Containers: Priority Queues

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A priority queue

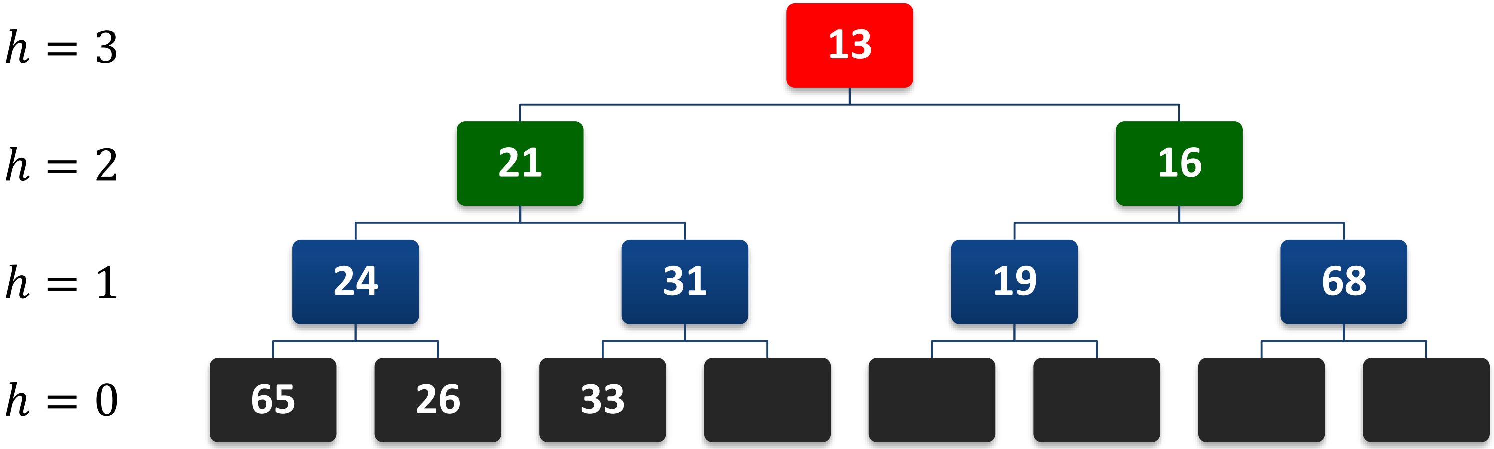
- A priority queue is a queue in which each element has a priority.

- Elements with higher priority are served before elements with lower priority.

- It can be implemented as a vector or a linked list. For a queue with n elements:
  - Insertion is $\mathrm{O}(n)$.
  - Extraction is $\mathrm{O}(1)$.

- A more efficient implementation can be proposed in which insertion and extraction are $\mathrm{O}(\log n)$: ***binary heap***.

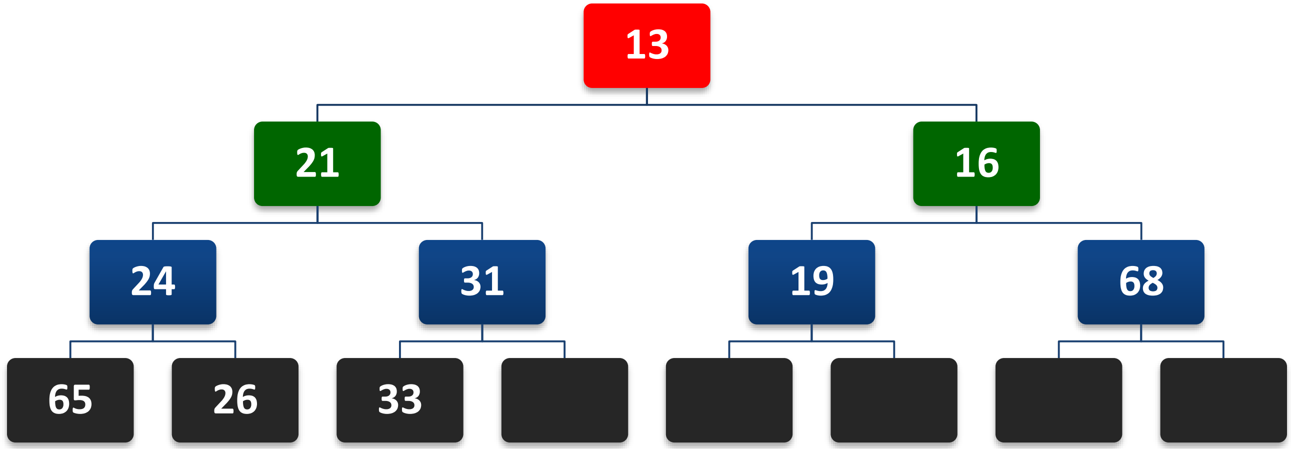
Binary Heap



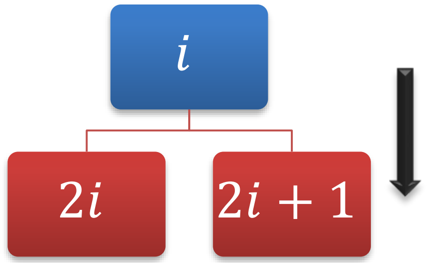
- Complete binary tree (except at the bottom level).
- Height h: between 2^h and $2^{h+1} - 1$ nodes.
- For N nodes, the height is $O(\log N)$.
- It can be represented in a vector.

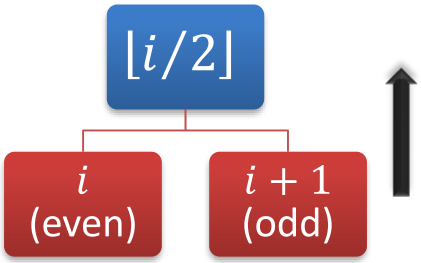
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	13	21	16	24	31	19	68	65	26	33					

Binary Heap



Locations in the vector:

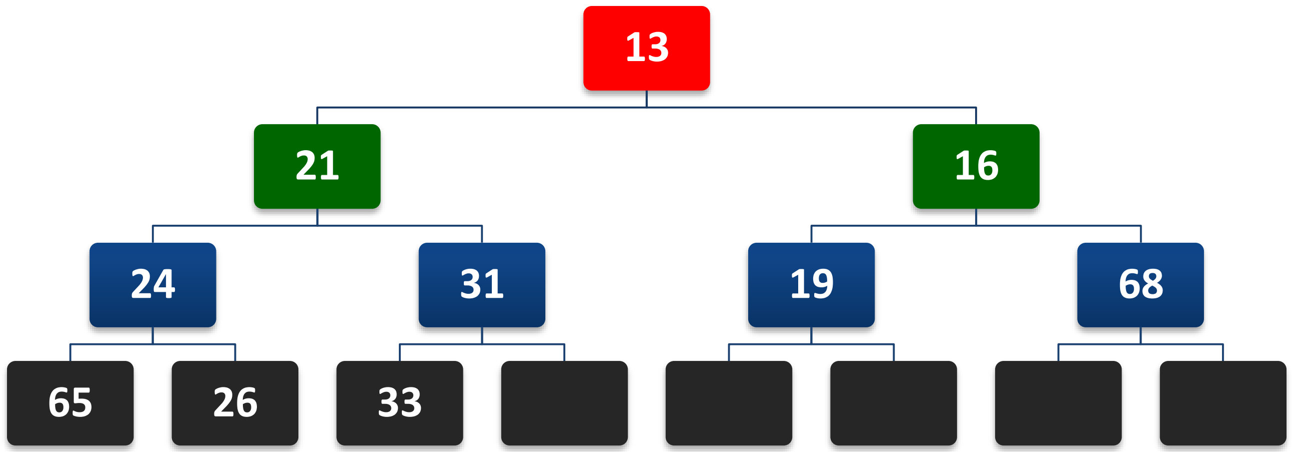




Heap-Order Property: the key of the parent of X is smaller than (or equal to) the key in X.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	13	21	16	24	31	19	68	65	26	33					

Binary Heap



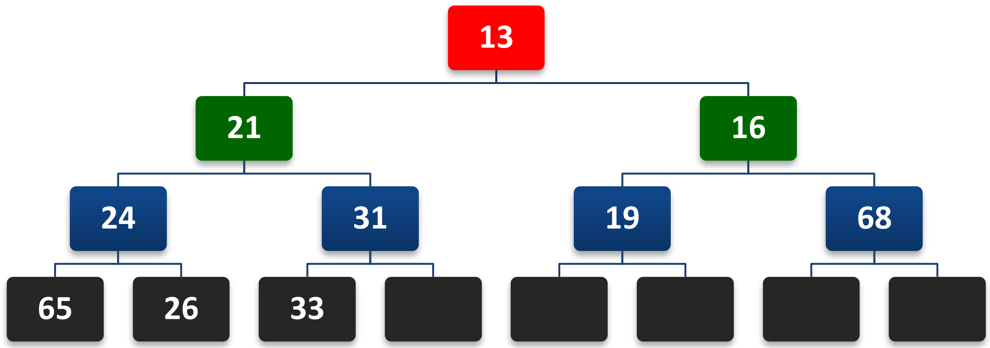
Two main operations on a binary heap:

- Insert a new element
- Remove the min element

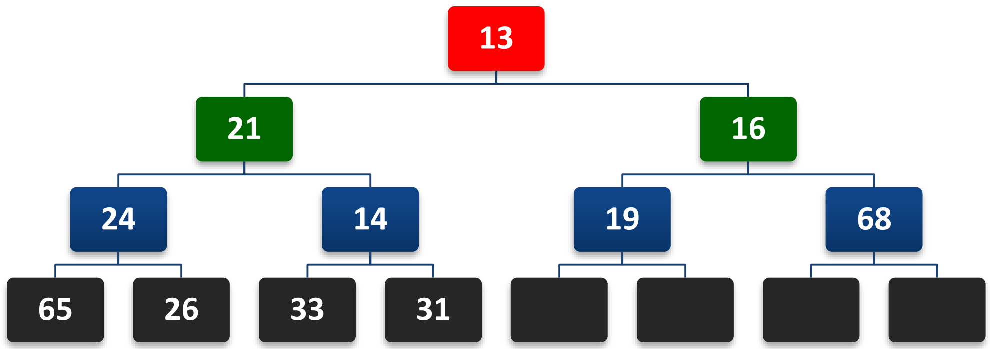
Both operations must preserve the properties of the binary heap:

- Completeness
- Heap-Order property

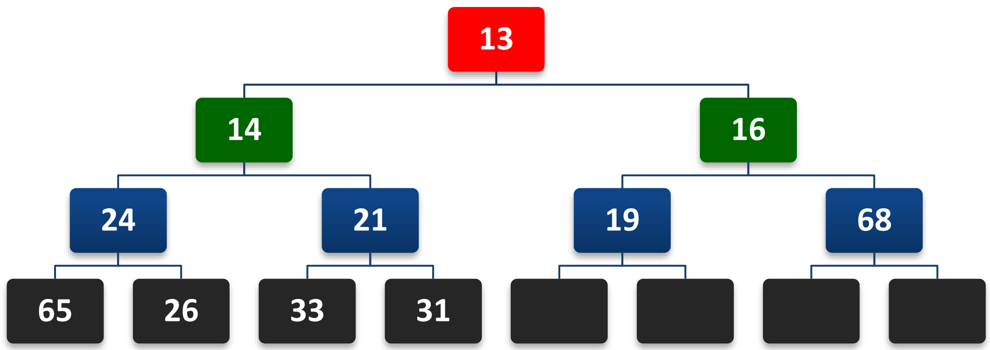
Binary Heap: insert 14



Insert in the last location

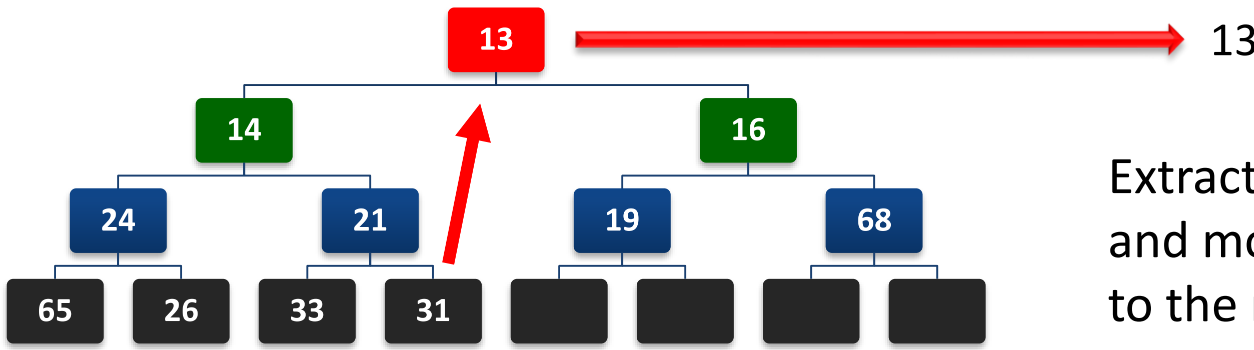


… and bubble up …

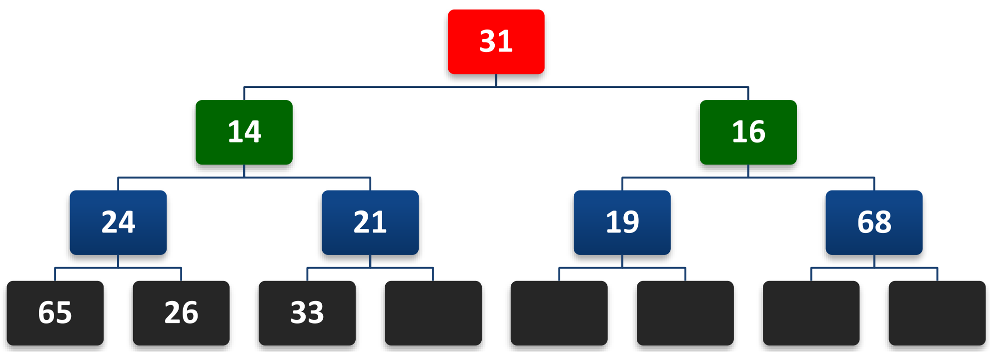


done !

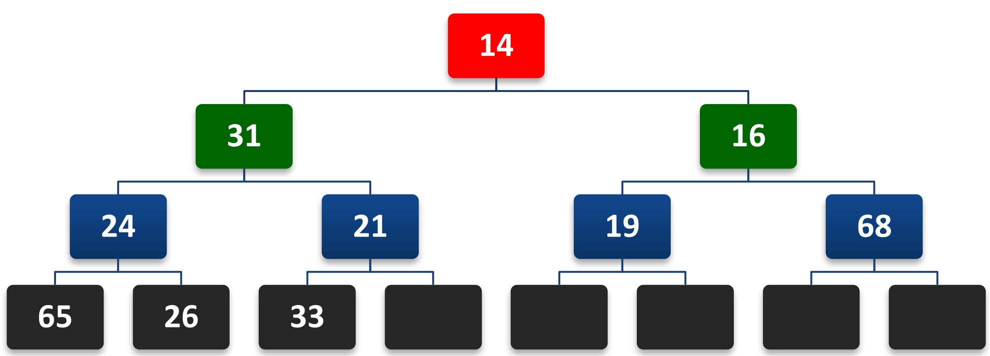
Binary Heap: remove min



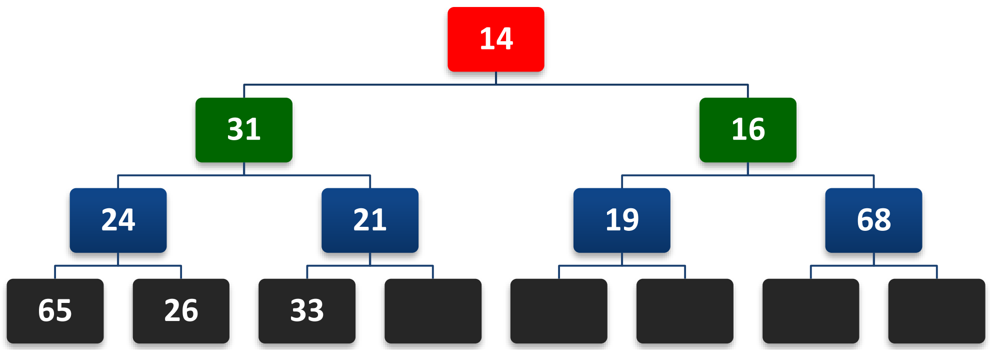
Extract the min element and move the last one to the root of the heap

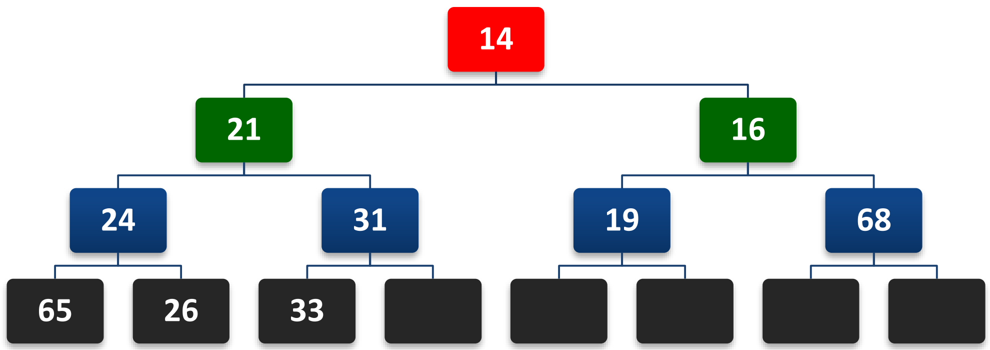


… and bubble down …



Binary Heap: remove min





done !

Binary Heap: complexity

- Bubble up/down operations do at most h swaps, where h is the height of the tree and

$$h = \lfloor \log_2 N \rfloor$$

- Therefore:
  - Getting the min element is $\mathrm{O}(1)$
  - Inserting a new element is $\mathrm{O}(\log N)$
  - Removing the min element is $\mathrm{O}(\log N)$

Binary Heap: other operations

- Let us assume that we have a method to know the location of every key in the heap.

- Increase/decrease key:
  - Modify the value of one element in the middle of the heap.
  - If decreased → bubble up.
  - If increased → bubble down.

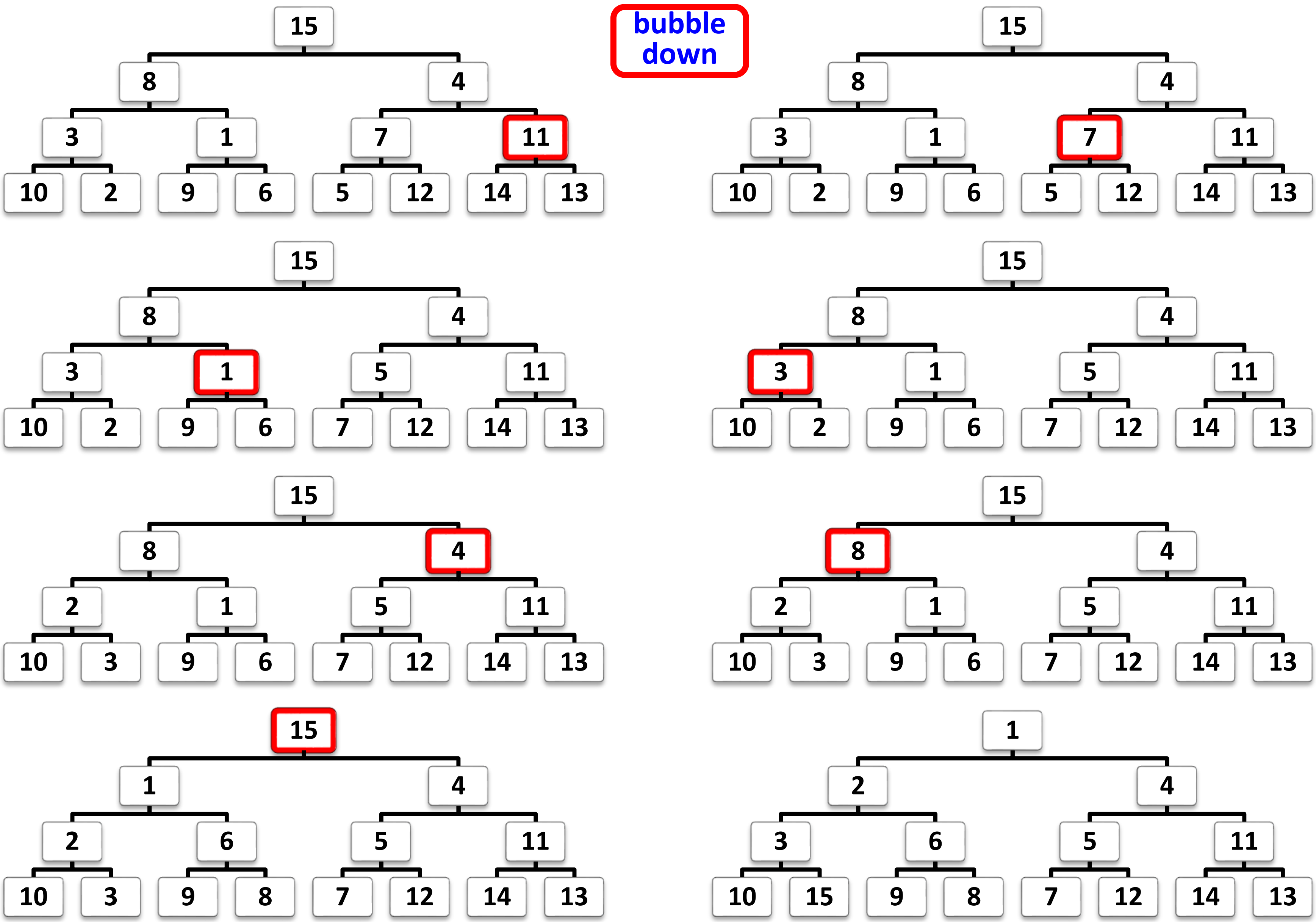
- Remove one element:
  - Set value to $-\infty$, bubble up and remove min element.

Building a heap from a set of elements

- Heaps are sometimes constructed from an initial collection of N elements. How much does it cost to create the heap?
  - Obvious method: do N insert operations.
  - Complexity: $\mathrm{O}(N \log N)$

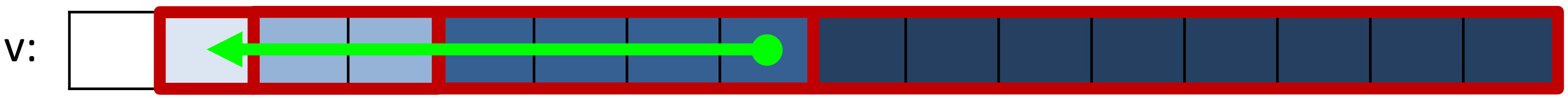
- Can it be done more efficiently?

Building a heap from a set of elements



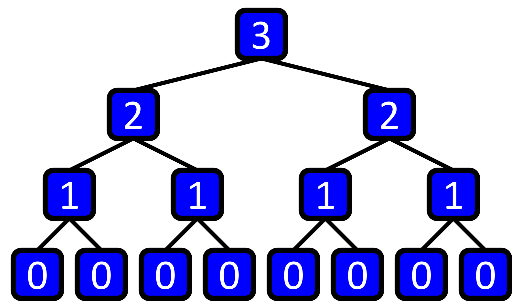
Building a heap: implementation

```
// Constructor from a collection of items
BinaryHeap(const vector<Elem>& items) {
  v.push_back(Elem()); // v is the vector holding the elements
  for (auto& e: items) v.push_back(e);
  for (int i = size()/2; i > 0; --i) bubble_down(i);
}
```



Sum of the heights of all nodes:

- 1 node with height h
- 2 nodes with height $h-1$
- 4 nodes with height $h-2$
- 2^i nodes with height $h-i$



$$S = \sum_{i=0}^{h-1} 2^i(h-i)$$

$$S = h + 2(h-1) + 4(h-2) + 8(h-3) + 16(h-4) + \cdots + 2^{h-1}(1)$$

$$2S = 2h + 4(h-1) + 8(h-2) + 16(h-3) + \cdots + 2^h(1)$$

Subtract the two equations:

$$S = -h + 2 + 4 + 8 + \cdots + 2^{h-1} + 2^h = \boxed{\left(2^{h+1}-1\right)} - (h+1) = O(N)$$

A heap can be built from a collection of items in linear time.

Heap sort

```
template <typename T>
void HeapSort(vector<T>& v) {
  BinaryHeap<T> heap(v);
  for (T& e: v) e = heap.remove_min();
}
```

- Complexity: $\mathrm{O}(n \log n)$
  - Building the heap: $\mathrm{O}(n)$
  - Each removal is $\mathrm{O}(\log n)$, executed n times.

EXERCISES

Exercise: insert/remove element

Given the binary heap implemented in the following vector, draw the tree represented by the vector.

	6	7	9	10	11	12	13	15	19	14	21	17	16

Execute the following sequence of operations

```
insert(8); remove_min(); insert(6); insert(18); remove_min();
```

and draw the tree after the execution of each operation.

Exercise: guess a and b

Consider the binary heap of integer keys implemented by the following vector:

	3	7	a	10	15	18	b	25	13	20	17	22	19

After executing the operations **`insert(8)`** and **`remove_min()`** the contents of the binary heap is:

	7	10	8	b	15	18	a	25	13	20	17	22	19

Discuss about the possible values of a and b. Assume there can never be two identical keys in the heap.

Exercise: the k-th element

The k-th element of n sorted vectors.

Let us consider n vectors sorted in ascending order.

Design an algorithm with cost $\Theta(k \log n + n)$ that finds the k-th global smallest element.

Exercise: bubble-up/down

Consider the following declaration for a Binary Heap:

```
template <typename T>  // T must be a comparable type
class BinaryHeap {
private:
  vector<Elem> v;  // Table for the heap (location 0 not used)

  // Bubbles up the element at location i
  void bubble_up(int i);

  // Bubbles down the element at location i
  void bubble_down(int i);
```

Give an implementation for the methods **bubble_up** and **bubble_down**.