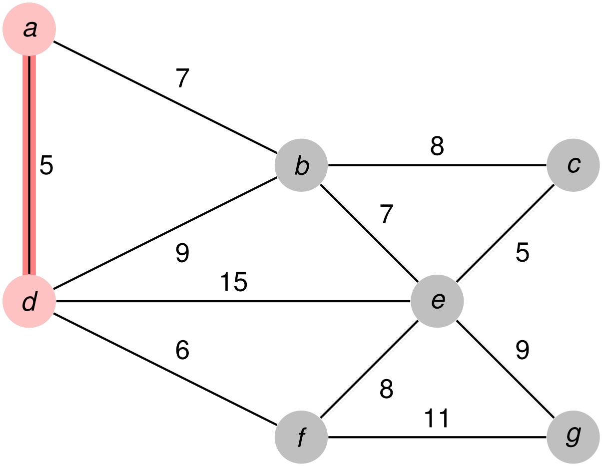

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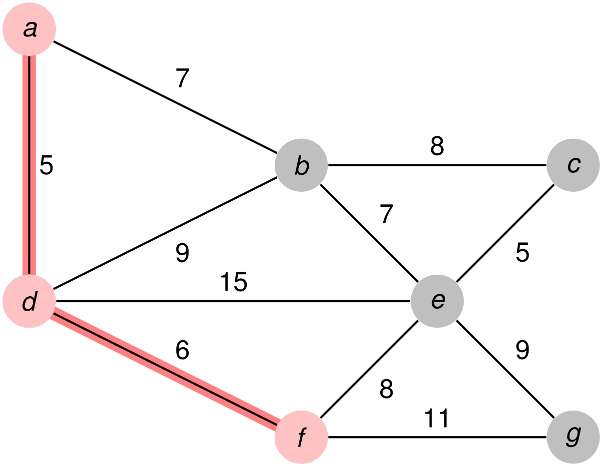

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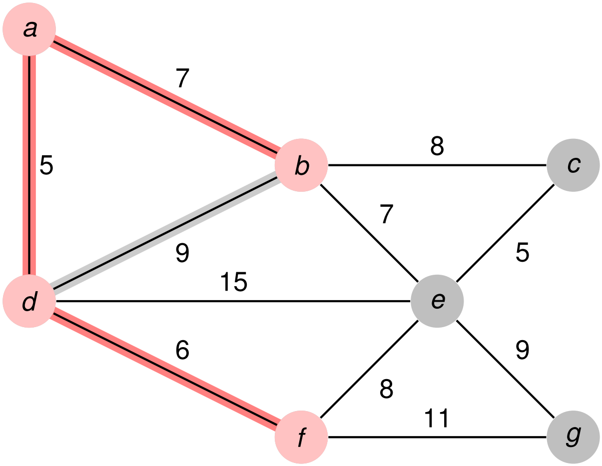

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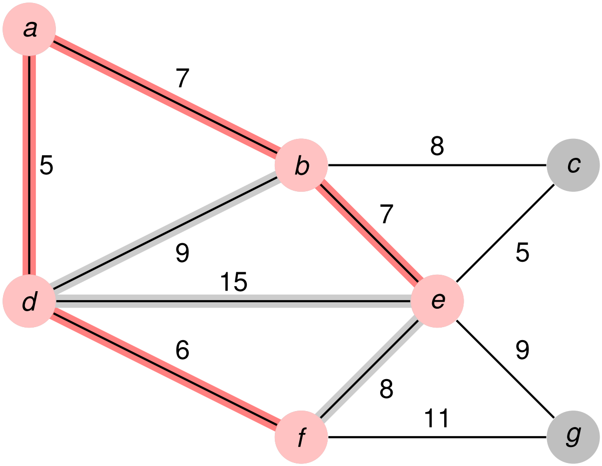

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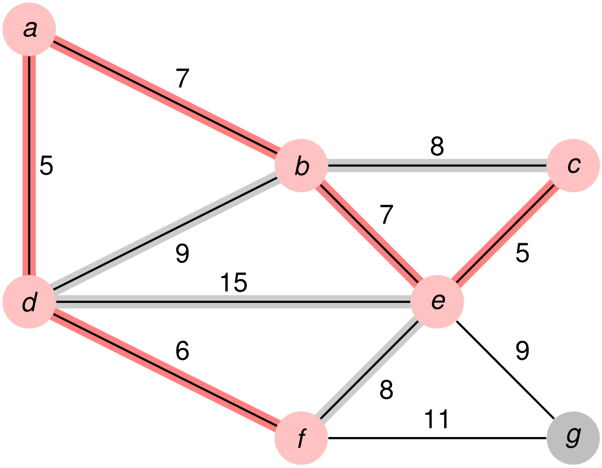

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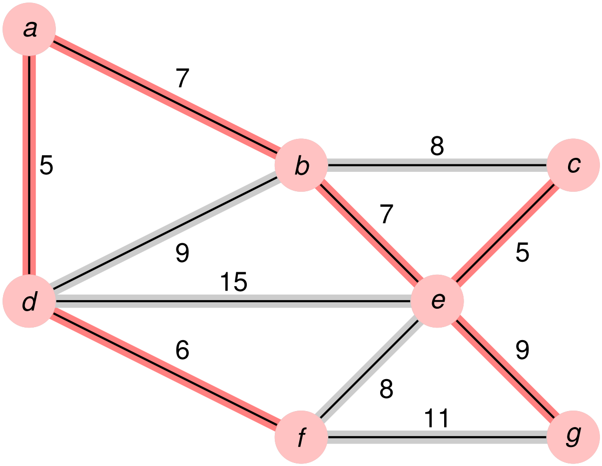

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Prim's Algorithm

```
typedef pair< double, pair<int, int> > WEdge;
typedef vector< vector< pair<double, int> > > WGraph;

int Prim(const WGraph& G, vector<pair<int,int>>& MST) {
  vector<bool> used(G.size(), false);
  priority_queue<WEdge,vector<WEdge>,greater<WEdge>> Q;
  used[0] = true;
  for (auto& x : G[0]) Q.push({x.first,{0,x.second}});
  int totalWeight = 0; uint sz = 1;
  while (sz < G.size()) {
    int u = Q.top().second.first;
    int v = Q.top().second.second;
    double wEdge = Q.top().first;  Q.pop();
    if (not used[v]) {
      used[v] = true;
      MST.push_back({u,v});
      totalWeight += wEdge; ++sz;
      for (auto e : G[v]) Q.push({e.first, {v, e.second}});
  }  }
  return totalWeight; }
```

Prim's Algorithm

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Prim's Algorithm runs in $O(m \log n)$ time

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In the implementation, the cost of the loop dominates overall cost.

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In total:

$$O(m \log m) + \sum_{x \in V} O(\deg(x)\ \log m) = O(m \log m) = O(m \log n)$$

□

Kruskal's Algorithm

In Kruskal's Algorithm, at each step we keep:

- a set of forests *SF* (our MST in construction)
- a set of pending edges *P* candidates to be added to *SF*

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Initially *SF* is empty and *P* contains all edges in the graph.
At each step, we add to *SF* the edge *e* in *P* of minimum weight that does not create a cycle in *SF*, and remove *e* from *P*.

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Initially SF is empty and P contains all edges in the graph.
At each step, we add to SF the edge e in P of minimum weight that does not create a cycle in SF, and remove e from P.

Next we will see the algorithm is correct, as it follows the previous recipe:

1. start with an empty set of edges A
2. define a cut of G which is respected by A
3. choose the edge e with minimum weight among those that cross the cut
4. add e to A

We only need to determine what is the cut of Kruskal's algorithm

Kruskal's Algorithm

Let $G = (V, E)$ be the graph for which we want to find an MST.

At each step, consider $\{u, v\}$ the edge added by Kruskal
(i.e. the edge in P of minimum weight that does not create a cycle in SF)

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At each step, consider $\{u, v\}$ the edge added by Kruskal
(i.e. the edge in P of minimum weight that does not create a cycle in SF)

We define the cut as follows:

$$\begin{aligned} C &= \{w \in V \mid u \text{ has a path to } w \text{ in } SF\} \\ C' &= V \setminus C \end{aligned}$$

$\{u, v\}$ clearly crosses the cut, as we know it does not create a cycle in SF.

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In general, any edge e crosses the cut iff it does not create a cycle in SF.

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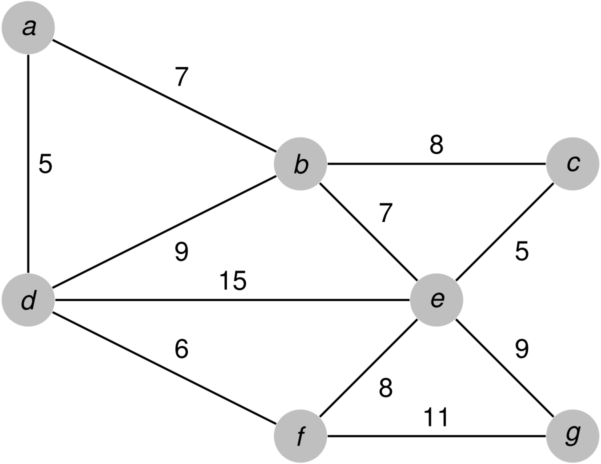
$\{u, v\}$ clearly crosses the cut, as we know it does not create a cycle in SF.

In general, any edge e crosses the cut iff it does not create a cycle in SF.

So $\{u, v\}$ is the edge with minimum weight among those crossing the cut.

Kruskal's Algorithm



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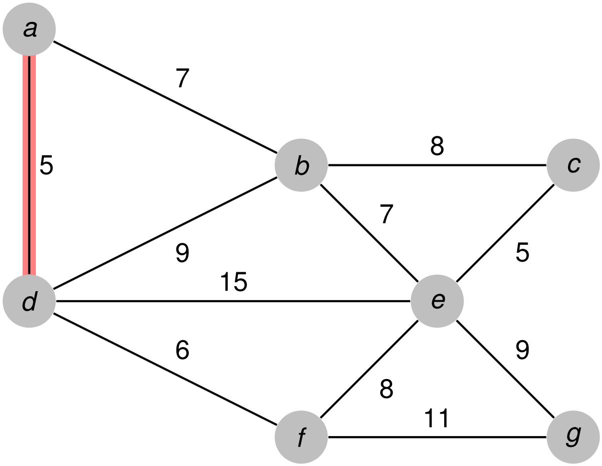

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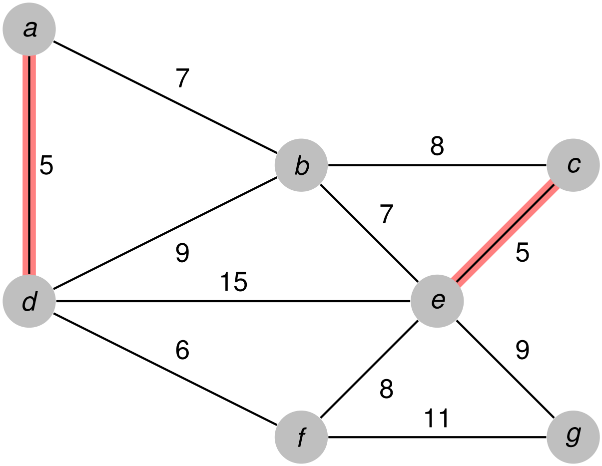

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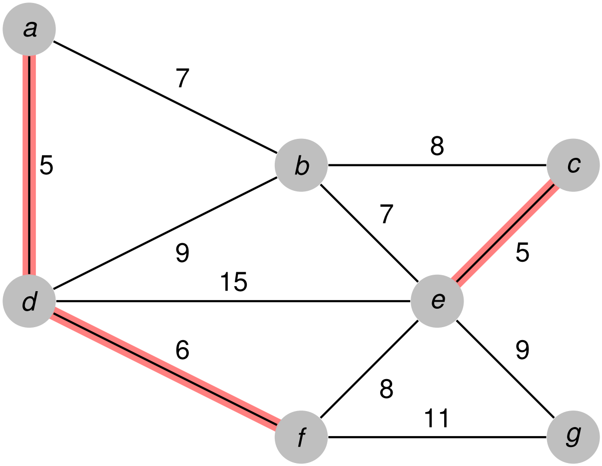

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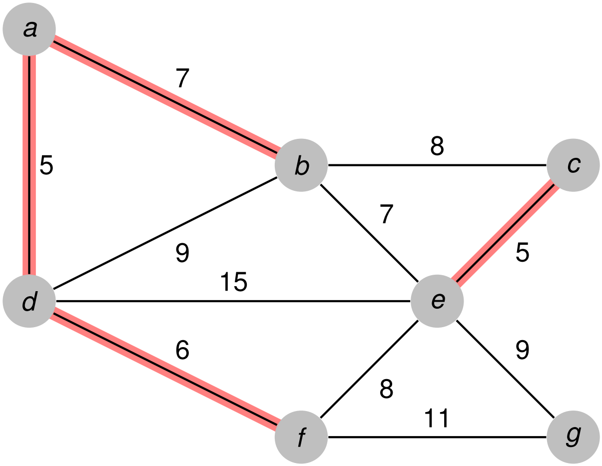

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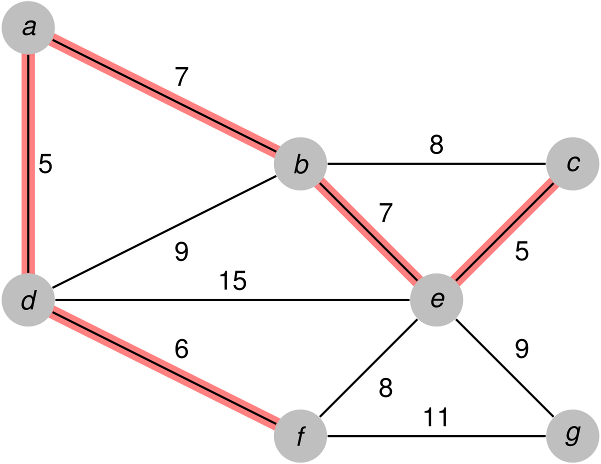

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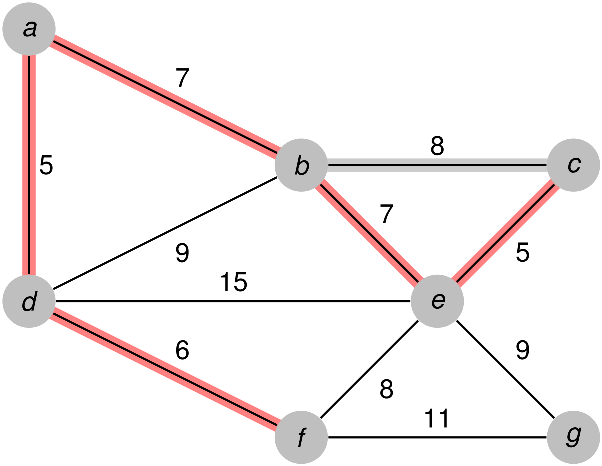

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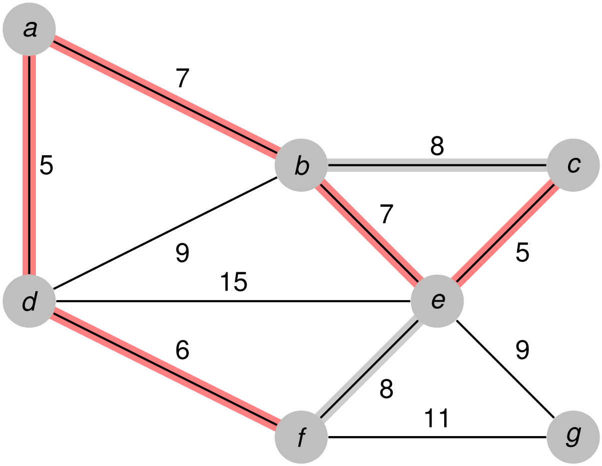

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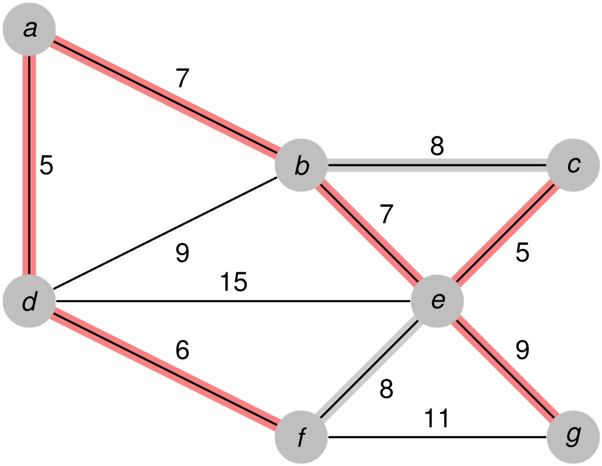

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Kruskal's Algorithm

```
typedef pair< double, pair<int, int> > WEdge;
typedef vector< vector< pair<double, int> > > WGraph;

int Kruskal(const WGraph& G, vector<pair<int,int>>& MST) {
  int n = G.size();
  UnionFind uf(n);
  int totalWeight = 0;
  priority_queue<WEdge,vector<WEdge>,greater<WEdge>> Q;
  for (int v = 0; v < n; ++v)
    for (auto& x: G[v]) if (v < x.second)
       Q.push({x.first,{v,x.second}});
  int sz = 0;
  while (sz < n - 1) {
    int u = Q.top().second.first;
    int v = Q.top().second.second;
    double wEdge = Q.top().first; Q.pop();
    if (uf.find(u) != uf.find(v)) {
      totalWeight += wEdge; ++sz;
      MST.push_back({u,v}); uf.merge(u,v);
    } }
  return totalWeight; }
```

Kruskal's Algorithm

```
 class UnionFind {
   vector<int> rep;
   vector<int> size; // only valid for representative elements
   vector<vector<int>> elementsInClass; // only valid for reps
public:
   UnionFind(int n) : rep(n), size(n), elementsInClass(n) {
     for (int i = 0; i < n; ++i)
       {rep[i] = i; size[i] = 1; elementsInClass[i] = {i}; } }
   int find (int e) {return rep[e];}
   void merge(int e1, int e2) {
     int r1 = find(e1); int r2 = find(e2);
     if (size[r1] > size[r2]) { // r1 new rep
       size[r1] += size[r2];
       for (auto& x: elementsInClass[r2])
         {rep[x] = r1; elementsInClass[r1].push_back(x);} }
     else { // r2 new rep
       size[r2] += size[r1];
       for (auto& x: elementsInClass[r1])
         {rep[x] = r2; elementsInClass[r2].push_back(x); } } }
 };
```

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Theorem

Kruskal's Algorithm runs in $O(m \log n)$ time

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Proof:

Let us consider the contribution to the cost of:

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priority_queue<WEdge,vector<WEdge>,greater<WEdge>> Q;
for (int v = 0; v < n; ++v)
  for (auto& x: G[v]) if (v < x.second)
     Q.push({x.first,{v,x.second}});
```

Adding m edges to Q takes $O(m \log m)$ time.

Kruskal's Algorithm

Proof (continued):

Let us consider the contribution to the cost of the loop:

```
int sz = 0;
while (sz < n - 1) {
  int u = Q.top().second.first;
  int v = Q.top().second.second;
  double wEdge = Q.top().first; Q.pop();
  if (uf.find(u) != uf.find(v)) {
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Kruskal's Algorithm

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- Code in red: The loop is executed at most m times.
  At each iteration we pop one element from Q,
  which takes $O(\log m)$ time, and request 2 Finds from the Union-Find.

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  At each iteration we pop one element from Q,
  which takes $O(\log m)$ time, and request 2 Finds from the Union-Find.
- Code in blue: Overall the algorithm makes $n - 1$ Merges.

All in all, the cost is $O(m \log m)$ plus the cost of $O(m)$ Finds and $O(n)$ Merges.

Proof (continued):

In the previous Union-Find implementation:

- **Find:** costs constant time

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- **Find:** costs constant time
- **Merge:** In a sequence of $O(n)$ Merges, every element can change representative at most $O(\log n)$ times because every time it changes, the size of its class at least doubles. Since each change needs $O(n)$ time, the Union-Find needs $O(n \log n)$ time to process the Merges.

Proof (continued):

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The total time is hence $O(m \log m) + O(m) + O(n \log n)$.

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The total time is hence $O(m \log m) + O(m) + O(n \log n)$.

The graph is connected (otherwise there is no spanning tree). So $m \geq n - 1$.

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- **Find:** costs constant time
- **Merge:** In a sequence of $O(n)$ Merges, every element can change representative at most $O(\log n)$ times because every time it changes, the size of its class at least doubles. Since each change needs $O(n)$ time, the Union-Find needs $O(n \log n)$ time to process the Merges.

The total time is hence $O(m \log m) + O(m) + O(n \log n)$.

The graph is connected (otherwise there is no spanning tree). So $m \geq n - 1$.

Also since m is $O(n^2)$ we know that $O(\log m) = O(\log n)$.

Kruskal's Algorithm

Proof (continued):

In the previous Union-Find implementation:

- **Find:** costs constant time
- **Merge:** In a sequence of $O(n)$ Merges, every element can change representative at most $O(\log n)$ times because every time it changes, the size of its class at least doubles. Since each change needs $O(n)$ time, the Union-Find needs $O(n \log n)$ time to process the Merges.

The total time is hence $O(m \log m) + O(m) + O(n \log n)$.

The graph is connected (otherwise there is no spanning tree). So $m \geq n - 1$.

Also since m is $O(n^2)$ we know that $O(\log m) = O(\log n)$.

Hence, Kruskal's Algorithm runs in $O(m \log n)$. □