Tema 1a: Sèries

- (a_n) monòtona $\Rightarrow \exists \lim_{n\to\infty}(a_n) \in \mathbb{R}\cup\{\pm\infty\}$
- (a_n) fitada $\Rightarrow \exists$ subsucc conv.
- (a_n) no fitada $\Rightarrow \exists$ subsucc div.
- (a_n) conv $\iff$ (a_n) succ de Cauchy.

• infinitèssim: (a_n) tq $\lim_(a_n) = 0$.

- si (a_n) infinitèssim $\Rightarrow log(1+a_n), sin(a_n), 1-cos(a_n), e^{a_n}-1, tg(a_n), |a_n|^\alpha$ on $\alpha>0$ tmb.
- si (a_n) infinitèssim $\Rightarrow log(1+a_n)\sim a_n$, $sin(a_n)\sim a_n$, $1-cos(a_n)\sim \frac{(a_n)^2}{2}$, $e^{a_n}-1\sim a_n$, $tg(a_n)\sim a_n$.
- $lg(lg(n)) \prec lg(n) \prec n^c \prec n \prec n^a \prec n^{lg(n)} \prec b^n \prec n! \prec n^n \prec b^{n^a}$ on $b>1, 0<c<1<a$.
- $\sum a_n$ conv $\Rightarrow \lim_{n\to\infty} a_n = 0$.

• sèries telesòpiques: $\sum a_n$ tq $a_n = b_n - b_{n+1} \forall n\in\mathbb{N} \Rightarrow S_n = b_0 - b_{n+1}$.

• sèries aritmetico-geomètriques: $\sum a_n$ tq $a_n = (dn+s)r^n$ on $d,r,s\in\mathbb{R}, |d|+|s|>0$

- $\lim_{n\to\infty} |dn+s||r|^n) = 0 \iff |r|<1$ $\Rightarrow \sum(dn+s)r^n = \frac{s}{1-r} + \frac{dr}{(1-r)^2}$.

Criteris

▶ **Dirichlet**: $\sum_{n=1}^{\infty} a_n b_n$ conv si:

i)$s_n = \sum_{k=1}^{n} b_k$ fitada

ii)(a_n) monòtona i $\lim(a_n) = 0$

▶ **Leibnitz**: $\sum_{n=0}^{\infty} (-1)^n a_n$ conv si (a_n) monòtona decreixent i $\lim(a_n) = 0$

Criteris per a sèries positius

▶ **Comparació**: $\exists n_0$ tq $\forall n \geq n_0, a_n \leq b_n$

i)si $\sum b_n < \infty \Rightarrow \sum a_n < \infty$

ii)si $\sum a_n = \infty \Rightarrow \sum b_n = \infty$

▶ **Comparació al límit**: $\forall n\geq n_0, a_n \leq b_n, \exists \lim \frac{a_n}{b_n} = l \in [0,\infty]$

i)si $l\in(0,\infty) \Rightarrow (a_n),(b_n)$ mateix caràcter

ii) + iii) si $l=0$ ó $l=\infty$ $\Rightarrow$C.Comparació

▶ **Quocient**: si $\exists \lim \frac{a_{n+1}}{a_n} = \alpha \in [0,\infty]$ $\Rightarrow$si $\alpha>1$ div i si $\alpha<1$ conv.

▶ **Raabe**: si $\exists(1-\frac{a_{n+1}}{a_n}) = l \in [0,\infty]$ $\Rightarrow$si $l<1$ div i si $l>1$ conv.

▶ **Logarítmic**: si $\exists \lim \frac{log(\frac{1}{a_n})}{log(n)} = l \in [0,\infty]$ $\Rightarrow$si $l<1$ div i si $l>1$ conv.

▶ **Condensació**: si a_n decreix. i $\lim a_n = 0$ $\Rightarrow \sum a_n$ i $\sum 2^n a_{2^n}$ mateix caràcter.

-$\frac{1}{2}\sum 2^n a_{2^n} \leq \sum a_n \leq \sum 2^n a_{2^n}$.

▶ **Arrel**: si $\exists \lim(a_n)^{\frac{1}{n}} = \alpha \in [0,\infty]$ $\Rightarrow$si $\alpha>1$ div i si $\alpha<1$ conv.

▶ **Integral**: si $f:[m_{\geq 0},\infty)\to\mathbb{R}$ tq $f\geq 0, \lim f(x) = 0$ i f decreix. Llavors:

i)+ii)si $\lim \int_m^n f <$ ó $= \infty \Rightarrow \sum_{n=m}^{\infty} f(n)$ c ó d

- $\lim \int_m^n f \leq \sum_{n=m}^{\infty} f(n) \leq f(m) + \lim \int_m^n f$

- si F primitiva de f. Llavors:

i)+ii)si $\lim F(n) <$ ó $= \infty \Rightarrow \sum_{n=m}^{\infty} f(n)$ c ó d

- $\lim F(n) - F(k) + \sum_{n=m}^{k-1} f(n) \leq \sum_{n=m}^{\infty} f(n) \leq \lim F(n) - F(k) + \sum_{n=m}^{k} f(n)$

• sèrie harmònica: $\sum_{n\geq 1} \frac{1}{n^\alpha}, \alpha\in\mathbb{R}$; c sii $\alpha>1$ i d sii $\alpha\leq 1$.

Tema 1b: Integrals impròpies

• funció localm. integ: $f\in\mathcal{L}_{loc}(I)$ si $f: I\to\mathbb{R}\in\mathcal{R}(J)\forall J\in I$ interval tancat i fitat.

- $f\mathscr{C}$ ó monòtona $\Rightarrow f\in\mathcal{L}_{loc}(I)$.

• integral impròpia de $f:[a,b)\to\mathbb{R}\in\mathcal{L}_{loc}(I)$ és $\int_a^b f = \lim_{x\to b^-}\int_a^x f$

i)de 1a espècie si $b=\infty$ i f fitada

ii)de 2a espècie si $b<\infty$ i f no fitada

iii)de 3a espècie si $b=\infty$ i f no fitada

- $f\mathcal{L}_{loc}(I)$, $\int_{-\infty}^{\infty} f$ c $\Rightarrow \exists \lim_{R\to\infty}\int_{-R}^{R} f = \int_{-\infty}^{\infty} f$.
- si $f\in\mathcal{L}_{loc}(I)$, $\int_a^\infty f$ conv i $\exists \lim_{x\to\infty} f(x)$ $\Rightarrow \lim_{x\to\infty} f(x) = 0$.

▶ **C.Cauchy**: $\int_a^b f$ conv $\iff$ $\forall\varepsilon>0$, $\exists a\leq c_0 < b$ tq $|\int_c^{\tilde{c}} f| \leq \varepsilon$, $\forall c,\tilde{c}\in[c_0,b)$.

• integral imp. abs. conv: si $|f|$ té integ imp conv.

- f abs conv $\Rightarrow$f conv.
- Comparació, Comparació al límit i Dirichlet.

Funcions Gamma i Beta

• Gamma: $\Gamma:(0,\infty)\to\mathbb{R}$ tq $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt > 0$

- Γ conv $\forall\alpha > -1$ - $\Gamma(x+1) = x\Gamma(x)$
- $\Gamma(n+1) = n!$ - $\Gamma(n+1) = n!$
- $\Gamma(n+\frac{1}{2}) = \frac{\sqrt{\pi}(2n)!}{n!2^{2n}}$ - $\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-s^2} = \sqrt{\pi}$

• Beta: $B:(0,\infty)^2\to\mathbb{R}$ tq $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$, funció simètrica

- $B(x,y) = 2\int_0^\infty (sin(\theta))^{2x-1}(cos(\theta))^{2y-1}d\theta$
- $B(x,y) = \int_0^\infty \frac{s^{x-1}}{(1+s)^{x+y}} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

Tema 2: Integració a $\mathbb{R}^n$

Intervals, Rectangles i Particions

• interval: conv connex $\in\mathbb{R}$.

• longitud: $l(I) = b-a$.

- $l(I) = l(I^o) = l(\tilde{(I)})$.

• rectangle: $R = I_1$x...xI_n on I_j interval $\forall j$. És degenerat si $\exists j\in\{1,...,n\}$ tq $l(I_j) = 0$.

• volum: $v(R) = l(I_1)$x...x$l(I_n)$.

• diàmetre: $\delta(R) = \sqrt{l(I_1)^2\text{x...x}l(I_n)^2}$.

- $\max_j\{l(I_j)\} \leq \delta(R) \leq \sqrt{n}\max_j\{l(I_j)\}$ $\Rightarrow v(R)\leq\delta(R^n)$.
- $v(Q) = l^n \Rightarrow \delta(Q) = l\sqrt{n}$.

• partició: $P = \{x_0,...,x_n\}$ de $[a,b]$ tq $a = x_0 < ... < x_n = b$. Un subinterval és $I_i = [x_i, x_{i+1}]$. Obs $I_i^\circ\cap I_j^\circ = \varnothing, \forall i\neq j$ $\Rightarrow [a,b] = \bigcup_{i=0}^{k-1} I_i \Rightarrow b-a = \sum_{i=0}^{k-1} l(I_i)$.

• diàmetre: $\delta(P) = \max_j\{l(I_j)\}$.

• partició més fina: P' que P si $P\subset P'$.

- si $P\subset P' \Rightarrow \delta(P')\leq\delta(P)$.
- $\forall P,P'$ particions de $[a,b]$, $\exists\tilde{P}$ partició de $[a,b]$ tq $P,P'\subset\tilde{P}$.

• diàmetre: $\delta(P) = \sqrt{\delta(P_1)^2\text{x...x}\delta(P_n)^2}$.

- $\max_j\{\delta(P_j)\} \leq \delta(P) \leq \sqrt{n}\max_j\{\delta(P_j)\}$ $\Rightarrow v(R)\leq\delta(P)^n$.
- $\forall\varepsilon, \exists P$ tq $\delta(P)\leq\varepsilon$.

Integració de funcions fitades

$R = [a_1,b_1]$x...x$[a_n,b_n]$, $f: R\to\mathbb{R}$ fitada.

• $M_{ij} = \sup_{x\in R_{ij}}\{f(x)\}$; $m_{ij} = \inf_{x\in R_{ij}}\{f(x)\}$.

• suma superior: $S(f,P) = \sum_R M_R v(R)$.

• suma inferior: $s(f,P) = \sum_R m_R v(R)$.

- si $P\subset P'$, $s(f,P)\leq s(f,P')\leq S(f,P')\leq S(f,P)$.

• integral superior: $\overline{\int}_R f = \inf_P S(f,P)$.

• integral inferior: $\underline{\int}_R f = \sup_P s(f,P)$.

- $m_R v(R) \leq s(f,P) \leq \underline{\int}_R f \leq \overline{\int}_R f \leq S(f,P) \leq M_R v(R)$.

• integrable Riemman: si $\underline{\int}_R f = \overline{\int}_R f$.

- $\int_{\tilde{R}} f = \int_{\partial R} f + \int_{R^\circ} f$; - $\int_{\partial R} f = 0$.
- $\forall\tilde{R}$ tq $R^\circ\subset\tilde{R}\subset R$; $f\in\mathcal{R}(R)$ sii $f\in\mathcal{R}(\tilde{R})$.

▶ **C.Darboux**: f fitada; $f\in\mathcal{R}(R)$ sii $\forall\varepsilon>0, \exists P$ tq $S(f,P) - s(f,P)\leq\varepsilon$. Llavors $\int_R f = \lim_{k\to\infty} S(f,P_k) = \lim_{k\to\infty} s(f,P_k)$.

- si $v(R) = 0$, $\forall f$ fitada és integ i té $\int_R f = 0$.
- si $f(x) = c$ ct $\Rightarrow f\in\mathcal{R}(R)$ i $\int_R f = cv(R)$.
- $f\in\mathscr{C}(R) \Rightarrow$f$\in\mathcal{R}(R)$.

• funció de Dirichlet: $d_{\alpha,\beta}:\mathbb{R}^n\to\mathbb{R}$ tq $d_{\alpha,\beta} = \begin{cases}\alpha \text{ si } x\in\mathbb{Q}.\\ \beta \text{ si } x\notin\mathbb{Q}.\end{cases}$. (fitada no integ.)

• suma de Riemman associada a $f, P, \{\xi_{i_j}\}_{j=1}^n$: $R(f,P,\{\xi_{i_j}\}_j) = \sum_j f(\xi_{i_j})v(R)$.

- $\lim_{\delta(R)\to 0} R(f,P,\{\xi_{i_j}\}_j) = k\in\mathbb{R}$ si $\forall\varepsilon, \exists P$ amb $\delta(P)\leq\varepsilon$ tq $\forall P\subset P', |R(f,P,\{\xi_{i_j}\}_j) - k|\leq\varepsilon$.

▶ **C.Riemman d'integrabilitat**: $f\in\mathcal{R}(R)$ sii $\exists \lim_{\delta(R)\to 0} R(f,P,\{\xi_{i_j}\}_j)$. Llavors és $= \int_R f$.

- si $f\in\mathscr{C}(R) \Rightarrow \int_R f = \lim_{\delta(R)\to 0} R(f,P,\{\xi_{i_j}\}_j)$.

Criteri de Lebesgue

• oscil·l. de f en A: $w(f,A) = \sup_{x,y\in A}\{|f(x) - f(y)|\}$.

- $w(f,A) = 0$ sii f ct; $w(f,A)\in\mathbb{R}$ sii f fitada.

• oscil·l. de f en a: $w(f,a) = \lim_{r\to 0} w(f, A\cap B_{(a,r)}) = \inf_{r>0} w(f, A\cap B_{(a,r)})$.

- f fitada; $\forall\varepsilon, \{x\in A | w(f,x) < \varepsilon\}$ és ob de A.
- f fitada, A tancat; $\forall\varepsilon, \{x\in A | w(f,x)\geq\varepsilon\}$ és tancat de A.
- f fitada, R tancat; si $\exists\varepsilon$ tq $w(f,x) < \varepsilon\forall x\in\mathbb{R}$ $\Rightarrow\exists P$ tq $S(f,P) - s(f,P) < \varepsilon v(R)$.

• contingut nul: si $\forall\epsilon\exists$ subrecubriment finit de A per rectangles tq $\sum_{j=1}^{m} v(R_j)\leq\epsilon$.

• mesura nul·la: si $\forall\epsilon\exists$ subrecubriment numm de A per rectangles tq $\sum_{j=1}^{\infty} v(R_j)\leq\epsilon$.

▶ **C.Lebesgue**: $f: R\to\mathbb{R}$ fitada, R tancat; $f\in\mathcal{R}(R) \iff f\in\mathscr{C}(R)$ cs.

- A $cn \Rightarrow$A fitat i mn; A cpt i mn$\Rightarrow$A cn.
- A $cn\Rightarrow$ cn; A mn $\Rightarrow A^\circ = \varnothing$.
- A cn, B fitat$\Rightarrow$AxB, BxA cn.

- A $mn \Rightarrow$AxR^k, $\mathbb{R}^k$xA mn.
- $f, g : R \to \mathbb{R}, D = \{x \in \mathbb{R} | f(x) \neq g(x)\}$cn $\Rightarrow f \in \mathcal{R}(R)$ sii $g \in \mathcal{R}(R)$. Llavors $\int_R f = \int_R g$.
- funció Lipschitziana: si $\exists L \in (0,1]$ tq $|f(x) - f(y)| \leq L|x-y|, \ \forall x, y \in A$.
- f Lip. sii $f' < \infty \Rightarrow$f Lip. $\forall$ $\overline{subintd}$'A.
- si A, f Lip. en $A \Rightarrow f$ diferenciable cs en A.
- funció localm. Lip en Ω: si $\forall x \in \Omega, \exists B_x \subset \Omega$ bola ob tq f Lip en B_x.
- f loc.Lip. $\Rightarrow f \in \mathscr{C}(\Omega)$; $\in \mathscr{C}^1(\Omega) \Rightarrow$f loc.Lip.
- f loc.Lip.$(\Omega), K$ cpt $\subset \Omega \Rightarrow f$ Lip. en K.
- si $m \geq n, f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ Lip:
  i)si $B \subset Amn//cn \Rightarrow$f(B) mn//cn.
  ii)si $n < m \Rightarrow f(A) mn$. I si A fitat $\Rightarrow f(A) cn$.
- si $m \geq n, f : A \subset \mathbb{R}^n \to \mathbb{R}^m \mathscr{C}^1$:
  i)si A $mn \Rightarrow$f(A) mn.
  ii)si $\bar{B} \subset A$ i B $cn \Rightarrow$f(B) cn.
  iii)si $n < m \Rightarrow f(A)$ mn. I si $\bar{B} \subset A$ i B fitat $\Rightarrow f(B) cn$.

Fubini

► **T.Fubini**: $R \subset \mathbb{R}^k, \tilde{R} \subset \mathbb{R}^m, f \in \mathcal{R}(R\text{x}\tilde{R})$; si $\phi : R \to \mathbb{R}, \psi : \tilde{R} \to \mathbb{R}$ tq $\forall x \in R, y \in \tilde{R}$, $\underline{\int}_{\tilde{R}} f_x \leq \phi(s) \leq \overline{\int}_{\tilde{R}} f_x$ i $\underline{\int}_R f_y \leq \psi(y) \leq \overline{\int}_R f_y$ $\Rightarrow \phi \in \mathcal{R}(R), \psi \in \mathcal{R}(\tilde{R})$ i $\int_R \phi = \int_R \text{x} \tilde{R} f = \int_{\tilde{R}} \psi$.
- $A = \{x \in R | f_x \notin \mathcal{R}(\tilde{R})\}$ mn en $\mathbb{R}^k$.

Integració en conj. mJ

- $\chi_{A \cap B} = \chi_A \chi_B$.
- $\chi_{A \cup B} = \max\{\chi_A, \chi_B\} = \chi_A + \chi_B - \chi_{AB}$.
- conjunt mesurable Jordan: A fitat i ∂A mn.
- A mJ sii A fitat i $\forall R \supset A$ cpt, $\exists \int_R \chi_A$.
- A mJ; $v(A) = \int_R \chi_A, \ \forall R \supset A$ tancat.
- $cn \Rightarrow mJ$. - si A mJ ó cpt: cn sii mn.
- A $mJ \Rightarrow A^\circ, \bar{A}, \partial A$ mJ. I si $A^\circ \subset B \subset A$ $\Rightarrow B$ mJ.
- extenció de f fitada en A fitat: $f^*(x) = f(x)$ si $x \in A$ i 0 si $x \notin A$.
- $f^* \in \mathcal{R}(R)$ sii $f \in \mathcal{R}(A)$. Llavors són iguals.
- $f \in \mathcal{R}(R) \Rightarrow \Gamma(f)$ cn.

► **C.Lebesgue**: $f : A \to \mathbb{R}$ fitada, A mJ; $f \in \mathcal{R}(A) \iff f \in \mathscr{C}(A)$ cs.
- f fitada i $\mathscr{C}$ en A $mJ \Rightarrow$f$\in \mathcal{R}(A)$.
- $m_f v(A) \leq \int_A f \leq M_f v(A)$.
- conjunt elemental: $E = \{x \in A | \phi(x) \leq \psi(x)\}$ on A mJ i $\phi, \psi \in \mathscr{C}$.
- $\tilde{E} \setminus \bar{E} = \Gamma(\phi) \cup \Gamma(\psi)$ cn; $\Rightarrow \tilde{E}$ mJ sii $\bar{E}$ mJ; $\Rightarrow v(\tilde{E}) = v(\bar{E}) = \int_A \phi - \psi$.
- unió finita de rectangles és conj elem.

Integració impròpia

- exhaustió de $E \subset \mathbb{R}^{n+1}$: $\{E_k\} mJ$ tq $E_k \subset E_{k+1} \subset E$ i $E = \bigcup_{k=1}^{\infty} E_k$.
- E $mJ, \{E_k\}$ exh, $f \in \mathcal{R}(E) \Rightarrow f|_{E_k} \in \mathcal{R}(E_k)$ i $\int_E f = \lim_{k\to\infty} \int_{E_k} f$.
- f té integral impròpia en E: si $\exists \{E_k\}$ exh mJ tq $f|_{E_k} \in \mathcal{R}(E_k)$ i $\forall \{\tilde{E}_k\}$ exh, $\lim_{k\to\infty} \int_{E_k} f = \lim_{k\to\infty} \int_{\tilde{E}} f$.
- f loc fitada en A ob: si $\forall x \in A, \exists R_x$ rect no deg tq $x \in R_x$ i f fitada en R_x.
- f loc fitada en A ob: si $\forall x \in A, \exists R_x$ rect no deg tq $x \in R_x^\circ$ i $f|_{R_x} \in \mathcal{R}(R_x)$.
- loc.Int $\Rightarrow$loc.Fitada.
- $f : A_{ob} \to \mathbb{R}$ són equiv:
  i)f loc. Integ.
  ii)$f|_K \in \mathcal{R}(K), \forall K$ cpt mJ $\subset A$.
  iii)f loc. Fitada i $\mathscr{C}$ cs en A.
- $f : A_{ob} \to \mathbb{R}, \geq 0$ loc.Integ $\Rightarrow f$ té integ.Imp. en A.
- $f : A_{ob} \to \mathbb{R}$ loc.Integ; f té integ.Imp. conv sii $|f|$ té integ.Imp. conv; $\Rightarrow |\int_A f| \leq \int_A |f|$.

Canvi de Variable

- difeomorfisme $\mathscr{C}^1(\Omega)$: $F : \Omega_{ob} \subset \mathbb{R}^n \to \mathbb{R}^n$ tq inj i $\det D_F \neq 0$.
- $F : \Omega \to \mathbb{R}$ difeo $\mathscr{C}^1$, A mJ tq $\bar{A} \subset \Omega$ $\Rightarrow F(A)$ mJ i $f \in \mathcal{R}(F(A))$ sii $(f \circ F)|\det D_F| \in \mathcal{R}(A)$.

► **T.Canvi Variable**: $F : \Omega_{ob} \subset \mathbb{R}^n \to \mathbb{R}^n$ difeo $\Rightarrow \forall A$ mJ tq $\bar{A} \subset \Omega$ i $\forall f \in \mathcal{R}(F(A))$, $\int_{F(A)} f = \int_A (f \circ F)|\det D_F| \in \mathcal{R}(A)$.

Tema 3: Integració en línia i superf.

Corbes

- camí/corba: $\alpha : I \subset \mathbb{R} \to \mathbb{R}^n \mathscr{C}$ tq $\alpha(t) = (\alpha_1(t), ..., \alpha_n(t))$.
- suport/traça: C_α, l'imatge d'α.
- si α inj $\Rightarrow C_\alpha \equiv$ corba.
- $\alpha : I \to \mathbb{R}^2, \alpha(t) = (\alpha_1(t), \alpha_2(t))$ $\Rightarrow \bar{\alpha} : I \to \mathbb{R}^3, \alpha(t) = (\alpha_1(t), \alpha_2(t), 0)$.
- corba tancada: si $\alpha(a) = \alpha(b)$.
- corba simple: si tancada i inj en $[a, b)$.
- corbes equiv.: $\alpha : I \to \mathbb{R}^n, \beta : J \to \mathbb{R}^n$ si $\exists \varphi : J \to I$ homeo tq $\beta = \alpha \circ \varphi$.
- α inj sii $\forall$ corba equiv és inj.
- si α, β equiv $\Rightarrow C_\alpha = C_\beta$.
- composició: $\alpha : [a, b] \to \mathbb{R}^n, \beta : [b, c] \to \mathbb{R}^n, \alpha(b) = \beta(b)$; $\alpha * \beta : [a, c] \to \mathbb{R}^n$ tq $\alpha * \beta = \alpha(t)$ si $t \in [a, b]$ i $\alpha * \beta = \beta(t)$ si $t \in (b, c]$.
- $\alpha, \beta \in \mathscr{C} \Rightarrow \alpha * \beta \in \mathscr{C}$; - $C_{\alpha*\beta} = C_\alpha \cup C_\beta$.
- si α, β inj i $C_\alpha \cap C_\beta = \{\alpha(b)\} \Rightarrow \alpha * \beta$ inj.
- $\alpha : I \to \mathbb{R}^n$; si $\alpha_j \in \mathscr{C}^{k(I)} \forall j \Rightarrow \alpha \in \mathscr{C}^k$ $\Rightarrow \alpha' \mathscr{C}^{k-1}(I)$ i $\alpha(t)' = (\alpha_1'(t), ., \alpha_n'(t))$ tg d'α.
- el tg d'α pot canviar de sgn segons param.
- $\alpha : I \to \mathbb{R}^n$; si $\alpha_j \in \mathscr{C}_s^{k(I)} \forall j \Rightarrow \alpha \in \mathscr{C}_s^{k(I)}$ $\Rightarrow \alpha \in \mathscr{C}$ i α' def en $I \setminus$ nº finit de pts.
- α regular si $\forall$ pt és pt regular: si $\alpha'(t) \neq 0$ (iParam).
- $\alpha_j \in \mathcal{R}[a, b], \forall j \Rightarrow \alpha \in \mathcal{R}[a, b]$ $\Rightarrow \int_a^b \alpha = (\int_a^b \alpha_1, ., \int_a^b \alpha_n)$.
- $\alpha \in \mathscr{C}_s^k \Rightarrow \int_{\tilde{a}}^{\tilde{b}} \alpha' = \alpha(\tilde{a}) - \alpha(\tilde{b}), \forall \tilde{a}, \tilde{b} \in [a, b]$.
- $\alpha \in \mathcal{R}[a, b] \Rightarrow |\alpha| \in \mathcal{R}[a, b]$ i $|\int_a^b \alpha| \leq \int_a^b |\alpha|$.

Longitud de corbes

- long. polig.: $l(\alpha, P) = \sum_{j=1}^{m} |\alpha(t_j) - \alpha(t_{j-1})|$.
- $P \subset P' \Rightarrow$l$(\alpha, P) \leq l(\alpha, P')$.
- long. d'α. $= l(\alpha) = \sup_P \{l(\alpha, P)\}$ (iParam).
- corba rectificable: si $l(\alpha)$ finita.
- $l(\alpha_j) \leq l(\alpha) \leq l(\alpha_1) + ... + l(\alpha_n)$.
- $\mathscr{C} \not\Rightarrow$ rectif.
- $\mathscr{C}^1$ ó $\mathscr{C}_s^1 \Rightarrow \alpha$ Lip $\Rightarrow \alpha$ rectif. i $l(\alpha) \leq L(b-a)$.
- si $\alpha \in \mathscr{C}_s^1[a, b] \Rightarrow l(\alpha) = \int_a^b |\alpha'(t)| dt$.

Integració en línia

- camp escalar en Ω: $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}, \mathscr{C}^k$.
- camp vectorial en Ω: $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n, \mathscr{C}^k$.
- si $\alpha \in \mathscr{C}_x^1[a, b], C_\alpha \subset \Omega, f : \Omega \to \mathbb{R}$ $\mathscr{C}^k$, $\int_\alpha f dl = \int_a^b f(\alpha(t))|\alpha'(t)| dt$: int. línia de f.
- $|\int_\alpha f dl| \leq \int_\alpha |f| dl \leq l(\alpha) \max_{x \in C_\alpha} \{f(x)\}$.
- si $\alpha \in \mathscr{C}_x^1[a, b], C_\alpha \subset \Omega, F : \Omega \to \mathbb{R}$ $\mathscr{C}^k$, $\int_\alpha F dl = \int_a^b < F(\alpha(t)), \alpha'(t) > dt$: int. línia/circulació de F (q el sgn dParam).
- $\int_\alpha f dl = \int_\alpha < f, t > dl = \int_\alpha f_t dl$.

Integració en superfície

- superf. regular σ: si D_σ té rang 2.
- àrea de S $a(S) = \int_\Omega |\sigma_u \text{x} \sigma_v| dudv$.
- $* = \{\Omega$ $mJ, \bar{\Omega}$ cpt$, \sigma \in \mathscr{C}^k(\bar{\Omega})$i inj$\}$; si $f : S \subset \bar{\Omega} \to \mathbb{R}, \mathscr{C}$, la int. de superf de f és $\int_S f dS = \int_\Omega f(\sigma(u, v))|\sigma(u) \text{x} \sigma(v)| dudv$.
- $|\int_S f dS| \leq \int_S |f| dS \leq a(S) \max_{x \in \bar{S}} \{f(x)\}$.
- $*$; si $F : S \to \mathbb{R}^3, \mathscr{C}, S = \sigma(\Omega)$ orientada, la int. de superf de F (q el sgn dParam) és $\int_S F dS = \int_\Omega < F(\sigma(u, v)), \sigma(u) \text{x} \sigma(v) > dudv$.
- $\int_S f dS = \int_S < f, n > dS = \int_S f_n$: flux de f//F.

Tema 4: Teoremes integrals

$u : \Omega \subset \mathbb{R}^n \to \mathbb{R}, f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$.

Camps i pot. escalars i vectorials

- gradient: $\nabla u = (\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n})$.
- rotacional: $rot(f) = \nabla$xf.
- divergència: $div(f) = \frac{\partial f}{\partial x_1} + ... + \frac{\partial f}{\partial x_n}$.
- laplacià: $div(\nabla f)$.
- camp gradient: f si $\exists u \in \mathscr{C}^1$pot. escalar: $f = \nabla u$.
- camp irrotacional: f si $rot(f) = 0$.
- camp solenoidal: f si $div(f) = 0$.
- pot. vector: g si $\exists f \in \mathscr{C}^1$ tq $rot(g) = f$.

► **R.Leibnitz**: $\nabla uv = u\nabla v + v\nabla u$ i $div(uf) = udiv(f) + < \nabla u, f >$.
- camp conservatiu: f si $f \in \mathscr{C}$ i $\forall \alpha \in \mathscr{C}_s^1[a, b]$ tq $C_\alpha \subset \Omega, \int_{C_\alpha} f dl = u(\alpha(b)) - u(\alpha(a))$.
- f conservatiu sii $\oint f dl = 0$ sii f gradient.
- f conservatiu i $f \in \mathscr{C}^1 \Rightarrow \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}, \forall i, j$.
- $f \in \mathscr{C}^1(\Omega, \mathbb{R}^2) \Rightarrow f = (f_1, f_2, 0)$, $rot(f) = (0, 0, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y})$.
- f gradient $\Rightarrow f$ irrot $(rot(\nabla) = 0)$.
- f rotacional $\Rightarrow f$ solenoidal $(div(rot) = 0)$.
- obert estrellat Ω: si $\exists x$ tq $(1-t)x_0 + tx \in \Omega$, $\forall x \in \Omega, t \in [0, 1]$.

► **L.Poincaré**: Ω ob estrellat $\subset \mathbb{R}^3, f \in \mathscr{C}^1$; si f solen. $\Rightarrow f$ rotacional i si f irrot $\Rightarrow f$ conservatiu.
- f irrot en ob estrellat $\Rightarrow u(x) = \int_0^1 < f(t\mathbf{x} + (1-t)x_0, r(\mathbf{x} - x_0) > dt$.
- f solen en ob estrellat $\Rightarrow g(x, y, z) = \int_0^1 (tf(t\mathbf{x} + (1-t)x_0 \text{ x } r(\mathbf{x} - x_0)) dt$.

Green, Stoke, Gauss

- corba de Jordan: traça corba param simple.

► **T.Corba Jordan**: si α corba Jordan $\Rightarrow \mathbb{R}^2 \setminus C_\alpha = \Omega_1 \cup \Omega_2$ on Ω_1 acotat i Ω_2 no.

► **T.Green**: $\Omega \subset \mathbb{R}^2$ domini elem, $f : \bar{\Omega} \to \mathbb{R}^2 \mathscr{C}^1$

$\Rightarrow \int_{\partial\Omega} fdl = \int_\Omega \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} dxdy = \int_\Omega rot(f)dS$.

- Green sii $\{\oint_{\partial\Omega} f_1 dl = -\int_\Omega \frac{\partial f_1}{\partial y} dxdy$ i $\oint_{\partial\Omega} f_2 dl = \int_\Omega \frac{\partial f_2}{\partial x} dS\}$.

► **T.Stokes**: $(S \cup \partial S) \subset \Omega \subset \mathbb{R}^3$ domini elem, $f : \bar{\Omega} \to \mathbb{R}^3 \mathscr{C}^1 \Rightarrow \oint_{\partial S} fdl = \int_S rot(f)dS$.

► **T.Gauss**: $V \subset \Omega \subset \mathbb{R}^3$ domini elem, $f : \bar{\Omega} \to \mathbb{R}^3 \mathscr{C}^1 \Rightarrow \oint_{\partial V} fdS = \int_V div(f)dV$.

- Gauss sii $\{\int_{\partial\Omega} f_i dl = \int_\Omega \frac{\partial f_i}{\partial x_i} dV, \forall i$.

- $vol(V) = \frac{1}{3}\int_{\partial V} fdS = \frac{1}{3}\int_{\partial V}(x,0,0)dS$.

- $a(S) = \frac{1}{2}\int_{\partial V}(-y,x)dS$.

Tema 5: Formes diferencials

• forma d'ordre k en Ω: $w : \Omega \to \bigwedge^k(\Omega)$ tq $w = \sum_{1\le i_1 < \ldots < i_k \le n} \alpha_{i_1,\ldots,i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$.

• producte exterior: $(w_1 \wedge w_2)(x) = w_1(x) \wedge w_2(x)$ (p+q)-forma $\mathscr{C}^m$ bilineal, anticomm i associativa.

• pull-back de F: $F^*(u) = (D_f(u))^*$ tq $F^*(fdx^{i_1} \wedge \ldots \wedge dx^{i_k}) = F^*(f)F^*(dx^{i_1} \wedge \ldots \wedge dx^{i_k})$.

• diferencial exterior: de w k-forma $\mathscr{C}^p$, és $d(w) = \sum_{1\le i_1<\ldots<i_k\le n} d\alpha_{i_1,\ldots,i_k} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ (k+1)-forma $\mathscr{C}^{p-1}$.

- $dy^1 \wedge \ldots \wedge dy^n = (detD_\psi)dx^1 \wedge \ldots \wedge dx^n$.

- $d(w \wedge \tilde{w}) = d(x) \wedge \tilde{w} + (-1)^k w \wedge d(\tilde{w})$.

- $d^2 = d \circ d = 0$.

- $F^* \circ d = d \circ F^* \Rightarrow F^*(dw) = d(F^*(w))$.

• forma tancada: si $dw = 0$.

• forma exacta: si $w = d\tilde{w}$.

- exacta $\Rightarrow$tancada.

► **L.Poincaré**: $\Omega \subset \mathbb{R}^n$ ob estrellat, w k-forma $\mathscr{C}^1(\Omega) \Rightarrow w = K(dw) + d(K(w))$.

• integral n-forma: $w = fdx^1 \wedge \ldots \wedge dx^n$ $\Rightarrow \int_\Omega w = \int_\Omega fdx^1 \wedge \ldots \wedge dx^n = \int_\Omega fdx^1 \ldots dx^n = \int_\Omega f$.

- $\int_\Omega w = \pm \int_{\tilde{\Omega}} F^*(w)$.

- $\sigma : \tilde{\Omega} \to \Omega, \mathscr{C}^1 \Rightarrow \int_\sigma w = \int_{\tilde{\Omega}} \sigma^*(w)$.

► **T.Stokes**: $\sigma(\tilde{\Omega}) = M \cup \partial M$, w k-forma $\mathscr{C}^1$ $\Rightarrow \int_{\partial M} w = \int_M dw$ (cal ∂M sigui corba tancada).

Altres

Taylor

$e^x = \sum_{n\ge0} \frac{x^n}{n!}$.

$\cos x = \sum_{n\ge0} (-1)^n \frac{x^{2n}}{(2n)!}$.

$\sin x = \sum_{n\ge0} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$.

$\log(1+x) = \sum_{n\ge1} (-1)^{n+1} \frac{x^n}{n}$.

$(1+x)^p = \sum_{n\ge0} \binom{p}{n} x^n$.

$(1+x)^{-1} = \sum_{n\ge0} (-1)^n x^n$.

$\cosh x = \sum_{n\ge0} \frac{x^{2n}}{(2n)!}$.

$\sinh x = \sum_{n\ge0} \frac{x^{2n+1}}{(2n+1)!}$.

$\arctan x = \sum_{n\ge0} (-1)^n \frac{x^{2n+1}}{2n+1}$.

Trigonometria

$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$.

$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$.

$\tan(a \pm b) = \frac{\tan(a)\pm\tan(b)}{1\mp\tan(a)\tan(b)}$

$\sin(a) + \sin(b) = 2\sin(\frac{a+b}{2})\cos(\frac{a-b}{2})$.

$\cos(a) + \cos(b) = 2\cos(\frac{a+b}{2})\cos(\frac{a-b}{2})$.

$2\cos(a)\cos(b) = \cos(a-b) + \cos(a+b)$

$2\sin(a)\sin(b) = \cos(a-b) - \cos(a+b)$

$2\sin(a)\cos(b) = \cos(a+b) + \cos(a-b)$

$2\cos(a)\sin(b) = \cos(a+b) - \cos(a-b)$

$\cos^2(a) = \frac{1+\cos(2a)}{2}$

$\sin^2(a) = \frac{1-\cos(2a)}{2}$

$\sin(\pi/2 - x) = \cos(x)$

$\cos(\pi/2 - x) = \sin(x)$