Unit 3. Discrete-time signals and systems in the frequency domain

2019-2020

Signals and Systems (DSE)

Fourier analysis for discrete-time signals

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- Fourier analysis
  - simplifies the study of linear and time invariant (LTI) systems
  - is useful to represent or analyze periodic phenomena

- Most signals in nature are continuous in time (or space) but data storage and processing are digital …

- In this unit we will extend Fourier analysis to signals and systems that are discrete in time

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Part 1: Discrete-time Fourier Transform (DTFT)

Extension of Fourier analysis to discrete-time signals and systems

- DTFT and inverse DTFT:
  - Definition, properties and FT basic sequences
  - Relationship between DTFT of $x[n]=x_a(nT)$ and the CTFT of $x_a(t)$

    (how to sample to avoid loss of information?)
- Response to discrete-time LTI systems
- DTFT of periodic sequences

DTFT and CTFT

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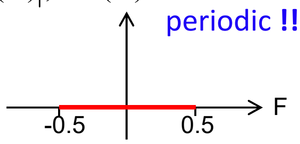
For a discrete-time signal $x[n]$:

$$X(F) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi Fn}$$

$|X(F)|, \sphericalangle X(F)$ Always 1-periodic **!!**

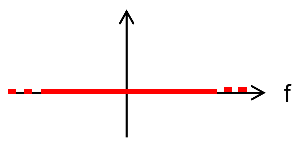


$$x[n] = \int_{\langle 1\rangle} X(F)e^{j2\pi Fn}dF$$

Continuous-time signal $x_a(t)$:

$$X_a(f) = \int_{-\infty}^{\infty} x_a(t)e^{-j2\pi ft}dt$$

$|X_a(f)|, \sphericalangle X_a(f)$



$$x_a(t) = \int_{-\infty}^{\infty} X_a(f)e^{j2\pi ft}df$$

Properties of DTFT

$x[n] \leftrightarrow X(F)$, $y[n] \leftrightarrow Y(F)$

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Similar properties to the continous-time Fourier transform (CTFT):

- Linearity: $\alpha \cdot x[n] + \beta \cdot y[n] \leftrightarrow \alpha \cdot X(F) + \beta \cdot Y(F)$
- Time inversion: $x[-n] \leftrightarrow X(-F)$
- $x[n] = x^*[n]$ (real) $\leftrightarrow X(F) = X^*(-F)$ (hermiticity)
- Time-shift: $x[n - n_o] \leftrightarrow e^{-j2\pi F n_o} X(F)$
- Modulation: $x[n] \cdot e^{j2\pi F_0 n} \leftrightarrow X(F - F_0)$
- Convolution: $x[n] * y[n] \leftrightarrow X(F)\,Y(F)$
- Frequency derivative: $-j2\pi n x[n] \leftrightarrow \frac{dX(F)}{dF}$

$X(F)$ is usually represented in (0,0.5) for real signals

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Properties of DTFT

$x[n] \leftrightarrow X(F)$, $y[n] \leftrightarrow Y(F)$

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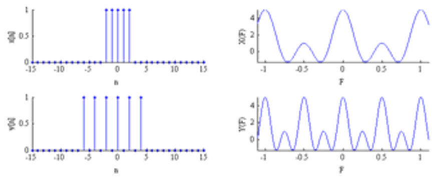
with some differences ...

- Time scale: $y[n] = \begin{cases} x\left[\frac{n}{N}\right] & \text{para } n = \dot{N} \\ 0 & \text{resto} \end{cases} \leftrightarrow Y(F){=}X(NF)$



- Parseval theorem: $\sum_{n=-\infty}^{\infty} x[n] \cdot y^*[n] = \int_{-0.5}^{0.5} X(F) Y^*(F) dF$

  If $x[n] = y[n]$: $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \int_{-0.5}^{0.5} |X(F)|^2 dF$$

- Product (windowing): $x[n]\,y[n] \leftrightarrow X(F) \circledast Y(F) = \int_{-0.5}^{0.5} X(F')Y(F - F')\,dF'$

  Periodic convolution

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DTFT of basic discrete-time signals

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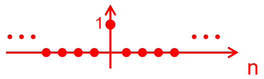
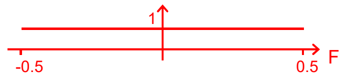
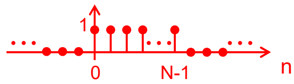
- x [n] =δ[n] ⟷ $X(F) = 1$





$$X(F) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j2\pi Fn} = \delta[0] e^{-j2\pi F0} = 1$$

- x[n] $\begin{cases} 1 & 0 \le n < N \\ 0 & n<0,\ n \ge N \end{cases}$ ⟷ $X(F) = e^{-j\pi F(N-1)} \dfrac{sin(\pi FN)}{sin(\pi F)}$



???

$$X(F) = \sum_{n=0}^{N-1} e^{-j2\pi Fn} = \frac{1 - e^{-j2\pi FN}}{1 - e^{-j2\pi F}} = \frac{e^{-j2\pi F\frac{N}{2}}\left(1 - e^{-j2\pi FN}\right)}{e^{-j2\pi F\frac{1}{2}}\left(1 - e^{-j2\pi FN}\right)} = e^{-j\pi F(N-1)} \frac{\sin(\pi FN)}{\sin(\pi F)}$$

"Analog sinc" versus "digital sinc"

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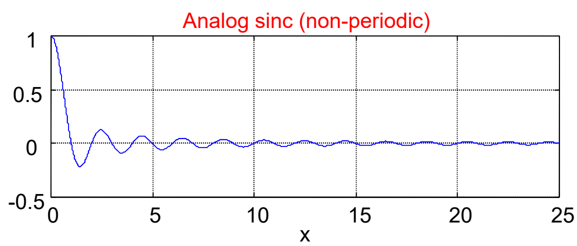
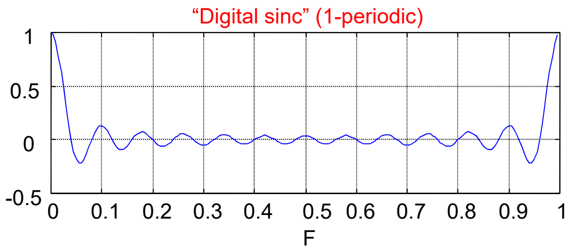
$$\frac{\sin(\pi x)}{\pi x}$$



$$\frac{\sin(\pi F 25)}{25 \sin(\pi F)}$$



DTFT of basic discrete-time signals

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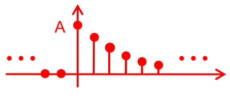
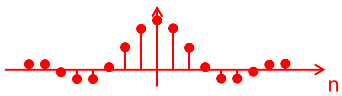
- x[n]=A·aⁿ·u[n], |a|<1 ⟷ $X(F) = \frac{A}{1 - a \cdot e^{-j2\pi F}}$

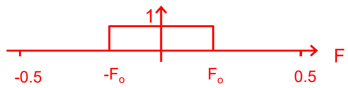


???

$$X(F) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j2\pi F n} = \sum_{n=0}^{\infty} \left(a \cdot e^{-j2\pi F}\right)^n = \frac{1}{1 - a \cdot e^{-j2\pi F}}$$

- $\frac{\sin(\pi 2F_0 n)}{\pi n} = 2F_0 \mathrm{sinc}(2F_0 n) \longleftrightarrow X(F) = \sum_{r=-\infty}^{\infty} \Pi\left(\frac{F-r}{2F_0}\right)$





Proof ???

DTFT of basic discrete-time signals

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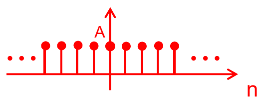
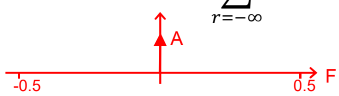
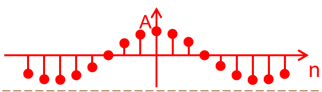
- x[n]=A ⟷ $X(F) = A \sum_{r=-\infty}^{\infty} \delta(F-r)$

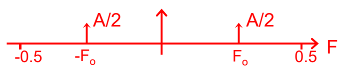




$$x[n] = \int_{-0.5}^{0.5} A\,\delta(F) e^{j2\pi F n}\, dF = A$$

- $x[n] = A\cos(2\pi F_0 n) \longleftrightarrow X(F) = \frac{A}{2} \sum_{r=-\infty}^{\infty} [\delta(F - F_o - r) + \delta(F + F_o - r)]$





$$x[n] = \int_{-0.5}^{0.5} \left(\frac{A}{2}\,\delta(F - F_0) + \frac{A}{2}\delta(F + F_0)\right) e^{j2\pi F n}\, dF = \left(\frac{A}{2}\, e^{j2\pi F_0 n} + e^{-j2\pi F_0 n}\right) = A\cos(2\pi F_0 n)$$

DTFT of basic discrete-time signals

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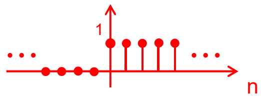
- x[n]=u[n] $\longleftrightarrow$ $X(F) = \frac{1}{2}\sum_{r=-\infty}^{\infty} \delta(F-r) + \frac{1}{1-e^{-j2\pi F}}$



$$u[n] = \frac{1}{2} + \frac{1}{2}\delta[n] + \frac{1}{2}\mathrm{sign}[n]$$

$$\mathrm{sign}[n] - \mathrm{sign}[n-1] = \delta[n] + \delta[n-1] \quad \Rightarrow \quad F\big[\mathrm{sign}[n]\big] = \frac{1+e^{-j2\pi F}}{1-e^{-j2\pi F}}$$

$$F\big[u[n]\big] = \frac{1}{2}\sum_{r=-\infty}^{\infty} \delta(F-r) + \frac{1}{2} + \frac{1}{2}\frac{1+e^{-j2\pi F}}{1-e^{-j2\pi F}} = \frac{1}{2}\sum_{r=-\infty}^{\infty} \delta(F-r) + \frac{1}{1-e^{-j2\pi F}}$$