$Polynomial\ Reduction -$
$Solving\ Homogeneous\ Equation -$
$Example - 2$

$Example\ 2$: $Solve\ the\ following\ recurrence\ equation\ using\ the\ polynomial\ reduction\ method$:

$t_n - 11t_{n-1} + 30t_{n-2} = 0$

$t_0 = 0$

$t_1 = 1$

$t_2 = 2$

<u>$Solution$:</u>

$Let\ t_n = r^n\ be\ a\ solution\ of\ this\ second - order\ recurrence\ equation.$

$Then\ we\ have: r^n - 11r^{n-1} + 30r^{n-2} = 0$

$If\ so, let\ us\ divide\ the\ entire\ equation\ by\ r^{n-2}\ seeing\ we\ have\ t_{n-2}.$

$$\Rightarrow \frac{r^n}{r^{n-2}} - 11 \times \frac{r^{n-1}}{r^{n-2}} + 30 \times \frac{r^{n-2}}{r^{n-2}} = 0$$

$$\Rightarrow r^{n-(n-2)} - 11 \times r^{n-1-(n-2)} + 30 \times r^{n-2-(n-2)} = 0$$

$$\Rightarrow r^2 - 11 \times r^1 + 30 \times r^0 = 0$$

$$\Rightarrow r^2 - 11r + 30 = 0$$

$By\ middle\ term\ factor\ we\ can\ get\ the\ roots$:

$\Rightarrow r^2 - 5r - 6r + 30 = 0$
$\Rightarrow r(r-5) - 6(r-5) = 0$
$\Rightarrow (r-5)(r-6) = 0$
$i.e., r - 5 = 0 \Rightarrow r = 5\ and\ r - 6 = 0 \Rightarrow r = 6$.
$Hence\ r\ is\ 5\ and\ 6.$

$Therefore, the\ roots\ \ of\ this\ characteristic\ equation\ are$
$5\ and\ 6$.

$Since\ , the\ roots\ are\ distinct\ , hence\ Case\ 1\ of\ Theorem\ is$
$applicable. Therefore, the\ general\ solution\ can\ be$
$given\ as\ follows$:

$t_n = c_1 5^n + c_2 6^n$

$One\ can\ verify\ the\ correctness\ of\ this\ solution\ by$ $substituting\ \ these\ 5^n\ or\ 6^n\ in\ the\ original\ equation.$

$Let\ us\ substitute\ 5^n\ in\ the\ original\ to\ get\ the\ following$ $equations:$

$r^n - 11r^{n-1} + 30r^{n-2} = 0,$

$Substituting\ 2^n\ on\ the\ above\ equation, we\ get:$

$\Rightarrow 5^n - 11 \times 5^{n-1} + 30 \times 5^{n-2}$

$\Longrightarrow 5^n - 11 \times 5^{n-1} + 6 \times 5 \times 5^{n-2}$

$\Longrightarrow 5^n - 11 \times 5^{n-1} + 6 \times 5^{n-2+1}$

$\Longrightarrow 5^n - 11 \times 5^{n-1} + 6 \times 5^{n-1}$

$\Longrightarrow 5^n + 5^{n-1} \times (-11 + 6)$

$\Longrightarrow 5^n + 5^{n-1} \times (-5)$

$\Longrightarrow 5^n - 5^{n-1+1}$

$\Longrightarrow 5^n - 5^n$

$\Longrightarrow 0$

$Similarly\ ,one\ can\ also\ verify\ by\ substituting\ 6^n\ in\ the\ original\ equation\ and\ check\ that\ is\ a\ correct\ solution$:

$r^n - 11r^{n-1} + 30r^{n-2} = 0,$

$Substituting\ 1^n\ on\ the\ above\ equation, we\ get$:

$\Rightarrow 6^n - 11 \times 6^{n-1} + 30 \times 6^{n-2}$

$\Rightarrow 6^n - 11 \times 6^{n-1} + 30 \times 6^{n-2}$

$\Longrightarrow 6^n - 11 \times 6^{n-1} + 6 \times 5 \times 6^{n-2}$

$\Longrightarrow 6^n - 11 \times 6^{n-1} + 5 \times 6^{n-2+1}$

$\Longrightarrow 6^n - 11 \times 6^{n-1} + 5 \times 6^{n-1}$

$\Longrightarrow 6^n + 6^{n-1} \times (-11 + 5)$

$\Longrightarrow 6^n + 6^{n-1} \times (-6)$

$\Longrightarrow 6^n - 6^{n-1+1}$

$\Longrightarrow 6^n - 6^n$

$\Longrightarrow 0$

$In\ this\ original\ solution\ t_n = c_1 5^n + c_2 6^n\ ,only\ the\ values\ of\ the\ constant\ c_1\ and\ c_2\ are\ unknown.$

$For\ determining\ these\ values\ ,substitute\ n = 0\ and$
$n = 1\ in\ the\ general\ equation\ t_n = c_1 5^n + c_2 6^n\ to\ get\ the$
$following\ equations$:

$t_0 = c_1 5^0 + c_2 6^0 = c_1 + c_2$
$\Longrightarrow c_1 + c_2 = 0$

and ,

$t_1 = c_1 5^1 + c_1 6^1 = 5c_1 + 6c_2$,
$\Longrightarrow 5c_1 + 6c_2 = 1$

$The\ initial\ values\ t_0\ and\ t_1\ are\ already\ given\ in\ the$
$problem\ as\ initial\ conditions.\ The\ obtained\ \ equations$
$are\ as\ follows$:

$Now\ we\ have\ to\ cancel\ out\ ,hence\ multiplying$ 5 $with$
$equation$: $c_1 + c_2 = 0\ we\ get\ 5c_1 + 5c_2 = 0$.

$\therefore\ \mathbf{5c_1 + 5c_2 = \quad 0}$

$\quad\mathbf{5c_1 + 6c_2 = \quad 1}$

$(-)\quad(-)\qquad(-)$

$\mathbf{0 - c_2 \quad = \ -1}$

$\boldsymbol{-c_2 = -1\ or\ c_2 = 1\ \ and\ replacing\ c_2 = 1\ in\ equation}$
$\boldsymbol{(c_1 + c_2 = 0)\ we\ get}$:

$\mathbf{1 + c_1 = 0 \ \ \Rightarrow c_1 = -1.}$

$\boldsymbol{Hence\ c_1 = -1\ and\ c_2 = 1}$.

$\boldsymbol{Substituting\ these\ values\ in\ the\ general\ equation, one\ gets\ the\ following}$:

$\boldsymbol{t_n = c_1 5^n + c_2 6^n}$

$\boldsymbol{t_n = -1 \times 5^n + 1 \times 6^n}$

$\boldsymbol{= -5^n + 6^n}$

$\boldsymbol{The\ solution\ -5^n + 6^n\ is\ the\ solution\ of\ the\ recurrence\ equation\ with\ respect\ to\ the\ initial\ condition.}$

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