$B.B.1.$ *Simplified Master Theorem*

The master theorem is used for solving divide − and − conquer recurrence relations.

While the master theorem doesnot solve all types of divide − and − conquer recurrences , it can solve the majority of recurrene equation.

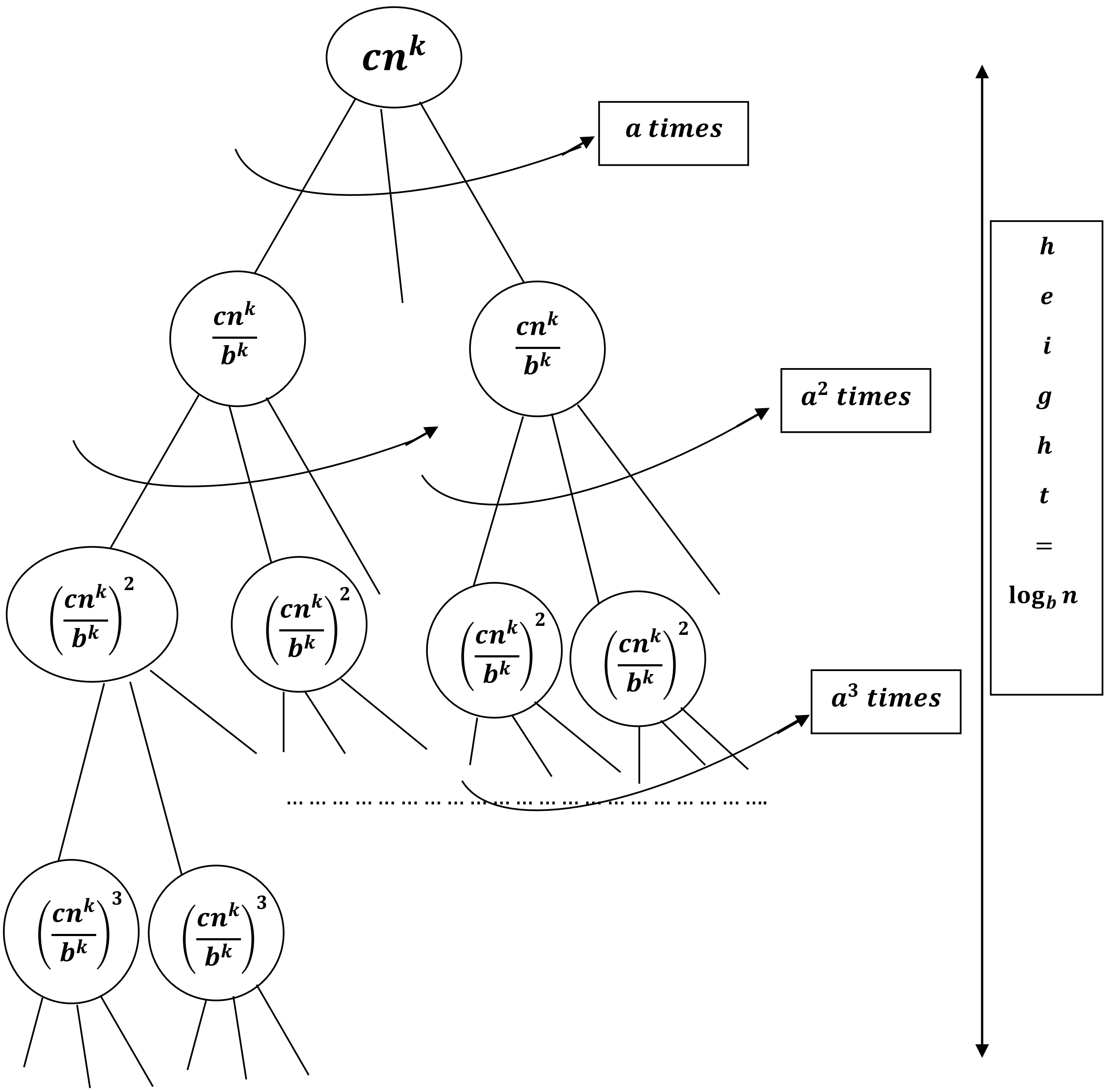
At preliminary stage , we are provided with the formula:

$$T(n) = aT\left(\frac{n}{b}\right) + \mathrm{cn^k}.$$

This implies that the problem is divided into `a` subproblems at every level and every subproblems is of the size

$$\frac{cn^k}{b^k}.$$

The total cost of the recurrence tree is calculated by adding the costs of all the levels .



cn^k
$a\ times$
$\frac{cn^k}{b^k}$
$\frac{cn^k}{b^k}$
$a^2\ times$
$\left(\frac{cn^k}{b^k}\right)^2$
$\left(\frac{cn^k}{b^k}\right)^2$
$\left(\frac{cn^k}{b^k}\right)^2$
$\left(\frac{cn^k}{b^k}\right)^2$
$a^3\ times$
$\left(\frac{cn^k}{b^k}\right)^3$
$\left(\frac{cn^k}{b^k}\right)^3$
$height = \log_b n$

$Level$	$No.of\ problems$	$Problem\ Size$	$Work\ done = Problem\ Size \times No.of\ Problems$
$\mathbf{0}$	$\mathbf{1}$	cn^k	$1 \times cn^k = cn^k$
$\mathbf{1}$	a	$\frac{cn^k}{b^k}$	$a \times \left(\frac{cn^k}{b^k}\right)$
$\mathbf{2}$	a^2	$\left(\frac{cn^k}{b^k}\right)^2$	$a^2 \times \left(\frac{cn^k}{b^k}\right)^2$
. .	. .	. .	. .
$n-1$	a^k	$\left(\frac{cn^k}{b^k}\right)^k$	$a^k \times \left(\frac{cn^k}{b^k}\right)^k$
$k = \log_b n$	$a^{\log_b n}$	$\left(\frac{cn^k}{b^k}\right)^{\log_b n}$	$a^{\log_b n} \times \left(\frac{cn^k}{b^k}\right)^{\log_b n}$

$\boldsymbol{Therefore\ , the\ total\ cost\ of\ the\ \ recurrence\ tree\ can\ be}$

$\boldsymbol{given\ as\ follows}$:

$$\boldsymbol{T(n) = cn^k + \ a \times \left(\frac{cn^k}{b^k}\right) + a^2 \times \left(\frac{cn^k}{b^k}\right)^2 + \cdots + a^{\log_b n} \times \left(\frac{cn^k}{b^k}\right)^{\log_b n}}$$

$$\boldsymbol{\Rightarrow cn^k \left(1 + \ a \times \left(\frac{1}{b^k}\right) + a^2 \times \left(\frac{1}{b^k}\right)^2 + \cdots + a^{\log_b n} \times \left(\frac{1}{b^k}\right)^{\log_b n}\right)}$$

$$\boldsymbol{\Rightarrow cn^k \left(1 + \ \left(\frac{a}{b^k}\right) + \left(\frac{a}{b^k}\right)^2 + \cdots + \left(\frac{a}{b^k}\right)^{\log_b n}\right)}$$

$\boldsymbol{if\ d = \left(\frac{a}{b^k}\right), then\ we\ can\ rewrite\ the\ equation\ as}$:

$$\boldsymbol{T(n) = \ cn^k\left(1 + \ (d) + (d)^2 + \cdots + (d)^{\log_b n}\right)}$$

$$\boldsymbol{Case\ 1:\rightarrow\ \frac{a}{b^k} < 1}$$

$\boldsymbol{When\ d < 1\ i.e. \frac{a}{b^k} < 1\ \ hence,}$

$\underline{\boldsymbol{To\ prove\ the\ geometric\ series\ is\ infinite:}}$

$$\lim_{n\to\infty}\left(1 + d + d^2 + d^3 + \cdots + d^{\log_b n}\right)$$

$\boldsymbol{As\ this\ is\ a\ geometric\ series\ which\ yields\ to}$:

$$= \lim_{n\to\infty}\left(\frac{d^{n+1}-1}{d-1}\right)$$

$$= \frac{1}{d-1} \times \lim_{n\to\infty} d^{n+1} - \lim_{n\to\infty} 1$$

$$= \frac{1}{d-1} \times (\infty - 1)$$

$$= \frac{1}{d-1} \times (\infty - 1)$$

$$= \frac{1}{d-1} \times \infty$$

$$= \infty$$

The series is infinite geometric series , as n goes infinite:

The series can be expressed as a common ratio \`d\` and a is 1 , Hence by formula of infinite geometric series we we get:

$$S = \frac{a}{1 - r} = \frac{1}{1 - d}.$$

<u>To prove the series is convergent:</u>

$$\lim_{n \to \infty} \left(\frac{1}{1 - d} \right) = \frac{1}{1 - 0} = 1$$, hence series is covergent i.e. finite and unique.

As the condition is $d < 1$, the series becomes convergent i.e. finite and unique i.e.:

$$1 + d + d^2 + \cdots = \frac{1}{1 - d}$$

Even if the sequence goes infinite , one can observe that the sequence reduces to constant factors only, when $d < 1$.

$$Therefore : cn^k \left[\frac{1}{1 - d} \right] = \frac{cn^k}{1 - d} = \Theta \left(\frac{cn^k}{1 - d} \right)$$

$$= \left(\frac{1}{1-d}\right) \times \Theta\left(\mathrm{cn^k}\right) = \Theta\left(\mathrm{cn^k}\right) = \Theta\left(\mathrm{n^k}\right).$$

$$\boldsymbol{Case\ 2: \rightarrow \frac{a}{b^k} = 1}$$

$\boldsymbol{When\ d = 1\ , this\ implies\ a = b^k.\ Therefore\ we\ will\ have}$:

$$T(n) = cn^k(1 + (d) + (d)^2 + \cdots + (d)^{\log_b n})$$

$\boldsymbol{Substuting\ d\ with\ 1\ we\ get}$:

$$T(n) = cn^k\left(1 + (1) + (1)^2 + \cdots + (1)^{\log_b n}\right)$$

$$T(n) = cn^k(1 + (1 + 1 + \cdots + \log_b n\ times))$$

$$T(n) = cn^k(1 + 1 \times \log_b n)$$

$$T(n) = cn^k(1 + \log_b n)$$

$$T(n) = \left(cn^k + cn^k \times \log_b n\right)$$

$Here\ dominant\ term\ is\ cn^k \times \log_b n\ , hence\ result\ is$:

$\Theta(n^k \log_b n)\ or\ \Theta(n^k \log n)$.

$Case\ 3: \rightarrow\ \dfrac{a}{b^k} > 1$

$When\ d\ \neq 1, this\ implies\ that\ \dfrac{a}{b^k} > 1\ , that\ is\ a > b^k, then$:

$$T(n) = \ cn^k(1 + \ (d) + (d)^2 + \cdots + (d)^{\log_b n})$$

$We\ can\ write\ this\ as$:

$$T(n) = \ cn^k \ \times d^{\log_b n} \underbrace{\left(\frac{1}{d^{\log_b n}} + \cdots + \frac{1}{d} + 1\right)}_{\boxed{Constant}}$$

$i.e., \left(\dfrac{1}{d^{\log_b n}} + \cdots + \dfrac{1}{d} + 1\right)\ will\ lead\ to\ constant\ , as\ the$

series converges to a finite , as n tends to inifinity , each term in the summation tends to 0 when $d > 1$.

If n tends to infinity , the terms , $\frac{1}{d^{\log_b n}}, \frac{1}{d^{\log_b n-1}}$, ... etc. will become very small .

We can write it as:

$$\frac{1}{d^{\log_b n}} + \frac{1}{d^{\log_b n-1}} + \cdots + \frac{1}{d} + 1$$

$$= \left(\frac{1}{d}\right)^{\log_b n} + \left(\frac{1}{d}\right)^{\log_b n-1} + \cdots + \left(\frac{1}{d}\right)^{1} + 1$$

Now lets consider common ratio of the series r, which is equal to $\frac{1}{d}$.

Since $0 < d < 1$ and n is infinite, we can apply the formula for the sum of an geometric series:

$$S = \frac{a}{1 - r}$$

$Substuting\ the\ values\ of\ a = \left(\frac{1}{d}\right)^{\log_b n}\quad and\ r = \frac{1}{d}\ into\ the$

$formula\ for\ the\ sum\ of\ an\ infinite\ geometric\ series$:

$$S = \frac{\left(\frac{1}{d}\right)^{\log_b n}}{1 - d}$$

$$S = \frac{\left(\frac{1}{d}\right)^{\log_b n}}{1 - \frac{1}{d}}$$

$$S = \frac{\left(\frac{1}{d}\right)^{\log_b n}}{\frac{d-1}{d}}$$

$$S = \left(\frac{1}{d}\right)^{\log_b n} \times \frac{d}{(d-1)}$$

$As\ n\ tends\ to\ inifinity:\ \left(\frac{1}{d}\right)^{\log_b n}\ approaches\ to\ zero,$

$since\ any\ positive\ base\ raised\ to\ a\ negative\ exponent$

$tends\ to\ zero.\ i.e. Suppose\ take\ an\ example$:

$$\left(\frac{1}{10}\right)^{\log_b 100}, then\ \log_{10} 100 = 2$$

$$\left(\frac{1}{10}\right)^{\log_b 100} = \left(\frac{1}{10}\right)^{2} = \frac{1}{100}\ , and\ the\ value\ is$$

$less\ than\ one, hence\ exponent\ is\ negative\ i.e.\ 10^{-2}.$

$**** Hence, the\ sum\ of\ series\ converges\ to\ a$
$constant\ value.\ \ ***$

$Note: The\ above\ expression\ can\ also\ be\ represented\ as:$

$$S = \frac{1}{d^{\log_b n}} \times \frac{d}{(d-1)} = \frac{1}{n^{\log_b d}} \times \frac{d}{(d-1)}\ also\ as:$$

$$S = \frac{1}{n^{\log_b d} \times \frac{(d-1)}{d}}$$

$Also\ we\ can\ write:$

$As\ n\ tends\ to\ infinity, term\ \frac{1}{n^{\log_b d}}\ approaches\ to\ zero\ ,$

$since\ any\ positive\ base\ raised\ to\ a\ negative\ exponent$

$tends\ to\ zero. Therefore, the\ sum\ of\ series\ converges\ to\ a\ constant\ value.$

$If\ we\ want\ an\ convergence\ test:$

$Convergence\ Test$

$$\frac{1}{d^{\log_b n}} + \frac{1}{d^{\log_b n-1}} + \cdots + \frac{1}{d} + 1$$

$Lets\ consider\ the\ series: a_n = \frac{1}{d^{\log_b n}}\ , to\ apply\ the\ limit\ comparison\ test, we\ need\ to\ find\ another\ series\ b_n\ , whose\ convergence\ behaviour\ is\ known.$

$Lets\ choose\ b_n = \frac{1}{n^{\in}}\ , where\ \epsilon\ is\ a\ positive\ constant.$
$Now\ we\ will\ compare\ the\ series\ a_n\ and\ b_n, by\ taking\ the\ limit\ as\ n\ tends\ to\ infinity:$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{1}{d^{\log_b n}}\right)}{\frac{1}{n^{\in}}}$$

$To\ simplify\ this\ expression, lets\ rewrite \frac{1}{d^{\log_b n}}\ as \frac{1}{n^{\log_b d}}$

$$= \lim_{n\to\infty} \frac{\frac{1}{n^{\log_b d}}}{\frac{1}{n^{\in}}}$$

$$= \lim_{n\to\infty} \frac{n^{\in}}{n^{\log_b d}}$$

Since $b > 1$ *and* $d > 1$ *, we know that* $\log_b d > 0$.
Therefore , as `n` tends to inifinity , $n^{\log_b d}$ *grows faster than* $n^{\in}$ *for any positive* $\in$.

As a result the limit: $\lim_{n\to\infty} \frac{n^{\in}}{n^{\log_b d}}$ *is equal to* 0 *for any positive* $\in$. *This implies that* a_n *and* b_n *have the same convergence behaviour* .

Since the series: $\sum_{n=1}^{\infty} b_n$ *with* $b_n = \frac{1}{n^{\in}}$ *converges* $\in > 1$,
We can conclude that the series $\sum_{n=1}^{\infty} a_n$ *also converges.*
Therefore the series : $\frac{1}{d^{\log_b n}} + \frac{1}{d^{\log_b n-1}} + \cdots + \frac{1}{d} + 1$
converges for $d > 1$ *and* $b > 1$.

$$\boldsymbol{Therefore\ keeping\ out\ the\ constant}:$$

$$\left(\frac{1}{d^{\log_b n}} + \cdots + \frac{1}{d} + 1\right)\ \boldsymbol{generated\ from}: T(n) = \ cn^k \ \times d^{\log_b n},$$

$$\boldsymbol{as\ it\ doesnot\ matter\ in\ time\ complexity, we\ continue}:$$

$$T(n) = \ cn^k \ \times d^{\log_b n}$$

$$or, T(n) = \ cn^k \ \times \left(\frac{a}{b^k}\right)^{\log_b n}$$

$$= cn^k \times \left(\frac{a^{\log_b n}}{b^{k \log_b n}}\right)$$

$$= cn^k \times \left(\frac{a^{\log_b n}}{n^k}\right) \left[\ b^{k \log_b n} = n^{k \log_b b}\right]$$

$$= cn^k \times \left(\frac{a^{\log_b n}}{n^k}\right) \left[\ b^{k \log_b n} = n^{k \log_b b}\right]$$

$$= c \times a^{\log_b n}$$

$$\in \ \Theta\left(n^{\log_b a}\right) \left[as\ a^{\log_b n}\ can\ also\ be\ written\ as\ n^{\log_b a}\right].$$

$$\boldsymbol{Thus\ we\ get\ a\ simplified\ master\ theorem\ as\ follows}:$$

$Simplified\ Master\ Theorem$:

$Let\ the\ time\ complexity\ function\ T(n)\ be\ a\ positive\ and\ eventually\ a\ non-decreasing\ function\ of\ the\ following\ form$:

$$T(n) = aT\left(\frac{n}{b}\right) + cn^k$$

$$T(1) = d$$

$Here\ a, d, b, k\ are\ all\ constants$.

$Here\ b \geq 2, k \geq 0, a > 0, c > 0\ and\ d \geq 0. The\ solution\ for\ the\ recurrence\ equation\ is\ given\ as\ follows$:

$Case\ 1: T(n) \in \Theta\left(n^k\right)\ if\ a < b^k$

$Case\ 2: T(n) \in \Theta\left(n^k \log n\right)\ if\ a = b^k$

$Case\ 3: T(n) \in \Theta\left(n^{\log_b a}\right)\ if\ a > b^k$

******