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3.4 Models de v.a. discretes

Si X i Y tenen la mateixa llei, escrivim $X \sim Y$

- UNIFORME $U(a_1, \dots, a_n)$ $(a_1, \dots, a_n \in \mathbb{R})$

$X \sim U(a_1, \dots, a_n)$ si $\operatorname{Im} X = \{a_1, \dots, a_n\}$ i $p(X = a_i) = \frac{1}{n}$

$\mathbb{E}[X] = \sum_{i=1}^{n} a_i / n$

- BERNOUILLI $B(p)$ $(p \in [0,1])$

$X \sim B(p)$ si $\operatorname{Im} X = \{0, 1\}$ i $p(X=1) = p$, $p(X=0) = 1-p$

$G_X(z) = (1-p) + pz$

$\mathbb{E}[X] = p$

$\mathbb{V}ar[X] = p(1-p)$

- **BINOMIAL** $\text{Bin}(n,p)$ $\left(n \in \mathbb{Z}, n \geq 1, p \in [0,1] \right)$

$X \sim \text{Bin}(n,p)$ si existeixen $X_1, \dots, X_n$ v.a. indeps, $X_i \sim B(p)$

i $X = X_1 + \dots + X_n$

$\text{Im}\, X = \{0, 1, \dots, n\}$ i $p(X=i) = \binom{n}{i} p^i (1-p)^{n-i}$

$G_X(z) = G_{X_1}(z)^n = \left((1-p) + pz\right)^n$

$\mathbb{E}[X] = np$

$\text{Var}[X] = np(1-p)$

Com és la suma de binomials $\text{Bin}(n_i, p)$?

- GEOMÈTRICA $\mathrm{Geom}(p)$ $(p \in [0,1])$

"nombre de repeticions d'una $B(p)$ fins al primer èxit"

$X \sim \mathrm{Geom}(p)$ si $\mathrm{Im}(X) = \{0, 1, \dots\}$ i $p(X=k) = (1-p)^{k-1} \cdot p$

$$G_X(z) = \sum_{n \geq 1} p(1-p)^{n-1} z^n = \frac{pz}{1-(1-p)z}$$

$$\mathbb{E}[X] = 1/p$$

$$\mathrm{Var}[X] = \frac{1-p}{p^2}$$

Si només volem comptar fracassos, tenim la $\mathrm{Geom}_0(p)$ (exercici 13)

- **BINOMIAL NEGATIVA** $BinN(p,r)$ $(p\in[0,1], r\in\mathbb{Z}, r\geq 1)$

"Ara volem comptar el nombre de repeticions de $B(p)$ fins a obtenir r èxits"

$X \sim BinN(p,r)$ si existeixen $Y_1,\dots,Y_r$ indeps, $Y_i \sim Geom(p)$

i $X = Y_1 + \dots + Y_r$

$Im\, X = \{r, r+1, \dots\}$ i $p(X=k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$

$$G_X(z) = \left(\frac{pz}{1-(1-p)z}\right)^r$$

$\mathbb{E}[X] = r/p$, $\mathbb{V}ar[X] = r(1-p)/p^2$

Igual que amb la Geom, podríem voler comptar només els fracassos abans de l'r-èsim èxit

- POISSON $Po(\lambda)$ $(\lambda \in \mathbb{R}, \lambda > 0)$

$X \sim Po(\lambda)$ si $Im(X) = \{0, 1, 2, \dots\}$ i $p(X = i) = \frac{1}{i!} e^{-\lambda} \lambda^i$

$$G_X(z) = \sum_{i \geq 0} \frac{1}{i!} e^{-\lambda} \lambda^i z^i = e^{-\lambda} e^{\lambda z} = e^{\lambda(z-1)}$$

$\mathbb{E}[X] = \lambda$, $\mathbb{V}ar[X] = \lambda$

Com és la suma de Poissons $Po(\lambda_i)$ indeps?

Obs: $Y_n \sim Bin(n, p)$ on $np \in \lambda$ (ctnt)

$$p(Y_n = k) = \binom{n}{k} p^k (1-p)^{n-k} = \binom{n}{k} \frac{1}{n^k} \cdot (np)^k (1-p)^{n-k}$$

$$\Big\downarrow n \to \infty$$

$$\frac{1}{k!} \cdot \lambda^k \cdot e^{-\lambda}$$