# $Polynomial\ Reduction -$
# $Solving\ Homogeneous\ Equation -$
# $Example - 1$

$Example\ 1$: $Solve\ the\ following\ recurrence\ equation\ using\ the\ polynomial\ reduction\ method$:

$t_n - 3t_{n-1} + 2t_{n-2} = 0\ for\ n > 0$
$t_0 = 0$
$t_1 = 1$

<u>$Solution$:</u>
$Let\ t_n = r^n\ be\ a\ solution\ of\ this\ second - order\ recurrence\ equation.$

$Then\ we\ have: r^n - 3r^{n-1} + 2r^{n-2} = 0$

$If\ so, let\ us\ divide\ the\ entire\ equation\ by\ r^{n-2}\ seeing\ we\ have\ t_{n-2}.$

$$\Rightarrow \frac{r^n}{r^{n-2}} - 3 \times \frac{r^{n-1}}{r^{n-2}} + 2 \times \frac{r^{n-2}}{r^{n-2}} = 0$$

$$\Rightarrow r^{n-(n-2)} - 3 \times r^{n-1-(n-2)} + 2 \times r^{n-2-(r-2)} = 0$$

$$\Rightarrow r^2 - 3 \times r^1 + 2 \times r^0 = 0$$

$$\Rightarrow r^2 - 3r + 2 = 0$$

***By middle term factor we can get the roots*:**

$$\Rightarrow r^2 - 2r - r + 2 = 0$$
$$\Rightarrow r(r-2) - 1(r-2) = 0$$
$$\Rightarrow (r-2)(r-1) = 0$$

$i.e., r - 2 = 0 \Rightarrow r = 2$ *and* $r - 1 = 0 \Rightarrow r = 1$ .

***Hence r is 2 and 1.***

***Therefore, the roots of this characteristic equation are 2 and 1 .***

***Since , the roots are distinct , hence Case 1 of Theorem is applicable. Therefore, the general solution can be given as follows*:**

$t_n = c_1 2^n + c_2 1^n$

$One\ can\ verify\ the\ correctness\ of\ this\ solution\ by\ substituting\ \ these\ 2^n\ or\ 1^n\ in\ the\ original\ equation.$

$Let\ us\ substitute\ 2^n\ in\ the\ original\ to\ get\ the\ following\ equations:$

$r^n - 3r^{n-1} + 2r^{n-2} = 0,$

$Substituting\ 2^n\ on\ the\ above\ equation, we\ get:$

$\Rightarrow 2^n - 3 \times 2^{n-1} + 2 \times 2^{n-2}$

$\Longrightarrow 2^n - 3 \times 2^{n-1} + 2^{n-2+1}$

$\Longrightarrow 2^n - 3 \times 2^{n-1} + 2^{n-1}$

$\Longrightarrow 2^n + 2^{n-1} \times (-3 + 1)$

$\Longrightarrow 2^n + 2^{n-1} \times (-2)$

$or, 2^n - 2\ \times 2^{n-1}$

$or, 2^n - 2^{n-1+1}$

$or, 2^n - 2^n$

$or, 0$

$Similarly$ , $one\ can\ also\ verify\ by\ substituting\ 1^n\ in\ the\ original\ equation\ and\ check\ that\ is\ a\ correct\ solution$:

$r^n - 3r^{n-1} + 2r^{n-2} = 0,$

$Substituting\ 1^n\ on\ the\ above\ equation, we\ get$:

$\Rightarrow 1^n - 3 \times 1^{n-1} + 2 \times 1^{n-2} = 0$

$Taking\ Left - Hand - Side$:

$\Rightarrow 1 - 3 \times 1 + 2 \times 1$ , $as\ anything\ power\ of\ 1\ results\ 1\ only$.

$\Rightarrow 1 - 3 + 2$

$\Rightarrow 3 - 3 = 0$

$In\ this\ original\ solution\ t_n = c_1 2^n + c_1 1^n$ , $only\ the\ values\ of\ the\ constant\ c_1\ and\ c_2\ are\ unknown$.

$For\ determining\ these\ values$ , $substitute\ n = 0\ and\ n = 1\ in\ the\ general\ equation\ t_n = c_1 2^n + c_2 1^n\ to\ get\ the\ following\ equations$:

$$t_0 = c_1 2^0 + c_2 1^0 = c_1 + c_2$$
$$\Rightarrow c_1 + c_2 = 0$$

*and* ,

$$t_1 = c_1 2^1 + c_1 1^1 = 2c_1 + c_2 \; ,$$
$$\Rightarrow 2c_1 + c_2 = 1$$

***The initial values*** $t_0$ ***and*** $t_1$ ***are already given in the problem as initial conditions. The obtained equations are as follows:***

$$\begin{array}{l} \therefore c_1 + c_2 = \quad 0 \\ \;\; 2c_1 + c_2 = \quad 1 \\ (-) \quad (-) \qquad (-) \\ \hline -c_1 + 0 \quad = \; -1 \\ \hline \end{array}$$

$-c_1 = -1$ ***or*** $c_1 = 1$ ***and replacing*** $c_1 = 1$ ***in equation*** $(c_1 + c_2 = 0)$ ***we get:***

$$1 + c_2 = 0 \;\; \Rightarrow c_2 = -1.$$

$\boldsymbol{Hence\ c_1 = 1\ and\ c_2 = -1}$ .

$\boldsymbol{Substituting\ these\ values\ in\ the\ general\ equation, one\ gets\ the\ following}$:

$$\boldsymbol{t_n = c_1 2^n + c_2 1^n}$$
$$\boldsymbol{t_n = 1 \times 2^n + (-1) \times 1^n}$$
$$\boldsymbol{= 2^n - 1^n}$$

$\boldsymbol{The\ solution\ 2^n - 1^n\ is\ the\ solution\ of\ the\ recurrence\ equation\ with\ respect\ to\ the\ initial\ condition.}$

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