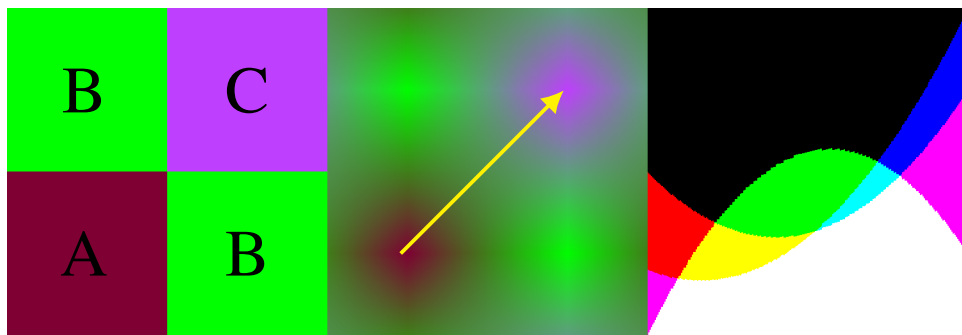


# GPU Efficient Texture Based Bezier Curve Evaluation

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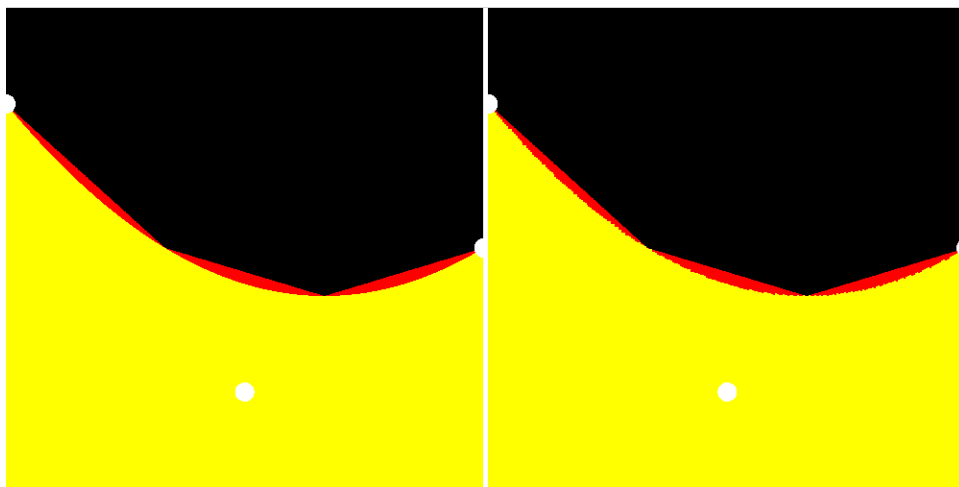
**Figure 1.** *Left:* 2x2 texture containing control points for a quadratic Bezier curve in each color channel. *Middle:* The texture as viewed with bilinear sampling. *Right:* The resulting curves when sampling along the yellow line (Alpha curve omitted).

## Abstract

Modern graphics techniques expose internal parameters to allow re-use and to help separate the technical implementation from artistic usage cases. A popular choice is to expose parameters as customizable curves and to quantize the curves into textures. Quantization leads to either lower quality results, or more texture memory being used to accommodate higher sampling frequencies. The technique presented in this paper leverages the capabilities of GPU texture samplers to allow more efficient storage and evaluation of both integral and rational Bezier curves of any order, resulting in higher fidelity for the same costs. Piecewise curves, B-Splines and Nurbs are addressed, and there are also limited applications towards vector graphics.

## 1. Introduction

There are two basic ways for a shader program to have customized curve data. One way is to evaluate the curve on the CPU at discrete intervals and put those points into a texture that can be used by the GPU. The other way is to pass curve control point



**Figure 2.** *Left:* A quadratic Bezier curve written to the green channel, and the baked out equivalent using 4 pixels written to the red channel. *Right:* The technique from this paper, also using 4 pixels, written to the green channel, and the baked out equivalent written to the red channel again for comparison.

data from the CPU to the shader program as shader constants, allowing the shader program to evaluate the curve at a given time  $t$ .

Baking curve points into a texture means that the shader program doesn't need to know what type of curve it was that made the data, nor how many control points it had, but comes at the loss of quality since there are a finite number of curve points sampled, and the data between points is just a linear interpolation between the samples. If higher quality is desired, you can increase the resolution of the texture to be able to trade accuracy for texture memory.

Passing control point data to a shader program as shader constants allows you to get high quality curve calculations, but comes at the cost of the shader program being written in a very specific way for the type of curve you want to use, as well as more shader program instructions to calculate specific curve points.

This paper shows a third method where:

- The control points of the curves are encoded in a texture.
- The texture sampler calculates points on the curve before the data reaches the shader program.
- It gives accuracy results closer to that of shader constant curves, while having less calculation overhead.
- The technique can support both integral and rational Bezier curves of any order and can also be used for piecewise curves.

- The curve type must be decided on in advance, like when using the shader constant method.
- There are limited applications towards vector graphics.

A quick comparison of the visual quality of these three techniques can be seen in [Figure 2](#).

### 1.1. Related Work

TODO: this

## 2. The Technique

The core of this technique is that the  $N$ -dimensional linear texture interpolation capabilities on the GPU can be mathematically equivalent to De Casteljau's algorithm for evaluating Bezier curves. We can use linear texture sampling on textures of increasing dimensionalities to have the texture sampler calculate points on curves of increasing order.

### 2.1. Intuition

One dimensional linear texture sampling is just the linear interpolation between two data points. This is equivalent to the De Casteljau algorithm when evaluating a linear curve of order 1. In both cases, it is just linear interpolation between two values.

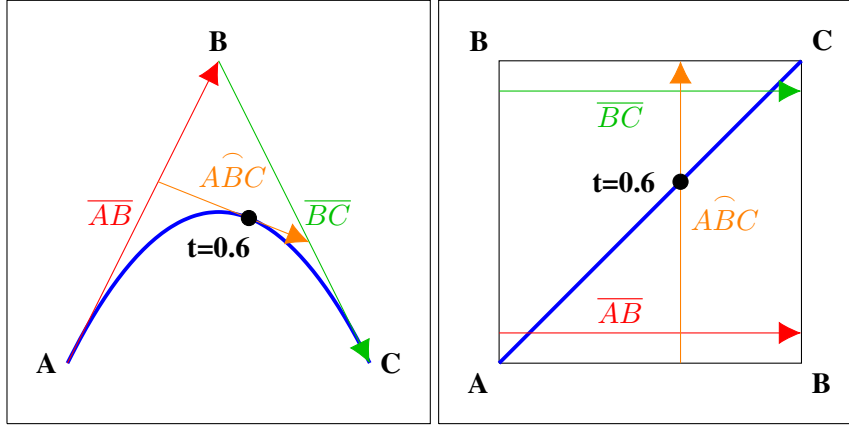
Two dimensional linear texture sampling (bilinear texture sampling) can be thought of as just the linear interpolation across the  $y$  axis, of two linear interpolations across the  $x$  axis. In other words, it is just the linear interpolation of two linear interpolations.

The De Casteljau algorithm for evaluating a quadratic curve (order 2) is also just to linear interpolate between two linear curves (order 1). If the control points of the quadratic curve are  $P_0, P_1, P_2$ , let's call the first interpolation  $\overline{P_0P_1}$  which is the interpolation between  $P_0$  and  $P_1$  for time  $t$ . The second linear interpolation is  $\overline{P_1P_2}$  and is the interpolation between  $P_1$  and  $P_2$  for time  $t$ . The final point on the curve will be the interpolation at time  $t$  between  $\overline{P_0P_1}$  and  $\overline{P_1P_2}$ , which we can call  $\widehat{P_0P_1P_2}$ .

We could set up a texture such that the two  $x$  axis interpolations would give us the values  $\overline{P_0P_1}$  and  $\overline{P_1P_2}$  respectively, and then the  $y$  axis interpolation between those values would give us  $\widehat{P_0P_1P_2}$ . When sampling, we would also use texture coordinates such that the weighting is the same for the interpolations across the  $x$  and  $y$  axis, so that it results in the same time  $t$  value for interpolation in both linear interpolations.

To further see how bilinear interpolation can be equivalent to evaluating a quadratic Bezier curve with the De Casteljau algorithm, [Figure 3](#) shows it visually, and you can also compare the GLSL source code in [Listing 1](#).

```
float QuadraticBezier (
```



**Figure 3.** *Left:* A quadratic Bezier curve evaluated at  $t=0.6$  for control points A,B,C using the De Casteljau algorithm. *Right:* The De Casteljau algorithm using bilinear interpolation. Note that in this case, the X axis is evaluated before the Y, but the Y axis could be evaluated before the X for the same results.

```

in float t,
in float P0, in float P1, in float P2
) {
    return mix(mix(P0,P1,t), mix(P1,P2,t), t);
}

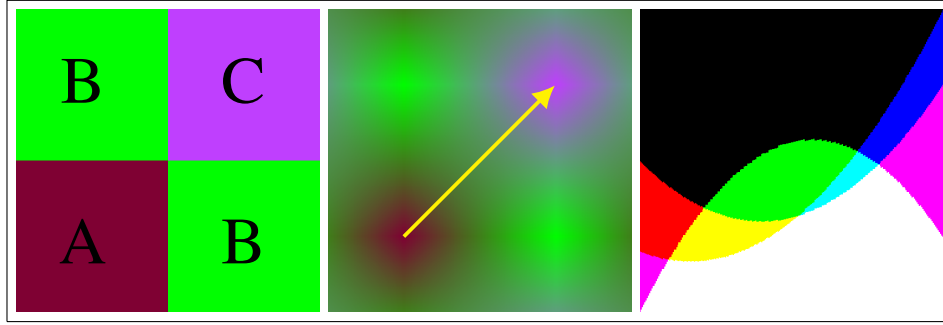
// This function can be equivalent to the function above
// by setting the parameters:
// u = t, v = t
// A = P0, B = P1, C = P1, D = P2
float BilinearInterpolate (
    in float u, in float v,
    in float A, in float B, in float C, in float D
) {
    return mix(mix(A,B,u), mix(C,D,u), v);
}

```

**Listing 1.** GLSL implementation of bilinear interpolation and the De Casteljau algorithm for a quadratic Bezier curve.

The pattern continues for higher texture dimensions and curve orders as well. Linear texture sampling of textures of dimension  $N$  is just the linear interpolation between two linear texture samples of dimension  $N - 1$ . Using the De Casteljau algorithm to evaluate a curve of order  $N$  just means that you take a linear interpolation between curves of order  $N - 1$ .

Linear texture sampling lets you control each value (each pixel), and the weighting of each level (dimension) of the interpolations, but you can set up the pixels and texture coordinates such that it evaluates a Bezier curve. The De Casteljau algorithm



**Figure 4.** *Left:* a 2x2 texture storing control points for a quadratic Bezier curve in each color channel. *Middle:* The same texture as viewed when using bilinear texture sampling. The yellow line indicates where texture samples are taken from to evaluate the quadratic Bezier curve. *Right:* The curves resulting from sampling along the yellow line (Alpha curve omitted).

is a subset of the  $N$ -dimensional linear texture sampling available on the GPU.

It can be seen that the actual pixel value in a bilinear sampled texture is in the middle of a pixel. That means that when calculating our texture coordinates for eg. a 2x2 texture, we must sample between pixel location  $(0.5, 0.5)$  and pixel location  $(1.5, 1.5)$  to get the correct results.

The common pixel formats contain four color channels Red, Green, Blue and Alpha which allows us to be able to evaluate four curves with a single texture read (Figure 4).

It is also possible to get points on curves of higher order by taking multiple texture reads and combining them either using the De Casteljau algorithm, or by using the Bernstein form equation of Bezier curves.

## 2.2. Mathematical Basis

The De Casteljau algorithm is equivalent to the Bernstein form of Bezier curves.

$$\mathbf{P}(t) = \sum_{i=0}^n \binom{n}{i} (1-t)^{n-i} t^i \mathbf{P}_i \quad (1)$$

That they are equivalent means that the Bernstein form for a given  $n$  must also be able to be evaluated by  $n$ -Linear interpolation. A linear curve should be able to be evaluated by linear interpolation, a quadratic curve should be able to be evaluated by bilinear interpolation, a cubic curve should be able to be evaluated by trilinear interpolation, and so on.

When  $n$  is 1, to make a linear Bezier curve, the equation we get out is in fact just linear interpolation, so we can see that it's true in that case.

$$\mathbf{P}(t) = \mathbf{P}_0(1 - t) + \mathbf{P}_1t \quad (2)$$

When  $n$  is 2, to make a quadratic Bezier curve with bilinear interpolation, the relationship is not as obvious.

$$\mathbf{P}(t) = \mathbf{P}_0(1 - t)^2 + \mathbf{P}_12(1 - t)t + \mathbf{P}_2t^2 \quad (3)$$

To see how that equation can be the same as bilinear interpolation, let's start with the equation for bilinear interpolation, which is just a linear interpolation between two other linear interpolations. We'll use  $t$  and  $u$  as the interpolation weights of each interpolation.

$$\begin{aligned} \mathbf{E}(u) &= \mathbf{P}_0 * (1 - u) + \mathbf{P}_1 * u \\ \mathbf{F}(u) &= \mathbf{P}_2 * (1 - u) + \mathbf{P}_3 * u \\ \mathbf{P}(t, u) &= \mathbf{E}(u) * (1 - t) + \mathbf{F}(u) * t \\ \mathbf{P}(t, u) &= (\mathbf{P}_0 * (1 - u) + \mathbf{P}_1 * u) * (1 - t) + (\mathbf{P}_2 * (1 - u) + \mathbf{P}_3 * u) * t \end{aligned} \quad (4)$$

In our usage case of bilinear interpolation to evaluate Bernstein polynomials,  $P_0$  will be  $A$ ,  $P_1$  and  $P_2$  will both be  $B$ , and  $P_3$  will be  $C$ . Also,  $t$  and  $u$  are the same in our usage case so we'll just replace  $u$  with  $t$ . We will also replace  $(1 - t)$  with  $s$  for readability.

$$\begin{aligned} \mathbf{P}(t) &= (\mathbf{A} * s + \mathbf{B} * t) * s + (\mathbf{B} * s + \mathbf{C} * t) * t \\ \mathbf{P}(t) &= \mathbf{A} * s^2 + \mathbf{B} * st + \mathbf{B} * st + \mathbf{C} * t^2 \\ \mathbf{P}(t) &= \mathbf{A} * s^2 + \mathbf{B} * 2st + \mathbf{C} * t^2 \end{aligned} \quad (5)$$

If we replace  $A, B, C$  with  $P_0, P_1, P_2$ , and  $s$  with  $(1 - t)$  we then get the familiar equation back out.

$$\mathbf{P}(t) = \mathbf{P}_0(1 - t)^2 + \mathbf{P}_12(1 - t)t + \mathbf{P}_2t^2 \quad (6)$$

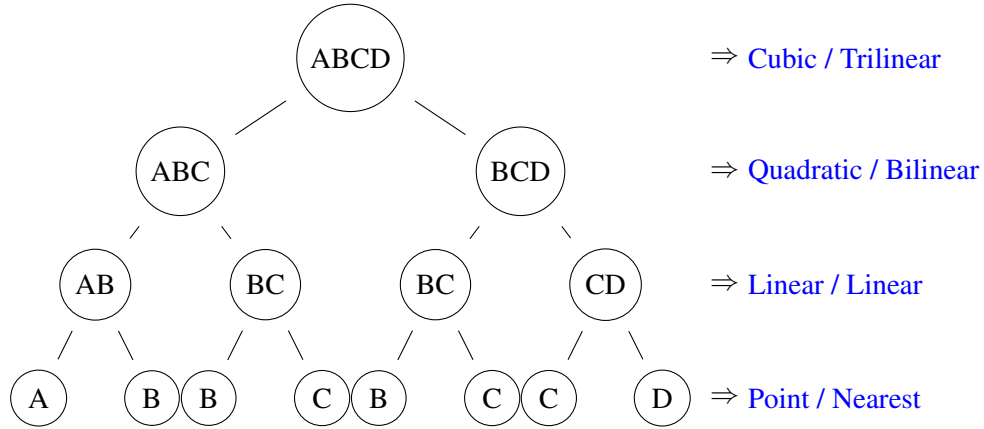
This pattern continues for trilinear interpolation and beyond.  $N$  dimensional linear texture sampling is just the linear interpolation between two linear texture samples of dimension  $N - 1$ . Bezier curves of order  $N$  are just the linear interpolation between two bezier curves of order  $N - 1$ .

### 2.3. Accuracy

While the Bernstein form of Bezier curves is equivalent to evaluating a Bezier curve with the De Casteljau algorithm, in practice there are differences that come up between the two. It is well known that the De Casteljau algorithm is slower but more numerically robust than evaluating the Bernstein form.



**Figure 5.** *Left:* Sampled curves. *Middle:* Zooming in on a curve calculated with shader program constants. *Right:* Zooming in on a curve calculated using the method in this paper



**Figure 6.** A tree showing the De Casteljau algorithm for a cubic curve. The labels on the right show what type of curves are evaluated at that level, as well as the  $N$  dimensional sampling that is required to evaluate nodes at that level with a single texture read.

The technique presented in this paper also has numerical differences in practice. While it's been shown that the multidimensional linear interpolation capabilities can be mathematically equivalent to the Bernstein form as well as the De Casteljau algorithm, there are mathematical precision issues due to current GPU architectures (Figure 5).

As shown in Figure 4, the quality can be significantly higher than a baked out curve for the same number of pixels however. There is more information on addressing accuracy in section 5.

TODO: give the details of the architecture imprecision sources and cite the two papers cited from the other place!

### 3. Texture Dimensionality

While this technique is able to be used with any texture dimensionality, there are different characteristics for each choice. Those characteristics will be explored here.

#### 3.1. One Dimensional Textures

Using one dimensional textures, the built in texture interpolation is only able to perform linear interpolation, which allows calculations of only 1st order (linear) Bezier curves (Figure 6).

Since the De Casteljau algorithm just interpolates between linear interpolations, we can take more texture samples and combine them in the shader program to get higher order curves. For an order  $N$  Bezier curve, the De Casteljau algorithm requires  $2^{N-1}$  linear interpolations for the linear level of the tree. You need two pixels to perform a linear interpolation, so it seems that we would need  $2^N$  pixels to store a curve of order  $N$  in a 1 dimensional textures. For a cubic curve, that would look like this:  $[P_0, P_1, P_1, P_2, P_1, P_2, P_2, P_3]$ .

We can do better than that though. Firstly, there are redundant lerps in the list for lerping between  $P_1$  and  $P_2$ . As the order of the curve increases, the number of redundancies increase as well. We don't want to have to pay the cost of pixel storage and texture samples for redundancies, so because we have  $N + 1$  control points and we only ever sample between  $P_i$  and  $P_{i+1}$ , we only need to store  $N$  lerps in the 1 dimensional texture, which would mean we would need  $N * 2$  pixels. A cubic curve would then look like this:  $[P_0, P_1, P_1, P_2, P_2, P_3]$ .

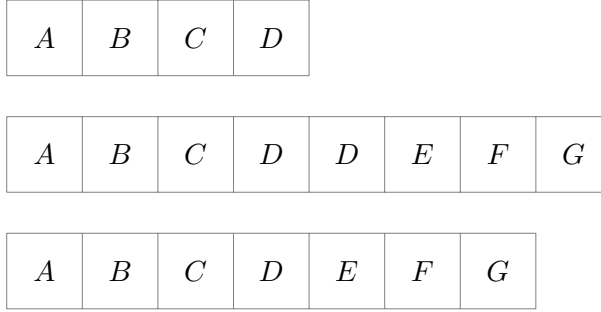
After eliminating redundant lerps, we can see that there are redundant data values in our texture.  $P_1$  and  $P_2$  each show up twice. As the order of the curve increases, we would notice the pattern that all control points show up twice except the end points. We can remove these redundancies and still be able to get the same information out. The result of removing these redundancies is that our texture only needs to contain each control point once, which means we only need  $N + 1$  pixels to encode a curve of order  $N$ . The cubic curve now looks like this:  $[P_0, P_1, P_2, P_3]$ .

If  $M$  piecewise curves are desired, we multiply the number of pixels used for one curve by  $M$ . The size of the one dimensional texture needed for  $M$  piecewise curves, each of degree  $N$  is  $M * (N + 1)$ . Two piecewise cubic curves encoded into a texture would be 8 pixels in size and look like this:  $[P_0, P_1, P_2, P_3, P_4, P_5, P_6, P_7]$

It's often the case that you want  $C0$  continuity for piecewise curves though, which in the example would give  $P_4$  the same value as  $P_3$ . If your piecewise curves always had  $C0$  continuity, you could remove the redundant control point for every curve after the first, which would make it so the size of the texture required became  $N * M + 1$  (Figure 7).

Taking  $N$  linearly interpolated texture samples from a one dimensional texture of size  $N + 1$  gives us the linear interpolated values we would need to continue to





**Figure 7.** *Top:* A cubic curve with control points  $A, B, C, D$  encoded in a 1d texture with a size of 4 pixels. *Middle:* Two piecewise curves encoded in a 1d texture. The control points for the first curve are  $A, B, C, D$  and the control points for the second curve are  $D, E, F, G$ . *Bottom:* With  $C0$  continuity, a redundant  $D$  can be removed to store these curves in 7 pixels instead of 8.

perform the De Casteljau algorithm. We can compute the final curve point either by doing that, or by plugging the values into the Bernstein form equation of Bezier curves.

In the case of a cubic curve with control points  $A, B, C, D$ , the one dimensional texture will be 4 pixels in size, and three texture reads will need to be done to get the interpolations for  $\overline{AB}, \overline{BC}, \overline{CD}$ . Those will then need to be combined using either the De Casteljau algorithm, or by using the Bernstein form of a quadratic Bezier curve ( $P = A * s^2 + B * st + C * t^2$ ) to elevate those three points from order 1 to order 3, to get the final point  $\widehat{ABCD}$ . Sample code of this example is provided in Listing 2.

```
// 4 pixel 1d texture with control points encoded: A,B,C,D
uniform sampler1D uSampler;
const float c_textureSize = 4.0;

vec4 CubicCurveFromTexture1D_DeCasteljau(in float t) {
    vec4 AB = texture(uSampler, (t + 0.5) / c_textureSize);
    vec4 BC = texture(uSampler, (t + 1.5) / c_textureSize);
    vec4 CD = texture(uSampler, (t + 2.5) / c_textureSize);
    vec4 ABC = mix(AB, BC, t);
    vec4 BCD = mix(BC, CD, t);
    return mix(ABC, BCD, t);
}

vec4 CubicCurveFromTexture1D_Bernstein(in float t) {
    vec4 AB = texture(uSampler, (t + 0.5) / c_textureSize);
    vec4 BC = texture(uSampler, (t + 1.5) / c_textureSize);
    vec4 CD = texture(uSampler, (t + 2.5) / c_textureSize);
    float s = (1 - t);
    float s2 = s * s;
```

```
float t2 = t * t;  
// Quadratic Bezier Curve = A*s*s + B*2*s*t + C*t*t  
return AB*s2 + BC*2.0*s*t + CD*t2;  
}
```

**Listing 2.** GLSL for evaluating a cubic curve encoded in a 4 pixel 1d texture. Linear texture sampling is used to evaluate the linear level of the De Casteljau algorithm, then the process is continued both with the De Casteljau algorithm, as well as the Bernstein form of a quadratic Bezier curve.

### 3.2. Two Dimensional Textures

Using two dimensional textures allows the texture interpolator to perform bilinear interpolation, which allows calculations of 2nd order (quadratic) Bezier curves (Figure 6).

This allows us to either calculate a quadratic bezier curve with a single texture read, or allows us to get the quadratic level values needed to make a higher order curve.

Knowing that a quadratic curve is just the linear interpolation of time  $t$  between the points on two linear curves at time  $t$ , we know that our two dimensional texture needs to set up to be able to make this happen. The result is that if we have a 2x2 texture, the top two pixels should be  $P_0$  and  $P_1$  and the bottom two pixels should be  $P_1$  and  $P_2$ . When we do a bilinear interpolation such that we plug  $t$  into both  $u$  and  $v$ , the end result will be that  $\overline{P_0P_1}$  and  $\overline{P_1P_2}$  will be calculated from the  $u$  axis interpolation, and then those two will be linearly interpolated across the  $v$  axis to give us the final result.

Knowing that we can linearly interpolate between two quadratic curves to create a cubic curve, and that we can combine three quadratic curves to create a quartic curve, the pattern seems to be that we need a texture size of  $(2, 2 * (N - 1))$  to encode a curve of order  $N$  into a two dimensional texture.

When actually creating that texture, we would once again see redundancies however. For a cubic curve with control points  $P_0, P_1, P_2, P_3$  we would see that the top row of pixels was  $P_0, P_1$ , then the next row was  $P_1, P_2$ . That would be our first quadratic curve. The third row would start our second quadratic curve which would be  $P_1, P_2$  and then the fourth and final row would be  $P_2, P_3$ . In this case,  $P_1, P_2$  is redundant. If we increased the order of the curve, the pattern that we would see is that every curve after the first curve would have a redundant row.

Removing that redundant row, it means that we start with a 2x2 texture for a quadratic curve, and then just add a row of pixels for each order above quadratic. That makes the size of our texture required be only  $(2, N)$ .

If  $M$  piecewise curves are desired, each of degree  $N$ , we can encode them across the  $X$  axis to require a texture size of  $(2 * M, N)$  (Figure 8).

		A	B	A	B	D	E
A	B	B	C	B	C	E	F
B	C	C	D	C	D	F	G

**Figure 8.** *Left:* A quadratic curve with control points  $A, B, C$  encoded in a 2d texture that is  $(2, 2)$  in size. *Middle:* A cubic curve with control points  $A, B, C, D$  encoded in a 2d texture that is  $(2, 3)$  in size. *Right:* Two piecewise curves encoded in a 2d texture that is  $(4, 3)$  in size. The control points for the first curve are  $A, B, C, D$  and the control points for the second curve are  $D, E, F, G$ .

```
// 2x4 2d texture, rounded to the nearest powers of 2 from 2x3.
uniform sampler2D uSampler;
const vec2 c_textureSize = vec2(2.0,4.0);

vec4 CubicCurveFromTexture2D_DeCasteljeau(in float t) {
    vec4 ABC = texture(uSampler, (vec2(0.5, 0.5)+t) / c_textureSize);
    vec4 BCD = texture(uSampler, (vec2(0.5, 1.5)+t) / c_textureSize);
    return mix(ABC, BCD, t);
}

vec4 CubicCurveFromTexture2D_Bernstein(in float t) {
    vec4 ABC = texture(uSampler, (vec2(0.5, 0.5)+t) / c_textureSize);
    vec4 BCD = texture(uSampler, (vec2(0.5, 1.5)+t) / c_textureSize);
    float s = (1 - t);
    // Linear Bezier Curve = A*s + B*t
    return ABC*s + BCD*t;
}
```

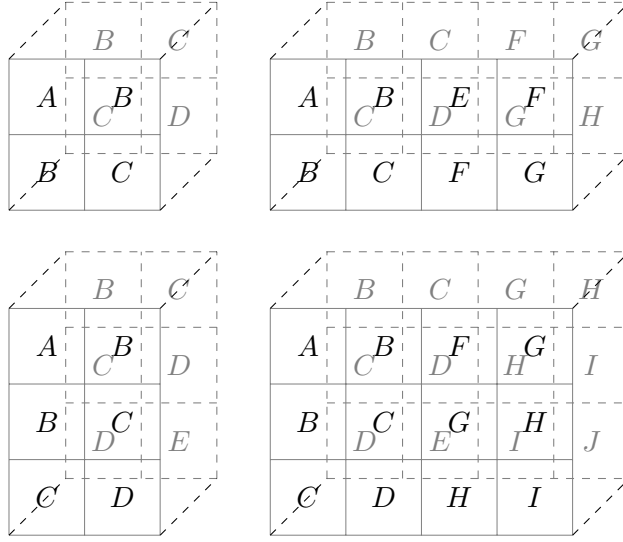
**Listing 3.** GLSL for evaluating a cubic curve encoded in a  $(2, 4)$  pixel 2d texture. Bilinear texture sampling used to evaluate the first two levels of the De Casteljau algorithm, then the process is continued both with the De Casteljau algorithm, as well as the Bernstein form of a linear Bezier curve (lerp).

### 3.3. Three Dimensional Textures

Using three dimensional textures (often referred to as volumetric textures) allows the texture interpolator to evaluate trilinear interpolation, which allows calculations of 3rd order (cubic) Bezier curves (Figure 6).

This allows us to either calculate a cubic bezier curve with a single texture read, or allows us to get the cubic level values needed to make a higher order curve.

Knowing that a cubic curve is just the linear interpolation of time  $t$  between the



**Figure 9.** *Top Left:* A cubic curve encoded in a  $(2, 2, 2)$  sized texture, decoded with a single trilinear texture sample. *Top Right:* Two piecewise cubic curves encoded in a  $(4, 2, 2)$  sized texture, decoded with a single trilinear texture sample. *Bottom Left:* A Quartic curve encoded in a  $(2, 3, 2)$  sized texture, decoded with two trilinear texture samples. *Bottom Right:* Two piecewise quartic curves encoded in a  $(4, 3, 2)$  sized texture, decoded with two trilinear texture samples.

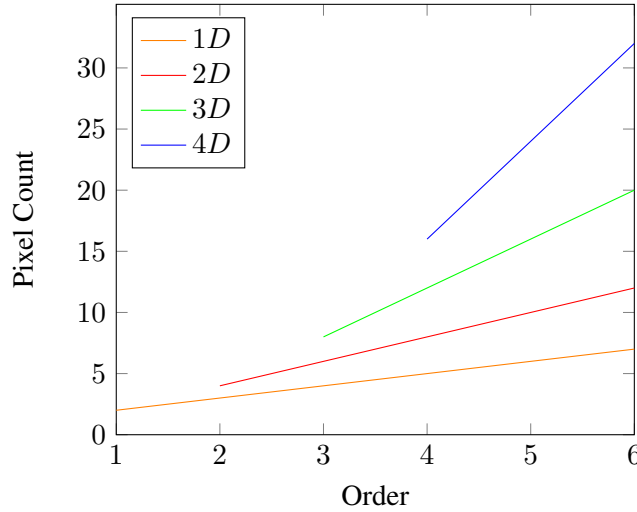
points on two quadratic curves at time  $t$ , we know that our three dimensional texture needs to be set up to be able to make this happen. With a  $2 \times 2 \times 2$  texture, the front  $2 \times 2$  pixels would encode a quadratic curve for  $P_0, P_1, P_2$  and the back  $2 \times 2$  pixels would encode a quadratic curve for  $P_1, P_2, P_3$ . Then, when we sample the texture, plugging  $t$  into  $u, v$  and  $w$ , the result will be that it will calculate the points on the quadratic curves  $P_0, P_1, P_2$  at time  $t$  and also  $P_1, P_2, P_3$  at time  $t$ , and then will interpolate across the  $w$  axis to give us the final result.

You can combine two cubic curves to make a quartic curve, or three cubic curves to make a quintic curve. If you were to put  $2 \times 2 \times 2$  cubic curves back to back, you would notice a redundancy like we saw in lower dimensionality textures. Instead of going that route though, we can just build on the redundancy removal from two dimensional textures.

The size of a three dimensional texture needed to encode a curve of degree  $N$  is  $(2, N - 1, 2)$ . If you wanted to store  $M$  piecewise curves, each of degree  $N$ , the formula becomes  $(2 * M, N - 1, 2)$  (Figure 9).

```
// 2x2x2 3d texture.
uniform sampler2D uSampler;
const vec3 c_textureSize = vec3(2.0, 2.0, 2.0);

vec4 CubicCurveFromTexture3D(in float t) {
```



**Figure 10.** Pixel count for curve order, with different texture dimensionalities.

```
return texture(uSampler, vec3(0.5 + t) / c_textureSize);
}
```

**Listing 4.** GLSL for evaluating a cubic curve encoded in a (2, 2, 2) pixel 3d texture. Trilinear texture sampling used to evaluate all three levels of the De Casteljau algorithm.

### 3.4. Summary of Texture Dimensionality

Increasing texture dimensionality allows you to evaluate the same order curve with fewer texture reads but comes at the cost of using more texture memory. We evaluated texture dimensions one, two and three, but the pattern continues for higher dimensions as well.

The table below is a comparison of texture dimensionality options.  $M$  is the number of piecewise curves being stored and  $N$  is the order of the curve. To get an idea of how dimensionality and order affects the size of the required textures, take a look at [Figure 10](#).

	Min. Order	# Samples	Dimensions	# Pixels
<b>1D</b>	1 (Linear)	$N$	$(M * (N + 1))$	$M * (N + 1) * 1$
<b>2D</b>	2 (Quadratic)	$N - 1$	$(2 * M, N)$	$M * (N + 0) * 2$
<b>3D</b>	3 (Cubic)	$N - 2$	$(2 * M, N - 1, 2)$	$M * (N - 1) * 4$
<b>4D</b>	4 (Quartic)	$N - 3$	$(2 * M, N - 2, 2, 2)$	$M * (N - 2) * 8$

## 4. Extensions

There are a few ways to extend this technique for more options.

#### 4.1. Combining Color Channels

If you encode Bezier curves into a texture which has multiple color channels - such as the standard four R,G,B,A - you can set up the textures such that once you read in the R,G,B,A, you can combine the channel values together such that you get a fewer number of curves per texture read, but that they are higher order.

For instance, if you had a 2x2 texture, you could encode the control points A,B,C into the red channel, and the control points B,C,D into the green channel. Then, after you did a texture read, you could do a lerp between the red and the green channel. Since the red and green channel each contain a quadratic curve, the result would be a cubic curve, defined by the control points A,B,C,D. However, you've used up two of the channels for a single curve, so are getting fewer curves per texture read, but they are higher order. The blue and the alpha channel could either be quadratic curves, or could also be encoded such that they could be combined together to provide another cubic curve.

You could also set it up such that R,G,B,A were meant to be combined together to make a quintic (degree 5) curve. You'd only get a single curve value per texture read, but it would be 3 degrees higher. You could combine the color channels either with De Casteljeaus algorithm, or with the cubic Bernstein Bezier form.

If you are combining  $M$  color channels, each of which are values from curves of order  $N$ , the resulting curve value will be order  $N+M-1$ . The intuitive explanation to this, is that each channel has the ability to add one more control point to the curve, regardless of what texture dimensionality you are working with.

An interesting side effect to combining color channels is that the accuracy of the resulting curve gets better. This is because you are doing some of the math in full precision floating point, instead of the limited precision of the fixed point math in the texture sampler.

#### 4.2. Piecewise Curves

As mentioned earlier, you can encode multiple Bezier curves end to end within textures of any dimensionality.

When you are in need of a curve with lots of control points, piecewise curves have several benefits over increasing the order of the curve.

The main benefit of piecewise curves are that they allow us to use lower order curves, which allows us to use lower dimensional textures, and have less computation in the shader programs.

Another benefit is that there can be discontinuities between curve segments, which is sometimes desired by artists and graphic designers.

B-splines can also be converted to piecewise Bezier curves using Boehm's algorithm, so even though this technique only explicitly deals with Bezier curves, B-splines could be converted to piecewise Bezier curves and could be used in that form.

Evaluating a piecewise curve with this technique is very simple. As an example, let's say that we have a texture that is 8x2, and encodes 4 quadratic Bezier curves end to end. The left most 2x2 pixels encode the quadratic curve used for time  $t \in [0, 0.25)$ . The next block of 2x2 pixels encode the quadratic curve used for time  $t \in [0.25, 0.5)$ . The third block encodes time  $t \in [0.5, 0.75)$ , and the last block encodes time  $t \in [0.75, 1.0]$ .

#### 4.3. Rational Bezier Curves

Rational Bezier curves are defined by the equation:

$$\mathbf{B}(t) = \frac{\sum_{i=0}^n \binom{n}{i} (1-t)^{n-i} t^i w_i \mathbf{P}_i}{\sum_{i=0}^n \binom{n}{i} (1-t)^{n-i} t^i w_i} \quad (7)$$

Looking closely, we can see that we essentially have one Bezier curve divided by another. The numerator has a weighting and a control point, while the denominator has only a weighting, but if we were to multiply the weighting into the control point in the numerator, we would essentially have one Bezier curve divided by another.

Using this technique, we can easily encode the numerator into one color channel, and the denominator into another.

If we wanted to get two rational Bezier curves per texture read, we could encode the two numerators into R and G, and the denominators into B and A.

Or, if we had three curves that all used the same weights, that would mean that the denominator was the same for all three curves. In this case, we could encode the three numerators into R,G,B and encode the denominator in A, which would give us three rational curves with a single texture read.

To compute the value of rational Bezier curves, you first take your texture samples, combining them as per normal if there were multiple taken (if needed), and then after that, you divide the numerators by the denominators to get the final values.

There is a limitation to encoding rational Bezier curves in this form due to the fact that texture values range from 0 to 1, but when using rational Bezier curves, weights greater than 1 are sometimes desired. The good news is that you can normalize the weights by dividing all weight values by the largest weight value, such that the largest weight value becomes 1. The result will be the same as if you had values greater than 1 in the weights.

Just as b-splines can be converted into piecewise Bezier curves, NURBS can also be converted into piecewise rational Bezier curves, again using Boehm's algorithm, which means these techniques can also apply to NURBS, if they are first preprocessed into rational Bezier curves.

#### 4.4. Multidimensional Bezier Curves

So far we have only talked about one dimensional - or explicit - bezier curves which take the form of  $y = f(x)$ . Multidimensional Bezier curves are evaluated separately per axis though, which means that you can encode information per axis in different color channels.

To name a few possibilities with a standard RGBA 2x2 texture, you could encode a fourth dimensional Bezier curve, or you could encode 2 two dimensional Bezier curves, or you could even encode a three dimensional rational Bezier curve using R,G,B for the numerator of each axis, and A as the denominator for all three curves.

### 5. Addressing Accuracy Issues

Even though there are accuracy limitations in modern GPU texture samplers, accuracy can be increased by doing more of the calculations in the shader program, which operate at full floating point precision.

A naive way to do this would be to use nearest neighbor texture sampling instead of linear, and do the linear interpolations yourself in the shader program. This would mean twice as many texture reads with a 1D texture, four times as many texture reads with a 2D texture, and eight times as many texture reads with a 3D texture! You would also be doing a lot more blending in the shader program, so your shader programs would become heavier weight. This solution is essentially doing software linear texture sampling.

A better approach that gives the same results would be to only do one nearest neighbor texture sample per control point in the texture, and then in the shader program, either use De Casteljeau's algorithm to combine them, or use the Bernstein form equation of Bezier curves. At this point, you are best off just using a 1D texture to store your control points in, since you don't need any of the built in interpolation.

These two techniques do all of the curve math in the shader program at full floating point precision and so are the highest quality, but are also the most computationally expensive.

Another way to address accuracy would be to do some of the work in the texture sampler, and some of the work in the shader program. This allows you to have a sliding scale for accuracy vs cost.

For instance, if you had a 2x2 image which encoded a quadratic Bezier curve, you might do two texture reads, letting the texture sampler do the  $x$  axis interpolation, and then you could linearly interpolate those results in the shader program in full precision floating point math. This would give you a result somewhere between full hardware and full software interpolation.

TODO: image to show hw vs sw vs hws

Also, if you want a higher order curve than your texture can provide with a single



read, forcing you to combine multiple curve points, combining those curve points in the shader program has the nice side benefit of increasing accuracy as well.

## 6. Limited Applications for Vector Graphics

This technique does have applications to vector graphics, but they are pretty limited admittedly. The main reason for the limitation is that this technique is good at calculating the point on a curve at time  $t$ , but when using curves in vector graphics situations, you usually do not know the time  $t$  that you want to know the curve point for, and it is usually difficult to find it.

TODO: mention loop and blinn's paper as an example of how it avoids the problem of finding the time 4.

However, there are certain special situations where you can easily get the time  $t$  value of the curve that you care about. Two of the most common situations for these are when rendering 1D explicit Bezier curves, and when rendering 1D explicit Bezier curves in polar form.

Explicit Bezier curves are curves which have scalar control points and are in the form of  $y = f(x)$ . For these curves, when you plug in a value for  $x$  you get a  $y$  value out. They are unable to make curves that have multiple  $y$  values for the same  $x$ , and the curves can't cross over themselves to make closed shapes. This does make them not as useful to vector graphics as 2D Bezier curves which are of the form  $(x, y) = f(t)$ , but they still do have some use cases. Rendering explicit bezier curves using this technique is very simple. If rendering a quad, you just choose either the  $x$  or  $y$  axis, normalize it from 0 to 1 and use that to calculate the other axis value by doing a texture sample. At this point, you can either choose a half space to color in, or you can calculate the gradient of the function (by using the built in `dFdx` style functions to get partial derivatives), and use that as an estimated distance to the curve.

TODO: picture of halfspace and gradient distance functions.

This idea can easily be extended to 1D Bezier curves rendered in polar form. If rendering a quad, you can use arc tangent to find the angle of the current pixel from the center, normalize that angle to 0 to 1, and then use that value to do a texture look up to calculate the value of the curve at that point. Once again you can then either choose a half space to color in, or you can use the gradient information to get an estimated distance to the curve, to draw a curved line.

TODO: do we need an image of polar form? maybe combine with picture above and reference the same thing

Another problem with using this technique for vector graphics, is that since the texture sampler is doing at least some of the calculations for us, we don't have access to the values of the control points, which can make some operations harder. Luckily, one property of Bezier curves is that affine transformations done to control points

is equivalent to affine transformations done to points on the curve. This means that we can do affine transformations on the curve points provided to us by the texture sampler, and it will be as if we had performed them on the control points themselves.

Lastly, in regards to vector graphics, this technique can be used to allow people to define custom color gradients. The benefit is that custom, non linear color gradients can be defined, and can be scaled up without decaying to the usual artifacts seen from linear texture sampling.

TODO: do we need a gradient (rect / polar?) image? TODO: need an image of the flag

## **7. Comparisons With Other Techniques**

TODO: real apples to apples comparisons with texture baking (can be any order!) as well as shader constants.

## **8. Performance Characteristics**

TODO: Show some real performance values. TODO: show difference between 3d texture doing cubic, vs 2d texture with 2 texture reads and shader program results. TODO: show results from multiple GPUs?

## **Future Work**

TODO: this

TODO: any play here with lobes and encoding curves in polar form?

## **Acknowledgements**

TODO: this

## **References**

TODO: this

## **Index of Supplemental Materials**

TODO: this

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Alan Wolfe, GPU Efficient Texture Based Bezier Curve Evaluation, *Journal of Computer Graphics Techniques (JCGT)*, vol. 3, no. 1, 1–1, 2014  
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