Stability of ExnerFoamH

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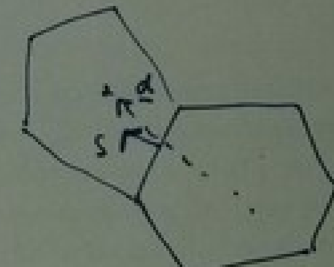
ExnerFoamH had stability problems. These were most obvious when running a stably stratified flow for a long time. Energy would very gradually increase. This is despite an upwind-biased, slightly dissipative advection scheme and a symmetric Hodge operator

$$U = HV$$



where $U = \rho \underline{u} \cdot \underline{S}$

$$V = \rho \underline{u} \cdot \underline{d}$$

This was related to the semi-implicit distretisation of the pressure gradient.

Old formulation (unstable)

$$\frac{\partial V}{\partial t} = -(\nabla \cdot \rho \underline{u}\underline{u}) \cdot \underline{d} + \rho \underline{g} \cdot \underline{d} - c_p \rho \Theta \nabla_d \Pi$$

where $\nabla_d \Pi = \nabla \Pi \cdot \underline{d}$

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$$\frac{\partial \rho}{\partial t} = - \nabla \cdot HV$$

Define explicit updates to V and U:
Assuming Crank-Nicholson with no off centering

$$V^* = V^n + \frac{\Delta t}{2}\left(-(\nabla\cdot\rho\underline{u}\underline{u})\cdot\underline{d} + \rho g\underline{d}\right)$$

$$U^* = HV^* - \frac{\Delta t}{2} c_p \rho\theta \, H_{off} \nabla_d \Pi^{\ell}$$ ℓ - lagged

where H_{off} is the off diagonal components of H

so that the continuity equation becomes

$$\frac{\partial\rho}{\partial t} = -\frac{1}{2}\left\{\nabla\cdot U^n + \underbrace{\nabla\cdot U^* - \nabla\cdot\frac{\Delta t}{2} c_p\rho\theta H_d \nabla_d \Pi^{n+1}}_{\text{at time } n+1}\right\}$$

where H_d is the diagonal part of H

$$H_d = \frac{\underline{d}\cdot\underline{S}}{|\underline{d}|^2}$$

With this formulation, a significant part of the pressure gradient is treated explicitly:

$$\frac{\Delta t}{2} c_p \rho\theta \, H_{off} \nabla_d \Pi^{\ell}$$

And the scheme can be unstable.
I do not want to work out how to put this into the matrix for the Laplacian. Instead I will take some of the diagonal out of the matrix and make it explicit. This is similar to how

non-orthogonal corrections are done in OpenFOAM standard. So I will define

H_c to be the central part of H. This is the diagonal matrix

$$H_c = \frac{|\underline{S}|}{|\underline{d}|}$$

And H_{corr} is the correction

$$H_{corr} V = HV - H_c V$$

Now we define the explicit part of U as:

$$U^* = HV^* - \frac{\Delta t}{2} c_p \rho \theta \, H_{corr} \nabla_d \Pi^{\ell}$$

And the continuity equation

$$\frac{\partial \rho}{\partial t} = -\frac{1}{2}\left\{ \nabla \cdot U^n + \nabla \cdot U^* - \nabla \cdot \frac{\Delta t}{2} c_p \rho \theta H_c \nabla_d \Pi^{n+1} \right\}$$

Now, ALL of the non-orthogonal correction is treated explicitly. The implicit solution is for an orthogonal grid, for which $H_{corr} = 0$ and $H_c = H_d$