$B.B.1. Simplified\ Master\ Theorem$

The master theorem is used for solving divide − and − conquer recurrence relations.

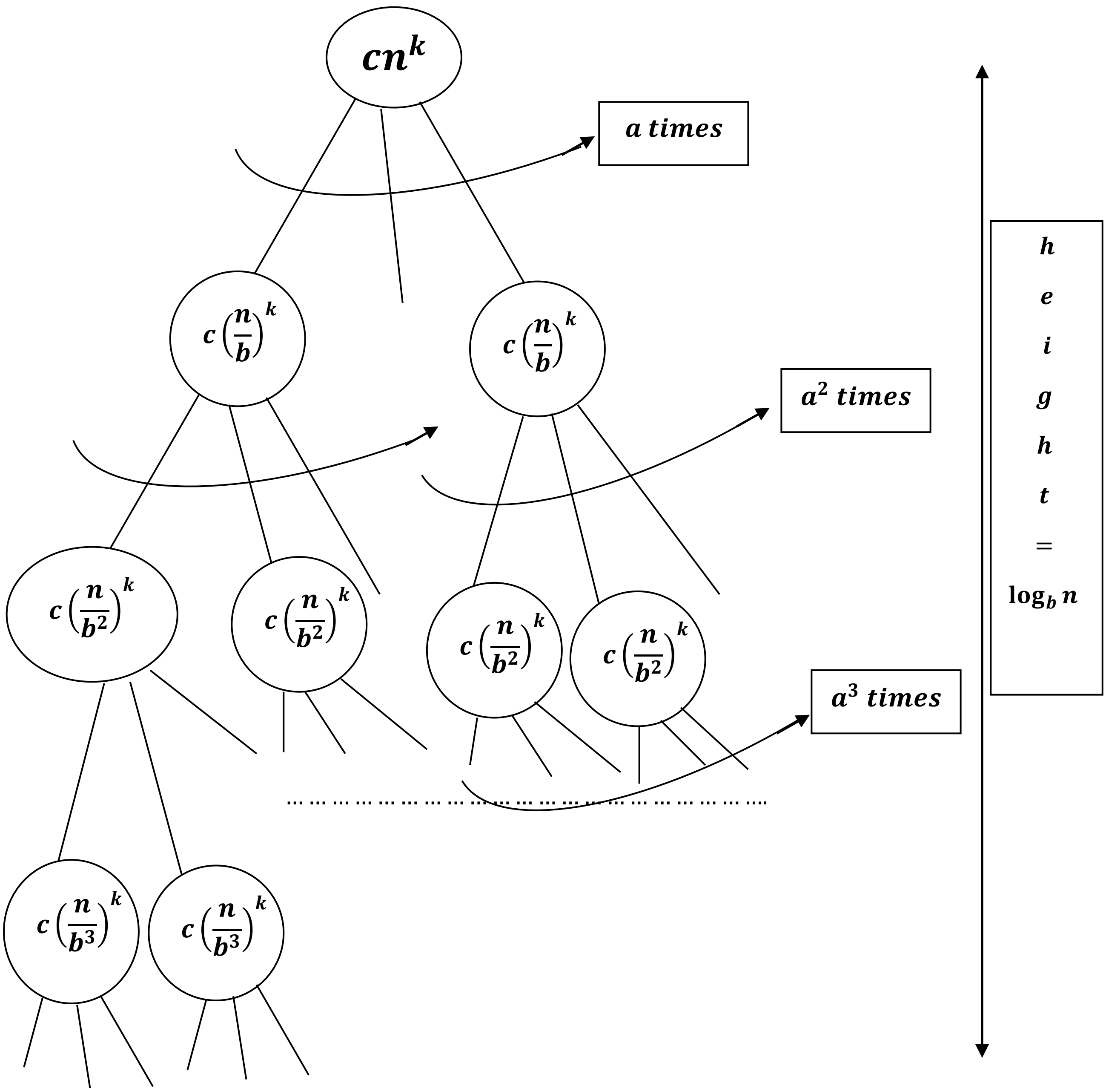
While the master theorem doesnot solve all types of divide − and − conquer recurrences ,it can solve the majority of recurrene equation.

At preliminary stage ,we are provided with the formula:

$$T(n) = aT\left(\frac{n}{b}\right) + \mathrm{cn^k}.$$

This implies that the problem is divided into \`a\` subproblems at every level and every subproblems is of the size $\frac{cn^k}{b^k}$.

The total cost of the recurrence tree is calculated by adding the costs of all the levels .



cn^k
$a\ times$
$c\left(\frac{n}{b}\right)^k$
$c\left(\frac{n}{b}\right)^k$
$a^2\ times$
$c\left(\frac{n}{b^2}\right)^k$
$c\left(\frac{n}{b^2}\right)^k$
$c\left(\frac{n}{b^2}\right)^k$
$c\left(\frac{n}{b^2}\right)^k$
$a^3\ times$
$c\left(\frac{n}{b^3}\right)^k$
$c\left(\frac{n}{b^3}\right)^k$
$height = \log_b n$

$\boldsymbol{Level}$	$\boldsymbol{No. of\ problems}$	$\boldsymbol{Problem\ Size}$	$\boldsymbol{Work\ done = Problem\ Size \times No. of\ Problems}$
$\mathbf{0}$	$\mathbf{1}$	$\boldsymbol{cn^k}$	$\boldsymbol{1 \times cn^k = cn^k}$
$\mathbf{1}$	$\boldsymbol{a}$	$\boldsymbol{\frac{cn^k}{b^k}}$	$\boldsymbol{a \times \left(\frac{cn^k}{b^k}\right)}$
$\mathbf{2}$	$\boldsymbol{a^2}$	$\boldsymbol{c\left(\frac{n}{b^2}\right)^k}$	$\boldsymbol{a^2 \times c\left(\frac{n}{b^2}\right)^k}$
. .	. .	. .	. .
$\boldsymbol{n-1}$	$\boldsymbol{a^k}$	$\boldsymbol{c\left(\frac{n}{b^k}\right)^k}$	$\boldsymbol{a^k \times c\left(\frac{n}{b^k}\right)^k}$
$\boldsymbol{k = \log_b n}$	$\boldsymbol{a^{\log_b n}}$	$\boldsymbol{\left(c\left(\frac{n}{b^k}\right)\right)^{\log_b n}}$	$\boldsymbol{a^{\log_b n} \times \left(c\left(\frac{n}{b^k}\right)\right)^{\log_b n}}$

$$\boldsymbol{Therefore\ , the\ total\ cost\ of\ the\ \ recurrence\ tree\ can\ be\ given\ as\ follows}:$$

$$T(n) = cn^k + \ a \times c\left(\frac{n}{b}\right)^k + a^2 \times c\left(\frac{n}{b^2}\right)^k + \cdots$$

$$+a^{\log_b n} \times \left(c\left(\frac{n}{b^k}\right)\right)^{\log_b n}$$

$$\Rightarrow cn^k\left(1 + \ a \times \left(\frac{1}{b^k}\right) + a^2 \times \frac{1}{(b^2)^k} + \cdots + a^{\log_b n} \times \left(\frac{1}{\left(b^k\right)^{\log_b n}}\right)\right)$$

$$\Rightarrow cn^k\left(1 + \ \left(\frac{a}{b^k}\right) + \frac{a^2}{(b^2)^k} + \cdots + \left(\frac{a^{\log_b n}}{\left(b^k\right)^{\log_b n}}\right)\right)$$

$$\boldsymbol{The\ same\ expression\ can\ be\ written\ as}:$$

$$\Rightarrow cn^k\left(1 + \ \left(\frac{a}{b^k}\right) + \left(\frac{a}{b^k}\right)^2 + \cdots + \left(\frac{a}{b^k}\right)^{\log_b n}\right)$$

$$\boldsymbol{if\ d} = \left(\frac{a}{b^k}\right), \boldsymbol{then\ we\ can\ rewrite\ the\ equation\ as}:$$

$$T(n) = cn^k\left(1 + (d) + (d)^2 + \cdots + (d)^{\log_b n}\right)$$

$$Case\ 1: \rightarrow \frac{a}{b^k} < 1$$

$When\ d < 1\ i.e. \frac{a}{b^k} < 1\ \ hence,$

<u>$To\ prove\ the\ geometric\ series\ is\ infinite:$</u>

$$\lim_{n \to \infty}\left(1 + d + d^2 + d^3 + \cdots + d^{\log_b n}\right)$$

$As\ this\ is\ a\ geometric\ series\ which\ yields\ to:$

$$= \lim_{n \to \infty}\left(\frac{d^{n+1} - 1}{d - 1}\right)$$

$$= \frac{1}{d-1} \times \lim_{n \to \infty} d^{n+1} - \lim_{n \to \infty} 1$$

$$= \frac{1}{d-1} \times (\infty - 1)$$

$$= \frac{1}{d-1} \times (\infty - 1)$$

$$= \frac{1}{d-1} \times \infty$$
$$= \infty$$

The series is infinite geometric series , as n goes infinite:

The series can be expressed as a common ratio `d` and a is **1** *, Hence by formula of infinite geometric series we we get*:

$$S = \frac{a}{1-r} = \frac{1}{1-d}.$$

<u>To prove the series is convergent</u>:

$$\lim_{n\to\infty}\left(\frac{1}{1-d}\right) = \frac{1}{1-0} = 1$$ *, hence series is covergent i.e. finite and unique.*

As the condition is $d < 1$ *, the series becomes convergent i.e. finite and unique i.e.*:

$$1 + d + d^2 + \cdots = \frac{1}{1-d}$$

Even if the sequence goes infinite , one can observe that the

$sequence\ reduces\ to\ constant\ factors\ only, when\ d < 1.$

$$Therefore : cn^k\left[\frac{1}{1-d}\right] = \frac{cn^k}{1-d} = \Theta\left(\frac{cn^k}{1-d}\right)$$

$$= \left(\frac{1}{1-d}\right) \times \Theta(\mathrm{cn^k}) = \Theta(\mathrm{cn^k}) = \Theta(\mathrm{n^k}).$$

$$Case\ 2: \rightarrow \frac{a}{b^k} = 1$$

$When\ d = 1, this\ implies\ a = b^k.\ Therefore\ we\ will\ have:$

$$T(n) = cn^k(1 + (d) + (d)^2 + \cdots + (d)^{\log_b n})$$

$Substuting\ d\ with\ 1\ we\ get:$

$$T(n) = cn^k\left(1 + (1) + (1)^2 + \cdots + (1)^{\log_b n}\right)$$

$$T(n) = cn^k(1 + (1 + 1 + \cdots + \log_b n\ times))$$

$$T(n) = cn^k(1 + 1 \times \log_b n)$$

$$T(n) = cn^k(1 + \log_b n)$$

$$T(n) = \left(cn^k + cn^k \times \log_b n\right)$$

$Here\ dominant\ term\ is\ cn^k \times \log_b n\ ,hence\ result\ is$:

$\Theta\left(n^k \log_b n\right)\ or\ \Theta\left(n^k \log n\right)$.

$$Case\ 3: \rightarrow\ \frac{a}{b^k} > 1$$

$When\ d \neq 1, this\ implies\ that\ \frac{a}{b^k} > 1\ , that\ is\ a > b^k, then$:

$$T(n) = cn^k(1 + (d) + (d)^2 + \cdots + (d)^{\log_b n})$$

$We\ can\ write\ this\ as$:

$$T(n) = cn^k \times d^{\log_b n} \underbrace{\left(\frac{1}{d^{\log_b n}} + \cdots + \frac{1}{d} + 1\right)}_{\boxed{Constant}}$$

$i.e., \left(\frac{1}{d^{\log_b n}} + \cdots + \frac{1}{d} + 1\right)$ *will lead to constant , as the series converges to a finite , as n tends to inifinity , each term in the summation tends to* 0 *when* $d > 1$.

If n tends to infinity , the terms , $\frac{1}{d^{\log_b n}}, \frac{1}{d^{\log_b n-1}}$, ... *etc. will become very small* .

We can write it as:

$$\frac{1}{d^{\log_b n}} + \frac{1}{d^{\log_b n-1}} + \cdots + \frac{1}{d} + 1$$

$$= \left(\frac{1}{d}\right)^{\log_b n} + \left(\frac{1}{d}\right)^{\log_b n-1} + \cdots + \left(\frac{1}{d}\right)^{1} + 1$$

Now lets consider common ratio of the series r, which is equal to $\frac{1}{d}$.

Since $0 < d < 1$ *and n is infinite, we can apply the formula for the sum of an geometric series*:

$$S = \frac{a}{1 - r}$$

$\boldsymbol{Substuting\ the\ values\ of\ a = \left(\frac{1}{d}\right)^{\log_b n}\quad and\ r = \frac{1}{d}\ into\ the}$

$\boldsymbol{formula\ for\ the\ sum\ of\ an\ infinite\ geometric\ series}$:

$$S = \frac{\left(\frac{1}{d}\right)^{\log_b n}}{1 - d}$$

$$S = \frac{\left(\frac{1}{d}\right)^{\log_b n}}{1 - \frac{1}{d}}$$

$$S = \frac{\left(\frac{1}{d}\right)^{\log_b n}}{\frac{d-1}{d}}$$

$$S = \left(\frac{1}{d}\right)^{\log_b n} \times \frac{d}{(d-1)}$$

$\boldsymbol{As\ n\ tends\ to\ inifinity: \left(\frac{1}{d}\right)^{\log_b n}\quad approaches\ to\ zero,}$

$\boldsymbol{since\ any\ positive\ base\ raised\ to\ a\ negative\ exponent}$

$\boldsymbol{tends\ to\ zero.\ i.e. Suppose\ take\ an\ example}$:

$$\left(\frac{1}{10}\right)^{\log_b 100},\ then\ \log_{10} 100 = 2$$

$$\left(\frac{1}{10}\right)^{\log_b 100} = \left(\frac{1}{10}\right)^{2} = \frac{1}{100}\ , and\ the\ value\ is$$

$\boldsymbol{less\ than\ one, hence\ exponent\ is\ negative\ i.e.\ 10^{-2}}$.

$**** \boldsymbol{Hence, the\ sum\ of\ series\ converges\ to\ a}$

$\boldsymbol{constant\ value.}\ \ ***$

$\boldsymbol{Note}$: $\boldsymbol{The\ above\ expression\ can\ also\ be\ represented\ as}$:

$$S = \frac{1}{d^{\log_b n}} \times \frac{d}{(d-1)} = \frac{1}{n^{\log_b d}} \times \frac{d}{(d-1)}\ also\ as:$$

$$S = \frac{1}{n^{\log_b d} \times \frac{(d-1)}{d}}$$

$\boldsymbol{Also\ we\ can\ write}$:

$As\ n\ tends\ to\ infinity, term\ \frac{1}{n^{\log_b d}}$ $approaches\ to\ zero$,

$since\ any\ positive\ base\ raised\ to\ a\ negative\ exponent$

$tends\ to\ zero. Therefore, the\ sum\ of\ series\ converges\ to\ a$

$constant\ value.$

$If\ we\ want\ an\ convergence\ test:$

$Convergence\ Test$

$$\frac{1}{d^{\log_b n}} + \frac{1}{d^{\log_b n-1}} + \cdots + \frac{1}{d} + 1$$

$Lets\ consider\ the\ series: a_n = \frac{1}{d^{\log_b n}}$ $, to\ apply\ the\ limit$

$comparison\ test, we\ need\ to\ find\ another\ series\ b_n$,

$whose\ convergence\ behaviour\ is\ known.$

$Lets\ choose\ b_n = \frac{1}{n^{\in}}$ $, where\ \epsilon\ is\ a\ positive\ constant.$

$Now\ we\ will\ compare\ the\ series\ a_n\ and\ b_n, by\ taking$

$the\ limit\ as\ n\ tends\ to\ infinity:$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{1}{d^{\log_b n}}\right)}{\frac{1}{n^{\in}}}$$

$$\boldsymbol{To\ simplify\ this\ expression, lets\ rewrite\ \frac{1}{d^{\log_b n}}\ as\ \frac{1}{n^{\log_b d}}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n^{\log_b d}}}{\frac{1}{n^{\in}}}$$

$$= \lim_{n\to\infty} \frac{n^{\in}}{n^{\log_b d}}$$

$\boldsymbol{Since\ b > 1\ and\ d > 1\ , we\ know\ that\ \log_b d > 0}$.

$\boldsymbol{Therefore\ , as\ `n`\ tends\ to\ inifinity\ , n^{\log_b d}\ grows\ faster}$

$\boldsymbol{than\ n^{\in}\ for\ any\ positive\ \in}$.

$\boldsymbol{As\ a\ result\ the\ limit}$: $\displaystyle \lim_{n\to\infty} \frac{n^{\in}}{n^{\log_b d}}$ $\boldsymbol{is\ equal\ to\ 0\ for\ any}$

$\boldsymbol{positive\ \in. This\ implies\ that\ a_n\ and\ b_n\ have\ the\ same}$

$\boldsymbol{convergence\ behaviour}$.

$\boldsymbol{Since\ the\ series}$: $\displaystyle \sum_{n=1}^{\infty} b_n$ $\boldsymbol{with\ b_n = \frac{1}{n^{\in}}\ converges\ \ \in > 1,}$

$\boldsymbol{We\ can\ conclude\ that\ the\ series}$ $\displaystyle \sum_{n=1}^{\infty} a_n$ $\boldsymbol{also\ converges.}$

$$\textit{Therefore the series}: \frac{1}{d^{\log_b n}} + \frac{1}{d^{\log_b n-1}} + \cdots + \frac{1}{d} + 1$$

converges for $d > 1$ *and* $b > 1$.

Therefore keeping out the constant:

$\left(\frac{1}{d^{\log_b n}} + \cdots + \frac{1}{d} + 1\right)$ *generated from*: $T(n) = cn^k \times d^{\log_b n}$,

as it doesnot matter in time complexity, we continue:

$$T(n) = cn^k \times d^{\log_b n}$$

$$or, T(n) = cn^k \times \left(\frac{a}{b^k}\right)^{\log_b n}$$

$$= cn^k \times \left(\frac{a^{\log_b n}}{b^{k \log_b n}}\right)$$

$$= cn^k \times \left(\frac{a^{\log_b n}}{n^k}\right) \left[\, b^{k \log_b n} = n^{k \log_b b}\right]$$

$$= cn^k \times \left(\frac{a^{\log_b n}}{n^k}\right) \left[\, b^{k \log_b n} = n^{k \log_b b}\right]$$

$$= c \times a^{\log_b n}$$

$\in \ \boldsymbol{\Theta}\left(\boldsymbol{n}^{\log_b a}\right) \left[\boldsymbol{as\ a^{\log_b n}\ can\ also\ be\ written\ as\ n^{\log_b a}}\right].$

$\boldsymbol{Thus\ we\ get\ a\ simplified\ master\ theorem\ as\ follows}$:

$\boldsymbol{Simplified\ Master\ Theorem}$:

$\boldsymbol{Let\ the\ time\ complexity\ function\ T(n)\ be\ a\ positive\ and\ eventually\ a\ non-decreasing\ function\ of\ the\ following\ form}$:

$$\boldsymbol{T(n) = aT\left(\frac{n}{b}\right) + cn^k}$$
$$\boldsymbol{T(1) = d}$$

$\boldsymbol{Here\ a, d, b, k\ are\ all\ constants}$.

$\boldsymbol{Here\ b \geq 2, k \geq 0, a > 0, c > 0\ and\ d \geq 0. The\ solution\ for\ the\ recurrence\ equation\ is\ given\ as\ follows}$:

$\boldsymbol{Case\ 1: T(n) \in \Theta\left(n^k\right)\ if\ a < b^k}$

$\boldsymbol{Case\ 2: T(n) \in \Theta\left(n^k \log n\right)\ if\ a = b^k}$

$\boldsymbol{Case\ 3: T(n) \in \Theta\left(n^{\log_b a}\right)\ if\ a > b^k}$

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