Divide And Conquer −
Akrabazzi − Theorem
Proof

The proof is given by Mohammad Akra and LOUAY BAZZI in On the Solution of Linear Recurrence Equations .

The same proof is depicted here:

The general solution for linear divide − and −conquer recurrences of the form ***provided***:

$$u_n = \sum_{i=1}^{k} a_i u_{\left\lfloor \frac{n}{b^i} \right\rfloor} + g(n).$$

As said in the proof that it can handle more cases than Master Method.

Introduction:

$$u_n = \begin{cases} u_0 & n = 0 \\ \sum_{i=1}^{k} a_i u_{\left\lfloor \frac{n}{b^i} \right\rfloor} + g(n) & n \geq 1 \end{cases}$$

$Where$

$\rightarrow u_0, a_i \in R^{*+}, \sum_{i=1}^{k} a_i \geq 1,$

R^{*+} *is positive real numbers excluding* **0**.

$\rightarrow b_i, k \in N, b_i \geq 2, k \geq 1$

$\rightarrow$ $g(x)$ *is defined for real values* x, *and is bounded, positive and nondecreasing function for all* $x \geq 0$

$\rightarrow$ *For all* $c > 1$, *there exist* $x_1, k_1 > 0$ *such that*

$g\left(\frac{x}{c}\right) \geq k_1 g(x)$, *for all* $x \geq x_1$.

$Theorem\ 1: Let\ u_n\ be\ a\ sequence\ .Let\ f(x) be\ a\ function$
$defined\ by$:

$$f(x) = \begin{cases} u_0 & x\epsilon[0,1) \\ \sum_{i=1}^{k} a_i f\left(\frac{x}{b_i}\right) + g(\lfloor x \rfloor) & x\epsilon[1,\infty) \end{cases}$$

$x\epsilon[0,1)$ $, is\ closed\ open\ neighbourhood\ i.e. 0\ to\ 1$.
$And\ x\epsilon[1,\infty) is\ also\ closed\ open\ neighbourhood$
$from\ 1\ to\ inifinite.$

$Then,$

1. $For\ all\ x \geq 0, f(x) = f(\lfloor x \rfloor).$
2. $For\ all\ n \geq 0\ , f(n) = u_n.$

$In\ other\ words, f(x) is\ a\ staircase\ function.$

$In\ mathematics\ , a\ step\ function(also\ called\ as$
$staircase\ function)\ is\ defined\ , as\ a\ precise\ constant$
$function, that\ has\ only\ a\ finite\ number\ of\ pieces\ with$

$\boldsymbol{Linear\ combinations}$.

$\boldsymbol{Hence\ f(x)\ is\ a\ staircase\ function\ which\ matches\ with\ u_n}$ $\boldsymbol{at\ integer\ values\ of\ `x`}$.

$\boldsymbol{In\ proving\ the\ above\ theorem\ we\ need\ a\ part\ i.e.}$

$\boldsymbol{Part\ 1}$: $\boldsymbol{If\ b \in N\ , b \geq 1\ and\ x \in R^+\ (positive\ real\ number)}$, $\boldsymbol{then}$:

$$\left\lfloor \frac{\mathbf{x}}{\mathbf{b}} \right\rfloor = \left\lfloor \frac{\lfloor \mathbf{x} \rfloor}{\mathbf{b}} \right\rfloor.$$

$\boldsymbol{Proof\ of\ Theorem\ 1}$:

$\boldsymbol{To\ prove\ that\ f(x) = f(\lfloor x \rfloor), we\ use\ strong\ induction}$.

$\boldsymbol{Note\ that\ for\ all\ x \in [0,1)\ we\ have\ f(x) = u_0\ and}$ $\boldsymbol{f(\lfloor x \rfloor) = f(0) = u_0}$.

$\boldsymbol{Hence\ , f(\lfloor x \rfloor) = f(x)}$.

$\boldsymbol{Now\ assume\ that\ f(\lfloor x \rfloor) = f(x)\ for\ all\ x \in [0,n), and\ let}$ $\boldsymbol{us\ prove\ that\ it\ is\ true\ for\ all\ x \in [n, n+1)}$.

$\boldsymbol{Consider,}$

$$f(x) = \sum_{i=1}^{k} a_i f\left(\frac{x}{b_i}\right) + g(\lfloor x \rfloor)$$

$\boldsymbol{Let\ x \epsilon [n, n+1), then}$

$$\frac{x}{b_i} \epsilon \left(0, \frac{n+1}{2}\right) \ since\ b_i \geq 2 .$$

$\boldsymbol{But}$

$$\left(0, \frac{n+1}{2}\right) \subset [0, n)\ for\ n \geq 1.$$

$\boldsymbol{Hence, we\ conclude\ that:}$

$$\frac{x}{b} \in [0, n), \frac{\lfloor x \rfloor}{b} \in [0, n)\ and\ \left\lfloor \frac{x}{b} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor}{b} \right\rfloor \epsilon [0, n)\ (from\ Part\ 1) .$$

$\boldsymbol{Therefore,}$

$$f\left(\frac{x}{b_i}\right) = f\left(\left\lfloor \frac{x}{b_i} \right\rfloor\right)\ (by\ assumption)$$

$$= f\left(\left\lfloor \frac{\lfloor \mathbf{x} \rfloor}{\mathbf{b_i}} \right\rfloor\right) \ (using\ Part\ 1)$$

$$= f\left(\frac{\lfloor x \rfloor}{b_i}\right) \ (by\ assumption)$$

$$Now\ replacing, in\ eqn \rightarrow\ f(x) = \sum_{i=1}^{k} a_i f\left(\frac{x}{b_i}\right) + g(\lfloor \mathbf{x} \rfloor),$$

we get:

$$f(x) = \sum_{i=1}^{k} a_i f\left(\frac{\lfloor x \rfloor}{b_i}\right) + g(\lfloor \mathbf{x} \rfloor)$$

$f(x) = f(\lfloor x \rfloor), for\ all\ x \geq 0.$

Which completes the proof of Part 1 of the theorem.

To prove that $f(n) = u_n$, *we use strong induction again.*

The equality holds trivially for $n = 0$. *Assume it is true for all* $m < n$ *and consider*:

$$f(x) = \sum_{i=1}^{k} a_i f\left(\frac{n}{b_i}\right) + g(\mathbf{n})$$

$Let\ n \geq 1. It\ has\ been\ proved\ from\ Part\ 1\ that$

$$f\left(\frac{n}{b_i}\right) = f\left(\left\lfloor\frac{n}{b_i}\right\rfloor\right).$$

$Now\ since$ $\left\lfloor\frac{n}{b_i}\right\rfloor \in [0, n)$ $it\ has\ been\ concluded\ that$

$$f\left(\left\lfloor\frac{n}{b_i}\right\rfloor\right) = u_{\left\lfloor\frac{n}{b_i}\right\rfloor} \quad .$$

$Replacing\ again\ the\ equation$ $\sum_{i=1}^{k} a_i f\left(\frac{n}{b_i}\right) + g(\mathbf{n}):$

$$f(n) = \sum_{i=1}^{k} a_i u_{\left\lfloor\frac{\mathrm{n}}{b^i}\right\rfloor} + g(n).$$

$But,$

$$u_n = \sum_{i=1}^{k} a_i u_{\left\lfloor\frac{\mathrm{n}}{b^i}\right\rfloor} + g(n)$$

$So, f(n) = u_n\ , which\ completes\ the\ proof.$

$$********* \ Theorem\ 1\ *********$$

$\boldsymbol{Definition\ 1: Let\ S\ be\ the\ set\ of\ all\ real\ of\ the\ real\ variable}$
$\boldsymbol{variable\ `x`\ satisfying\ the\ following\ conditions:}$

$\boldsymbol{1. for\ all\ x \geq 0, f(x)\ is\ bounded.}$
$\boldsymbol{2. for\ all\ x \geq 0\ f(x)\ is\ non - decreasing\ .}$
$\boldsymbol{3. for\ all\ c > 1\ , there\ exist\ x_1, k_2 > 0\ such\ that\ for\ all}$
$\boldsymbol{x \geq\ x_1, f\left(\frac{x}{c}\right) \geq k_1 f(x),}$

$\boldsymbol{Theorem\ 2: (\ The\ Order\ Transform) Let\ P\{\ \}\ be\ a\ mapping\ that}$
$\boldsymbol{f(x) \in\ S\ a\ real - valued\ function\ F(s,p) of\ the\ real}$
$\boldsymbol{variables\ s\ \in\ R^{+} (positive\ real\ numbers) and}$
$\boldsymbol{p, defined\ by:}$

$$\boldsymbol{F(s,p) = P\{f(x)\} \equiv \int_1^s f(u) u^{-p-1} du}$$

$\boldsymbol{\equiv \rightarrow is\ \ known\ as : identical\ to}$

$\boldsymbol{Then\ P\{\ \}\ satisfies\ the\ following\ properties:}$

$\boldsymbol{1. P\{\ \}\ exists.}$
$\boldsymbol{2. P\{\ \}\ is\ linear.}$
$\boldsymbol{3. P\{\ \}\ is\ one - to - one.}$
$\boldsymbol{4. (Scaling\ property)\ Let\ f(x) \in S, F(s,p) = P\{f(x)\}, a \in R}$
$\boldsymbol{and\ a > 1. Then,}$

$$p\left\{f\left(\frac{x}{a}\right)\right\} = a^{-p}F(s,p) - \Theta_s\left(\frac{f(s)}{s^p}\right) + \Theta_s(1),$$

$Where\ \Theta_s\big(h(s,p)\big)\ is\ a\ function\ bounded\ between$

$c_1(p)h(s,p)\ and\ c_2(p)h(s,p), \quad for\ some\ positive$

$functions\ c_1(p), c_2(p), for\ all\ s > s_0\ , for\ all\ p.$

$The\ scaling\ property\ states\ that\ when\ we\ scale\ the\ function\ f(x)$

$by\ a\ factor\ of \frac{1}{a}\ , the\ property\ P\ of\ the\ function\ is\ affected\ in$

$the\ following\ way:$

$1. The\ term\ a^{-p}F(s,p) captures\ the\ change\ in\ the\ property\ due$

$to\ the\ scaling\ . It\ represents\ a\ scaled\ version\ of\ the\ original$

$property\ F(s,p), where\ the\ exponent\ p\ is\ scaled\ by\ a\ factor$

$of\ -1\ and\ the\ amplitude\ is\ scaled\ by\ a\ factor\ of\ a^{-p}.$

$The\ amplitude\ refers\ to\ the\ change\ in\ magnitude\ or\ size\ of\ the$

$property\ after\ scaling. It\ captures\ how\ the\ property\ is\ stretched$

$or\ compressed\ when\ the\ scaling\ factor\ is\ applied\ . A\ larger$

$amplitude\ indicates\ a\ greater\ scaling\ effect, while\ a\ smaller$

$amplitude\ indicates\ a\ lesser\ scaling\ effect.$

$By\ multiplying\ the\ original\ property\ F(s,p) by\ scaling$

$factor\ a^{-p}, the\ amplitude\ of\ the\ property\ is\ adjusted$

accordingly to account the scaling of the argument. This allows us to compare and analyze the behavior of the property before and after the scaling operation.

2. *The term* $\Theta_s\left(\frac{f(s)}{s^p}\right)$ *represents the impact of the function* $f(x)$ *itself at the scale `s`. It captures the asymptotic behavior of* $f(x)$ *relative to* s^p*. The function* $h(s,p) = \frac{f(s)}{s^p}$ *is bounded between* $c_1(p)h(s,p)$ *and* $c_2(p)h(s,p)$*, where* $c_1(p)$ *and* $c_2(p)$ *are positive functions of* p*. And* c_1 *and* c_2 *are constants. In other words, for all values of* s *greater than some threshold* s_0 *and for all values of* p*, the function* $\Theta_s(h(s,p))$ *lies between* $c_1(p)h(s,p)$ *and* $c_2(p)h(s,p)$*.*

3. *The term* $\Theta_s(1)$ *represents any constant term that arises due to scaling, which doesnot depends* $f(x)$ *or* s*.*

Overall, the scaling property provides a relationship between

$\textit{the property P of a function } f(x) \textit{ and the scaled function } \left(\frac{x}{a}\right)$

$\textit{when the argument is scaled by a factor of } \frac{1}{a}. \textit{ It shows}$

how the property changes in terms of scaling exponents, amplitude, and the asymptotic behavior of the function.

The `scaling` refers to the transformation of a function or its argument by a factor.

When we say "scaling the argument of the function $f(x)$ *by a factor* $\frac{1}{a}$*", it means multiplying the argument `x` of the function* $f(x)$ *by* $\frac{1}{a}$*. This scaling factor stretches or compresses the input value of the function.*

Proof of the theorem:

1. *Since* f *is bounded and the range of the integral is finite, then then* $P\{\ \}$ *exists.*

2. *Linearity of the transform is trivial. Which means that the property or characteristic of the transform satisfies the properties of linearity in a straightforward and obvious manner, without requiring complex or intricate*

$proofs\ by\ complex\ mathematical\ reasoning\ .A\ linear$
$transform\ staisfies\ two\ perties$:

1. $Additivity$: $The\ transform\ of\ the\ sum\ of\ two\ inputs\ is$
$is\ equal\ to\ the\ sum\ of\ the\ transforms\ of\ the\ individual$
$inputs. Mathematically\ ,for\ a\ transform\ P\ and$
$inputs\ f(x) and\ g(x), linearity\ can\ be\ expressed\ as$:

$$P\{f(x) + g(x)\} = P\{f(x)\} + P\{g(x)\}\ .$$

2. $Scalar\ Multiplication$: $The\ transform\ of\ a\ scalar\ multiple$
$of\ an\ input\ is\ equal\ to\ the\ scalar\ multiple\ of\ the$
$transform\ of\ the\ input. Mathematically\ ,for\ a\ transform$
$P, an\ input\ f(x)\ and\ scalar\ c, linearity\ can\ be\ expressed\ as$:
$P\{cf(x)\} = cP\{f(x)\}\ .$

3. $Let\ f_1(x), f_2(x)\ \epsilon\ S\ and\ let\ P\{f_1(x)\} = P\{f_2(x)\}. Then,$

$$\int_1^s f_1(u)u^{-p-1}\,du = \int_1^s f_2(u)u^{-p-1}\,du$$

$$\Rightarrow \frac{\partial}{\partial s}\int_1^s f_1(u)u^{-p-1}\,du = \frac{\partial}{\partial s}\int_1^s f_2(u)u^{-p-1}\,du$$

$$\Rightarrow \frac{\partial}{\partial s}[f_1(u)u^{-p-1}]_1^s = \frac{\partial}{\partial s}[f_2(u)u^{-p-1}]_1^s$$

Computing boundaries:

$$\int_a^b f(x)dx = F(b) - F(a) = \lim_{x\to b^-} F(x) - \lim_{x\to a^+} F(x)$$

$$\lim_{x\to s-} f_1(u)u^{-p-1} = f_1(s)s^{-p-1}$$

$$\lim_{x\to 1+} f_1(u)u^{-p-1} = f_1(1)1^{-p-1} = f_1(1)$$

As $f_1(1)$ *is* $[0,1)$ *hence lets consider* $f_1(1)$ *is* 0 *here.*

Therefore, $f_1(s)s^{-p-1} - f_1(1) = f_1(s)s^{-p-1}$

Same goes for other part:

i.e. we get $f_2(s)s^{-p-1}$

Now, The derivative of an integral is equal to the integrand multiplied by the derivative of the upper limit.

$Here\ integrands\ are:\ f_1(u)u^{-p-1}\ and\ f_2(u)u^{-p-1}\, du$,

$and\ upper\ limit\ is\ `s`\ here.$

$Hence\ we\ get$:

$$f_1(s)s^{-p-1} \times \frac{\partial}{\partial s}(s) = f_2(s)s^{-p-1} \times \frac{\partial}{\partial s}(s)$$

$Hence\ we\ get$:

$f_1(s)s^{-p-1} = f_2(s)s^{-p-1}$

$\therefore f_1(s) = f_2(s)[As\ s^{-p-1}\ gets\ eliminated\ from\ both\ the$

$sides]$

$which\ completes\ the\ proof\ that\ P\{\}\ is\ one - to - one.$

$4. To\ proof\ scaling\ property:$

$$F_1(s,p) = \int_1^s f\left(\frac{u}{a}\right) u^{-p-1} du$$

$Making\ change\ of\ variable\ v = \frac{u}{a}$,

$i.e. when\ u = 1, v = \frac{1}{a}\ in\ lower\ limit.$

$In\ upper\ limit\ u = s\ , then\ v\ becomes \frac{s}{a}.$

$v = \frac{u}{a} \implies u = av$

$\implies du = a \times dv\ .$

$After\ replacing\ all\ of\ these\ we\ obtain$:

$$\Rightarrow \int_{\frac{1}{a}}^{\frac{s}{a}} f(v)(av)^{-p-1} adv$$

$$\Rightarrow \int_{\frac{1}{a}}^{\frac{s}{a}} f(v)(a^{-p-1}v^{-p-1}) adv$$

$$\Rightarrow \int_{\frac{1}{a}}^{\frac{s}{a}} f(v)(a^{-p}a^{-1}v^{-p-1}) adv$$

$$\Rightarrow \int_{\frac{1}{a}}^{\frac{s}{a}} f(v)\left(a^{-p} \times \frac{1}{a} \times\ v^{-p-1}\right) adv$$

$$\Rightarrow \int_{\frac{1}{a}}^{\frac{s}{a}} f(v)(a^{-p}\ v^{-p-1}) dv$$

$$\Rightarrow a^{-p}\int_{\frac{1}{a}}^{\frac{s}{a}} f(v)(v^{-p-1})dv$$

$We\ can\ write\ it\ as$:

$$\Rightarrow a^{-p}\left[\int_{1}^{s}-\int_{\frac{s}{a}}^{s}+\int_{\frac{1}{a}}^{1} f(v)v^{-p-1}dv\right]$$

$And\ it\ represents$:

$$\Longrightarrow a^{-p}\int_{1}^{s} f(v)v^{-p-1}dv - a^{-p}\int_{\frac{s}{a}}^{s} f(v)v^{-p-1}dv$$
$$+ a^{-p}\int_{\frac{1}{a}}^{1} f(v)v^{-p-1}dv$$

$We\ know$:

$F_1(s,p) = \int_{1}^{s} f\left(\frac{u}{a}\right)u^{-p-1}du\ \ and\ v = \frac{u}{a}\ , hence$:

$$\Longrightarrow a^{-p}F(s,p) - a^{-p}\int_{\frac{s}{a}}^{s} f(v)v^{-p-1}dv + a^{-p}\int_{\frac{1}{a}}^{1} f(v)v^{-p-1}dv$$

$\boldsymbol{Similarly}$,

$$a^{-p}\int_{\frac{1}{a}}^{1} f(v)v^{-p-1}dv$$

$\boldsymbol{We\ can\ write\ it\ as}$:

$$\Rightarrow a^{-p}\times[f(v)v^{-p-1}]_{\frac{1}{a}}^{1}$$

$\boldsymbol{Applying\ power\ rule}$: $\int x^{a}dx=\dfrac{x^{a+1}}{a+1}, a\neq 1$

$$\Rightarrow a^{-p}\times\left[f(v)\,\frac{v^{-p-1+1}}{-p-1+1}\right]_{\frac{1}{a}}^{1}$$

$$\Rightarrow a^{-p}\times\left[f(v)\,\frac{v^{-p}}{-p}\right]_{\frac{1}{a}}^{1}$$

$$\Rightarrow a^{-p}\times-\frac{1}{p}[f(v)v^{-p}]_{\frac{1}{a}}^{1}$$

$\boldsymbol{By\ computation\ of\ boundaries}$:

$$\int_a^b f(x)dx = F(b) - F(a) = \lim_{x\to b^-} F(x) - \lim_{x\to a^+} F(x)$$

$$\Rightarrow a^{-p} \times -\frac{1}{p}\left(\lim_{v\to 1^+} f(v)v^{-p} - \lim_{v\to\left(\frac{1}{a}\right)^{-1}} f(v)v^{-p}\right)$$

$$\Rightarrow a^{-p} \times -\frac{1}{p}\left(f(1)1^{-p-1} - f\left(\frac{1}{a}\right)\left(\frac{1}{a}\right)^{-p}\right)$$

$$\Rightarrow a^{-p} \times -\frac{1}{p}\left(f(1) - f\left(\frac{1}{a}\right)\left(\frac{1}{a}\right)^{-p}\right)$$

We can rewrite it as:

$$\Rightarrow a^{-p} \times -\frac{1}{p}\left(f(1) - f\left(\frac{1}{a}\right)\left(\frac{1}{a^{-p}}\right)\right)$$

$$\Rightarrow a^{-p} \times -\frac{1}{p}\left(f(1) - f\left(\frac{1}{a}\right)(a^{p})\right)$$

$$\Rightarrow \left(\left(f(1) \times a^{-p} \times -\frac{1}{p}\right) - \left(f\left(\frac{1}{a}\right) \times a^{p} \times a^{-p} \times -\frac{1}{p}\right)\right)$$

$$\Rightarrow \left(f(1) \times a^{-p} \times -\frac{1}{p}\right) - \left(f\left(\frac{1}{a}\right) \times -\frac{1}{p}\right)$$

$\boldsymbol{We\ can\ write\ it\ as}$:

$$\Rightarrow \left(f(1) \times \frac{1}{a^p} \times -\frac{1}{p}\right) - \left(f\left(\frac{1}{a}\right) \times -\frac{1}{p}\right)$$

$\boldsymbol{Now\ let\ us\ test\ converginity}$:

$\underline{\boldsymbol{Left\ side}}$:

$$\lim_{p\to\infty}\left(f(1) \times \frac{1}{a^p} \times -\frac{1}{p}\right) , a > 1$$

$$= \left(f(1) \times \frac{\lim\limits_{p\to\infty}(1)}{\lim\limits_{p\to\infty}(a^p)} \times -\frac{\lim\limits_{p\to\infty}(1)}{\lim\limits_{p\to\infty}(p)} \right)$$

$$= \left(f(1) \times \frac{1}{\infty} \times -\frac{1}{\infty} \right)$$

$$= (f(1) \times 0)$$

$\underline{\boldsymbol{Right\ side}}$:

$$\lim_{p\to\infty}\left(f\left(\frac{1}{a}\right) \times -\frac{1}{p}\right) , a > 1$$

$$= \left(f\left(\frac{1}{a}\right) \times -\frac{\lim_{p\to\infty}(1)}{\lim_{p\to\infty}(p)} \right)$$

$$= f\left(\frac{1}{a}\right) \times -\frac{1}{\infty}$$

$$= f\left(\frac{1}{a}\right) \times 0$$

$$= 0$$

$Hence\ 0 - 0 = 0\ , therefore,$

$Hence\ as\ `p`\ approaches\ to\ infinity\ , the\ equation$

$converges\ to\ zero, as\ a > 1.$

$Therefore\ , if\ we\ replace\ `p`\ with\ `s`\ we\ get\ the\ same$

$result\ i.e. when\ s\ tends\ to\ \infty, the\ above\ equation$

$converges\ to\ zero.$

$Hence\ we\ can\ write:\ a^{-p} \int_{\frac{1}{a}}^{1} f(v) v^{-p-1} dv = \Theta_s(1)$

$Now\ we\ are\ left\ with:$

$$\Rightarrow a^{-p}F(s,p) - a^{-p}\int_{\frac{s}{a}}^{s} f(v)v^{-p-1}dv + \Theta_s(1)$$

Hence lets investigate the asymptotic behavior of

$a^{-p}\int_{\frac{s}{a}}^{s} f(v)v^{-p-1}dv$ ***, with respect to `s`***

$A)$ ***For all*** $v > 0, f(v)$ ***is a non − decreasing function, and***

$B)$ ***There exist*** $k_1, s_1 > 0$ ***such that*** $k_1(v) \leq f\left(\frac{v}{a}\right)$ ***for all***

$v \geq s_1$

Therefore, for all $v \in \left[\frac{s}{a}, s\right]$ ***we have:***

$$f\left(\frac{s}{a}\right) \leq f(v) \leq f(s).$$

Since for $s \geq s_1$ ***we have*** $k_1 f(s) \leq f\left(\frac{s}{a}\right)$***, then***

$k_1 f(s) \leq f(v) \leq f(s)$ ***for all*** $s > s_1$

$$\Rightarrow k_1 \frac{f(s)}{v^{p+1}} \leq \frac{f(v)}{v^{p+1}} \leq \frac{f(s)}{v^{p+1}} \text{ for all } s > s_1,$$

$$\Rightarrow k_1 f(s)\int_{\frac{s}{a}}^{s}\frac{1}{v^{p+1}}dv \le \int_{\frac{s}{a}}^{s} f(v)\frac{1}{v^{p+1}}d(v) \le f(s)\int_{\frac{s}{a}}^{s}\frac{1}{v^{p+1}}dv$$

$for\ all\ s > s_1,$

$We\ have\ two\ cases\ to\ consider, the\ case\ of\ p\ \neq 0\ and$

$the\ case\ of\ p\ =\ 0\ .$

<u>$A)\ If\ p\ \neq 0\ ,we\ get$:</u>

$$\int_{\frac{s}{a}}^{s}\frac{1}{v^{p+1}}dv = \left[\frac{1}{v^{p+1}}\right]_{\frac{s}{a}}^{s}$$

$Applying\ exponent\ rule$:

$$= [v^{-p-1}]_{\frac{s}{a}}^{s}$$

$Applying\ power\ rule$: $\int x^a dx = \dfrac{x^{a+1}}{a+1}, a \neq 1$

$$= \left[\frac{v^{-p-1+1}}{-p-1+1}\right]_{\frac{s}{a}}^{s}$$

$$= \left[\frac{v^{-p}}{-p}\right]_{\frac{s}{a}}^{s}$$

$$= -\frac{1}{p}[v^{-p}]_{\frac{s}{a}}^{s}$$

$$= -\frac{1}{p}\left[\frac{1}{v^{p}}\right]_{\frac{s}{a}}^{s}$$

$\boldsymbol{By\ computation\ of\ boundaries}$:

$$\int_{a}^{b} f(x)dx = F(b) - F(a) = \lim_{x \to b^{-}} F(x) - \lim_{x \to a^{+}} F(x)$$

$\boldsymbol{Replacing\ v\ with\ s\ and\ \frac{s}{a}\ we\ get}$:

$$\lim_{s \to s^{-}}\left[\frac{1}{s^{p}}\right] = \frac{1}{s^{p}}\,(will\ remain\ same)$$

$$\lim_{s \to \left(\frac{s}{a}\right)^{+}}\left[\frac{1}{s^{p}}\right] = \frac{1}{\left(\frac{s}{a}\right)^{p}} = \frac{1}{\frac{s^{p}}{a^{p}}}$$

$\boldsymbol{Now,}$

$$F(b) - F(a) = -\frac{1}{p}\left(\frac{1}{s^{p}} - \frac{1}{\frac{s^{p}}{a^{p}}}\right) = -\frac{1}{p}\left(\frac{1}{s^{p}} - \frac{a^{p}}{s^{p}}\right) = -\frac{1}{p}\left(\frac{1 - a^{p}}{s^{p}}\right)$$

$$= \frac{1}{s^p}\left(\frac{a^p - 1}{P}\right)$$

Therefore,

Replacing the equation :

$$k_1 f(s) \int_{\frac{s}{a}}^{s} \frac{1}{v^{p+1}} dv \leq \int_{\frac{s}{a}}^{s} f(v) \frac{1}{v^{p+1}} d(v) \leq f(s) \int_{\frac{s}{a}}^{s} \frac{1}{v^{p+1}} dv$$

we get:

$$k_1 \frac{f(s)}{s^p} \left(\frac{a^p - 1}{p}\right) \leq \int_{\frac{s}{a}}^{s} f(v) \frac{1}{v^{p+1}} d(v) \leq \frac{f(s)}{s^p} \left(\frac{a^p - 1}{p}\right)$$

for all $s > s_1$,

$$i.e., \int_{\frac{s}{a}}^{s} f(v) \frac{1}{v^{p+1}} dv = \Theta_s\left(\frac{f(s)}{s^p}\right)\left[as \left(\frac{a^p - 1}{p}\right) \text{ is constant }\right]$$

$$\left(And \left(\frac{a^p - 1}{p}\right) > 0, \qquad a > 1\right)$$

(B) If p = 0 then:

$$\int_{\frac{s}{a}}^{s} \frac{1}{v} d(v)$$

$$[\log v]_{\frac{s}{a}}^{s} \left[\int \frac{1}{v} dv = \log v\right]$$

Computing the boundaries:

$$\int_a^b f(x)dx = F(b) - F(a) = \lim_{x \to b^-} F(x) - \lim_{x \to a^+} F(x)$$

$$\lim_{v \to (s)^-} \log v = \log s$$

$$\lim_{v \to \left(\frac{s}{a}\right)^+} \log v = \log\left(\frac{s}{a}\right)$$

$$Now, \log s - \log \frac{s}{a}$$

$$Applying\ logarithmic\ rule \to \log x - \log y = \log \frac{x}{y}\ , we\ get:$$

$$\Rightarrow \log \frac{s}{\frac{s}{a}} = \log(s \times \frac{a}{s}) = \log a$$

Hence ,

$As\ p = 0,$

$$\int_{\frac{s}{a}}^{s} \frac{1}{v^{p+1}} d(v) = \int_{\frac{s}{a}}^{s} \frac{1}{v^{0+1}} d(v) \ \int_{\frac{s}{a}}^{s} \frac{1}{v} d(v) = \ \log s - \log \frac{s}{a}$$

$$\log a = \frac{\log a}{s^p} \ as\ we\ doing\ in\ respect\ of\ \Theta\left(\frac{1}{s^p}\right).$$

$Replacing\ in$:

$$k_1 f(s) \int_{\frac{s}{a}}^{s} \frac{1}{v^{p+1}} dv \leq \int_{\frac{s}{a}}^{s} f(v) \frac{1}{v^{p+1}} d(v) \leq f(s) \int_{\frac{s}{a}}^{s} \frac{1}{v^{p+1}} dv$$

$We\ get$:

$$k_1 \log a \frac{f(s)}{s^p} \leq \int_{\frac{s}{a}}^{s} f(v) \frac{1}{v^{p+1}} d(v) \leq \log a \frac{f(s)}{s^p}$$

$$i.e. \int_{\frac{s}{a}}^{s} f(v) \frac{1}{v^{p+1}} d(v) = \ \Theta\left(\frac{f(s)}{s^p}\right)$$

$$or, \int_{\frac{s}{a}}^{s} \frac{f(v)}{v^{p+1}} d(v) = \ \Theta\left(\frac{f(s)}{s^p}\right)$$

$Note$: $\log a\ \ is\ a\ constant\ and\ \ \log a > 0$.

$Thus\ replacing\ in\ eqn$:

$$a^{-p}F(s,p) - a^{-p}\int_{\frac{s}{a}}^{s} f(v)v^{-p-1}dv + \Theta_s(1)$$

$$\Rightarrow\ a^{-p}F(s,p) - a^{-p}\int_{\frac{s}{a}}^{s} \frac{f(v)}{v^{p+1}}dv + \Theta_s(1)$$

$\boldsymbol{We\ get}$:

$$\Rightarrow\ a^{-p}F(s,p) - \Theta\left(\frac{f(s)}{s^p}\right) + \Theta_s(1)$$

$\boldsymbol{Which\ completes\ the\ proof.}$

$\boldsymbol{Part\ 2: Let\ f(x)\ be\ a\ function\ defined\ as\ in\ Theorem\ 1.}$

$\boldsymbol{In\ other\ words,}$

$$f(x) = \begin{cases} u_0 & x \in [0,1) \\ \displaystyle\sum_{i=1}^{k} a_i f\left(\left\lfloor \frac{x}{b_i} \right\rfloor\right) + g(\lfloor x \rfloor) & x \in [1,\infty) \end{cases}$$

$Where$:

- $u_0, a_i \in R^{*+}, \sum_{i=1}^{k} a_i \geq 1$
- $b_i\ ,\ k \in N,\ b_i \geq 2, k \geq 1$
- $g(x) is\ a\ bounded, positive\ and\ nondecreasing\ function\ for\ all\ x\ \geq\ 0.$
- $for\ all\ c\ >\ 1, there\ exist\ x_1, k_1\ >\ 0\ such\ that\ g\left(\frac{x}{c}\right) \geq k_1 g(x), for\ all\ x\ \geq\ x_1.$

$Then,$

$1. g(\lfloor x \rfloor) \in S$

$2. f(x) \in S$

$\underline{Proof}$:

$1. Note\ that\ g(x) \in S\ by\ definition.$

$Hence\ , g(\lfloor x \rfloor)\ \ is\ clearly\ boundedand\ non-decreasing.$
$Moreover\ , g(\lfloor x \rfloor)\ is\ positive.$

$There\ remains\ to\ prove\ that$:

$for\ c >\ 1\ , there\ exists\ x_2, k_2, such\ that\ g\left(\left\lfloor \frac{x}{2} \right\rfloor\right) \geq k_2 g(\lfloor x \rfloor\,)$
$for\ all\ x \geq x_2\ .$

$$\text{Let } c > 1 \text{ be given. Let } x > \max\left\{x_1 + c + 1, \frac{c^2}{c-1} + 1\right\} \text{ then}$$

the following three useful inequalities can be derived:

$$x > x_1 + c + 1$$

$$\Rightarrow \lfloor x \rfloor > x_1 + c$$

$$\Rightarrow \lfloor x \rfloor - c > x_1 ------ (a)$$

On the other hand,

$$x > \frac{c^2}{c-1} + 1$$

$$\Rightarrow \lfloor x \rfloor > \frac{c^2}{c-1} + 1$$

$$\Rightarrow \lfloor x \rfloor > \frac{c^2}{c-1}$$

$$\Rightarrow \lfloor x \rfloor (c-1) > c^2$$

$$\Rightarrow \frac{\lfloor x \rfloor (c-1)}{c} > c$$

$$\Rightarrow \frac{c\lfloor x \rfloor - \lfloor x \rfloor}{c} > c$$

$$\Rightarrow \lfloor x \rfloor - \frac{\lfloor x \rfloor}{c} > c$$

$$\Longrightarrow \lfloor x \rfloor - c > \frac{\lfloor x \rfloor}{c} - - - - -(b)$$

Finally,

$$x > x_1 + c + 1$$

$$\Longrightarrow \lfloor x \rfloor > x_1 - - - - - - - - - - - - - - -(c)$$

Using above inequalities of $(a), (b), (c)$, *we conclude* :

$$g\left(\left\lfloor \frac{x}{c} \right\rfloor\right) = g\left(\frac{\lfloor x \rfloor}{c}\right) (\textit{using Part } 1)$$

$$\geq g\left(\frac{[x]}{c} - 1\right) \ (\textit{since } g \textit{ is a non} - \textit{decreasing})$$

$$= g\left(\frac{\lfloor x \rfloor - c}{c}\right)$$

$$\geq k_1 g(\lfloor x \rfloor - c) \ \big(\textit{using Inequality } (a)\big)$$

$$> k_1 g\left(\frac{\lfloor x \rfloor}{c}\right)\big(\textit{using Inequality } (b)\big)$$

$$> (k_1)^2 \ g(\lfloor x \rfloor) \ \big(\textit{using Inequality } (c)\big)$$

Therefore, for all $c > 1$, *such that*

$$x_2 = max\left(x_1 + c + 1, \frac{c^2}{c-1} + 1\right) > 0, k_2 = (k_1)^2 > 0 \text{ , such that}$$

$$g\left(\frac{\lfloor x \rfloor}{c}\right) \geq k_2 g(\lfloor x \rfloor), \text{which completes the proof that } g(\lfloor x \rfloor) \in S.$$

$2.$ ***To prove that $f(x) \in S$, note that \`$f$\` is defined in terms of a finite sum of positive terms each of which is bounded and also positive. So, \`$f$\` is bounded and positive.***

Also according to Theroem 1, $f(x) = f(\lfloor x \rfloor)$.

So, it is sufficient to prove that:

$(A) f(n)$ ***is non − decreasing for all*** $n > 0$,

(B) ***for all*** $c > 1$, ***there exists*** n_3 , k_3 ***such that***

$f\left(\frac{n}{c}\right) \geq k_3 f(n)$ ***for all*** $n > n_3$.

To prove $f(n)$ is non − decreasing , we use strong induction . Note that :

$f(0) = u_0$,

$$f(1) = \sum_{i=1}^{k} a_i u_0 + g(1) \geq u_0$$

$$\left(since \sum\nolimits_{i=1}^{k} a_i \geq 1 \; and \; g(1) \geq 0 \right)$$

Assume that for all $k < n$ *we have* $f(k) \geq f(k-1)$, *and let us prove that* $f(n) \geq f(n-1)$. *Consider*

$$f(n) = \sum_{i=1}^{k} a_i f\left(\frac{n}{b^i}\right) + g(n)$$

$$= \sum_{i=1}^{k} a_i f\left(\left\lfloor \frac{n}{b^i} \right\rfloor\right) + g(n)$$

$$\geq \sum_{i=1}^{k} a_i f\left(\left\lfloor \frac{n-1}{b^i} \right\rfloor\right) + g(n-1) \;\; (g(n) \; is \; nondecreasing)$$

$$= \sum_{i=1}^{k} a_i f\left(\left\lfloor \frac{n-1}{b^i} \right\rfloor\right) + g(n-1)$$

$$= f(n-1)$$

$So, for\ all\ n \geq 1\ we\ have\ f(n) \geq f(n-1). Therefore\ , f\ is$

$is\ non - decreasing. To\ prove\ the\ regularity\ condition,$

$we\ use\ strong\ induction\ again. Using\ the\ results\ of\ the$

$previous\ Part, there\ exist\ k_2, x_2, such\ that$

$g\left(\left\lfloor \frac{x}{c} \right\rfloor\right) \geq k_2 g(\lfloor x \rfloor) for\ all\ x \geq x_2, c > 1\ . Consider:$

$$n_0 = \lfloor x_2 \rfloor,$$

$$k_3 = min\left\{\frac{f(0)}{f(n_0)}, k_2\right\}.$$

$f\ is\ nondecreasing\ and\ strictly\ positive. So\ for\ all$

$m \in [0, n_0)$

$$f\left(\frac{m}{c}\right) \geq f(0)$$

$$\geq f(0)\frac{f(m)}{f(n_0)}$$

$$= f(m)\frac{f(0)}{f(n_0)}$$

$$\geq f(m)k_3$$

$i.e., for\ all\ m \in [0, n_0] we\ have\ f\left(\frac{m}{c}\right) \geq k_3 f(m).$

$Now\ assume\ that\ for\ all\ m \in [0, n) where\ n > n_0\ we\ have$

$f\left(\frac{m}{c}\right) \geq k_3 f(m)$, and let us prove that $f\left(\frac{n}{c}\right) \geq k_3 f(n)$.

Since $n > n_0$, we have $\frac{n}{c} > 1$. So,

$$f\left(\frac{n}{c}\right) = \sum_{i=1}^{k} f\left(\frac{\frac{n}{a_i}}{c}\right) + g\left(\left\lfloor\frac{n}{c}\right\rfloor\right)$$

$$= \sum_{i=1}^{k} f\left(\left\lfloor\frac{\left(\frac{n}{a_i}\right)}{c}\right\rfloor\right) + g\left(\left\lfloor\frac{n}{c}\right\rfloor\right) \quad \left(\text{since } f(n) = f(\lfloor n \rfloor)\right)$$

$$= \sum_{i=1}^{k} f\left(\left\lfloor\frac{\left\lfloor\frac{n}{a^i}\right\rfloor}{c}\right\rfloor\right) + g\left(\left\lfloor\frac{n}{c}\right\rfloor\right) \quad \left(\text{since } \left\lfloor\frac{n}{c}\right\rfloor = \left\lfloor\frac{\lfloor n \rfloor}{c}\right\rfloor\right)$$

$$= \sum_{i=1}^{k} f\left(\frac{\left\lfloor\frac{n}{a^i}\right\rfloor}{c}\right) + g\left(\left\lfloor\frac{n}{c}\right\rfloor\right) \quad \left(\text{since } f(n) = f(\lfloor n \rfloor)\right)$$

$$\geq \sum_{i=1}^{k} k_3 f\left(\frac{\left\lfloor\frac{n}{a^i}\right\rfloor}{c}\right) + g\left(\left\lfloor\frac{n}{c}\right\rfloor\right) \quad \left(\text{since } f(n) = f(\lfloor n \rfloor)\right)$$

$$= k_3 f(n).$$

As a result ,

$for\ all\ c > 1$, $there\ exist\ such\ that\ n_3 = 0$,

$k_3 = min\left\{\frac{f(0)}{f(n_0)}, k_2\right\}$ $such\ that\ f\left(\frac{n}{c}\right) \geq k_3 f(n)\ for\ all$

$n \geq n_3$,

$which\ completes\ the\ proof.$

$Theorem\ 3 : Let\ f(x)\ be\ a\ function\ defined\ as\ in$
$Theorem\ 1. Let\ p_0\ be\ the\ real\ solution\ of\ the\ equation$
$\sum_{i=1}^{k} a_i b_i^{-p} = 1. Then\ p_0\ always\ exists\ \ and\ is\ unique\ and$
$positive. Furthermore,$

$$f(x) = \Theta(x^{p_0}) + \Theta\left(x^{p_0}\int_{x_1}^{x} \frac{g(u)}{u^{p_0+1}} d(u)\right)$$

$for\ x_1\ large\ enough.$

$Proof$:

$Since\ g(x) \in S, then\ according\ Part\ 1\ both\ g(\lfloor x \rfloor) and$
$f(x)\ belong\ to\ S. Hence, both\ functions\ possess\ an\ Order$
$transform\ . Rewriting\ the\ definition\ of\ f(x)$,

$$f(x) = \sum_{i=1}^{k} a_i f\left(\frac{x}{b_i}\right) + g(\lfloor x \rfloor)\ for\ all\ x > 1$$

$$P\{f(x)\} = P\left\{\sum_{i=1}^{k} a_i f\left(\frac{x}{b_i}\right) + g(\lfloor x \rfloor) \quad\right\} for\ all\ s > 1$$

$$P\{f(x)\} = \sum_{i=1}^{k} a_i P\left\{f\left(\frac{x}{b_i}\right)\right\} + \int_1^s g\left(\frac{\lfloor u \rfloor}{u^{p+1}}\right) du$$

$$F(s,p) = \sum_{i=1}^{k} a_i \left(b_i^{-p}\, F(s,p\,) - \Theta\left(\frac{f(s)}{s^p}\right) + \Theta_s(1)\right) + \int_1^s g\left(\frac{\lfloor u \rfloor}{u^{p+1}}\right)\, du$$

$$F(s,p) - \sum_{i=1}^{k} a_i \left(b_i^{-p}\, F(s,p\,)\right) + \Theta\left(\frac{f(s)}{s^p}\right) = \int_1^s g\left(\frac{\lfloor u \rfloor}{u^{p+1}}\right)\, du + \Theta_s(1)$$

$Taking\ F(s,p)\ common$:

$$F(s,p)\left(1 - \sum_{i=1}^{k} a_i\left(b_i^{-p}\right) \quad\right) + \Theta\left(\frac{f(s)}{s^p}\right) = \int_1^s g\left(\frac{\lfloor u \rfloor}{u^{p+1}}\right)\, du + \Theta_s(1)$$

$$\boldsymbol{Let\ h(p) = 1 - \sum_{i=1}^{k} a_i(b_i^{-p})\ .Then:}$$

$$\boldsymbol{h(0) = 1 - \sum_{i=1}^{k} a_i(b_i^{0}) \implies 1 - \sum_{i=1}^{k} a_i \leq 0}$$

$$\boldsymbol{\lim_{p\to\infty} h(p) = \lim_{p\to\infty}(1) - \sum_{i=1}^{k} a_i \lim_{p\to\infty}(b_i^{-p})}$$

$$\boldsymbol{\implies 1 - \sum_{i=1}^{k} a_i \lim_{p\to\infty}\left(\frac{1}{b_i^{p}}\right)}$$

$$\boldsymbol{\implies 1 - \sum_{i=1}^{k} a_i \left(\frac{1}{\infty_i}\right)}$$

$$\boldsymbol{\implies 1 - \sum_{i=1}^{k} a_i \times 0}$$

$$\boldsymbol{\implies 1 - 0}$$

$$\boldsymbol{\implies 1}$$

And

$$\lim_{p\to\infty} h(p) = 1 > 0$$

$$\frac{d}{dp}h(p) = \frac{d}{dp}\left(1 - \sum_{i=1}^{k} a_i(b_i^{-p})\right)$$

$$\Rightarrow \frac{d}{dp}(1) - \frac{d}{dp}\left(\sum_{i=1}^{k} a_i(b_i^{-p})\right)$$

$$\Rightarrow 0 - \frac{d}{dp}\left(\sum_{i=1}^{k} a_i(b_i^{-p})\right)\left[\frac{d}{dx}(c) = 0\,, where\ c\ is\ constant\right]$$

$$Applying\ multiplication\ chain\ rule:$$

$$\frac{d}{dx}f(x)g(x) = f(x) \times \frac{d}{dx}g(x) + g(x) \times \frac{d}{dx}f(x)\,, we\ get:$$

$$= -1\left(\sum_{i=1}^{k} a_i \times \frac{d}{dp}(b_i^{-p}) + (b_i^{-p}) \times \frac{d}{dp}\left(\sum_{i=1}^{k} a_i\right)\right)$$

$$Note: \sum_{i=1}^{k} a_i \quad is\ constant\,, hence:$$

$$= -\mathbf{1}\left(\sum_{i=1}^{k} a_i \times \frac{d}{dp}\left(b_i^{-p}\right) + \left(b_i^{-p}\right) \times \mathbf{0}\right)$$

$$= -\mathbf{1}\left(\sum_{i=1}^{k} a_i \times \frac{d}{dp}\left(b_i^{-p}\right)\right)$$

$$Lets\ deduce\ \frac{d}{dp}\left(b_i^{-p}\right):$$

$$Applying\ exponent\ rule : a^b = e^{b \log a}$$

$$Hence\ , b^{-p} = e^{(-p)\log b}$$

$$= \frac{d}{dp}\left(e^{(-p)\log b}\right)$$

$$Applying\ chain\ rule : \frac{df(u)}{dx} = \frac{df}{du} \times \frac{du}{dx}\ we\ get:$$

$$f = e^u, u = (-p)\log b$$

$$= \frac{d}{du}\left(e^u\right)\frac{d}{dp}(-p)\log b$$

$$= e^{u}\frac{d}{dp}(-p)\log b \quad \left[as\frac{d}{du}e^{u} = e^{u}\right]$$

$$= e^{u}\frac{d}{dp}(-p)\log b \quad \left[as\frac{d}{du}e^{u} = e^{u}\right]$$

$Substuting\ back\ u\ =\ (-p)\ log\ b$

$$= e^{(-p)\,log\,b}\ \ \frac{d}{dp}(-p)\log b$$

$$= e^{(-p)\,log\,b}\ \ \left[\log b \times \frac{d}{dp}(-p)\right]$$

$$= e^{(-p)\,log\,b}\ \ \left[\log b \times -\frac{dp}{dp}\right]$$

$$= e^{(-p)\,log\,b}\ \ [\log b \times -1]$$

$$= -e^{(-p)\,log\,b}\ \ \log b$$

$$= -\left(e^{log\,b}\ \right)^{-p}\log b$$

$This\ is\ actually\ \log_{e} b\ \ ,hence\ e^{\log_{e} b} = b\ as, a^{\log_{a} b} = b$

$$= -(b)^{-p} \log b$$

$$= -b^{-p} \log b$$

$\boldsymbol{Hence,}$

$$= -1\left(\sum_{i=1}^{k} a_i \times \frac{d}{dp}(b_i^{-p})\right)$$

$$= -1\left(\sum_{i=1}^{k} a_i \times -b_i^{-p} \log b_i\right)$$

$$= \sum_{i=1}^{k} a_i\ (\log b_i) b_i^{-p}$$

$\boldsymbol{Hence:}$

$$\frac{d}{dp}h(p) = \frac{d}{dp}\left(1 - \sum_{i=1}^{k} a_i(b_i^{-p})\right) = \sum_{i=1}^{k} a_i\ (\log b_i) b_i^{-p} > 0$$

$\boldsymbol{for\ all\ p\ (b_i \geq 2, a_i > 0).}$

$\boldsymbol{So\ , h(p) = 0\ has\ a\ unique\ positive\ solution\ p_0.}$

$\boldsymbol{We\ get,}$

$$\boldsymbol{F(s,p)\left(1-\sum_{i=1}^{k} a_i(b_i^{-p}) \right)+\Theta\left(\frac{f(s)}{s^p}\right) = \int_1^s g\left(\frac{\lfloor u\rfloor}{u^{p+1}}\right) du+\Theta_s(1)}$$

$$\boldsymbol{\Longrightarrow F(s,p)h(p)+\Theta\left(\frac{f(s)}{s^p}\right) = \int_1^s g\left(\frac{\lfloor u\rfloor}{u^{p+1}}\right) du+\Theta_s(1)}$$

$$\boldsymbol{\Longrightarrow F(s,p)\times 0+\Theta\left(\frac{f(s)}{s^p}\right) = \int_1^s g\left(\frac{\lfloor u\rfloor}{u^{p+1}}\right) du+\Theta_s(1)}$$

$$\boldsymbol{\Longrightarrow 0+\Theta\left(\frac{f(s)}{s^p}\right) = \int_1^s g\left(\frac{\lfloor u\rfloor}{u^{p+1}}\right) du+\Theta_s(1)}$$

$$\boldsymbol{\Longrightarrow \Theta\left(\frac{f(s)}{s^p}\right) = \int_1^s g\left(\frac{\lfloor u\rfloor}{u^{p+1}}\right) du+\Theta_s(1)}$$

***Now replacing p with p_0 we get*:**

$$\boldsymbol{i.e., \Theta\left(\frac{f(s)}{s^{p_0}}\right) = \int_1^s g\left(\frac{\lfloor u\rfloor}{u^{p_0+1}}\right) du+\Theta_s(1)}$$

$$\boldsymbol{i.e., \frac{\Theta\big(f(s)\big)}{\Theta(s^{p_0})} = \int_1^s g\left(\frac{\lfloor u\rfloor}{u^{p_0+1}}\right) du+\Theta_s(1)}$$

$$\Rightarrow \Theta\big(f(s)\big) = \Theta\left(s^{p_0}\int_1^s g\left(\frac{\lfloor u\rfloor}{u^{p_0+1}}\right)\,du\ \right) + \Theta_s(s^{p_0})$$

Replacing s with x, so that $\Theta\big(f(s)\big) = f(x)$, ***we get:***

$$\Rightarrow f(x) = \Theta\left(x^{p_0}\int_1^x g\left(\frac{\lfloor u\rfloor}{u^{p_0+1}}\right)\,du\ \right) + \Theta(x^{p_0})$$

Since $g(x)\ \epsilon\ S$ ***and*** $g(x)$ ***in non − decreasing , then:***

for all $x \geq 1$, $g\left(\frac{x}{2}\right) \leq g(\lfloor x\rfloor)$ ***and***

There exist $k_1, x_1 > 0$ ***such that*** $g\left(\frac{x}{2}\right) \geq k_1 g(x)$ ***for all*** $x > x_1$

In other words for all $x > x_1$, $k_1 g(x) < g(\lfloor x\rfloor) < g(x)$,

$$i.e.,\ k_1\int_{x_1}^x \frac{g(u)}{u^{p_0+1}}\,du \leq \int_{x_1}^x \frac{g(\lfloor u\rfloor)}{u^{p_0+1}}\,du \leq \int_{x_1}^x \frac{g(u)}{u^{p_0+1}}\,du$$

$$i.e.,\ x^{p_0}\int_{x_1}^x \left(\frac{g\lfloor u\rfloor}{u^{p_0+1}}\right)du\ = \Theta\left(x^{p_0}\int_{x_1}^x\left(\frac{g(u)}{u^{p_0+1}}\right)du\ \right)$$

Replacing it in above equation

$$f(x) = \boldsymbol{\Theta}\left(x^{p_0}\int_1^x \left(\frac{g\lfloor u\rfloor}{u^{p_0+1}}\right) du\right) + \boldsymbol{\Theta}\left(x^{p_0}\int_{x_1}^x \left(\frac{g(u)}{u^{p_0+1}}\right) du\right) + \boldsymbol{\Theta}(x^{p_0})$$

$$f(x) = \boldsymbol{\Theta}\left(x^{p_0}\int_{x_1}^x \left(\frac{g(u)}{u^{p_0+1}}\right) du\right) + \boldsymbol{\Theta}(x^{p_0})$$

which completes the proof .

Theorem 4 : Let $f(x)$ be a function defined in Theorem 1. Let p_0 be the unique solution of the characteristic equation:

1. if there exist $\varepsilon > 0$ such that $g(x) = O(x^{p_0-\varepsilon})$, then $f(x) = \Theta(x^{p_0})$.

2. If there exist $\varepsilon > 0$ such that $g(x) = \Omega(x^{p_0+\varepsilon})$ and $\frac{g(x)}{x^{p_0+\varepsilon}}$ is a non − decreasing function then $f(x) = \Theta\big(g(x)\big)$.

3.. If $g(x) = \Theta(x^{p_0})$ then $f(x) = \Theta(x^{p_0}\log x)$

Proof

Suppose there exist $\varepsilon > 0$ such that $g(x) = O(x^{p_0-\varepsilon})$, i.e., there exists $x_0, k > 0$, such that $g(x) < kx^{p_0-\varepsilon}$, for all $x > x_0$.

Let $x_1 > x_0$, then for all $x > x_1$ we have :

$$\int_{x_1}^{x} \frac{g(u)}{u^{p_0+1}} d(u) \leq k \int_{x_1}^{x} \frac{ku^{p_0-\varepsilon}}{u^{p_0+1}} du$$

$$= k \int_{x_1}^{x} \frac{1}{u^{p_0+1}} du$$

$$= k \int_{x_1}^{x} \frac{1}{u^{p_0+1}} du$$

$$= k \int_{x_1}^{x} u^{-p_0-1} du$$

$\boldsymbol{Lets\ make\ the\ power\ rule\ of\ integral}$: $\int x^p = \frac{x^{p+1}}{p+1}$

$\boldsymbol{Hence}$:

$$= k \int_{x_1}^{x} \frac{u^{-p_0-1+1}}{-p_0-1+1} du$$

$$= k \int_{x_1}^{x} \frac{u^{-p_0}}{-p_0} du$$

$$= k \times -\frac{1}{p^0} \int_{x_1}^{x} u^{-p_0} du$$

$\boldsymbol{Let\ u = x, and\ p_0 = \varepsilon\ , then}$:

$$= \boldsymbol{k} \times -\frac{\mathbf{1}}{\boldsymbol{\varepsilon}} \int_{x_1}^{x} \boldsymbol{x^{-\varepsilon} dx}$$

$\boldsymbol{By\ computation\ of\ boundaries}$:

$$\int_a^b f(x)dx = F(b) - F(a) = \lim_{x \to b^-} F(x) - \lim_{x \to a^+} F(x)$$

$$= \boldsymbol{k} \times -\frac{\mathbf{1}}{\boldsymbol{\varepsilon}} \left[\boldsymbol{x^{-\varepsilon}}\right]_{x_1}^{x}$$

$$\lim_{x \to x^-} x^{-\varepsilon} = x^{-\varepsilon} \ \ and \ \lim_{x \to x_1^+} x^{-\varepsilon} = x_1^{-\varepsilon}$$

$$= \boldsymbol{k} \times -\frac{\mathbf{1}}{\boldsymbol{\varepsilon}} \ (\boldsymbol{x^{-\varepsilon} - x_1^{-\varepsilon}})$$

$$= \boldsymbol{k} \times -\frac{\mathbf{1}}{\boldsymbol{\varepsilon}} \ \left((-\mathbf{1})\boldsymbol{x_1^{-\varepsilon} - x^{-\varepsilon}}\right)$$

$$= \frac{\boldsymbol{k}}{\boldsymbol{\varepsilon}} \ (\boldsymbol{x_1^{-\varepsilon} - x^{-\varepsilon}})$$

$$= \frac{k}{\varepsilon}\left(\frac{1}{x_1^{\varepsilon}} - \frac{1}{x^{-\varepsilon}}\right)$$

$$< \frac{k}{\varepsilon} \times \frac{1}{x_1^{\varepsilon}}$$

$$= \mathbf{O}(1)$$

But we proved that :

$$f(x) = \Theta(x^{p_0}) + \Theta\left(x^{p_0}\int_1^s \left(\frac{g(u)}{u^{p_0+1}}\right) du\right)$$

Therefore replacing we get:

$$f(x) = \Theta(x^{p_0}) + \Theta\big(x^{p_0} \times \mathbf{O}(1)\big) = \Theta(x^{p_0}) + \Theta(x^{p_0})$$
$$= \Theta(x^{p_0}).$$

---

2. *Suppose there exists* $\varepsilon > 0$ *such that* $g(x) = \Omega(x^{p_0+\varepsilon})$ *and* $\frac{g(x)}{x^{p_0+\varepsilon}}$ *is a non* $-$ *dereasing function for large* x. *Let* $\phi(x) = \frac{g(x)}{x^{p_0+\varepsilon}}$. *Then*:

$$g(x) = \phi(x)x^{p_0+\varepsilon}$$

$$i.e., x^{p_0} \int_{x_1}^{x} \frac{g(u)}{u^{p_0+1}} du = x^{p_0} \int_{x_1}^{x} \frac{\phi(u)u^{p_0+\varepsilon}}{u^{p_0+1}} du$$

$$= x^{p_0} \int_{x_1}^{x} \phi(u)u^{p_0+\varepsilon-p_0-1} du$$

$$= x^{p_0} \int_{x_1}^{x} \phi(u)u^{\varepsilon-1} du$$

for x_1 *large enough to make* $\phi(x)$ *non decreasing. Therefore, for all* $x > x_1$:

$$x^{p_0} \int_{x_1}^{x} \frac{g(u)}{u^{p_0+1}} du < x^{p_0}\, \phi(x) \int_{x_1}^{x} u^{\varepsilon-1} du$$

Let $u = x$, *we get*:

$$x^{p_0}\, \phi(x) \int_{x_1}^{x} x^{\varepsilon-1} dx$$

Applying power rule: $\int x^a dx = \frac{x^{a+1}}{a+1}, a \neq 1$

$$= x^{p_0}\, \phi(x) \int_{x_1}^{x} \frac{x^{\varepsilon-1+1}}{\varepsilon-1+1} dx$$

$$= \boldsymbol{x^{p_0}\,\phi(x)\int_{x_1}^{x}\frac{x^{\varepsilon}}{\varepsilon}dx}$$

$$\boldsymbol{Taking\ out\,\frac{1}{\varepsilon}\ we\ get}:$$

$$= \boldsymbol{x^{p_0}\,\phi(x)\times\frac{1}{\varepsilon}\int_{x_1}^{x}x^{\varepsilon}dx}$$

$$= \boldsymbol{x^{p_0}\,\phi(x)\times\frac{1}{\varepsilon}\times[x^{\varepsilon}]_{x_1}^{x}}$$

$$\boldsymbol{By\ computation\ of\ boundaries}:$$

$$\boldsymbol{\int_{a}^{b}f(x)dx = F(b) - F(a) = \lim_{x\to b^-}F(x) - \lim_{x\to a^+}F(x)}$$

$$\boldsymbol{\lim_{x\to x^-}(x^{\varepsilon}) = x^{\varepsilon}\ and\ \lim_{x\to x_1^+}(x^{\varepsilon}) = x_1^{\varepsilon}}$$

$$\boldsymbol{Hence}:$$

$$= \boldsymbol{x^{p_0}\,\phi(x)\times\frac{1}{\varepsilon}\times(x^{\varepsilon} - x_1^{\varepsilon})}$$

$$< x^{p_0}\,\phi(x) \times \frac{1}{\varepsilon} \times (x^{\varepsilon})$$

$$< \phi(x)\, x^{p_0+\varepsilon} \times \frac{1}{\varepsilon}$$

$We\ know\ \phi(x)\, x^{p_0+\varepsilon} = g(x)$

$$< g(x) \times \frac{1}{\varepsilon}$$

$$= \mathrm{O}\big(g(x)\big)$$

$Further\ more,$

$$x^{p_0} \int_{x_1}^{x} \left(\frac{g(u)}{u^{p_0+1}}\right) du = x^{p_0} \int_{\frac{x}{a}}^{x} \left(\frac{g(u)}{u^{p_0+1}}\right) du + x^{p_0} \int_{x_1}^{\frac{x}{a}} \left(\frac{g(u)}{u^{p_0+1}}\right) du$$

$But\ g(x) \in S, so:$

$$x^{p_0} \int_{\frac{x}{a}}^{x} \left(\frac{g(u)}{u^{p_0+1}}\right) du = \Theta\big(g(x)\big)[we\ can\ prove\ like\ above].$$

$By\ scaling\ property:$

$$\int_{\frac{x}{a}}^{x}\left(\frac{g(u)}{u^{p_0+1}}\right) du = \Theta\left(\frac{g(x)}{x^{p_0}}\right)$$

Therefore,

$$x^{p_0}\int_{x_1}^{x}\left(\frac{g(u)}{u^{p_0+1}}\right) du = x^{p_0}\int_{\frac{x}{a}}^{x}\left(\frac{g(u)}{u^{p_0+1}}\right) du + x^{p_0}\int_{x_1}^{\frac{x}{a}}\left(\frac{g(u)}{u^{p_0+1}}\right) du$$

$$\Longrightarrow x^{p_0}\int_{x_1}^{x}\left(\frac{g(u)}{u^{p_0+1}}\right) du = x^{p_0}\int_{\frac{x}{a}}^{x}\left(\frac{g(u)}{u^{p_0+1}}\right) du + x^{p_0}\int_{x_1}^{\frac{x}{a}}\left(\frac{g(u)}{u^{p_0+1}}\right) du$$

$$\Longrightarrow x^{p_0}\int_{x_1}^{x}\left(\frac{g(u)}{u^{p_0+1}}\right) du = \Theta\big(g(x)\big) + x^{p_0}\int_{x_1}^{\frac{x}{a}}\left(\frac{g(u)}{u^{p_0+1}}\right) du$$

$$\Longrightarrow x^{p_0}\int_{x_1}^{x}\left(\frac{g(u)}{u^{p_0+1}}\right) du = \Omega\big(g(x)\big)$$

As we got:

$$x^{p_0}\int_{x_1}^{x}\frac{g(u)}{u^{p_0+1}} du = O\big(g(x)\big)$$

$$x^{p_0} \int_{x_1}^{x} \frac{g(u)}{u^{p_0+1}} du = \Omega\big(g(x)\big)$$

Hence,

$$x^{p_0} \int_{x_1}^{x} \frac{g(u)}{u^{p_0+1}} du = \Theta\big(g(x)\big)$$

But we proved in Theorem 3 that:

$$\Rightarrow f(x) = \Theta(x^{p_0}) + \Theta\left(x^{p_0} \int_{x_1}^{x} \frac{g(u)}{u^{p_0+1}} d(u)\right)$$

Hence by replacing we get:

$$f(x) = \Theta\Big(\Big(\Theta\big(g(x)\big)\Big)\Big) + \Theta(x^{p_0})$$

$$\Rightarrow \Theta\big(\mathrm{g}(\mathrm{x})\big) + \Theta(x^{p_0}) \ \ [\textit{Asymptotic Multiplication Property}]$$

$$\Rightarrow \Theta\big(\mathrm{g}(\mathrm{x})\big) \textit{ since } g(x) = \Omega(x^{p_0+\varepsilon})$$

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3. *Suppose* $g(x) = \Theta(x^{p_0})$. *Replacing the result of theorem* 3 *we get*:

$$f(x) = \Theta(x^{p_0}) + \Theta\left(x^{p_0}\int_{x_1}^{x}\frac{g(u)}{u^{p_0+1}}d(u)\right)$$

$$f(x) = \Theta\left(x^{p_0}\int_{x_1}^{x}\frac{1}{u}\,du\ \right) + \Theta_s(x^{p_0})$$

When $g(x)$ *is assymptotically equivalent to* x^{p_0}.

Then $\int_{x_1}^{x}\frac{1}{u}$ *is assymptotically equivalent to* x^{p_0+1}.

This is because :

$\int\frac{1}{u}du = \log u$ *or* $\log x$ *, And* $\log x$ *is assymptotically equivalent to* x^{p_0+1} *for large value of* x.

We can check it , i.e.

$$f(x) = \Theta\left(x^{p_0}\int_{x_1}^{s}\left(\frac{g(u)}{u^{p_0+1}}\right)du\ \right) + \Theta(x^{p_0})$$

Hence if $u = x$ *then*:

$$= \boldsymbol{\Theta}\left(x^{p_0}\int_{x_1}^{x}\left(\frac{g(x)}{x^{p_0+1}}\right)\,dx\ \right) + \boldsymbol{\Theta}(x^{p_0})$$

$$= \boldsymbol{\Theta}\left(x^{p_0}\int_{x_1}^{x}\left(\frac{x^{p_0}}{x^{p_0+1}}\right)\,dx\ \right) + \boldsymbol{\Theta}(x^{p_0})$$

$$= \boldsymbol{\Theta}\left(x^{p_0}\int_{x_1}^{x}(x^{p_0-p_0-1})\,dx\ \right) + \boldsymbol{\Theta}(x^{p_0})$$

$$= \boldsymbol{\Theta}\left(x^{p_0}\int_{x_1}^{x}(x^{-1})\,dx\ \right) + \boldsymbol{\Theta}(x^{p_0})$$

$$= \boldsymbol{\Theta}\left(x^{p_0}\int_{x_1}^{x}\left(\frac{1}{x}\right)\,dx\ \right) + \boldsymbol{\Theta}(x^{p_0})$$

or,

$$= \boldsymbol{\Theta}\left(x^{p_0}\int_{x_1}^{x}\left(\frac{1}{u}\right)\,du\ \right) + \boldsymbol{\Theta}(x^{p_0})$$

$\boldsymbol{Continuing\ from}: \boldsymbol{\Theta}\left(x^{p_0}\int_{x_1}^{x}\left(\frac{1}{x}\right)\,dx\ \right) + \boldsymbol{\Theta}(x^{p_0})$

$$f(x) = \boldsymbol{\Theta}\left(x^{p_0}\int_{x_1}^{x}\frac{1}{x}\,dx\ \right) + \boldsymbol{\Theta}_s(x^{p_0})$$

$$\boldsymbol{We\ know}, \int \frac{1}{x}dx = \log x$$

$\boldsymbol{Then}$:

$$= \Theta\left(x^{p_0}[\log x]_{x_1}^{x}\right) + \Theta_s(x^{p_0})$$

$$= \Theta\left(x^{p_0}[\log x]_{x_1}^{x}\right) + \Theta_s(x^{p_0})$$

$\boldsymbol{By\ computation\ of\ boundaries}$:

$$\int_a^b f(x)dx = F(b) - F(a) = \lim_{x \to b^-} F(x) - \lim_{x \to a^+} F(x)$$

$$\lim_{x \to x^-} \log x = \log x \ \ and \lim_{x \to x_1^+} \log x = \log x_1$$

$\boldsymbol{Hence}$,

$$= \Theta\left(x^{p_0}\left(\log x - \log x_1\right)\right) + \Theta_s(x^{p_0})$$

$$= \Theta(x^{p_0} \log x - x^{p_0} \log x_1) + \Theta_s(x^{p_0})$$

$$= \Theta(x^{p_0} \log x - x^{p_0} \log x_1)$$

$\boldsymbol{It\ doesnot\ matter\ of\ x\ or\ x_1\ , hence\ the\ result\ will\ be}$:

$= \Theta(x^{p_0} \log x)$

Hence it completes the proof.

Corollary* 1: *Let* u_n *be a sequence as in Equation* 1. *Then,

$$u_n = \Theta(n^{p_0}) + \Theta\left(n^{p_0} \int_{n_1}^{n} \frac{g(u)}{u^{p_0+1}} du\right) \textit{for } n_1 \textit{ large enough,}$$

where p_0 ***is the real solution of the*** $\sum_{i=1}^{k} a_i b_i^{-p} = 1$ ***which always exists and is unique and positive. Futhermore,***

1. ***if there exists*** $\varepsilon > 0$ ***such that*** $g(x) = O(x^{p_0-\varepsilon})$ ***then*** $u_n = \Theta(n^{p_0})$.

2. ***if there exists*** $\varepsilon > 0$ ***such that*** $g(x) = \Omega(x^{p_0+\varepsilon})$ ***and*** $\frac{g(x)}{x^{p_0+\varepsilon}}$ ***is a non − decreasing function, then*** $u_n = \Theta(g(n))$.

3. ***if*** $g(x) = \Theta(x^{p_0})$ ***then*** $u_n = \Theta(n^{p_0} \log n)$

Proof*: *We can prove these conditions by above process.

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