CHAPTER 16

Foundations Revisited

The reader should picture a street mime juggling non-existent balls. As the mime continues, the action of juggling slowly brings the balls into existence, at first in dim outline and then into solid reality. — T.W. Körner[1]

An ordered field F is **Dedekind complete** if every nonempty subset which is bounded above has a least upper bound (or "supremum").

Exercise 16.0: Show that an ordered field is Dedekind complete iff every nonempty subset which is bounded below has a greatest upper bound (or "infimum").

Our initial definition of $\mathbb{R}$ was precisely that it was a Dedekind complete ordered field. Practically speaking, this is a great foundation for honors calculus and real analysis, because it contains all the information we need to know about $\mathbb{R}$.

In other words, we have put a neat little black box around our foundational problems. Real analysis works perfectly well without ever having to look in the box. But curiosity is a fundamental part of mathematics, and at some point most of us will want to look in the box. This chapter is for those who have reached that point, i.e., who want to understand a proof of the following theorem.

Theorem 16.1. *(Black Box Theorem)*
a) There is a Dedekind complete ordered field.
b) If F_1 and F_2 are Dedekind complete ordered fields, they are isomorphic*: that is, there is a bijection $\Phi : F_1 \to F_2$ such that:*
(i) For all $x, y \in F_1$, $\Phi(x + y) = \Phi(x) + \Phi(y)$.
(ii) For all $x, y \in F_1$, $\Phi(xy) = \Phi(x)\Phi(y)$.
(iii) For all $x, y \in F_1$, $x \leq y \iff \Phi(x) \leq \Phi(y)$.
c) The isomorphism Φ of part b) is unique*: there is exactly one such map between any two Dedekind complete ordered fields.*

The Black Box Theorem explains why we never needed any further axioms of $\mathbb{R}$ beyond the fact that it is a Dedekind complete ordered field: there is exactly one such structure, up to isomorphism.[2]

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[2] The student unfamiliar with the notion of "isomorphism" should think of it as nothing else than a relabelling of the points of $\mathbb{R}$. For instance consider the x-axis $\mathbb{R}_x = \{(a, 0) \mid a \in \mathbb{R}\}$ and the y-axis $\mathbb{R}_y = \{(0, a) \mid a \in \mathbb{R}\}$ in the plane. These are two copies of $\mathbb{R}$. Are they "the same"? Not in a hard-nosed set-theoretic sense: they are different subsets of the plane. But they are essentially the same: the bijection Φ which carries $(a, 0) \mapsto (0, a)$ preserves the addition, multiplication and order relation. So really we have two slightly different presentations of the same

We will prove the Black Box Theorem...eventually. But rather than taking the most direct possible route we broaden our focus to a study of the structure of ordered fields, not just $\mathbb{Q}$ and $\mathbb{R}$.

1. Ordered Fields

1.1. Basic Definitions.

In this section we revisit the considerations of §1.2 from a somewhat different perspective. Before we listed certain **ordered field axioms**, but the perspective there was that we were collecting true, and basic, facts about the real numbers for use in our work with them. This time our perspective is to study and understand the collection of all ordered fields. One of our main goals is to *construct* the real numbers $\mathbb{R}$ in terms of the rational numbers $\mathbb{Q}$ and to understand this in terms of a more general process, **completion**, which can be applied in any ordered field.

A **field** is a set F endowed with two binary operations $+$ and $\cdot$ which satisfy all of the field axioms (P0) through (P9). To a first approximation, these axioms simply encode the usual rules of addition, subtraction, multiplication and division of numbers, so any field can be thought of as a kind of "generalized number system". The most important basic examples are the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$. But there are other examples which seem farther removed from the "usual numbers": e.g. finite fields like $\mathbb{F}_2 = \{0, 1\}$ are smaller than what we normally think of as a number system, whereas the set $\mathbb{R}(t)$ of all rational functions (with real coefficients) is a field whose elements are naturally regarded as functions, not as numbers.

Field theory is an active branch of mathematical research, with several texts and thousands of papers devoted to it (e.g. [**FT**]). Nevertheless the very simple properties of fields established in § 1.2.1 will be sufficient for our needs here, in part because we are not interested in fields *per se* but rather **ordered fields**. An ordered field is a field equipped with the additional structure of a **total order relation**, namely a binary relation $\leq$ which satisfies:

(TO1) Reflexivity: for all $x \in F$, $x \leq x$.
(TO2) Antisymmetry: for all $x, y \in F$, if $x \leq y$ and $y \leq x$, then $x = y$.
(TO3) Transitivity: for all $x, y, z \in F$, if $x \leq y$ and $y \leq z$ then $x \leq z$.
(TO4) Totality: for all $x, y \in F$, either $x \leq y$ or $y \leq x$.

Given a total order, we may define a **strict ordering** $x < y$ by $x \leq y$ but $x \neq y$. This is (as we well know!) natural and useful. From an abstract perspective the key remark is that it is essentially equivalent to a total ordering. Namely, any strict ordering coming from a total ordering satisfies the following modified versions of the above axioms:

(TR) Trichotomy: for all $x, y \in F$, exactly one of the following holds:

$$x < y; \;\; x = y; \;\; \text{or } x > y.$$

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essential structure. An arbitrary isomorphism is no more than this, except that the "relabelling map" Φ might be more complicated.

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essential structure. An arbitrary isomorphism is no more than this, except that the "relabelling map" Φ might be more complicated.

(TO3): Transitivity: for all $x, y, z \in F$, if $x < y$ and $y < z$ then $x < z$.

Exercise 16.1: Let $(F, \leq)$ be a total order relation on the set F.
a) Show that the associated strict ordering satisfies (TR) and (TO3).
b) Let $<$ be a binary relation on F satisfying (TR) and (TO3). Define $x \leq y$ by $x < y$ or $x = y$. Show that this gives a total ordering on F.
c) Show that the processes of passing from a total ordering to its strict ordering and from a strict ordering to its total ordering are mutually inverse: doing one followed by the other, in either order, brings us back where we started.

In light of the preceding exercise we may (and shall) in fact pass from a total ordering to its strict ordering and back without explicit comment.

An **ordered field** is a field $(F, +, \cdot)$ equipped with a total ordering $\leq$ which is **compatible** with the field structure in the sense of satisfying the following two familiar axioms:

(P11) If $x, y > 0$, then $x + y > 0$.
(P12) If $x, y > 0$, then $xy > 0$.

We refer the reader back to § to 1.2.2 for the most elementary consequences of these axioms, e.g. Proposition 1.11.

Now we introduce a new idea. Let F and F' be ordered fields. (To be formally corrrect, we should speak of "the ordered field $(F, +, \cdot, <)$", and similarly for F'. In practice this extra notation weighs us down without any advantages in clarity or precision.) An **ordered field homomorphism** is a map $f : F \to F'$ satisfying all of the following:

(FH1) For all $x, y \in F$, $f(x + y) = f(x) + f(y)$.
(FH2) For all $x, y \in F$, $f(xy) = f(x)f(y)$.
(FH3) $f(1) = 1$.
(OFH) For all $x \in F$, if $x > 0$, then $f(x) > 0$.

Exercise 16.2: Show that for any homomorphism of ordered fields, $f(0) = 0$.

Because of Exercise 16.2 it is natural to wonder whether we really need (FH3) or whether in fact $f(1) = 1$ follows automatically from the other axioms. The answer is a resounding *no*. Indeed, in the absence of (FH3), for any ordered fields F and F' we could get a homomorphism betweeen them simply by mapping every $x \in F$ to 0. This is not an interesting map,[3] and as we shall see the theory is cleaner for not allowing it.

If F and F' are fields, then a **field homomorphism** is a map $f : F \to F'$ sayisfying (FH1) through (FH3).

PROPOSITION 16.2. *Every field homomorphism $f : F \to F'$ is injective.*

[3]Nothing can come of nothing. – King Lear

Proof. Assume to the contrary that there are $x \neq y \in F$ with $f(x) = f(y)$. Then $f(x-y) = f(x) - f(y) = 0$. Since $x \neq y$, $x - y \neq 0$, and thus we have a multiplicative inverse $\frac{1}{x-y}$. Then

$$1 = f(1) = f((x-y)\frac{1}{x-y}) = f(x-y)f(\frac{1}{x-y}) = 0 \cdot f(\frac{1}{x-y}) = 0,$$

contradicting axiom (P0). Note the use of (FH3)! □

A **field isomorphism** is a field homomorphism $f : F \to F'$ which has an inverse: i.e., there is a field homomorphism $f' : F' \to F$ such that $f' \circ f = 1_F$, $f \circ f' = 1_{F'}$.

Exercise 16.3: Show that for a field homomorphism $f : F \to F'$, TFAE:
(i) f is a field isomorphism.
(ii) f is bijective.[4]
(iii) f is surjective.

Exercise 16.4: Let $f : F \to F'$ be a field homomorphism. Show that f induces a field isomorphism $F \to f(F)$.

A **subfield** of a field K is a subset of K which is a field under the binary operations $+$ and $\cdot$ on K. For example, $\mathbb{Q}$ is a subfield of $\mathbb{R}$ and $\mathbb{R}$ is a subfield of $\mathbb{C}$.

Exercise 16.5: Let K be a field and F a subset of K. Show that F is a subfield iff all of the following hold:
(SF1) $0, 1 \in F$. (In particular F is nonempty.)
(SF2) For all $x, y \in F$, $x + y, x - y, xy \in F$.
(SF3) If $x \in F$ and $x \neq 0$, then $\frac{1}{x} \in F$.

Thus whenever we have a field homomorphism $f : F \to F'$, we get an isomorphic copy of F as a subfield of F', namely $f(F)$. Because of this it is safe to think of of F itself as a subfield of F', and this perspective is often useful.

Exercise 16.6: Let K be an ordered field, and let $F \subset K$ be a subfield. Show that restricting the total ordering $\leq$ on K to F endows F with the structure of an ordered field in such a way that the inclusion map $\iota : F \to K$ is a homomorphism of ordered fields.

We say that an element a of a field F is **a sum of squares** if there is $n \in \mathbb{Z}^+$ and $x_1, \ldots, x_n \in F$ such that $a = x_1^2 + \ldots + x_n^2$. Notice that in any field 0 and 1 are sums of squares.

Exercise 16.7: Let F be a field.
a) Let $x \in F$ be a sum of squares. Show that for any compatible ordering $\leq$ on F,

[4]Some philistines take this as the definition of a field isomorphism. Without getting into too much detail about it, this is really the wrong definition. It happens to be equivalent to the right definition for fields, but it has an analogue for other types of isomorphisms of mathematical structure which is not always true. E.g. this definition suggests that an isomorphism of topological spaces should be a continuous bijection, and this is truly weaker than the correct definition, namely a continuous map between topological spaces which admits a continuous inverse.

$x \geq 0$. b) Suppose that there is a nonzero element $x \in F$ such that both x and $-x$ are sums of squares. Show that there *does not* exist an ordering $\leq$ on F compatible with the field structure.
c) Deduce that the field $\mathbb{C}$ of complex numbers admits no compatible ordering.

Exercise 16.8: Let F be a field.
a) Suppose that F has the following property: for each nonzero $x \in F$, exactly one of x and $-x$ is a sum of squares. Show that F admits a unique compatible ordering: namely $x \geq 0$ iff x is a sum of squares.
b) Show that the standard ordering on $\mathbb{R}$ is the only compatible ordering.
c) Show that the standard ordering on $\mathbb{Q}$ is the only compatible ordering.

Exercise 16.9: A field F is **formally real** if -1 is not a sum of squares in F.
a) Suppose that F admits a compatible ordering. Show that F is formally real. (Hint: show the contrapositive.)
b) We suppose that $1 + 1 \neq 0 \in F$. (If not, then $-1 = 1$ is a sum of squares, so by part a) F admits no compatible ordering.) remarkable theorem of Artin-Schreier asserts for $x \in F$, the following are equivalent:
(i) For every compatible ordering $\leq$ on F, $x \leq 0$.
(ii) x is a sum of squares.
Deduce from this theorem that if F is formally real then it admits at least one ordering. (Hint: (i) holds vacuously if there are no compatible orderings!)

Proposition 16.3. *For every ordered field F, there is a unique ordered field homomorphism $\iota : \mathbb{Q} \to F$.*

Proof. Step 1: Since F can be ordered, the sum of positive elements is positive: in particular the sum of positive elements is not zero. Thus for any positive integer n, $1 + \ldots + 1$ (n times) is not zero. The map ι which sends 0 to 0 and each positive integer to $\iota(n) = 1 + \ldots + 1$ (n times) therefore gives an injective map $\mathbb{N} \to F$. We can extend this map to negative integers by mapping $-n$ to the additive inverse of $1 + \ldots + 1$ (n times). Further, for each positive integer n, since $\iota(n) \neq 0$, it has a multiplicative inverse $\frac{1}{\iota(n)}$, and we may map $\frac{m}{n} \mapsto \iota(m)\frac{1}{\iota(n)}$. It is now straightforward (but not completely trivial) to check that this map $\iota : \mathbb{Q} \to F$ is a field homomorphism: we leave this verification to the reader.
Step 2: In fact we had no choice in the matter: the map ι is the *unique* field homomorphism from $\mathbb{Q}$ to F. We leave this to the reader.
Step 3: We must check that if $x = \frac{m}{n} \in \mathbb{Q}$ is positive, then so is $\iota(a) = \iota(m)\frac{1}{\iota(n)}$. But this is easy: $\iota(m)$ is $1 + \ldots + 1$, so it is a sum of positive elements and is thus positive. Similarly $\iota(n)$ is positive, and thus so is its reciprocal $\frac{1}{\iota(n)}$. Finally, $\iota(\frac{m}{n}) = \iota(m)\frac{1}{\iota(n)}$ is a product of two positive elements, hence positive. $\square$

In light of this result, for any ordered field F, we may view $\mathbb{Q}$ as a subfield.

1.2. Some Topology of Ordered Fields.

Let $(K, \leq)$ be a linearly ordered set, and let $F \subset K$ be a subset. We say that F is **cofinal** in K if for all $x \in K$, there is $y \in F$ with $y > x$. In fact this is precisely the concept that in the case of subsets of $\mathbb{R}$ we call **unbounded above**. We use this terminology as an indication that this property can behave a bit differently in a non-Archimedean field.

By definition, an ordered field K is Archimedean if $\mathbb{Z}$ is cofinal in K. Equivalently, K is Archimedean iff $\mathbb{Q}$ is cofinal in K.

Let F be a subfield of the ordered field F. We say that F is **dense** in K if for all $x < y \in K$, there is $z \in F$ such that $x < z < y$.

LEMMA 16.4. *Let K be an ordered field, and let F be a subfield of K. If F is dense in K, then F is cofinal in K.*

PROOF. We show the contrapositive: suppose F is not cofinal in K: there is $x \in K$ such that for all $y \in F$, $y \leq x$. Then the interval $(x, x+1)$ contains no points of F, so F is not dense in K. $\square$

More generally, let $f : F \to F'$ be a homomorphism of ordered fields. We say that f is **cofinal** if the image $f(F)$ is a cofinal subfield of F'. We say that f is **dense** if the image $f(F)$ is a dense subfield of F'.

Exercise 16.10: Let K be a subfield of F.
a) Suppose that for every $\alpha \in F$, there is a sequence $\{x_n\}$ of elements of K such that $x_n \to \alpha$. Show that K is a dense subfield of F.
b) Does the converse of part a) hold? (Hint: no, but counterexamples are not so easy to come by.)

LEMMA 16.5. *For a homomorphism $f : F \to F'$ of ordered fields, TFAE:*
(i) f is cofinal.
(ii) For every positive $\epsilon' \in F'$, there is a positive $\delta \in F$ such that $f(\delta) < \epsilon'$.

Exercise 16.11: Prove Lemma 16.5. (Hint: take reciprocals!)

LEMMA 16.6. *For an ordered field F, the following are equivalent:*
(i) F is Archimedean.
(ii) $\mathbb{Q}$ is a dense subfield of F.

PROOF. (i) $\implies$ (ii): Suppose F is Archimedean and let $x < y \in F$. Let $n \in \mathbb{Z}^+$ be such that $\frac{1}{y-x} < n$; then $0 < \frac{1}{n} < y - x$, so

$$x < x + \frac{1}{n} < y.$$

(ii) $\implies$ (i): If $\mathbb{Q}$ is dense in F then by Lemma 16.4, $\mathbb{Q}$ is cofinal in F. $\square$

Let S be a subset of an ordered field F. For $s \in S$, we say that S is **discrete at s** if there is a positive $\epsilon \in F$ such that $(s - \epsilon, s + \epsilon) \cap S = \{s\}$. We say that S is **discrete** if it is discrete at s for all $s \in S$. An ordered field homomorphism $f : F \to F'$ is **discrete** if $f(F)$ is a discrete subset of F'.

PROPOSITION 16.7. *(Dorais's Dichotomy) Let $f : F \to F'$ be an ordered field homomorphism. Then* exactly one *of the following holds:*
(i) f is cofinal.
(ii) f is discrete.

PROOF. We will use the characterization of cofinality from Lemma 16.5.

Suppose f is cofinal. Then $0 \in f(F)$ and for every positive $\epsilon' \in F'$, there is a positive $\delta \in F$ such that $f(\delta) \in (-\epsilon, \epsilon)$. It follows that $f(F)$ is not discrete.

Suppose f is not cofinal: there is $\epsilon' \in F'$ such that $(-\epsilon', \epsilon') \cap f(F) = \{0\}$: $f(F)$ is discrete at 0. Now let $x \in F$. Suppose there is $y \in F$ such that $x - \epsilon < y < x + \epsilon$. Then $y - x \in (\epsilon, \epsilon)$. By what we've just seen we must have $y - x = 0$, i.e., $y = x$. Thus $f(F)$ is discrete at each of its elements, so it is discrete. $\square$

For any element x in an ordered field, we can define $|x|$ in the usual way, i.e., x if $x \geq 0$ and $-x$ if $x \leq 0$.

Exercise 16.12: Let $f : F \to F'$ be an ordered field homomorphism. Show that for all $x \in F$, $|f(x)| = f(|x|)$. (The absolute value on the left hand side is taking place in F'; the one on the right hand side is taking place in F.)

It is now possible to carry over our definitions of convergent sequencess, Cauchy sequences and continuous functions to the context of ordered fields and homomorphisms between them. We repeat the basic definitions. In many cases the proofs are exactly the same as in the case $F = \mathbb{R}$ treated in loving detail in this text; when this is so we leave the proofs to the reader. However, there are some things which *do not carry over* to the context of all ordered fields, and we treat these in some detail.

First recall that for any set S, a sequence $\{x_n\}$ in S is given by a function $f : \mathbb{Z}^+ \to S$; we write x_n in place of $f(n)$.

Let $\{x_n\}$ be a sequence in an ordered field F and let $x \in F$. We say that x_n **converges to** x and write $x_n \to x$ if: for all positive $\epsilon \in F$, there is $N \in \mathbb{Z}^+$ such that for all $n \geq N$, $|x_n - x| < \epsilon$. We say that a sequence $\{x_n\}$ is **convergent** if $x_n \to x$ for some $x \in F$.

We observe that this is verbatim the same as the definition in $\mathbb{R}$. At the same itme, the fact that our "small ϵ" is now an element of the ordered field F rather than (necessarily) a real number has certain surprising implications. The following exercises exhibits one.

Exercise 16.13: For an ordered field F, show that the following are equivalent:
(i) The sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ is convergent in F.
(ii) F is Archimedean.

LEMMA 16.8. *Let $\{x_n\}$ be a sequence in an ordered field F, and let $x, y \in F$. If $x_n \to x$ and $x_n \to y$, then $x = y$.*

PROOF. Left to the reader. $\square$

Thus if $x_n \to x$ we may call x **the limit** of the sequence: it is unique.

A sequence $\{x_n\}$ in an ordered field F is **Cauchy** if for all positive $\epsilon \in F$ there is $N \in \mathbb{Z}^+$ such that for all $m, n \geq N$, $|x_m - x_n| < \epsilon$.

PROPOSITION 16.9. *A Cauchy sequence in an ordered field which admits a convergent subsequence is itself convergent.*

PROOF. Let F be an ordered field, let $\{x_n\}$ be a Cauchy sequence in F, and let $\{x_{n_k}\}$ be a subsequence converging to $x \in F$. Let $\epsilon > 0$. Since $\{x_n\}$ is Cauchy, there is $N \in \mathbb{Z}^+$ such that for all $m, n \geq N$, $|x_m - x_n| < \frac{\epsilon}{2}$. Also there is $K \geq N$ such that for all $k \geq K$, $|x_{n_k} - x| < \frac{\epsilon}{2}$. If $n \geq K$, then since $n_K \geq K$ we have

$$|x_n - x| \leq |x_n - x_{n_K}| + |x_{n_K} - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

so $x_n \to x$. □

Lemma 16.10. *A convergent sequence in an ordered field is Cauchy.*

Proof. Left to the reader. □

On the other hand, in an arbitrary ordered field a Cauchy sequence need not be convergent. For instance the "Babylonian sequence" $\{x_n\}$ of §VII.2.2 is a sequence of rational numbers converging to $\sqrt{2} \in \mathbb{R}$. Since $\sqrt{2} \notin \mathbb{Q}$ and limits in $\mathbb{R}$ are unique, viewed as a sequence *in the ordered field* $\mathbb{Q}$, the sequence $\{x_n\}$ is not convergent. It is however Cauchy. More generally we have the following result.

Proposition 16.11. *Let $f : F \to F'$ be an ordered field homomorphism, and let $\{x_n\}$ be a sequence in F. If the sequence $\{f(x_n)\}$ is Cauchy in F', then $\{x_n\}$ is Cauchy in F.*

Proof. We show the contrapositive: suppose $\{x_n\}$ is *not* Cauchy in F. Then there is some positive $\epsilon \in F$ such that for all $N \in \mathbb{Z}^+$ there are $m, n \geq N$ such that $|x_m - x_n| \geq \epsilon$. Then

$$f(\epsilon) \leq f(|x_m - x_n|) = |f(x_m - x_n)| = |f(x_m) - f(x_n)|.$$

Since $\epsilon' = f(\epsilon)$ is a positive element of F', this shows that $\{f(x_n)\}$ is not Cauchy. □

Thus we get a method for producing nonconvergent Cauchy sequences in an ordered field F: find a sequence $\{x_n\}$ in F and a homomorphism $f : F \to F'$ such that $f(x_n)$ converges to an element of $F' \setminus f(F)$. In fact every nonconvergent Cauchy sequence arises this way, as we will see later on.

Let F and F' be ordered fields, and let $f : F \to F'$ be a map between them. We say that f is **continuous at** $c \in F$ if for all positive $\epsilon' \in F'$, there is a positive $\delta \in F$ such that for all $x \in F$ with $|x - c| < \delta$, $|f(x) - f(c)| < \epsilon'$. We say that $f : F \to F'$ is **continuous** if it is continuous at every $c \in F$. We say that $f : F \to F'$ is **uniformly continuous** if for all positive $\epsilon' \in F'$ there is a positive $\delta \in F$ such that for all $x, x' \in F$, if $|x - x'| < \delta$ then $|f(x) - f(x')| < \epsilon'$.

Lemma 16.12. *Let $f : F \to F'$ be a function.*
a) If f is continuous and x_n is a sequence in F which converges to $x \in F$, then $f(x_n) \to f(x)$ in F'.
b) If f is uniformly continuous and $\{x_n\}$ is a Cauchy sequence in F, then $\{f(x_n)\}$ is a Cauchy sequence in F'.

Proof. Left to the reader. □

Warning: When $F = F' = \mathbb{R}$, the converse of Lemma 16.6a) holds (Theorem 10.5): a map which preserves limits of convergent sequences is necessarily continuous. This *does not hold* in general; but unfortunately counterexamples lie beyond the scope of this text.

THEOREM 16.13. *(F. Dorais) Let $f : F \to F'$ be a homomorphism of ordered fields. The following are equivalent:*
(i) f is uniformly continuous.
(ii) f is continuous.
(iii) f is continuous at 0.
(iv) f is cofinal.

PROOF. (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) is immediate.
(iii) $\Longleftrightarrow$ (iv): By Lemma 16.5, $f(F)$ is cofinal in F' iff for every positive ϵ' in F', there is a positive δ in F such that $f(\delta) < \epsilon'$. Suppose this holds, and let ϵ' be positive in F', and choose a positive $\delta \in F$ as above. Then for $x \in F$, if $|x| = |x - 0| < \delta$, then

$$|f(x)| = f(|x|) < f(\delta) = |f(\delta) - f(0)| < \epsilon'.$$

This shows f is continuous at 0. Conversely, if f is continuous at 0, then for each positive $\epsilon' \in F$ there is a positive $\delta \in F$ such that if $|x| < \delta$, $|f(x)| < \epsilon'$. Thus $\frac{\delta}{2}$ is a positive element of F such that $\frac{f(\delta)}{2} < \epsilon'$, so $f(F')$ is cofinal in F by Lemma 16.5.
(iii) $\Longrightarrow$ (i): Let ϵ' be a positive element of F'; since f is continuous at 0 there is a positive $\delta \in F$ such that for all $x \in F$ with $|x| < \delta$, $|f(x)| < \epsilon'$. So, if $x, y \in F$ are such that $|x - x'| < \delta$, then

$$|f(x) - f(y)| = |f(x - y)| < \epsilon'.$$

□

1.3. A Non-Archimedean Ordered Field.

Let $K = \mathbb{R}((t))$ be the field of **formal Laurent series** with $\mathbb{R}$-coefficients: an element of K is a formal sum $\sum_{n \in \mathbb{Z}} a_n t^n$ where there exists $N \in \mathbb{Z}$ such that $a_n = 0$ for all $n < N$. We add such series term by term and multiply them in the same way that we multiply polynomials.

Exercise 16.14: Show that $K = \mathbb{R}((t))$ is a field.

We need to equip K with an ordering; equivalently, we need to specify a set of positive elements. For every nonzero element $x \in K$, we take $v(x)$ to be the smallest $n \in \mathbb{Z}$ such that $a_n \neq 0$. Then we say that x is positive if the coefficient $a_{v(x)}$ of the smallest nonzero term is a positive real number. It is straightforward to see that the sum and product of positive elements is positive and that for each nonzero $x \in K$, exactly one of x and $-x$ is positive, so this gives an ordering on K in the usual way: we decree that $x < y$ iff $y - x$ is positive.

We observe that this ordering is non-Archimedean. Indeed, the element $\frac{1}{t}$ is positive – its one nonzero coefficient is 1, which is a positive real number – and infinitely large: for any $n \in \mathbb{Z}$, $\frac{1}{t} - n$ is still positive – recall that we look to the smallest degree coefficient to check positivity – so $\frac{1}{t} > n$ for all n.

Next we observe that the set $\{\frac{1}{t^n}\}$ is unbounded in K. Taking reciprocals, it follows that the sequence $\{t^n\}$ converges to 0 in K: explicitly, given any $\epsilon > 0$ – here ϵ is not necessarily a real number but any positive element of F! – for all sufficiently large n we have that $\frac{1}{t^n} > \frac{1}{\epsilon}$, so $|t^n| = t^n < \epsilon$. We will use this fact to give a simple explicit description of all convergent sequences in F. First, realize that a sequence in K consists of, for each $m \in \mathbb{Z}^+$ a formal Laurent series $x_m = \sum_{n \in \mathbb{Z}} a_{m,n} t^n$, so in fact for each $n \in \mathbb{Z}$ we have a real sequence $\{a_{m,n}\}_{m=1}^{\infty}$. Now consider the following conditions on a sequence $\{x_m\}$ in K:

(i) There is an integer N such that for all $m \in \mathbb{Z}^+$ and $n < N$, $a_{m,n} = 0$, and
(ii) For each $n \in \mathbb{Z}$ the sequence $a_{m,n}$ is eventually constant: i.e., for all sufficiently large m, $a_{m,n} = C_n \in \mathbb{R}$. (Because of (i) we must have $C_n = 0$ for all $n < N$.)

Then condition (i) is equivalent to boundedness of the sequence.

I claim that if the sequence converges – say $x_m \to x = \sum_{n=N}^{\infty} a_n t^n \in F$ – then (i) and (ii) both hold. Indeed convergent sequences are bounded, so (i) holds. Then for all $n \geq N$, $a_{m,n}$ is eventually constant in m iff $a_{m,n} - a_n$ is eventually constant in m, so we may consider $x_m - x$ instead of x_m and thus we may assume that $x_m \to 0$ and try to show that for each fixed n, $a_{m,n}$ is eventually equal to 0. As above, this holds iff for all $k \geq 0$, there exists M_k such that for all $m \geq M_k$, $|x_m| \leq t^k$. This latter condition holds iff the coefficient $a_{m,n}$ of t^n in x_n is zero for all $N < k$. Thus, for all $m \geq M_k$, $a_{m,-N} = a_{m,-N+1} = \ldots = a_{m,k-1} = 0$, which is what we wanted to show.

Conversely, suppose (i) and (ii) hold. Then since for all $n \geq N$ the sequence $a_{m,n}$ is eventually constant, we may define a_n to be this eventual value, and an argument very similar to the above shows that $x_m \to x = \sum_{n \geq N} a_n t^n$.

Next I claim that if a sequence $\{x_n\}$ is Cauchy, then it satisfies (i) and (ii) above, hence is convergent. Again (i) is immediate because every Cauchy sequence is bounded. The Cauchy condition here says: for all $k \geq 0$, there exists M_k such that for all $m, m' \geq M_k$ we have $|x_m - x'_m| \leq t^k$, or equivalently, for all $n < k$, $a_{m,n} - a_{m',n} = 0$. In other words this shows that for each fixed $n < k$ and all $m \geq M_k$, the sequence $a_{m,n}$ is constant, so in particular for all $n \geq N$ the sequence $a_{m,n}$ is eventually constant in m, so the sequence x_m converges.

Exercise 16.15: Show that the subfield $\mathbb{R}((t^2))$ of $\mathbb{R}((t))$ is cofinal but not dense.

2. The Sequential Completion

2.1. Sequentially Complete Fields.

An ordered field F is **sequentially complete** if every Cauchy sequence in F converges to an element of F. We have seen that a Dedekind complete ordered field is sequentially complete. Here we wish to examine the converse.

Theorem 16.14. *For an* Archimedean *ordered field F, TFAE:*
(i) F is ***Dedekind complete****.*
(ii) F is ***sequentially complete****: every Cauchy sequence converges.*

Proof. The implication (i) $\implies$ (ii) is the content of Theorem 10.32, since the Bolzano-Weierstrass Theorem holds in any ordered field satisfying (LUB).
(ii) $\implies$ (i): Let $S \subset F$ be nonempty and bounded above, and write $\mathcal{U}(S)$ for the set of least upper bounds of S. Our strategy will be to construct a decreasing Cauchy sequence in $\mathcal{U}(S)$ and show that its limit is $\sup S$.

Let $a \in S$ and $b \in \mathcal{U}(S)$. Using the Archimedean property, we choose a negative integer $m < a$ and a positive integer $M > b$, so

$$m < a \leq b \leq M.$$

For each $n \in \mathbb{Z}^+$, we define

$$S_n = \{k \in \mathbb{Z} \mid \frac{k}{2^n} \in \mathcal{U}(A) \text{ and } k \leq 2^n M\}.$$

Every element of S_n lies in the interval $[2^n m, 2^n M]$ and $2^n M \in S_n$, so each S_n is finite and nonempty. Put $k_n = \min S_n$ and $a_n = \frac{k_n}{2^n}$, so $\frac{2k_n}{2^{n+1}} = \frac{k_n}{2^n} \in \mathcal{U}(S)$ wihle $\frac{2k_n-2}{2^{n+1}} = \frac{k_n-1}{2^n} \notin \mathcal{U}(S)$. It follows that we have either $k_{n+1} = 2k_n$ or $k_{n+1} = 2k_n - 1$ and thus either $a_{n+1} = a_n$ or $a_{n+1} = a_n - \frac{1}{2^{n+1}}$. In particular $\{a_n\}$ is decreasing. For all $1 \leq m < n$ we have

$$0 \leq a_m - a_n = (a_m - a_{m+1}) + (a_{m+1} - a_{m+2}) + \ldots + (a_{n-1} - a_n)$$
$$\leq 2^{-(m+1)} + \ldots + 2^{-n} = 2^{-m}.$$

Thus $\{a_n\}$ is Cauchy, hence by our assumption on F $a_n \to L \in F$.

We CLAIM $L = \sup(S)$. Seeking a contradiction we suppose that $L \notin \mathcal{U}(S)$. Then there exists $x \in S$ such that $L < x$, and thus there exists $n \in \mathbb{Z}^+$ such that

$$a_n - L = |a_n - L| < x - L.$$

It follows that $a_n < x$, contradicting $a_n \in \mathcal{U}(S)$. So $L \in \mathcal{U}(S)$. Finally, if there exists $L' \in \mathcal{U}(S)$ with $L' < L$, then (using the Archimedean property) choose $n \in \mathbb{Z}^+$ with $\frac{1}{2^n} < L - L'$, and then

$$a_n - \frac{1}{2^n} \geq L - \frac{1}{2^n} > L',$$

so $a_n - \frac{1}{2^n} = \frac{k_n-1}{2^n} \in \mathcal{U}(S)$, contradicting the minimality of k_n. $\square$

The proof of (ii) $\implies$ (i) in Theorem 16.14 above is taken from [**HS**] by way of [**Ha11**]. It is rather unexpectedly complicated, but I do not know a simpler proof at this level. However, if one is willing to introduce the notion of convergent and Cauchy **nets**, then one can show first that in an Archimedean ordered field, the convergence of all Cauchy sequences implies the convergence of all Cauchy nets, and second use the hypothesis that all Cauchy nets converge to give a proof which is (in my opinion of course) more conceptually transparent. This is the approach taken in my (more advanced) notes on Field Theory [**FT**].

In fact there are (many!) non-Archimedean sequentially complete ordered fields. We will attempt to describe two very different examples of such fields here. We hasten to add that this is material that the majority of working research mathematicians are happily unfamiliar with, and which is thus extremely rarely covered in undergraduate courses. Only the exceptionally curious need the next section.

2.2. Sequential Completion I: Statement and Applications.

We will now establish one of our main results: for every ordered field F, there is a sequentially complete ordered field $\mathcal{R}$ and a homomorphism $f : F \to \mathcal{R}$.

In fact we can, and will prove, even more than this. The point is that there will be many (nonisomorphic) sequentially complete fields into which any given ordered field embeds. For example, when we construct the real numbers we will have an embedding $\mathbb{Q} \hookrightarrow \mathbb{R}$. But we also have an embedding $\mathbb{R} \hookrightarrow \mathbb{R}((t))$, so taking the composite gives an embedding $\mathbb{Q} \to \mathbb{R}((t))$. (There is no way that $\mathbb{R}$ and $\mathbb{R}((t))$ are isomorphic, since the former is Archimedean and the latter is not.)

We would like a general definition which allows us to prefer the embedding $\mathbb{Q} \hookrightarrow \mathbb{R}$ to the embedding $\mathbb{Q} \hookrightarrow \mathbb{R}((t))$. The key observation is that, since $\mathbb{R}$ is Archimedean, the embedding of $\mathbb{Q}$ into $\mathbb{R}$ is dense, whereas since $\mathbb{R}((t))$ is not Archimedean, the embedding of $\mathbb{Q}$ into $\mathbb{R}$ is not dense. This leads to the following important definition.

A **sequential completion** of an ordered field F is a dense embedding $F \hookrightarrow \mathcal{R}$ into a sequentially complete ordered field.

Lemma 16.15. *For an ordered field F, the following are equivalent.*
(i) F is Dedekind complete.
(ii) The inclusion $\iota : \mathbb{Q} \hookrightarrow F$ makes F into a sequential completion of $\mathbb{Q}$.

Proof. (i) $\implies$ (ii): By Theorem 16.14, F is sequentially complete and Archimedean. By Lemma 16.6, $\mathbb{Q} = \iota(\mathbb{Q})$ is a dense subfield of F, and it follows that F is a sequential completion of $\mathbb{Q}$. $\square$

We will prove that every ordered field admits a sequential completion. And again, we will in fact prove a bit more.

Theorem 16.16. *Let F be an ordered field.*
a) F admits a sequential completion $\iota : F \to \mathcal{R}$.
b) If L is any sequentially complete ordered field and $f : F \to L$ is a cofinal ordered field homomorphism, then there is a unique ordered field homomorphism $g : \mathcal{R} \to L$ such that $f = g \circ \iota$.

Corollary 16.17. *Two sequential completions of the same ordered field are isomorphic.*

Proof. Let $\iota_1 : F \to \mathcal{R}_1$ and $\iota_2 : F \to \mathcal{R}_2$ be two sequential completions. Applying Theorem 16.16 with $\mathcal{R} = \mathcal{R}_1$ and $f = \iota_2 : F \to \mathcal{R}_2$, we get a unique homomorphism $g : \mathcal{R}_1 \to \mathcal{R}_2$ such that $\iota_2 = g \circ \iota_1$. Interchanging the roles of $\mathcal{R}_1$ and $\mathcal{R}_2$ we also get a unique homomorphism $g' : \mathcal{R}_2 \to \mathcal{R}_1$ such that $\iota_1 = g' \circ \iota_2$.

Now consider $g' \circ g : \mathcal{R}_1 \to \mathcal{R}_\infty$. We have

$$(g' \circ g) \circ \iota_1 = g' \circ (g \circ \iota_1) = g' \circ \iota_2 = \iota_1.$$

Applying Theorem 16.16 with $L = \mathcal{R} = \mathcal{R}_1$ we get that there is a *unique* homomorphism $G : \mathcal{R}_\infty \to \mathcal{R}_1$ such that $G \circ \iota_1 = \iota_1$, but clearly the identity map $1_{\mathcal{R}_1}$ has this property. Thus we must have $g' \circ g = 1_{\mathcal{R}_1}$. Similarly considering $g \circ g' : \mathcal{R}_2 \to \mathcal{R}_2$, then in view of

$$(g \circ g') \circ \iota_2 = g \circ (g' \circ \iota_2) = g \circ \iota_1 = \iota_2,$$

we deduce that $g \circ g' = 1_{\mathcal{R}_2}$. In other words, g and g' are mutually inverse isomorphisms...so $\mathcal{R}_1$ and $\mathcal{R}_2$ are isomorphic. $\square$

Applying Theorem 16.16 to the ordered field $\mathbb{Q}$, we get a sequential completion $\mathcal{R}$ of $\mathbb{Q}$. Since $\mathcal{R}$ is Archimedean and sequentially complete, by Theorem 16.14, $\mathcal{R}$ is Dedekind complete. Conversely, by Lemma 16.15any Dedekind complete ordered field $\mathcal{R}'$ is isomorphic to $\mathcal{R}$. Thus the existence and uniqueness statements of Theorem 16.16 imply the existence and uniqueness up to isomorphism of a Dedekind complete ordered field.

The uniqueness statement can be strengthened: let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two Dedekind complete ordered fields. We claim that not only are they isomorphic, but that the isomorphism between them is unique. Indeed, for $i = 1, 2$ let $\iota_i : \mathbb{Q} \to \mathcal{R}_i$ be the inclusion maps. We saw above that there is a unique map $g : \mathcal{R}_1 \to \mathcal{R}_2$ such that $g \circ \iota_1 = \iota_2$ and this g is an isomorphism. But any isomorphism $h : \mathcal{R}_1 \to \mathcal{R}_2$ will satisfy $h \circ \iota_1 = \iota_2$, since in fact there is exactly one embedding from $\mathbb{Q}$ into any ordered field. Thus whereas in general there is an isomorphim g between two sequential completions of a given ordered field F which is unique such that blah blah blah (more precisely, such that $g \circ \iota_1 = \iota_2$), in this case the "blah blah blah" is vacuous and the isomorphism is unique full stop.

In abstract mathematics, uniqueness up to a *unique* isomorphism is as close to identical as we can reasonably ask for two structures to be. (Even the "horizontal copy of $\mathbb{R}$" and the "vertical copy of $\mathbb{R}$" are different sets, but the obvious isomorphism between them is the only isomorphism, so no trouble can arise by identifying the two.) We denote this unique field by $\mathbb{R}$ and call it the real numbers...of course.

2.3. Sequential Completion II: The Proof.

Now we are properly motivated to roll up our sleeves and endure the rather lengthy, technical proof of Theorem 16.16. The essential idea (which is indeed due to A.L. Cauchy) is to build the sequential completion directly from the set $\mathcal{C}$ of all Cauchy sequences in F.

We can observe that $\mathcal{C}$ itself has some structure reminiscent of an ordered field but that things do not quite work out: it is somehow *too large* to itself be an ordered field. Namely, it makes perfectly good sense to add, subtract and multiply Cauchy sequences in F. For that matter, it makes perfectly good sense to add, subtract and multiply arbitrary sequences in F: we simply put

$$\{x_n\} + \{y_n\} = \{x_n + y_n\},$$
$$\{x_n\} - \{y_n\} = \{x_n - y_n\},$$
$$\{x_n\} \cdot \{y_n\} = \{x_n \cdot y_n\}.$$

It remains to check that these operations take Cauchy sequences to Cauchy sequences. At the very beginning of our study of sequences we showed this for *convergent* sequences (in $\mathbb{R}$, but the proofs certainly did not use any form of the completeness axiom). It is no more difficult to establish the analogue for Cauchy sequences in F.

LEMMA 16.18. *Let F be any ordered field, and let $a_\bullet$, $b_\bullet$ be Cauchy sequences. Then $a_\bullet + b_\bullet$ and $a_\bullet \cdot b_\bullet$ are both Cauchy.*

PROOF. Since $a_\bullet$ and $b_\bullet$ are both Cauchy, for $\epsilon > 0$ there is $N \in \mathbb{Z}^+$ such that for $m, n \geq N$, $|a_m - a_n| < \epsilon$ and and $|b_m - b_n| < \epsilon$. Then

$$|(a_m + b_m) - (a_n + b_n)| \leq |a_m - a_n| + |b_m - b_n| < 2\epsilon.$$

Further, since the sequences are Cauchy, they are bounded: there are $M_a, M_b \in F$ such that $|a_n| \leq M_a$ and $|b_n| \leq M_b$ for all $n \in \mathbb{Z}^+$. Then for $m, n \geq N$,

$$|a_m b_m - a_n b_n| \leq |a_m - a_n||b_m| + |a_n||b_m - b_n| \leq (M_a + M_b)\epsilon.$$

$\square$

So does this addition and multiplication endow $\mathcal{C}$ with the structure of a field? There is an additive identity, namely the sequence with $x_n = 0$ for all n. There is also a multiplicative identity, namely the sequence with $x_n = 1$ for all n. It all works well until we get to multiplicative inverses.

Exercise 16.16: Let $\{x_n\}$ be a sequence in the ordered field F.
a) Show that there is a sequence $\{y_n\}$ with $\{x_n\} \cdot \{y_n\} = \{1\}$ if and only if for all $n \in \mathbb{Z}^+$, $x_n \neq 0$.
b) Show that if $\{x_n\}$ is Cauchy and $x_n \neq 0$ for all n, then its inverse $\{\frac{1}{x_n}\}$ is again a Cauchy sequence.

Thus there are plenty of Cauchy sequences other than the constantly zero sequence which do not have multiplicative inverses: e.g. $(0, 1, 1, 1, \ldots)$, or indeed any constant sequence which takes the value 0 at least once. Thus $\mathcal{C}$ has many good algebraic properties, but it is not the case that every nonzero element has a multiplicative inverse, so it is not a field.[5]

We also have some order structure on $\mathcal{C}$. For instance, it is tempting to define $\{x_n\} > \{y_n\}$ if $x_n > y_n$ for all n. This turns out not to be a good definition in the sense that it does not lead to a trichotomy: there will be unequal Cauchy sequences $\{x_n\}$ and $\{y_n\}$ for which neither is less than the other, e.g.

$$\{x_n\} = \{0, 1, 1, \ldots\}, \ \{y_n\} = \{1, 0, 0, \ldots\}.$$

As in the definition of convergence, it is more fruitful to pay attention to what a Cauchy sequence is doing *eventually*. Exploiting this idea we can get a sort of trichotomy result.

Lemma 16.19. *(Cauchy Trichotomy) For a Cauchy sequence $\{x_n\}$ in an ordered field F, exactly one of the following holds:*
(i) There is a positive element $\epsilon \in F$ and $N \in \mathbb{Z}^+$ such that $x_n \geq \epsilon$ for all $n \geq N$.
(ii) There is a positive element $\epsilon \in F$ and $N \in \mathbb{Z}^+$ such that $x_n \leq -\epsilon$ for all $n \geq N$.
(iii) x_n converges to 0.

Proof. It is easy to see that the conditions are mutually exclusive. Let us suppose that (iii) does not hold: thus there is $\epsilon > 0$ and a subsequence $\{x_{n_k}\}$ such that $|x_{n_k}| \geq \epsilon$ for all $k \in \mathbb{Z}^+$. By passing to a further subsequence we may assume either that $x_{n_k} \geq \epsilon$ for all k or $x_{n_k} \leq -\epsilon$ for all k. Let us suppose that the former holds and show that this implies (i): if so, replacing x by $-x$ shows that the latter alternative implies (ii). Since $\{x_n\}$ is Cauchy, there is $N \in \mathbb{Z}^+$ such that $|x_m - x_n| \leq \frac{\epsilon}{2}$ for all $m, n \geq N$. Putting these two conditions together we get $x_n \geq \frac{\epsilon}{2}$ for all $n \geq N$. $\square$

[5]for those who know some abstract algebra: what we've shown is that $(\mathcal{C}, +, \cdot)$ is a **commutative ring**. There is a very general algebraic method which, when given a commutative ring, will yield a collection of fields defined in terms of that ring. The present construction is indeed an instance of this.

Unfortunately this is not quite the kind of trichotomy which defines a total order relation: we have some elements that we regard as positive – case (i) above, some elements that we regard as negative – case (ii) above – but for an order relation only the zero element should be neither positive nor negative, whereas case (iii) above includes the much larger collection of elements *converging* to zero.

Lemma 16.19 suggests that if we could somehow "squash down" the subset of Cauchy sequences which converge to 0 to a single point, then we would actually get a total order relation. This business of "squashing subsets to a point" is formalized in mathematics (more so in algebra and topology than the part of mathematics we've been studying for most of this text!) by an **equivalence relation**. Rather than providing a logically complete but pedagogically useless whirlwind tour of equivalence relations, we will simply assume that the reader is familiar with them.[6] Namely, we regard any two Cauchy sequences which converge to 0 as equivalent. We are left with the question of when to regard two Cauchy sequences which do not converge to zero as equivalent. We could simply "not squash them", i.e., declare two such sequences to be equivalent exactly when they are equal. But a little exploration shows that this won't work: we'll get a total order relation but it won't interact well with the algebraic structure. For instance, consider the Cauchy sequences

$$\{x_n\} = (0, 0, 1, 1, 1, \ldots), \{y_n\} = (1, 1, 1, 1, \ldots).$$

The difference $\{x_n\} - \{y_n\}$ converges to 0 so is getting identified with 0. Thus we should identify $\{x_n\}$ and $\{y_n\}$ as well. This leads to the following key definitions.

Let $\mathcal{Z}$ be the set of all sequences in F which converge to 0; convergent sequences are Cauchy, so certainly $\mathcal{Z} \subset \mathcal{C}$. For two Cauchy sequences $a_\bullet, b_\bullet \in \mathcal{C}$, we put

$$a_\bullet \sim b_\bullet \iff a_\bullet - b_\bullet \in \mathcal{Z}.$$

In words, two Cauchy sequences are equivalent iff their difference converges to zero.

Exercise 16.17 (if you know abstract algebra): Show that $\mathcal{Z}$ is a maximal ideal in the commutative ring $\mathcal{C}$. Why is this an exciting sign that we're on the right track?

Exercise 16.18: Let $\{x_n\}$ be a Cauchy sequence in F, and let $\{x_{n_k}\}$ be any subsequence. Show that $\{x_n\} \sim \{x_{n_k}\}$.

Now we define $\mathcal{R}$ as $\mathcal{C}/\sim$, that is, the set of equivalence classes of Cauchy sequences. This will be the underlying set of our sequential completion. It remains to endow it with all the rest of the structure. The idea here is that when we pass to a quotient by an equivalence relation we can try to simply carry over the structure we already had, but at every step we must check that the operations are **well-defined**, meaning they are independent of the chosen equivalence class. At no point are these verifications difficult, but we admit they can be somewhat tedious.

Let us check the addition and multiplication induced well-defined operations on

[6] At UGA they are covered in the "transitional" Math 3200 course. The reader who has made it through most of this text will have no problem learning this concept.

the set $\mathcal{R}$ of equivalence classes. This means: if we have four Cauchy sequences $a_\bullet, b_\bullet, c_\bullet, d_\bullet$ and $a_\bullet \sim c_\bullet$, $b_\bullet \sim d_\bullet$, then

$$a_\bullet + b_\bullet \sim c_\bullet + d_\bullet, \;\; a_\bullet b_\bullet \sim c_\bullet d_\bullet.$$

All right: since $a_\bullet \sim c_\bullet$ and $b_\bullet \sim d_\bullet$, $a_\bullet - c_\bullet \to 0$ and $b_\bullet - d_\bullet \to 0$, so

$$(a_\bullet + b_\bullet - (c_\bullet + d_\bullet) = (a_\bullet - c_\bullet) + (b_\bullet - d_\bullet) \to 0 + 0 = 0,$$

so $a_\bullet + b_\bullet \sim c_\bullet + d_\bullet$. Similarly,

$$a_\bullet b_\bullet - c_\bullet d_\bullet = (a_\bullet - c_\bullet)b_\bullet + (b_\bullet - d_\bullet)c_\bullet,$$

and this converges to 0 because $a_\bullet - c_\bullet, b_\bullet - d_\bullet \to 0$ and $b_\bullet, c_\bullet$ are bounded. Thus we have equipped our set $\mathcal{R}$ with two binary operations $+$ and $\cdot$.

Proposition 16.20. *$(\mathcal{R}, +, \cdot)$ is a field.*

Proof. Most of these axioms are completely straightforward (but, yes, somewhat tedious) to verify and are left to the reader as exercises. Let us single out:
(P3) The additive identity is $[0_\bullet]$, the class of the constant sequence 0.
(P7) The multiplicative identity is $[1_\bullet]$, the class of the constant sequence 1.
(P8) Suppose that $x \in \mathcal{R} \setminus \{[0_\bullet]\}$, and let $x_\bullet$ be any Cauchy sequence representing x. Then we must have $x_n \neq 0$ for all sufficiently large n: indeed, otherwise we would have $0_\bullet = (0, 0, 0, \ldots)$ as a subsequence, and if a subsequence of a Cauchy sequence converges to 0, then the Cauchy sequence itself converges to 0, contradiction. Suppose $x_n \neq 0$ for all $n > N$. Then define $y_\bullet$ by $y_n = 0$ for all $1 \leq n \leq N$ (or whatever you want: it doesn't matter) and $y_n = \frac{1}{x_n}$ for $n > N$. Then $x_n y_n = 1$ for all $n > N$, so $x_\bullet y_\bullet$ differs from $1_\bullet$ by a sequence which is convergent to zero: $[x_\bullet][y_\bullet = [1_\bullet] = 1$, so $y = [y_\bullet]$ is the multiplicative inverse of $x = [x_\bullet]$. $\square$

We now equip $\mathcal{R}$ with an ordering. For $a_\bullet, b_\bullet \in \mathcal{C}$, we put $[a_\bullet] > [b_\bullet]$ if there is a positive element ϵ in F such that $a_n - b_n \geq \epsilon$ for all sufficiently large n. We claim that this is well-defined independent of the representatives $a_\bullet$ and $b_\bullet$ chosen. Indeed, if $x_\bullet$ and $y_\bullet$ converge to zero, then for sufficiently large n we have $|x_n - y_n| < \frac{\epsilon}{2}$ and then

$$a_n + x_n) - (b_n + y_n) = (a_n - b_n) + (x_n - y_n) \geq \frac{\epsilon}{2}.$$

Theorem 16.21. *The strict ordering $<$ on $\mathcal{R}$ endows it with the structure of an ordered field.*

Proof. Step 1: The trichotomy property follows from Lemma 16.19. The transitive property is easy: if $[a_\bullet] < [b_\bullet]$ and $[b_\bullet] < [c_\bullet]$, then there is $N \in \mathbb{Z}^+$ and $\epsilon_1, \epsilon_2 > 0$ such that for all $n \geq N$, $b_n - a_n \geq \epsilon_1$ and $c_n - b_n \geq \epsilon_2$, hence $c_n - a_n = (c_n - b_n) + (b_n - a_n) \geq \epsilon_1 + \epsilon_2$. Thus $<$ is a strict ordering on $\mathcal{R}$.
Step 2: If $[a_\bullet], [b_\bullet] > 0$, then as above there is $N \in \mathbb{Z}^+$ and $\epsilon_1, \epsilon_2 > 0$ such that for all $n \geq N$, $a_n \geq \epsilon_1$ and $b_n \geq \epsilon_2$. Then for all $n \geq N$, $a_n + b_n \geq \epsilon_1 + \epsilon_2$ and $a_n b_n \geq \epsilon_1 \epsilon_2$. $\square$

Proposition 16.22. *For an ordered field F, let $\iota : F \to \mathcal{R}$ by sending $x \in F$ to the class of the constant sequence $(x, x, \ldots)$. Then ι is a dense homomorphism of ordered fields.*

Proof. Step 1: It is immediate that ι is a field homomorphism. Further, if $x > 0$, then $(x, x, \ldots) > 0$: indeed, we can take $\epsilon = x$. So ι is a homomorphism of ordered fields.

Step 2: Let $x = [a_\bullet]$ in $\mathcal{R}$. We claim that $\iota(a_n) \to x$. Indeed, if $\epsilon = [\epsilon_\bullet]$ is a positive element of $\mathcal{R}$, then there is $e > 0$ in F such that $\epsilon_n \geq \epsilon$ for all sufficiently large n. Since $\{a_n\}$ is Cauchy in F, there is $N \in \mathbb{Z}^+$ such that for all $m, n \geq N$,

$$|a_n - a_m| < \frac{e}{2}.$$

Then for sufficiently large m,

$$\epsilon_m - |a_n - a_m| \geq e - |a_n - a_m| \geq \frac{e}{2},$$

which shows that for all $n \geq N$, $|\iota(a_n) - x| < \epsilon$. Step 3: It remains to show that ι is dense, so let $a = [a_\bullet] < b = [b_\bullet] \in \mathcal{R}$. Let $m = \frac{a+b}{2}$ and let $\epsilon = \frac{b-a}{2}$; thus the interval of width ϵ centered at m has as its endpoints a and b. By Step 2, there is a sequence $\{x_n\} \in F$ such that $\iota(x_n) \to m$. By definition then, for all sufficiently large n we have$|\iota(x_n) - m| < \epsilon$, and thus $\iota(x_n) \in (a, b)$. □

Finally, we prove Theorem 16.16 in the following explicit form.

THEOREM 16.23. *For any ordered field F, the homomorphism $\iota : F \to \mathcal{R} = \mathcal{C}/\mathcal{Z}$ is a sequential completion. Moreover, if L is a sequentially complete field and $f : F \to L$ is a cofinal ordered field homomorphism, then there is a unique ordered field homomorphism $g : \mathcal{R} \to L$ such that $f = g \circ \iota$.*

PROOF. We already know that $\iota : F \to \mathcal{R}$ is a dense ordered field homomorphism. It remains to check first that $\mathcal{R}$ is sequentially complete and second that ι satisfies the "universal property" of Theorem 16.16b).
Step 1: Let $\{x_n\}$ be a Cauchy sequence in $\mathcal{R}$; we must show that it is convergent. By Proposition 16.9 it suffices to check this after passing to a subsequence. If $\{x_n\}$ has only finitely many distinct points, we have a constant subsequence, which is certainly convergent. Otherwise $\{x_n\}$ has infinitely many distinct points, so that after passage to a subsequence we may assume they are all distinct. Since by Proposition 16.22 ι is a dense homomorphism, for all $n \in \mathbb{Z}^+$ there is $y_n \in F$ such that y_n lies strictly between x_n and x_{n+1}. It follows that for any $n, k \in \mathbb{Z}^+$, y_{n+k} lies in any interval containing $x_n, x_{n+1}, \ldots, x_{n+k+1}$. Since $\{x_n\}$ is Cauchy, this forces $\{y_n\}$ be to Cauchy, and thus $y = [\{y_n\}] \in \mathcal{R}$. We also have $x_n - y_n \to 0$, so $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = y$.
Step 2: Let L be a sequentially complete ordered field, and let $f : F \to L$ be a cofinal ordered field homomorphism. We must show that there is a unique ordered field homomorphism $g : \mathcal{R} \to L$ such that $f = g \circ \iota$.
Uniqueness: Let $g : \mathcal{R} \to L$ be such that $f = g \circ \iota$. We will see explicitly what g must be on each $x \in \mathcal{R}$. Write $x = [a_\bullet]$. Since $\{a_n\}$ is Cauchy in F and f is cofinal, f is uniformly continuous, so $\{f(a_n)\}$ is Cauchy in L, hence convergent, say to $y \in L$. Since f is cofinal, is g, hence g is uniformly continuous. Since $\iota(a_n) \to x$, $f(a_n) = g(\iota(a_n)) \to g(x)$. Since $f(a_n) \to y$ and limits are unique, we must have

$$g(x) = y = \lim_{n\to\infty} f(a_n).$$

Existence: We must show that putting $g(x) = y$ as above defines an ordered field homomorphism from $\mathcal{R}$ to L. If $x_1 = [a_n]$ and $x_2 = [b_n]$, let $y_1 = \lim_{n\to\infty} f(a_n)$ and $y_2 = \lim_{n\to\infty} f(b_n)$. Then $a_n + b_n \to x_1 + x_2$ and $f(a_n + b_n) = f(a_n) + f(b_n) \to y_1 + y_2$, so $g(x_1 + x_2) = y_1 + y_2 = g(x_1) + g(x_2)$. Similarly, $a_n b_n \to x_1 x_2$ and $f(a_n b_n) = f(a_n) f(b_n) \to y_1 y_2$, so $g(x_1 x_2) = y_1 y_2 = g(x_1) g(x_2)$. Finally, if

$x = [a_\bullet] > 0$, then there is a positive $\epsilon \in F$ such that $a_n \geq \epsilon$ for all sufficiently large n, so

$$g(x) = \lim_{n\to\infty} f(a_n) \geq \lim_{n\to\infty} f(\epsilon) = f(\epsilon) > 0.$$

$\square$

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