Hinging Hyperplane Regression

A Presentation by

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Introduction

- Hinging Hyperplanes (HHs) were proposed by Breiman (1993) in 1993 as versatile piece-wise linear models for regression, classification, and function approximation tasks. They offer flexibility and are particularly useful when dealing with complex data patterns, making them a valuable tool in machine learning and statistical modeling.

Introduction

- Hinging Hyperplanes (HHs) were proposed by Breiman (1993) in 1993 as versatile piece-wise linear models for regression, classification, and function approximation tasks. They offer flexibility and are particularly useful when dealing with complex data patterns, making them a valuable tool in machine learning and statistical modeling.
- There has been a large activity during the past years in the field of nonlinear function approximation. Many interesting results have been reported in connection with, for example the projection pursuit regression in neural network and the recent wavelets approach .These methods are closely related to the hinging hyperplane HH model investigated here.

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Setup and Definitions

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Hinge Function and Hinge

A *hinge function* $y = h(x)$,consists of two hyperplanes continuously joined together. Taking (intercept term),$x_o \equiv 1$ and using $\cdot$ to denote the inner product of two vectors, if the two hyperplanes are given by:

$$y = \beta^+ \cdot \boldsymbol{x} \qquad y = \beta^- \cdot \boldsymbol{x}$$

they are joined together on $\{\boldsymbol{x} : (\beta^+ - \beta^-) \cdot \boldsymbol{x} = 0\}$ and we refer to $\Delta = (\beta^+ - \beta^-)$, or any multiple of Δ, as the *hinge* for the function.

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Explicit form of the Hinge

The explicit form of the hinge function is either
$\max(\beta^+ \cdot \boldsymbol{x}, \beta^- \cdot \boldsymbol{x})$ *or* $\min(\beta^+ \cdot \boldsymbol{x}, \beta^- \cdot \boldsymbol{x})$

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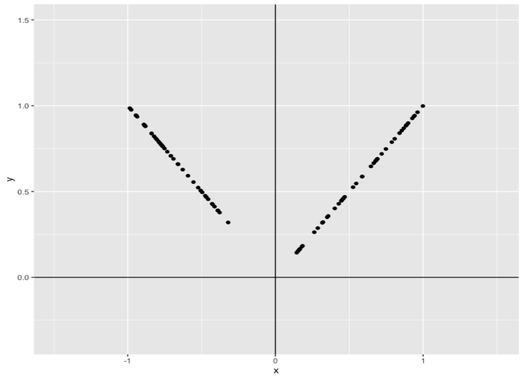
7. Now, let at k-th step we get $\Delta^{(k)}$ as our updated hinge.We find $cos(\Delta^{(k)}, \Delta^{(k-1)})$. If it becomes ≥0.99 we stop other wise we go to step 2 and continue our process.

Hinge Finding Algorithm

Example

Suppose in a 2 dimensional space a hinge function $y = h(\boldsymbol{x})$ exists and we have the observations $(y_1, x_1), \ldots, (y_n, x_n)$ from $y = |x|$. Upon plotting in 2D let the data set looks like:



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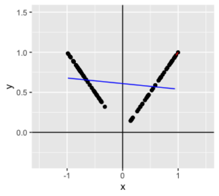

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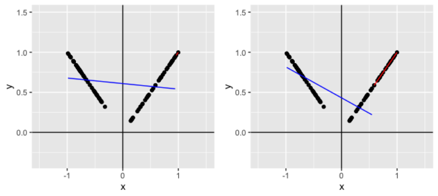

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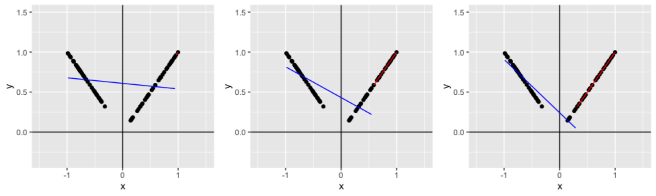

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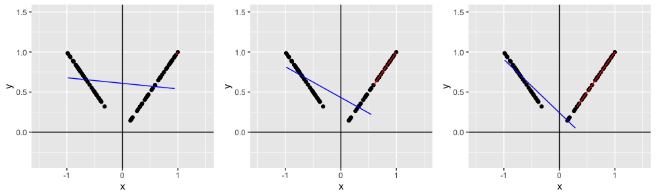

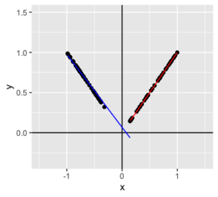

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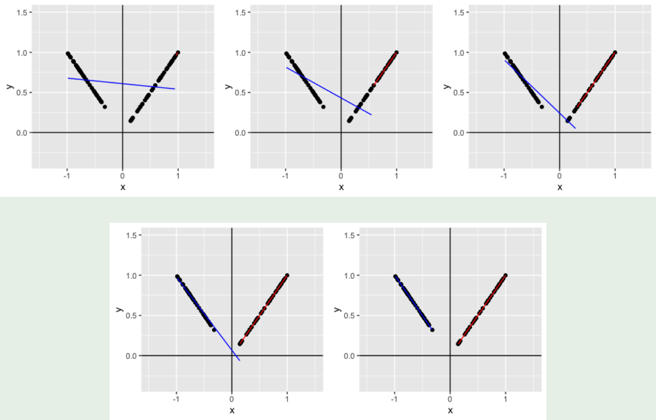

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- In such cases we try to get the relation between the response variable and the co-variate using regression.But instead of using simple linear models we approach for non-linear regression techniques
- Hinging Hyperplane is one of the choices for non-linear regression methods and it fits a piece-wise continuous function to the data

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- Now we can think any function as a addition of multiple hinge functions.
- We fit hinge function to get one corner point but in non-linear functions we may need multiple corner point to approximate the function
- In the following slides, we shall discuss the method to fit multiple hinge functions.

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- Refit h_2 again on the new updated residual data $\{\tilde{y}_{[1]}, \boldsymbol{x}\}$
- Update h_1 and h_2 in this process until there is no significant change in residual sum of squares in the iterations

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Algorithm for fitting K hinge functions

Now, we shall describe the algorithm to fit K many hinge functions to our data, assuming we have fitted $(K-1)$ many hinge functions to our data

1. We have residuals $\tilde{y}_{[1,2,\dots,(K-1)]} = f(x) - \sum_{i=1}^{K-1} h_i(x)$

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7. In this procedure we update all the K hinge functions by refitting

Noiseless Function Approximation

Algorithm for fitting K hinge functions

- Repeat steps 4-7 until there is no significant change in the RSS.

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Theorem:Convergence of Multiple Hinge Algorithm

Let P be any measure with compact support on E^M and $f(x)$ any sufficiently smooth function.Let $\tilde{f}(w)$ be the Fourier transformation of $f(x)$ If support of P contained in the sphere of radius R,and if

$$\int ||w||^2 |f(w)|\, dw = c < \infty,$$

then there are hinge $h_1, h_2 ...., h_K$ such that

$$\left|\left| f - \sum_{i=1}^{K} h_i \right|\right|^2 \leq \frac{(2R)^4 c^2}{K}$$

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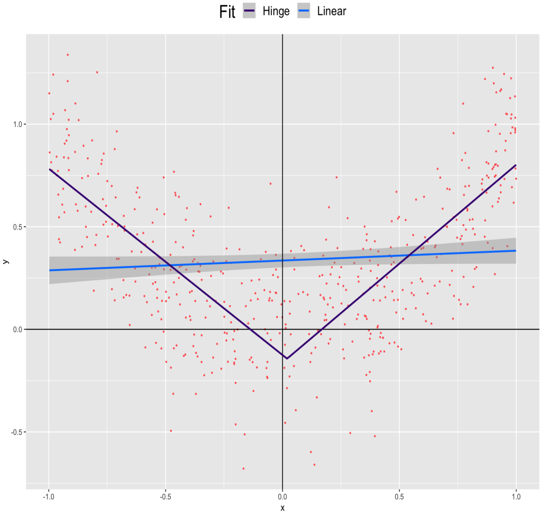
$y = x^2 + \epsilon$,$\epsilon \sim N(0, 1/4)$



- SLR solution:

$$\hat{y} = \begin{pmatrix} 0.335 \\ 0.047 \end{pmatrix}^T \cdot \boldsymbol{x}$$

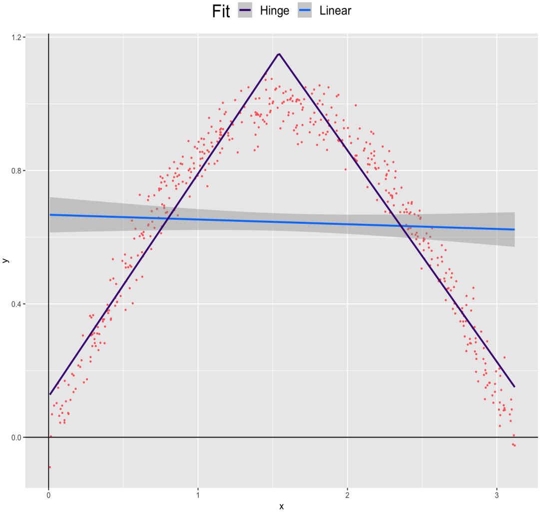
$RSS = 70.80669$

- HHR solution:

$$\hat{y} = \max \left\{ \begin{pmatrix} -0.125 \\ -0.908 \end{pmatrix}^T \cdot \boldsymbol{x}, \begin{pmatrix} -0.162 \\ 0.964 \end{pmatrix}^T \cdot \boldsymbol{x} \right\}$$

$RSS = 33.659$

$y = sin(x) + \epsilon \ , \epsilon \sim N(0, 1/4)$



- SLR solution:

$$\hat{y} = \begin{pmatrix} 0.667 \\ -0.014 \end{pmatrix}^T \cdot \boldsymbol{x}$$

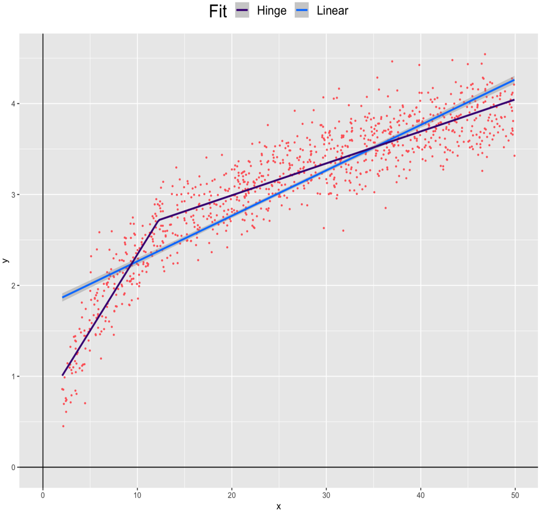
$RSS = 44.10121$

- HHR solution:

$$\hat{y} = \min \left\{ \begin{pmatrix} 0.122 \\ 0.668 \end{pmatrix}^T \cdot \boldsymbol{x}, \begin{pmatrix} 2.128 \\ -0.634 \end{pmatrix}^T \cdot \boldsymbol{x} \right\}$$

$RSS = 3.33$

$y = log(x) + \epsilon \ ,\epsilon \sim N(0, 1/4)$



- SLR solution:

$$\hat{y} = \begin{pmatrix} 1.764 \\ 0.050 \end{pmatrix}^T \cdot \boldsymbol{x}$$

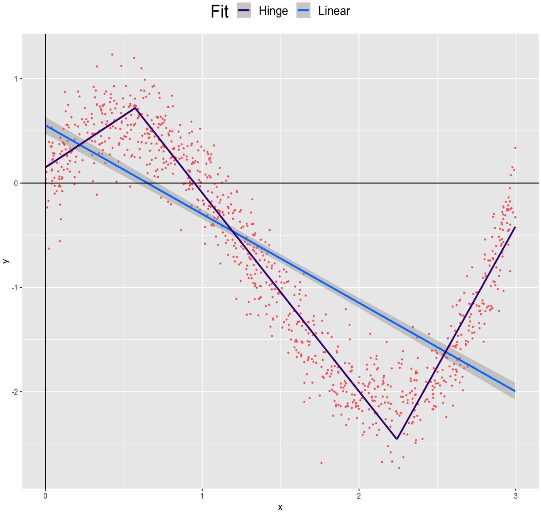
$RSS = 122.452$

- HHR solution:

$$\hat{y} = \min \left\{ \begin{pmatrix} 0.660 \\ 0.167 \end{pmatrix}^T \cdot \boldsymbol{x}, \begin{pmatrix} 2.287 \\ 0.035 \end{pmatrix}^T \cdot \boldsymbol{x} \right\}$$

$RSS = 66.72372$

$y = x(x-1)(x-3) + \epsilon$,$\epsilon \sim N(0, 1/4)$



- SLR solution:

$$\hat{y} = \begin{pmatrix} 0.5529 \\ -0.8506 \end{pmatrix}^T \cdot \boldsymbol{x}$$

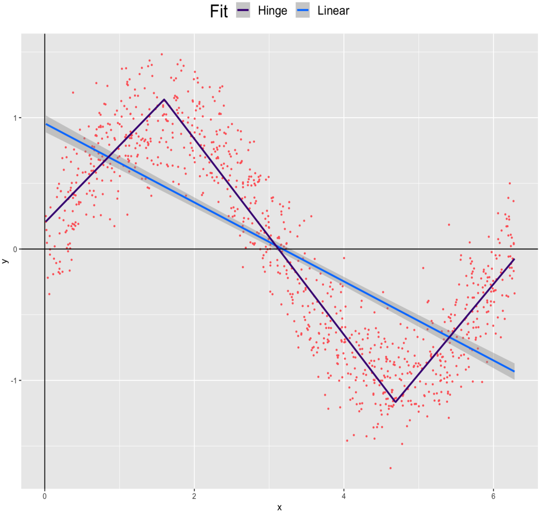
$RSS = 435.1966$

- HHR solution:

$$\hat{y} = \min\left\{ \begin{pmatrix} -0.538 \\ 1.772 \end{pmatrix}^T \cdot \boldsymbol{x}, \begin{pmatrix} 1.118 \\ -1.124 \end{pmatrix}^T \cdot \boldsymbol{x} \right\}$$
$$+ \max\left\{ \begin{pmatrix} 0.689 \\ -0.778 \end{pmatrix}^T \cdot \boldsymbol{x}, \begin{pmatrix} -9.619 \\ 3.821 \end{pmatrix}^T \cdot \boldsymbol{x} \right\}$$

$RSS = 73.407$

$y = sin(x) + \epsilon \ , \epsilon \sim N(0, 1/4), x \in [0, 2\pi]$



- SLR solution:

$$\hat{y} = \begin{pmatrix} 0.956 \\ -0.301 \end{pmatrix}^T \cdot \boldsymbol{x}$$

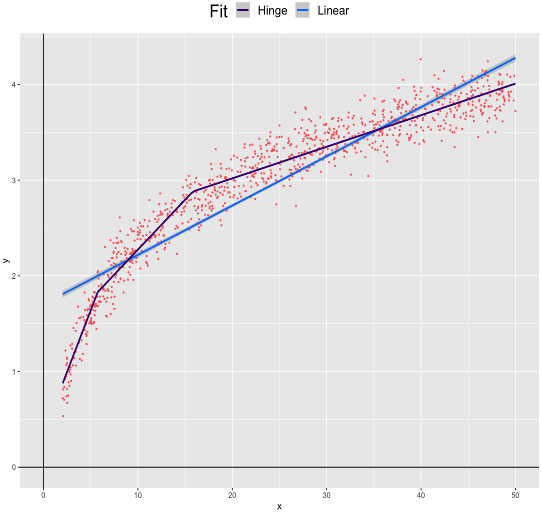
$RSS = 259.7786$

- HHR solution:

$$\hat{y} = \min\left\{ \begin{pmatrix} -0.206 \\ 0.827 \end{pmatrix}^T \cdot \boldsymbol{x}, \begin{pmatrix} 1.867 \\ -0.505 \end{pmatrix}^T \cdot \boldsymbol{x} \right\} + \max\left\{ \begin{pmatrix} 0.462 \\ -0.240 \end{pmatrix}^T \cdot \boldsymbol{x}, \begin{pmatrix} -6.250 \\ 1.191 \end{pmatrix}^T \cdot \boldsymbol{x} \right\}$$

$RSS = 70.15768$

$y = \log(x) + \epsilon$,$\epsilon \sim N(0, 1/6)$



- SLR solution:

$$\hat{y} = \begin{pmatrix} 0.317 \\ 0.20 \end{pmatrix}^T \cdot \boldsymbol{x}$$

$RSS = 71.693$

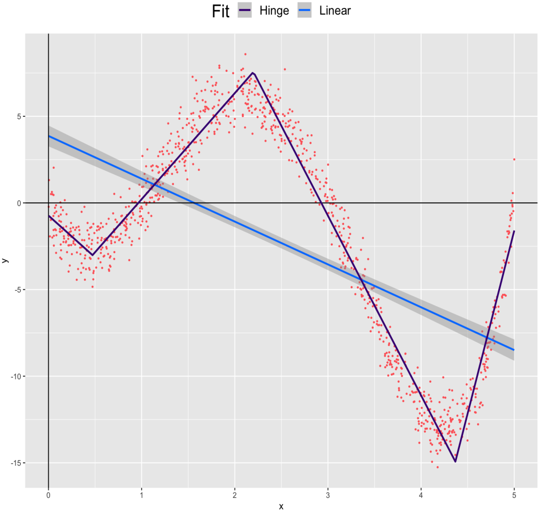
- HHR solution:

$$\hat{y} = \min\left\{ \begin{pmatrix} 0.926 \\ 0.201 \end{pmatrix}^T \cdot \boldsymbol{x}, \begin{pmatrix} 1.803 \\ 0.048 \end{pmatrix}^T \cdot \boldsymbol{x} \right\}$$

$$+ \min\left\{ \begin{pmatrix} 0.554 \\ -0.015 \end{pmatrix}^T \cdot \boldsymbol{x}, \begin{pmatrix} -0.572 \\ 0.055 \end{pmatrix}^T \cdot \boldsymbol{x} \right\}$$

$RSS = 29.08094$

$y = x(x-1)(x-3)(x-5) + \epsilon \ , \epsilon \sim N(0,1)$



- SLR solution:

$$\hat{y} = \begin{pmatrix} 3.866 \\ -2.472 \end{pmatrix}^T \cdot \boldsymbol{x}$$

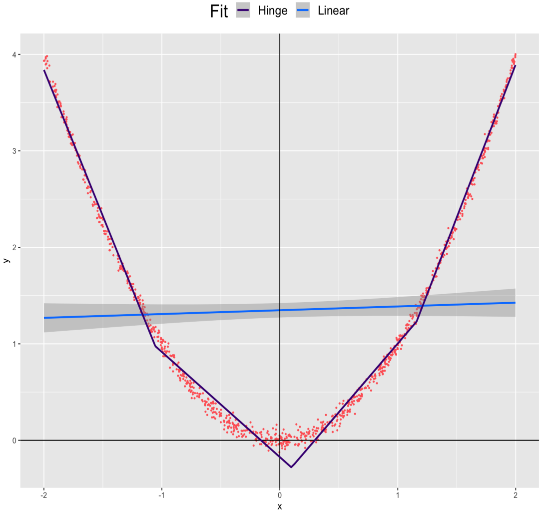
$RSS = 24600.87$

- HHR solution:

$$\hat{y} = \min\left\{ \begin{pmatrix} 29.374 \\ -9.355 \end{pmatrix}^T \cdot \boldsymbol{x}, \begin{pmatrix} -6.916 \\ 7.142 \end{pmatrix}^T \cdot \boldsymbol{x} \right\}$$
$$+ \max\left\{ \begin{pmatrix} 2.011 \\ -1.313 \end{pmatrix}^T \cdot \boldsymbol{x}, \begin{pmatrix} -135.478 \\ 30.163 \end{pmatrix}^T \cdot \boldsymbol{x} \right\}$$
$$+ \max\left\{ \begin{pmatrix} 4.174 \\ -10.654 \end{pmatrix}^T \cdot \boldsymbol{x}, \begin{pmatrix} -1.022 \\ 0.294 \end{pmatrix}^T \cdot \boldsymbol{x} \right\}$$

$RSS = 1439.839$

$y = x^2 + \epsilon, \epsilon \sim N(0, 1/20)$



- SLR solution:

$$\hat{y} = \begin{pmatrix} 1.348 \\ 0.039 \end{pmatrix}^T \cdot \boldsymbol{x}$$

$RSS = 1447.701$

- HHR solution:

$$\hat{y} = \max \left\{ \begin{pmatrix} -0.047 \\ 1.369 \end{pmatrix}^T \cdot \boldsymbol{x}, \begin{pmatrix} 0.202 \\ -1.146 \end{pmatrix}^T \cdot \boldsymbol{x} \right\}$$
$$+ \max \left\{ \begin{pmatrix} -0.158 \\ -0.209 \end{pmatrix}^T \cdot \boldsymbol{x}, \begin{pmatrix} -2.183 \\ 1.537 \end{pmatrix}^T \cdot \boldsymbol{x} \right\}$$
$$+ \max \left\{ \begin{pmatrix} -0.217 \\ 0.265 \end{pmatrix}^T \cdot \boldsymbol{x}, \begin{pmatrix} -2.268 \\ -1.677 \end{pmatrix}^T \cdot \boldsymbol{x} \right\}$$

$RSS = 10.17562$

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- We increase number of hinges and we notice at which K there is no significant change in PE_{GCV} further. We choose that first K as our total number of hinges.

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Derivation of Algorithm

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$$V_N(\boldsymbol{\beta}) = \frac{1}{2}\sum_{i=1}^{N}(y_i - h(x_i;\boldsymbol{\beta}))^2 \qquad \text{where,} \quad \boldsymbol{\beta} = \begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix}$$

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- We want to get an estimate of $\boldsymbol{\beta}$ such that

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Finding MLE

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Finding MLE

- We try to find the MLE of $\boldsymbol{\beta}$ using Newton-Raphson Method now
- So we will need to calculate the first derivative and the hessian of V_N w.r.t $\boldsymbol{\beta}$

The first derivative

$$\nabla V_N = \begin{bmatrix} \frac{\partial}{\partial \beta_+} V_N \\ \frac{\partial}{\partial \beta_-} V_N \end{bmatrix} = \begin{bmatrix} -\sum_{\boldsymbol{x}_i \in S_+} \boldsymbol{x}_i (y_i - \boldsymbol{x}_i^T \beta_+) \\ -\sum_{\boldsymbol{x}_i \in S_-} \boldsymbol{x}_i (y_i - \boldsymbol{x}_i^T \beta_-) \end{bmatrix}$$

The Hessian

$$\nabla^2 V_N = \begin{bmatrix} \sum_{\boldsymbol{x}_i \in S_+} \boldsymbol{x}_i \boldsymbol{x}_i^T & 0 \\ 0 & \sum_{\boldsymbol{x}_i \in S_-} \boldsymbol{x}_i \boldsymbol{x}_i^T \end{bmatrix}$$

Newton-Raphson updating step

- Now, we will update $\boldsymbol{\beta}$ using Newton-Raphson method. Let at k th step we have $\boldsymbol{\beta}_k$

Update

$$\boldsymbol{\beta}^{(k+1)} = \boldsymbol{\beta}^{(k)} - (\nabla^2 V_N)^{-1} \nabla V_N$$

$$= \boldsymbol{\beta}^{(k)} + \begin{bmatrix} (\sum_{\boldsymbol{x}_i \in S_+} \boldsymbol{x}_i \boldsymbol{x}_i^T)^{-1} \sum_{\boldsymbol{x}_i \in S_+} \boldsymbol{x}_i y_i - \boldsymbol{\beta}_+^{(k)} \\ (\sum_{\boldsymbol{x}_i \in S_-} \boldsymbol{x}_i \boldsymbol{x}_i^T)^{-1} \sum_{\boldsymbol{x}_i \in S_-} \boldsymbol{x}_i y_i - \boldsymbol{\beta}_-^{(k)} \end{bmatrix}$$

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$$\boldsymbol{\beta}^{(k+1)} = \boldsymbol{\beta}^{(k)} + (\boldsymbol{\beta}_{Br}^{(k)} - \boldsymbol{\beta}^{(k)})$$

- Notice that using Newton-Raphson we get the same update for $\boldsymbol{\beta}^{(k)}$ as we got in case of HFA

Damped Newton-Raphson to update

In damped Newton-Raphson we modify the step length and take relatively small step size

$$\boldsymbol{\beta}^{(k+1)} = \boldsymbol{\beta}^{(k)} + \mu(\boldsymbol{\beta}_{Br}^{(k)} - \boldsymbol{\beta}^{(k)})$$

we will take μ to be $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$..... and make step size small so that the method converge.

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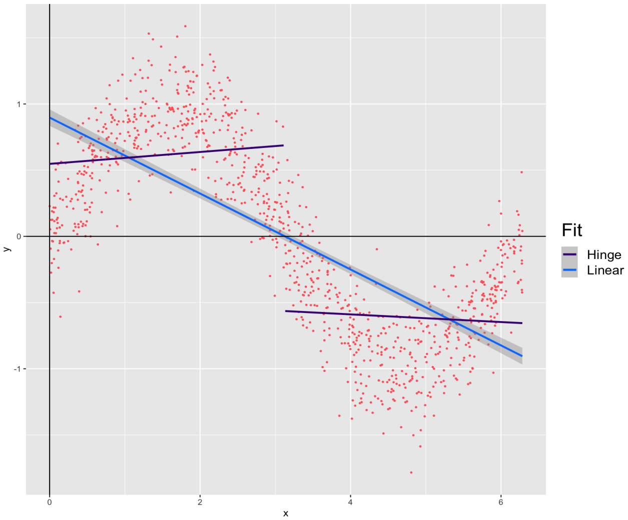
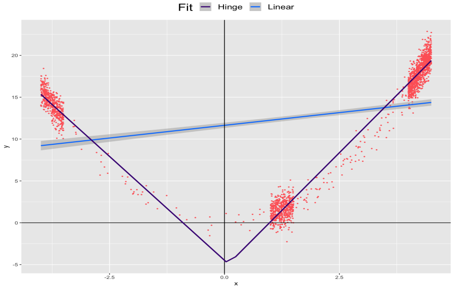
Futility:Convergence problem with any initial hinge

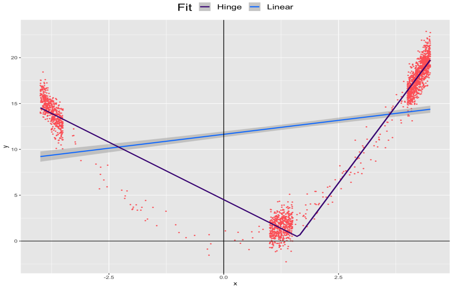


Example

- Here we have simulated data from $\{x, sin(x) + \epsilon\}$
- If we take our initial hinge at $x = \pi$, this HFA will never converge
- Breiman's algorithm of finding hinge fails here

Futility:Unstable Hinge Positions





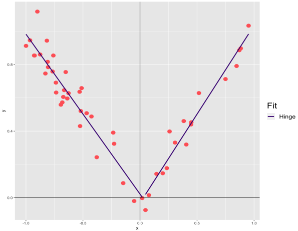
Example

- Suppose we have data from $\{x, x^2 + \epsilon\}$ and it is non-uniform over the intervals
- Then we get different convergent points. Hence the hinge becomes unstable.

Futility:Sensitive to outliers

Example

We have taken data from $\{x, |x| + \epsilon\}$.We have shown how it effects when there are outliers.



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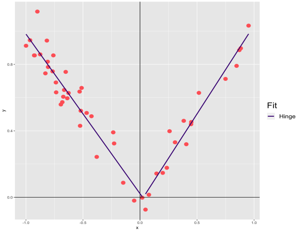

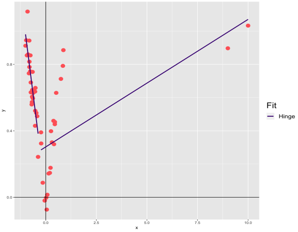

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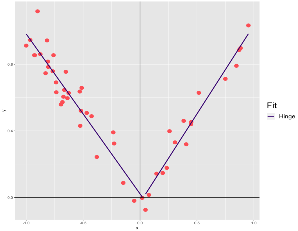

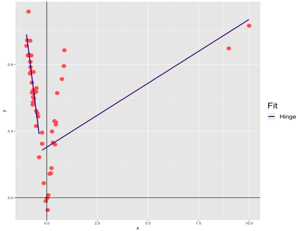

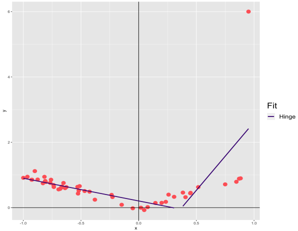

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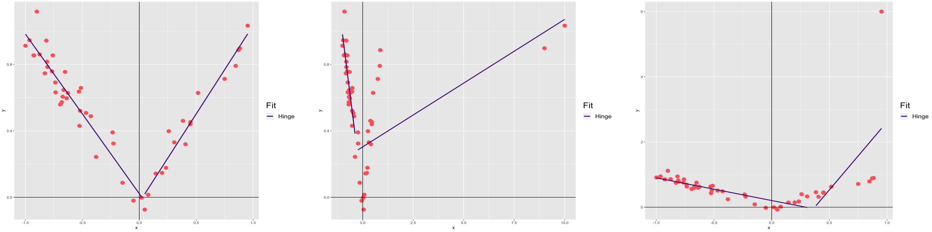

Reason

We know that SLR is sensitive to outliers. As we are doing SLR in the partitioned sets in HHR , HHR becomes sensitive to outliers

Futility

- This method is computationally intensive
- Breiman's algorithm may lead to singularity problem on intermediate steps of HHR and may not converge

References

L. Breiman. Hinging hyperplanes for regression, classification, and function approximation. *IEEE Trans. Inf. Theory*, 39:999–1013, 1993. URL https://api.semanticscholar.org/CorpusID:12319558.

Thank You