

Time Series

3. Non-Stationary Processes

Josep A. Sanchez-Espigares

Department of Statistics and Operations Research
Universitat Politècnica de Catalunya
Barcelona, Spain



UNIVERSITAT POLITÈCNICA
DE CATALUNYA
BARCELONATECH

- Stationarity and Invertibility
- ARIMA models: $ARIMA(p, d, q)$
- Seasonal ARIMA models: $SARIMA(p, d, q)(P, D, Q)_s$

Under certain conditions of stationarity, the ARMA(p, q) models can be expressed as an AR(∞) or MA(∞) model:

$$(1 - \phi_1 B - \dots - \phi_p B^p)X_t = (1 + \theta_1 B + \dots + \theta_q B^q)Z_t$$

Expression as an AR(∞):

$$\frac{1 - \phi_1 B - \dots - \phi_p B^p}{1 + \theta_1 B + \dots + \theta_q B^q} X_t = (1 - \pi_1 B - \pi_2 B^2 - \dots) X_t = Z_t$$

Expression as an MA(∞):

$$X_t = \frac{1 + \theta_1 B + \dots + \theta_q B^q}{1 - \phi_1 B - \dots - \phi_p B^p} z_t = (1 + \psi_1 B + \psi_2 B^2 + \dots) Z_t$$

- The π weights come from the power expansion of the rational function in B :
$$\pi(B) = \frac{\phi_p(B)}{\theta_q(B)}$$
- The expansion will converge ($\sum_{i=0}^{\infty} \pi_i^2 < \infty$) \Leftrightarrow The module of all roots in $\theta_q(B)$ are greater than one
- This means that all complex roots of the characteristic polynomial of the MA part lie outside the unit circle.
- This condition implies that the model is **invertible**

Example: $MA(1)$ $X_t = (1 + \theta B)Z_t$

Expression as an $AR(\infty)$:

$$\frac{1}{1 + \theta B} X_t = Z_t$$

$$(1 - \theta B + \theta^2 B^2 - \dots + (-1)^k \theta^k B^k + \dots) X_t = Z_t$$

In this case: $\pi_k = (-1)^k \theta^k$

The model is **invertible** $\Leftrightarrow |B| = |-\frac{1}{\theta}| > 1 \Leftrightarrow |\theta| < 1$

- The ψ weights come from the power expansion of the rational function in B : $\psi(B) = \frac{\theta_q(B)}{\phi_p(B)}$
- The expansion will converge ($\sum_{i=0}^{\infty} \psi_i^2 < \infty$) \Leftrightarrow The module of all roots in $\phi_p(B)$ are greater than one
- This means that all complex roots of the characteristic polynomial of the AR part lie outside the unit circle.
- This condition implies that the model is **causal** (stationary)

Example: $AR(1)$ $(1 - \phi B)X_t = Z_t$

Expression as an $MA(\infty)$:

$$X_t = \frac{1}{1 - \phi B} Z_t$$

$$X_t = (1 + \phi B + \phi^2 B^2 + \dots + \phi^k B^k + \dots) Z_t$$

In this case: $\psi_k = \phi^k$

The model is **causal** $\Leftrightarrow |B| = |\frac{1}{\phi}| > 1 \Leftrightarrow |\phi| < 1$

- All $AR(p)$ models are **invertible**:

$$\pi_k = \phi_k \quad k = 1 \dots p \Rightarrow \sum_{i=0}^{\infty} \pi_i^2 = \sum_{i=0}^p \pi_i^2 < \infty$$

- All $MA(q)$ models are **causal**:

$$\psi_k = \theta_k \quad k = 1 \dots q \Rightarrow \sum_{i=0}^{\infty} \psi_i^2 = \sum_{i=0}^q \psi_i^2 < \infty$$

- An $ARMA(p, q)$ model will be...:
 - Invertible \Leftrightarrow All roots of $\theta_q(B)$ have module greater than one
 - Causal \Leftrightarrow All roots of $\phi_p(B)$ have module greater than one

For an **causal** and **invertible** $ARMA(p, q)$ model:

- Expression as an $AR(\infty)$ can be truncated when the π_k weight is very small. This is useful to calculate the point prediction by using past observations and the π -weights
- Expression as an $MA(\infty)$ can be truncated when the ψ_k weight is very small. This is useful to calculate the variance of the prediction by using the ψ -weights

Example: $ARMA(1, 2)$ $X_t = 0.8X_{t-1} + Z_t - 3Z_{t-1} + 2Z_{t-2}$

$$(1 - 0.8B)X_t = (1 - 3B + 2B^2)Z_t$$

Roots of the AR-polynomial: $1 - 0.8B = 0 \Rightarrow B = 1/0.8 = 1.25 > 1$

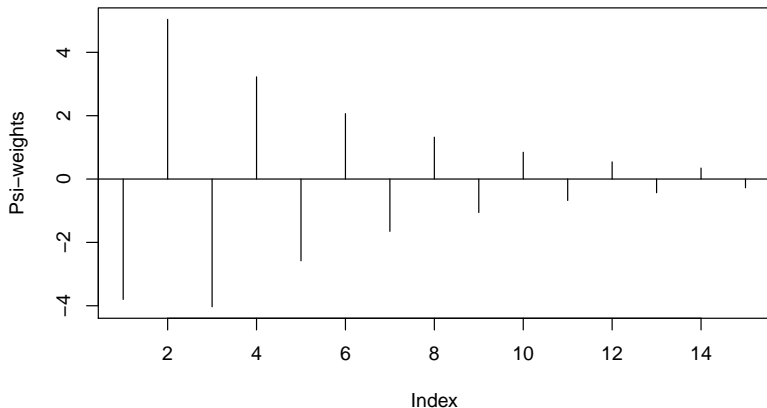
```
polyroot(c(1,-0.8))
```

```
## [1] 1.25+0i
```

```
Mod(polyroot(c(1,-0.8)))
```

```
## [1] 1.25
```

The model $X_t = 0.8X_{t-1} + z_t - 3Z_{t-1} + 2Z_{t-2}$ is **causal**



Example: $ARMA(1, 2)$ $x_t = 0.8x_{t-1} + z_t - 3z_{t-1} + 2z_{t-2}$

$$(1 - 0.8B)x_t = (1 - 3B + 2B^2)z_t$$

Roots of the MA-polynomial: $1 - 3B + 2B^2 = 0 \Rightarrow B = \{0.5, 1\}$

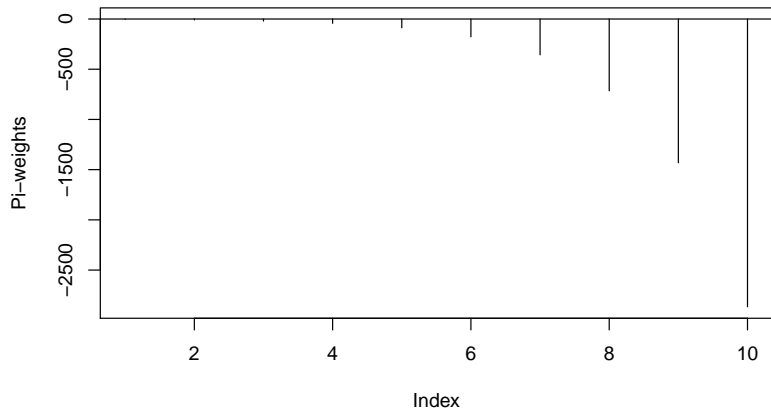
```
polyroot(c(1,-3,2))
```

```
## [1] 0.5+0i 1.0-0i
```

```
Mod(polyroot(c(1,-3,2)))
```

```
## [1] 0.5 1.0
```

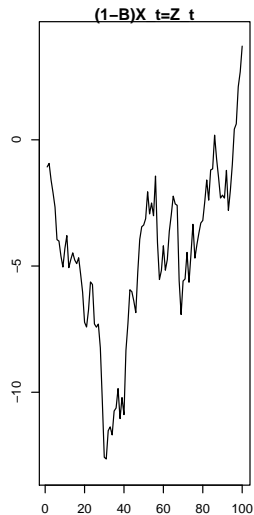
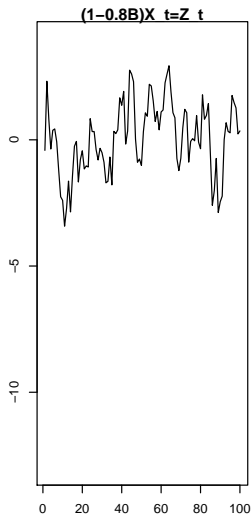
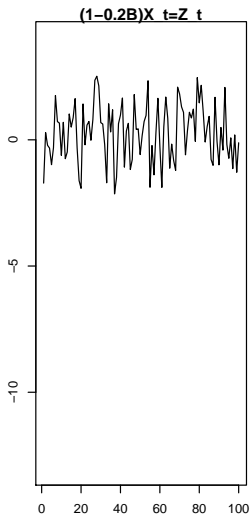
The model $X_t = 0.8X_{t-1} + z_t - 3Z_{t-1} + 2Z_{t-2}$ is not **invertible**



Non-Stationary models with unit roots

Let's consider several $AR(1)$ models

$$(1 - 0.2B)X_t = Z_t \quad (1 - 0.8B)X_t = Z_t \quad (1 - B)X_t = Z_t$$



Random Walk:

$$X_t = X_{t-1} + Z_t \quad (1 - B)X_t = Z_t \quad Z_t \sim WN(0, \sigma_Z^2)$$

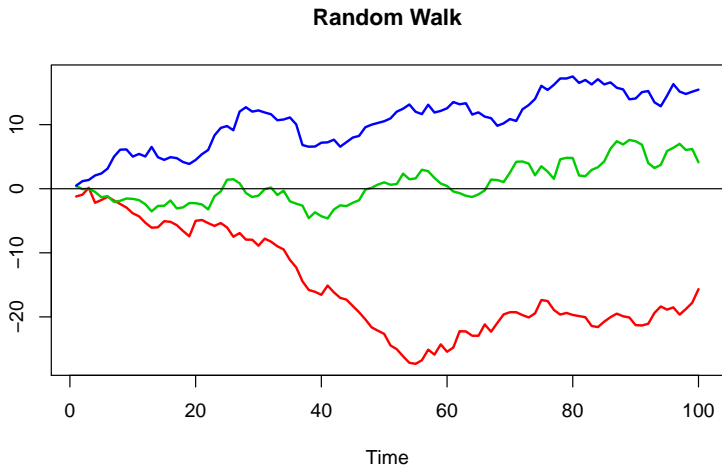
$$X_t = Z_t + Z_{t-1} + \dots + Z_1$$

Non stationary process:

$$V(X_t) = t\sigma_Z^2$$

$$\gamma(X_t, X_{t+k}) = t\sigma_Z^2$$

Random Walk:



Non-stationary $ARMA(p, q)$ with 1 unit root:

$$\phi_p(B)X_t = \theta_q(B)Z_t$$

$$\phi_p(B) = 1 - \phi_1 B - \dots - \phi_p B^p = (1 - \phi'_1 B - \dots - \phi'_{p-1} B)(1 - B)$$

x_t is a process with 1 unit root: Integrated of order 1 ($x_t \sim I(1)$)

This means that $W_t = (1 - B)X_t$ is an stationary $ARMA(p - 1, q)$ process

x_t is an $ARIMA(p - 1, 1, q)$

Non-stationary $ARMA(p, q)$ with d unit roots:

$$\phi_p(B)X_t = \theta_q(B)Z_t$$

$$\phi_p(B) = 1 - \phi_1 B - \dots - \phi_p B^p = (1 - \phi'_1 B - \dots - \phi'_{p-d} B)(1 - B)^d$$

X_t is a process with d unit roots: Integrated of order d ($X_t \sim I(d)$)

This means that $W_t = (1 - B)^d X_t$ is an stationary $ARMA(p - d, q)$ process

X_t is an $ARIMA(p - d, d, q)$

Non-stationary $ARIMA(p, d, q)$ with d unit roots:

$$\begin{aligned}\phi_p(B)(1-B)^d X_t &= \theta_q(B)Z_t \\ (1 - \phi_1 B - \dots - \phi_p B^p)(1-B)^d X_t &= (1 + \theta_1 B + \dots + \theta_q B^q)Z_t\end{aligned}$$

In the Identification step:

- d = number of regular differences to reach stationarity
- p, q = analysis of ACF and PCF of the transformed series

$ARMA(1, 1)$ with $\phi = 1 \Rightarrow$ Non-stationary and invertible (if $|\theta| < 1$)

$$(1 - B)X_t = (1 + \theta B)Z_t$$

The π -weights show an exponential decay:

$$X_t = (\theta + 1)X_{t-1} - \theta(\theta + 1)X_{t-2} + \theta^2(\theta + 1)X_{t-3} + \dots + Z_t$$

This model is the basis for the **EWMA** filter (Exponential Weighted Moving Average)

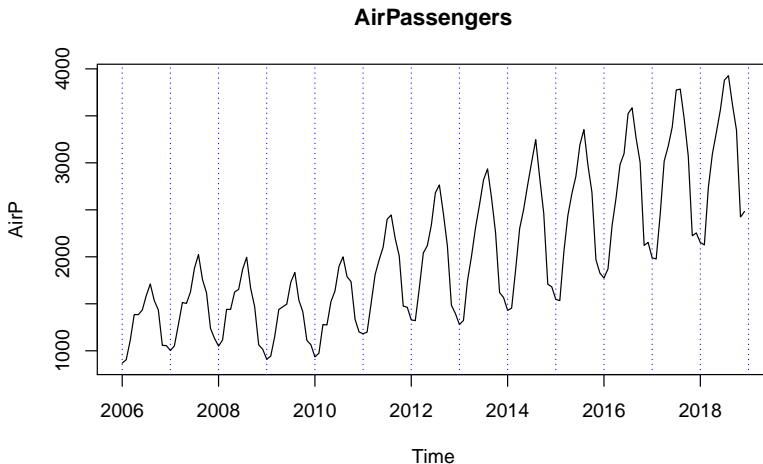
AirP: Number of monthly passengers (in thousands) of international air flights at El Prat between January 2006 and December 2018

Source: Ministry of Public Works of Spain (<http://www.fomento.gob.es/BE/?nivel=2&orden=03000000>)

##		Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
##	2006	869	905	1110	1386	1384	1433	1587	1711	1533	1435	1058	1055
##	2007	1003	1052	1279	1514	1504	1625	1878	2023	1758	1617	1237	1132
##	2008	1050	1115	1441	1440	1625	1653	1867	1995	1665	1468	1063	1016
##	2009	910	943	1154	1439	1468	1497	1730	1833	1540	1413	1111	1064
##	2010	935	974	1279	1275	1524	1636	1898	2000	1789	1736	1336	1200
##	2011	1181	1199	1504	1812	1967	2103	2400	2445	2195	2010	1475	1463
##	2012	1329	1321	1665	2041	2118	2337	2680	2765	2462	2115	1485	1395
##	2013	1279	1322	1744	2005	2308	2554	2819	2936	2625	2255	1619	1566
##	2014	1430	1454	1865	2301	2506	2767	3011	3246	2838	2461	1709	1680
##	2015	1547	1534	2049	2437	2671	2856	3191	3353	2977	2686	1968	1826
##	2016	1775	1871	2327	2623	2982	3098	3523	3586	3263	3005	2122	2153
##	2017	1989	1977	2448	3019	3177	3381	3775	3784	3462	3068	2224	2254
##	2018	2152	2127	2737	3094	3325	3566	3880	3927	3619	3349	2424	2482

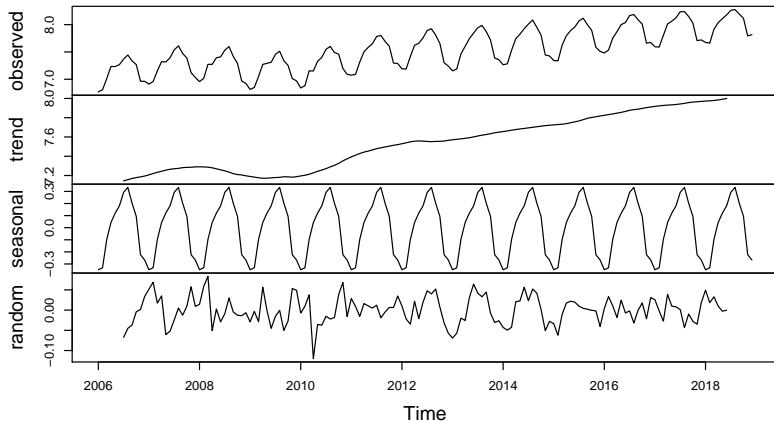
Seasonal Model: AirPassengers

$X_t = \text{Increasing Variance} + \text{Linear Trend} + \text{Seasonal component} + \text{stationary process}$



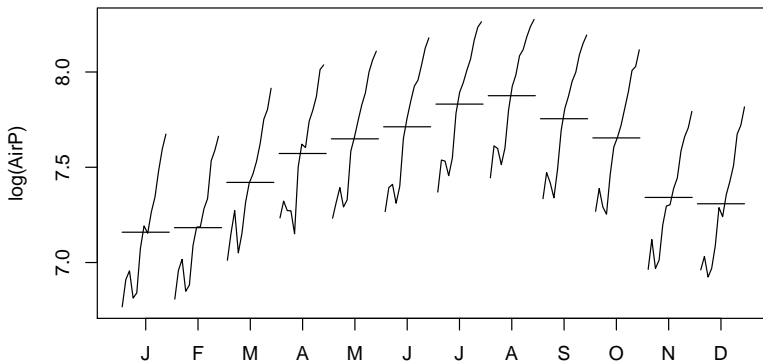
$\log(X_t) = \text{Linear Trend} + \text{Seasonal component} + \text{stationary process}$

Decomposition of additive time series

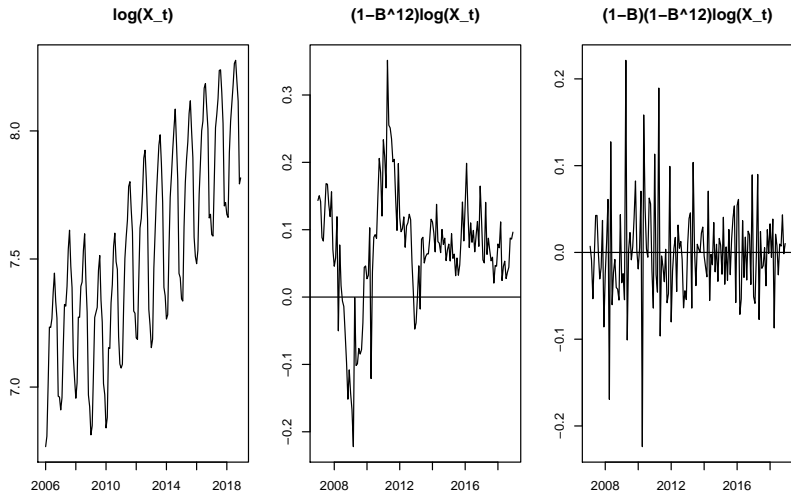


Seasonal Model: AirPassengers

$\log(X_t) = \text{Linear Trend} + \text{Seasonal component} + \text{stationary process}$



Seasonal Model: AirPassengers

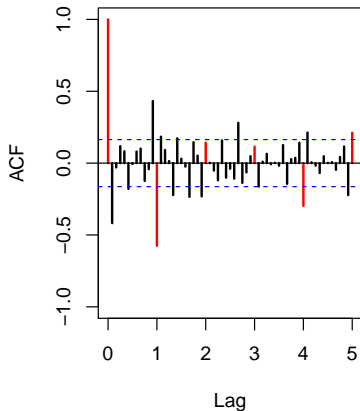


Transformation into an stationary series:

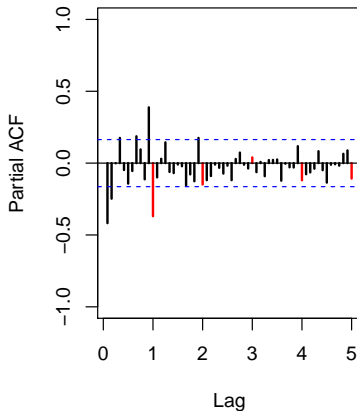
$$W_t = (1 - B)(1 - B^{12})\log(X_t)$$

- Logarithm scale: to stabilize the variance
- Seasonal difference: remove the seasonal pattern (and perhaps a global linear trend: $(1 - B^{12}) = (1 - B)(1 + B + \dots + B^{11})$)
- Regular difference: to reach a constant mean

Series d1d12lnAirP



Series d1d12lnAirP



Non-stationary Seasonal $ARIMA(p, d, q)(P, D, Q)_s$ (or SARIMA):

$$\phi_p(B)\Phi_P(B^s)(1-B)^d(1-B^s)^D X_t = \theta_q(B)\Theta_Q(B^s)Z_t$$

$$\begin{aligned}(1 - \phi_1 B - \dots - \phi_p B^p)(1 - \Phi_1 B^s - \dots - \Phi_P B^s P)(1 - B)^d(1 - B^s)^D X_t \\ = (1 + \theta_1 B + \dots + \theta_q B^q)(1 + \Theta_1 B^s + \dots + \Theta_Q B^s Q)Z_t\end{aligned}$$

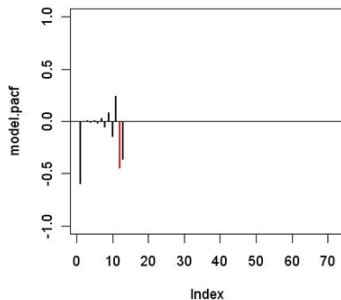
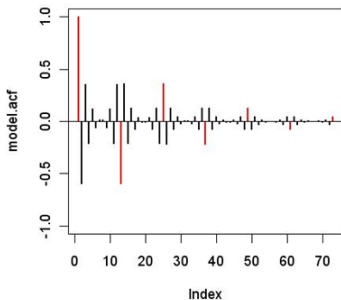
In the Identification step:

- d = number of regular differences to reach stationarity
- D = number of seasonal differences to reach stationarity (usually 0 or 1)
- p, q = analysis of ACF and PCF (only first lags)
- P, Q = analysis of ACF and PCF (only lags multiple of s)

Example 1: The following Theoretical ACF and PACF belong to models $ARMA(p, q)(P, Q)_{12}$ of this form:

$$(1 - \phi B)(1 - \Phi B^{12})W_t = (1 + \theta B)(1 + \Theta B^{12})Z_t$$

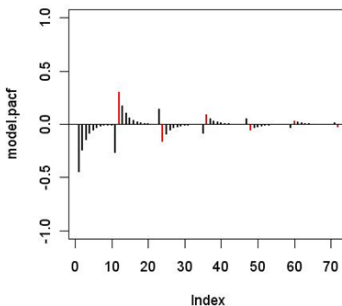
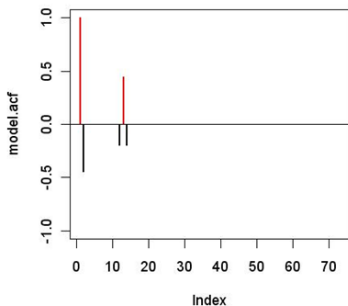
and $\phi, \Phi, \theta, \Theta \in \{-0.6, 0, 0.6\}$



Example 2: The following Theoretical ACF and PACF belong to models $ARMA(p, q)(P, Q)_{12}$ of this form:

$$(1 - \phi B)(1 - \Phi B^{12})W_t = (1 + \theta B)(1 + \Theta B^{12})Z_t$$

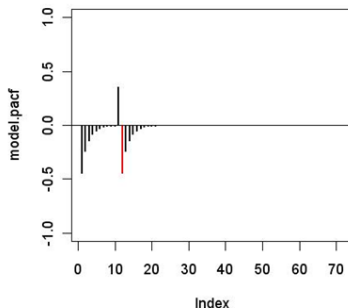
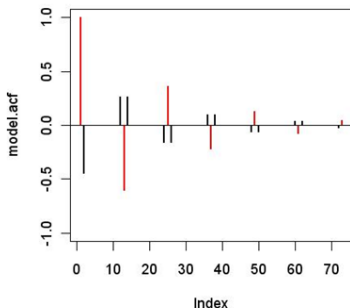
and $\phi, \Phi, \theta, \Theta \in \{-0.6, 0, 0.6\}$



Example 3: The following Theoretical ACF and PACF belong to models $ARMA(p, q)(P, Q)_{12}$ of this form:

$$(1 - \phi B)(1 - \Phi B^{12})W_t = (1 + \theta B)(1 + \Theta B^{12})Z_t$$

and $\phi, \Phi, \theta, \Theta \in \{-0.6, 0, 0.6\}$



Example 1: $(1 - 0.6B)(1 - 0.6B^{12})W_t = Z_t$

Example 2: $W_t = (1 - 0.6B)(1 + 0.6B^{12})Z_t$

Example 3: $(1 + 0.6B^{12})W_t = (1 - 0.6B)Z_t$