$Guess\ And\ Verify\ Method - Example - 3$

$Example\ 3$: $Use\ the\ guess - and - verify\ method\ and\ solve\ the\ following\ recurrence\ equation$:

$T(n) = 3T\left(\frac{n}{2}\right)$ $subjected\ to\ initial\ condition.$

$T(1) = 1$

$Note\ that\ n\ is\ greater\ than\ 1\ and\ a\ power\ of\ 2.$

<u>$Solution$</u>:

$As\ n > 1\ and\ a\ power\ of\ 2\ , it\ can\ 2, 4, 8\ and\ so\ on.$

$T(1) = 1$

$T(2) = 3T\left(\frac{2}{2}\right) = 3 \times T(1) = 3 \times 1 = 3.$

$T(4) = 3T\left(\frac{4}{2}\right) = 3 \times T(2) = 3 \times 3 = 3^2.$

$T(8) = 3T\left(\frac{8}{2}\right) = 3 \times T(4) = 3 \times 3^2 = 3^3.$

$T(16) = 3T\left(\frac{16}{2}\right) = 3 \times T(8) = 3 \times 3^3 = 3^4.$

Guess:

$\boldsymbol{One\ can\ observe\ that\ as\ `n`\ increases, the\ power\ of\ 3\ increases\ incrementally. Therefore\ the\ guessed\ solution\ for\ this\ recurrence\ would\ be\ the\ following}$:

$$\boldsymbol{t_n = 3^{\log_2 n}}$$

$\boldsymbol{if\ n > 1\ and\ n\ is\ power\ of\ 2\ , we\ can\ get\ the\ values}$:

$$\boldsymbol{\Longrightarrow 3^{\log_2 2} = 3}$$
$$\boldsymbol{\Longrightarrow 3^{\log_2 4} = 9 = 3^2}$$
$$\boldsymbol{\Longrightarrow 3^{\log_2 8} = 27 = 3^3}$$
$$\boldsymbol{\Longrightarrow 3^{\log_2 16} = 81 = 3^4}$$

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Verify:

$\boldsymbol{Now\ we\ can\ verify\ using\ mathematical\ induction}$:

$\boldsymbol{P(1) = 3^{\log_2 1} = 3^0 = 1\ also\ T(1) = 1\ , hence\ true}$.

$\boldsymbol{For\ 2n\ , the\ solution\ should\ be\ (as\ n\ is\ a\ power\ of\ 2)}$

$$\boldsymbol{t_{2n} = 3 \log_2 2n}$$

$\boldsymbol{Substituting\ 2n\ in\ place\ of\ n\ in\ the\ original\ equation}$:

$$T(2n) = 3T\left(\frac{2n}{2}\right) = 3T(n)$$

$$= 3 \times\ 3^{\log_2 n} = 3^{1+\log_2 n}$$

$$= 3^{\log_2 2+\log_2 n}\ (\because\ 1\ \textit{can be written as}\ \log_2 2\ )$$

$$= 3^{\log_2 2n}(\because\ \log_a x + \log_a y = \log_a xy)$$

$\underline{P(m)}$	$\underline{P(m+1)}$
$3^{\log_2 2} = 3$	$3^{\log_2 4} = 9$
$3^{\log_2 4} = 9$	$3^{\log_2 8} = 27$
$3^{\log_2 8} = 27$	$3^{\log_2 16} = 81$
$3^{\log_2 16} = 81$	

$\boldsymbol{Hence}\ P(m) = 3^{\log_2 m}\ \boldsymbol{is\ true\ then}\ P(m+1) = 3^{1+\log_2 m}$ $\boldsymbol{is\ also\ true}$.

$\boldsymbol{Hence\ by\ induction, recurrence\ equation}, t_n = 3^{\log_2 n}$ $\boldsymbol{is\ true}$.

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