$$Generating\ Function -$$
$$Example - 3$$

$Example$: $Solve\ the\ following\ recurrence\ equation\ of\ the\ towers\ of\ Hanoi\ method\ using\ the\ generating\ function\ method$:

$t_0 = 0$

$t_n - 2t_{n-1} = 1\ for\ n = 1, 2, 3, \dots..$

<u>$Solution$:</u>

$The\ corresponding\ generating\ function\ for\ this\ recurrence\ equation\ would\ be$:

$$G(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$
$$2z \times G(z) = 2a_{1-1}z^1 + 2a_{2-1}z^2 + 2a_{3-1}z^3$$

$$
\begin{aligned}
\boldsymbol{G(z)} &= \boldsymbol{a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots} \\
\boldsymbol{5z \times G(z)} &= \qquad \boldsymbol{2a_0 z + 2a_1 z^2 + 2a_2 z^3}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{G(z) - \big(2z \times G(z)\big)} &= \boldsymbol{a_0 + (a_1 z - 2a_0 z) + (a_2 z^2 - 2a_1 z^2)} \\
&\quad \boldsymbol{+(a_3 z^3 - 2a_2 z^3) + \cdots}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{(1-2z)\big(G(z)\big)} &= \boldsymbol{a_0 + (a_1 - 2a_0)z + (a_2 - 2a_1)z^2 +} \\
&\quad \boldsymbol{(a_3 - 2a_2)z^3 + \cdots}
\end{aligned}
$$

As given $\boldsymbol{t_n - 2t_{n-1} = 1}$ ***, therefore,***

$$\boldsymbol{(1-2z)\big(G(z)\big) = a_0 + 1z + 1z^2 + 1z^3 + \cdots}$$

And $\boldsymbol{t_0 = 0}$ ***, then:***

$$\boldsymbol{(1-2z)\big(G(z)\big) = 0 + z + z^2 + z^3 + \cdots}$$

$$\boldsymbol{(1-2z)\big(G(z)\big) = z + z^2 + z^3 + \cdots}$$

$$\boldsymbol{(1-2z)\big(G(z)\big) = z(1 + z + z^2 + \cdots)}$$

$$\sum_{r=0}^{\infty} z^r = \frac{1}{1-z}$$

$$(1-2z)\big(G(z)\big) = z(z^0 + z^1 + z^2 + \cdots)$$

$$(1-2z)\big(G(z)\big) = z\left(\sum_{r=0}^{\infty} z^r\right)$$

$$(1-2z)\big(G(z)\big) = z\left(\frac{1}{1-z}\right)$$

$$(1-2z)\big(G(z)\big) = \frac{z}{1-z}$$

$$G(z) = \frac{z}{(1-z)(1-2z)}$$

$Using\ the\ partial\ function$:

$We\ observe\ (1-z) - (1-2z) = 2z - z = z, hence$

$Therefore,$

$$\frac{(1-z)}{(1-z)(1-2z)} - \frac{(1-2z)}{(1-z)(1-2z)}$$

$$= \frac{\mathbf{1}}{(\mathbf{1}-\mathbf{2}z)} - \frac{\mathbf{1}}{(\mathbf{1}-z)}$$

Now,

$$\sum_{n=0}^{\infty} 2^n z^n = 2z^0 + 4z^1 + 8z^2 + \cdots = \frac{1}{1-2z}$$

Also,

$$\sum_{n=0}^{\infty} z^n = \frac{1}{(1-z)}$$

Hence,

$$= \sum_{n=0}^{\infty} 2^n z^n - \sum_{n=0}^{\infty} z^n$$

$$= 2^n - 1$$

It is the general form of recurrence equation.

Thus generating functions are useful in finding solutions of many recurrence equations.

Even if the normal methods fail, the generating function method can be relied upon.

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