

## ***Generating Function – Example – 3***

***Example: Solve the following recurrence equation of the towers of Hanoi method using the generating function method:***

$$t_0 = 0$$

$$t_n - 2t_{n-1} = 1 \text{ for } n = 1, 2, 3, \dots$$

**Solution:**

***The corresponding generating function for this recurrence equation would be :***

$$G(z) = a_0 + a_1z + a_2z^2 + a_3z^3 + \dots$$

$$2z \times G(z) = 2a_{1-1}z^1 + 2a_{2-1}z^2 + 2a_{3-1}z^3$$

$$G(z) = a_0 + a_1z + a_2z^2 + a_3z^3 + \dots$$

$$5z \times G(z) = 2a_0z + 2a_1z^2 + 2a_2z^3$$

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$$G(z) - (2z \times G(z)) = a_0 + (a_1z - 2a_0z) + (a_2z^2 - 2a_1z^2) + (a_3z^3 - 2a_2z^3) + \dots$$


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$$(1 - 2z)(G(z)) = a_0 + (a_1 - 2a_0)z + (a_2 - 2a_1)z^2 + (a_3 - 2a_2)z^3 + \dots$$

*As given  $t_n - 2t_{n-1} = 1$ , therefore,*

$$(1 - 2z)(G(z)) = a_0 + 1z + 1z^2 + 1z^3 + \dots$$

*And  $t_0 = 0$ , then:*

$$(1 - 2z)(G(z)) = 0 + z + z^2 + z^3 + \dots$$

$$(1 - 2z)(G(z)) = z + z^2 + z^3 + \dots$$

$$(1 - 2z)(G(z)) = z(1 + z + z^2 + \dots)$$

$$\sum_{r=0}^{\infty} z^r = \frac{1}{1-z}$$

$$(1-2z)(G(z)) = z(z^0 + z^1 + z^2 + \dots)$$

$$(1-2z)(G(z)) = z\left(\sum_{r=0}^{\infty} z^r\right)$$

$$(1-2z)(G(z)) = z\left(\frac{1}{1-z}\right)$$

$$(1-2z)(G(z)) = \frac{z}{1-z}$$

$$G(z) = \frac{z}{(1-z)(1-2z)}$$

***Using the partial function:***

***We observe  $(1-z) - (1-2z) = 2z - z = z$ , hence***

***Therefore,***

$$\frac{(1-z)}{(1-z)(1-2z)} - \frac{(1-2z)}{(1-z)(1-2z)}$$

$$= \frac{1}{(1-2z)} - \frac{1}{(1-z)}$$

*Now,*

$$\sum_{n=0}^{\infty} 2^n z^n = 2z^0 + 4z^1 + 8z^2 + \dots = \frac{1}{1-2z}$$

*Also,*

$$\sum_{n=0}^{\infty} z^n = \frac{1}{(1-z)}$$

*Hence,*

$$= \sum_{n=0}^{\infty} 2^n z^n - \sum_{n=0}^{\infty} z^n$$

$$= 2^n - 1$$

*It is the general form of recurrence equation.*

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