<u>Problem</u>: Set $T_t g(x) = \tilde{g}(x-t)$, $\tilde{g}(x) = \begin{cases} g(x) & x \geq 0 \\ 0 & x < 0 \end{cases}$

Assume $f \in C^0([0,T]; L^2(0,1)) =: \mathbb{X}$

$$\|f\|_{\mathbb{X}} := \sup_{t\in[0,T]} \|f(\cdot,t)\|_{L^2(0,1)} < +\infty$$

Set $v(x,t) := \int_0^t (T_{t-s} f(\cdot,s))(x)\,ds$.

We know $v(\cdot,t) \in L^2(0,1)\ \forall\, t>0$.

Let us show that, $\forall\, 0<t<T$,

$$\lim_{\delta\to 0} \frac{1}{\delta}\int_0^\delta |v(x,t)|\,dx = 0 \qquad (1)$$

[this is a reasonable way to interpret "$v(0,t)=0$"]

$(T_{t-s} f(\cdot,s))(x) = f(x-(t-s), s)$ assuming that $f(x,t) = 0\ \forall\, x<0$ (say $f \equiv \tilde{f}$).

Take $t>0$ and assume $0<\delta<t$. Then,

$$\frac{1}{\delta}\int_0^\delta |v(x,t)|\,dx \leq \frac{1}{\delta}\int_0^\delta\int_0^t |f(x-(t-s),s)|\,ds\,dx$$

$$= \frac{1}{\delta}\int_0^\delta\int_0^t \chi_{\{x-t+s>0\}} |f(x-(t-s),s)|\,ds\,dx$$

(Fubini)

$$= \frac{1}{\delta}\underbrace{\int_0^t\int_0^\delta \chi_{\{x-t+s>0\}}}_{\left\{\begin{smallmatrix} x-t+s>0 \\ 0<x<\delta<t \\ 0<s<t\end{smallmatrix}\right\} \Longrightarrow \left\{\begin{smallmatrix} t-s<x<\delta \\ s>t-x>t-\delta\end{smallmatrix}\right\}} |f(x-(t-s),s)|\,dx\,ds$$

$$= \frac{1}{\delta}\int_{t-\delta}^t\int_{t-s}^\delta |f(x-(t-s),s)|\,dx\,ds$$

(CS)

$$\leq \frac{1}{\delta}\int_{t-\delta}^t \underbrace{|\delta-(t-s)|^{1/2}}_{\leq \delta} \Big(\int_{t-s}^\delta |f(x-(t-s),s)|^2\,dx\Big)^{1/2} ds$$

$(0 < x-(t-s) < \delta - t + s < \delta - t + t = \delta < 1)$

$$\leq \frac{1}{\delta}\int_{t-\delta}^{t} \delta^{1/2} \underbrace{\|f(\cdot,s)\|_{L^2(0,1)}}_{\leq \|f\|_{X}} ds \qquad \text{(since } s\in(t-\delta,t),\ \delta<t<T.)$$

$$\leq \delta^{1/2}\|f\|_{X}\, \frac{1}{\delta}\int_{t-\delta}^{t} ds = \delta^{1/2}\|f\|_{X}$$

This proves (1).