U3

Part 2: Discrete Fourier transform (DFT)

- The DTFT provides a continuous function: It is not suitable for computer processing
- The DFT provides a sampling representation of the frequency domain

Definition of DFT and inverse DFT

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The discrete Fourier transform (DFT) of size N converts finite sequences $x[0], \ldots, x[N-1]$ of N complex numbers into other finite sequences $X[0], \ldots, X[N-1]$

TIME DOMAIN

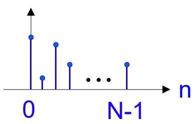
$$X_N[k] = \sum_{n=0}^{N-1} x_N[n] e^{-j2\pi \frac{k}{N} n}, \quad 0 \le k \le N-1$$

FREQUENCY DOMAIN

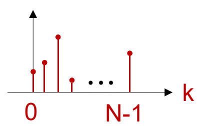
$x_N[n]$: sequence of length N

$\xrightarrow{\text{DFT}_N}$

$X_N[k]$: sequence of length N



$\xleftarrow{\text{IDFT}_N}$



$$x_N[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_N[k] e^{j2\pi \frac{k}{N} n}, \quad 0 \le n \le N-1$$

Relation between DFT and DTFT (1)

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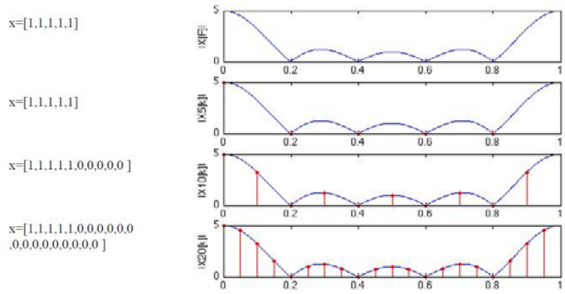
- The DFT of size N of a signal $x[n]$ of length $L \leq N$:

$$X_N[k] = DFT_N\{x[n]\} = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{N} n} = X(F)\big|_{F=\frac{k}{N}}, \quad 0 \leq k \leq N-1$$

k: discrete frequency variable/index

Sampling of the Fourier transform with N points in the interval $[0,1)$

1 not included!



Implicit windowing of the DFT

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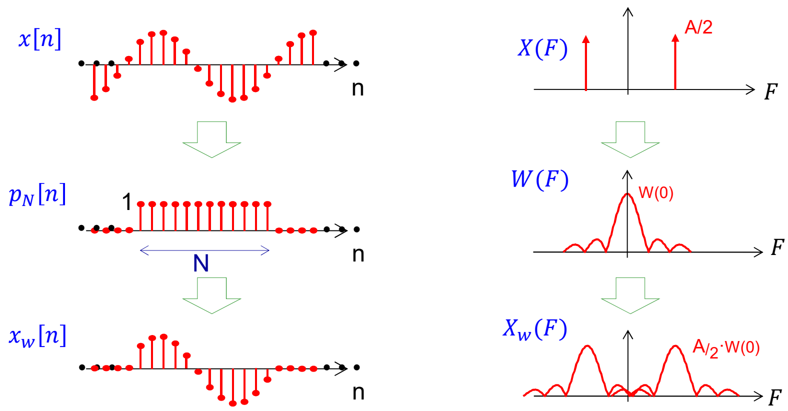
Given a general signal, an implicit windowing in the interval $[0, N-1]$ is applied when computing the DFT:



$$x_w[n] = x[n] \cdot p_N[n]$$

$$X_w(F) = \int_{-0.5}^{0.5} X(F')W(F'-F)\,dF$$

Relation between DFT and DTFT (2)

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- The DFT of size N of a signal $x[n]$ of length $L > N$:

Define $x_w[n] = x[n] \cdot p_N[n] \leftrightarrow X_w(F) = X(F) \circledast W(F)$

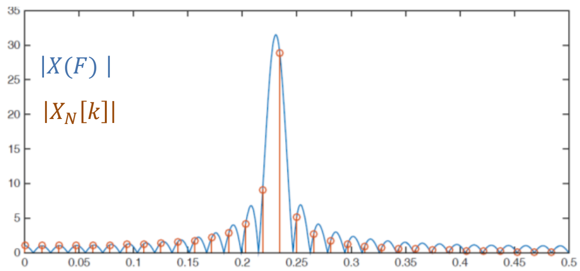
$$X_N[k] = DFT_N\{x[n]\} = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{N} n} = X(F) \circledast \mathrm{W(F)}\big|_{F=\frac{k}{N}}, \quad 0 \le k \le N-1$$

k: discrete frequency variable/index

Now $X_N[k]$ is composed of N points of $X_w(F) \neq \mathrm{X(F)}$ in the interval $[0,1)$

$x[n] = \cos(2\pi F_0 n)$

$F_0 = \frac{3}{13}$, $N = 64$



Windowing of signals for DFT computation

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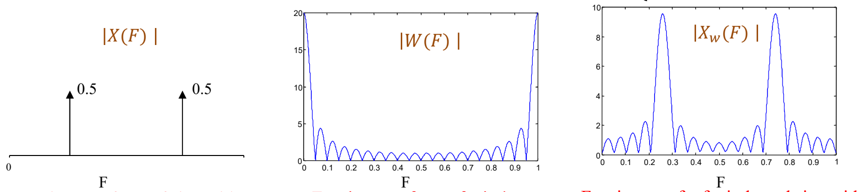
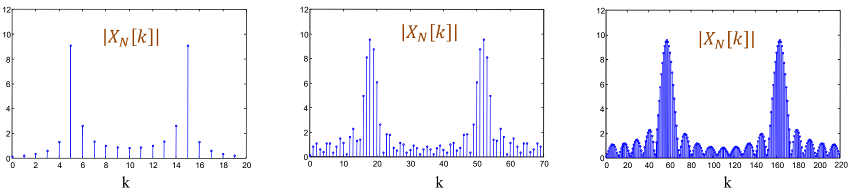
Keeping fixed the size of the window and increasing the number of points of DFT (*zero-padding*):

$x[n] = \cos(2\pi \frac{5.2}{20} n)$ $\quad x_w[n] = x[n] \cdot p_L[n] = \begin{cases} x[n], 0 \le n < 20 \\ 0, \quad otherwise \end{cases}$



Fourier transform of sinusoid — Fourier transform of window — Fourier transf. of windowed sinusoid



D.F.T. of length N=20 — D.F.T. of length N=70 — D.F.T. of length N=220

Windowing of signals for DFT computation

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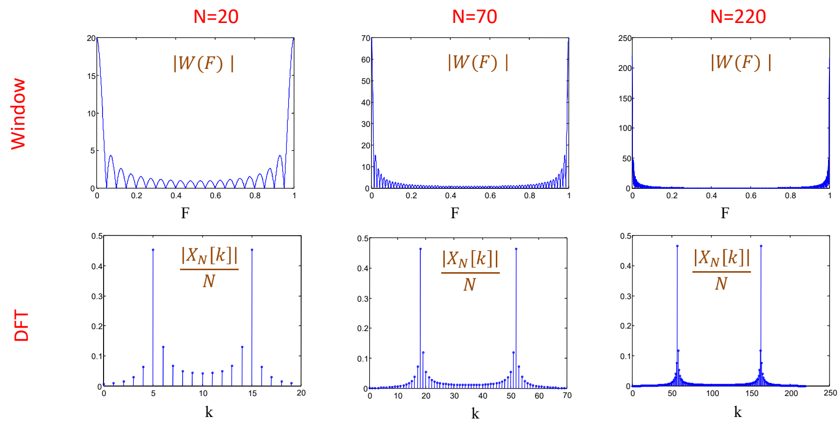
Increase the size of the window (and the size of the DFT):

$x[n] = \cos(2\pi \frac{5.2}{20} n)$ $\qquad$ $x_w[n] = x[n] \cdot p_N[n] = \begin{cases} x[n], 0 \le n < N \\ 0, \quad otherwise \end{cases}$



Inverse DFT (proof)

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$$IDFT_N[X_N[k]] = \frac{1}{N}\sum_{k=0}^{N-1} X_N[k] e^{j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{l=0}^{N-1} x_N[l] e^{-j\frac{2\pi}{N}kl}\right) e^{j\frac{2\pi}{N}kn} = \sum_{l=0}^{N-1} x_N[l] \underbrace{\left(\frac{1}{N}\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(n-l)}\right)}$$

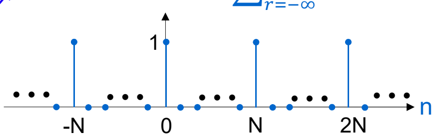
$$\boxed{\frac{1}{N}\frac{1-e^{j2\pi(n-l)}}{1-e^{j\frac{2\pi}{N}(n-l)}} = \begin{cases} 1, & n-l = r\cdot N \\ 0, & \text{otherwise} \end{cases}}$$

$$= \sum_{l=0}^{N-1} x_N[l] \sum_{r=-\infty}^{\infty} \delta[n-l-rN] = \sum_{l=0}^{N-1} x_N[l] t_N[n-l]$$

$$= x_N[n] * t_N[n] = \sum_{r=-\infty}^{\infty} x_N[n-rN] = \tilde{x}_N[n] = x_N[n]$$

$0 \le n \le N-1$
$x_N[n] = 0, n < 0, n \ge N$

$t_N[n] = \sum_{r=-\infty}^{\infty} \delta[n-rN]$



Definition: $\tilde{a}_N[n]$ periodic extension of a sequence $a[n]$ with period N $\Rightarrow$ $\tilde{a}_N[n] = a[n] * t_N[n] = \sum_{r=-\infty}^{\infty} a[n-rN]$

Frequency sampling

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- Let us assume that we have a general signal x[n] (taking values at any time instant n, and not only in [0,N-1]):
  1. We calculate its Fourier transform: $X(F) = \sum_{l=-\infty}^{\infty} x[l] e^{-j2\pi F l}$
  2. We sample the Fourier transform in the interval $F \in [0,1)$ with N samples: $X[k] = X(F)|_{F=\frac{k}{N}}, \quad 0 \leq k \leq N-1$
  3. We apply the IDFT of N points (use similar proof as before):

$$IDFT_N\{X[k]\} = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{l=-\infty}^{\infty} x[l] e^{-j\frac{2\pi}{N}kl}\right) e^{j\frac{2\pi}{N}kn}$$

$$= \sum_{l=-\infty}^{\infty} x[l] \left(\frac{1}{N}\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(n-l)}\right) = x[n] * t_N[n] = \sum_{r=-\infty}^{\infty} x[n-rN] = \tilde{x}_N[n]$$

In general $\tilde{x}_N[n] \neq x[n]$ in the interval $n \in [0, N-1]$ (*temporal aliasing*):

--- if the signal has non-zero samples out of the interval $[0, N-1]$, the periodic extension is not equal to the original signal ---

Check yourself (1)

U3

Consider $x[n] = [\underline{1}, -1, 0, 1, 1]$, $L = 5$.
Take the DFT of size N of $x[n]$, then the IDFT of size N is:

- For $N = 6$
- For $N = 4$
- For $N = 3$

Check yourself (2)

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Consider $x[n] = [\underline{1}, -1, 0, 1, 1]$, $L = 5$. Take N samples of the Fourier transform of $x[n]$, then the IDFT of size N is:

- For $N = 6$
- For $N = 4$
- For $N = 3$