

# Randomly solving 2-SAT.

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**a)** Let  $X_n$  be the number of satisfied clauses after  $n$  iterations of the loop in (2) (we will refer to it as time  $n$ ). Given the execution up to time  $n - 1$ , is it always true that  $\mathbb{E}[X_n] \geq X_{n-1}$ ?

**Solution**

The claim is not true. Let us prove it by a counterexample: let  $\phi$  be the following CNF formula,

$$\phi = (x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_4) \wedge (\bar{x}_1 \vee x_3) \wedge (\bar{x}_4 \vee \bar{x}_3),$$

and  $\mathbf{x} = (0, 1, 0, 1)$  the assignment at time  $n - 1$ . Then, it follows that  $X_{n-1} = 4$ , and the expectancy of the variable  $X_n$  can be calculated as follows; whether  $x_1$  swaps its value, and then  $X_n = 3$ , or  $x_3$  swaps its value and  $X_n = 4$ . This gives an expectancy

$$\mathbb{E}[X_n] = \frac{1}{2}3 + \frac{1}{2}4 = \frac{7}{2} < 4 = X_{n-1}.$$

**b)** Since  $\phi$  is satisfiable, let  $\mathbf{x}^*$  be an arbitrary satisfying assignment. Let  $Y_n$  be the number of variables in  $\mathbf{x}$  whose value coincides with the one in  $\mathbf{x}^*$  at time  $n$ . Given the execution up to time  $n - 1$ , is it always true that  $\mathbb{E}[Y_n] \geq Y_{n-1}$ ?

**Solution**

In order to prove this, we will first take a look at how does  $Y_n$  behave with respect to  $Y_{n-1}$ . Since the algorithm only changes variables in violated clauses, we will take  $Z_n^i$  to be the random variable that counts how many of the variables in clause  $i$  have the same value in  $\mathbf{x}$  and in  $\mathbf{x}^*$ . So, clause  $i$  can be thought of as  $(x_{j_1} \vee x_{j_2})$ . In  $\mathbf{x}^*$  this can only be (1,1), (0,1), or (1,0), as a consequence of a clause being a disjunction, and in  $\mathbf{x}$  this variables have to be (0,0), because the clause is violated. Then, suppose the first step of the loop selects this clause; now it can only change the value of one of the two variables. So, the only possible outcomes of  $Y_n$  are

$$Y_n = \begin{cases} Y_{n-1} + 1 \\ Y_{n-1} - 1 \end{cases},$$

and this will depend on the value of  $Z_n^i$ . As clause  $i$  was violated at time  $n - 1$ ,  $Z_{n-1}^i$  could have been 0 or 1, and it can at time  $n$  equal zero, one or two:

$$Z_n^i = \begin{cases} 0 & \text{with probability } \frac{1}{2}, \text{ if } Z_{n-1}^i = 1 \\ 1 & \text{with probability } 1, \text{ if } Z_{n-1}^i = 0 \\ 2 & \text{with probability } \frac{1}{2}, \text{ if } Z_{n-1}^i = 1 \end{cases}.$$

Taking this into account, it is more probable that  $Z_n^i$  increases than that it decreases, and so is the case for  $Y_n$ . Then, the expectancy of  $Y_n$  is greater than (or equal to) the value of  $Y_{n-1}$ .

**c)** Argue that if  $Y_n = k$ , then RAND2SAT terminates at time  $n$ . Is the converse true?

**Solution**

Since  $Y_n = k$ , it means that all variables in  $\mathbf{x}$  are equal to  $\mathbf{x}^*$ . As  $\mathbf{x}^*$  is an arbitrary satisfying assignment, it means that  $\phi(\mathbf{x}^*) = 1$ , and as  $\phi \in \text{CNF-2-SAT}$ , it follows that it has no violated clauses. Then, as described by the algorithm, the loop in (2) stops when no clauses are violated. Hence, when  $Y_n = k$ , the algorithm

halts at this moment (*time n*).

The converse is not true. Take for example the following CNF-2-SAT formula  $\phi'$ ,

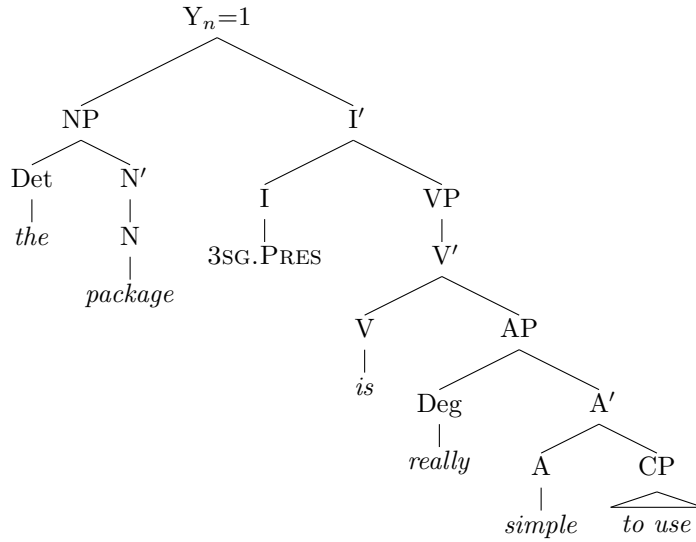
$$\phi' = (x_1 \vee x_2) \wedge (x_2 \vee x_3);$$

it has 4 different satisfying assignments,  $a_1 = (1, 1, 0)$ ,  $a_2 = (1, 0, 1)$ ,  $a_3 = (0, 1, 1)$ , and  $a_4 = (0, 1, 0)$ . If we take  $\mathbf{x}^*$  to be one of these, say  $a_3$ , the loop in (2) could possibly reach  $a_4$  before reaching  $a_3$  and halt as a consequence of  $a_4$  being a satisfying assignment. If that were the case,  $Y_n$  would be 2 instead of 3 (the  $k$  in this example), and so the converse can't be true.

d) Is  $Y_n$  a Markov chain?

**Solution**

It's not. In order to prove this, we are going to consider a counterexample. First, recall that in order for  $Y_n$  to be a Markov chain, it is needed that  $Y_{n-1}$  gives us all the information to determine the possible outputs of  $Y_n$ .



e) Design a Markov chain  $Z_n$  such that  $Y_n \geq Z_n$ .

**Solution**

f) Use  $Z_n$  to prove that the expected running time of RAND2SAT is at most  $k^2$ .

**Solution**

\*g) We modify RAND2SAT to stop in bounded time as follows. Let  $l \in \mathbb{Z}$ . If after  $2lk^2$  iterations of the loop in (2) we have not halted, we break the loop and return the current assignment  $\mathbf{x}$ . Prove that the output of the modified RAND2SAT is a satisfying assignment with probability at least  $1 - 2^{-l}$ .

**Solution**