Randomly solving 2-SAT.

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a) Let X_n be the number of satisfied clauses after n iterations of the loop in (2) (we will refer to it as time n). Given the execution up top time $n-1$, is it always true that $\mathbb{E}\left[X_n\right] \geq X_{n-1}$?
Solution
The claim is not true. Let us prove it by a counterexample: let ϕ be the following CNF formula:

$$\phi = (x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge (\bar{x_1} \vee \bar{x_4}) \wedge (\bar{x_1} \vee x_3) \wedge (\bar{x_4} \vee \bar{x_3}),$$

and $\mathbf{x} = (0,1,0,1)$ the assignment at time $n-1$. Then, it follows that $X_{n-1} = 4$, and the expectancy of the variable X_n can be calculated as follows; whether x_1 swaps its value, and then $X_n = 3$, or x_3 swaps its value and $X_n = 4$. This gives an expectancy

$$\mathbb{E}\left[X_n\right] = \frac{1}{2}3 + \frac{1}{2}4 = \frac{7}{2} < 4 = X_{n-1}.$$

b) Since ϕ is satisfiable, let $\mathbf{x}^*$ be an arbitrary satisfying assignment. Let Y_n be the number of variables in $\mathbf{x}$ whose value coincides with the one in $\mathbf{x}^*$ at time n. Given the execution up to time $n-1$, is it always true that $\mathbb{E}\left[Y_n\right] \geq X_{n-1}$?
Solution

c) Argue that if $Y_n = k$, then Rand2SAT terminates at time n. Is the converse true?
Solution
Since $Y_n = k$, it means that all variables in $\mathbf{x}$ are equal to $\mathbf{x}^*$. As $\mathbf{x}^*$ is an arbitrary satisfying assignment, it means that $\phi(\mathbf{x}^*) = 1$, and as $\phi \in$ CNF-2-SAT, it follows that is has no violated clauses. Then, as described by the algorithm, the loop in (2) stops when no clauses are violated. Hence, when $Y_n = k$, the algorithm halts at this moment (*time* n).

d) Is Y_n a Markov chain?
Solution

e) Design a Markov chain Z_n such that $Y_n \geq Z_n$.
Solution

f) Use Z_n to prove that the expected running time of Rand2SAT is at most k^2.
Solution

***g)** We modify Rand2SAT to stop in bounded time as follows. Let $l \in \mathbb{Z}$. If after $2lk^2$ iterations of the loop in (2) we have not halted, we break the loop and return the current assignment $\mathbf{x}$. Prove that the output of the modified Rand2SAT is a satisfying assignment with probability at least $1 - 2^{-l}$.
Solution