Generating Function

Introduction

Generating functions is another way of representing the complexity functions of algorithms.

This was introduced by the famous mathematician De Moire.

Abraham De Moire used generating functions to derive the Fibonacci numbers. Leonardo Euler used these functions for partition integers; Pierre Simon Laplace used them extensively and in 1754 published his work on the calculus of generating functions. Generating functions have many applications such as in counting and solving recurrence equations.

The primary use of a generating function is to solve recurrence equations , even when the recurrence is very complex.

Formally , a generating function represents a given sequence as a power series.

$Let\ a_0, a_1, a_2, a_3, \dots, a_n\ be\ finite\ sequence\ of\ numbers, the\ generating\ function\ can\ be\ expressed\ as\ a\ power\ series\ in\ the\ following\ way$:

$$G(z) = \sum_{k=0}^{n} a_k z^k$$

$$= a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + \cdots + a_k z^k$$

$$= \sum_{k=0}^{\infty} a_k z^k$$

$Here\ `z`\ is\ an\ intermidiate\ variable\ .The\ closed\ form\ of\ this\ sequence\ is\ given\ as\ follows$:

$$a_n = \frac{1}{1 - az}$$

$For\ example, for\ the\ sequence\ 1, 3, 5, 7, \dots, the\ generating\ function\ can\ be\ written\ as\ follows$:

$$G(z) = 1 + 3z + 5z^2 + 7z^3 + \cdots$$

$\boldsymbol{Lets\ observe\ some\ closed\ form\ of\ some\ important}$

$\boldsymbol{Generating\ function}$:

$$\sum_{r=0}^{\infty} x^r = x^0 + x^1 + x^2 + \cdots = 1 + x + x^2 + \cdots = \frac{1}{1-x}$$

$\boldsymbol{Where\ a_0, a_1, \ldots. = 1}$

$\boldsymbol{Similarly},$

$$\sum_{r=0}^{\infty} (-1)^r x^r = (-1)^0 \times x^0 + (-1)^1 \times x^1 + (-1)^2 \times x^2 + \cdots$$

$$= 1 - x + x^2 - x^3 + \cdots = \frac{1}{1-(-1)x} = \frac{1}{1+x}$$

$\boldsymbol{Where\ a_0, a_1, \ldots. = -1}$

$\boldsymbol{Similarly},$

$$\sum_{r=0}^{\infty} (r+1)x^r$$

$$= (0+1) \times x^0 + (1+1) \times x^1 + (2+1) \times x^2 + \cdots$$

$$= 1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2}$$

$\boldsymbol{Where\ a_0, a_1, \ldots. = r + 1}$

$\boldsymbol{Also,}$

$$\sum_{r=0}^{\infty}(-1)(r+1)x^r = \frac{1}{(1+(-1)x)^{\,2}} = \frac{1}{(1+x)^2}$$

Other some very useful closed form of generating function is given below:

$$\left.\begin{aligned} &1.\ (a+x)^n = \sum_{r=0}^{n} n_{C_r} a^{n-r} x^r \\ &2.\ (a-x)^n = \sum_{r=0}^{n} (-1) n_{C_r} a^{n-r} x^r \end{aligned}\right\} \rightarrow \boxed{\textit{Binomial Theorem}}$$

$$3.\ \frac{1}{(1-x)^n} = 1 + \sum_{r=1}^{\infty} \frac{n(n+1)(n+2)\ \dots.(n+r-1)}{r!} \times x^r$$

$$4.\ \frac{1}{(1+x)^n}$$

$$= 1 + \sum_{r=1}^{\infty} (-1)^r \frac{n(n+1)(n+2)\ \dots.(n+r-1)}{r!} \times x^r$$

$$\mathbf{5.}\ \log(1+x) = \sum_{r=0}^{\infty} \frac{x^r}{r!}$$

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