Calculus — Summary

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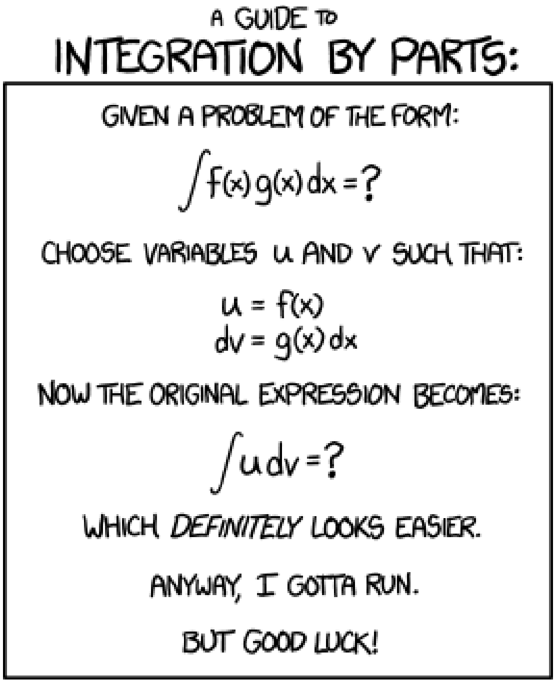


Abstract

This document contains a summary of the Calculus course given in the first year of Computer Science and Engineering. This is *not* a definitive guide, and might contain errors. Please send an email to "dany@atlasdev.nl". This summary is distributed under the MIT license.

Contents

1 Limits

Limits are the foundation of calculus. A lot of other things commonly used in calculus are made with the help of limits. For example the derivative of x^2 is $2x$. This can be proven by using limits. A limit of a function is the value of the function when it approaches a value.

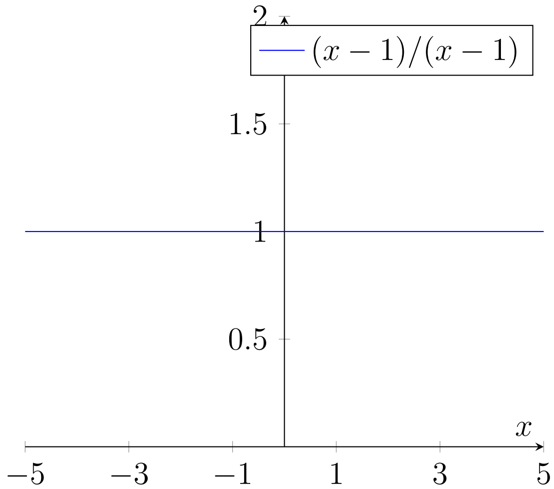
1.1 Limit law

Given two limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ we can say that:

$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
$\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
$\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$
$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) * \lim_{x\to a} g(x)$
$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \text{if} \lim_{x\to a} g(x) \neq 0$

Table 1: Limit laws

1.2 Finding a limit



The simplest way of approximating the limit, is to evaluate the function with an x close to the limit you want. Take for example the function graphed above. $f(1)$ is obviously not defined, you cannot solve it that way. But you can approximate it by filling 1.00000001 into the function. This gives 1, which is the limit.

1.3 Doing it properly

This does not always work, so it's better to rewrite the function. For example:

$$f(x) = \frac{x-1}{x-1} = \frac{1}{1} = 1$$

Obviously this is a simple function, other functions will not be this simple. For example, solving the following function will look like this:

$$\begin{aligned}\lim_{x \to 0} \frac{\sqrt{x^2+9}-3}{x^2} &= \lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2} * \frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3} \\ &= \lim_{x\to 0} \frac{(x^2+9)-9}{x^2(\sqrt{x^2+9}+3)} \\ &= \lim_{x\to 0} \frac{1}{\sqrt{x^2+9}+3} \\ &= \frac{1}{\sqrt{0^2+9}+3} \\ &= \frac{1}{6}\end{aligned} \tag{1}$$

1.4 Continuity

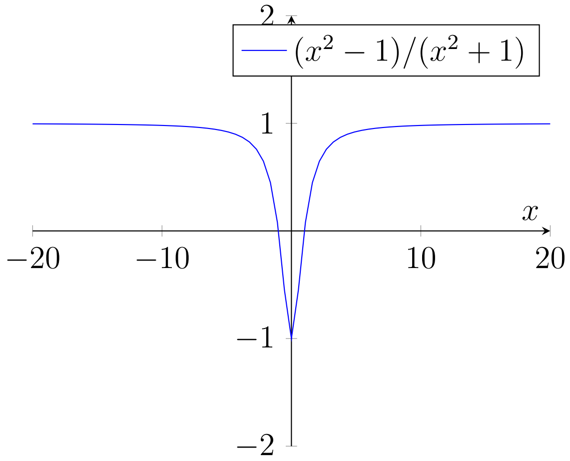
A function is **continuous** iff

- $f(a)$ is defined
- $\lim_{x\to a} f(x)$ exits
- $\lim_{x\to a} f(x) = f(a)$

So, f is continuous at a if $f(x)$ approaches $f(a)$. If this is not the case we can say that te function is **discontinuous at a**.

1.5 Limits at infinity

If you take an infinity positive or negative number as the limit you can get the horizontal asymptotes of a function. Take for example the following function:



This function probably has a horizontal limit at $y = 1$. But you don't know for sure. You can use the limit of infinity to prove them. As constants don't really matter when approaching infinity you can remove them. So you'll get the following:

$$\lim_{x\to\infty} \frac{x^2-1}{x^2+1} = \frac{\infty^2-1}{\infty^2+1} = \frac{\infty^2}{\infty^2} = \frac{1}{1} = 1$$

This gets the horizontal asymptote on the right. If you want to get the limit on the left you'll have to use negative infinity.

$$\lim_{x\to-\infty} \frac{x^2-1}{x^2+1} = \frac{-\infty^2-1}{-\infty^2+1} = \frac{-\infty^2}{-\infty^2} = \frac{1}{1} = 1$$

In this example it's also $y = 1$, as both asymptotes are the same. One important thing to keep in mind when solving infinite limits is that $\frac{1}{\infty} = 0$.

2 Differentiation

Differentiation is a really important concept of calculus. With this you can calculate the slope of a function in a specific point.

2.1 The product rule

The product rule is a simple rule to differentiate two functions which are in a product. The rule is as following:

$$(f * g)'(x) = f'(x) * g(x) + f(x) * g'(x)$$

2.2 The quotient rule

The quotient rule is a rule to differentiate a function which contains the quotient of two differentiable functions. It works as following:

$$(\frac{f}{g})'(x) = \frac{g(x) * f'(x) - g'(x) * f(x)}{g'(x)^2}$$

2.3 The chain rule

Some functions cannot be differentiated that easily. You will need the chain rule for that. Take the next function:

$$f(x) = \sqrt{x^2 + 1}$$

To apply the chain rule you need to substitute a part of the function with $u(x)$. You should choose a part which, when removed, makes the function way easier. The previous function would look like this:

$$f(x) = \sqrt{u(x)}$$
$$u(x) = x^2 + 1$$

Now you can differentiate the functions separately. This would look as following:

$$f'(x) = \frac{1}{2\sqrt{u(x)}} * u'(x) = \frac{1}{2\sqrt{x^2 + 1}} * 2x$$

You can apply this to more than the square root, like power functions. A more general rule for the chain rule is as following:

$$(f \circ g)'(x) = f'(g(x)) * g'(x)$$

2.4 Implicit differentiation

Till now we have assumed that we differentiate over a **explicit function**. This means it has the form of $f(x) = y$. But this is not always the case. Take the following example:

$$x^3 + y^3 = 6xy$$

This is *not* an easy function to differentiate. With implicit differentiation you differentiate both sides with respect to x. We can differentiate y^3 with the chain rule. So y^3 will become $3y^2 * \frac{dy}{dx}$. Trying to do that with the example function gives you:

$$(x^3 + y^3)' = (6xy)'$$
$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$
$$y^2\frac{dy}{dx} - 2x\frac{dy}{dx} = 2y - x^2$$
$$\frac{dy}{dx}(y^2 - 2x) = 2y - x^2$$
$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$$

2.5 Linear approximation

If you zoom into a function with the respective tangent line itĺl start to look like that function. With this you can approximate the value of a function with a tangent line close to that value. The following is called the **linearization** of f in a:

$$l(x) = f(a) + f'(a)(x - a)$$

For example, if we want to calculate $f(x) = \sqrt{x}$ with $x = 4.36$. We take an a we know close to that point, 4 would make sense. This gives the following function:

$$l(x) = 2 + \frac{1}{4}(x - 4) = 1 + \frac{x}{4}$$

Now we can fill in our x into $l(x)$.

$$l(4.36) = 1 + \frac{4.36}{4} = 1 + \frac{109}{100} = 2.09 \approx \sqrt{4.36}$$