Probability and Statistics 2

Data Science Engineering

Session 6: Poisson Process

Poisson Process. Interarrival Times. Combine and Split Poisson Process. Queues.

The Poisson process is one of the simplest continuous time processes, and still has wide applications. Poisson processes are used to count events that occur randomly in time and the interarrival times follow an exponential law.

1. Poisson Process

Recall that a continuous time random process is a family $X(t)$ of random variables where $t \in \mathbb{R}$, usually $t \in [0, \infty)$. There are several equivalent definitions of a Poisson process.

Definition 1.1 (Poisson Process)**.** *A random process $N(t), t \in [0, \infty)$, where each $N(t)$ is a discrete random variable taking nonnegative integer values, is a Poisson process with rate λ if*

(1) $N(0) = 0$.
(2) *the process has independent increments: for each sequence $0 \leq t_1 < \cdots < t_k$, the random variables $N(t_1) - N(0), N(t_2) - N(t_1), \ldots, N(t_k) - N(t_{k-1})$ are independent.*
(3) *for each interval (t, t') with $t \leq t'$, $N(t') - N(t)$ is a Poisson random variable $N(t') - N(t) \sim Pois(\lambda(t' - t))$.*

It follows from the definition that a Poisson process $N(t)$ is a Markovian continuous process: for all $s, t \geq 0$,

$$\Pr(N(s+t) = n | N(u), 0 \leq u \leq t) = \Pr(N(s+t) = n | N(t)).$$

The mean value of the process is

$$\mathbb{E}(N(t)) = \lambda t,$$

and the autocorrelation function, for $0 < t < t'$,

$$R(t, t') = \mathbb{E}(N(t)N(t')) = \lambda t + \lambda^2 t t'.$$

2. Interarrival times

Let X_n be the time of the n–arrival in a Poisson (or more generally, counting) process:

$$X_n = \inf\{t : N(t) = n\}.$$

The interarrival times are the random variables $T_1, T_2, \ldots$ where

$$T_n = X_n - X_{n-1}.$$

Theorem 2.1 (Interarrival times)**.** *Let $N(t)$ be a Poisson process with rate λ. Then*

$$T_n \sim Exp(\lambda),$$

and the variables $T_1, T_2, \ldots$ are independent.

Recall that the sum of independent exponential variables follows a Gamma distribution, so that $X_n \sim Gamma(n, \lambda)$.

One interesting property of the arrival times is its behaviour under conditioning on the number of arrivals.

Theorem 2.2 (Conditional arrival times)**.** *Let $N(t)$ be a Poisson process with rate λ. Conditional to $N(t) = 1$, the first arrival time X_1 has a uniform distribution on $(0, t)$.*

3. Combining and splitting Poisson processes

Recall that the sum of independent Poisson random variables is again Poisson. In terms of Poisson processes this property is reflected as follows.

Theorem 3.1 (Combining Poisson processes)**.** *Let $N_1(t), N_2(t)$ be independent Poisson processes with rates λ_1 and λ_2 respectively. Then $N(t) = N_1(t) + N_2(t)$ is a Poisson random process of rate $\lambda_1 + \lambda_2$.*

The somewhat opposite procedure of splitting a Poisson process according to a random selection leads also to a Poisson process.

Theorem 3.2 (Splitting a Poisson process)**.** *Let $N(t)$ be a Poisson process with rate λ. Suppose that each event counted by $N(t)$ is selected with probability p independently of the other events. Then the counting process $N_1(t)$ of the selected events is a Poisson process of rate λp.*

4. The $M/M/1$ queue

One of the applications of the Poisson process is in Queueing Theory. The simplest example is the following one. Suppose that customers arrive to a service according to a Poisson process of rate λ. Once in the service, the customers line up till they are served.

Suppose that each customer is served in a time $T \sim Exp(\mu)$, independently of the other customers. The question is how the number of customers in the line evolves in time.

Definition 4.1 (M/M/1 queue)**.** *A queue is $M/M/1$ if there is an only server, the interarrival time of customers are independent identically distributed with exponential distribution $Exp(\lambda)$, and the service time is also a sequence of independent identically distributed random variables with exponential distribution $Exp(\mu)$.*

The graphical representation of the queue is

$$\begin{tikzcd} 0 \arrow[r, bend left, "\lambda"] & 1 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 2 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 3 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 4 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 5 \arrow[l, bend left, "\mu"] & \cdots \end{tikzcd}$$

Let $M(t)$ be the number of customers in a $M/M/1$ line at time t.

Theorem 4.2. *Let $M(t)$ be the number of customers in a $M/M/1$ line at time t. If $\lambda < \mu$ then $M(t)$ is a Markov process with a stationary distribution $(\pi_0, \pi_1, \ldots)$ with*

$$\pi_0 = 1 - (\lambda/\mu) \quad \textit{and } \pi_k = (1 - (\lambda/\mu))(\lambda/\mu)^k.$$

The expected number of customers on the line at equilibrium is

$$\frac{\lambda}{\mu - \lambda}.$$

5. Exercises and Problems

(1) The number of tasks arriving at a processor follows a Poisson process with rate 1 task per minute. Compute the probabilities of the following events:
   (a) exactly two tasks arrive in the first minute.
   (b) there is no task in the first 2 minutes.
   (c) there is at least one task in the first 2 minutes.
   (d) the first task arrives during the third minute.
   (e) the first task arrives during the third minute, if the second task has arrived at time 4:00 exactly.
   (f) exactly 11 tasks arrive during the first 10 minutes.
   (g) the time between the first two arrivals is shorter than the time of the first arrival.

(2) Let $N(t)$ be a counting random process with $N(0) = 0$. Suppose that the interarrival times $T_1, T_2, \ldots$ are independent identically distributed random variables, each $T_i \sim Exp(\lambda)$. Show that $N(t)$ is a Poisson process. (Hint: the cdf of Gamma(k, λ) is $1 - e^{-\lambda t}\left(1 + \lambda t + \frac{(\lambda t)^2}{2} + \cdots + \frac{(\lambda t)^{k-1}}{(k-1)!}\right)$)

(3) Red cars arrive in a toll according to a Poisson process of rate 1 per minute and white cars according to an independent Poisson process of rate 2 per minute.

(a) What is the probability that no car arrives in the first two minutes?
(b) What is the probability that the first car arrives during the third minute?
(c) What is the probability that the first car to arrive is red?

(4) Bachelor students go to office hours according to a Poisson process N_1 with rate $\lambda_1 = 3$ per hour. Master students go to office hours according to a Poisson process N_2 with rate $\lambda_2 = 2$ per hour. What is the probability that the second master student arrives before the third bachelor student?

(5) (Impatient hitchhikers) Alice and Bob are hitchhiking. Cars arrive as a Poisson process with rate λ_C. Alice is first in line for a ride. Moreover, after $\mathrm{Exp}(\lambda_A)$ time, Alice leaves, and after $\mathrm{Exp}(\lambda_B)$ time, Bob leaves. Compute the probability that Alice is picked up before she leaves, and the same for Bob.

(6) (Finite queues) A line of customers waiting for service works as follows: there is a fixed number n and rates λ and μ such that,
- If there are less than n customers, a new one joins the line with rate λ.
- If the line is nonempty, a customer is served with rate μ.
- Otherwise line remains unchanged.

Let $X(t)$ be the number of customers at time t.
(a) Draw the graphic representation of the chain.
(b) Find the stationary distribution of the chain. (Hint: $\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}$ for $|x| < 1$.)
(c) In the long run, determine the fraction of time that the server is occupied.
(d) In the long run, determine the probability a client cannot enter the system

(7) (Multiple servers) A tourist information office has two service desks. The arrivals are distributed according to a Poisson point process with expectation 50 per hour. The service time is distributed according to an exponential random variable with mean 2 minutes.
(a) Draw the graphical representation of the chain.
(b) Find the stationary distribution.
(c) Assume that if the desks are empty, a customer chooses a random one. In the long run, determine the fraction of time that a server is waiting for customers.

(8) (Impatient customers) Modify the $M/M/1$ with $\lambda = \mu = 1$ queue as following: when a customer arrives and there are i people in the line, the customer stays with probability $1/(i+1)$.
(a) Draw the graphical representation of the chain.
(b) Find the stationary distribution.
(c) In the long run, find the probability that there is at least one customer in the line (the client being served is not considered to be in the line)

(d) In the long run, find the probability that when a customer arrives, they decide to stay and join the queue.