

Polynomial Reduction – Solving Non – Homogeneous Equation – Example

***Solve the recurrence equation $t_n - 3t_{n-1} = n - 1$
for $n = 0$ subject to the following initial
conditions: $t_0 = 0$; $t_1 = 1$; $t_2 = 2$.***

Solution

***1. Obtain the characteristic equation for the corresponding
homogeneous part of the recurrence equation as
follows:***

$$t_n - 3t_{n-1} = 0$$

***Let $t_n = r^n$ be a solution of this second – order
recurrence equation.***

Then we have : $r^n - 3r^{n-1} = 0$

If so, let us divide the entire equation by r^{n-1} seeing we have t_{n-1} :

$$\Rightarrow \frac{r^n}{r^{n-1}} - 3 \times \frac{r^{n-1}}{r^{n-1}} = 0$$

$$\Rightarrow r^{n-(n-1)} - 3 \times r^{n-1-(n-1)} = 0$$

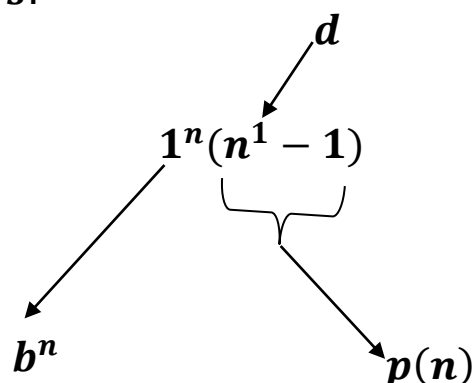
$$\Rightarrow r^1 - 3 \times r^0 = 0$$

$$\Rightarrow r - 3 = 0$$

$$\therefore r = 3$$

2. The non – homogeneous part of the recurrence equation is $n - 1$. Rewrite this part by comparing it with the form $b^n p(n)$.

This equation $n - 1$ can be written in the form $b^n p(n)$ as follows:



Here `d` is degree of polynomial.

Thus, one can observe here that `b` is 1 and the degree of the polynomial $(n - 1)$ is 1.

Therefore , the corresponding equations based on the procedure are as follows:

Substituting `n` with `r` in $(n - 1)$ and $d = 1$, we get:

$$(r - b)^{d+1} = (r - 1)^{1+1} = (r - 1)^2$$

The roots are $r = 1, 1$

Thus the roots are 3, 1, and 1, which are not distinct.

Therefore , Case 2 can be applied.

The general solution would be as follows:

$$t_n = c_1 \times 3^n + c_2 \times 1^n + c_3 \times n \times 1^n$$

For $n = 0$:

$$t_0 = c_1 3^0 + c_2 1^0 + c_3 (0) 1^0$$

$$t_0 = c_1 + c_2 = 0 \text{ --- (i)}$$

For $n = 1$:

$$t_1 = c_1 3^1 + c_2 1^1 + c_3 (1) 1^1 = 1$$

$$\Rightarrow t_1 = 3c_1 + c_2 + c_3 = 1 \text{ --- (ii)}$$

For $n = 2$:

$$t_2 = c_1 3^2 + c_2 1^2 + c_3 (2) 1^2 = 2$$

$$\Rightarrow t_2 = 9c_1 + c_2 + 2c_3 = 2 \text{ --- (iii)}$$

\therefore The equations are as follows :

$$c_1 + c_2 = 0 \text{ --- (i)}$$

$$3c_1 + c_2 + c_3 = 1 \text{ --- (ii)}$$

$$9c_1 + c_2 + 2c_3 = 2 \text{ --- (iii)}$$

From (i) we get $c_1 = -c_2$

Now replacing $-c_2$ in (ii) we get:

$$3(-c_2) + c_2 + c_3 = 1$$

$$\Rightarrow -3c_2 + c_2 + c_3 = 1$$

$$\Rightarrow -2c_2 + c_3 = 1$$

$$\Rightarrow -2c_2 = 1 - c_3$$

$$\Rightarrow 2c_2 = c_3 - 1$$

$$\Rightarrow c_2 = \frac{(c_3 - 1)}{2}$$

Substituting c_2 in $c_1 = -c_2$ we get:

$$c_1 = -\frac{(c_3 - 1)}{2} = \frac{(1 - c_3)}{2}$$

Hence relacing c_1 and c_2 in (iii)we get:

$$\Rightarrow \frac{9(1-c_3)}{2} + \frac{(c_3-1)}{2} + 2c_3 = 2$$

$$\Rightarrow \frac{9-9c_3}{2} + \frac{(c_3-1)}{2} + 2c_3 = 2$$

$$\Rightarrow \frac{9-9c_3+c_3-1}{2} + 2c_3 = 2$$

$$\Rightarrow \frac{8-8c_3}{2} + 2c_3 = 2$$

$$\Rightarrow \frac{8-8c_3+4c_3}{2} = 2$$

$$\Rightarrow \frac{8-4c_3}{2} = 2$$

$$\Rightarrow 8-4c_3 = 4$$

$$\Rightarrow -4c_3 = 4-8$$

$$\Rightarrow -4c_3 = -4$$

$$\Rightarrow c_3 = 1$$

$$\text{Hence, } c_2 = \frac{1-1}{2} = 0 \text{ and } c_1 = \frac{(1-1)}{2} = 0 .$$

Hence, we get $c_1 = 0, c_2 = 0$ and $c_3 = 1$

Therefore the specific solution of the recurrence equation with respect to the initial conditions is as follows:

$$t_n = (0) \times 3^n + (0)1^n + (1) \times n \times 1^n$$
