24. $M_n = M_{n\times n}(\mathbb{R})$. $S_n \subseteq M_n$ i $A_n \subseteq M_n$ els subconjunts de les matrius simètriques i antisimètriques, respectivament. Comproveu que S_n i A_n són subespais vectorials de M_n. Trobeu la seva dimensió i demostreu que $M_n = S_n \oplus A_n$.

S_n

$$\cancel{S_n = \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \middle| \, x_{11}, \ldots, x_{nn} \in \mathbb{R} \right\}}$$

$$S_n = \left\{ A \in M_n \mid A^t = A \right\} = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \middle| \, a_{ij} = a_{ji} \ \forall i,j \right\}.$$

<u>Propietats:</u>

- Siguin $A, B \in S_n$, aleshores:

$$A + B = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} + \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & \cdots & a_{nn}+b_{nn} \end{pmatrix}$$

suma commutativa a $\mathbb{R}$.

$$= \begin{pmatrix} b_{11}+a_{11} & \cdots & b_{1n}+a_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1}+a_{n1} & \cdots & b_{nn}+a_{nn} \end{pmatrix} = B + A \in S_n. \blacksquare$$

- Siguin $A, B, C \in S_n$. Aleshores:

$$(A+B)+C = \left(\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} + \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} \right) + \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & \cdots & a_{nn}+b_{nn} \end{pmatrix} + \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11}+c_{11} & \cdots & a_{1n}+b_{1n}+c_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1}+b_{n1}+c_{n1} & \cdots & a_{nn}+b_{nn}+c_{nn} \end{pmatrix}$$

suma associativa a $\mathbb{R}$

$$= \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} + \left(\begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} + \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} \right) = A + (B+C).$$

$\in S_n \blacksquare$

- Sigui $A \in S_n$. Aleshores, si existeix $B \in S_n$: $A+B=A$, $B=\vec{O}_{S_n}$.

Agafem $B=\begin{pmatrix} 0 & \overset{n)}{\cdots} & 0 \\ \vdots^{n)} & \ddots^{n)} & \vdots^{n)} \\ 0 & \underset{n)}{\cdots} & 0 \end{pmatrix}$. $\boxed{\text{Es dóna que } b_{ij}=b_{ji}\ \forall i,j}$ $\Rightarrow B \in S_n$.

Aleshores, $A+B=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}+\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}=$

$=\begin{pmatrix} a_{11}+0 & \cdots & a_{1n}+0 \\ \vdots & \ddots & \vdots \\ a_{n1}+0 & \cdots & a_{nn}+0 \end{pmatrix}=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}=A.$

$\Rightarrow B=\vec{O}_{S_n}$ ∎

- Sigui $A \in S_n$. Aleshores, si existeix $B \in S_n$: $A+B=\vec{O}_{S_n}$, $B=-A$ ∎.

Agafem $B=\begin{pmatrix} -a_{11} & \cdots & -a_{1n} \\ \vdots & \ddots & \vdots \\ -a_{n1} & \cdots & -a_{nn} \end{pmatrix}$. $\boxed{\text{Es dóna que } b_{ij}=b_{ji}\ \forall i,j}$ $\Rightarrow B \in S_n$.

$A+B=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}+\begin{pmatrix} -a_{11} & \cdots & -a_{1n} \\ \vdots & \ddots & \vdots \\ -a_{n1} & \cdots & -a_{nn} \end{pmatrix}$

$=\begin{pmatrix} a_{11}-a_{11} & \cdots & a_{1n}-a_{1n} \\ \vdots & & \vdots \\ a_{n1}-a_{n1} & \cdots & a_{nn}-a_{nn} \end{pmatrix}=\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}=$

$=\vec{O}_{S_n} \Rightarrow B=-A$ ∎

- Sigui $A \in S_n$ i $B \in S_n$, i $c \in \mathbb{R}$. Aleshores:

$c(A+B)=c\begin{pmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & \cdots & a_{nn}+b_{nn} \end{pmatrix}=\begin{pmatrix} ca_{11}+cb_{11} & \cdots & ca_{1n}+cb_{1n} \\ \vdots & \ddots & \vdots \\ ca_{n1}+cb_{n1} & \cdots & ca_{nn}+cb_{nn} \end{pmatrix}=$

$=cA+cB$ ∎ $\quad (cA \in S_n \Leftrightarrow ca_{ij}=ca_{ji}\ \forall i,j \Leftrightarrow a_{ij}=a_{ji}\ \forall i,j.$ ∎$)$

- Sigui $A \in S_n$ i $c,d \in \mathbb{R}$. Aleshores:

$(c+d)A=(c+d)\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}=\begin{pmatrix} (c+d)a_{11} & \cdots & (c+d)a_{1n} \\ \vdots & \ddots & \vdots \\ (c+d)a_{n1} & \cdots & (c+d)a_{nn} \end{pmatrix}=$

$=\begin{pmatrix} ca_{11}+da_{11} & \cdots & ca_{1n}+da_{1n} \\ \vdots & \ddots & \vdots \\ ca_{n1}+da_{n1} & \cdots & ca_{nn}+da_{nn} \end{pmatrix}=cA+dA$ ∎

- Sigui $A \in S_n$ i $c, d \in \mathbb{R}$. Aleshores:

$$c \cdot (dA) = c\begin{pmatrix} da_{11} & \cdots & da_{1n} \\ \vdots & \ddots & \vdots \\ da_{n1} & \cdots & da_{nn} \end{pmatrix} = \begin{pmatrix} cda_{11} & \cdots & cda_{1n} \\ \vdots & \ddots & \vdots \\ cda_{n1} & \cdots & cda_{nn} \end{pmatrix} =$$

$$= (cd)A \blacksquare$$

- Sigui $A \in S_n$. Aleshores:

$$1 \cdot A = 1 \cdot \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = A \blacksquare$$

$\Longrightarrow$ S_n és un espai vectorial i un subespai vectorial de M_n.

Dim S_n:

Els vectors $A_{12} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & & \ddots & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$, $A_{13} = \begin{pmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 0 & & & \vdots \\ 0 & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$, $\ldots$, $A_{1n} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$,

$A_{23} = \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & & \ddots & & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$, $\ldots$, $A_{n-1\,n} = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & & 0 \\ \vdots & & 0 & 1 \\ 0 & \cdots & 1 & 0 \end{pmatrix}$, i $A_{11} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$,

$A_{22} = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$, $\ldots$, $A_{nn} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & & 0\ 0 \\ 0 & \cdots & 0\ 1 \end{pmatrix}$, generen S_n i són linealment independents. $\to$ són base d'S_n. N'hi ha:

- n per la diagonal.
- $n-1$ per la primera fila/columna.
- $n-2$ per la segona columna/fila.

$\vdots$

- 1 per la $(n-1)$-èsima fila/columna.

$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. Per tant, $\dim S_n$ és:

$$\boxed{\dim S_n = \frac{n(n+1)}{2}}$$

A_n

$$A_n=\left\{A\in M_n \mid A^t=-A\right\}=\left\{\begin{pmatrix} a_{11} & \cdots & a_{1n}\\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn}\end{pmatrix} \middle|\ a_{ij}=-a_{ji}\ \forall i,j,\ a_{ii}=0\ \forall i\right\}.$$

<u>Suma:</u> Siguin $A,B\in A_n$. Aleshores:

$$A+B=\begin{pmatrix} 0 & a_{12} & \cdots & a_{1n}\\ a_{21} & \ddots & & \vdots \\ \vdots & & 0 & a_{n-1,n} \\ a_{n1} & \cdots & a_{n,n-1} & 0\end{pmatrix}+\begin{pmatrix} 0 & b_{12} & \cdots & b_{1n}\\ b_{21} & \ddots & & \vdots \\ \vdots & & 0 & b_{n-1,n} \\ b_{n1} & \cdots & b_{n,n-1} & 0\end{pmatrix}=\begin{pmatrix} 0 & b_{12}+a_{12} & \cdots & b_{1n}+a_{1n}\\ -a_{12}-b_{12} & 0 & & \vdots \\ \vdots & & \ddots & \\ -b_{1n}-a_{1n} & \cdots & & 0\end{pmatrix}\in A_n \blacksquare$$

<u>Producte:</u> Siguin $A\in A_n$ i $c\in\mathbb{R}$. Aleshores:

$$cA=\begin{pmatrix} 0 & ca_{12} & \cdots & ca_{1n}\\ -ca_{12} & \ddots & & \vdots \\ \vdots & & \ddots & \\ -ca_{1n} & \cdots & & 0\end{pmatrix}\in A_n \blacksquare$$

<u>Propietats:</u>

- Siguin $A,B\in A_n$. Aleshores:

$$A+B=\begin{pmatrix} 0 & \cdots & a_{1n}\\ \vdots & \ddots & \vdots \\ -a_{1n} & \cdots & 0\end{pmatrix}+\begin{pmatrix} 0 & \cdots & b_{1n}\\ \vdots & \ddots & \vdots \\ -b_{1n} & \cdots & 0\end{pmatrix}=\begin{pmatrix} 0 & \cdots & a_{1n}+b_{1n}\\ \vdots & \ddots & \vdots \\ -a_{1n}-b_{1n} & \cdots & 0\end{pmatrix}=$$

suma commutativa a $\mathbb{R}$

$$=\begin{pmatrix} 0 & \cdots & b_{1n}+a_{1n}\\ \vdots & \ddots & \vdots \\ -b_{1n}-a_{1n} & \cdots & 0\end{pmatrix}=B+A \blacksquare$$

- Siguin $A,B,C\in A_n$. Aleshores:

$$(A+B)+C=\left(\begin{pmatrix} 0 & \cdots & a_{1n}\\ \vdots & \ddots & \vdots \\ -a_{1n} & \cdots & 0\end{pmatrix}+\begin{pmatrix} 0 & \cdots & b_{1n}\\ \vdots & \ddots & \vdots \\ -b_{1n} & \cdots & 0\end{pmatrix}\right)+\begin{pmatrix} 0 & \cdots & c_{1n}\\ \vdots & \ddots & \vdots \\ -c_{1n} & \cdots & 0\end{pmatrix}=$$

suma associativa a $\mathbb{R}$

$$\begin{pmatrix} 0 & \cdots & a_{1n}+b_{1n}\\ \vdots & \ddots & \vdots \\ -a_{1n}-b_{1n} & \cdots & 0\end{pmatrix}+\begin{pmatrix} 0 & \cdots & c_{1n}\\ \vdots & \ddots & \vdots \\ -c_{1n} & \cdots & 0\end{pmatrix}=\begin{pmatrix} 0 & \cdots & a_{1n}+b_{1n}+c_{1n}\\ \vdots & \ddots & \vdots \\ -a_{1n}-b_{1n}-c_{1n} & \cdots & 0\end{pmatrix}$$

$$=\begin{pmatrix} 0 & \cdots & a_{1n}\\ \vdots & \ddots & \vdots \\ -a_{1n} & \cdots & 0\end{pmatrix}+\left(\begin{pmatrix} 0 & \cdots & b_{1n}\\ \vdots & \ddots & \vdots \\ -b_{1n} & \cdots & 0\end{pmatrix}+\begin{pmatrix} 0 & \cdots & c_{1n}\\ \vdots & \ddots & \vdots \\ -c_{1n} & \cdots & 0\end{pmatrix}\right)=A+(B+C) \blacksquare$$

• Sigui $A \in A_n$. Aleshores, si existeix $B \in A_n$: $A+B=A$, $B=\vec{0}_{A_n}$

Agafem $B=\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in A_n$. Aleshores $A+B=\begin{pmatrix} 0 & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ -a_{1n} & \cdots & 0 \end{pmatrix}+\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}=$

$$=\begin{pmatrix} 0 & \cdots & a_{1n}+0 \\ \vdots & \ddots & \vdots \\ -a_{1n}+0 & \cdots & 0 \end{pmatrix}=\begin{pmatrix} 0 & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ -a_{1n} & \cdots & 0 \end{pmatrix}=A$$

$$\Longrightarrow B=\vec{0}_{A_n}.$$

• Sigui $A \in A_n$. Aleshores, si existeix $B \in A_n$: $A+B=\vec{0}_{A_n}$, $B=-A$.

Agafem $B=\begin{pmatrix} 0 & \cdots & -a_{1n} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & 0 \end{pmatrix}$. Aleshores $A+B=\begin{pmatrix} 0 & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ -a_{1n} & \cdots & 0 \end{pmatrix}+\begin{pmatrix} 0 & \cdots & -a_{1n} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & 0 \end{pmatrix}$

$$=\begin{pmatrix} 0 & \cdots & a_{1n}-a_{1n} \\ \vdots & \ddots & \vdots \\ -a_{1n}+a_{1n} & \cdots & 0 \end{pmatrix}=\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}=\vec{0}_{A_n} \Longrightarrow B=-A \ \blacksquare$$

• Siguin $A, B \in A_n$ i $c \in \mathbb{R}$. Aleshores:

$$c(A+B)=c\begin{pmatrix} 0 & \cdots & a_{1n}+b_{1n} \\ \vdots & \ddots & \vdots \\ -a_{1n}-b_{1n} & \cdots & 0 \end{pmatrix}=\begin{pmatrix} 0 & \cdots & ca_{1n}+cb_{1n} \\ \vdots & \ddots & \vdots \\ -ca_{1n}-cb_{1n} & \cdots & 0 \end{pmatrix}=cA+cB \ \blacksquare$$

• Siguin $A \in A_n$ i $c, d \in \mathbb{R}$. Aleshores:

$$(c+d)A=\begin{pmatrix} 0 & \cdots & (c+d)a_{1n} \\ \vdots & \ddots & \vdots \\ (c+d)(-a_{1n}) & \cdots & 0 \end{pmatrix}=\begin{pmatrix} 0 & \cdots & ca_{1n} \\ \vdots & \ddots & \vdots \\ -ca_{1n} & \cdots & 0 \end{pmatrix}+\begin{pmatrix} 0 & \cdots & da_{1n} \\ \vdots & \ddots & \vdots \\ -da_{1n} & \cdots & 0 \end{pmatrix}=cA+dA \ \blacksquare$$

$$c\cdot(dA)=\begin{pmatrix} 0 & \cdots & c(da_{1n}) \\ \vdots & \ddots & \vdots \\ c(-da_{1n}) & \cdots & 0 \end{pmatrix}=\begin{pmatrix} 0 & \cdots & cda_{1n} \\ \vdots & \ddots & \vdots \\ cd(-a_{1n}) & \cdots & 0 \end{pmatrix}=(cd)A \ \blacksquare$$

• Sigui $A \in A_n$. Aleshores:

$$1\cdot A=\begin{pmatrix} 0 & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ -a_{1n} & \cdots & 0 \end{pmatrix}=A \ \blacksquare$$

$\Longrightarrow A_n$ és un subespai vectorial de M_n.

Dim A_n:

Els vectors $B_{1,2} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & & & \vdots \\ 0 & & \ddots & & \vdots \\ \vdots & & & & 0 \\ 0 & \cdots & & & 0 \end{pmatrix}$, $B_{1,3} = \begin{pmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & \cdots & & 0 \\ -1 & & & & & \vdots \\ 0 & & & \ddots & & \vdots \\ \vdots & & & & & 0 \\ 0 & \cdots & & & & 0 \end{pmatrix}$, ...,

$B_{1,n} = \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & & \vdots \\ -1 & 0 & \cdots & & 0 \end{pmatrix}$, ..., $B_{n-1,n} = \begin{pmatrix} 0 & \cdots & & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & & 0 \\ \vdots & & 0 & 1 \\ 0 & \cdots 0 & -1 & 0 \end{pmatrix}$, generen A_n i són l.i.

$\Longrightarrow$ són base d'A_n. N'hi ha:

- $n-1$ per la primera fila/columna
- $n-2$ per la segona fila/columna.

$\vdots$

- 1 per la $(n-1)$-èsima fila/columna.

$$\sum_{k=1}^{n-1} k = \frac{(n-1)(n-1+1)}{2} = \frac{n(n-1)}{2}.$$

Per tant, dim A_n és: $\dim A_n = \dfrac{n(n-1)}{2}$.

Per demostrar que $M_n = S_n \oplus A_n$, necessitem:

$$\boxed{\begin{array}{l} (1)\ M_n = S_n + A_n. \\ (2)\ S_n \cap A_n = \{\vec{0}\}. \end{array}}$$

(1) Per demostrar la igualtat només cal demostrar que $M_n \subseteq S_n + A_n$, ja que l'altra inclusió és sempre certa (teorema). Volem veure que donada $A \in M_n$ podem escriure-la ~~[illegible]~~ com a suma d'un vector d'S_n i un d'A_n.

Sigui $A \in M_n$. Aleshores, podem escriure A de la manera següent:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \overbrace{\underbrace{a_{11}A_{11} + \cdots + a_{nn}A_{nn}}_{\sum_{i=1}^{n} a_{ii}A_{ii}} + \frac{1}{2}\sum_{i=1}^{n-1}\left(\sum_{j=i+1}^{n-1} a_{ij}A_{ij}\right)}^{\text{vector d'}S_n} +$$

$$+ \underbrace{\frac{1}{2}\sum_{j=1}^{n-1}\left(\sum_{i=j+1}^{n-1} a_{ij}B_{ij}\right)}_{\text{vector d'}A_n} + \underbrace{\frac{1}{2}\sum_{i=1}^{n-1}\left(\sum_{j=i+1}^{n-1} a_{ij}B_{ij}\right)}_{\text{vector d'}S_n} + \underbrace{\frac{1}{2}\sum_{j=1}^{n-1}\left(\sum_{i=j+1}^{n-1} a_{ij}A_{ij}\right)}_{\text{Vector d'}A_n}.$$

$\Longrightarrow M_n \subseteq A_n + S_n$ ■

(2) Per veure que la intersecció és efectivament el conjunt del vector $\vec{0}$, n'hi ha prou amb escriure la definició d'S_n i A_n:

$$S_n = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \middle| \ a_{ij} = a_{ji} \ \forall i,j \right\} \text{ i } A_n = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \middle| \ a_{ij} = -a_{ji} \ \forall i,j \right\}$$

Aleshores, si $A \in S_n \cap A_n$, els elements d'A són de la forma:

$$a_{ij} = a_{ji} = -a_{ji}.$$

Per tant tots els a_{ij} són 0 i per tant, $A_n \cap S_n = \{\vec{0}\}$.

(1), (2) $\Longrightarrow M_n = S_n \oplus A_n$ ■

Ho podem corroborar amb la fórmula de Grassmann:

$$\dim(A_n \cap S_n) + \dim(A_n + S_n) = \dim A_n + \dim S_n$$

$$0 + n^2 = \frac{n(n+1)}{2} + \frac{n(n-1)}{2} = \frac{n(n+1+n-1)}{2} = \frac{2n^2}{2} = n^2.$$