$C.1.\ Recurrence\ Tree\ Method - Example\ 3$

$Solve\ the\ following\ recurrence\ equation:$

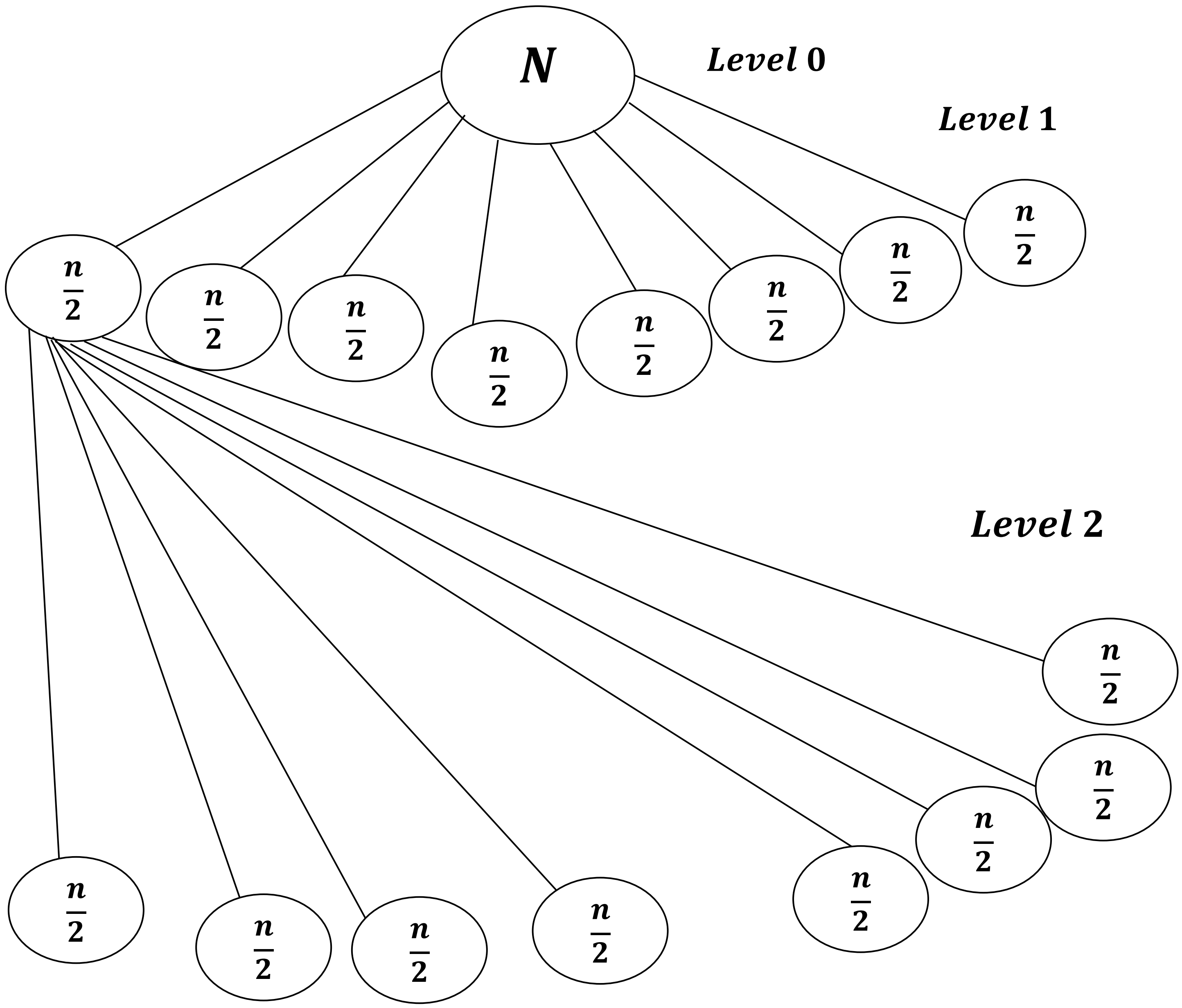
$$t_n = \begin{cases} 1 & for\ n = 1 \\ 8T\left(\frac{n}{2}\right) & for\ n > 1 \end{cases}$$

$\underline{Solution}:$

$Initially\ the\ input\ is\ n. It\ is\ sub - divided\ into\ eight\ sub - problems. In\ the\ next\ level, each\ of\ these\ eight\ sub - problems\ is\ again\ divided\ into\ eight\ sub\ problems(\ so\ a\ total\ of\ 8\ \times 8 = 64\ sub\ problems).$

$Here\ 8T\ represents\ 8\ subdivision\ and \frac{n}{2}\ represents \frac{n}{2}\ increase\ of\ problem\ size.$

$This\ process\ is\ continued\ till\ a\ pattern\ is\ obtained.$



$Recurrence\ tree\ at\ levels\ 1\ and\ 2\ (\ and\ only\ one\ problem\ division\ is\ shown)$

$Level$	$No. of\ problems$	$Problem\ Size$	$Work\ done = No. of\ problems \times problem\ size$
0	1	n	$1 \times n = n$
1	8	$\frac{n}{8}$	$8 \times \frac{n}{8} = n$
2	8^2	$\frac{n}{8^2}$	$8^2 \times \frac{n}{8^2} = n$
.	.	.	.
.	.	.	.
k	8^k	$\frac{n}{8^k}$	$8^k \times \frac{n}{8^k} = n$
$\log_2 n$	$8^{\log_2 n}$	1	$8^{\log_2 n} \times 1 = n^3$

$The\ problem\ size\ reduces\ to\ 1\ as\ T(n) = 1$
$for\ n = 1\ i.e.\ T(1) = 1\ or\ t_1 = 1.$

$It\ can\ be\ observed\ that\ at\ every\ level\ a\ problem\ is\ divided\ into\ eight\ subproblems\ or\ nodes\ is\ increasing\ in\ the\ following\ pattern: 1, 8, 64, \ldots \left(8^0, 8^1, 8^2, \ldots., 8^i\right).$

$\boldsymbol{The\ problem\ size\ is\ decreasing\ in\ a\ geometric\ series\ as\ follows}$: $\left(n, \frac{n}{8}, \frac{n}{8^2}, \dots, \frac{n}{8^k}, \dots, 1\right)$. $\boldsymbol{Based\ on\ the\ table, the\ amount\ of\ work\ done\ at\ the}$ $\log_2 n$ $\boldsymbol{can\ be\ calculated\ as\ follows}$:

$8^{\log_2 n} \times T(1)$

$= (2^3)^{\log_2 n} \times 1$

$= \left(2^{\log_2 n}\right)^3$

$= (n)^3 \ \left[a^{\log_a b} = b\right]$

$= n^3$

$\boldsymbol{Alternatively,}$

$8^{\log_2 n} \times T(1)$

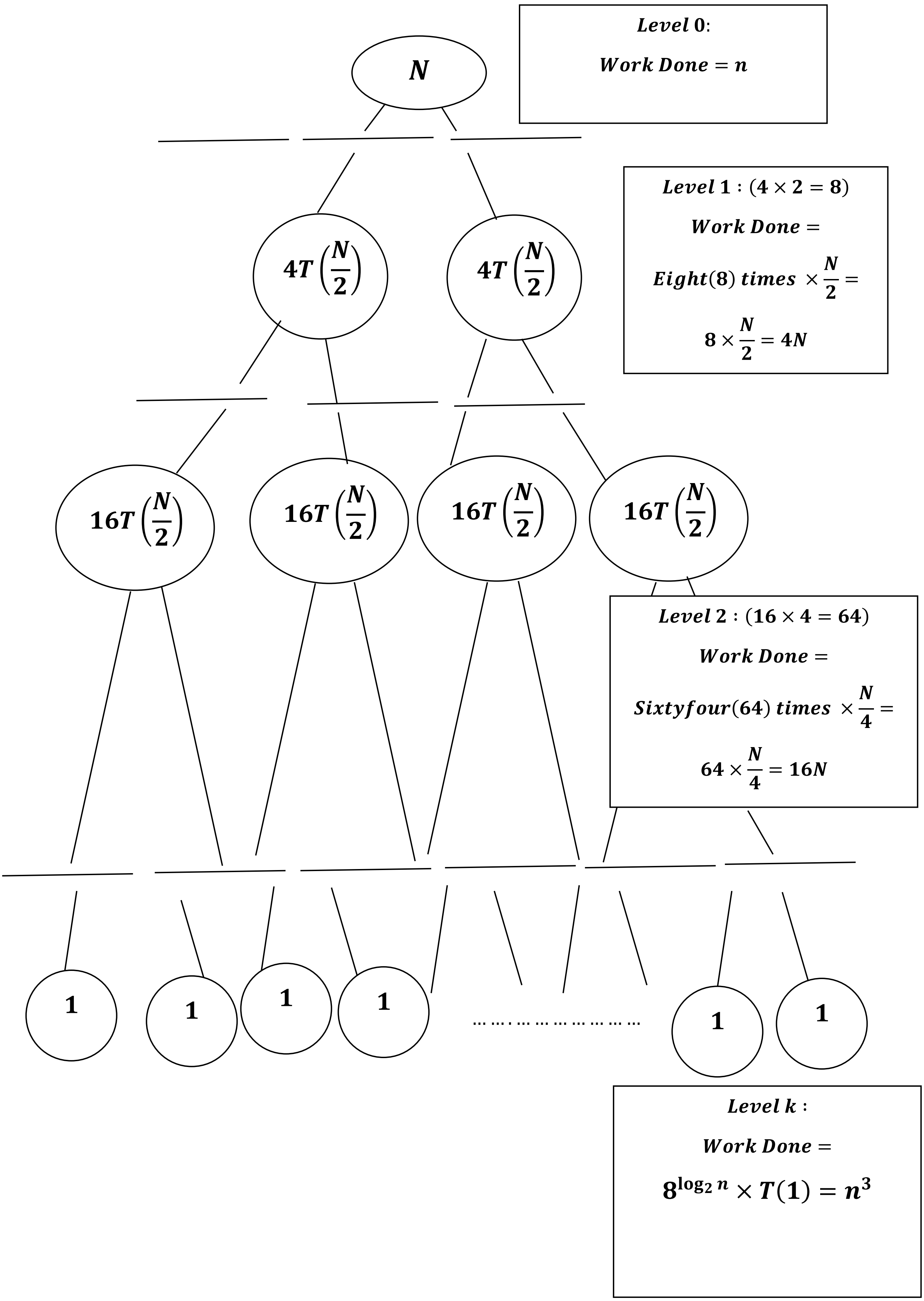
$= (2^3)^{\log_2 n} \times 1$

$= \left(2^{\log_2 n}\right)^3$

$= \left(n^{\log_2 2}\right)^3 \ \left[a^{\log_a b} = b^{\log_a a} = b\right]$

$= n^3$

$\boldsymbol{Now\ to\ calculate\ amount\ of\ work\ done\ at\ other\ level\ lets\ divide\ the\ above\ again}$:


N
$Level\ 0$:
$Work\ Done = n$
$4T\left(\frac{N}{2}\right)$
$4T\left(\frac{N}{2}\right)$
$Level\ 1 : (4 \times 2 = 8)$
$Work\ Done =$
$Eight(8)\ times\ \times \frac{N}{2} =$
$8 \times \frac{N}{2} = 4N$
$16T\left(\frac{N}{2}\right)$
$16T\left(\frac{N}{2}\right)$
$16T\left(\frac{N}{2}\right)$
$16T\left(\frac{N}{2}\right)$
$Level\ 2 : (16 \times 4 = 64)$
$Work\ Done =$
$Sixtyfour(64)\ times\ \times \frac{N}{4} =$
$64 \times \frac{N}{4} = 16N$
1
1
1
1
1
1
$Level\ k$:
$Work\ Done =$
$8^{\log_2 n} \times T(1) = n^3$

$\boldsymbol{Re - writing\ the\ above\ table\ as}$:

$Level$	$No.of\ problems$	$Problem\ Size$	$Work\ done = No.of\ problems \times problem\ size$
0	1	n	$1 \times n = n$
1	8	$\frac{n}{2}$	$8 \times \frac{n}{2} = 4n$
2	8^2	$\frac{n}{4}$	$8^2 \times \frac{n}{4} = 16n$
.	.	.	.
.	.	.	.
k	8^k	$\frac{n}{2^k}$	$8^k \times \frac{n}{2^k} = n \times 2^{2k}$
$\log_2 n$	$8^{\log_2 n}$	1	$8^{\log_2 n} \times 1 = n^3$

$Hence\ at\ level\ 0\ , work\ done\ is\ N. At\ the\ next\ level\ , the\ cost$ $is\ 8\ times\ \left(\frac{N}{2}\right) = 4N\ ; at\ level\ 2\ , the\ cost\ is\ 64\ times\ \left(\frac{N}{4}\right)$ $= 16N\ and\ so\ on.$

$\boldsymbol{The\ work\ done\ is\ \ increasing\ in\ the\ following\ pattern}$:

$\mathbf{1, 4, 16, \ldots . .}\ \boldsymbol{Therefore\ the\ total\ cost\ is\ the\ work\ done}$

$\boldsymbol{at\ the\ last\ level\ and\ work\ done\ at\ all\ other\ level}(\mathbf{0, 1, 2, 3}, \ldots,$

$(\log_2 \boldsymbol{n} - \mathbf{1})$.

$\boldsymbol{Thus}$, $\boldsymbol{the\ total\ cost\ of\ the\ tree\ can\ be\ esitmated\ as}$

$\boldsymbol{follows}$:

$$\Rightarrow \sum_{i=0}^{\log_2 n-1} 4^i n + 8^{\log_2 n} \times T(1)$$

$\boldsymbol{Where}$, $8^{log_2 n} \times T(1)$ $\boldsymbol{is\ the\ last\ level\ for}$ $\log_2 n$.

$\boldsymbol{and\ for}$ $\log_2 n - 1$ $\boldsymbol{i.e.till}$ $\log_2 n - 1$ $\boldsymbol{we\ have}$: $\sum_{i=0}^{\log_2 n-1} 4^i n$

$\boldsymbol{Hence\ we\ got}$: $\sum_{i=0}^{\log_2 n-1} 4^i n + 8^{\log_2 n} \times T(1)$

$\boldsymbol{And\ we\ know}$: $8^{\log_2 n} \times T(1) = n^3$, $\boldsymbol{hence}$:

$$\Rightarrow \sum_{i=0}^{\log_2 n-1} 4^i n + n^3$$

$$\Rightarrow n \times \sum_{i=0}^{\log_2 n-1} 4^i + n^3$$

And we know the geometric series :

$$\sum_{k=1}^{n} k = 1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

And hence for $\sum_{i=0}^{\log_2 n-1} 4^i$ ***, we get***

$$\Rightarrow n \times \left(\frac{4^{(\log_2 n-1)+1} - 1}{4 - 1} \right) + n^3$$

$$\Rightarrow n \times \left(\frac{4^{\log_2 n} - 1}{4 - 1} \right) + n^3$$

$$\Rightarrow n \times \left(\frac{(n)^2 - 1}{4 - 1} \right) + n^3 \left[\therefore 4^{\log_2 n} = \left(2^{\log_2 n}\right)^2 = n^2 \right]$$

$$\Rightarrow n \times \left(\frac{n^2 - 1}{3} \right) + n^3$$

$$\implies \frac{n^3 - n}{3} + n^3$$

$$\implies \frac{n^3 - n + 3n^3}{3}$$

$$\implies \frac{4n^3 - n}{3}$$

$$\textit{Hence total cost of the tree as}: \Theta\left(\frac{4n^3 - n}{3}\right)$$

$$\implies \Theta\ \left(\frac{4n^3}{3} - \frac{n}{3}\right)$$

$$\implies \Theta\left(\frac{4n^3}{3}\right)$$

$$\implies \frac{4}{3} \times \Theta(n^3)$$

$$\implies \Theta(n^3)$$

Alternative way.

Something is divisibe by 8 *also are divisible by* 4 *or are multiples of* 4.

The multiple 4 *is the approach earlier, but if we keep the original table* .

$Level$	$No. of\ problems$	$Problem\ Size$	$Work\ done = No. of\ problems \times problem\ size$
$\mathbf{0}$	$\mathbf{1}$	$\boldsymbol{n}$	$\mathbf{1} \times \boldsymbol{n} = \boldsymbol{n}$
$\mathbf{1}$	$\mathbf{8}$	$\frac{\boldsymbol{n}}{\mathbf{8}}$	$\mathbf{8} \times \frac{\boldsymbol{n}}{\mathbf{8}} = \boldsymbol{n}$
$\mathbf{2}$	$\mathbf{8^2}$	$\frac{\boldsymbol{n}}{\mathbf{8^2}}$	$\mathbf{8^2} \times \frac{\boldsymbol{n}}{\mathbf{8^2}} = \boldsymbol{n}$
.	.	.	.
.	.	.	.
$\boldsymbol{k}$	$\mathbf{8}^{\boldsymbol{k}}$	$\frac{\boldsymbol{n}}{\mathbf{8}^{\boldsymbol{k}}}$	$\mathbf{8}^{\boldsymbol{k}} \times \frac{\boldsymbol{n}}{\mathbf{8}^{\boldsymbol{k}}} = \boldsymbol{n}$
$\log_2 \boldsymbol{n}$	$\mathbf{8}^{\log_2 \boldsymbol{n}}$	$\mathbf{1}$	$\mathbf{8}^{\log_2 \boldsymbol{n}} \times \mathbf{1} = \boldsymbol{n}^{\mathbf{3}}$

$$\textit{Hence,}$$
$$= (n + n + n + \cdots + \log_2 n - 1 \; times) + n^3$$
$$= \mathbf{n} \times (\log_2 n - 1) + n^3$$
$$= \mathbf{n} \log_2 n - n + n^3$$
$$= \Theta(\mathbf{n} \log_2 n - n + n^3)$$

$$\textit{Now let us view the exponential rates of growth:}$$
$$2^{2n} < n! < 4^{2n} < 2^n < n^3 < n^2 < nlogn < log(n!) < n$$
$$< 2^{logn} < log^2 n < \sqrt{logn} < loglogn < 1$$

$$\textit{And complexities from fastest to slowest:}$$

$$\Theta(1) < \Theta(log\, n) < \Theta(\sqrt{n}\,) < \Theta(n) < \Theta(nlogn) < \Theta(n^2) <$$
$$\Theta(n^3) < \Theta(2^n) < \Theta(n!) < \Theta(2^{2n}) < \Theta(2^{logn}) < \Theta(loglogn)$$
$$< \Theta(3^n) < \Theta(n^n)$$

$$\textit{We can write it oppositely:}$$

$$\Theta(n^n) > \Theta(3^n) > \Theta(loglogn) > \Theta(2^{logn}) > \Theta(2^{2n}) > \Theta(n!)$$
$$> \Theta(2^n) > \Theta(n^3) > \Theta(n^2) > \Theta(nlogn) > \Theta(n) > \Theta(\sqrt{n}\,) >$$
$$\Theta(log\, n) > \Theta(1)$$

$\boldsymbol{We\ can\ examine}$:

$\boldsymbol{\Theta}(\ \mathbf{n}\,\mathbf{log_2}\,\boldsymbol{n}-\boldsymbol{n}+\boldsymbol{n^3})$

$\Longrightarrow \boldsymbol{\Theta}(\boldsymbol{n^3}) > \boldsymbol{\Theta}(\boldsymbol{n}\,\mathbf{log_2}\,\boldsymbol{n}) > \boldsymbol{\Theta}(\boldsymbol{n})$

$\Longrightarrow \boldsymbol{\Theta}(\boldsymbol{n^3})\ \boldsymbol{is\ the\ anwer}.$

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