2 Estimation Theory

2.1: Introduction to Estimation Theory

Estimation Theory

2.1

1. **Introduction to Estimation Theory**
    - Assessing Estimator Performance
    - Minimum Variance Unbiased Estimator
    - Function Estimation
2. **Cramer-Rao Bound and Efficient Estimator**
    - Cramer-Rao Bound
    - Examples
3. **Maximum Likelihood & Maximum a Posteriori Estimator**
    - Classical estimation: Maximum Likelihood Estimator
    - The Bayesian framework: Maximum a Posteriori Estimator

Introduction to Estimation Theory

2.1

1. **Introduction**
   - The relevance of model selection
2. **Assessing Estimator Performance**
   - Bias, Variance and MSE
   - Vector notation
3. **Minimum Variance Unbiased Estimator**
   - Derivative of a scalar function with respect to a vector
   - MVU: Lagrange optimization
4. **Function Estimation**
   - Autocorrelation
   - Power Spectral Density

Usefulness of Parameter Estimation

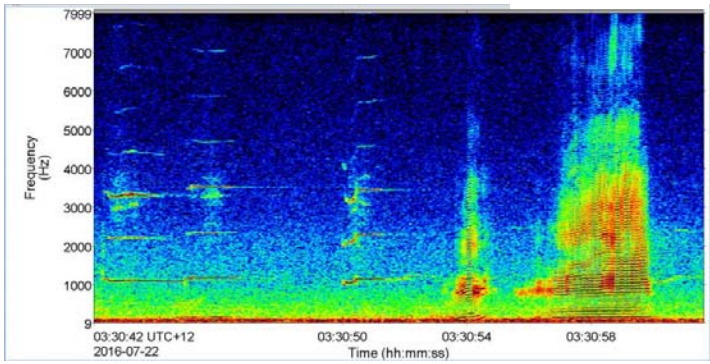
2.1

Given an N- point data set $\{x[1], x[2], \cdots, x[N]\}$ which **depends on an unknown parameter** θ (or set of parameters $\underline{\boldsymbol{\theta}}$), we wish to determine θ based on the data, through the definition of an estimator:

$$\hat{\theta} = g(x[1], x[2], \cdots, x[N]) = g(\underline{\mathbf{x}})$$

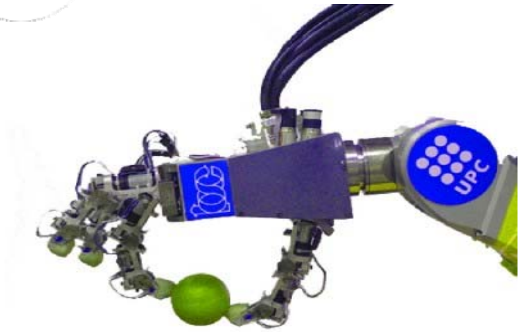
where $g(.)$ is some function.

Spectrogram and sound clip of a humpback whale



http://www.lab.upc.edu

Multifingered hand for dexterity and versatility



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Model selection (I)

2.1

The dependence of the available data ($\underline{\mathbf{x}}$) with respect to the parameters ($\underline{\boldsymbol{\theta}}$) is captured by the **model** that is proposed.

As data is random in nature, we represent it by its **probability density function** (pdf):

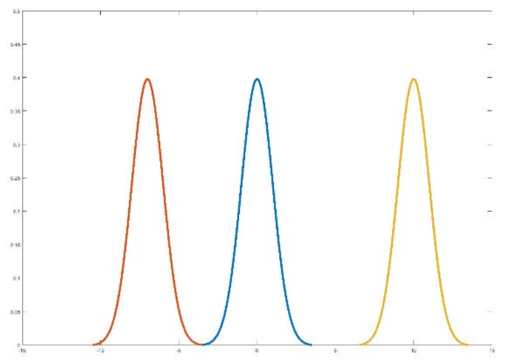
$$f_{\underline{\mathbf{x}}}\left(x[1], x[2], \cdots, x[N]; \underline{\boldsymbol{\theta}}\right) = f_{\underline{\mathbf{x}}}\left(\underline{\mathbf{x}}; \underline{\boldsymbol{\theta}}\right)$$

The **pdf is parameterized** by the unknown (vector of) parameter(s) $\underline{\boldsymbol{\theta}}$.

Case 1: We are given a pdf.

For instance, $N = 1$ ($x[1] = x$) and θ is the mean, the pdf could be:

$$f_x(x; \theta) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left[-\frac{(x-\theta)^2}{2\sigma^2}\right]$$



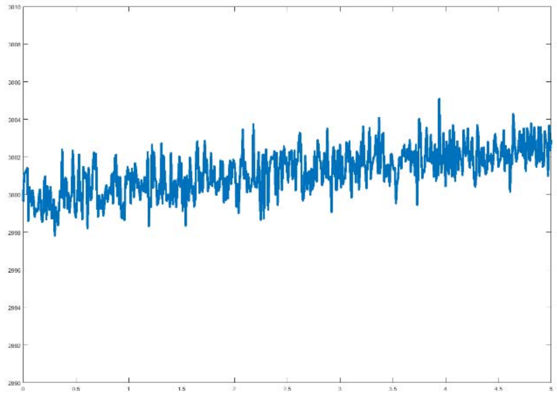
Example of three Gaussian pdf's parameterized with respect to θ

Model selection (II)

2.1

Case 2: Usually, we are given data and we have to **chose a model**:

1) Models should be **consistent** with the **problem** and **previous knowledge**
2) Models should be **mathematically tractable**



(1)

$$x[n] = A + Bn + w[n]$$

(2)

$$f_{\underline{w}}(\underline{\mathbf{x}}; \underline{\boldsymbol{\theta}}) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=1}^{N}(x[n] - A - Bn)^2\right]$$

Case 3: **Bayessian approach**: We can assume that the parameters to be estimated are random variables (instead of deterministic but unknown):

- The knowledge about its pdf can be included

$$f(\underline{\mathbf{x}}, \underline{\boldsymbol{\theta}}) = f(\underline{\mathbf{x}} / \underline{\boldsymbol{\theta}})\ f(\underline{\boldsymbol{\theta}})$$

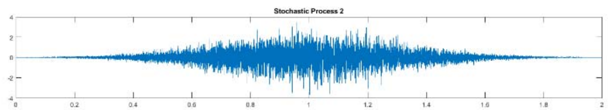
Several estimators are possible (I)

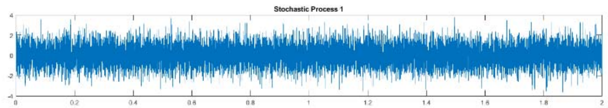
2.1

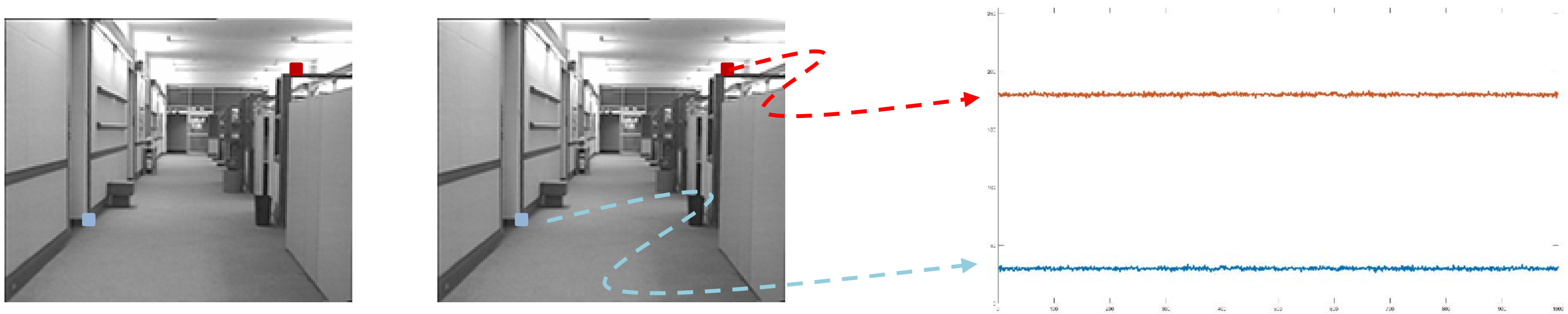
In several situations, we want to **estimate the mean value of a random process** that can be modeled as a constant value (θ) embedded in stationary white noise ($W[n]$):

$$X[n] = \theta + W[n]$$

- **White noise**: Each sample has a probability distribution with zero mean and finite variance, and samples are statistically independent and $r_w[n,l] = \sigma_w^2[n]\delta[l]$
- **Stationary white noise**: All variance samples have the same value and the autocorrelation function is $r_w[l] = \sigma_w^2\delta[l]$







The value of two pixels through time (n)

Several estimators are possible (II)

2.1

How can we **estimate the mean value of a random process** given a set of observations (N) of a single realization?

We can propose different estimators:

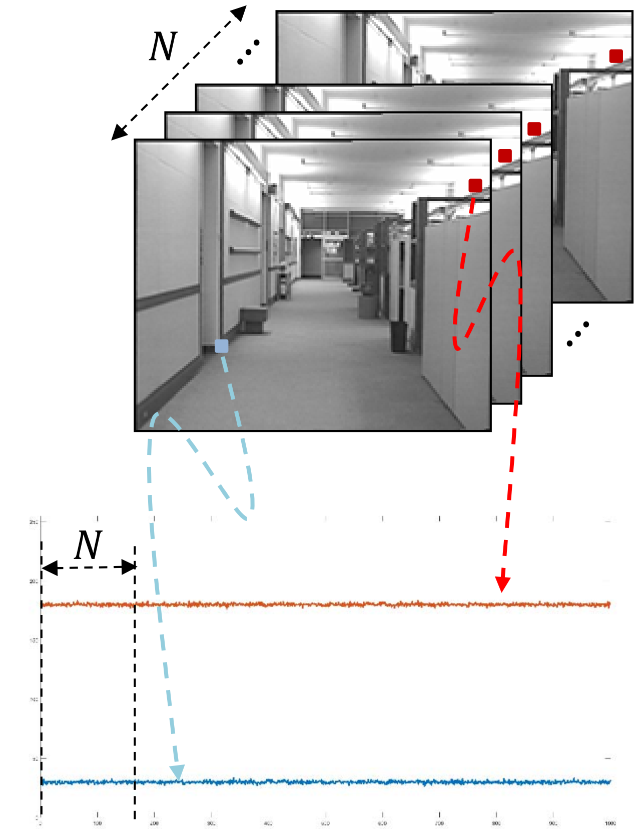
$$\hat{m}_X^{(1)} = \frac{1}{N}\sum_{n=1}^{N} x[n]$$

$$\hat{m}_X^{(2)} = \mathrm{median}(x[1], x[2], \cdots, x[N])$$

$$\hat{m}_X^{(3)} = \frac{\max(x[1], \cdots, x[N]) + \min(x[1], \cdots, x[N])}{2}$$

We need to assess **the performance of the estimators** to decide which one should be used.

$$\boxed{X[n] = \theta + W[n]}$$



Estimation of the mean value based on N samples

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Assessing the estimator

2.1

How can we **estimate the mean value of a random process** given a set of samples (N) of a single realization?

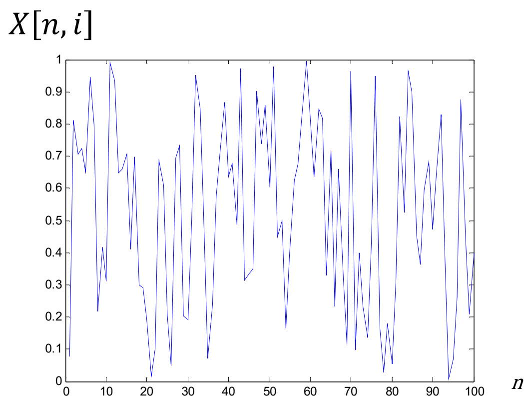
$$X[n] = \theta + W[n]$$

Let us assume that we select the average of the available samples (**sample mean**) as estimate of the mean value of the process. For this selection to be correct, we have to assume that:

$$\hat{m}_X^{(1)} = \frac{1}{N}\sum_{n=1}^{N} x[n]$$

- **Stationarity**: The parameter to be estimated does not change through time
- **Ergodicity**: Any realization of the process ($X[n,i]$) assumes the statistical properties of the whole process

$$m_x[n] = m_x$$



$$m_X = E\{X[n]\} = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} X[n,i]$$

Any estimator is a random variable

2.1

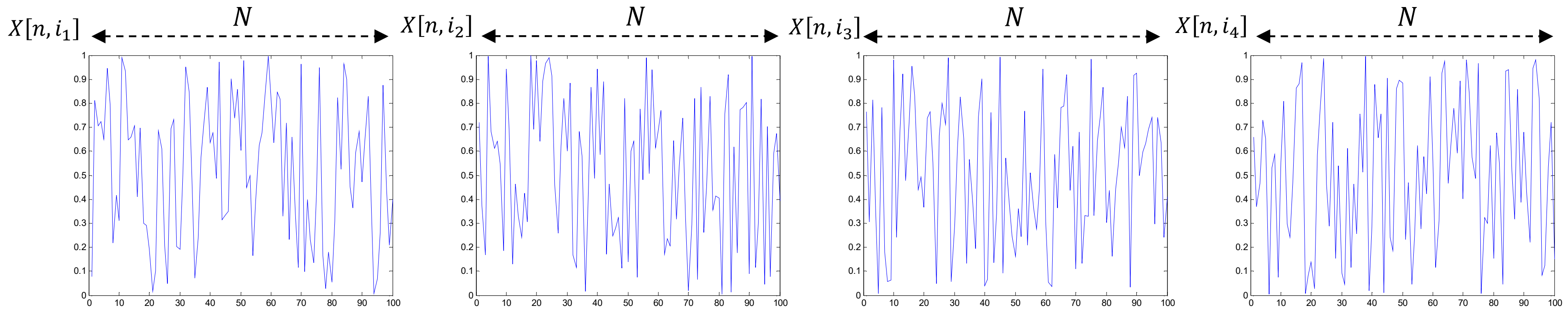
Estimators operate on the **samples of a given realization**. The estimated value depends on:

- The **available realization** ($X[n,i]$)
- The **selected window** (n, N)

$$X[n] = \theta + W[n]$$

$$\hat{m}_x^{(1)} = \frac{1}{N}\sum_{n=1}^{N} X[n,i]$$

Thus, **any estimator is a random variable!**



$\hat{m}_x^{(1)}(0,i_1) = 0.5205$	$\hat{m}_x^{(1)}(0,i_2) = 0.5217$	$\hat{m}_x^{(1)}(0,i_3) = 0.5045$	$\hat{m}_x^{(1)}(0,i_4) = 0.5100$
$\hat{m}_x^{(3)}(0,i_1) = 0.5010$	$\hat{m}_x^{(3)}(0,i_2) = 0.5007$	$\hat{m}_x^{(3)}(0,i_3) = 0.5008$	$\hat{m}_x^{(3)}(0,i_4) = 0.4990$

Four realizations of a process with samples uniformly distributed between 0 and 1

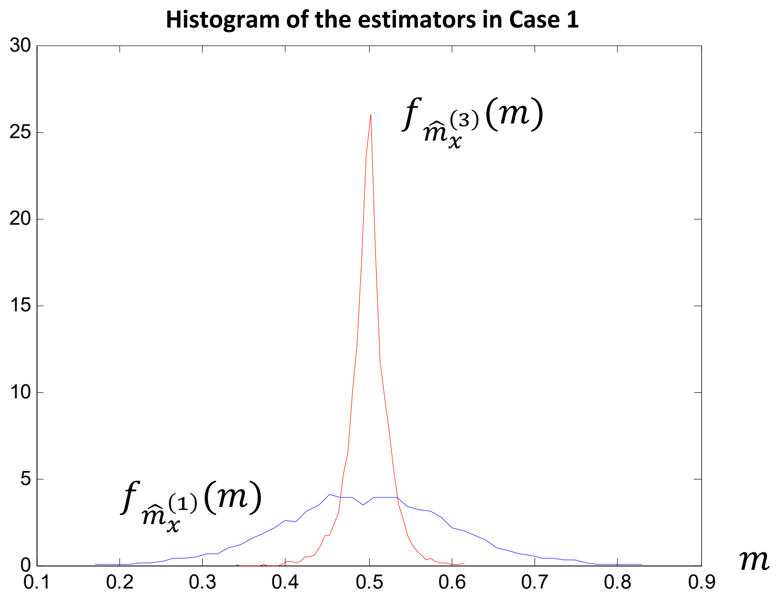
PDF of an estimator

Examples: Estimators as **random variables** (pdf):

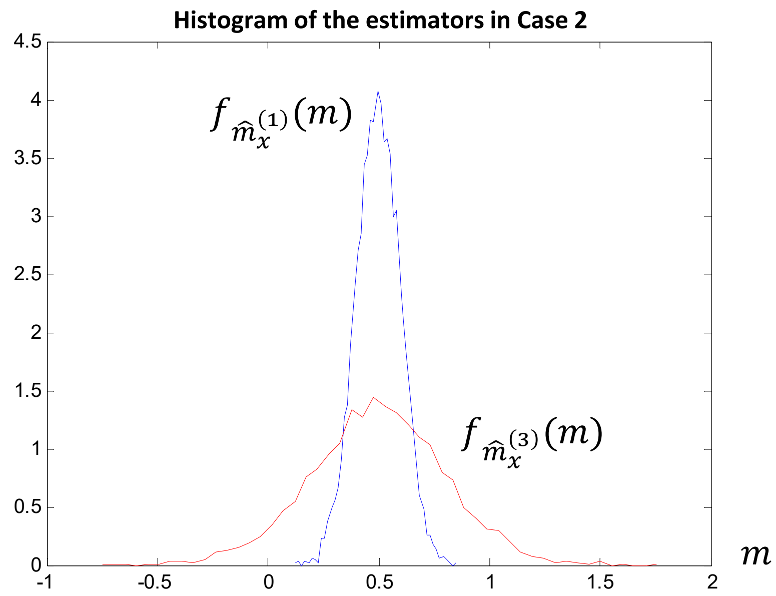
- **Case 1:** 5000 realizations of 100 samples from a white **uniform** process ($m_X = 0.5, \sigma_X^2 = 1$)
- **Case 2:** 5000 realizations of 100 samples from a white **Gaussian** process ($m_X = 0.5, \sigma_X^2 = 1$)

$$\widehat{m}_x^{(1)} = \frac{1}{N}\sum_{n=1}^{N} x[n] = \frac{1}{N}\underline{\mathbf{1}}^T\underline{\mathbf{x}}$$

$$\widehat{m}_x^{(3)} = \frac{\max(\underline{\mathbf{x}}) + \min(\underline{\mathbf{x}})}{2}$$



Estimators behave differently with the same data



Estimators behave differently when the pdf varies

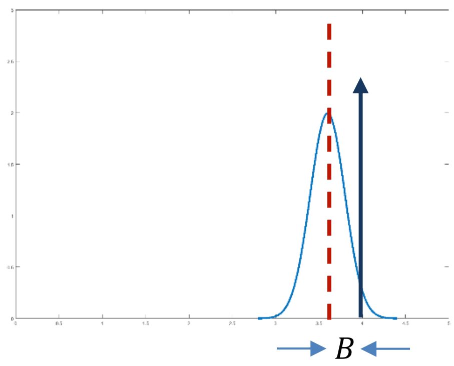
Features of an estimator (I)

2.1

The **bias of an estimator** is the difference between the expected value of the estimator and the true value of the parameter being estimated:

- Estimations delivered by a biased estimator are **consistently different** from the parameter to be estimated
- An estimator without biased is named **unbiased**

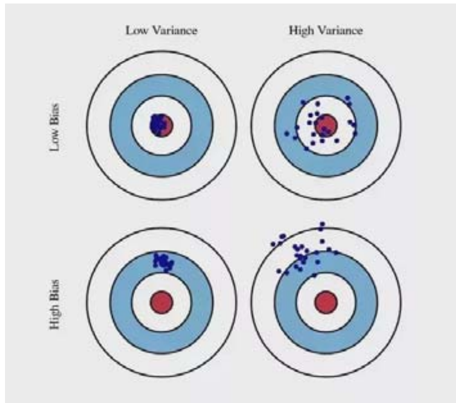
$$B(\hat{\theta}) = |\theta - E\{\hat{\theta}\}|$$



- ❑ Given the signal model ($X[n] = \theta + W[n]$), calculate the bias of the estimator.

  Note: $W[n]$ is a stationary, white noise.

$$\hat{\theta}_N = \frac{1}{N}\sum_{n=1}^{N} x[n]$$



Graphical comparison of bias and variance

Bias of the estimator

2.1

$$B(\hat{\theta}_N) = \left| \theta - E\{\hat{\theta}_N\} \right| = \qquad \left[\hat{\theta}_N = \frac{1}{N}\sum_{n=1}^{N} x[n]\right]$$

$$= \left| \theta - E\left\{ \frac{1}{N}\sum_{n=1}^{N} x[n] \right\} \right| = \qquad \left[x[n] = \theta + w[n]\right]$$

$$= \left| \theta - E\left\{ \frac{1}{N}\sum_{n=1}^{N} (\theta + w[n]) \right\} \right| = \qquad [\text{OPERAD. LIN.}]$$

$$= \left| \theta - \frac{1}{N}\sum_{n=1}^{N} E\{\theta + w[n]\} \right| = \qquad [\theta \ \text{DETERM.}]$$

$$= \left| \theta - \frac{1}{N}\sum_{n=1}^{N} (\theta + E\{w[n]\}) \right| = \qquad [E\{w[n]\} = \emptyset]$$

$$= \left| \theta - \frac{1}{N}\sum_{n=1}^{N} (\theta + \emptyset) \right| = \left| \theta - \frac{1}{N} N\theta \right| = \emptyset$$

Features of an estimator (II)

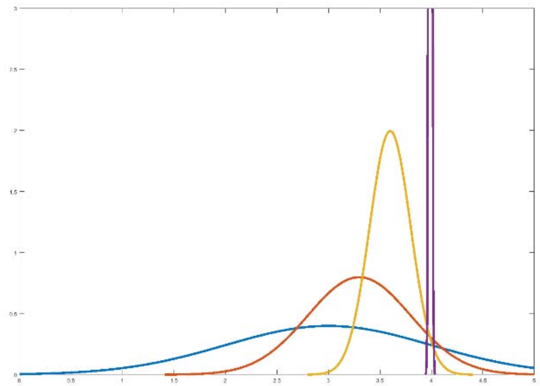
2.1

The unbiased constrain is desirable and, among all unbiased estimators, that of minimum **variance** is preferred (**Minimum Variance Unbiased: MVU**)

$$\sigma_{\hat{\theta}}^2 = E\left\{\left(\hat{\theta} - E\{\hat{\theta}\}\right)^2\right\}$$

An estimator is **consistent** if, as the number of samples (N) increases, the resulting sequence of estimates converges to θ

$$\lim_{N\to\infty} E\{\hat{\theta}\} \longrightarrow \theta \qquad \lim_{N\to\infty} \sigma_{\hat{\theta}}^2 \longrightarrow 0$$



A consistent estimator of $\theta = 4$

- ❑ Given the signal model ($X[n] = \theta + W[n]$), calculate the variance of the estimator $\hat{\theta}_N$. Is it consistent?
  Note: $W[n]$ is a stationary, white noise.

$$\hat{\theta}_N = \frac{1}{N}\sum_{n=1}^{N} x[n]$$

Variance of the estimator (I)

$$\sigma^2_{\hat{\theta}_N} = E\left\{\left(\hat{\theta}_N - E\{\hat{\theta}_N\}\right)^2\right\} = \qquad \left[E\{\hat{\theta}_N\} = \theta\right]$$

$$= E\left\{\left(\hat{\theta}_N - \theta\right)^2\right\} = \qquad \left[\hat{\theta}_N = \frac{1}{N}\sum_{n=1}^{N} x[n]\right]$$

$$= E\left\{\left(\frac{1}{N}\sum_{n=1}^{N} x[n] - \theta\right)^2\right\} = \qquad \left[x[n] = \theta + w[n]\right]$$

$$= E\left\{\left(\frac{1}{N}\sum_{n=1}^{N}\left(\theta + w[n]\right) - \theta\right)^2\right\} = \qquad \left[\text{SIMPLIF. EL } (\cdot)^2\right]$$

$$= E\left\{\left(\frac{1}{N}\cdot N\theta + \frac{1}{N}\sum_{n=1}^{N} w[n] - \theta\right)^2\right\} =$$

$$= E\left\{\frac{1}{N^2}\left(\sum_{n=1}^{N} w[n]\right)^2\right\} = \qquad \left[\Sigma \text{ ÍNDEX DIF}\right]$$

Variance of the estimator (II)

2.1

$$\sigma^2_{\hat{\theta}_N} = E\left\{\left(\hat{\theta}_N - E\{\hat{\theta}_N\}\right)^2\right\} = \qquad \left[E\{\hat{\theta}_N\} = \theta\right]$$

$$= E\left\{\frac{1}{N^2}\sum_{n=1}^{N} w[n] \sum_{m=1}^{N} w[m]\right\} =$$

$$= \frac{1}{N^2} E\left\{\sum_{n=1}^{N}\sum_{m=1}^{N} w[n]\,w[m]\right\} = \qquad \left[\text{OPERAD. LINEAL}\right]$$

$$= \frac{1}{N^2}\sum_{n=1}^{N}\sum_{m=1}^{N} E\{w[n]\,w[m]\} = \qquad \left[r_w[l] = \sigma_w^2\,\delta[l]\right]$$

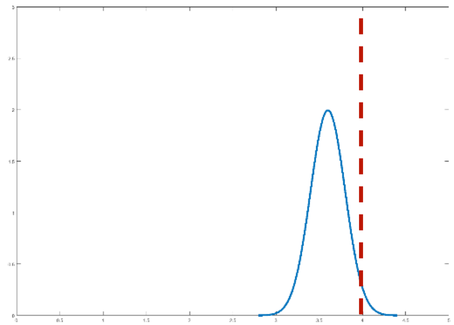
$$= \frac{1}{N^2}\sum_{n=1}^{N}\sum_{m=1}^{N} \delta[n-m]\,\sigma_w^2 = \qquad \left[\sum_{n=1}^{N}\sum_{m=1}^{N}\delta[n-m] = N\right]$$

$$= \frac{1}{N^2}\cdot N\sigma_w^2 \Rightarrow \boxed{\sigma^2_{\hat{\theta}_N} = \frac{\sigma_w^2}{N}} \Leftarrow \text{CONSISTENT}$$

Features of an estimator (III)

2.1

If the estimator is biased, the dispersion of the estimations with respect to the actual value to be estimated (θ) is not the variance but the **Mean Square Error** of the estimator ($MSE(\hat{\theta})$)



- The MSE can be a measure of assessment for a given estimator, but to define an estimator **optimizing the MSE usually leads to unrealizable estimators**

$$\sigma_{\hat{\theta}}^2 = E\left\{\left(\hat{\theta} - E\{\hat{\theta}\}\right)^2\right\} = [\text{Biased}] \neq E\left\{\left(\hat{\theta} - \theta\right)^2\right\} = MSE\left(\hat{\theta}\right)$$

❑ Demonstrate that, for a given estimator ($\hat{\theta}$),

$$MSE\left(\hat{\theta}\right) = \sigma_{\hat{\theta}}^2 + B^2\left(\hat{\theta}\right)$$

MSE of the estimator

2.1

$$MSE(\hat{\theta}) = E\{(\hat{\theta}-\theta)^2\} = E\{(\hat{\theta}-E\{\hat{\theta}\}-\theta+E\{\hat{\theta}\})^2\} =$$

$$= E\{((\hat{\theta}-E\{\hat{\theta}\})-(\theta-E\{\hat{\theta}\}))^2\} =$$

$$= E\{(\hat{\theta}-E\{\hat{\theta}\})^2-2(\hat{\theta}-E\{\hat{\theta}\})(\theta-E\{\hat{\theta}\})+(\theta-E\{\hat{\theta}\})^2\} =$$

$$= E\{(\hat{\theta}-E\{\hat{\theta}\})^2\}-2E\{(\hat{\theta}-E\{\hat{\theta}\})(\theta-E\{\hat{\theta}\})\}+E\{(\theta-E\{\hat{\theta}\})^2\} =$$

$$= \sigma_{\hat{\theta}}^2 - 2E\{(\hat{\theta}-E\{\hat{\theta}\})(\theta-E\{\hat{\theta}\})\} + B^2(\hat{\theta}) =$$

$$= \sigma_{\hat{\theta}}^2 + B^2(\hat{\theta}) - 2E\{\hat{\theta}-E\{\hat{\theta}\}\}(\theta-E\{\hat{\theta}\}) =$$

$$= \sigma_{\hat{\theta}}^2 + B^2(\hat{\theta}) - 2(E\{\hat{\theta}\}-E\{E\{\hat{\theta}\}\})(\theta-E\{\hat{\theta}\}) =$$

$$= [E\{E\{\hat{\theta}\}\} = E\{\hat{\theta}\}] = \sigma_{\hat{\theta}}^2 + B^2(\hat{\theta}) \leftarrow MSE(\hat{\theta}) = \sigma_{\hat{\theta}}^2 + B^2(\hat{\theta})$$

Exercise 2.1

2.1

2.1. Suppose that we have N samples of a random process $X[n] = A + W[n]$, where A is an unknown constant and $W[n]$ is a stationary, white noise (σ_W^2). Consider the modified estimator $\check{A}$ for some constant a. Find a which results in the minimum MSE and discuss the result.

$$\boxed{\check{A} = a\frac{1}{N}\sum_{n=1}^{N} x[n]}$$

$$\check{A} = a\frac{1}{N}\sum_{n=1}^{N} x[n] = a\hat{\theta}_N \quad \Rightarrow \quad \begin{cases} m_{\check{A}} = a m_{\hat{\theta}_N} = aA \\ \sigma_{\check{A}}^2 = a^2\sigma_{\hat{\theta}}^2 = a^2\dfrac{\sigma_W^2}{N} \end{cases}$$

$$MSE(\hat{\theta}) = \sigma_{\hat{\theta}}^2 + B^2(\hat{\theta}) \quad \Rightarrow \quad MSE(\check{A}) = a^2\frac{\sigma_W^2}{N} + (a-1)^2A^2$$

$$\frac{dMSE(\check{A})}{da} = 0 \quad \Rightarrow \quad \boxed{a_{\text{opt}} = \frac{A^2}{A^2 + \sigma_W^2/N}}$$

"The MSE can be a measure of assessment for a given estimator, but to define an estimator optimizing the MSE usually leads to unrealizable estimators"

Exercise 2.2

2.2. Suppose that we have N uncorrelated samples of a stationary random process denoted by $X[n]$. From these samples, we estimate the mean m_X and the variance σ_X^2 of the process using the following expressions. Determine:

a) The reasoning behind the definition of the variance estimator
b) The bias of the variance estimator σ_X^2.
c) An unbiased estimator for the variance of the process σ_X^2.

$$\hat{m}_X = \frac{1}{N}\sum_{n=1}^{N} x[n]$$

$$\hat{\sigma}_X^2 = \frac{1}{N}\sum_{n=1}^{N} (x[n] - \hat{m}_X)^2$$

Exercise 2.2

a) WE NEED TO USE $\hat{m}_x$ SINCE WE DO NOT KNOW THE ACTUAL m_x VALUE

b) $B(\hat{\sigma}_x^2) = \left| \sigma_x^2 - E\{\hat{\sigma}_x^2\} \right| \Rightarrow \sigma_x^2$ AND $E\{\hat{\sigma}_x^2\}$

$$\sigma_x^2 = E\left\{[x[n] - m_x]^2\right\} = \left[x[n] = m_x + w[n]\right] =$$

$$= E\left\{[m_x + w[n] - m_x]^2\right\} = E\left\{w^2[n]\right\} = \sigma_w^2$$

$$E\{\hat{\sigma}_x^2\} = E\left\{\frac{1}{N}\sum_n (x[n] - \hat{m}_x)^2\right\} = \left[\text{I NEED } \sigma_x^2\right] =$$

$$= E\left\{\frac{1}{N}\sum_n (x[n] - m_x + m_x - \hat{m}_x)^2\right\} =$$

$$= E\left\{\frac{1}{N}\sum_n \left((x[n] - m_x) - (\hat{m}_x - m_x)\right)^2\right\} =$$

Exercise 2.2

$$(1)\quad E\{(x[n]-m_x)^2\} = \sigma_x^2 = \sigma_w^2$$

$$(3)\quad E\{(\hat{m}_x - m_x)^2\} = \left[E\{\hat{m}_x\} = m_x\right] = \sigma^2[\hat{m}_x] = \frac{\sigma_w^2}{N}$$

$$(2)\quad E\{(x[n]-m_x)(\hat{m}_x - m_x)\} = \left[x[n]\ ;\ \hat{m}_x\right] =$$

$$= E\left\{(m_x + w[n] - m_x)\left(\frac{1}{N}\sum_k (m_x + w[k] - m_x)\right)\right\} =$$

$$= E\left\{w[n]\,\frac{1}{N}\sum_k w[k]\right\} = \frac{1}{N}\sum_k E\{w[n]\,w[k]\} =$$

$$= \left[r_w[l] = \sigma_w^2\,\delta[l]\right] = \frac{1}{N}\sum_k \sigma_w^2\,\delta[n-k].$$

Exercise 2.2

$$(1)+(2)+(3) \Rightarrow E\{\hat{\sigma}_x^2\} = \frac{1}{N}\sum_n\left(\sigma_\omega^2 - \frac{2}{N}\sum_k \sigma_\omega^2\,\delta[n-k] + \frac{\sigma_\omega^2}{N}\right) =$$

$$= \sigma_\omega^2 - \frac{2\sigma_\omega^2}{N}\sum_n\sum_k \delta[n-k] + \frac{\sigma_\omega^2}{N} = \sigma_\omega^2 - \frac{2}{N}\sigma_\omega^2 + \frac{\sigma_\omega^2}{N} =$$

$$E\{\hat{\sigma}_\omega^2\} = \frac{N-1}{N}\sigma_\omega^2 = \frac{N-1}{N}\sigma_x^2 \leftarrow \text{BIASED}$$

c) $$\tilde{\sigma}_x^2 = \frac{1}{N-1}\sum_{n=0}^{N-1}\left(x[n]-m_x\right)^2 = \frac{N}{N-1}\tilde{\sigma}_x^2$$

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The sample mean estimator as a filter

2.1

The previous **sample mean** estimator can be interpreted as a **filter** and, this way, we can generalize the study of its properties:

$$\hat{\theta}_N = \frac{1}{N}\sum_{n=1}^{N} x[n] = \frac{1}{N}\underline{1}^T\underline{x} \quad \Rightarrow \quad \boxed{\hat{\theta}_N = \underline{h}^T\underline{x}}$$

◀ Estimator **linear** in the (N) data

❑ Given the signal model $(X[n] = \theta + W[n])$, analyze the **bias** of the estimator $\hat{\theta}_N$.

Note: $W[n]$ is a stationary, white noise

$$B(\hat{\theta}_N) = 0 \quad \Rightarrow \quad \boxed{\underline{h}^T\underline{1} = 1}$$

◀ Only **zero-mean** has been imposed

One possible solution is the previous result: $\underline{h}^T = \frac{1}{N}\underline{1}^T$

Bias of the estimator

2.1

$$B(\hat{\theta}_N) = |\theta - E\{\hat{\theta}_N\}| \Rightarrow E\{\hat{\theta}_N\}$$

$$E\{\hat{\theta}_N\} = E\{\underline{h}^T \underline{x}\} = \underline{h}^T E\{\underline{x}\} = [\underline{x} = \underline{\theta} + \underline{w}] =$$

$$= \underline{h}^T E\{\underline{\theta} + \underline{w}\} = [\theta \text{ DETERM.}] = \underline{h}^T [\underline{\theta} + E\{\underline{w}\}] =$$

$$= [E\{\underline{w}\} = \underline{0}] = \underline{h}^T \cdot \underline{\theta} = \theta\, \underline{h}^T \cdot \underline{1}$$

$$B(\hat{\theta}_N) = |\theta - \theta\, \underline{h}^T \cdot \underline{1}| = |\theta [1 - \underline{h}^T \underline{1}]|$$

$$B(\hat{\theta}_N) = \emptyset \Rightarrow \underline{h}^T \cdot \underline{1} = 1 \Rightarrow \text{FOR INSTANCE } \underline{h}^T = \frac{1}{N} \underline{1}$$

The sample mean estimator as a filter

2.1

The previous **sample mean** estimator can be interpreted as a **filter** and, this way, we can generalize the study of its properties:

$$\hat{\theta}_N = \frac{1}{N}\sum_{n=1}^{N} x[n] = \frac{1}{N}\underline{1}^T\underline{x} \quad \Rightarrow \quad \boxed{\hat{\theta}_N = \underline{h}^T\underline{x}}$$

◀ Estimator **linear** in the (N) data

❑ Given the signal model ($X[n] = \theta + W[n]$), analyze the **variance** of the estimator $\hat{\theta}_N$. Note: $W[n]$ is a stationary, white noise

$$\boxed{\sigma^2_{\hat{\theta}_N} = \underline{h}^T\underline{\underline{R}}_w\underline{h}}$$

◀ It is the **variance** (power) **of the filtered noise**

◀ Only **unbiased (zero-mean noise)** has been imposed

➢ Evaluate this result for a **stationary**, white noise and $\underline{h} = \frac{1}{N}\underline{1}$

Variance of the estimator

2.1

$$\hat{\sigma}^2_{\hat{\theta}_N} = E\{(\hat{\theta}_N - E\{\hat{\theta}_N\})^2\} =$$

$$= [\text{UNBIASED CASE}] = [\underline{h}^T \cdot \underline{1} = 1] = [E\{\hat{\theta}_N\} = \theta] =$$

$$E\{(\hat{\theta}_N - \theta)^2\} = E\{(\underline{h}^T \cdot \underline{x} - \theta)^2\} = [\underline{x} = \underline{\theta} + \underline{w}] =$$

$$= E\{(\underline{h}^T(\underline{\theta} + \underline{w}) - \theta)^2\} = E\{(\underline{h}^T \cdot \underline{\theta} + \underline{h}^T \underline{w} - \underline{\theta})^2\} =$$

$$= [\underline{h}^T \cdot \underline{\theta} = \theta \cdot \underline{h}^T \cdot \underline{1} = \theta] = E\{(\theta + \underline{h}^T \cdot \underline{w} - \theta)^2\} =$$

$$= E\{(\underline{h}^T \cdot \underline{w})^2\} = E\{\underline{h}^T \cdot \underline{w} \cdot \underline{h}^T \cdot \underline{w}\} = E\{\underline{h}^T \cdot \underline{w}\, \underline{w}^T \cdot \underline{h}\} =$$

$$= [\underline{h} \text{ DETERM.}] = \underline{h}^T E\{\underline{w} \cdot \underline{w}^T\} \cdot \underline{h} = \underline{h}^T \underline{\underline{R}}_w \cdot \underline{h}$$

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A family of estimators

2.1

After generalizing the **sample mean** as a **filter**, we have obtained **a family of unbiased linear estimators** of the mean of a random process, for which we have the expression of their variance:

$$\hat{\theta}_N = \underline{\mathrm{h}}^T \underline{\mathrm{x}} \quad \Rightarrow \quad \text{Unbiased if: } \underline{\mathrm{h}}^T \underline{1} = 1$$

$$\sigma^2_{\hat{\theta}_N} = \underline{\mathrm{h}}^T \underline{\underline{\mathrm{R}}}_w \underline{\mathrm{h}}$$

Note: We imposed zero-mean noise ($\hat{\theta}_N$) and the use of the unbiased estimator ($\sigma^2_{\hat{\theta}_N}$)

To obtain the **Minimum Variance Unbiased (MVU)** estimator, we should solve the following problem of optimization with constraints :

$$\min_{\underline{\mathrm{h}}} \left(\underline{\mathrm{h}}^T \underline{\underline{\mathrm{R}}}_w \underline{\mathrm{h}} \right)$$

$$\text{subject to } \underline{\mathrm{h}}^T \underline{1} = 1$$

This optimization problem is formulated through **Lagrange multipliers**

Lagrange multipliers

2.1

The method of **Lagrange multipliers** allows an optimization problem with constraints to be solved **without explicit parameterization** in terms of the constraints.

Given a function $f(\underline{x})$ that we want to optimize subject to a constraint (described by another function) $g(\underline{x})$, we can define a **Lagrange function** (or **Lagrangian**) $\mathcal{L}(\underline{x}, \lambda)$ whose first derivatives are zero at the solutions of the original constrained problem.

Note: The theory of Lagrange multipliers will be studied in the *Mathematical Optimization* course.

$$\left.\begin{array}{l} \underset{\underline{x}}{\text{optimize}}\ f(\underline{x}) \\ \text{subject to } g(\underline{x}) = 0 \end{array}\right\} \Rightarrow \underline{\mathcal{L}(\underline{x}, \lambda) = f(\underline{x}) - \lambda g(\underline{x})} \Rightarrow \left\{\begin{array}{l} \nabla_{\underline{x}} \mathcal{L}(\underline{x}, \lambda) = \underline{0} \\ \partial \mathcal{L}(\underline{x}, \lambda)/\partial \lambda = 0 \end{array}\right.$$

It is necessary to **derivate a scalar function with respect to a vector.**

Derivate a scalar wrt a vector: rules (I)

2.1

Given a scalar function $f(\underline{\mathrm{x}}) \in \mathbb{R}$ with $\underline{\mathrm{x}} \in \mathbb{R}^N$, we define:

$$\nabla_{\underline{\mathrm{x}}} f(\underline{\mathrm{x}}) = \left(\frac{\partial f(\underline{\mathrm{x}})}{\partial x_1}, \frac{\partial f(\underline{\mathrm{x}})}{\partial x_2}, \cdots, \frac{\partial f(\underline{\mathrm{x}})}{\partial x_N} \right)^T \in \mathbb{R}^N$$

Given this definition, the most common cases that we will work with are:

$$\nabla_{\underline{\mathrm{x}}} \left(\underline{\mathrm{h}}^T \underline{\mathrm{x}} \right) = \nabla_{\underline{\mathrm{x}}} \left(\sum_{i=1}^{N} h_i x_i \right) = \left(\frac{\partial \sum_{i=1}^{N} h_i x_i}{\partial x_1}, \frac{\partial \sum_{i=1}^{N} h_i x_i}{\partial x_2}, \cdots, \frac{\partial \sum_{i=1}^{N} h_i x_i}{\partial x_N} \right)^T$$

$$\nabla_{\underline{\mathrm{x}}} \left(\underline{\mathrm{h}}^T \underline{\mathrm{x}} \right) = (h_1, h_2, \cdots, h_N)^T = \underline{\mathrm{h}}$$

$$\boxed{\nabla_{\underline{\mathrm{x}}} \left(\underline{\mathrm{h}}^T \underline{\mathrm{x}} \right) = \underline{\mathrm{h}}}$$

In the same way, we can obtain:

$$\boxed{\nabla_{\underline{\mathrm{x}}} \left(\underline{\mathrm{x}}^T \underline{\mathrm{h}} \right) = \underline{\mathrm{h}}}$$

Derivate a scalar wrt a vector: rules (II)

2.1

Given this definition, the most common cases that we will work with are (cont.):

$$\nabla_{\underline{x}}\left(\underline{z}^T \underline{\underline{A}}\,\underline{x}\right) = \left[\underline{z}^T \underline{\underline{A}} = \underline{v}^T\right] = \nabla_{\underline{x}}\left(\underline{v}^T \underline{x}\right) = \underline{v} = \left(\underline{v}^T\right)^T = \left(\underline{z}^T \underline{\underline{A}}\right)^T = \underline{\underline{A}}^T \underline{z}$$

In the same way: $\nabla_{\underline{x}}\left(\underline{x}^T \underline{\underline{A}}\,\underline{z}\right) = \nabla_{\underline{x}}\left(\underline{x}^T \underline{v}\right) = \underline{v} = \underline{\underline{A}}\,\underline{z}$

$$\boxed{\nabla_{\underline{x}}\left(\underline{z}^T \underline{\underline{A}}\,\underline{x}\right) = \underline{\underline{A}}^T \underline{z}} \qquad \boxed{\nabla_{\underline{x}}\left(\underline{x}^T \underline{\underline{A}}\,\underline{z}\right) = \underline{\underline{A}}\,\underline{z}}$$

If we have a symmetric matrix ($\underline{\underline{A}}^T = \underline{\underline{A}}$), such as a correlation matrix, it can be shown that:

$$\boxed{\nabla_{\underline{x}}\left(\underline{x}^T \underline{\underline{A}}\,\underline{x}\right) = 2\underline{\underline{A}}\,\underline{x}}$$

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Minimum Variance Unbiased Estimator

2.1

To obtain the **Minimum Variance Unbiased (MVU)** estimator, we should solve the following problem of optimization with constraints :

$$\min_{\underline{\mathrm{h}}} \left(\underline{\mathrm{h}}^T \underline{\underline{\mathrm{R}}}_w \underline{\mathrm{h}}\right)$$

◀ Minimizing the variance of the estimator

$$\text{subject to } \underline{\mathrm{h}}^T \underline{1} = 1$$

◀ Imposing the estimator to be unbiased

Note: Only unbiased estimator and zero-mean noise were imposed to obtain these results

- ❑ Given the signal model ($X[n] = \theta + W[n]$), find the **MVU estimator** for the parameter θ.

$$\underline{\mathrm{h}} = \frac{\underline{\underline{\mathrm{R}}}_w^{-1} \underline{1}}{\underline{1}^T \underline{\underline{\mathrm{R}}}_w^{-1} \underline{1}}$$

In the case of stationary, white noise $\Rightarrow$ $\underline{\mathrm{h}} = \frac{1}{N} \underline{1}$

Minimum Variance Unbiased Estimator

$$\left.\begin{array}{l} \min\limits_{\underline{h}} \ \underline{h}^T \cdot \underline{\underline{R}}_w \cdot \underline{h} \\ \underline{h}^T \cdot \underline{1} = 1 \end{array}\right\} \quad \mathcal{L}(\underline{h}, \lambda) = \underline{h}^T \underline{\underline{R}}_w \underline{h} - \lambda(\underline{h}^T \underline{1} - 1)$$

$$\nabla \mathcal{L}(\underline{h}, \lambda) = \emptyset \Rightarrow \begin{cases} \nabla_{\underline{h}} \mathcal{L}(\underline{h}, \lambda) = \emptyset \\ \dfrac{\partial \mathcal{L}}{\partial \lambda} = 0 \end{cases}$$

$$\nabla_{\underline{h}} \left[\underline{h}^T \underline{\underline{R}}_w \underline{h} - \lambda(\underline{h}^T \underline{1} - 1) \right] = 2 \underline{\underline{R}}_w \cdot \underline{h} - \lambda [\underline{1}] = \underline{\emptyset} \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \underline{h}^T \cdot \underline{1} - 1 = \emptyset \quad \longleftarrow \text{PREVIOUS CONSTRAINT} \qquad (2)$$

$$(1) \quad 2 \underline{\underline{R}}_w \cdot \underline{h} = \lambda \underline{1} \Rightarrow \underline{h} = \frac{\lambda}{2} \underline{\underline{R}}_w^{-1} \underline{1} \qquad (3)$$

Minimum Variance Unbiased Estimator

$$\nabla_{\underline{h}} \left[\underline{h}^T \underline{\underline{R}}_w \underline{h} - \lambda \left(\underline{h}^T \underline{1} - 1 \right) \right] = 2 \underline{\underline{R}}_w \cdot \underline{h} - \lambda \left[\underline{1} \right] = \underline{\emptyset} \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \underline{h}^T \cdot \underline{1} - 1 = \emptyset \quad \longleftarrow \text{PREVIOUS CONSTRAINT} \quad (2)$$

$$(1) \quad 2 \underline{\underline{R}}_w \cdot \underline{h} = \lambda \underline{1} \Rightarrow \underline{h} = \frac{\lambda}{2} \underline{\underline{R}}_w^{-1} \underline{1} \quad (3)$$

$$(3) \rightarrow (2) \quad \left[\frac{\lambda}{2} \underline{\underline{R}}_w^{-1} \cdot \underline{1} \right]^T \cdot \underline{1} = 1 \Rightarrow \frac{\lambda}{2} \underline{1}^T \underline{\underline{R}}_w^{-1} \underline{1} = 1 \Rightarrow$$

$$\Rightarrow \quad \lambda = \frac{2}{\underline{1}^T \underline{\underline{R}}_w^{-1} \underline{1}} \quad \text{LAGRANGE MULTIPLIER} \quad (4)$$

$$(4) \rightarrow (3) \quad \underline{h} = \frac{\lambda}{2} \underline{\underline{R}}_w^{-1} \underline{1} \Rightarrow \underline{h} = \frac{\underline{\underline{R}}_w^{-1} \cdot \underline{1}}{\underline{1}^T \underline{\underline{R}}_w^{-1} \underline{1}}$$

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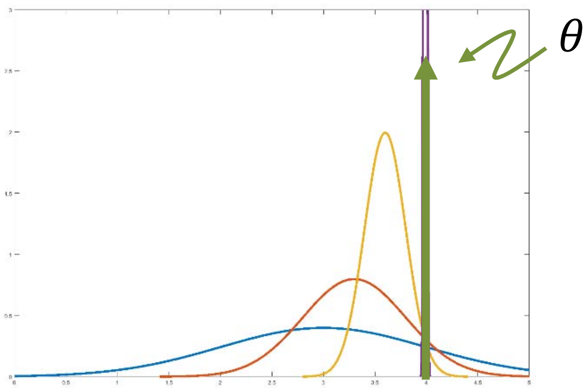
Estimation of the correlation function

2.1

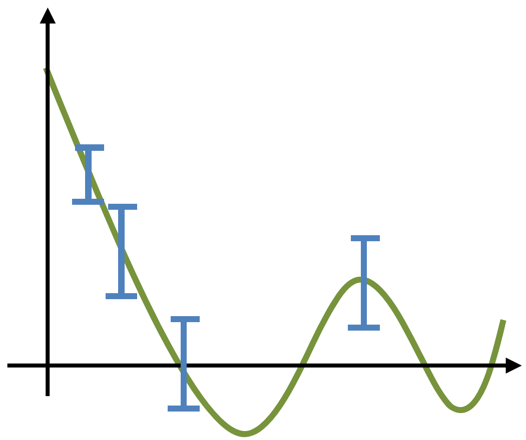
In some cases, we want to **estimate a function** rather than a single parameter. Common cases are:

- The **autocorrelation** function of a process
- The **spectral density function** of a process

When estimating a parameter, the used **estimator** becomes a **random variable**. Therefore, when estimating a function (an ordered set of parameters) the **estimator** becomes a **random process** (an ordered set of random variables).



Different estimators of θ

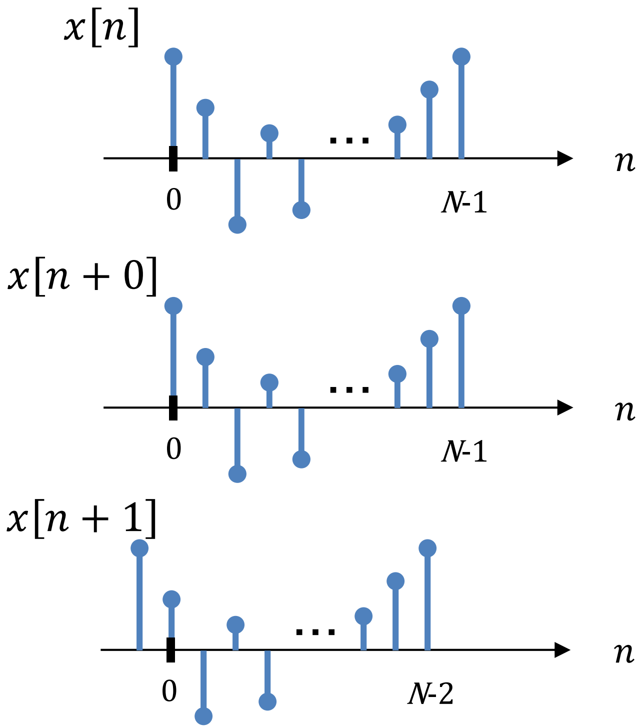


Estimation of the correlation function

2.1

Given N samples $\{x[0], x[1], \cdots, x[N-1]\}$ of a realization of an ergodic process $X[n]$, we want to estimate the autocorrelation of that process:

$$r_x[l] = E\{X[n+l]X[n]\}$$



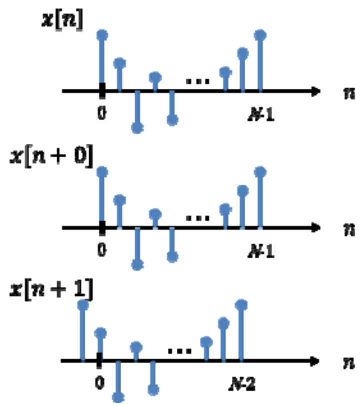
Let us analyze how to estimate each lag (l) of the autocorrelation function:

$$\check{r}_x[0] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]x[n]$$

$$\check{r}_x[1] = \frac{1}{N-1}\sum_{n=0}^{N-2} x[n+1]x[n]$$

Estimation of the correlation function

$$\check{r}_x[l] = \begin{cases} \dfrac{1}{N-l}\sum_{n=0}^{N-l-1} x[n+l]x[n] & 0 \le l \le N-1 \\ \dfrac{1}{N-|l|}\sum_{n=|l|}^{N-1} x[n+l]x[n] & -N+1 \le l \le 0 \end{cases}$$



- As the correlation function is symmetric ($r_x[l] = r_x[-l]$) the second expression (for negative lags) is not computed.
- The $\check{r}_x[l]$ estimator is **unbiased**
- Therefore, $MSE(\check{r}_x[l]) = \sigma^2(\check{r}_x[l])$
- However, the value of $\sigma^2(\check{r}_x[l])$ is not known.
  - Only it has been approximated for specific cases of random processes

Unbiased estimator of the correlation

2.1

- **Unbiased estimator**: let us analyze the bias of the estimator, using only the expression for the positive lags:

$$E\{\{\check{r}_x[l]\}\} = E\left\{\frac{1}{N-l}\sum_{n=0}^{N-l-1} x[n+l]x[n]\right\} = \frac{1}{N-l}\sum_{n=0}^{N-l-1} E\{x[n+l]x[n]\}$$

$$E\{\{\check{r}_x[l]\}\} = \frac{1}{N-l}\sum_{n=0}^{N-l-1} r_x[l] = r_x[l]$$

$$\boxed{E\{\{\check{r}_x[l]\}\} = r_x[l]}$$

- **The variance** has been only demonstrated for the Gaussian case and $N \gg l$

$$\sigma^2(\check{r}_x[l]) = \frac{N}{(N-|l|)^2}\sum_{K=-\infty}^{\infty}\left(r_x^2[k] + r_x[k+l] + r_x[k-l]\right)$$

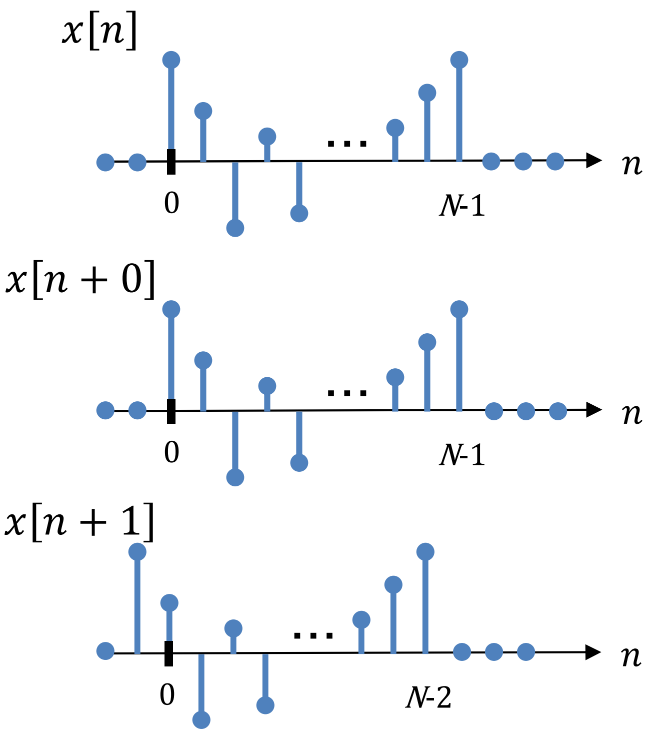
- **Common behavior**:
  - The **variance increases** with $|l|$
  - The estimator is **consistent**

$$\boxed{\lim_{N\to\infty} \sigma^2(\check{r}_x[l]) \longrightarrow 0}$$

Improving the variance behavior

To remove the dependency of the variance with the lag ($|l|$), a new estimator for the autocorrelation is proposed:

$$\hat{r}_x[l] = \begin{cases} \dfrac{1}{N}\displaystyle\sum_{n=0}^{N-l-1} x[n+l]x[n] & 0 \le l \le N-1 \\ \dfrac{1}{N}\displaystyle\sum_{n=|l|}^{N-1} x[n+l]x[n] & -N+1 \le l \le 0 \end{cases}$$



Both estimators are related :

$$\hat{r}_x[l] = \frac{N-|l|}{N}\check{r}_x[l]$$

- The new estimator is **biased**
- It reduces the **variance**
- It reduces the **MSE**

Interpretation of the bias

2.1

The biased estimator is **linearly related** with the unbiased one:

$$\hat{r}_x[l] = \frac{N-|l|}{N}\check{r}_x[l]$$

- The mean value is:
  - A **bias** has been introduced

$$E\{\{\hat{r}_x[l]\}\} = \frac{N-|l|}{N} r_x[l]$$

- The variance has decreased:
  - **Independent** of $|l|$
  - **Consistent**

$$\sigma^2(\hat{r}_x[l]) = \frac{1}{N}\sum_{K=-\infty}^{\infty}\left(r_x^2[k] + r_x[k+l] + r_x[k-l]\right)$$

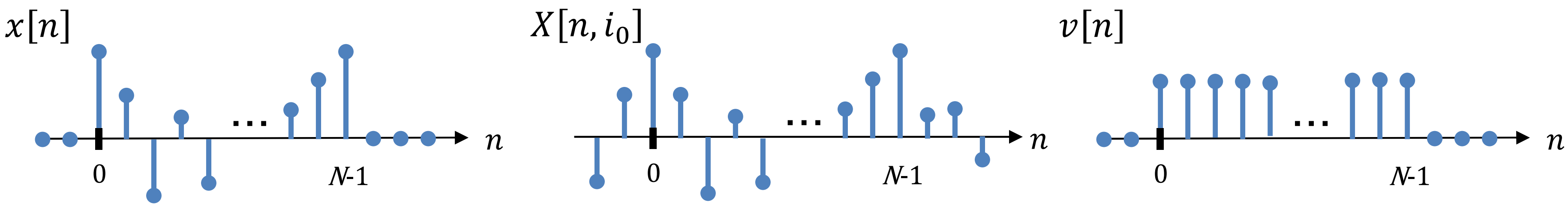
- It can be shown that the MSE has decreased:

$$MSE(\hat{r}_x[l]) < MSE(\check{r}_x[l])$$

Assessment of the biased estimator

2.1

The available N samples can be modelled as having a **whole realization** of the processed that has been **windowed**:

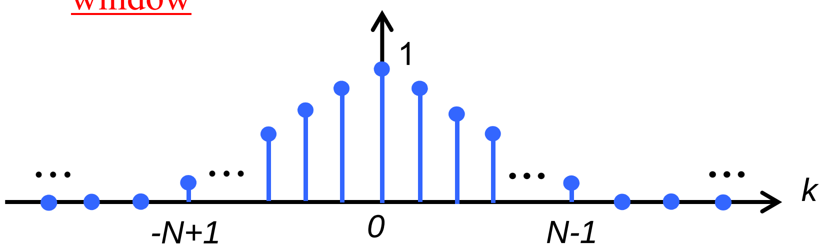


A (consistent) square window upon the data samples ($v[n]$) produces a **triangular window** ($w[l]$) upon the mean of the correlation samples:

$$E\{\{\hat{r}_x[l]\}\} = \frac{N-|l|}{N} r_x[l]$$

Autocorrelation window $\rightarrow$

$$w[l] = \frac{1}{N} v[k] * v[-k]$$



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