$\boldsymbol{Guess\ And\ Verify\ Method - Example - 2}$

$\boldsymbol{Example\ 2}$: $\boldsymbol{Solve\ the\ recurrence\ equation\ t_n = t_{n-1} + n^2}$ $\boldsymbol{using\ the\ guess-and-verify\ method. Where\ t_1 = 1}$.

<u>$\boldsymbol{Solution}$:</u>

$\boldsymbol{For\ guessing\ the\ solution, we\ substitute\ the\ values}$ $\boldsymbol{of\ `n`}$,

$$t_1 = 1$$

$$t_2 = t_{2-1} + 2^2 = 1 + 2^2$$

$$t_3 = t_{3-1} + 3^2 = 1 + 2^2 + 3^2$$

$$t_4 = t_{4-1} + 4^2 = 1 + 2^2 + 3^2 + 4^2$$

$\dots..$

$$t_n = t_{n-1} + n^2 = 1 + 2^2 + 3^2 + 4^2 + \cdots + n^2$$

$$it\ gives, \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$Hence, P(n) = \frac{n(n+1)(2n+1)}{6}$$

$$\boldsymbol{Therefore\ for\ P(1) = \frac{1(1+1)(2\times 1+1)}{6}}$$

$$= \frac{2\times 3}{6}$$

$$= \frac{6}{6}$$

$= \mathbf{1}$, ***which is obviously true as*** $\mathbf{1^2 = 1}$, ***when*** $\boldsymbol{n = 1}$.

Let $\boldsymbol{P(m)}$ ***be true that is***

$$\mathbf{1^2 + 2^2 + 3^2 + \cdots + \boldsymbol{m}^2 = \frac{\boldsymbol{m}(\boldsymbol{m}+1)(2\boldsymbol{m}+1)}{6} - - - (\boldsymbol{i})}$$

is true, then we shall prove $\boldsymbol{P(m+1)}$, ***that is***:

$$\mathbf{1^2 + 2^2 + 3^2 + \cdots + (\boldsymbol{m}+1)^2 = \sum_{\boldsymbol{i}=1}^{\boldsymbol{m}+1} \boldsymbol{i}^2}$$

$$\frac{(\boldsymbol{m}+\mathbf{1})\big((\boldsymbol{m}+\mathbf{1})+\mathbf{1}\big)(\mathbf{2}(\boldsymbol{m}+\mathbf{1})+\mathbf{1})}{\mathbf{6}}$$

$$= \frac{(\boldsymbol{m}+\mathbf{1})(\boldsymbol{m}+\mathbf{2})(\mathbf{2}\boldsymbol{m}+\mathbf{3})}{\mathbf{6}} - - - (\boldsymbol{ii})$$

$Adding\ (m+1)^2\ on\ both\ side\ of\ (i), we\ get$:

$$1^2+2^2+3^2+\cdots+m^2+\ (m+1)^2=$$

$$\frac{m(m+1)(2m+1)}{6}+(m+1)^2$$

$$=\frac{m(m+1)(2m+1)}{6}+(m+1)^2$$

$$=\frac{6\times(m+1)^2+m(m+1)(2m+1)}{6}$$

$$=(m+1)\left\{\frac{6(m+1)+m(2m+1)}{6}\right\}$$

$$=(m+1)\left\{\frac{6m+6+2m^2+m}{6}\right\}$$

$$=(m+1)\left\{\frac{2m^2+7m+6}{6}\right\}$$

$Splitting\ the\ middle\ term\ , also\ known\ as\ middle\ term\ factor$:

$$=(m+1)\left\{\frac{2m^2+4m+3m+6}{6}\right\}$$

$$= (m+1)\left\{\frac{2m(m+2)+3(m+2)}{6}\right\}$$

$$= (m+1)\left\{\frac{2m(m+2)+3(m+2)}{6}\right\}$$

$$= (m+1)\left\{\frac{(2m+3)+(m+2)}{6}\right\}$$

$$= \frac{(m+1)(2m+3)+(m+2)}{6} --- (iii)$$

$Hence\ by\ (iii), we\ prove\ (ii).$

$Thus\ we\ prove\ that\ (i)\ P(1) is\ true.$

$(ii)\ if\ P(m)\ is\ true\ then\ P(m+1)\ is\ true\ .$

$Hence\ by\ induction\ method\ , general\ solution\ for$

$recurrence\ equation\ is: \dfrac{n(n+1)(2n+1)}{6}.$

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