Properties of DFT (1)

U3

The DFT properties are the properties of the relationship of the periodic extensions interpreted in the intervals 0 ≤ n, k ≤ N-1

$$x[n] \xleftrightarrow{DFT_N} X_N[k] \qquad k = 0, \dots, N-1$$

$$\tilde{x}_N[n] = x[n] * \sum_{r=-\infty}^{\infty} \delta[n - rN] = \sum_{r=-\infty}^{\infty} x[n - rN]$$ Periodic extension

In the following, we will asume that $x[n]$=0 for n<0 and $n \geq N$ (also for $y[n]$)

Properties of DFT (2) $x[n] \xleftrightarrow{DFT_N} X_N[k]$

U3

The DFT properties are the properties of the relationship of the periodic extensions interpreted in the intervals $0 \leq n, k \leq N-1$

- Linearity: $\alpha \cdot x[n] + \beta \cdot y[n] \xleftrightarrow{DFT_N} \alpha \cdot X_N[k] + \beta \cdot Y_N[k]$
- Duality: $X_N[n] \xleftrightarrow{DFT_N} N \cdot \tilde{x}_N[-k]$
- Symmetry for real sequences: $x[n] \in \mathbb{R} \Rightarrow X_N[k] = \tilde{X}_N^*[-k]$
- Parseval: $E_x = \sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X_N[k]|^2$

Properties of DFT (3) $x[n] \overset{DFT_N}{\longleftrightarrow} X_N[k]$

U3

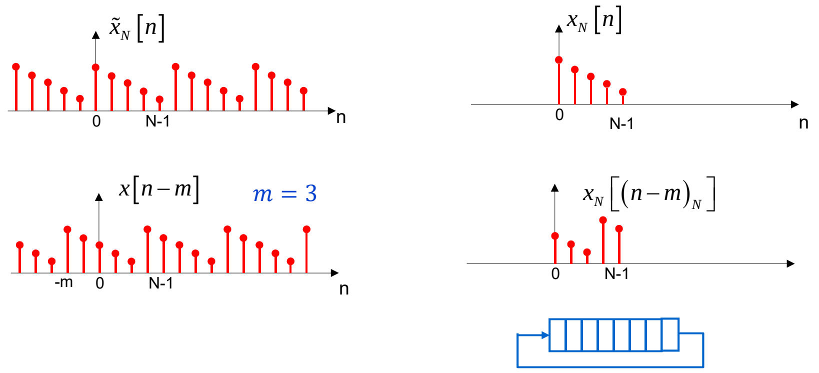
The DFT properties are the properties of the relationship of the periodic extensions interpreted in the intervals $0 \leq n, k \leq N-1$

- Frequency shift: $x[n]e^{j2\pi\frac{k_0}{N}n} \overset{DFT_N}{\longleftrightarrow} \tilde{X}_N[k-k_0]$
- Product of sequences: $x[n] \cdot y[n] \overset{DFT_N}{\longleftrightarrow} \frac{1}{N} X_N[k] \odot Y_N[k]$
- Circular delay: $\tilde{x}_N[n-n_0] \overset{DFT_N}{\longleftrightarrow} X_N[k]e^{-j\frac{2\pi}{N}kn_0}$
- Circular convolution: $a[n] \odot b[n] \overset{DFT_N}{\longleftrightarrow} A_N[k]B_N[k]$

Shifting a periodic signal

U3

- Shifting a periodic signal is equivalent to a circular shift (circular delay) of a period

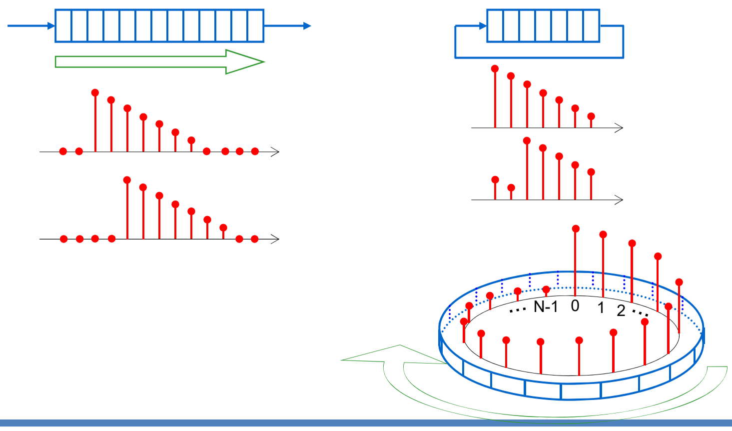


Circular delay

U3

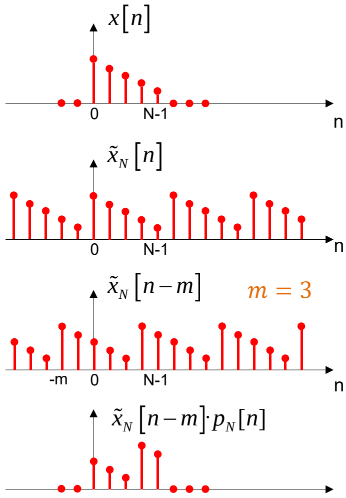
$$x[n] \overset{DFT_N}{\longleftrightarrow} X_N[k]$$

$$\tilde{x}_N[n-n_0] \overset{DFT_N}{\longleftrightarrow} X_N[k]e^{-j2\pi\frac{k}{N}n_0}, \qquad 0 \le n,k \le N-1$$



Circular delay (proof)

U3



Proof:

Remember that

$$\frac{1}{N}\sum_{k=0}^{N-1} X_N[k]e^{j2\pi\frac{k}{N}n} = \tilde{x}_N[n]$$

Then

$$\frac{1}{N}\sum_{k=0}^{N-1} X_N[k]e^{j2\pi\frac{k}{N}(n-n_0)} = \tilde{x}_N[n-n_0]$$

Circular (or periodic) convolution

U3

The circular (or periodic) convolution of two periodic sequences, $\tilde{x}_N[n]$ and $\tilde{y}_N[n]$ of period N, is defined as:

$$\tilde{x}_N[n] \circledast \tilde{y}_N[n] = \tilde{x}_N[n] \circledast \tilde{y}_N[n] = \sum_{k=0}^{N-1} \tilde{x}_N[n]\tilde{y}_N[n-k]$$

- The result is a periodic signal of period N
- The notation can be extended to the circular convolution of two signals defined between 0 and $N-1$: $x[n] \circledast y[n] = \sum_{k=0}^{N-1} x[n]\tilde{y}_N[n-k]$ with $\tilde{y}_N[n]$ the periodic extension of $y[n]$
- The circular convolution can be also computed as: $x[n] \circledast y[n] = \sum_{k=0}^{N-1} x[n]\tilde{y}_N[n-k] = x[n] * y[n] * \sum_{r=-\infty}^{\infty} \delta[n-rN]$

Circular convolution

U3

$$a[n] \xleftrightarrow{DFT_N} A_N[k]$$

$$b[n] \xleftrightarrow{DFT_N} B_N[k]$$

$$\tilde{c}_N[n] = a[n] \, N \, b[n] \xleftrightarrow{DFT_N} A_N[k]B_N[k]$$

$$\boxed{\begin{aligned}\tilde{c}_N[n] &= a[n] \, N \, b[n] = c[n] * t_N[n] = \sum_{r=-\infty}^{\infty} c[n-rN] \\ &= \sum_{m=0}^{N-1} a[m] \cdot \tilde{b}_N[n-m] = \sum_{m=0}^{N-1} b[m] \cdot \tilde{a}_N[n-m]\end{aligned}}$$

$$c[n] = a[n] * b[n]$$

Proof:

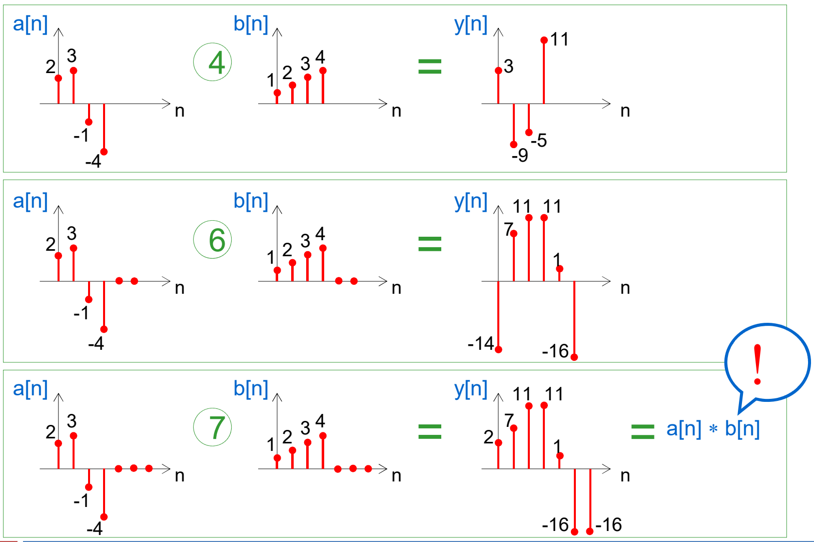
$$\frac{1}{N}\sum_{k=0}^{N-1} A_N[k]B_N[k]e^{j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=0}^{N-1} a[m]e^{-j\frac{2\pi}{N}km}\right) \cdot B_N[k]e^{j\frac{2\pi}{N}kn}$$

$$= \sum_{m=0}^{N-1} a[m] \frac{1}{N}\sum_{k=0}^{N-1} B_N[k]e^{j\frac{2\pi}{N}k(n-m)} = \sum_{m=0}^{N-1} a[m]\,\tilde{b}_N[n-m]$$

More examples

$$y[n] = \tilde{c}_N[n] = a[n] \circledN b[n] = \sum_{m=0}^{N-1} a[m]\cdot \tilde{b}_N[n-m]$$

U3



Filtering with DFT (1)

U3

$$z[n] = x[n] * y[n]$$

$L_z = L_x + L_y - 1$ L_x L_y

$$z[n] = x[n] * y[n] \xleftrightarrow{\text{Fourier transform}} Z(F) = X(F)\cdot Y(F)$$

$$z[n] \xleftrightarrow{DFT_{L_z}} Z[k] = Z\left(\tfrac{F}{L_z}k\right) = X\left(\tfrac{F}{L_z}k\right)\cdot Y\left(\tfrac{F}{L_z}k\right)$$

$$x[n] \xleftrightarrow{DFT_{L_z}} X[k] = X\left(\tfrac{F}{L_z}k\right)$$

$$y[n] \xleftrightarrow{DFT_{L_z}} Y[k] = Y\left(\tfrac{F}{L_z}k\right)$$

$$\boxed{z[n] = x[n] * y[n] \xleftrightarrow{DFT_{L_x+L_y-1}} Z[k] = X[k]\cdot Y[k]}$$

Filtering with DFT (2)

U3

- Computation of linear convolutions based on DFT. Procedure:
  1. Extend signals x[n], y[n] with zeros up to length $L \geq L_x + L_y - 1$
  2. DFT of length L of x[n], y[n] $\Rightarrow$ X[k], Y[k]
  3. Z[k]=X[k]·Y[k] k= 0 ,…, L-1
  4. IDFT of length L of Z[k] $\Rightarrow$ z[n] = x[n] $*$ y[n] = x[n] Ⓛ y[n]

- Computational cost (number of complex multiplications):
  - Direct computation of N-points DFT/IDFT: $\sim N^2$
  - Computation of DFT/IDFT based on N-points Fast Fourier Transform (FFT): $\sim N\log_2 N$

N	N^2	$N \cdot \log_2 N$
4	16	8
1.024	1.048.576	10.240

Exercise 3.17

U3

Consider a sequence $x[n]$, which ones are true?

a) $x[n] = DFT_N^{-1}\left\{X(F)|_{F=\frac{k}{N}}\right\} \quad 0 \leq n \leq N-1$

b) For $x[n] = p_N[n]$, $|X(F)|_{F=\frac{k}{N}} = N\delta[k] \;\; 0 \leq k \leq N-1$

c) For $x[n]$ real, $X(F)|_{F=\frac{k}{N}}$ real

d) $DFT_N^{-1}\left\{X_N[k]e^{-j2\pi\frac{k}{N}}\right\} = \begin{cases} x[N-1], & n=0 \\ x[n-1], & 1 \leq n \leq N-1 \end{cases}$

Exercise 3.19

U3

Consider the sequence $x[n] = \{\underline{1}, 2,2,2,0, \ldots\}$, find $y[n]$

a) $Y_N[k] = X_N[k]e^{-j2\pi\frac{k}{N}}$, N=4

b) $Y_N[k] = X_N[k]e^{-j4\pi\frac{k}{N}}$, N=5

c) $Y_N[k] = X_N[k]e^{j4\pi\frac{k}{N}}$, N=6

d) $Y_N[k] = X_N[k]e^{j4\pi\frac{k}{N}}$, N=4

Exercise 3.28 a

U3

Consider the sequence $x[n] = \{\underline{a}, b, c, d, e, f\}$, the $\mathrm{DFT}_6\{\underline{a}^*, b^*, c^*, d^*, e^*, f^*, 0, \ldots\}$ is

a) $[\underline{A}^*, B^*, C^*, D^*, E^*, F^*]$

b) $[\underline{F}^*, E^*, D^*, C^*, B^*, A^*]$

c) $[\underline{A}^*, F^*, E^*, D^*, C^*, B^*]$

d) $[-\underline{F}^*, -E^*, -D^*, -C^*, -B^*, -A^*]$

e) $[\underline{F}^*, E^*, D^*, A^*, B^*, C^*]$