ALGEBRA
5. DIAGONALIZATION

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OUTLINE

Outline

Motivation

Throughout the chapter we work on a $\mathbb{K}$-e.v. E of finite dimension n.

- Goal: **compute powers of matrices**.
- If M is a diagonal matrix, then it is easy to compute M^m for any $m \in \mathbb{N}$.
- If M is not diagonal, is there a change of basis that converts it to a diagonal matrix?

Definition

We say that an endomorphism $f : E \to E$ is diagonalizable in $\mathbb{K}$ if there exists a basis $\mathbf{v}$ of E such that $M_{\mathbf{v}}(f)$ is a diagonal matrix $D \in \mathcal{M}_n(\mathbb{K})$.

In other words, f is diagonalizable in $\mathbb{K}$ if there exists an invertible matrix $P \in \mathcal{M}_n(\mathbb{K})$ such that

$$P^{-1} M_{\mathbf{e}}(f) P$$

is a diagonal matrix (P can be thought as a change of basis matrix).

- An $n \times n$ matrix diagonalizes if there exists an invertible matrix P such that $P^{-1}MP$ is a diagonal matrix.
- If M diagonalizes, then $M = PDP^{-1}$ for a certain diagonal matrix D. Hence, M^m can be easily computed:

$$M^m = PDP^{-1}PDP^{-1} \dots PDP^{-1}PDP^{-1} = PD^mP^{-1}.$$

Remark: If $M_{\mathbf{v}}(f)$ is diagonal, then $f(v_i) = d_i v_i$ ($d_i = i$th value in the diagonal).

DEFINITION

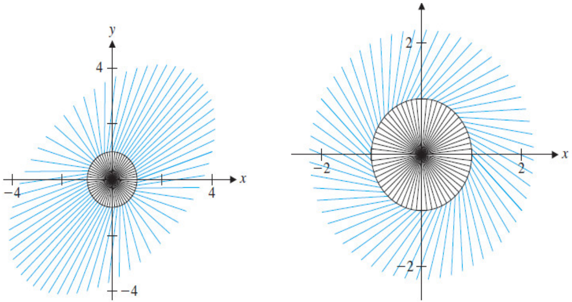
Let $f \in End(E)$. A vector $u \neq 0 \in E$ is an eigenvector (VEP) of f if $f(u) = \lambda\, u$ for some $\lambda \in \mathbb{K}$. In this case, we say that λ is an eigenvalue (VAP) of f and that u is an eigenvector with eigenvalue λ.

EXAMPLE

Consider the endomorphism of $\mathbb{R}^2$ given by $f(x, y) = (5x, 2y)$. Then, $e_1 = (1, 0)$ is an eigenvector of f with eigenvalue 1, and $e_2 = (0, 1)$ is an eigenvector of f with eigenvalue 2.
The standard matrix of f is

$$M_e(f) = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}.$$

In black v; in **blue** $f(v)$.



[D.Poole]

A map with eigenvectors $[(1,1)]$ and $[(1,-1)]$

A map with no eigenvectors.

Let f be an endomorphism of E.

Lemma

- *$u \in E$ is a VEP of VAP $\lambda \in \mathbb{K} \Leftrightarrow u \in \mathrm{Nuc}(f - \lambda Id)$ and $u \neq 0$.*
- *$\lambda \in \mathbb{K}$ is an eigenvalue of f $\Leftrightarrow \det(f - \lambda Id) = 0$.*

Definition

If λ is a VAP of f, we call E_λ the subspace $\mathrm{Nuc}(f - \lambda Id) \subseteq E$. This is the subspace formed by all VEP's of VAP λ plus $\mathbf{0}$.

Remark: A vector $u \neq 0$ is a VEP with VAP $0 \Leftrightarrow u \in \mathrm{Nuc}(f)$.
The spectrum of f is the set of all its eigenvalues and is denoted by $\sigma(M)$.

DEFINITION

The characteristic polynomial of $A \in \mathcal{M}_n(\mathbb{K})$ is

$$P_A(x) := det(A - xId) = \begin{vmatrix} a_{1,1} - x & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} - x & \dots & a_{2,n} \\ \dots & \dots & & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} - x \end{vmatrix}.$$

If $f \in End(E)$, the characteristic polynomial of f is $p_A(x)$ where $A = M_{\mathbf{u}}(f)$ for some basis **u**.

PROPOSITION

1. *$P_f(x)$ does not depend on the basis $\boldsymbol{u}$ chosen.*
2. *$P_f(x)$ is a polynomial of degree n,*
   $P_f(x) = c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0$*. Moreover, if*
   $M_{\boldsymbol{u}}(f) = (a_{i,j})$, $\Rightarrow$
$$c_n = (-1)^n,$$
$$c_{n-1} = (-1)^{n-1}\,\mathrm{tr}(f) = (-1)^{n-1}(a_{1,1} + \ldots + a_{n,n}),$$
$$c_0 = \det(f).$$
3. *In particular,* det *and* tr *are the same for any matrix of the endomorphism f.*
4. *The roots of $P_f(x)$ are the eigenvalues of f, that is,*
   $\lambda \in \mathbb{K}$ *is an eigenvalue of f* $\Leftrightarrow P_f(\lambda) = 0$.

Algebraic and geometric multiplicity of an eigenvalue

Definition

If λ is an eigenvalue of f, the algebraic multiplicity of λ , denoted by a_λ, is the multiplicity as a root of $P_f(x)$.

Definition

The geometric multiplicity of λ, denoted by g_λ, is the dimension of the vector subspace $\mathrm{Nuc}(f - \lambda Id)$, that is, $n - rk(A - \lambda I)$.

Proposition

For every eigenvalue λ, we have $1 \leq g_\lambda \leq a_\lambda$.

Outline

Let $f \in End(E)$. Then,

LEMMA

- *If u, v are VEP's of different VAP's $\Rightarrow u, v$ are l.i.*
- *If $\lambda_1, \ldots, \lambda_r$ are different VAP's $\Rightarrow$ the sum $E_{\lambda_1} + \ldots + E_{\lambda_r}$ is a direct sum,*

$$E_{\lambda_1} + \ldots + E_{\lambda_r} = E_{\lambda_1} \oplus \ldots \oplus E_{\lambda_r}$$

COROLLARY

If all the roots of $P_f(x)$ are in $\mathbb{K}$ and simple ($a_{\lambda_i} = 1$ for each λ_i), then f diagonalizes.

Diagonalization theorem

Theorem (Diagonalization)

An endomorphism f of E is diagonalizable in $\mathbb{K}$ if and only if

1. *$P_f(x)$ has all its roots $\lambda_1, \ldots, \lambda_k$ in $\mathbb{K}$ (P_f fully decomposes in $\mathbb{K}$) and*
2. *for every VAP λ_i, the algebraic multiplicity and geometric multiplicity are equal: $g_{\lambda_i} = a_{\lambda_i}$.*

If it diagonalizes, it does so in a basis of VEP's.

Given an endomorphism f of $\mathbb{R}^n$, let A be its standard matrix.

1. Compute the characteristic polynomial $P_f(x) = det(A - x\,Id)$.
2. Compute the roots $\lambda_1, \ldots, \lambda_k$ of $P_f(x)$: if some $\lambda_i \notin \mathbb{R} \Rightarrow f$ does not diagonalize in $\mathbb{R}$. Otherwise,
3. For each eigenvalue λ_i, compute its algebraic multiplicity as a root of $P_f(x)$, a_{λ_i}.
4. For each λ_i, compute Nuc$(A - \lambda_i Id)$: this is the set of all eigenvectors of f with eigenvalue λ_i. The dimension of this space is the geometric multiplicity g_{λ_i} of λ_i.
5. If $\lambda_i \in \mathbb{R}$ and $g_{\lambda_i} = a_{\lambda_i}$ for each eigenvalue λ_i, then f diagonalizes.

In this case, for each eigenvalue λ_i, let $\{v_1^i, \dots, v_{a_{\lambda_i}}^i\}$ be a basis for $\mathrm{Nuc}(A - \lambda_i\, Id)$. Then,

1. $\mathbf{v} = \bigcup_{i=1}^k \{v_1^i, \dots, v_{a_{\lambda_i}}^i\}$ is a basis of $\mathbb{R}^n$.
2. $M_{\mathbf{v}}(f)$ is a diagonal matrix:

$$M_{\mathbf{v}}(f) = D = \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_k \end{pmatrix}.$$

Recall that $M_{\mathbf{v}}(f)$ can be computed by doing a change of basis: if **e** is the standard basis of $\mathbb{R}^n$ and , then

$$A_{\mathbf{e}\to\mathbf{v}}\, A\, A_{\mathbf{v}\to\mathbf{e}} = D.$$

(Equivalently, $A_{\mathbf{v}\to\mathbf{e}}\, D\, A_{\mathbf{e}\to\mathbf{v}} = A$).

REAL MATRICES WITH COMPLEX EIGENVALUES

If we allow to work in $\mathbb{K} = \mathbb{C}$ then,

THEOREM

An endomorfism f is diagonalizable ***in*** $\mathbb{C}$ *if and only if*

- *for every VAP λ_i, the algebraic multiplicity and geometric multiplicity are equal: $g_{\lambda_i} = a_{\lambda_i}$.*

If A is a **real** matrix and we allow diagonalization in $\mathbb{K} = \mathbb{C}$, then VAP's and VEP's go "in pairs":

- $p_A(x) \in \mathbb{R}[x] \Rightarrow \lambda$ is a VAP of A if and only if $\bar{\lambda}$ is also a VAP.
- If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $w \in \mathbb{C}^n$ is a VEP with VAP λ, then we can write $w = u + iv$ where $u, v \in \mathbb{R}^n$.
- Then, $\bar{w} := u - iv$ is an eigenvector with eigenvalue $\bar{\lambda}$.

If we want to work only in $\mathbb{R}$, the one can rearrange complex VAP's and VEP's in conjugate pairs to obtain a "diagonalization" of A in 2×2 blocks.

OUTLINE

LEMMA

Let $f \in End(E)$. Then, there exists a basis $\mathbf{u}$ in which $M_{\mathbf{u}}$ is triangular if and only if $P_f(x)$ fully decomposes in $\mathbb{K}$.

In this case the triangular matrix has the VAP's $\lambda_1, \ldots, \lambda_n$ (repeated if necessary) of f in the diagonal and

$$\det(A) = \lambda_1 \ldots \lambda_n,$$

$$tr(A) = \lambda_1 + \ldots + \lambda_n.$$

First decomposition theorem

Theorem (First decomposition)

Let f in $End(E)$ such that

$$P_f(x) = (-1)^n (x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2}\dots(x-\lambda_r)^{m_r}.$$

Then $\dim \mathrm{Nuc}(f-\lambda_i Id)^{m_i} = m_i$ *for each i and*

$$E = \mathrm{Nuc}(f-\lambda_1 Id)^{m_1} \oplus \dots \oplus \mathrm{Nuc}(f-\lambda_r Id)^{m_r}.$$

Lemma

Let $f \in End(E)$, $\lambda \neq \mu \in \mathbb{K}$ and let $k \in \mathbb{N}_{>0}$. Then,

- *the endomorphisms $f-\lambda Id$ and $f-\mu Id$ commute,*
- $V = \mathrm{Nuc}(f-\lambda Id)^k$ *is invariant by $f-\mu Id$*
- *$f-\mu Id$ restricted to V is bijective.*

Cayley-Hamilton theorem

Definition

A polynomial $q(x)$ is an annihilator polynomial for f if the endomorphism $q(f)$ is 0 (equivalently, $q(A) = 0$ if A is the matrix of f in any basis).

Theorem (Cayley-Hamilton)

The characteristic polynomial $P_f(x)$ of f is an annihilator polynomial for f, $p_f(f) = 0$.

Consequences: if A is the standard matrix of $f \in \mathbb{R}^n$ and $P_f(x) = a_0 + a_1 x + \ldots + a_n x^n$, then

- $a_0 Id + a_1 A + \ldots + a_n A^n = 0$
- A^n can be computed as a linear combination of $Id, A, A^2, \ldots, A^{n-1}$
- If A is invertible $\Rightarrow A^{-1}$ can be computed as a linear combination of $Id, A, \ldots, A^{n-1}$

Minimal polynomial

Definition

There exists a monic annihilator polynomial $m_f(x)$ such that any other annihilator polynomial for f is multiple of it. This polynomial is called the minimal annihilator polynomial.

Properties of $m_f(X)$:

- $m_f(X)$ is (up to a scalar) the annihilator polynomial of smallest degree for f.
- $m_f(X) | P_f(X)$
- $\lambda \in \mathbb{K}$ is a zero of $P_f(x)$ if and only if it is a zero of $m_f(X)$.
- If $P_f(x) = (-1)^n (x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2}\dots(x-\lambda_r)^{m_r} \Rightarrow$

$$m_f(x) = (x-\lambda_1)^{n_1}(x-\lambda_2)^{n_2}\dots(x-\lambda_r)^{n_r} \quad \text{with} \quad n_i \le m_i.$$

- How do we find n_i? n_i is the smallest exponent k such that

$$\{\mathbf{0}\} \subsetneq \mathrm{Nuc}(f-\lambda_i Id) \subsetneq \cdots \subsetneq \mathrm{Nuc}(f-\lambda_i Id)^k = \mathrm{Nuc}(f-\lambda_i Id)^{k+1} = \dots$$

OUTLINE

If a matrix does not diagonalize in $\mathbb{C}$, one can do a change of basis to obtain a **triangular** matrix and one can reduce it even more to obtain its **Jordan canonical form**; that is, a block-diagonal matrix formed by blocks of type:

$$\begin{pmatrix} \lambda & 0 & \dots & 0 \\ 1 & \lambda & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \vdots & \\ 0 & \dots & 1 & \lambda \end{pmatrix}$$

In the exercise list there are some examples of how to obtain a Jordan form.

Study of $A^k x$ when $k \to \infty$

Let $A \in \mathcal{M}_n(\mathbb{R})$, with $A = PDP^{-1}$, $D = diag(\lambda_1, \ldots, \lambda_n)$, $P =$ change-of-basis $= A_{\mathbf{v}\to\mathbf{e}}$, $\mathbf{v} = \{v_1, \ldots, v_n\}$. Then,

- $A^k = PD^kP^{-1}$.
- If $x = c_1 v_1 + \cdots + c_n v_n \Rightarrow A^k x = c_1 \lambda_1^k v_1 + \cdots + c_n \lambda_n^k v_n$
- If $|\lambda_1| > |\lambda_i|$, then λ_1^k grows faster than λ_i^k so, if $c_1 \neq 0$,

$$A^k x \sim c_1 \lambda_1^k v_1 \quad \text{for } k \text{ big, and}$$

- This is the technical basis to efficiently compute VAPs (and VEPs) $\rightsquigarrow$ Algebra Lineal Numèrica (Q2).

Definition

A homogeneous linear discrete dynamical system is a matrix equation of the form

$$\mathbf{x}(k+1) = A\mathbf{x}(k), \quad k \in \mathbb{N},$$

where A is an $n \times n$ square matrix , and

$$\mathbf{x}(k) = \begin{pmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{pmatrix} \in \mathbb{R}^n.$$

The vector $\mathbf{x}(0)$ is called an initial condition.
A solution (or trajectory) is a collection of vectors $\{\mathbf{x}(k)\}_{k\geq 0}$ such that each $\mathbf{x}(k)$ satisfies the equation above.

Lemma

The solutions to $\mathbf{x}(k+1) = A\mathbf{x}(k)$ are $\{\mathbf{x}(k) = A^k\mathbf{x}(0)\}_k$.

Definition

A (column) stochastic matrix is a non-negative $n \times n$ matrix whose columns sum to 1.

A similar definition can be made for rows.
If A is a stochastic matrix we have:

- 1 is an eigenvalue of A.
- If x sums to 1, then Ax still sums to 1.

We say that a matrix A is positive if all its entries are > 0.

Theorem (Perron-Frobenius theorem)

Let A be a positive $n \times n$ matrix then A has a real eigenvalue λ_1 with the following properties:

- $\lambda_1 > 0$
- $\lambda_1 > |\lambda|$ *for any other eigenvalue* λ *(*λ_1 *is dominant).*
- λ_1 *has algebraic multiplicity 1.*
- λ_1 *has a corresponding positive eigenvector,* v*,* $Null(A - \lambda_1 Id) = [v]$.
- *There are no other positive eigenvectors for other eigenvalues.*

This theorem also holds for certain types of non-negative matrices.

Perron-Frobenius for stochastic matrices

Theorem

If A is a stochastic matrix, then 1 is VAP and, if A is positive,

- *$1 > |\lambda|$ for any other VAP λ*
- *1 has a positive VEP v.*
- *no other VAP has positive eigenvectors.*
- *If we take v to* ***sum to 1****, then we have*

$$\lim A^k = (v\, v \ldots v)$$

$$\textit{and} \qquad \lim A^k \boldsymbol{x} = v$$

for ***any*** *positive vector $\boldsymbol{x}$ that sums to 1.*

$\rightsquigarrow$ *This is the theory behind Google Pagerank algorithm.*

If f is a diagonalizable endomorphism with standard matrix $A \in M_n(\mathbb{R})$, so that $A = S\,D\,S^{-1}$, where D diagonal: $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, $S = A_{\mathbf{v}\to\mathbf{e}}$, and $\mathbf{v} = \{v_1, \ldots, v_n\}$ is the corresponding basis of eigenvectors. Then, we define the exponential of the matrix A:

$$e^A = S\,e^D\,S^{-1} = S \begin{pmatrix} e^{\lambda_1} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e^{\lambda_n} \end{pmatrix} S^{-1}$$

and this coincides with $\sum_{n\geq 0} \frac{A^n}{n!}$.

Outline

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