Randomly solving 2-SAT.

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a) Let X_n be the number of satisfied clauses after n iterations of the loop in (2) (we will refer to it as time n). Given the execution up top time $n-1$, is it always true that $\mathbb{E}[X_n] \geq X_{n-1}$?
Solution
The claim is not true. Let us prove it by a counterexample: let ϕ be the following CNF formula,

$$\phi = (x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge (\bar{x_1} \vee \bar{x_4}) \wedge (\bar{x_1} \vee x_3) \wedge (\bar{x_4} \vee \bar{x_3}),$$

and $\mathbf{x} = (0,1,0,1)$ the assignment at time $n-1$. Then, it follows that $X_{n-1} = 4$, and the expectancy of the variable X_n can be calculated as follows; whether x_1 swaps its value, and then $X_n = 3$, or x_3 swaps its value and $X_n = 4$. This gives an expectancy

$$\mathbb{E}[X_n] = \frac{1}{2}3 + \frac{1}{2}4 = \frac{7}{2} < 4 = X_{n-1}.$$

b) Since ϕ is satisfiable, let $\mathbf{x}^*$ be an arbitrary satisfying assignment. Let Y_n be the number of variables in $\mathbf{x}$ whose value coincides with the one in $\mathbf{x}^*$ at time n. Given the execution up to time $n-1$, is it always true that $\mathbb{E}[Y_n] \geq Y_{n-1}$?
Solution
In order to prove this, we will first take a look at how does Y_n behave with respect to Y_{n-1}. Since the algorithm only changes variables in violated clauses, we will take Z_n^i to be the random variable that counts how many of the variables in clause i have the same value in $\mathbf{x}$ and in $\mathbf{x}^*$. So, clause i can be thought of as $(x_{j_1} \vee x_{j_2})$. In $\mathbf{x}^*$ this can only be (1,1), (0,1), or (1,0), as a consequence of a clause being a disjunction, and in $\mathbf{x}$ this variables have to be (0,0), because the clause is violated. Then, suppose the first step of the loop selects this clause; now it can only change the value of one of the two variables. So, the only possible outcomes of Y_n are

$$Y_n = \begin{cases} Y_{n-1} + 1 \\ Y_{n-1} - 1 \end{cases},$$

and this will depend on the value of Z_n^i. As clause i was violated at time $n-1$, Z_{n-1}^i could have been 0 or 1, and it can at time n equal zero, one or two:

$$Z_n^i = \begin{cases} 0 \text{ with probability } \frac{1}{2}, \text{ if } Z_{n-1}^i = 1 \\ 1 \text{ with probability } 1, \text{ if } Z_{n-1}^i = 0 \\ 2 \text{ with probability } \frac{1}{2}, \text{ if } Z_{n-1}^i = 1 \end{cases}.$$

Taking this into account, it is more probable that Z_n^i increases than that it decreases, and so is the case for Y_n. Then, the expectancy of Y_n is greater than (or equal to) the value of Y_{n-1}.

c) Argue that if $Y_n = k$, then RAND2SAT terminates at time n. Is the converse true?
Solution
Since $Y_n = k$, it means that all variables in $\mathbf{x}$ are equal to $\mathbf{x}^*$. As $\mathbf{x}^*$ is an arbitrary satisfying assignment, it means that $\phi(\mathbf{x}^*) = 1$, and as $\phi \in$ CNF-2-SAT, it follows that is has no violated clauses. Then, as described by the algorithm, the loop in (2) stops when no clauses are violated. Hence, when $Y_n = k$, the algorithm halts at this moment (*time* n).
The converse is not true. Take for example the following CNF-2-SAT formula ϕ',

$$\phi' = (x_1 \vee x_2) \wedge (x_2 \vee x_3);$$

it has 4 different satisfying assignments, $a_1 = (1, 1, 0)$, $a_2 = (1, 0, 1)$, $a_3 = (0, 1, 1)$, and $a_4 = (0, 1, 0)$. If we take $\mathbf{x}^*$ to be one of these, say a_3, the loop in (2) could possibly reach a_4 before reaching a_3 and halt as a consequence of a_4 being a satisfying assignment. If that were the case, Y_n would be 2 instead of 3 (the k in this example), and so the converse can't be true.

d) Is Y_n a Markov chain?
Solution
It's not. In order to prove this, we are going to consider a counterexample. First, recall that in order for Y_n to be a Markov chain, it is needed that Y_{n-1} gives us all the information to determine the possible outputs of Y_n. From the following CNF formula,

$$(x_1 \vee x_2) \wedge (x_1 \vee \bar{x_3}) \wedge (x_4 \vee x_2) \wedge (x_4 \vee x_3)$$

the starting assignment $\mathbf{x} = (0, 0, 0, 0)$ and the satisfying assignment $\mathbf{x}^* = (1, 1, 0, 1)$. If we draw a part of the tree of possibilities,

$$\begin{array}{cccc} & Y_1{=}1 & & \\ Y_2{=}2 & Y_2{=}2 & Y_2{=}2 & Y_2{=}0 \\ | & & & | \\ Y_3{=}1 & & & Y_3{=}1 \\ | & & & | \\ Y_4{=}2 & & & Y_4{=}2 \end{array}$$

we can clearly see that the event $\{Y_4 = 2\}$ can be reached from two paths that can have different probabilities; for example, the transition between $Y_2 = 2$ and $Y_3 = 1$ could be reached by switching the value of either of the variables that coincide with $\mathbf{x}^*$. So, the following happens

$$\Pr(Y_4 = 2|_{Y_3=1,Y_2=0}) \neq \Pr(Y_4 = 2|_{Y_3=1,Y_2=2}),$$

which should be equal in a Markov chain as, by definition, in a Markov chain the following should happen:

$$\Pr(Y_n = k_n|_{Y_{n-1}=k_{n-1},Y_{n-2}=k_{n-2}\dots,Y_1=k_1}) = \Pr(Y_n = k_n|_{Y_{n-1}=k_{n-1}}) = \Pr(Y_n = k_n|_{Y_{n-1}=k_{n-1},Y_{n-2}=l_{n-2}\dots,Y_1=l_1})$$

for any $k_1, \dots, k_n, l_1, \dots, l_n \in \operatorname{Im} Y$. This ends the proof.

e) Design a Markov chain Z_n such that $Y_n \geq Z_n$.
Solution
As we have seen before, the worst case in which Y_n increases is with probability $\frac{1}{2}$. Then, we can define the following Markov chain:

$$Z_n = \begin{cases} Z_{n-1} + 1 \text{ with probability } \frac{1}{2} \\ Z_{n-1} - 1 \text{ with probability } \frac{1}{2} \\ 1 \text{ if } Z_{n-1} = 0, \text{ with probability } 1 \\ k \text{ if } Z_{n-1} = k, \text{ with probability } 1 \end{cases}$$

starting at $Z_0 = 0$, similar to the Gambler's ruin. Now, by definition, $Z_n \leq Y_n$.

f) Use Z_n to prove that the expected running time of RAND2SAT is at most k^2.
Solution
We will now calculate the expected running time of RAND2SAT by calculating the expected hitting time of Z_n to state k, which by definition acts as an upper bound to the same expected hitting time of Y_n, starting at 0. Recall that the expected hitting times $\{\tau_i\}$ of a Markov chain obey the following recurrence equation,

$$\tau_i = 1 + p\tau_{i+1} + q\tau_{i-1}.$$

In this case, $p = q = \dfrac{1}{2}$ (except for τ_0, where the probability of increasing is 1), so the terms of this recurrence will be

$$\tau_0 = 1 + \tau_1 \quad \tau_1 = 1 + \frac{1}{2}\tau_2 + \frac{1}{2}\tau_0 \quad \tau_2 = 1 + \frac{1}{2}\tau_1 + \frac{1}{2}\tau_3 \quad (\cdots) \quad \tau_k = 0.$$

If we expand τ_1, we can see that

$$\tau_1 = 1 + \frac{1}{2}\tau_2 + \frac{1}{2}(1+\tau_1) \implies \tau_1 = 1 + 2 + \tau_2 \implies \tau_2 = 1 + \frac{1}{2}\tau_3 + \frac{1}{2}(3+\tau_2) \implies \tau_2 = 2 + 3 + \tau_3 \quad \cdots$$

Then, we can see by induction that

$$\tau_i = \tau_{i+1} + 2i + 1,$$

and as by definition $\tau_k = 0$, we have that $\tau_0 = 1 + \tau_1 = 1 + (3 + \tau_2) = 1 + (3 + (5 + \tau_3)) = \cdots$. It is known that the sum of the first m odd numbers is m^2, and as τ_k is zero, $\tau_0 = \sum_{i=0}^{k-1} 2i + 1 = k^2$. As a consequence, RAND2SAT cannot run more than k^2 iterations.

***g)** We modify RAND2SAT to stop in bounded time as follows. Let $l \in \mathbb{Z}$. If after $2lk^2$ iterations of the loop in (2) we have not halted, we break the loop and return the current assignment $\mathbf{x}$. Prove that the output of the modified RAND2SAT is a satisfying assignment with probability at least $1 - 2^{-l}$.
Solution
Let T_i be the random variable that counts the time (number of iterations) that it takes for Y_n to reach k. Then, because of **f)**, it follows that $\mathbb{E}[T_i] = k^2$. Using Markov's inequality, we can see that

$$\Pr(T_i \geq 2k^2) \leq \frac{k^2}{2k^2} \implies \Pr(T_i \geq 2k^2) \leq \frac{1}{2}.$$

Then, we define the random indicator variable I_{T_i} to be

$$I_{T_i} = \begin{cases} 0 \text{ if } T_i < 2k^2 \\ 1 \text{ if } T_i \geq 2k^2 \end{cases} .$$

We have that $I_{T_i} \sim B\left(\frac{1}{2}\right)$. If $\mathcal{T} = \sum_{i=1}^{l} I_{T_i}$, then $\mathcal{T} \geq \mathcal{N} \sim \text{Bin}\left(l, \frac{1}{2}\right)$, and then

$$\Pr(\mathcal{N} = 0) = \binom{l}{0}\left(\frac{1}{2}\right)^l = \frac{1}{2^l}.$$

Now we have finished, as $\Pr(\mathcal{N} > 0) = 1 - \Pr(\mathcal{N} = 0) = 1 - 2^{-l}$.