Tema 1a: Sèries

- (a_n) monòtona $\implies \exists \lim_{n\to\infty}(a_n) \in \mathbb{R}\cup\{\pm\infty\}$
- (a_n) fitada $\implies \exists$ subsucc conv.
- (a_n) no fitada $\implies \exists$ subsucc div.
- (a_n) conv $\iff$ (a_n) succ de Cauchy.

• infinitèssim: (a_n) tq $\lim_(a_n) = 0$.

- si (a_n) infinitèssim $\implies \log(1+a_n), sin(a_n), 1-cos(a_n), e^{a_n}-1, tg(a_n), |a_n|^\alpha$ on $\alpha>0$ tmb.
- si (a_n) infinitèssim $\implies \log(1+a_n)\sim a_n$, $sin(a_n)\sim a_n$, $1-cos(a_n)\sim\frac{(a_n)^2}{2}$, $e^{a_n}-1\sim a_n$, $tg(a_n)\sim a_n$.
- $lg(lg(n)) \prec lg(n) \prec n^c \prec n \prec n^a \prec n^{lg(n)} \prec b^n \prec n! \prec n^n \prec b^{n^a}$ on $b>1, 0<c<1<a$.
- $\sum a_n$ conv $\implies \lim_{n\to\infty} a_n = 0$.

• sèries telesòpiques: $\sum a_n$ tq $a_n = b_n - b_{n+1} \forall n\in\mathbb{N} \implies S_n = b_0 - b_{n+1}$.

• sèries aritmetico-geomètriques: $\sum a_n$ tq $a_n = (dn+s)r^n$ on $d,r,s\in\mathbb{R}, |d|+|s|>0$

- $\lim_{n\to\infty} |dn+s||r|^n) = 0 \iff |r|<1$
$\implies \sum(dn+s)r^n = \frac{s}{1-r}+\frac{dr}{(1-r)^2}$.

Criteris

▶ **Dirichlet**: $\sum_{n=1}^{\infty} a_n b_n$ conv si:

i)$s_n = \sum_{k=1}^{n} b_k$ fitada

ii)(a_n) monòtona i $\lim(a_n) = 0$

▶ **Leibnitz**: $\sum_{n=0}^{\infty}(-1)^n a_n$ conv si (a_n) monòtona decreixent i $\lim(a_n) = 0$

Criteris per a sèries positius

▶ **Comparació**: $\exists n_0$ tq $\forall n\geq n_0, a_n\leq b_n$

i)si $\sum b_n<\infty \implies \sum a_n<\infty$

ii)si $\sum a_n=\infty \implies \sum b_n=\infty$

▶ **Comparació al límit**:
$\forall n\geq n_0, a_n\leq b_n, \exists\lim\frac{a_n}{b_n} = l\in[0,\infty]$

i)si $l\in(0,\infty) \implies (a_n),(b_n)$ mateix caràcter

ii) + iii) si $l=0$ ó $l=\infty \implies C.Comparaci$

▶ **Quocient**: si $\exists\lim\frac{a_{n+1}}{a_n} = \alpha\in[0,\infty]$
$\implies si\alpha>1$ div i si $\alpha<1$ conv.

▶ **Raabe**: si $\exists\lim n(1-\frac{a_{n+1}}{a_n}) = l\in[0,\infty]$
$\implies sil<1divisil>1conv.$

▶ **Logarítmic**: si $\exists\lim\frac{log(\frac{1}{a_n})}{log(n)} = l\in[0,\infty]$
$\implies sil<1divisil>1conv.$

▶ **Condensació**: si a_n decreix. i $\lim a_n = 0$
$\implies \sum a_n$ i $\sum 2^n a_{2^n}$ mateix caràcter.
-$\frac{1}{2}\sum 2^n a_{2^n} \leq \sum a_n \leq \sum 2^n a_{2^n}$.

▶ **Arrel**: si $\exists\lim(a_n)^{\frac{1}{n}} = \alpha\in[0,\infty]$
$\implies si\alpha>1$ div i si $\alpha<1$ conv.

▶ **Integral**: si $f:[m_{\geq 0},\infty)\to\mathbb{R}$ tq $f\geq 0, \lim f(x)=0$ i f decreix. Llavors:

i)+ii)si $\lim_{n\to\infty}\int_m^n f <$ ó $=\infty \implies \sum_{n=m}^{\infty} f(n)$ c ó d

- $\lim_{n\to\infty}\int_m^n f \leq \sum_{n=m}^{\infty} f(n) \leq f(m) + \lim_{n\to\infty}\int_m^n f$
- si F primitiva de f. Llavors:

i)+ii)si $\lim F(n) <$ ó $=\infty \implies \sum_{n=m}^{\infty} f(n)$ c ó d

- $\lim F(n) - F(k) + \sum_{n=m}^{k-1} f(n) \leq \sum_{n=m}^{\infty} f(n) \leq \lim F(n) - F(k) + \sum_{n=m}^{k} f(n)$

• sèrie harmònica: $\sum_{n\geq 1}\frac{1}{n^\alpha}, \alpha\in\mathbb{R}$; c sii $\alpha>1$ i d sii $\alpha\leq 1$.

Tema 1b: Integrals impròpies

• funció localm. integ: $f\in\mathcal{L}_{loc}(I)$ si $f:I\to\mathbb{R}\in\mathcal{R}(J)\forall J\in I$ interval tancat i fitat.

- $f\mathscr{C}$ ó monòtona $\implies$ f$\in\mathcal{L}_{loc}(I)$.

• integral impròpia de $f:[a,b)\to\mathbb{R}\in\mathcal{L}_{loc}(I)$ és $\int_a^b f = \lim_{x\to b^-}\int_a^x f$

i)de 1a espècie si $b=\infty$ i f fitada

ii)de 2a espècie si $b<\infty$ i f no fitada

iii)de 3a espècie si $b=\infty$ i f no fitada

- $f\mathcal{L}_{loc}(I), \int_{-\infty}^{\infty} f$ c $\implies \exists\lim_{R\to\infty}\int_{-R}^{R} f = \int_{-\infty}^{\infty} f$.
- si $f\in\mathcal{L}_{loc}(I), \int_a^\infty f$ conv i $\exists\lim_{x\to\infty} f(x) \implies \lim_{x\to\infty} f(x) = 0$.

▶ **C.Cauchy**: $\int_a^b f$ conv $\iff \forall\varepsilon>0, \exists a\leq c_0<b$ tq $|\int_c^{\tilde{c}} f|\leq\varepsilon, \forall c,\tilde{c}\in[c_0,b)$.

• integral imp. abs. conv: si $|f|$ té integ imp conv.

- f abs conv $\implies fconv.$
- Comparació, Comparació al límit i Dirichlet.

Funcions Gamma i Beta

• Gamma: $\Gamma:(0,\infty)\to\mathbb{R}$ tq $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt>0$

- Γ conv $\forall\alpha>-1$ - $\Gamma(x+1) = x\Gamma(x)$
- $\Gamma(n+1) = n!$ - $\Gamma(n+1) = n!$
- $\Gamma(n+\frac{1}{2}) = \frac{\sqrt{\pi}(2n)!}{n!2^{2n}}$ - $\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-s^2} = \sqrt{\pi}$

• Beta: $B:(0,\infty)^2\to\mathbb{R}$ tq $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$, funció simètrica

- $B(x,y) = 2\int_0^\infty(sin(\theta))^{2x-1}(cos(\theta))^{2y-1}d\theta$
- $B(x,y) = \int_0^\infty\frac{s^{x-1}}{(1+s)^{x+y}} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

Tema 2: Integració a $\mathbb{R}^n$

Intervals, Rectangles i Particions

• interval: conv connex $\in\mathbb{R}$.

• longitud: $l(I) = b-a$.

- $l(I) = l(I^o) = l(\tilde{(I)})$.

• rectangle: $R = I_1$x...xI_n on I_j interval $\forall j$. És degenerat si $\exists j\in\{1,...,n\}$ tq $l(I_j)=0$.

• volum: $v(R) = l(I_1)$x...x$l(I_n)$.

• diàmetre: $\delta(R) = \sqrt{l(I_1)^2\text{x...x}l(I_n)^2}$.

- $\max_j\{l(I_j)\}\leq\delta(R)\leq\sqrt{n}\max_j\{l(I_j)\}$
$\implies$ v(R) $\leq\delta(R^n)$.
- $v(Q) = l^n \implies \delta(Q) = l\sqrt{n}$.

• partició: $P=\{x_0,...,x_n\}$ de $[a,b]$ tq $a=x_0<...<x_n=b$. Un subinterval és $I_i=[x_i,x_{i+1}]$. Obs $I_i^\circ\cap I_j^\circ=\varnothing,\forall i\neq j$
$\implies$ [a,b] $= \bigcup_{i=0}^{k-1} I_i \implies$ b-a $= \sum_{i=0}^{k-1} l(I_i)$.

• diàmetre: $\delta(P) = \max_j\{l(I_j)\}$.

• partició més fina: P' que P si $P\subset P'$.

- si $P\subset P' \implies \delta(P')\leq\delta(P)$.
- $\forall P,P'$ particions de $[a,b]$, $\exists\tilde{P}$ partició de $[a,b]$ tq $P,P'\subset\tilde{P}$.

• diàmetre: $\delta(P) = \sqrt{\delta(P_1)^2\text{x...x}\delta(P_n)^2}$.

- $\max_j\{\delta(P_j)\}\leq\delta(P)\leq\sqrt{n}\max_j\{\delta(P_j)\}$
$\implies$ v(R) $\leq\delta(P)^n$.
- $\forall\varepsilon,\exists P$ tq $\delta(P)\leq\varepsilon$.

Integració de funcions fitades

$R=[a_1,b_1]$x...x$[a_n,b_n]$, $f:R\to\mathbb{R}$ fitada.

• $M_{ij} = \sup_{x\in R_{ij}}\{f(x)\}$; $m_{ij} = \inf_{x\in R_{ij}}\{f(x)\}$.

• suma superior: $S(f,P) = \sum_R M_R v(R)$.

• suma inferior: $s(f,P) = \sum_R m_R v(R)$.

- si $P\subset P'$, $s(f,P)\leq s(f,P')\leq S(f,P')\leq S(f,P)$.

• integral superior: $\bar{\int}_R f = \inf_P S(f,P)$.

• integral inferior: $\underline{\int}_R f = \sup_P s(f,P)$.

- $m_R v(R)\leq s(f,P)\leq\underline{\int}_R f\leq\bar{\int}_R f\leq S(f,P)\leq M_R v(R)$.

• integrable Riemman: si $\underline{\int}_R f = \bar{\int}_R f$.

- $\int_{\bar{R}} f = \int_{\partial R} f + \int_{R^\circ} f$; - $\int_{\partial R} f = 0$.
- $\forall\tilde{R}$ tq $R^\circ\subset\tilde{R}\subset R$; $f\in\mathcal{R}(R)$ sii $f\in\mathcal{R}(\tilde{R})$.

▶ **C.Darboux**: f fitada; $f\in\mathcal{R}(R)$ sii $\forall\varepsilon>0,\exists P$ tq $S(f,P)-s(f,P)\leq\varepsilon$. Llavors $\int_R f = \lim_{k\to\infty} S(f,P_k) = \lim_{k\to\infty} s(f,P_k)$.

- si $v(R)=0$, $\forall f$ fitada és integ i té $\int_R f=0$.
- si $f(x)=c$ ct $\implies$ f$\in\mathcal{R}(R)$ i $\int_R f = cv(R)$.
- $f\in\mathscr{C}(R) \implies f\in\mathcal{R}(R)$.

• funció de Dirichlet: $d_{\alpha,\beta}:\mathbb{R}^n\to\mathbb{R}$ tq $d_{\alpha,\beta} = \begin{cases}\alpha \text{ si } x\in\mathbb{Q}.\\ \beta \text{ si } x\notin\mathbb{Q}.\end{cases}$. (fitada no integ.)

• suma de Riemman associada a $f,P,\{\xi_{i_j}\}_{j=1}^n$: $R(f,P,\{\xi_{i_j}\}_j) = \sum_j f(\xi_{i_j})v(R)$.

- $\lim_{\delta(R)\to 0} R(f,P,\{\xi_{i_j}\}_j) = k\in\mathbb{R}$ si $\forall\varepsilon,\exists P$ amb $\delta(P)\leq\varepsilon$ tq $\forall P\subset P', |R(f,P,\{\xi_{i_j}\}_j)-k|\leq\varepsilon$.

▶ **C.Riemman d'integrabilitat**: $f\in\mathcal{R}(R)$ sii $\exists\lim_{\delta(R)\to 0} R(f,P,\{\xi_{i_j}\}_j)$. Llavors és $=\int_R f$.

- si $f\in\mathscr{C}(R)$
$\implies \int_R f = \lim_{\delta(R)\to 0} R(f,P,\{\xi_{i_j}\}_j)$.

Criteri de Lebesgue

• oscil·l. de f en A: $w(f,A) = \sup_{x,y\in A}\{|f(x)-f(y)|\}$.

- $w(f,A)=0$ sii f ct; $w(f,A)\in\mathbb{R}$ sii f fitada.

• oscil·l. de f en a: $w(f,a) = \lim_{r\to 0} w(f,A\cap B_{(a,r)}) = \inf_{r>0} w(f,A\cap B_{(a,r)})$.

- f fitada; $\forall\varepsilon,\{x\in A|w(f,x)<\varepsilon\}$ és ob de A.
- f fitada, A tancat; $\forall\varepsilon,\{x\in A|w(f,x)\geq\varepsilon\}$ és tancat de A.
- f fitada, R tancat; si $\exists\varepsilon$ tq $w(f,x)<\varepsilon\forall x\in\mathbb{R}$
$\implies\exists P$ tq $S(f,P)-s(f,P)<\varepsilon v(R)$.

• contingut nul: si $\forall\epsilon\exists$ subrecubriment finit de A per rectangles tq $\sum_{j=1}^{m} v(R_j)\leq\epsilon$.

• mesura nul·la: si $\forall\epsilon\exists$ subrecubriment numm de A per rectangles tq $\sum_{j=1}^{\infty} v(R_j) \le \epsilon$.

▶ **C.Lebesgue**: $f : R \to \mathbb{R}$ fitada, R tancat; $f \in \mathcal{R}(R) \iff f \in \mathscr{C}(R)$ *cs*.

- A *cn* $\implies$ A fitat i *mn*;
A cpt i *mn* $\implies$ A *cn*.
- A *cn* $\implies$ cn; A mn $\implies$ $A^\circ = \varnothing$.
- A *cn*, B fitat $\implies$ AxB, BxA*cn*.
- A *mn* $\implies$ Ax$\mathbb{R}^k$, $\mathbb{R}^k$xA mn.
- $f, g : R \to \mathbb{R}, D = \{x \in \mathbb{R} | f(x) \neq g(x)\}$cn $\implies$ f$\in \mathcal{R}(R)$ sii $g \in \mathcal{R}(R)$. Llavors $\int_R f = \int_R g$.

• funció Lipschitziana: si $\exists L \in (0,1]$ tq $|f(x) - f(y)| \le L|x-y|$, $\forall x, y \in A$.

- f Lip. sii $f' < \infty \implies f Lip.\forall\, \overline{subintd'}$A.
- si A, f Lip. en $A \implies$ f$diferenciablecsen$A.

• funció localm. Lip en Ω: si $\forall x \in \Omega, \exists B_x \subset \Omega$ bola ob tq f Lip en B_x.

- f loc.Lip. $\implies$ f$\in \mathscr{C}(\Omega)$; $\in \mathscr{C}^1(\Omega)$ $\implies$ $f loc.Lip.$
- f loc.Lip.$(\Omega), K$ cpt $\subset \Omega$ $\implies$ f$Lip.en$K.
- si $m \ge n, f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ Lip:
  i)si $B \subset Amn//cn \implies f(B)mn//cn.$
  ii)si $n < m$ $\implies$ f(A) mn.$IsiA fitat$ $\implies$ f(A) cn.
- si $m \ge n, f : A \subset \mathbb{R}^n \to \mathbb{R}^m \mathscr{C}^1$:
  i)si A *mn* $\implies$ $f(A)mn.$
  ii)si $\bar{B} \subset A$ i B *cn* $\implies$ $f(B)cn.$
  iii)si $n < m$ $\implies$ f(A) mn.$Isi$$\subset A$ i B fitat $\implies$ f(B) cn.

Fubini

▶ **T.Fubini**: $R \subset \mathbb{R}^k, \tilde{R} \subset \mathbb{R}^m, f \in \mathcal{R}(R\text{x}\tilde{R})$; si $\phi : R \to \mathbb{R}, \psi : \tilde{R} \to \mathbb{R}$ tq $\forall x \in R, y \in \tilde{R}$, $\underline{\int}_{\tilde{R}} f_x \le \phi(s) \le \overline{\int}_{\tilde{R}} f_x$ i $\underline{\int}_R f_y \le \psi(y) \le \overline{\int}_R f_y$ $\implies \phi \in \mathcal{R}(R), \psi \in \mathcal{R}(\tilde{R})$ i $\int_R \phi = \int_{R\text{x}\tilde{R}} f = \int_{\tilde{R}} \psi$.

- $A = \{x \in R | f_x \notin \mathcal{R}(\tilde{R})\}$ mn en $\mathbb{R}^k$.

Integració en conj. mJ

- $\chi_{A \cap B} = \chi_A \chi_B$.
- $\chi_{A \cup B} = \max\{\chi_A, \chi_B\} = \chi_A + \chi_B - \chi_{AB}$.

• conjunt mesurable Jordan: A fitat i ∂A *mn*.

- A *mJ* sii A fitat i $\forall R \supset A$ cpt, $\exists \int_R \chi_A$.
- A *mJ*; $v(A) = \int_R \chi_A$, $\forall R \supset A$ tancat.
- *cn* $\implies$ mJ. -*si*A*mJcpt* :cn*sii*mn.
- A *mJ* $\implies$ $A^\circ, \bar{A}, \partial A$ *mJ*. I si $A^\circ \subset B \subset A$ $\implies$ B mJ.

• extensió de f fitada en A fitat: $f^*(x) = f(x)$ si $x \in A$ i 0 si $x \notin A$.

- $f^* \in \mathcal{R}(R)$ sii $f \in \mathcal{R}(A)$. Llavors són iguals.
- $f \in \mathcal{R}(R) \implies \Gamma(f)$ cn.

▶ **C.Lebesgue**: $f : A \to \mathbb{R}$ fitada, A *mJ*; $f \in \mathcal{R}(A) \iff f \in \mathscr{C}(A)$ *cs*.

- f fitada i $\mathscr{C}$ en A *mJ* $\implies$ $f \in \mathcal{R}(A)$.
- $m_f v(A) \le \int_A f \le M_f v(A)$.

• conjunt elemental: $E = \{x \in A | \phi(x) \le \psi(x)\}$ on A *mJ* i $\phi, \psi \in \mathscr{C}$.

- $\tilde{E} \setminus \bar{E} = \Gamma(\phi) \cup \Gamma(\psi)$ *cn*; $\implies$ mJ*sii* mJ; $\implies$ v() = v() = $\int_A \phi - \psi$.
- unió finita de rectangles és conj elem.

Integració impròpia

• exhaustió de $E \subset \mathbb{R}^{n+1}$: $\{E_k\} mJ$ tq $E_k \subset E_{k+1} \subset E$ i $E = \bigcup_{k=1}^{\infty} E_k$.

- E *mJ*, $\{E_k\}$ exh, $f \in \mathcal{R}(E) \implies$ f$|_{E_k} \in \mathcal{R}(E_k)$ i $\int_E f = \lim_{k\to\infty} \int_{E_k} f$.

• f té integral impròpia en E: si $\exists\{E_k\}$ exh *mJ* tq $f|_{E_k} \in \mathcal{R}(E_k)$ i $\forall\{\tilde{E}_k\}$ exh, $\lim_{k\to\infty} \int_{E_k} f = \lim_{k\to\infty} \int_{\tilde{E}} f$.

• f loc fitada en A ob: si $\forall x \in A, \exists R_x$ rect no deg tq $x \in R_x$ i f fitada en R_x.

• f loc fitada en A ob: si $\forall x \in A, \exists R_x$ rect no deg tq $x \in R_x^\circ$ i $f|_{R_x} \in \mathcal{R}(R_x)$.

- loc.Int $\implies$ *loc.Fitada.*
- $f : A_{ob} \to \mathbb{R}$ són equiv:
  i)f loc. Integ.
  ii)$f|_K \in \mathcal{R}(K), \forall K$ cpt mJ $\subset A$.
  iii)f loc. Fitada i $\mathscr{C}$ *cs* en A.
- $f : A_{ob} \to \mathbb{R}, \ge 0$ loc.Integ $\implies$ f*tinteg.Imp.en*A.
- $f : A_{ob} \to \mathbb{R}$ loc.Integ; f té integ.Imp. conv sii $|f|$ té integ.Imp. conv; $\implies |\int_A f| \le \int_A |f|$.

Canvi de Variable

• difeomorfisme $\mathscr{C}^1(\Omega)$: $F : \Omega_{ob} \subset \mathbb{R}^n \to \mathbb{R}^n$ tq inj i det $D_F \neq 0$.

- $F : \Omega \to \mathbb{R}$ difeo $\mathscr{C}^1$, A *mJ* tq $\bar{A} \subset \Omega$ $\implies$ F(A) mJ*i*f$\in \mathcal{R}(F(A))$ sii $(f \circ F)|\det D_F| \in \mathcal{R}(A)$.

▶ **T.Canvi Variable**: $F : \Omega_{ob} \subset \mathbb{R}^n \to \mathbb{R}^n$ difeo $\implies \forall A$ *mJ* tq $\bar{A} \subset \Omega$ i $\forall f \in \mathcal{R}(F(A))$, $\int_{F(A)} f = \int_A (f \circ F)|\det D_F| \in \mathcal{R}(A)$.

Tema 3: Integració en línia i superf.

Corbes

• camí/corba: $\alpha : I \subset \mathbb{R} \to \mathbb{R}^n \mathscr{C}$ tq $\alpha(t) = (\alpha_1(t), ..., \alpha_n(t))$.

• suport/traça: C_α, l'imatge d'α.

- si α inj $\implies$ C$_\alpha \equiv$ corba.
- $\alpha : I \to \mathbb{R}^2, \alpha(t) = (\alpha_1(t), \alpha_2(t))$ $\implies$:I$\to \mathbb{R}^3, \alpha(t) = (\alpha_1(t), \alpha_2(t), 0)$.

• corba tancada: si $\alpha(a) = \alpha(b)$.

• corba simple: si tancada i inj en $[a, b)$.

• corbes equiv.: $\alpha : I \to \mathbb{R}^n, \beta : J \to \mathbb{R}^n$ si $\exists \varphi : J \to I$ homeo tq $\beta = \alpha \circ \varphi$.

- α inj sii $\forall$ corba equiv és inj.
- si α, β equiv $\implies$ C$_\alpha = C_\beta$.

• composició: $\alpha : [a,b] \to \mathbb{R}^n, \beta : [b,c] \to \mathbb{R}^n, \alpha(b) = \beta(b)$; $\alpha * \beta : [a,c] \to \mathbb{R}^n$ tq $\alpha * \beta = \alpha(t)$ si $t \in [a,b]$ i $\alpha * \beta = \beta(t)$ si $t \in (b,c]$.

- $\alpha, \beta \in \mathscr{C}$ $\implies$ $\alpha * \beta \in \mathscr{C}$; - $C_{\alpha*\beta} = C_\alpha \cup C_\beta$.
- si α, β inj i $C_\alpha \cap C_\beta = \{\alpha(b)\}$ $\implies$ $\alpha * \beta$ inj.
- $\alpha : I \to \mathbb{R}^n$; si $\alpha_j \in \mathscr{C}^{k(I)} \forall j$ $\implies \alpha \in \mathscr{C}^k$ $\implies \alpha' \mathscr{C}^{k-1}(I)$ i $\alpha(t)' = (\alpha_1'(t), ., \alpha_n'(t))$ tg d'α.
- el tg d'α pot canviar de sgn segons param.
- $\alpha : I \to \mathbb{R}^n$; si $\alpha_j \in \mathscr{C}_s^{k(I)} \forall j$ $\implies \alpha \in \mathscr{C}_s^{k(I)}$ $\implies \alpha \in \mathscr{C}$ i α' def en $I \setminus$ nº finit de pts.

• α regular si $\forall$ pt és pt regular: si $\alpha'(t) \neq 0$ (iParam).

- $\alpha_j \in \mathcal{R}[a,b], \forall j$ $\implies \alpha \in \mathcal{R}[a,b]$ $\implies \int_a^b \alpha = (\int_a^b \alpha_1, ., \int_a^b \alpha_n)$.
- $\alpha \in \mathscr{C}_s^k$ $\implies \int_{\tilde{a}}^{\tilde{b}} \alpha' = \alpha(\tilde{a}) - \alpha(\tilde{b}), \forall \tilde{a}, \tilde{b} \in [a,b]$.
- $\alpha \in \mathcal{R}[a,b]$ $\implies |\alpha| \in \mathcal{R}[a,b]$ i $|\int_a^b \alpha| \le \int_a^b |\alpha|$.

Longitud de corbes

• long. polig.: $l(\alpha, P) = \sum_{j=1}^{m} |\alpha(t_j) - \alpha(t_{j-1})|$.

- $P \subset P'$ $\implies l(\alpha, P) \le l(\alpha, P')$.

• long. d'α. $= l(\alpha) = \sup_P\{l(\alpha, P)\}$ (iParam).

• corba rectificable: si $l(\alpha)$ finita.

- $l(\alpha_j) \le l(\alpha) \le l(\alpha_1) + ... + l(\alpha_n)$.
- $\mathscr{C} \not\Rightarrow$ rectif.
- $\mathscr{C}^1$ ó $\mathscr{C}_s^1$ $\implies \alpha$ Lip $\implies \alpha$ rectif. i $l(\alpha) \le L(b-a)$.
- si $\alpha \in \mathscr{C}_s^1[a,b] \implies$ l$(\alpha) = \int_a^b |\alpha'(t)| dt$.

Integració en línia

• camp escalar en Ω: $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}, \mathscr{C}^k$.

• camp vectorial en Ω: $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n, \mathscr{C}^k$.

- si $\alpha \in \mathscr{C}_x^1[a,b], C_\alpha \subset \Omega, f : \Omega \to \mathbb{R}\ \mathscr{C}^k$, $\int_\alpha f dl = \int_a^b f(\alpha(t))|\alpha'(t)| dt$: int. línia de f.
- $|\int_\alpha f dl| \le \int_\alpha |f| dl \le l(\alpha) \max_{x \in C_\alpha}\{f(x)\}$.
- si $\alpha \in \mathscr{C}_x^1[a,b], C_\alpha \subset \Omega, F : \Omega \to \mathbb{R}\ \mathscr{C}^k$, $\int_\alpha F dl = \int_a^b < F(\alpha(t)), \alpha'(t) > dt$: int. línia/circulació de F (q el sgn dParam).
- $\int_\alpha f dl = \int_\alpha < f, t > dl = \int_\alpha f_t dl$.

Integració en superfície

• superf. regular σ: si D_σ té rang 2.

• àrea de S $a(S) = \int_\Omega |\sigma_u \text{x} \sigma_v| dudv$.

- $* = \{\Omega\ mJ, \bar{\Omega}$ cpt$, \sigma \in \mathscr{C}^k(\bar{\Omega})$i inj$\}$; si $f : S \subset \bar{\Omega} \to \mathbb{R}, \mathscr{C}$, la int. de superf de f és $\int_S f dS = \int_\Omega f(\sigma(u,v))|\sigma(u)\text{x}\sigma(v)| dudv$.
- $|\int_S f dS| \le \int_S |f| dS \le a(S) \max_{x \in \bar{S}}\{f(x)\}$.
- $*$; si $F : S \to \mathbb{R}^3, \mathscr{C}, S = \sigma(\Omega)$ orientada, la int. de superf de F (q el sgn dParam) és $\int_S F dS = \int_\Omega < F(\sigma(u,v)), \sigma(u)\text{x}\sigma(v) > dudv$.
- $\int_S f dS = \int_S < f, n > dS = \int_S f_n$: flux de f//F.

Tema 4: Teoremes integrals

$u : \Omega \subset \mathbb{R}^n \to \mathbb{R}, f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$.

Camps i pot. escalars i vectorials

• gradient: $\nabla u = (\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n})$.

• rotacional: $rot(f) = \nabla \text{x} f$.

• divergència: $div(f) = \frac{\partial f}{\partial x_1} + ... + \frac{\partial f}{\partial x_n}$.

• laplacià: $div(\nabla f)$.

• camp gradient: f si $\exists u \in \mathscr{C}^1$pot. escalar: $f = \nabla u$.

• camp irrotacional: f si $rot(f) = 0$.

• camp solenoidal: f si $div(f) = 0$.

• pot. vector: g si $\exists f \in \mathscr{C}^1$ tq $rot(g) = f$.

▶ **R.Leibnitz**: $\nabla uv = u\nabla v + v\nabla u$ i $div(uf) = udiv(f) + < \nabla u, f >$.

• camp conservatiu: f si $f \in \mathscr{C}$ i $\forall \alpha \in \mathscr{C}_s^1[a,b]$ tq $C_\alpha \subset \Omega$, $\int_{C_\alpha} f dl = u(\alpha(b)) - u(\alpha(a))$.

- f conservatiu sii $\oint f dl = 0$ sii f gradient.
- f conservatiu i $f \in \mathscr{C}^1$ $\implies \partial f_{i_{\partial x_j = \frac{\partial f_j}{\partial x_i}, \forall i,j}}$.
- $f \in \mathscr{C}^1(\Omega, \mathbb{R}^2)$ $\implies$ f $= (f_1, f_2, 0)$, $rot(f) = (0, 0, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y})$.

- f gradient $\implies f irrot(\text{rot}(\nabla)=0)$.
- f rotacional $\implies f solenoidal(\text{div}(\text{rot})=0)$.

• obert estrellat Ω: si $\exists x$ tq $(1-t)x_0+tx\in\Omega$, $\forall x\in\Omega, t\in[0,1]$.

▶ **L.Poincaré**: Ω ob estrellat $\subset\mathbb{R}^3, f\in\mathscr{C}^1$; si f solen. $\implies f rotacional isi f irrot \implies f conservatiu.$

- f irrot en ob estrellat $\implies$ u(x) $=\int_0^1 < f(t\mathbf{x}+(1-t)x_0, r(\mathbf{x}-x_0)> dt$.
- f solen en ob estrellat $\implies$ g(x,y,z) $=\int_0^1 (tf(t\mathbf{x}+(1-t)x_0 \text{ x } r(\mathbf{x}-x_0))dt$.

Green, Stoke, Gauss

• corba de Jordan: traça corba param simple.

▶ **T.Corba Jordan**: si α corba Jordan $\implies \mathbb{R}^2\setminus C_\alpha=\Omega_1\cup\Omega_2$ on Ω_1 acotat i Ω_2 no.

▶ **T.Green**: $\Omega\subset\mathbb{R}^2$ domini elem, $f:\bar\Omega\to\mathbb{R}^2\mathscr{C}^1$ $\implies \int_{\partial\Omega} fdl=\int_\Omega \frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}dxdy=\int_\Omega rot(f)dS$.

- Green sii $\{\oint_{\partial\Omega} f_1 dl=-\int_\Omega \frac{\partial f_1}{\partial y}dxdy$ i $\oint_{\partial\Omega} f_2 dl=\int_\Omega \frac{\partial f_2}{\partial x}dS\}$.

▶ **T.Stokes**: $(S\cup\partial S)\subset\Omega\subset\mathbb{R}^3$ domini elem, $f:\bar\Omega\to\mathbb{R}^3\mathscr{C}^1 \implies \oint_{\partial S} fdl=\int_S rot(f)dS$.

▶ **T.Gauss**: $V\subset\Omega\subset\mathbb{R}^3$ domini elem, $f:\bar\Omega\to\mathbb{R}^3\mathscr{C}^1 \implies \oint_{\partial V} fdS=\int_V div(f)dV$.

- Gauss sii $\{\int_{\partial\Omega} f_i dl=\int_\Omega \frac{\partial f_i}{\partial x_i}dV, \forall i$.
- $vol(V)=\frac{1}{3}\int_{\partial V} fdS=\frac{1}{3}\int_{\partial V}(x,0,0)dS$.
- $a(S)=\frac{1}{2}\int_{\partial S}(-y,x)dS$.

Tema 5: Formes diferencials

• forma d'ordre k en Ω: $w:\Omega\to\bigwedge^k(\Omega)$ tq $w=\sum_{1\le i_1<\ldots<i_k\le n}\alpha_{i_1,\ldots,i_k}dx^{i_1}\wedge\ldots\wedge dx^{i_k}$.

• producte exterior: $(w_1\wedge w_2)(x)=w_1(x)\wedge w_2(x)$ (p+q)-forma $\mathscr{C}^m$ bilineal, anticomm i associativa.

• pull-back de F: $F^*(u)=(D_f(u))^*$ tq $F^*(fdx^{i_1}\wedge\ldots\wedge dx^{i_k})=F^*(f)F^*(dx^{i_1}\wedge\ldots\wedge dx^{i_k})$.

• diferencial exterior: de w k-forma $\mathscr{C}^p$, és $d(w)=\sum_{1\le i_1<\ldots<i_k\le n} d\alpha_{i_1,\ldots,i_k}\wedge dx^{i_1}\wedge\ldots\wedge dx^{i_k}$ (k+1)-forma $\mathscr{C}^{p-1}$.

- $dy^1\wedge\ldots\wedge dy^n=(detD_\psi)dx^1\wedge\ldots\wedge dx^n$.
- $d(w\wedge\tilde w)=d(x)\wedge\tilde w+(-1)^k w\wedge d(\tilde w)$.
- $d^2=d\circ d=0$.
- $F^*\circ d=d\circ F^* \implies F^*(dw)=d(F^*(w))$.

• forma tancada: si $dw=0$.

• forma exacta: si $w=d\tilde w$.

- exacta $\implies$ *tancada*.

▶ **L.Poincaré**: $\Omega\subset\mathbb{R}^n$ ob estrellat, w k-forma $\mathscr{C}^1(\Omega) \implies$ w=K(dw)+d(K(w)).

• integral n-forma: $w=fdx^1\wedge\ldots\wedge dx^n \implies \int_\Omega w=\int_\Omega fdx^1\wedge\ldots\wedge dx^n=\int_\Omega fdx^1\ldots dx^n=\int_\Omega f$.

- $\int_\Omega w=\pm\int_{\tilde\Omega} F^*(w)$.
- $\sigma:\tilde\Omega\to\Omega, \mathscr{C}^1 \implies \int_\sigma w=\int_{\tilde\Omega}\sigma^*(w)$.

▶ **T.Stokes**: $\sigma(\tilde\Omega)=M\cup\partial M$, w k-forma $\mathscr{C}^1$ $\implies \int_{\partial M} w=\int_M dw$ (cal ∂M sigui corba tancada).

Altres

Taylor

$e^x=\sum_{n\ge0}\frac{x^n}{n!}$.
$\cos x=\sum_{n\ge0}(-1)^n\frac{x^{2n}}{(2n)!}$.
$\sin x=\sum_{n\ge0}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$.
$\log(1+x)=\sum_{n\ge1}(-1)^{n+1}\frac{x^n}{n}$.
$(1+x)^p=\sum_{n\ge0}\binom{p}{n}x^n$.
$(1+x)^{-1}=\sum_{n\ge0}(-1)^n x^n$.
$\cosh x=\sum_{n\ge0}\frac{x^{2n}}{(2n)!}$.
$\sinh x=\sum_{n\ge0}\frac{x^{2n+1}}{(2n+1)!}$.
$\arctan x=\sum_{n\ge0}(-1)^n\frac{x^{2n+1}}{2n+1}$.

Trigonometria

$\sin(a\pm b)=\sin(a)\cos(b)\pm\cos(a)\sin(b)$.
$\cos(a\pm b)=\cos(a)\cos(b)\mp\sin(a)\sin(b)$.
$\tan(a\pm b)=\frac{\tan(a)\pm\tan(b)}{1\mp\tan(a)\tan(b)}$
$\sin(a)+\sin(b)=2\sin(\frac{a+b}{2})\cos(\frac{a-b}{2})$.
$\cos(a)+\cos(b)=2\cos(\frac{a+b}{2})\cos(\frac{a-b}{2})$.
$2\cos(a)\cos(b)=\cos(a-b)+\cos(a+b)$
$2\sin(a)\sin(b)=\cos(a-b)-\cos(a+b)$
$2\sin(a)\cos(b)=\cos(a+b)+\cos(a-b)$
$2\cos(a)\sin(b)=\cos(a+b)-\cos(a-b)$
$\cos^2(a)=\frac{1+\cos(2a)}{2}$
$\sin^2(a)=\frac{1-\cos(2a)}{2}$
$\sin(\pi/2-x)=\cos(x)$
$\cos(\pi/2-x)=\sin(x)$

Integrals útils

$\int_1^{+\infty}\frac{1}{x^\alpha}dx$ conv $\iff \alpha>1$, i val $\frac{1}{\alpha-1}$.
$\int_0^1\frac{1}{x^\alpha}dx$ conv $\iff \alpha<1$, i val $\frac{1}{1-\alpha}$.
$\int_0^{+\infty}e^{-\alpha t}dt$ conv $\iff \alpha>0$, i val $\frac{1}{\alpha}$.
$\int_1^{+\infty}\frac{1}{x^\alpha}dx$ conv $\iff \alpha>1$, i val $\frac{1}{\alpha-1}$.

Criteris per límits

•Stolz: (b_n) est. monòtona, $\{\lim b_n=\pm\infty$ o bé $\lim a_n=\lim b_n=0\} i \lim\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=L\in[-\infty,+\infty] \implies \lim\frac{a_n}{b_n}=L$.

•Arrel-Quocient: (a_n) no nul·la $\ge n_0.\exists\lim\left|\frac{a_{n+1}}{a_n}\right|=L \implies \lim\sqrt[n]{|a_n|}=L$