Algebra
4. Linear maps

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Outline

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Definition

A linear map (or linear transformation) between two $\mathbb{K}-$e.v E and F is a map that preserves linear combinations. More precisely,

Definition

$f : E \longrightarrow F$ is a linear map if

1. $f(u+v) = f(u) + f(v)$ for all $u, v \in E$, and
2. $f(cv) = cf(v)$ for any $c \in \mathbb{K}$ and any $v \in E$.

Examples

- $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ where $f(x,y) = (x+2y, 3x, y-x)$
- $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where $f(x,y) = (-y, x)$ (*rotation* of $\pi/2$ centered at $(0,0)$)
- $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ $f(v) = \lambda . v$ for some $\lambda \in \mathbb{K}$ (*homothety*).
- $f : E \longrightarrow F$, $f(v) = \mathbf{0}$ $\forall v \in E$ is called *zero* map.
- $f : E \longrightarrow E$ $f(v) = v$ is called *identity* map Id.
- Example of maps that are not linear

Let $f : E \longrightarrow F$ be a map between $\mathbb{K}$-e.v. Then:

- f linear $\Leftrightarrow f(c_1 v_1 + \cdots + c_k v_k) = c_1 f(v_1) + \cdots + c_k f(v_k)$ $\forall v_1, \ldots, v_k \in E$ and $c_1, \ldots, c_k \in \mathbb{K}$.
- f linear $\Rightarrow f(\mathbf{0}) = \mathbf{0}$.

A linear map f is determined by the **image of a basis** (any basis):

PROPOSITION

Given a basis $\{u_1, \ldots, u_n\}$ of E and any set of vectors $v_1, \ldots, v_n \in F$, there exists a unique linear map $f : E \longrightarrow F$ such that $f(u_i) = v_i \ \forall i$.

- Basic example of linear map: If $A \in \mathcal{M}_{m\times n}(\mathbb{K})$, the map $f:\mathbb{K}^n \longrightarrow \mathbb{K}^m$ defined by

$$v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto f(v) = A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

- **All** linear maps $f:\mathbb{K}^n \longrightarrow \mathbb{K}^m$ are of this type: in **standard coordinates** they are defined as degree 1 homogeneous polynomials:

$$(x_1,\ldots,x_n) \mapsto (a_{1,1}x_1+\ldots+a_{1,n}x_n, \cdots, a_{m,1}x_1+\ldots+a_{m,n}x_n)$$

  and f corresponds to $v \mapsto Av$ where $A=(a_{i,j})$; the ith column of A is $f(e_i)$.
- The standard matrix $M(f)$ of a linear map $f:\mathbb{K}^n \longrightarrow \mathbb{K}^m$ is the $m\times n$ matrix whose columns are the vectors $f(e_i)$:

$$M(f) = (f(e_1)\cdots f(e_n))$$

OUTLINE

Definitions

Let $f : E \longrightarrow F$ be a map between $\mathbb{K}$-e.v.

- f is injective if different vectors always have different images ($f(u) = f(v)$ implies $u = v$).
- f is surjective if every vector v in F is the image of a certain vector $u \in E$, $v = f(u)$.
- The set of all images of vectors is called the image or range of f,

$$\mathrm{Im}(f) = \{v \in F \mid v = f(u) \text{ for some u } \in E\} \subseteq F$$

- f is surjective if and only if $\mathrm{Im}(f) = F$.
- f is bijective if it is at the same time injective and surjective. A bijective linear map is called an isomorphism.

Let $f : E \longrightarrow F$ be a linear map.

Definition

The null space or kernel (*nucli*) of f is the subspace

$$\mathrm{Nuc}(f) = \{v \in E \mid f(v) = \mathbf{0}\} = f^{-1}(\{\mathbf{0}\}) \subset E.$$

Theorem

A linear map f is injective if and only if $\mathrm{Nuc}(f) = \{\mathbf{0}\}$.

If $f : \mathbb{K}^n \longrightarrow \mathbb{K}^m$ is a linear map and A is its standard matrix, then

- $\mathrm{Nuc}(f) = \{v \in \mathbb{K}^n \mid f(v) = 0\} = \{x \in \mathbb{K}^n \mid Ax = 0\}$.
- $\dim \mathrm{Nuc}(f) = n - \mathrm{rank}(A)$.
- f is injective $\Leftrightarrow \mathrm{rank}(A) = n$ (=number of columns).
- f injective $\Rightarrow n \leq m$.

Image and preimage of a subspace

Let $f : E \longrightarrow F$ be a linear map.

Definition

The image of $V \subseteq E$ is the set

$$f(V) := \{w \in F \,|\, w = f(u) \text{ for some } u \in V\}.$$

- If V is a subspace $\Rightarrow f(V)$ is also a subspace.
- If $V = [u_1, \dots, u_d] \subset E \Rightarrow f(V) = [f(u_1), \dots, f(u_d)] \subset F$.
- If $u_1, \dots, u_d$ are linearly independent, $f(u_1), \dots, f(u_d)$ do **NOT** need to be l.i.
- $\mathrm{Im}(f) = f(E) = [f(u_1), \dots, f(u_n)]$ if $\{u_1, \dots, u_n\}$ is a basis of E.
- $\dim \mathrm{Im}(f)$ is called the rank of f.

Image for $f : \mathbb{K}^n \longrightarrow \mathbb{K}^m$

Let $f : \mathbb{K}^n \longrightarrow \mathbb{K}^m$ be a linear map and let A be its standard matrix. Then,

- $\mathrm{Im}(f) = [\text{columns of A}]$.
- $\dim \mathrm{Im}(f) = \mathrm{rank}(A)$.
- f is surjective if and only if $\mathrm{rank}(A) = m$ (= number of rows).
- f surjective $\Rightarrow m \leq n$.

Let $f : E \longrightarrow F$ be a linear map.

Definition

The preimage of $W \subseteq F$ is $f^{-1}(W) := \{u \in E \mid f(u) \in W\} \subseteq E$.

Lemma

1. *If $u \in E$ and $v \in F$ satisfy $f(u) = v$, then*

$$f^{-1}(v) = \{u + w \,|\, w \in \mathrm{Nuc}(f)\}.$$

2. *If W is a subspace, so is $f^{-1}(W)$.*

Outline

Let $f : E \longrightarrow F$ and $g : F \longrightarrow G$ be linear maps, the composition of g with f is the linear map $g \circ f : E \longrightarrow G$ defined as:

$$\begin{array}{rcccl} g \circ f : E & \xrightarrow{f} & F & \xrightarrow{g} & G \\ v & \mapsto & f(v) & \mapsto & (g \circ f)(v) := g(f(v)) \end{array}.$$

If $f : \mathbb{K}^n \longrightarrow \mathbb{K}^m$ has standard matrix A and $g : \mathbb{K}^m \longrightarrow \mathbb{K}^p$ has standard matrix $B \Rightarrow$ the standard matrix of $g \circ f$ is

$$M(g \circ f) = \mathbf{BA}.$$

If $f : E \longrightarrow F$ is a linear map, we say that $g : F \longrightarrow E$ is the inverse of f (denoted as $g = f^{-1}$) if

$$g \circ f = f \circ g = Id.$$

Note: f is invertible $\Leftrightarrow$ f is bijective.

Invertible linear maps are called isomorphisms. Two $\mathbb{K}$-ev. are isomorphic if there exists an isomorphism $f : E \longrightarrow F$; in this case we use the notation $E \cong F$.

Properties:

- If f is iso. $\Rightarrow$ f^{-1} is a linear map.
- If $f : \mathbb{K}^n \longrightarrow \mathbb{K}^n$ is iso. and has standard matrix $A \Rightarrow$ $M(f^{-1}) = A^{-1}$.
- If f has inverse map f^{-1}, then the preimage $f^{-1}(W)$ of a subspace W coincides with its image by f^{-1}.

Theorem (First theorem of isomorphism)

If $f : E \longrightarrow F$ is a linear map, then the map

$$\begin{array}{rcl} \overline{f} : E/\operatorname{Nuc}(f) & \longrightarrow & \operatorname{Im}(f) \\ \overline{v} & \mapsto & \overline{f}(\overline{v}) := f(v) \end{array}$$

is well defined and is an isomorphism, $E/\operatorname{Nuc}(f) \cong \operatorname{Im}(f)$.

Corollary

Let $f : E \longrightarrow F$ be a linear map and assume that E has finite dimension. Then, $\operatorname{Nuc}(f)$ and $\operatorname{Im}(f)$ have finite dimension and

$$\dim \operatorname{Nuc}(f) + \dim \operatorname{Im}(f) = \dim E$$

If $f : E \longrightarrow F$ is a linear map between vector spaces of finite dimension, then:

- f is injective $\Leftrightarrow \mathrm{Nuc}(f) = \{\mathbf{0}\} \Leftrightarrow \dim \mathrm{Im}(f) = \dim E$.
- f is surjective $\Leftrightarrow \dim \mathrm{Im}(f) = \dim F \Leftrightarrow$ $\dim \mathrm{Nuc}(f) = \dim E - \dim F$.
- f is bijective $\Leftrightarrow \dim E = \dim F$ and $\mathrm{Nuc}(f) = \{\mathbf{0}\} \Leftrightarrow \dim E = \dim F$ and $\dim \mathrm{Im}(f) = \dim F$.
- If $\dim E = \dim F$, then f is bijective $\Leftrightarrow$ injective $\Leftrightarrow$ surjective.

Characterizations of inj./surj. maps

If $f : E \longrightarrow F$ is a linear map between vector spaces of finite dimension, then:

- f is injective $\Leftrightarrow \mathrm{Nuc}(f) = \{\mathbf{0}\} \Leftrightarrow \dim \mathrm{Im}(f) = \dim E$.
- f is surjective $\Leftrightarrow \dim \mathrm{Im}(f) = \dim F \Leftrightarrow \dim \mathrm{Nuc}(f) = \dim E - \dim F$.
- f is bijective $\Leftrightarrow \dim E = \dim F$ and $\mathrm{Nuc}(f) = \{\mathbf{0}\} \Leftrightarrow \dim E = \dim F$ and $\dim \mathrm{Im}(f) = \dim F$.
- If $\dim E = \dim F$, then f is bijective $\Leftrightarrow$ injective $\Leftrightarrow$ surjective.

If $f : E \longrightarrow F$ is a linear map between vector spaces of finite dimension, then:

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- f is surjective $\Leftrightarrow \dim \mathrm{Im}(f) = \dim F \Leftrightarrow \dim \mathrm{Nuc}(f) = \dim E - \dim F$.
- f is bijective $\Leftrightarrow \dim E = \dim F$ and $\mathrm{Nuc}(f) = \{\mathbf{0}\} \Leftrightarrow \dim E = \dim F$ and $\dim \mathrm{Im}(f) = \dim F$.
- If $\dim E = \dim F$, then f is bijective $\Leftrightarrow$ injective $\Leftrightarrow$ surjective.

If $f : E \longrightarrow F$ is a linear map between vector spaces of finite dimension, then:

- f is injective $\Leftrightarrow \text{Nuc}(f) = \{\mathbf{0}\} \Leftrightarrow \dim \text{Im}(f) = \dim E$.
- f is surjective $\Leftrightarrow \dim \text{Im}(f) = \dim F \Leftrightarrow \dim \text{Nuc}(f) = \dim E - \dim F$.
- f is bijective $\Leftrightarrow \dim E = \dim F$ and $\text{Nuc}(f) = \{\mathbf{0}\} \Leftrightarrow \dim E = \dim F$ and $\dim \text{Im}(f) = \dim F$.
- If $\dim E = \dim F$, then f is bijective $\Leftrightarrow$ injective $\Leftrightarrow$ surjective.

If $f : E \longrightarrow F$ is a linear map between vector spaces of finite dimension, then:

- f is injective $\Leftrightarrow \operatorname{Nuc}(f) = \{\mathbf{0}\} \Leftrightarrow \dim \operatorname{Im}(f) = \dim E$.
- f is surjective $\Leftrightarrow \dim \operatorname{Im}(f) = \dim F \Leftrightarrow \dim \operatorname{Nuc}(f) = \dim E - \dim F$.
- f is bijective $\Leftrightarrow \dim E = \dim F$ and $\operatorname{Nuc}(f) = \{\mathbf{0}\} \Leftrightarrow \dim E = \dim F$ and $\dim \operatorname{Im}(f) = \dim F$.
- If $\dim E = \dim F$, then f is bijective $\Leftrightarrow$ injective $\Leftrightarrow$ surjective.

PROPOSITION

If $\dim(E) = n$ *and* $B = \{v_1, \ldots, v_n\}$ *is a basis of* E*, then*

$$\begin{array}{ccc} E & \longrightarrow & \mathbb{K}^n \\ v & \mapsto & v_B \end{array}$$

is an isomorphism.

THEOREM

If E *and* F *are two* $\mathbb{K}$*-e.v of finite dimension, then*

$$E \cong F \Leftrightarrow \dim(E) = \dim(F).$$

OUTLINE

Consider now linear maps $f : E \longrightarrow F$ between $\mathbb{K}$-e.v. of finite dimension, $n = \dim E$, $m = \dim F$. Let $\mathbf{u} = \{u_1, \dots, u_n\}$ and $\mathbf{v} = \{v_1, \dots, v_m\}$ be bases of E and F (resp.).

Definition

The matrix of f in bases $\mathbf{u}$,$\mathbf{v}$ is the $m \times n$ matrix whose columns are the coordinates of $f(u_1), \dots, f(u_n)$ in the basis $\mathbf{v}$:

$$M_{\mathbf{u},\mathbf{v}}(f) = \left(f(u_1)_{\mathbf{v}} \ \cdots \ f(u_n)_{\mathbf{v}} \right).$$

Properties:

- If $E = \mathbb{K}^n$, $F = \mathbb{K}^m$ and $\mathbf{u}$, $\mathbf{v}$ are the standard bases $\Rightarrow$ this matrix is the *standard matrix* we have already introduced, $M(f)$.
- If $M_{\mathbf{u},\mathbf{v}}(f) = (a_{i,j})_{i,j} \Rightarrow f(u_j) = \sum_i a_{i,j} v_i$.
- $M_{\mathbf{u},\mathbf{v}}(f)(w_{\mathbf{u}}) = (f(w))_{\mathbf{v}}$.
- $M_{\mathbf{u},\mathbf{v}}(g \circ f) = M_{\mathbf{w},\mathbf{v}}(g) M_{\mathbf{u},\mathbf{w}}(f)$,

$$g \circ f : E_{\mathbf{u}} \ \underset{M_{\mathbf{u},\mathbf{w}}(f)}{\xrightarrow{f}} \ F_{\mathbf{w}} \ \underset{M_{\mathbf{w},\mathbf{v}}(g)}{\xrightarrow{g}} \ G_{\mathbf{v}} \ .$$

If $A_{\mathbf{u},\mathbf{e}}$ is the change-of-basis matrix from **u** to **e**, then this matrix can be thought as the matrix of the Identity map in certain basis:

$$A_{\mathbf{u}\to\mathbf{e}} = M_{\mathbf{u},\mathbf{e}}(Id).$$

Note: The matrix of the identity map is the Identity matrix if we put the same basis at both sides.

If $A_{\mathbf{u}\to\mathbf{u'}}$ is the change-of-basis matrix from **u** to **u'**, and $A_{\mathbf{v}\to\mathbf{e'}}$ is the change-of-basis matrix from **v** to **v'**, then:

$$M_{\mathbf{u'},\mathbf{v'}}(f) = A_{\mathbf{v}\to\mathbf{v'}}\, M_{\mathbf{u},\mathbf{v}}(f)\, A^{-1}_{\mathbf{u}\to\mathbf{u'}}\,,$$

$$M_{\mathbf{u},\mathbf{v}}(f) = A^{-1}_{\mathbf{v}\to\mathbf{v'}}\, M_{\mathbf{u'},\mathbf{v'}}(f)\, A_{\mathbf{u}\to\mathbf{u'}}\,.$$

Outline

An endomorphism is a linear map from E to itself.

Notation

- $End(E) = \{f : E \longrightarrow E \mid f \text{ linear map }\}$.
- If $f \in End(E)$ and $\mathbf{u} = \{u_1, \dots, u_n\}$ is a basis of E, we denote by $M_{\mathbf{u}}(f)$ the matrix $M_{\mathbf{u},\mathbf{u}}(f)$.
- Using composition we can define f^m for any $m \in \mathbb{N}$:

$$f^m = f \circ \overset{m)}{\dots} \circ f.$$

DEFINITION

The determinant of an endomorphism $f \in End(E)$ (E of finite dimension) is the determinant of its matrix in any basis $\mathbf{u}$,

$$\det(f) = \det(M_{\mathbf{u}}(f)).$$

This does not depend on the basis and

$$\det(g \circ f) = \det g \det f.$$

The set of linear maps between $\mathbb{K}$-e.v, E, F is denoted as $L(E, F)$. This is a $\mathbb{K}$-e.v with the usual sum and product by scalars of maps: if $f, g \in L(E, F)$ and $c \in \mathbb{K}$,

+ $f + g$ is the map $(f + g)(v) := f(v) + g(v)$, $v \in E$.
· $c \cdot f$ is the map $(c \cdot f)(v) := cf(v)$, $v \in E$.

Theorem

Let $\boldsymbol{u} = \{u_1, \ldots, u_n\}$ and $\boldsymbol{v} = \{u_1, \ldots, u_m\}$ be bases of E and F, respectively. Then the map

$$\begin{array}{rcl} \varphi : L(E, F) & \longrightarrow & \mathcal{M}_{m\times n}(\mathbb{K}) \\ f & \mapsto & M_{\boldsymbol{u},\boldsymbol{v}}(f) \end{array}$$

is an isomorphism.

Let $p(X) = a_m X^m + a_{m-1} X^{m-1} + \cdots + a_1 X + a_0$ be a polynomial in $\mathbb{K}[X]$; let $A \in \mathcal{M}_n(\mathbb{K})$ and $f \in End(E)$. We define

$$p(A) := a_m A^m + a_{m-1} A^{m-1} + \cdots + a_1 A + a_0 Id_n \in \mathcal{M}_n(\mathbb{K}),$$

$$p(f) := a_m f^m + a_{m-1} f^{m-1} + \cdots + a_1 f + a_0 Id \in End(E).$$

PROPOSITION

If $\boldsymbol{u}$ is a basis of E and $A = M_{\boldsymbol{u}}(f)$, then

$$M_{\boldsymbol{u}}(p(f)) = p(A).$$

Let $f \in End(E)$ and $F \subseteq E$ be a subspace.

DEFINITION

F is f-invariant (or invariant by f) if $f(F) \subseteq F$.
In this case we define the restriction of f to F, as the endomorphism $f_{|F}$ $\in End(F)$ defined by $f_{|F}(v) := f(v)$.

PROPOSITION

Let $\boldsymbol{u} = \{u_1 \dots u_n\}$ be a basis of E obtained by extension of a basis $B = \{u_1, \dots, u_d\}$ of a subspace $F \subset E$. Then F is f-invariant if and only if

$$M_{\boldsymbol{u}}(f) = \left(\begin{array}{c|c} A & * \\ \hline \boldsymbol{0} & * \end{array} \right),$$

where $A \in \mathcal{M}_d(\mathbb{K})$. In this case, $A = M_B(f_{|F})$.

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Dual space

Definition

If E is a $\mathbb{K}$-e.v., we define the dual space E^* as the $\mathbb{K}$-e.v $L(E, \mathbb{K})$. Its elements are called linear forms instead of vectors.

If $\mathbf{u} = \{u_1, \ldots, u_n\}$ is a basis of E, for each $i \in \{1, \ldots, n\}$ we call $u_i^* : E \to \mathbb{K}$ the unique linear map such that

$$u_i^*(u_j) = \begin{cases} 0, & \text{if } j \neq i \\ 1, & \text{if } j = i \end{cases}, \text{ for all } j \in \{1, \ldots, n\}.$$

Hence, $u_i^*(\sum x_j u_j) = x_i$.

Proposition

1. $\boldsymbol{u}^* = \{u_1^*, \ldots, u_n^*\}$ *is a basis of* E^**, called the dual basis of* $\boldsymbol{u}$.
2. *The coordinates of* $\omega \in E^*$ *in basis* $\boldsymbol{u}^*$ *are* $\omega(u_1), \ldots, \omega(u_n)$.
3. *The coordinates of* $v \in E$ *in basis* $\boldsymbol{u}$ *are* $u_1^*(v), \ldots, u_n^*(v)$.

In particular, if E has finite dimension $\Rightarrow \dim E^* = \dim E$.

Let $f \in L(E,F)$. We define the dual map f^* as the map

$$\begin{array}{rcl} f^* : F^* & \longrightarrow & E^* \\ \omega & \mapsto & f^*(\omega) := \omega \circ f \end{array}.$$

Note: $f^* \in L(F^*, E^*)$.

Proposition

If $\dim E, F < \infty$ *and* $\boldsymbol{u}$, $\boldsymbol{v}$ *are basis of* E, F *respectively, then*

$$M_{\boldsymbol{v}^*,\boldsymbol{u}^*}(f^*) = (M_{\boldsymbol{u},\boldsymbol{v}}(f))^t.$$

Corollary

If $\dim E < \infty$ *and* $\boldsymbol{u}$, $\boldsymbol{v}$ *are two basis of* E, *then*

$$A_{\boldsymbol{v}^*\to\boldsymbol{u}^*} = \left(A^{-1}_{\boldsymbol{v}\to\boldsymbol{u}}\right)^t.$$

Let E be a $\mathbb{K}$-e.v of finite dimension. For each $u \in E$, consider the map $\Phi_u : E^* \to \mathbb{K}$ defined as

$$\Phi_u(\omega) = \omega(u), \qquad \text{for each } \omega \in E^*.$$

Then, $\Phi_u \in E^{**} = (E^*)^*$, $\forall u \in E$.

THEOREM

$$\begin{array}{rcl} \Phi : E & \longrightarrow & E^{**} \\ u & \mapsto & \Phi(u) := \Phi_u \end{array}$$

is an isomorphism, $E \cong E^{**}$.

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