Chapter 6. Finite Automata

Algorithmics and Programming III

FIB

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Motivation

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  for (int j = 0; j < p.size(); ++j)
    if (p[j] != t[i+j]) return false;
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  for (int i = 0; i + p.size() <= t.size(); ++i)
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- In the worst case it makes $\Theta(|p| \cdot |t|)$ comparisons of characters
- But it is rather naive: it does not use info of previous attempts

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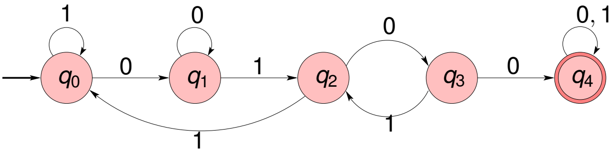
- Let us assume the text only contains binary digits
- For searching e.g. $p = 0100$ we can use the following finite automaton:



- This automaton accepts exactly the texts that contain p

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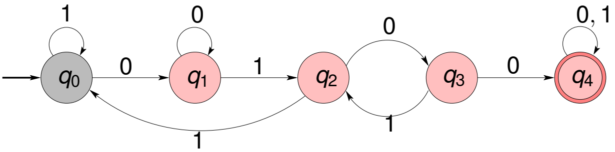

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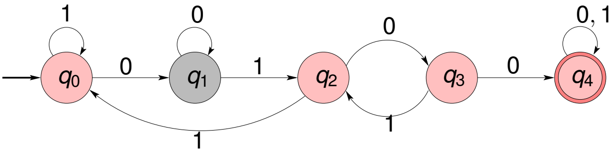

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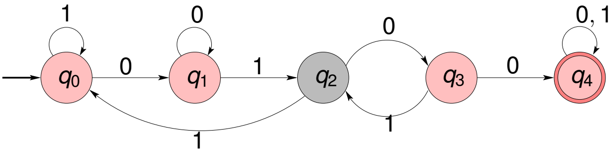

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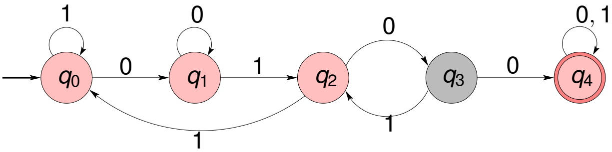

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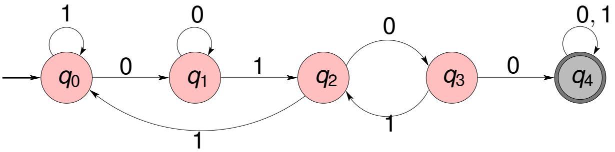

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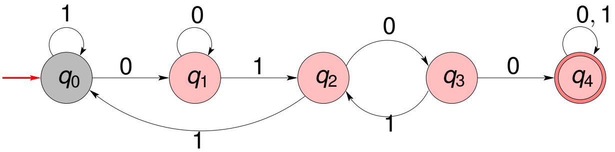

- It has one initial state (indicated by the arrow without source node)

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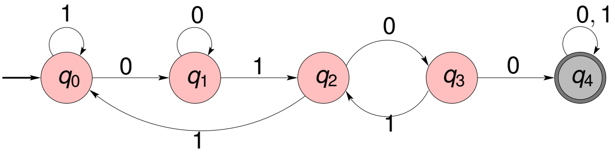

- It has one accepting state (indicated by the double circle)

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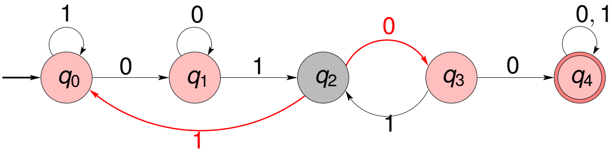

- Each state has two transitions, one for each symbol

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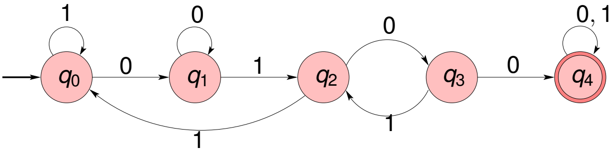

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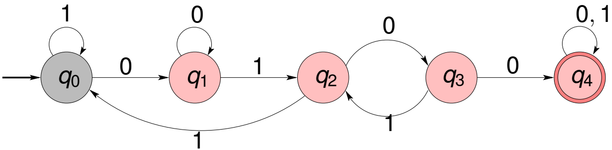

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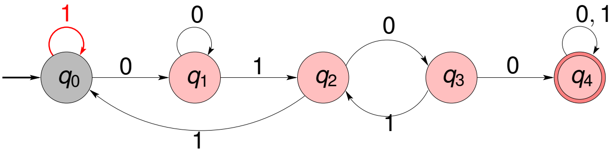

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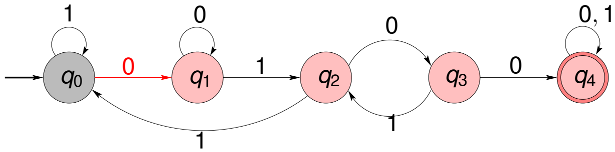

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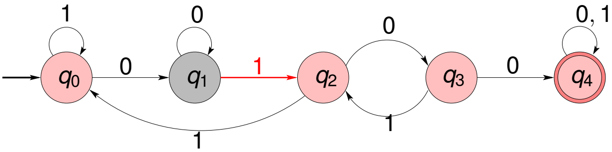

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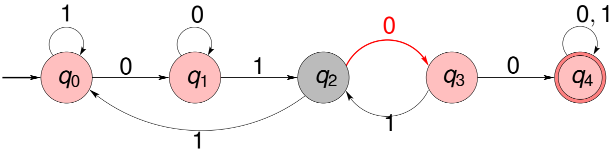

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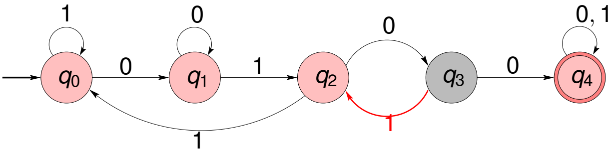

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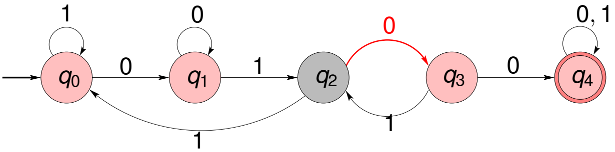

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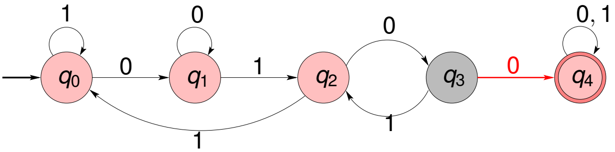

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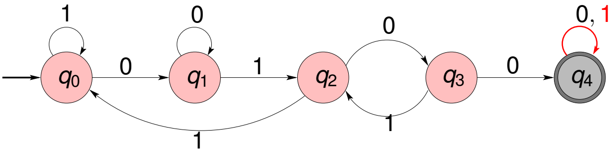

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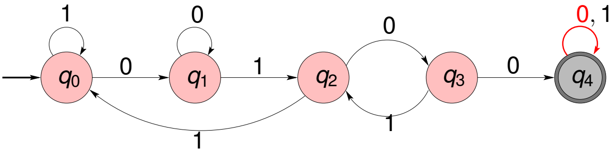

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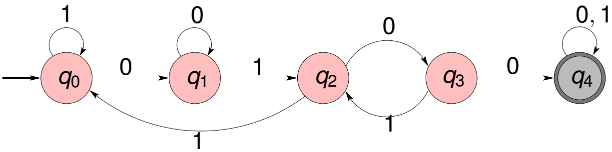

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- For instance let us run the automaton on the text 101010010
- Since in the end we are at an accepting state, the text is accepted

Motivation

- It can be proved that for any pattern p
  one can build a finite automaton recognizing p in time $\Theta(|p|)$
- For processing a text t this automaton takes exactly $|t|$ steps
- Algorithm for pattern search based on finite automata costs $\Theta(|p| + |t|)$
- Compare with the worst-case cost $\Theta(|p| \cdot |t|)$ of the naive algorithm!

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- An alphabet will be usually represented with the letter Σ

Words

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- For example:
  - $\texttt{FIB}$ is a word over the Latin alphabet $\{\texttt{A}, \texttt{B}, \texttt{C}, \ldots, \texttt{X}, \texttt{Y}, \texttt{Z}\}$
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- The length of a word ω, denoted $|\omega|$, is the number of its symbols; e.g.,
  - $|\texttt{FIB}| = 3$
  - $|\texttt{2019}| = 4$
  - $|\texttt{010100}| = 6$
  - $|\lambda| = 0$

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- For example, over the alphabet $\Sigma = \{0, 1\}$, we can consider the language $\mathcal{L}$ of all words that contain an occurrence of 0100:

$$\mathcal{L} = \{\lambda, 0100, 00100, 10100, 01000, 01001, \ldots\}$$

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- $\mathcal{L} = \emptyset$ and $\mathcal{L} = \{\lambda\}$ are other examples of languages

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  - $F \subseteq Q$, called the set of final or accepting states

- $(Q, \Sigma, \delta, q_0, F)$ is an example of DFA, where:
  - $Q = \{q_0, q_1, q_2, q_3, q_4\}$
  - $\Sigma = \{0, 1\}$
  - δ is described by the following transition table:

	q_0	q_1	q_2	q_3	q_4
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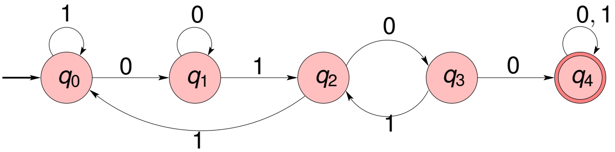
- An alternative representation of the automaton with a transition diagram:



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- If q is a state and a is a symbol,
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  - for any $q \in Q$, $\delta(q, \lambda) = q$
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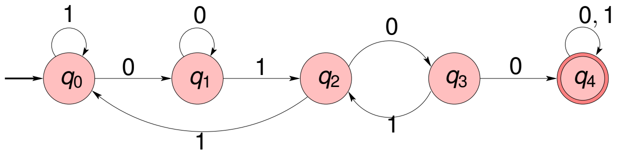

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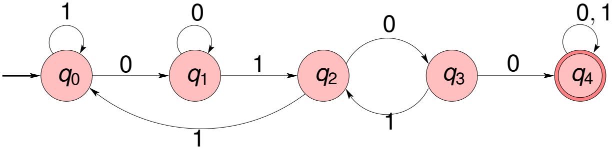

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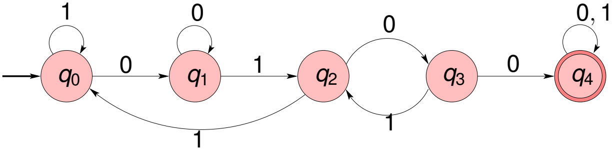

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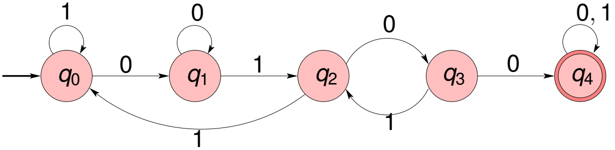

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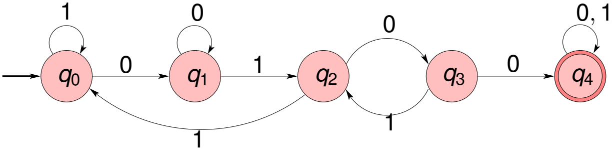

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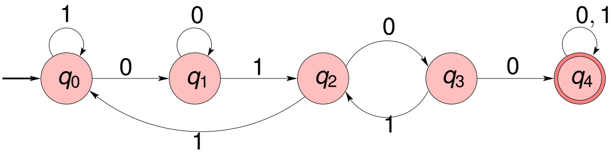

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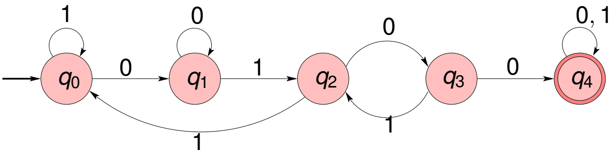

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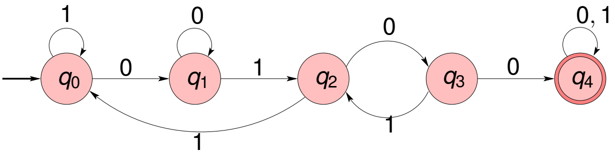

  consists of all words that contain an occurrence of 0100
- A language $\mathcal{L}$ is called regular if it is the language of some DFA
- So the language of all words containing an occurrence of 0100 is regular

Nondeterministic Finite Automata

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  - there is a single initial state
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- In nondeterministic finite automata (NFA) this is no longer true
- NFA's have the same expressive power as DFA's:
  a language is accepted by some DFA iff it is accepted by some NFA
- But NFA's are usually more compact and easier to design

Nondeterministic Finite Automata

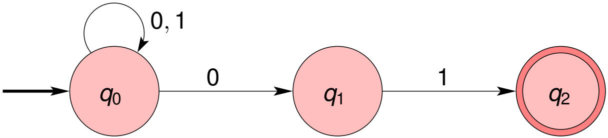
- The difference between DFA's and NFA's is in the transition function δ

  In NFA's:

  - δ is a function that takes a state and an input symbol as arguments (as in DFA's)
  - δ returns a set of zero, one, or more states
    (rather than returning exactly one state, as DFA's do)

Nondeterministic Finite Automata

- For example, this NFA accepts exactly the words that end in 01:



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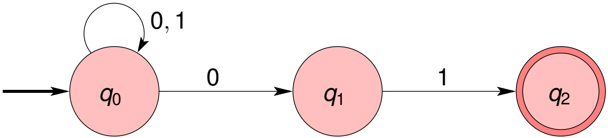

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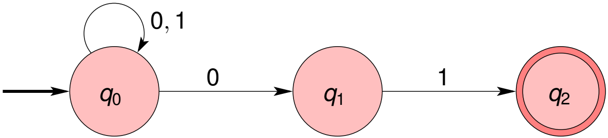

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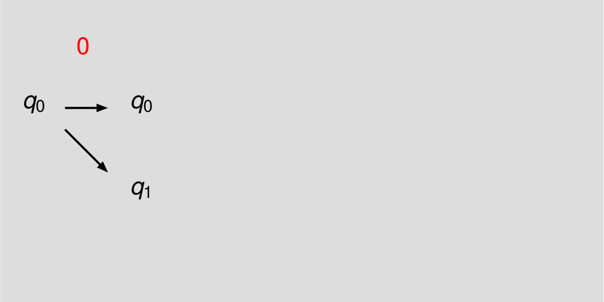

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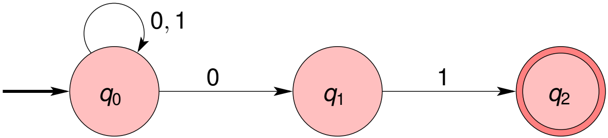

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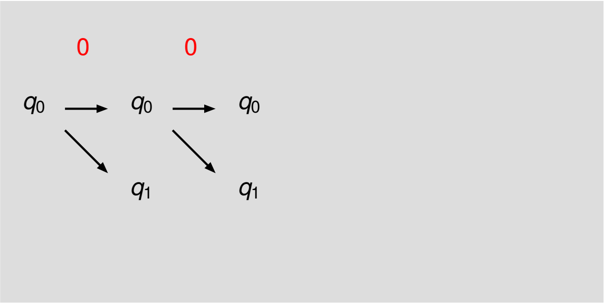

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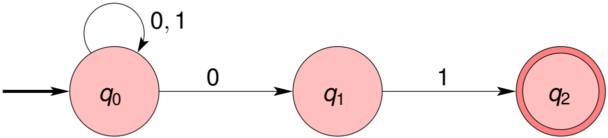

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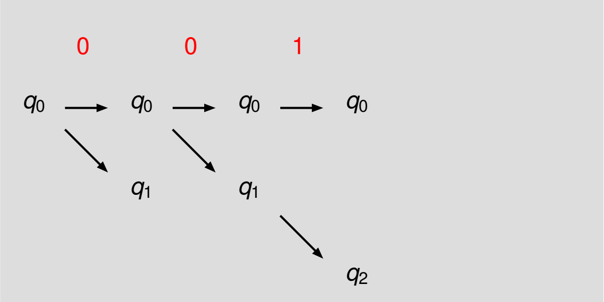

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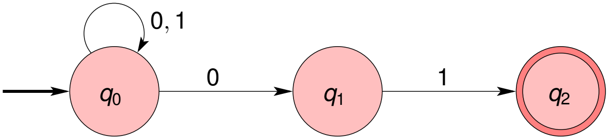

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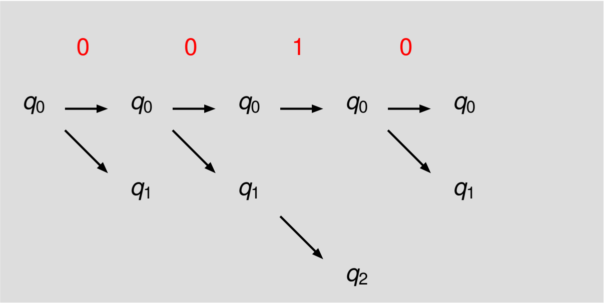

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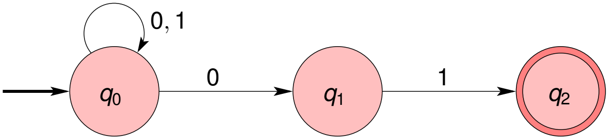

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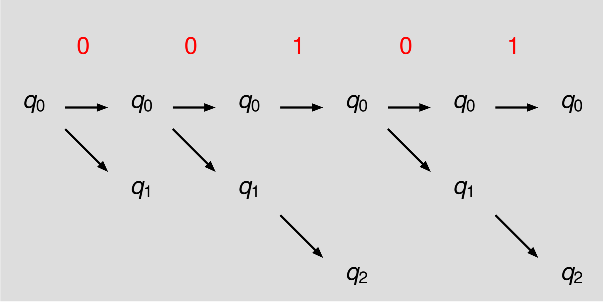

Nondeterministic Finite Automata

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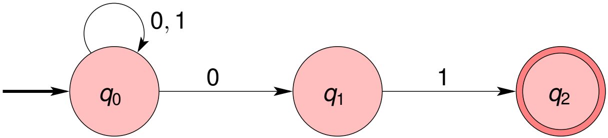

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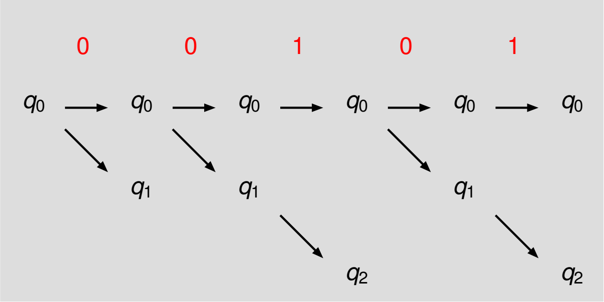

- Threads are run simultaneously, so the NFA can be in several states

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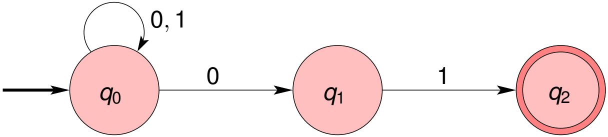
  - $q_0 \in Q$, called the initial state
  - $F \subseteq Q$, called the set of final or accepting states

Nondeterministic Finite Automata

- $(Q, \Sigma, \delta, q_0, F)$ is an example of NFA, where:
  - $Q = \{q_0, q_1, q_2\}$
  - $\Sigma = \{0, 1\}$
  - the initial state is q_0
  - $F = \{q_4\}$
  - δ is described by the following transition table:

	q_0	q_1	q_2
0	$\{q_0, q_1\}$	$\emptyset$	$\emptyset$
1	$\{q_0\}$	$\{q_2\}$	$\emptyset$

- The representation of the same NFA with a transition diagram:



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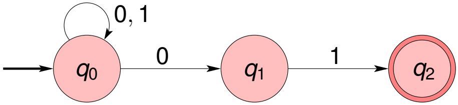

$\delta(q_0, 00101)$

Nondeterministic Finite Automata

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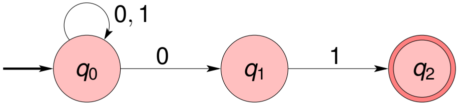

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&= \delta(q_0, 0101) \cup \delta(q_1, 0101)
\end{aligned}$$

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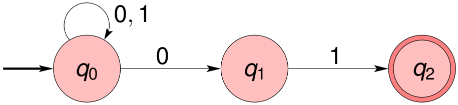

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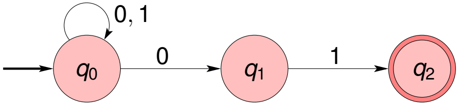

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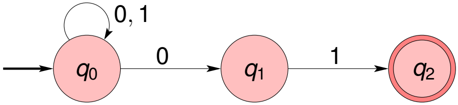

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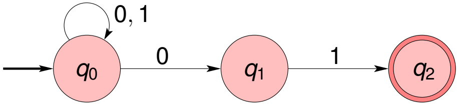

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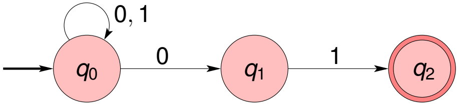

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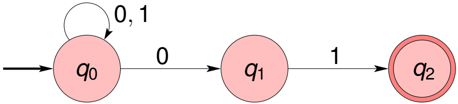

$$
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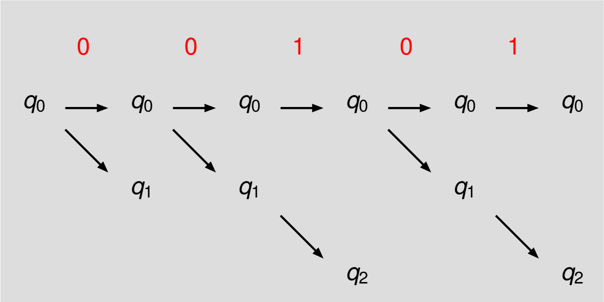
$\delta(q, \omega)$ is the column of states after reading ω if 1st column consists of q only



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- Intuitively, an NFA accepts a word ω if we can choose the next states while reading ω so that we go from the start state to an accepting state
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Subset Construction

- Given an NFA N, the subset construction allows constructing a DFA D that accepts the same language as N
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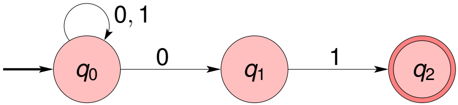
  - F_D is the set of $S \subseteq Q_N$ such that $S \cap F_N \neq \emptyset$ (S is an accepting state of D if it contains an accepting state of N)

Subset Construction

- Given an NFA N, the subset construction allows constructing a DFA D that accepts the same language as N
- Let $N = (Q_N, \Sigma, \delta_N, q_0, F_N)$ be an NFA
- Let $D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$ be a DFA where:
  - The input alphabets are the same
  - $Q_D = 2^{Q_N}$, i.e., states of Q_D are sets of states (subsets) of Q_N
  - The initial state of D is a singleton consisting of the initial state of N
  - F_D is the set of $S \subseteq Q_N$ such that $S \cap F_N \neq \emptyset$ (S is an accepting state of D if it contains an accepting state of N)
  - For each $S \subseteq Q_N$ and for each $a \in \Sigma$,

$$\delta_D(S, a) = \bigcup_{q \in S} \delta_N(q, a)$$

For example, if *N* is



then the transition table of *D* is as follows:

	0	1
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$

Subset Construction

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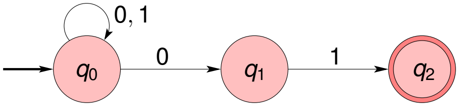

then the transition table of D is as follows:

	0	1
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$

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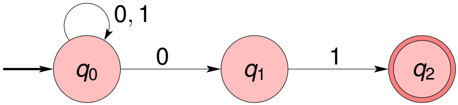

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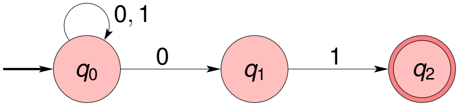

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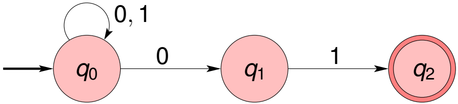
$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$
$\{q_0, q_2\}$	$\{q_0, q_1\}$	$\{q_0\}$

States of D that are unreachable from $\{q_0\}$ can be ignored, so the number of states of D is usually smaller than $2^{|Q_N|}$

- Procedure for constructing the transition table
  1. Add transitions from $\{q_0\}$ to the table
  2. For each new state S, add transitions from S to the table

Subset Construction

For example, if N is



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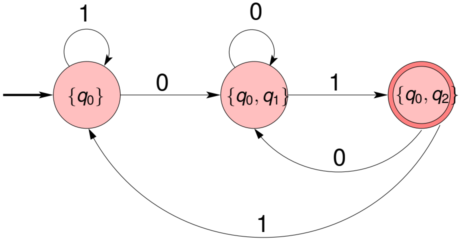

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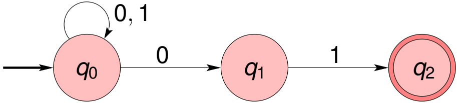

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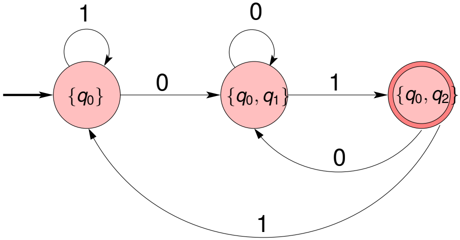
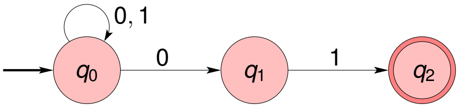


This is a proper DFA!

Though entries in table are sets, states of the automaton **are** sets

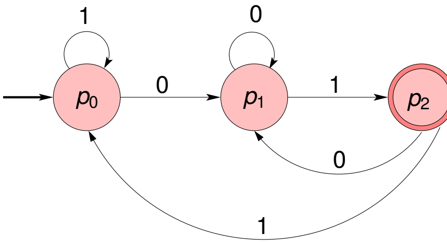
Subset Construction

For example, if N is



then the transition table of D is as follows:

	0	1
p_0	p_1	p_0
p_1	p_1	p_2
p_2	p_1	p_0



Subset Construction

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After reading word ω,
D is in the state that is the set of NFA states that N would be in after reading ω

But the accepting states of D are the sets that include an accepting state of N, and N also accepts if it gets into at least one of its accepting states.

So D and N accept exactly the same words.

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Theorem

A language $\mathcal{L}$ is accepted by some NFA iff $\mathcal{L}$ is accepted by some DFA

I.e., a language $\mathcal{L}$ is accepted by some NFA iff $\mathcal{L}$ is regular

Finite Automata with λ-Transitions

- Next we will give yet another extension of finite automata
- Now an NFA will be allowed to make a transition spontaneously, without consuming any input symbol
- These transitions are called λ-transitions, as λ stands for the empty word
- They do not expand the class of languages accepted by finite automata, but they do give us some added “programming convenience”

Finite Automata with λ-Transitions

- View the automaton as accepting the sequences of labels along paths from the start state to an accepting state

Finite Automata with λ-Transitions

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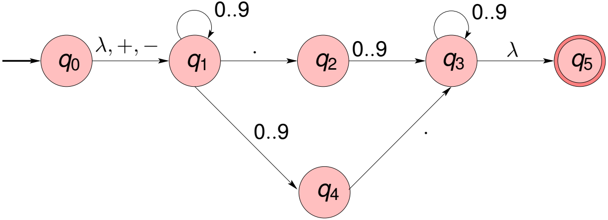
  But each λ is invisible: it contributes nothing to the word along the path

- E.g., the following automaton



  accepts decimal numbers consisting of:

  1. An optional $+$ or $-$ sign,
  2. A string of digits,
  3. A decimal point, and
  4. Another string of digits.

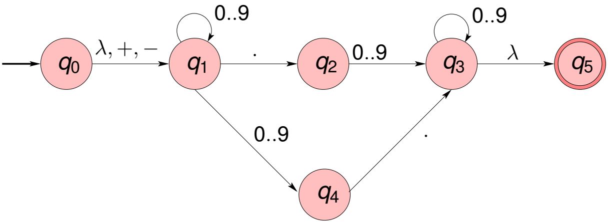
  Strings (2) or (4) can be empty, but at least one is not

Finite Automata with λ-Transitions

- A λ-nondeterministic finite automaton (λ-NFA) consists of:
  - Q, a finite set of states
  - Σ, an alphabet (called input alphabet)
  - δ, a transition function from $Q \times (\Sigma \cup \{\lambda\})$ to 2^Q
  - $q_0 \in Q$, called the initial state
  - $F \subseteq Q$, called the set of final or accepting states

Finite Automata with λ-Transitions

The transition table of



is

	λ	$+, -$	.	0..9
q_0	$\{q_1\}$	$\{q_1\}$	$\emptyset$	$\emptyset$
q_1	$\emptyset$	$\emptyset$	$\{q_2\}$	$\{q_1, q_4\}$
q_2	$\emptyset$	$\emptyset$	$\emptyset$	$\{q_3\}$
q_3	$\{q_5\}$	$\emptyset$	$\emptyset$	$\{q_3\}$
q_4	$\emptyset$	$\emptyset$	$\{q_3\}$	$\emptyset$
q_5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Finite Automata with λ-Transitions

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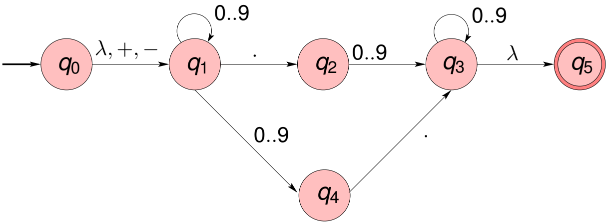
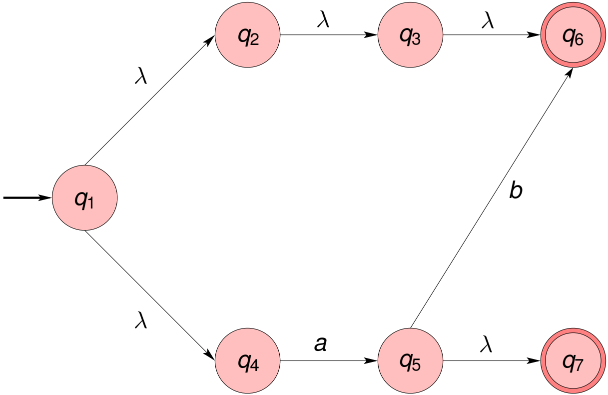


each state is its own λ-closure, except for q_0 and q_3:

- $\Lambda(q_0) = \{q_0, q_1\}$
- $\Lambda(q_3) = \{q_3, q_5\}$

Finite Automata with λ-Transitions



- $\Lambda(q_1) = \{q_1, q_2, q_3, q_4, q_6\}$
- $\Lambda(q_2) = \{q_2, q_3, q_6\}$
- $\Lambda(q_4) = \{q_4\}$
- ...

Finite Automata with λ-Transitions

- As usual we will extend transition functions from symbols to words
- If q is a state and ω is a word,
  then $\hat{\delta}(q, \omega)$ will be the states we can reach from q after reading word ω
  but taking into account that λ-transitions do not consume symbols

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- Given $\delta : Q \times (\Sigma \cup \{\lambda\}) \to 2^Q$, we extend it to $\hat{\delta} : Q \times \Sigma^* \to 2^Q$:
  - for any $q \in Q$, $\hat{\delta}(q, \lambda) = \Lambda(q)$
  - for any $q \in Q$ and word of the form $a\,\omega$,

$$\hat{\delta}(q, a\,\omega) = \bigcup_{p \in \Lambda(q)} \bigcup_{r \in \delta(p,a)} \hat{\delta}(r, \omega)$$

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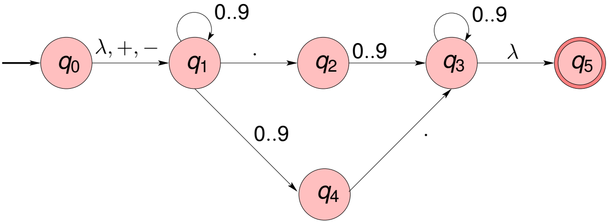
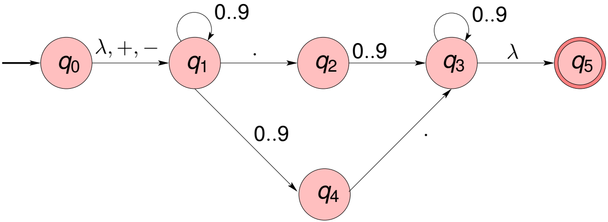
- Note the difference between
  $\delta(q, a)$ (the states we can move after reading symbol a in one transition)
  $\hat{\delta}(q, a)$ (allowing λ-transitions before/after reading symbol a)

Finite Automata with λ-Transitions



- $\hat{\delta}(q_0, 5.6)$

Finite Automata with λ-Transitions



- $\hat{\delta}(q_0, 5.6)$
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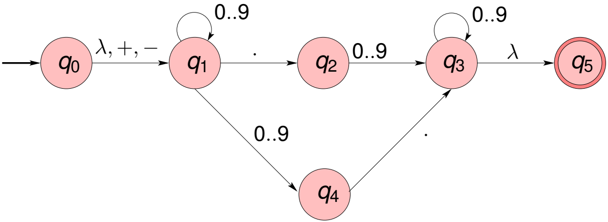
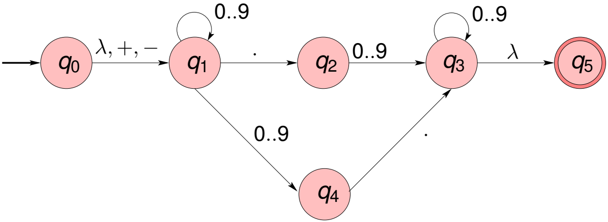
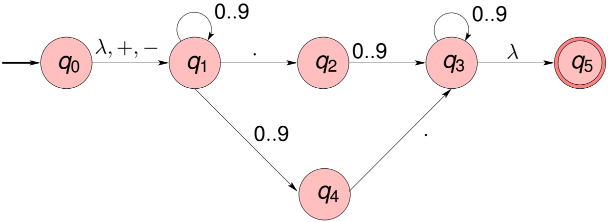


- $\hat{\delta}(q_0, 5.6)$
  - $\Lambda(q_0) = \{q_0, q_1\}$
  - $\delta(q_0, 5) = \emptyset$



- $\hat{\delta}(q_0, 5.6)$
  - $\Lambda(q_0) = \{q_0, q_1\}$
  - $\delta(q_0, 5) = \emptyset$
  - $\delta(q_1, 5) = \{q_1, q_4\}$

Finite Automata with λ-Transitions



- $\hat{\delta}(q_0, 5.6) = \hat{\delta}(q_1, .6) \cup \hat{\delta}(q_4, .6)$
  - $\Lambda(q_0) = \{q_0, q_1\}$
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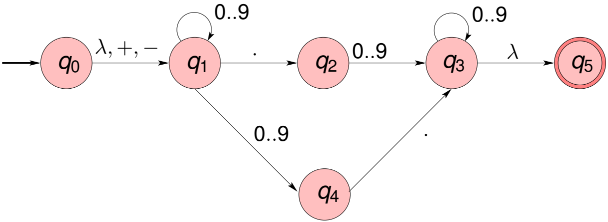
Finite Automata with λ-Transitions



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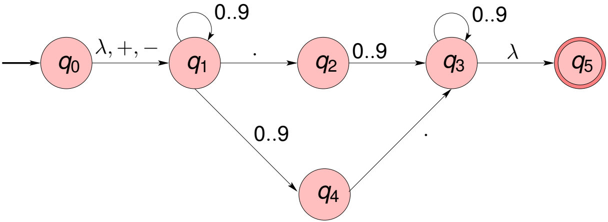
Finite Automata with λ-Transitions



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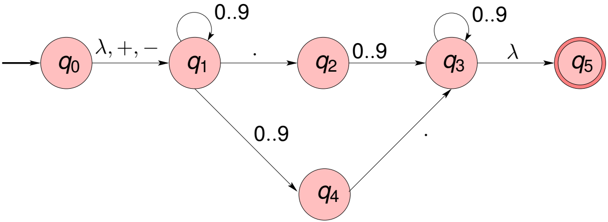

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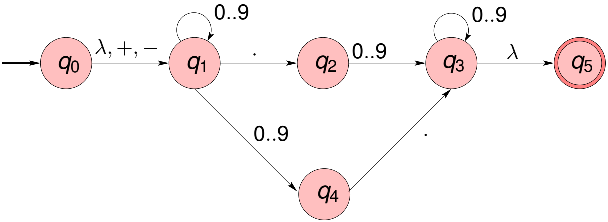

- $\hat{\delta}(q_0, 5.6) = \hat{\delta}(q_2, 6) \cup \hat{\delta}(q_4, .6)$
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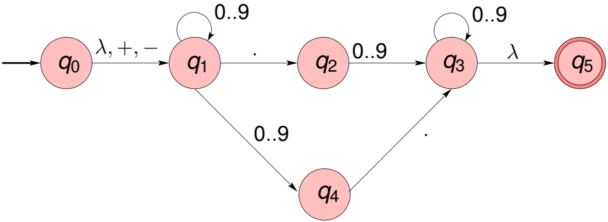
Finite Automata with λ-Transitions



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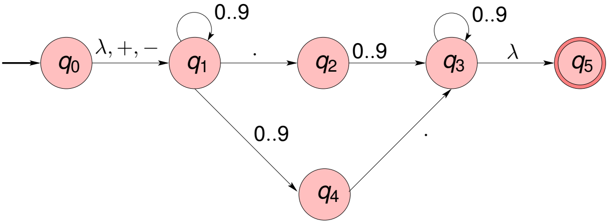
Finite Automata with λ-Transitions



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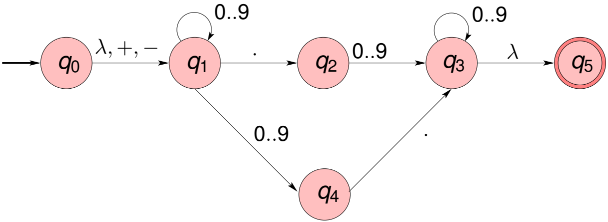

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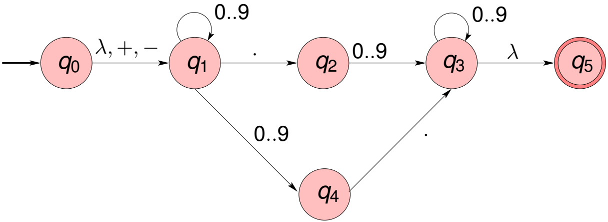
Finite Automata with λ-Transitions



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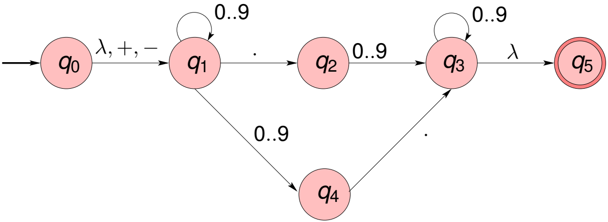

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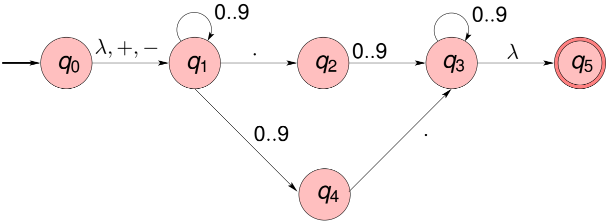
Finite Automata with λ-Transitions



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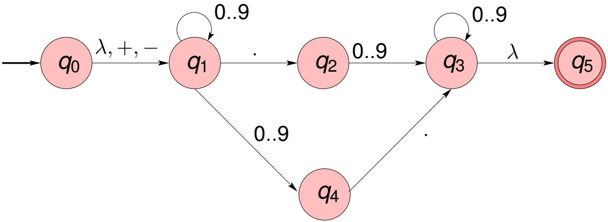

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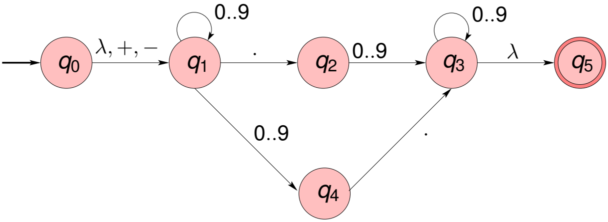

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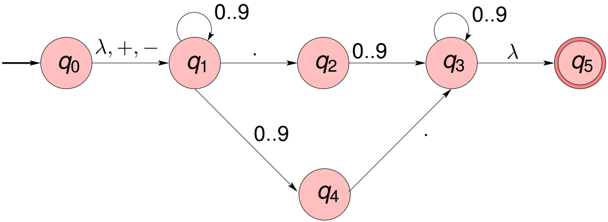

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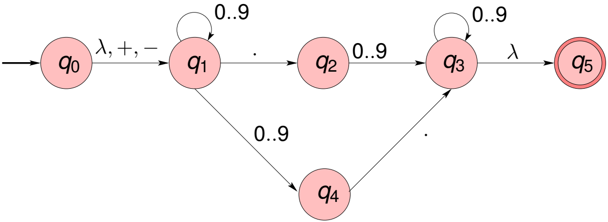
Finite Automata with λ-Transitions



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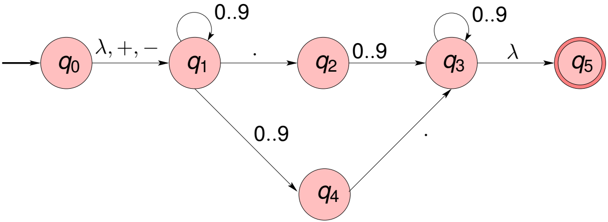

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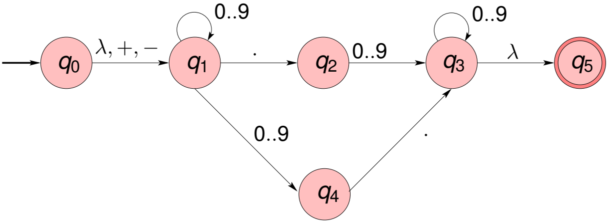

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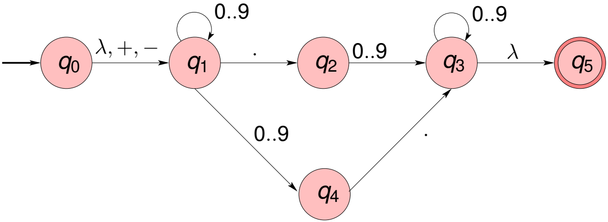
Finite Automata with λ-Transitions



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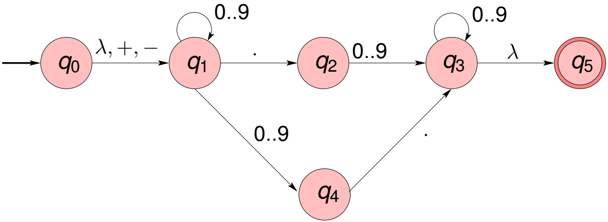
  - $\delta(q_3, 6) = \{q_3\}$
  - $\delta(q_5, 6) = \emptyset$

Finite Automata with λ-Transitions



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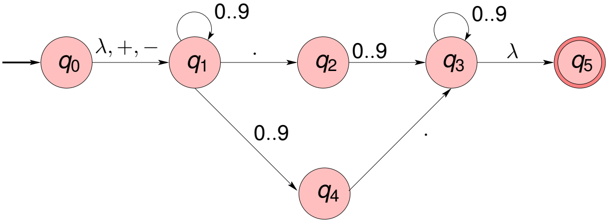
- For example, for the automaton A



  since $\hat{\delta}(q_0, 5.6) = \{q_3, q_5\}$ we conclude that $5.6 \in L(A)$

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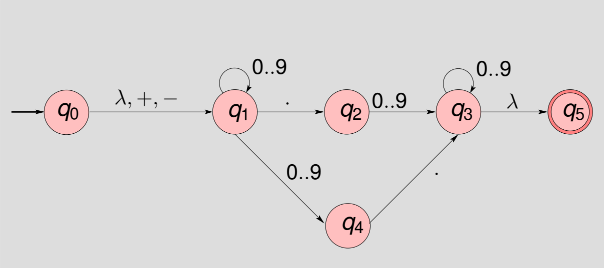

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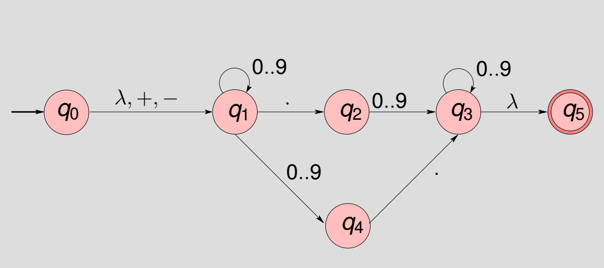

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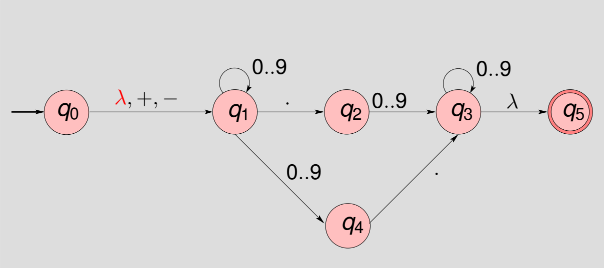

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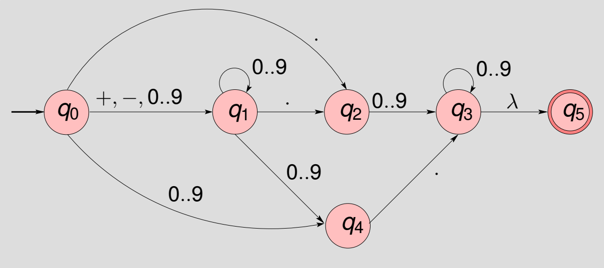

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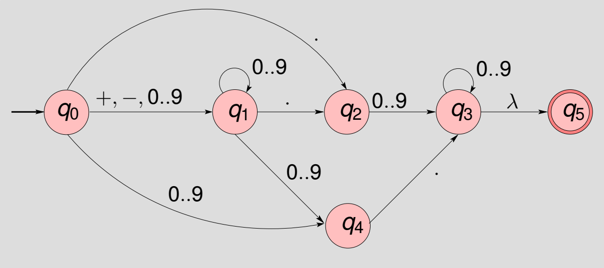

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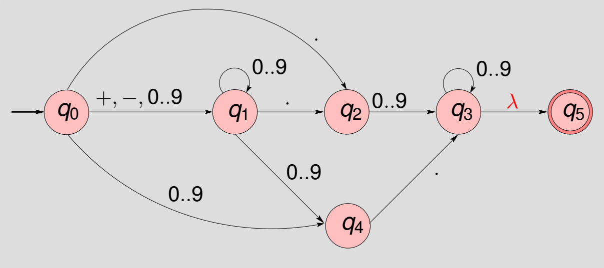

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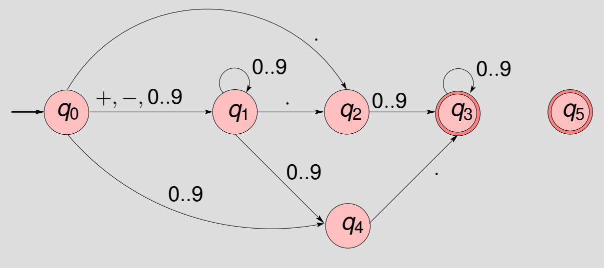

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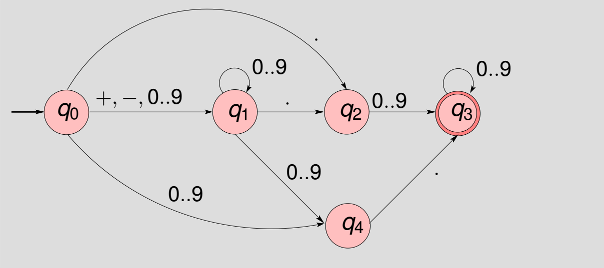

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$$\delta_N(p, a) = \bigcup_{q \in \Lambda(p)} \delta_A(q, a)$$

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- Both automata accept the same language: $L(A) = L(N)$

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- Finite automata can be viewed as mechanisms for defining languages
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- Both describe exactly the same class of languages: regular languages
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- Next we will see that regexps can be easily translated into λ-NFA's
  (which in turn can be translated into NFA's,
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- For example:

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[a-z][a-z]*(\.[a-z][a-z]*)*@(est\.fib|estudiant)\.upc\.edu
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is a regexp for recognizing the emails of students at AP3

(written in Linux *extended regular expression notation*)

  - `[a-z]` represents any character `a`, `b`, `c`, ..., `x`, `y`, `z`
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- To find the lines of file `p.txt` containing an email, using `grep`:

```
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- The star (or Kleene closure) of L is $L^* = \{\lambda\} \cup L \cup L^2 \cup L^3 \cup \cdots = \bigcup_{i=0}^{\infty} L^i$
- It is the set of the strings that can be formed by taking any number of strings from L, possibly with repetitions, and concatenating them
- If $L = \{00, 01\}$, then $L^* = \{\lambda, 00, 01, 0000, 0100, 0001, 0101, \ldots\}$

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- Let us consider all possible cases according to the definition of regexps

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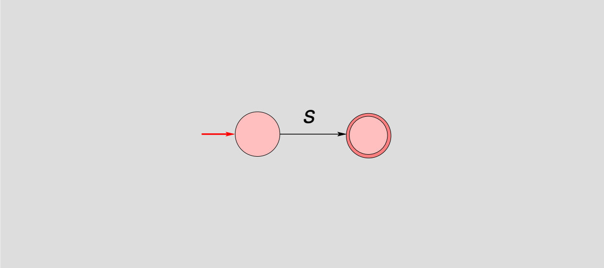

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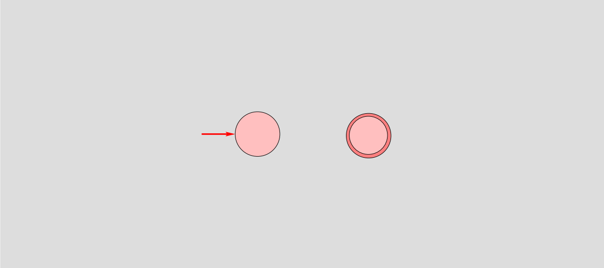
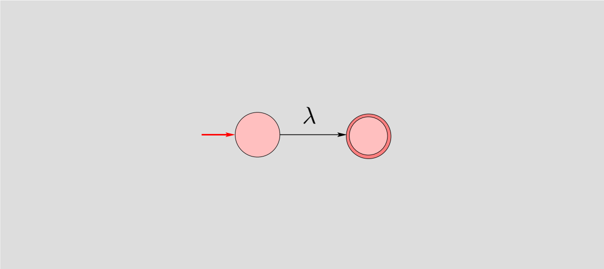
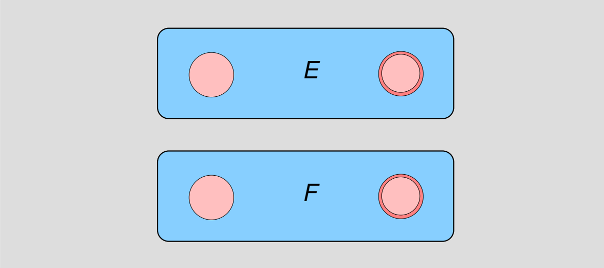


- Case $R = \lambda$



- Case $R = E + F$ for some regexps E, F



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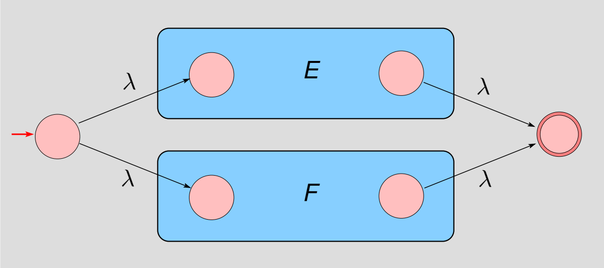

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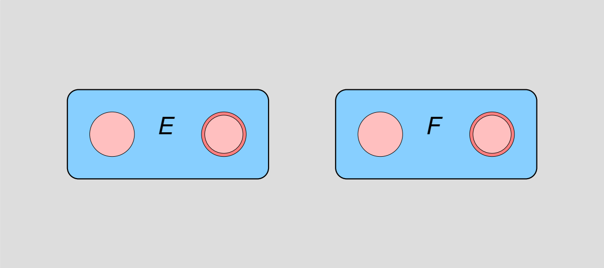

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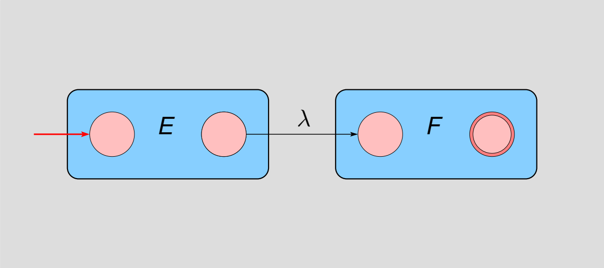

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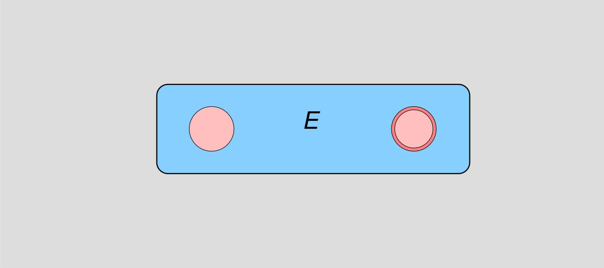

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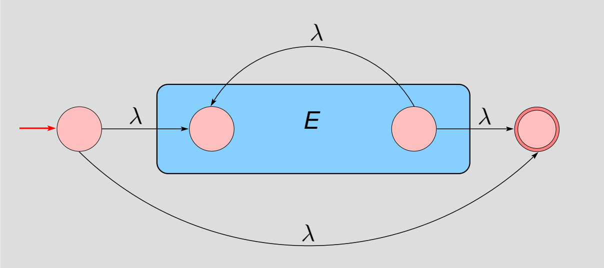

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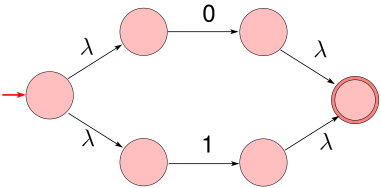

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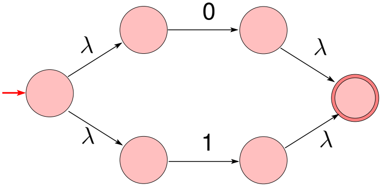

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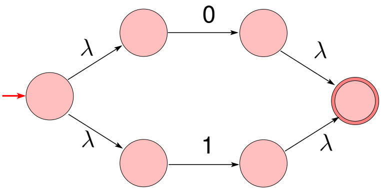

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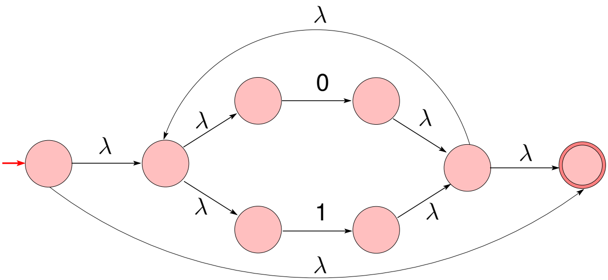

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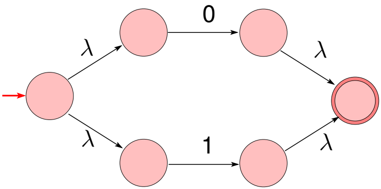

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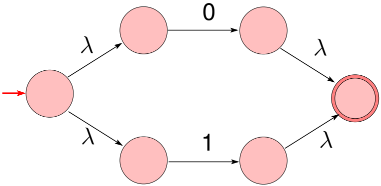

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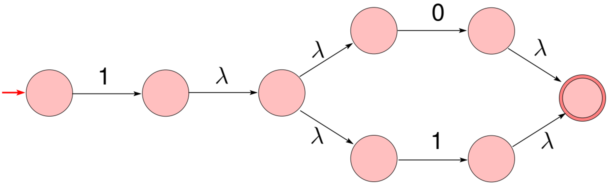

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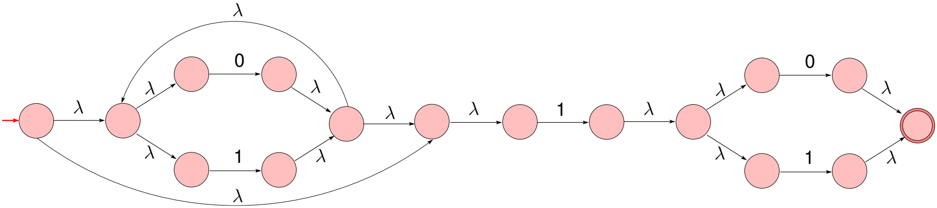

Regular Expressions

In Linux extended regular expression (ERE) notation:

- Character classes represent large sets of characters succinctly
  - The symbol `.` (dot) stands for any character (except newline)
  - The sequence `[` $a_1 a_2 \ldots a_n$ `]` stands for regexp $a_1 + a_2 + ... + a_n$
  - Between `[]` we can put a range of the form $x - y$
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  For example, `[A-Za-z0-9]` represents an alphanumeric character
- Additional operators sometimes make it easier to express what we want
  - The operator `|` is used in place of $+$ to denote union
  - The operator `?` means *"zero or one of"*
  - The operator `+` means *"one or more of"*
  - The operator `{`n`}` means *"n copies of"*

  E.g., `[+-]?[0-9]+\.[0-9]{2}` are numbers with 2 decimal digits

- For a more extensive explanation of the ERE notation, type `man grep`

Regular Expressions

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- ERE notation (or similar with slight changes) is used:
  - In lexical-analyzer generators, such as `lex` or `flex`
  - In Linux tools for finding patterns in text, such as `grep` (short for `global search regular expression & print`)
  - In text editors, such as `emacs`, `sed`, ...

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- Next: how to find an equivalent DFA with the minimum number of states

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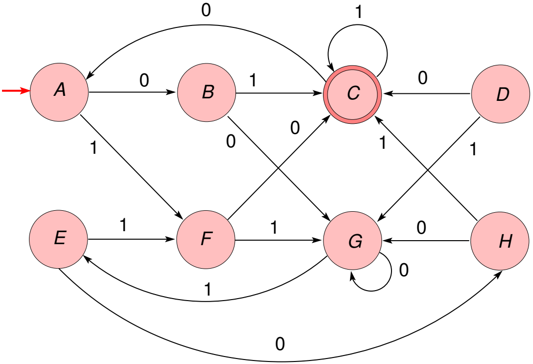
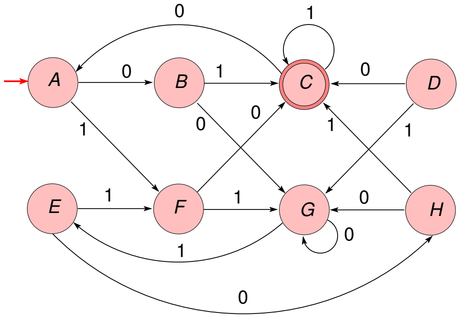
- If two states are not equivalent, then we say they are distinguishable

Testing Equivalence of States



- C and H are distinguishable since one is accepting and the other is not
- So are E and F, as on input 0, E and F go to states C and H, resp.
- So are A and G, as on input 1, E and F go to states C and H, resp.

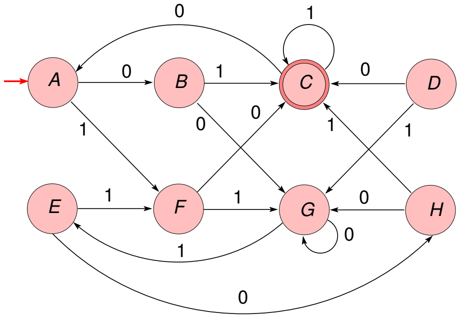


Algorithm for testing equivalence of states

1. For every pair of states p and q such that one is accepting and the other is not, mark (p, q) as distinguishable
2. For every pair of states (p, q) and symbol a, if $\delta(p, a)$ and $\delta(q, a)$ are distinguishable then mark (p, q) as distinguishable
3. Repeat 2. till no new pairs of states are marked

B							
C							
D							
E							
F							
G							
H							
	A	B	C	D	E	F	G

Testing Equivalence of States



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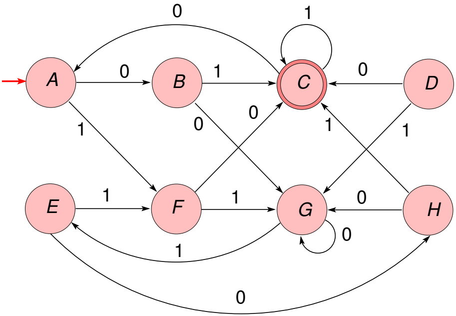
	A	B	C	D	E	F	G
B							
C	λ	λ					
D			λ				
E			λ				
F			λ				
G			λ				
H			λ				



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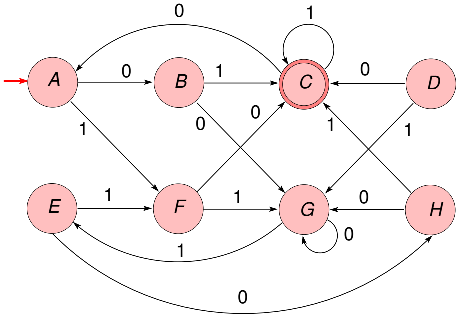
B	1						
C	λ	λ					
D	0	1	λ				
E		1	λ	0			
F	0	1	λ		0		
G		1	λ	0		0	
H	1		λ	0	1	0	1
	A	B	C	D	E	F	G



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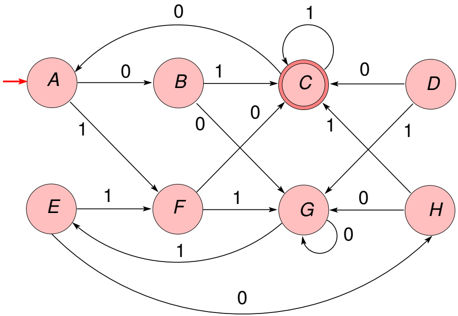

	A	B	C	D	E	F	G
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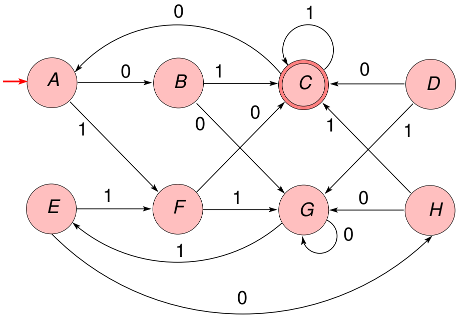

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C	λ	λ					
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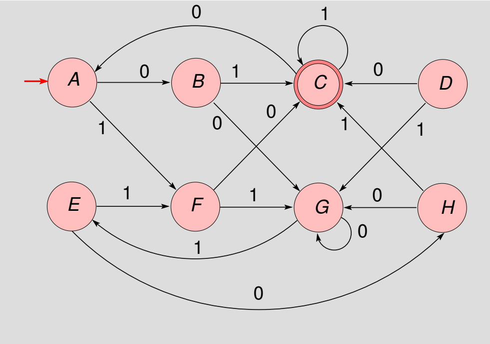
G	1	1	λ	0	1	0	
H	1		λ	0	1	0	1
	A	*B*	*C*	*D*	*E*	*F*	*G*

- We can partition the states into equivalence classes
  - A, E
  - B, H
  - D, F
  - C
  - G

Testing Equivalence of States



Equivalence classes

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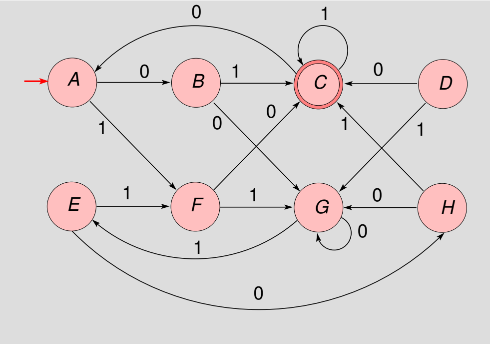
- We can consider the DFA were states are these equivalence classes



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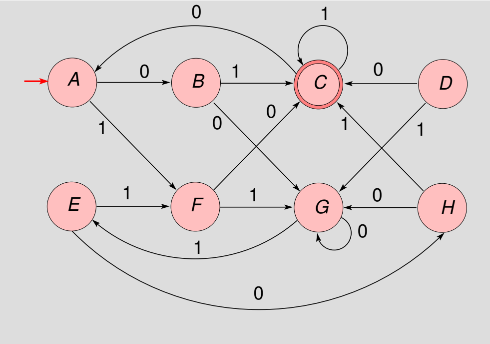

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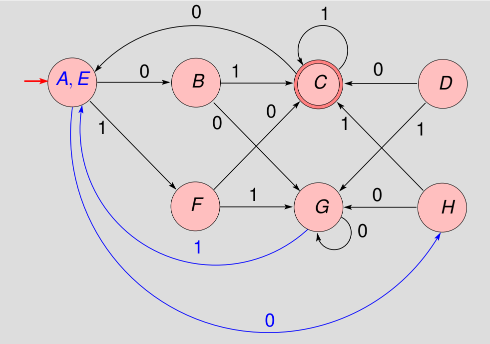

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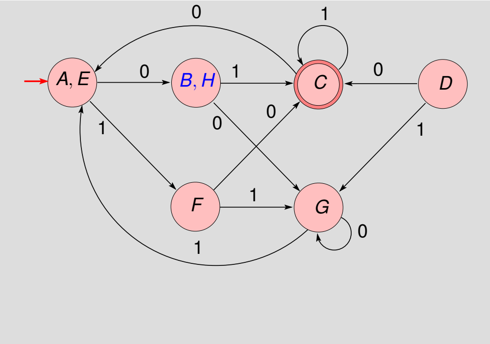

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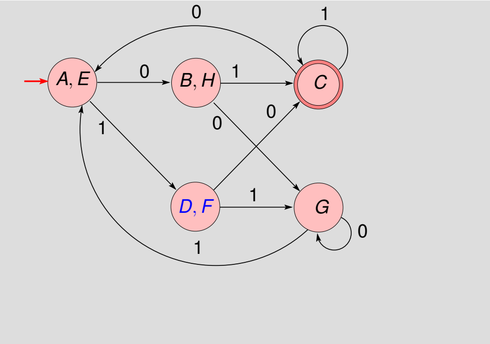

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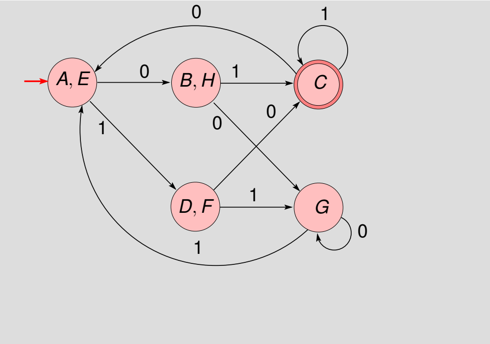

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Quotient Automaton

- Given a DFA A, the quotient automaton of A merges equivalent states
- Let $A = (Q_A, \Sigma, \delta_A, q_0, F_A)$ be a DFA
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  - For each $C \in Q_M$ and for each $a \in \Sigma$,
    $\delta_M(C, a)$ is the class of $\delta_A(q, a)$, with q any state whose class is C

Properties of the quotient automaton: (stated without proof)

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  if B is a DFA such that $L(B) = L(A)$ then $|Q_B| \geq |Q_M|$
- There is a one-to-one correspondence
  between the states of any minimum-state equivalent automaton and M