$Polynomial\ Reduction -$
$Solving\ Non - Homogeneous$
$Equation - Example$

$Solve\ the\ recurrence\ equation\ t_n - 3t_{n-1} = n - 1$
$for\ n = 0\ subject\ to\ the\ following\ initial$
$conditions: t_0 = 0\ ; t_1 = 1; t_2 = 2.$

<u>$Solution$</u>

$1. Obtain\ the\ characteristic\ equation\ \ for\ the\ corresponding$
$homogeneous\ part\ of\ the\ recurrence\ equation\ as$
$follows:$

$$t_n - 3t_{n-1} = 0$$

$Let\ t_n = r^n\ be\ a\ solution\ of\ this\ second - order$
$recurrence\ equation.$

$Then\ we\ have : r^n - 3r^{n-1} = 0$

If so, let us divide the entire equation by r^{n-1} seeing we have t_{n-1}:

$$\Rightarrow \frac{r^n}{r^{n-1}} - 3 \times \frac{r^{n-1}}{r^{n-1}} = 0$$

$$\Rightarrow r^{n-(n-1)} - 3 \times r^{n-1-(n-1)} = 0$$

$$\Rightarrow r^1 - 3 \times r^0 = 0$$

$$\Rightarrow r - 3 = 0$$

$$\therefore r = 3$$

2. The non-homogeneous part of the recurrence equation is $n - 1$. Rewrite this part by comparing it with the form $b^n p(n)$.

This equation $n - 1$ can be written in the form $b^n p(n)$ as follows:

$$\underbrace{1^n}_{b^n}\underbrace{(n^{\overset{d}{1}} - 1)}_{p(n)}$$

Here `d` is degree of polynomial.

Thus, one can observe here that `b` is 1 and the degree of the polynomial $(n-1)$ is 1.

Therefore, the corresponding equations based on the procedure are as follows:

Substituting `n` with `r` in $(n-1)$ and $d = 1$, we get:

$$(r-b)^{d+1} = (r-1)^{1+1} = (r-1)^2$$

The roots are $r = 1, 1$

Thus the roots are 3, 1, and 1, which are not distinct. Therefore, Case 2 can be applied.

The general solution would be as follows:

$$t_n = c_1 \times 3^n + c_2 \times 1^n + c_3 \times n \times 1^n$$

For $n = 0$:

$$t_0 = c_1 3^0 + c_2 1^0 + c_3 (0) 1^0$$

$$\boldsymbol{t_0 = c_1 + c_2 = 0 - - - (i)}$$

$\boldsymbol{For\ n = 1}$:

$$\boldsymbol{t_1 = c_1 3^1 + c_2 1^1 + c_3 (1) 1^1 = 1}$$

$$\boldsymbol{\Longrightarrow t_1 = 3c_1 + c_2 + c_3 = 1 - - - (ii)}$$

$\boldsymbol{For\ n = 2}$:

$$\boldsymbol{t_2 = c_1 3^2 + c_2 1^2 + c_3 (2) 1^2 = 2}$$

$$\boldsymbol{\Longrightarrow t_2 = 9c_1 + c_2 + 2c_3 = 2 - - - (iii)}$$

$\boldsymbol{\therefore The\ equations\ are\ as\ follows:}$

$$\boldsymbol{c_1 + c_2 = 0 - - - (i)}$$
$$\boldsymbol{3c_1 + c_2 + c_3 = 1 - - - (ii)}$$
$$\boldsymbol{9c_1 + c_2 + 2c_3 = 2 - - - (iii)}$$

$\boldsymbol{From\ (i)\ we\ get\ c_1 = -c_2}$

$\boldsymbol{Now\ replacing - c_2\ in\ (ii)\ we\ get}$:

$$3(-c_2) + c_2 + c_3 = 1$$

$$\Rightarrow -3c_2 + c_2 + c_3 = 1$$

$$\Rightarrow -2c_2 + c_3 = 1$$

$$\Rightarrow -2c_2 = 1 - c_3$$

$$\Rightarrow 2c_2 = c_3 - 1$$

$$\Rightarrow c_2 = \frac{(c_3 - 1)}{2}$$

$\boldsymbol{Substituting\ c_2\ in\ c_1 = -c_2\ we\ get}$:

$$c_1 = -\frac{(c_3 - 1)}{2} = \frac{(1 - c_3)}{2}$$

$\boldsymbol{Hence\ relacing\ c_1\ and\ c_2\ in\ (iii) we\ get}$:

$$\Longrightarrow \frac{9(1-c_3)}{2} + \frac{(c_3-1)}{2} + 2c_3 = 2$$

$$\Longrightarrow \frac{9-9c_3}{2} + \frac{(c_3-1)}{2} + 2c_3 = 2$$

$$\Longrightarrow \frac{9-9c_3+c_3-1}{2} + 2c_3 = 2$$

$$\Longrightarrow \frac{8-8c_3}{2} + 2c_3 = 2$$

$$\Longrightarrow \frac{8-8c_3+4c_3}{2} = 2$$

$$\Longrightarrow \frac{8-4c_3}{2} = 2$$

$$\Longrightarrow 8-4c_3 = 4$$

$$\Longrightarrow -4c_3 = 4-8$$

$$\Longrightarrow -4c_3 = -4$$

$$\Longrightarrow c_3 = 1$$

$$Hence, c_2 = \frac{1-1}{2} = 0 \; and \; c_1 = \frac{(1-1)}{2} = 0\,.$$

$\boldsymbol{Hence, we\ get\ c_1 = 0, c_2 = 0\ and\ c_3 = 1}$

$\boldsymbol{Therefore\ the\ specific\ solution\ of\ the\ recurrence\ equation\ with\ respect\ to\ the\ initital\ conditions\ is\ as\ follows}$:

$$\boldsymbol{t_n = (0) \times 3^n + (0)1^n + (1) \times n \times 1^n}$$

******