Time Series
2. Stationary Processes. ARMA Models

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Outline

- AR(p) models
- MA(q) models
- ARMA(p,q) models

Gaussian process $\{ X_t \}$, $t = 1, \cdots, n$

$$X = (X_t) \sim \text{Multivariate Normal/Gaussian distribution}$$

$$E(X) = \mu = (\mu_1, \cdots, \mu_n)'$$

Covariance matrix (nxn) is

$$V(X) = \Gamma = \{\gamma(t_i, t_j) | t, j = 1, \cdots, n\}$$

And the multivariate Normal density function can be written as

$$f(x) = (2\pi)^{-n/2} |\Gamma|^{-1/2} exp\{-1/2(x-\mu)^{'} \Gamma^{-1}(x-\mu)\}$$

where $|\Gamma| \equiv$ determinant

General Stochastic model for a time series:

$$X_t = G(X_{t-1}, X_{t-2}, ..., Z_{t-1}, Z_{t-2}, ...) + Z_t \quad Z_t \sim WN(0, \sigma_z)$$

The most easy mathematical function G(.) is the linear combination of the components:

Linear Stochastic model for a time series:

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + ... + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + ... + Z_t \quad Z_t \sim WN(0, \sigma_Z)$$

$$\phi_1, \phi_2, ..., \theta_1, \theta_2, ... \in \mathbf{R}$$

The variable at time t is a linear combination of past observations and disturbances plus a new disturbance independent form the past

AR(p) models

A **pth-order autoregressive model**, or AR(p), takes the form:

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + Z_t$$

$$(1 - \phi_1 B - \phi_2 B^2 - \cdots + \phi_p B^p) X_t = Z_t$$

where

X_t is stationary and $\phi_1, \cdots, \phi_p$ are the parameters (constants)

Z_t is a Gaussian white noise ($E(Z_t) = 0$, $V(Z_t) = \sigma_Z^2$)

B is the Backshift operator: $BX_t = X_{t-1}$

p is the lag of the farthest observation included

An **AR(p) model** is a regression model with lagged values of the dependent variable in the independent variable positions, hence the name **Auto-Regressive model**.

AR(p) models

- If $\mu \neq 0$, then

$$(X_t - \mu) = \phi_1(X_{t-1} - \mu) + \cdots + \phi_p(X_{t-p} - \mu) + Z_t$$

or

$$X_t = \alpha + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + z_t$$

where the constant term is:

$$\alpha = \mu(1 - \phi_1 - \cdots - \phi_p)$$

Or a more concise expression:

$$\phi_p(B)X_t = Z_t$$

where $\phi(B) = (1 - \phi_1 B - \cdots - \phi_p B^p)$ is the characteristic autoregressive polynomial of order p.

AR(1) process, considering $\mu = 0$:

$$(1-\phi B)X_t = Z_t, \quad Z_t \sim N(0, \sigma_Z^2).$$

Or equivalently:

$$X_t = \phi X_{t-1} + Z_t, \quad Z_t \sim N(0, \sigma_Z^2).$$

Autocovariance function derivation:

$$\gamma(0) = E[X_t^2] = E[(\phi X_{t-1} + Z_t)^2] = \phi^2\gamma(0) + \sigma_Z^2 \Rightarrow \gamma(0) = \frac{\sigma_Z^2}{1-\phi^2}$$

$$\gamma(1) = E[X_t X_{t-1}] = E[(\phi X_{t-1} + z_t)X_{t-1}] = \phi\gamma(0)$$

$$\vdots$$

$$\gamma(h) = E[X_t X_{t-h}] = E[(\phi X_{t-1} + Z_t)X_{t-h}] = \phi\gamma(h-1) = \phi^h\gamma(0)$$

Reminder: The noise at time t is independent of the past:
$E[Z_t X_s] = E[Z_t Z_s] = 0 \quad s < t$.

AR(p) models

AR(1) process: $(1-\phi B)X_t = Z_t$

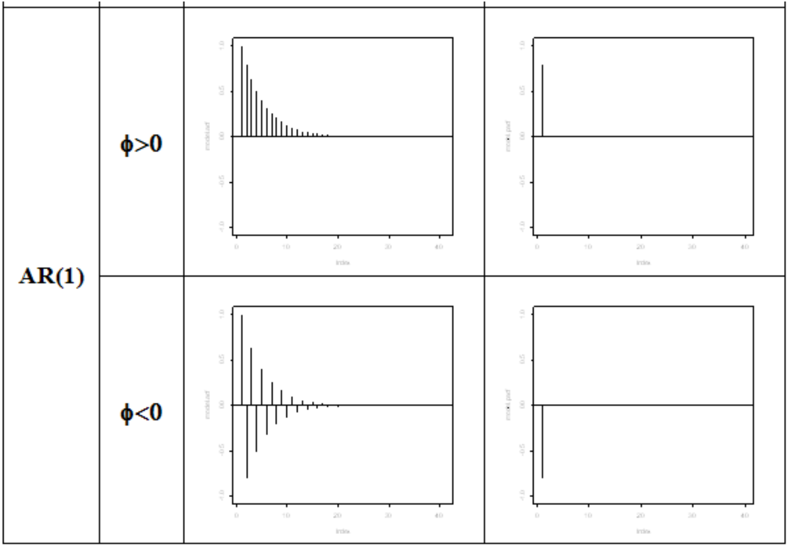
Autocorrelation function:

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \phi$$
$$\vdots$$
$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \phi^h$$

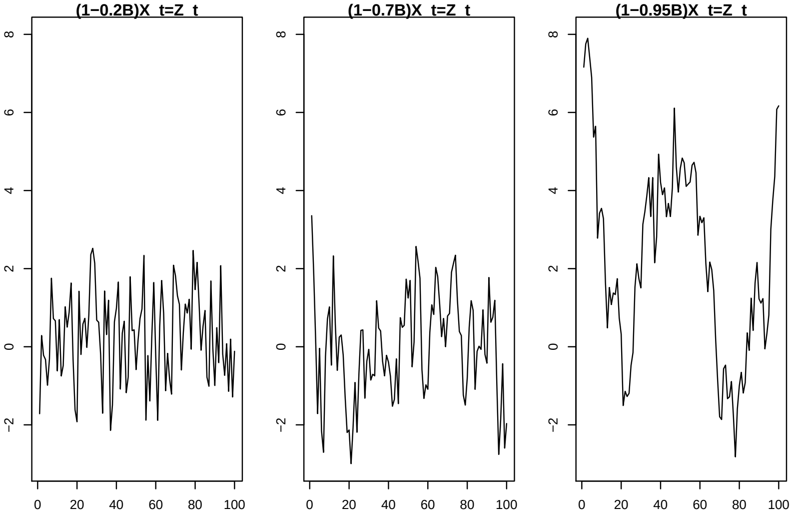
Recursion: $\rho(h) = \phi\rho(h-1)$
Partial Autocorrelation function:

$$\phi_{1,1} = \phi$$
$$\vdots$$
$$\phi_{h,h} = 0 \quad h > 1$$

AR(1) process: $(1 - \phi B)X_t = Z_t$



AR(1) models



AR(p) models

AR(2) process: $(1 - \phi_1 B - \phi_2 B^2)X_t = Z_t$

Considering $\mu = 0$, then

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-1} + Z_t \quad Z_t \sim N(0, \sigma_Z^2)$$

Autocovariance function:

$$\gamma(0) = E[X_t^2] = (\phi_1^2 + \phi_2^2)\gamma(0) + 2\phi_1\phi_2\gamma(1) + \sigma_Z^2$$
$$\gamma(1) = E[X_t X_{t-1}] = \phi_1\gamma(0) + \phi_2\gamma(1)$$
$$\vdots$$
$$\gamma(h) = E[X_t X_{t-h}] = \phi_1\gamma(h-1) + \phi_2\gamma(h-2) \quad h > 1$$

MA(q) models

A **qth-order moving average model**, or MA(q), takes the form:

$$X_t = z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \cdots + \theta_q Z_{t-q}$$

In other words,

$$X_t = \theta_q(B) Z_t$$

where

$$\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q$$

is the moving average operator or characteristic polynomial.

An **MA(q) mode**l is a regression model with the dependent variable, X_t, depending on previous values of the errors rather than on the variable itself.

MA(q) models

MA(1) process, considering $\mu = 0$:

$$X_t = (1 + \theta B)Z_t, \quad Z_t \sim N(0, \sigma_Z^2)$$

Or equivalently:

$$X_t = Z_t + \theta Z_{t-1}, \quad Z_t \sim N(0, \sigma_Z^2)$$

Autocovariance function derivation:

$$\gamma(0) = E[X_t^2] = E[(Z_t + \theta Z_{t-1})^2] = (1 + \theta^2)\sigma_Z^2$$

$$\gamma(1) = E[X_t X_{t-1}] = E[(Z_t + \theta Z_{t-1})(Z_{t-1} + \theta Z_{t-2})] = \theta\sigma_Z^2$$

$$\vdots$$

$$\gamma(h) = E[X_t X_{t-h}] = 0 \quad h > 1$$

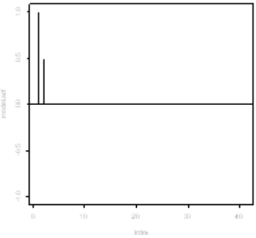
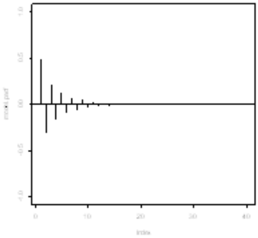
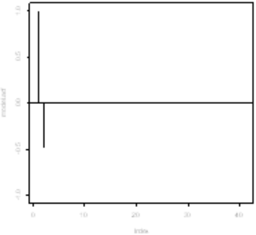
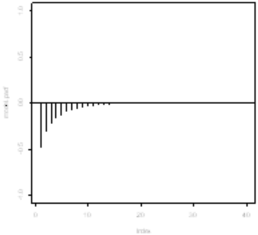
MA(q) models

MA(1) process: $X_t = (1 + \theta B) Z_t$**.**

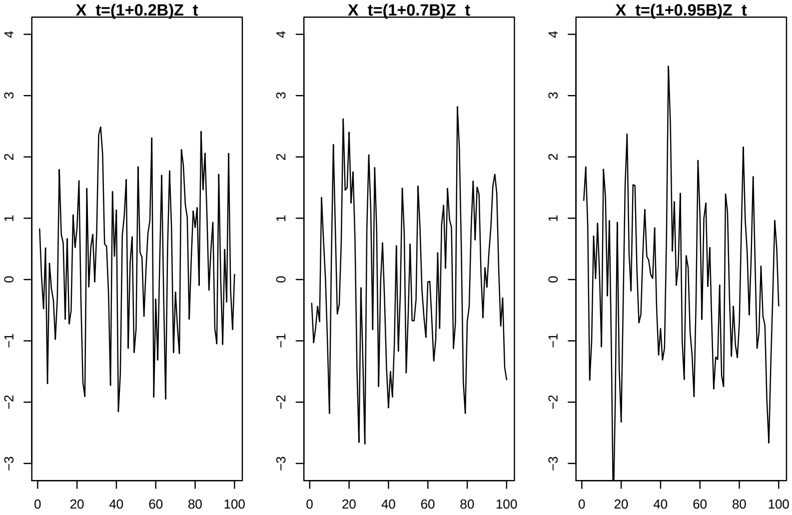
Autocorrelation function:

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\theta}{1+\theta^2}$$
$$\vdots$$
$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = 0 \quad h > 1$$

MA(1): $x_t = (1 + \theta B) z_t$

MA(1)	$\theta > 0$		
	$\theta < 0$		

MA(1) models



ARMA(p,q) models

A times series $\{X_t; t = 0, \pm 1, \pm 2, \cdots\}$ is an **AutoRegressive Moving Average model**, ARMA (p,q), if it is stationary and

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$$

with $\phi_p \neq 0$, $\theta_q \neq 0$ and $\sigma_z^2 > 0$

The parameters p and q are called the autoregressive and the moving average orders, respectively.

ARMA(p,q) models

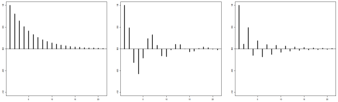
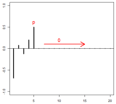
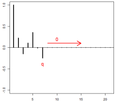
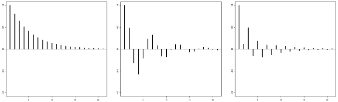
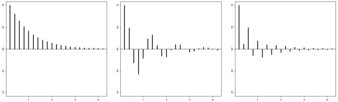
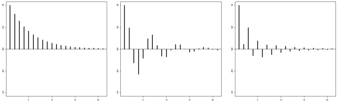
If X_t has a nonzero mean μ and $\alpha = \mu(1 - \phi_1 - \cdots - \phi_p)$, the model del will be:

$$X_t = \alpha + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$$

When $q = 0$, the model is an AR(p).
When $p = 0$, the model is a MA(q)

ARMA(p,q) model can be written in concise form as,

$$\phi(B)_p X_t = \theta_q(B) Z_t$$

MODEL	ACF	PACF
AR(p)	Decreasing patterns (∞ lags not null) 	$p =$ Last lag not null 
MA(q)	$q =$ Last lag not null 	Decreasing patterns (∞ lags not null) 
ARMA(p,q)	Decreasing patterns (∞ lags not null) 	Decreasing patterns (∞ lags not null) 

Expression of an $ARMA(p, q)$ model, using characteristic polynomials:

$$\phi(B)x_t = \theta(B)z_t$$

Under certain conditions, ARMA(p, q) models can be expressed as an AR(∞) or MA(∞)model.

Expression as an AR(∞):

$$\frac{\phi(B)}{\theta(B)}x_t = \pi(B)x_t = z_t$$

Expression as an MA(∞):

$$x_t = \frac{\theta(B)}{\phi(B)}z_t = \psi(B)z_t$$

Expression of an $ARMA(p,q)$ model, using characteristic polynomials:

$$(1-\phi_1 B - \dots - \phi_p B^p)x_t = (1+\theta_1 B + \dots + \theta_q B^q)z_t$$

Under certain conditions, $ARMA(p,q)$ models can be expressed as a pure AR(∞) or MA(∞) process.

Expression as an AR(∞):

$$\frac{1-\phi_1 B - \dots - \phi_p B^p}{1+\theta_1 B + \dots + \theta_q B^q}x_t = (1-\pi_1 B - \pi_2 B^2 - \dots)x_t = z_t$$

Expression as an MA(∞):

$$x_t = \frac{1+\theta_1 B + \dots + \theta_q B^q}{1-\phi_1 B - \dots - \phi_p B^p}z_t = (1+\psi_1 B + \psi_2 B^2 + \dots)z_t$$