

Divide And Conquer – AkraBazzi – Theorem Proof

***The proof is given by Mohammad Akra and LOUAY BAZZI
in On the Solution of Linear Recurrence Equations .***

The same proof is depicted here:

**The general solution for linear divide – and
–conquer recurrences of the form *provided*:**

$$u_n = \sum_{i=1}^k a_i u_{\lfloor \frac{n}{b^i} \rfloor} + g(n).$$

***As said in the proof that it can handle more cases than
Master Method.***

Introduction:

$$u_n = \begin{cases} u_0 & n = 0 \\ \sum_{i=1}^k a_i u_{\lfloor \frac{n}{b^i} \rfloor} + g(n) & n \geq 1 \end{cases}$$

Where

$$\rightarrow u_0, a_i \in R^{*+}, \sum_{i=1}^k a_i \geq 1,$$

R^{*+} is positive real numbers excluding 0.

$$\rightarrow b_i, k \in N, b_i \geq 2, k \geq 1$$

$g(x)$ is defined for real values x , and is bounded, positive and nondecreasing function for all $x \geq 0$

\rightarrow For all $c > 1$, there exist $x_1, k_1 > 0$ such that

$$g\left(\frac{x}{c}\right) \geq k_1 g(x), \text{ for all } x \geq x_1.$$

Theorem 1: Let u_n be a sequence . Let $f(x)$ be a function defined by:

$$f(x) = \begin{cases} u_0 & x \in [0, 1) \\ \sum_{i=1}^k a_i f\left(\frac{x}{b_i}\right) + g([x]) & x \in [1, \infty) \end{cases}$$

$x \in [0, 1)$, is closed open neighbourhood i. e. 0 to 1 .

And $x \in [1, \infty)$ is also closed open neighbourhood from 1 to infinite.

Then,

1. For all $x \geq 0$, $f(x) = f([x])$.

2. For all $n \geq 0$, $f(n) = u_n$.

In other words, $f(x)$ is a staircase function.

In mathematics , a step function (also called as staircase function) is defined , as a piecewise constant function, that has only a finite number of pieces with

Linear combinations.

Hence $f(x)$ is a staircase function which matches with u_n at integer values of x .

In proving the above theorem we need a part i. e.

Part 1: If $b \in \mathbb{N}$, $b \geq 1$ and $x \in \mathbb{R}^+$ (positive real number), then:

$$\left\lfloor \frac{x}{b} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor}{b} \right\rfloor.$$

Proof of Theorem 1:

To prove that $f(x) = f(\lfloor x \rfloor)$, we use strong induction.

Note that for all $x \in [0, 1)$ we have $f(x) = u_0$ and $f(\lfloor x \rfloor) = f(0) = u_0$.

Hence, $f(\lfloor x \rfloor) = f(x)$.

Now assume that $f(\lfloor x \rfloor) = f(x)$ for all $x \in [0, n)$, and let us prove that it is true for all $x \in [n, n + 1)$.

Consider,

$$f(x) = \sum_{i=1}^k a_i f\left(\frac{x}{b_i}\right) + g(\lfloor x \rfloor)$$

Let $x \in [n, n + 1)$, then

$$\frac{x}{b_i} \in \left(0, \frac{n+1}{2}\right) \text{ since } b_i \geq 2.$$

But

$$\left(0, \frac{n+1}{2}\right) \subset [0, n) \text{ for } n \geq 1.$$

Hence, we conclude that:

$$\frac{x}{b} \in [0, n), \frac{\lfloor x \rfloor}{b} \in [0, n) \text{ and } \left\lfloor \frac{x}{b} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor}{b} \right\rfloor \in [0, n) \text{ (from Part 1).}$$

Therefore ,

$$f\left(\frac{x}{b_i}\right) = f\left(\left\lfloor \frac{x}{b_i} \right\rfloor\right) \text{ (by assumption)}$$

$$= f\left(\left\lfloor \frac{\lfloor x \rfloor}{b_i} \right\rfloor\right) \text{ (using Part 1)}$$

$$= f\left(\frac{\lfloor x \rfloor}{b_i}\right) \text{ (by assumption)}$$

Now replacing, in eqn $\rightarrow f(x) = \sum_{i=1}^k a_i f\left(\frac{x}{b_i}\right) + g(\lfloor x \rfloor)$,

we get:

$$f(x) = \sum_{i=1}^k a_i f\left(\frac{\lfloor x \rfloor}{b_i}\right) + g(\lfloor x \rfloor)$$

$$f(x) = f(\lfloor x \rfloor), \text{ for all } x \geq 0.$$

Which completes the proof of Part 1 of the theorem.

To prove that $f(n) = u_n$, we use strong induction again.

The equality holds trivially for $n = 0$. Assume it is true for all $m < n$ and consider:

$$f(x) = \sum_{i=1}^k a_i f\left(\frac{n}{b_i}\right) + g(n)$$

Let $n \geq 1$. It has been proved from Part 1 that

$$f\left(\frac{n}{b_i}\right) = f\left(\left\lfloor \frac{n}{b_i} \right\rfloor\right).$$

Now since $\left\lfloor \frac{n}{b_i} \right\rfloor \in [0, n)$ it has been concluded that

$$f\left(\left\lfloor \frac{n}{b_i} \right\rfloor\right) = u_{\left\lfloor \frac{n}{b_i} \right\rfloor}.$$

Replacing again the equation $\sum_{i=1}^k a_i f\left(\frac{n}{b_i}\right) + g(n)$:

$$f(n) = \sum_{i=1}^k a_i u_{\left\lfloor \frac{n}{b^i} \right\rfloor} + g(n).$$

But,

$$u_n = \sum_{i=1}^k a_i u_{\left\lfloor \frac{n}{b^i} \right\rfloor} + g(n)$$

So, $f(n) = u_n$, which completes the proof.

******* Theorem 1 *******

Definition 1: Let S be the set of all real of the real variable variable x satisfying the following conditions:

1. for all $x \geq 0$, $f(x)$ is bounded.
2. for all $x \geq 0$ $f(x)$ is non – decreasing .
3. for all $c > 1$, there exist $x_1, k_2 > 0$ such that for all $x \geq x_1, f\left(\frac{x}{c}\right) \geq k_1 f(x)$,

Theorem 2: (The Order Transform) Let $P\{ \}$ be a mapping that $f(x) \in S$ a real – valued function $F(s, p)$ of the real variables $s \in R^+$ (positive real numbers) and p , defined by:

$$F(s, p) = P\{f(x)\} \equiv \int_1^s f(u)u^{-p-1}du$$

$\equiv \rightarrow$ is known as : identical to

Then $P\{ \}$ satisfies the following properties:

1. $P\{ \}$ exists.
2. $P\{ \}$ is linear.
3. $P\{ \}$ is one – to – one.
4. (Scaling property) Let $f(x) \in S, F(s, p) = P\{f(x)\}, a \in R$ and $a > 1$. Then,

$$p \left\{ f \left(\frac{x}{a} \right) \right\} = a^{-p} F(s, p) - \Theta_s \left(\frac{f(s)}{s^p} \right) + \Theta_s(1),$$

Where $\Theta_s(h(s, p))$ is a function bounded between $c_1(p)h(s, p)$ and $c_2(p)h(s, p)$, for some positive functions $c_1(p), c_2(p)$, for all $s > s_0$, for all p .

The scaling property states that when we scale the function $f(x)$ by a factor of $\frac{1}{a}$, the property P of the function is affected in the following way:

1. The term $a^{-p}F(s, p)$ captures the change in the property due to the scaling. It represents a scaled version of the original property $F(s, p)$, where the exponent p is scaled by a factor of -1 and the amplitude is scaled by a factor of a^{-p} .

The amplitude refers to the change in magnitude or size of the property after scaling. It captures how the property is stretched or compressed when the scaling factor is applied. A larger amplitude indicates a greater scaling effect, while a smaller amplitude indicates a lesser scaling effect.

By multiplying the original property $F(s, p)$ by scaling factor a^{-p} , the amplitude of the property is adjusted

accordingly to account the scaling of the argument. This allows us to compare and analyze the behavior of the property before and after the scaling operation.

2. The term $\Theta_s\left(\frac{f(s)}{s^p}\right)$ represents the impact of the function $f(x)$ itself at the scale `s`. It captures the asymptotic behavior of $f(x)$ relative to s^p . The function $h(s, p) = \frac{f(s)}{s^p}$ is bounded between $c_1(p)h(s, p)$ and $c_2(p)h(s, p)$, where $c_1(p)$ and $c_2(p)$ are positive functions of p . And c_1 and c_2 are constants. In other words, for all values of s greater than some threshold s_0 and for all values of p , the function $\Theta_s(h(s, p))$ lies between $c_1(p)h(s, p)$ and $c_2(p)h(s, p)$.

3. The term $\Theta_s(1)$ represents any constant term that arises due to scaling, which does not depend on $f(x)$ or s .

Overall, the scaling property provides a relationship between

the property P of a function $f(x)$ and the scaled function $\left(\frac{x}{a}\right)$ when the argument is scaled by a factor of $\frac{1}{a}$. It shows how the property changes in terms of scaling exponents, amplitude, and the asymptotic behavior of the function.

The `scaling` refers to the transformation of a function or its argument by a factor .

When we say "scaling the argument of the function $f(x)$ by a factor $\frac{1}{a}$ ", it means multiplying the argument `x` of the function $f(x)$ by $\frac{1}{a}$. This scaling factor stretches or compresses the input value of the function.

Proof of the theorem:

1. Since f is bounded and the range of the integral is finite, then $P\{\}$ exists.

2. Linearity of the transform is trivial. Which means that the property or characteristic of the transform satisfies the properties of linearity in a straightforward and obvious manner, without requiring complex or intricate

proofs by complex mathematical reasoning . A linear transform staisfies two perties :

1. Additivity: The transform of the sum of two inputs is is equal to the sum of the transforms of the individual inputs. Mathematically , for a transform P and inputs $f(x)$ and $g(x)$, linearity can be expressed as:

$$P\{f(x) + g(x)\} = P\{f(x)\} + P\{g(x)\} .$$

2. Scalar Multiplication: The transform of a scalar multiple of an input is equal to the scalar multiple of the transform of the input. Mathematically , for a transform P, an input $f(x)$ and scalar c , linearity can be expressed as:

$$P\{cf(x)\} = cP\{f(x)\} .$$

3. Let $f_1(x), f_2(x) \in S$ and let $P\{f_1(x)\} = P\{f_2(x)\}$. Then,

$$\int_1^s f_1(u)u^{-p-1} du = \int_1^s f_2(u)u^{-p-1} du$$

$$\Rightarrow \frac{\partial}{\partial s} \int_1^s f_1(u)u^{-p-1} du = \frac{\partial}{\partial s} \int_1^s f_2(u)u^{-p-1} du$$

$$\Rightarrow \frac{\partial}{\partial s} [f_1(u)u^{-p-1}]_1^s = \frac{\partial}{\partial s} [f_2(u)u^{-p-1}]_1^s$$

Computing boundaries:

$$\int_a^b f(x)dx = F(b) - F(a) = \lim_{x \rightarrow b^-} F(x) - \lim_{x \rightarrow a^+} F(x)$$

$$\lim_{x \rightarrow s^-} f_1(u)u^{-p-1} = f_1(s)s^{-p-1}$$

$$\lim_{x \rightarrow 1^+} f_1(u)u^{-p-1} = f_1(1)1^{-p-1} = f_1(1)$$

As $f_1(1)$ is $[0, 1)$ hence lets consider $f_1(1)$ is 0 here.

$$\text{Therefore, } f_1(s)s^{-p-1} - f(1) = f_1(s)s^{-p-1}$$

Same goes for other part:

$$\text{i. e. we get } f_2(s)s^{-p-1}$$

Now, The derivative of an integral is equal to the integrand multiplied by the derivative of the upper limit.

Here integrands are : $f_1(u)u^{-p-1}$ and $f_2(u)u^{-p-1} du$, and upper limit is 's' here.

Hence we get:

$$f_1(s)s^{-p-1} \times \frac{\partial}{\partial s}(s) = f_2(s)s^{-p-1} \times \frac{\partial}{\partial s}(s)$$

Hence we get:

$$f_1(s)s^{-p-1} = f_2(s)s^{-p-1}$$

$\therefore f_1(s) = f_2(s)$ [As s^{-p-1} gets eliminated from both the sides]

which completes the proof that $P\{\}$ is one – to – one.

4. To proof scaling property :

$$F_1(s, p) = \int_1^s f\left(\frac{u}{a}\right) u^{-p-1} du$$

Making change of variable $v = \frac{u}{a}$,

i. e. when $u = 1$, $v = \frac{1}{a}$ in lower limit .

In upper limit $u = s$, then v becomes $\frac{s}{a}$.

$$v = \frac{u}{a} \Rightarrow u = av$$

$$\Rightarrow du = a \times dv .$$

After replacing all of these we obtain:

$$\Rightarrow \int_{\frac{1}{a}}^{\frac{s}{a}} f(v)(av)^{-p-1}adv$$

$$\Rightarrow \int_{\frac{1}{a}}^{\frac{s}{a}} f(v)(a^{-p-1}v^{-p-1})adv$$

$$\Rightarrow \int_{\frac{1}{a}}^{\frac{s}{a}} f(v)(a^{-p}a^{-1}v^{-p-1})adv$$

$$\Rightarrow \int_{\frac{1}{a}}^{\frac{s}{a}} f(v) \left(a^{-p} \times \frac{1}{a} \times v^{-p-1} \right) adv$$

$$\Rightarrow \int_{\frac{1}{a}}^{\frac{s}{a}} f(v)(a^{-p} v^{-p-1})dv$$

$$\Rightarrow a^{-p} \int_{\frac{1}{a}}^{\frac{s}{a}} f(v)(v^{-p-1})dv$$

We can write it as:

$$\Rightarrow a^{-p} \left[\int_1^s - \int_{\frac{s}{a}}^s + \int_{\frac{1}{a}}^1 f(v)v^{-p-1}dv \right]$$

And it represents :

$$\begin{aligned} \Rightarrow a^{-p} \int_1^s f(v)v^{-p-1}dv - a^{-p} \int_{\frac{s}{a}}^s f(v)v^{-p-1}dv \\ + a^{-p} \int_{\frac{1}{a}}^1 f(v)v^{-p-1}dv \end{aligned}$$

We know:

$$F_1(s, p) = \int_1^s f\left(\frac{u}{a}\right) u^{-p-1} du \text{ and } v = \frac{u}{a}, \text{ hence:}$$

$$\Rightarrow a^{-p} F(s, p) - a^{-p} \int_{\frac{s}{a}}^s f(v)v^{-p-1}dv + a^{-p} \int_{\frac{1}{a}}^1 f(v)v^{-p-1}dv$$

Similarly ,

$$a^{-p} \int_{\frac{1}{a}}^1 f(v) v^{-p-1} dv$$

We can write it as:

$$\Rightarrow a^{-p} \times [f(v) v^{-p-1}]_{\frac{1}{a}}^1$$

$$\text{Applying power rule: } \int x^a dx = \frac{x^{a+1}}{a+1}, a \neq -1$$

$$\Rightarrow a^{-p} \times \left[f(v) \frac{v^{-p-1+1}}{-p-1+1} \right]_{\frac{1}{a}}^1$$

$$\Rightarrow a^{-p} \times \left[f(v) \frac{v^{-p}}{-p} \right]_{\frac{1}{a}}^1$$

$$\Rightarrow a^{-p} \times -\frac{1}{p} [f(v) v^{-p}]_{\frac{1}{a}}^1$$

By computation of boundaries:

$$\int_a^b f(x)dx = F(b) - F(a) = \lim_{x \rightarrow b^-} F(x) - \lim_{x \rightarrow a^+} F(x)$$

$$\Rightarrow a^{-p} \times -\frac{1}{p} \left(\lim_{v \rightarrow 1^+} f(v)v^{-p} - \lim_{v \rightarrow \left(\frac{1}{a}\right)^-} f(v)v^{-p} \right)$$

$$\Rightarrow a^{-p} \times -\frac{1}{p} \left(f(1)1^{-p-1} - f\left(\frac{1}{a}\right)\left(\frac{1}{a}\right)^{-p} \right)$$

$$\Rightarrow a^{-p} \times -\frac{1}{p} \left(f(1) - f\left(\frac{1}{a}\right)\left(\frac{1}{a}\right)^{-p} \right)$$

We can rewrite it as:

$$\Rightarrow a^{-p} \times -\frac{1}{p} \left(f(1) - f\left(\frac{1}{a}\right)\left(\frac{1}{a^{-p}}\right) \right)$$

$$\Rightarrow a^{-p} \times -\frac{1}{p} \left(f(1) - f\left(\frac{1}{a}\right)(a^p) \right)$$

$$\Rightarrow \left(\left(f(1) \times a^{-p} \times -\frac{1}{p} \right) - \left(f\left(\frac{1}{a}\right) \times a^p \times a^{-p} \times -\frac{1}{p} \right) \right)$$

$$\Rightarrow \left(f(1) \times a^{-p} \times -\frac{1}{p} \right) - \left(f\left(\frac{1}{a}\right) \times -\frac{1}{p} \right)$$

We can write it as:

$$\Rightarrow \left(f(1) \times \frac{1}{a^p} \times -\frac{1}{p} \right) - \left(f\left(\frac{1}{a}\right) \times -\frac{1}{p} \right)$$

Now let us test converginty :

Left side:

$$\lim_{p \rightarrow \infty} \left(f(1) \times \frac{1}{a^p} \times -\frac{1}{p} \right), a > 1$$

$$= \left(f(1) \times \frac{\lim_{p \rightarrow \infty} (1)}{\lim_{p \rightarrow \infty} (a^p)} \times -\frac{\lim_{p \rightarrow \infty} (1)}{\lim_{p \rightarrow \infty} (p)} \right)$$

$$= \left(f(1) \times \frac{1}{\infty} \times -\frac{1}{\infty} \right)$$

$$= (f(1) \times 0)$$

Right side:

$$\lim_{p \rightarrow \infty} \left(f\left(\frac{1}{a}\right) \times -\frac{1}{p} \right), a > 1$$

$$= \left(f\left(\frac{1}{a}\right) \times -\frac{\lim_{p \rightarrow \infty}(1)}{\lim_{p \rightarrow \infty}(p)} \right)$$

$$= f\left(\frac{1}{a}\right) \times -\frac{1}{\infty}$$

$$= f\left(\frac{1}{a}\right) \times 0$$

$$= 0$$

Hence $0 - 0 = 0$, therefore,

Hence as `p` approaches to infinity , the equation converges to zero, as $a > 1$.

Therefore , if we replace `p` with `s` we get the same result i. e. when s tends to ∞ , the above equation converges to zero.

Hence we can write : $a^{-p} \int_{\frac{1}{a}}^1 f(v)v^{-p-1}dv = \Theta_s(1)$

Now we are left with :

$$\Rightarrow a^{-p}F(s, p) - a^{-p} \int_{\frac{s}{a}}^s f(v)v^{-p-1}dv + \Theta_s(1)$$

Hence lets investigate the asymptotic behavior of

$$a^{-p} \int_{\frac{s}{a}}^s f(v)v^{-p-1}dv, \text{ with respect to `s`}$$

A) For all $v > 0$, $f(v)$ is a non – decreasing function, and

B) There exist $k_1, s_1 > 0$ such that $k_1(v) \leq f\left(\frac{v}{a}\right)$ for all $v \geq s_1$

Therefore, for all $v \in \left[\frac{s}{a}, s\right]$ we have:

$$f\left(\frac{s}{a}\right) \leq f(v) \leq f(s).$$

Since for $s \geq s_1$ we have $k_1 f(s) \leq f\left(\frac{s}{a}\right)$, then

$$k_1 f(s) \leq f(v) \leq f(s) \text{ for all } s > s_1$$

$$\Rightarrow k_1 \frac{f(s)}{v^{p+1}} \leq \frac{f(v)}{v^{p+1}} \leq \frac{f(s)}{v^{p+1}} \text{ for all } s > s_1,$$

$$\Rightarrow k_1 f(s) \int_{\frac{s}{a}}^s \frac{1}{v^{p+1}} dv \leq \int_{\frac{s}{a}}^s f(v) \frac{1}{v^{p+1}} d(v) \leq f(s) \int_{\frac{s}{a}}^s \frac{1}{v^{p+1}} dv$$

for all $s > s_1$,

We have two cases to consider, the case of $p \neq 0$ and the case of $p = 0$.

A) If $p \neq 0$, we get:

$$\int_{\frac{s}{a}}^s \frac{1}{v^{p+1}} dv = \left[\frac{1}{v^{p+1}} \right]_{\frac{s}{a}}^s$$

Applying exponent rule:

$$= [v^{-p-1}]_{\frac{s}{a}}^s$$

Applying power rule: $\int x^a dx = \frac{x^{a+1}}{a+1}, a \neq -1$

$$= \left[\frac{v^{-p-1+1}}{-p-1+1} \right]_{\frac{s}{a}}^s$$

$$= \left[\frac{v^{-p}}{-p} \right]_{\frac{s}{a}}^s$$

$$= -\frac{1}{p} [v^{-p}]_{\frac{s}{a}}$$

$$= -\frac{1}{p} \left[\frac{1}{v^p} \right]_{\frac{s}{a}}^s$$

By computation of boundaries:

$$\int_a^b f(x) dx = F(b) - F(a) = \lim_{x \rightarrow b^-} F(x) - \lim_{x \rightarrow a^+} F(x)$$

Replacing v with s and $\frac{s}{a}$ we get:

$$\lim_{s \rightarrow s^-} \left[\frac{1}{s^p} \right] = \frac{1}{s^p} \text{ (will remain same)}$$

$$\lim_{s \rightarrow \left(\frac{s}{a}\right)^+} \left[\frac{1}{s^p} \right] = \frac{1}{\left(\frac{s}{a}\right)^p} = \frac{1}{\frac{s^p}{a^p}}$$

Now,

$$F(b) - F(a) = -\frac{1}{p} \left(\frac{1}{s^p} - \frac{1}{\frac{s^p}{a^p}} \right) = -\frac{1}{p} \left(\frac{1}{s^p} - \frac{a^p}{s^p} \right) = -\frac{1}{p} \left(\frac{1 - a^p}{s^p} \right)$$

$$= \frac{1}{s^p} \left(\frac{a^p - 1}{p} \right)$$

Therefore,

Replacing the equation :

$$k_1 f(s) \int_{\frac{s}{a}}^s \frac{1}{v^{p+1}} dv \leq \int_{\frac{s}{a}}^s f(v) \frac{1}{v^{p+1}} d(v) \leq f(s) \int_{\frac{s}{a}}^s \frac{1}{v^{p+1}} dv$$

we get:

$$k_1 \frac{f(s)}{s^p} \left(\frac{a^p - 1}{p} \right) \leq \int_{\frac{s}{a}}^s f(v) \frac{1}{v^{p+1}} d(v) \leq \frac{f(s)}{s^p} \left(\frac{a^p - 1}{p} \right)$$

for all $s > s_1$,

$$i. e., \int_{\frac{s}{a}}^s f(v) \frac{1}{v^{p+1}} dv = \Theta_s \left(\frac{f(s)}{s^p} \right) \left[as \left(\frac{a^p - 1}{p} \right) \text{ is constant} \right]$$

$$\left(And \left(\frac{a^p - 1}{p} \right) > 0, \quad a > 1 \right)$$

(B) If $p = 0$ then:

$$\int_{\frac{s}{a}}^s \frac{1}{v} d(v)$$

$$[\log v]_{\frac{s}{a}}^s \left[\int \frac{1}{v} dv = \log v \right]$$

Computing the boundaries:

$$\int_a^b f(x) dx = F(b) - F(a) = \lim_{x \rightarrow b^-} F(x) - \lim_{x \rightarrow a^+} F(x)$$

$$\lim_{v \rightarrow (s)^-} \log v = \log s$$

$$\lim_{v \rightarrow (\frac{s}{a})^+} \log v = \log \left(\frac{s}{a} \right)$$

$$\text{Now, } \log s - \log \frac{s}{a}$$

Applying logarithmic rule $\rightarrow \log x - \log y = \log \frac{x}{y}$, we get:

$$\Rightarrow \log \frac{s}{\frac{s}{a}} = \log \left(s \times \frac{a}{s} \right) = \log a$$

Hence ,

As $p = 0$,

$$\int_{\frac{s}{a}}^s \frac{1}{v^{p+1}} d(v) = \int_{\frac{s}{a}}^s \frac{1}{v^{0+1}} d(v) = \int_{\frac{s}{a}}^s \frac{1}{v} d(v) = \log s - \log \frac{s}{a}$$

$\log a = \frac{\log a}{s^p}$ as we doing in respect of $\Theta\left(\frac{1}{s^p}\right)$.

Replacing in :

$$k_1 f(s) \int_{\frac{s}{a}}^s \frac{1}{v^{p+1}} dv \leq \int_{\frac{s}{a}}^s f(v) \frac{1}{v^{p+1}} d(v) \leq f(s) \int_{\frac{s}{a}}^s \frac{1}{v^{p+1}} dv$$

We get:

$$k_1 \log a \frac{f(s)}{s^p} \leq \int_{\frac{s}{a}}^s f(v) \frac{1}{v^{p+1}} d(v) \leq \log a \frac{f(s)}{s^p}$$

$$i. e. \int_{\frac{s}{a}}^s f(v) \frac{1}{v^{p+1}} d(v) = \Theta\left(\frac{f(s)}{s^p}\right)$$

$$or, \int_{\frac{s}{a}}^s \frac{f(v)}{v^{p+1}} d(v) = \Theta\left(\frac{f(s)}{s^p}\right)$$

Note: $\log a$ is a constant and $\log a > 0$.

Thus replacing in eqn :

$$a^{-p}F(s, p) - a^{-p} \int_{\frac{s}{a}}^s f(v) v^{-p-1} dv + \Theta_s(1)$$

$$\Rightarrow a^{-p}F(s, p) - a^{-p} \int_{\frac{s}{a}}^s \frac{f(v)}{v^{p+1}} dv + \Theta_s(1)$$

We get:

$$\Rightarrow a^{-p}F(s, p) - \Theta\left(\frac{f(s)}{s^p}\right) + \Theta_s(1)$$

Which completes the proof.

Part 2: Let $f(x)$ be a function defined as in Theorem 1.

In other words,

$$f(x) = \begin{cases} u_0 & \mathbf{x} \in [0, 1) \\ \sum_{i=1}^k a_i f\left(\left\lfloor \frac{x}{b_i} \right\rfloor\right) + g(\lfloor x \rfloor) & \mathbf{x} \in [1, \infty) \end{cases}$$

Where:

- $u_0, a_i \in R^{*+}, \sum_{i=1}^k a_i \geq 1$
- $b_i, k \in N, b_i \geq 2, k \geq 1$
- $g(x)$ is a bounded, positive and nondecreasing function for all $x \geq 0$.
- for all $c > 1$, there exist $x_1, k_1 > 0$ such that $g\left(\frac{x}{c}\right) \geq k_1 g(x)$, for all $x \geq x_1$.

Then,

1. $g(\lfloor x \rfloor) \in S$

2. $f(x) \in S$

Proof:

1. Note that $g(x) \in S$ by definition.

Hence, $g(\lfloor x \rfloor)$ is clearly bounded and non-decreasing. Moreover, $g(\lfloor x \rfloor)$ is positive.

There remains to prove that:

for $c > 1$, there exists x_2, k_2 , such that $g\left(\left\lfloor \frac{x}{2} \right\rfloor\right) \geq k_2 g(\lfloor x \rfloor)$ for all $x \geq x_2$.

Let $c > 1$ be given. Let $x > \max \left\{ x_1 + c + 1, \frac{c^2}{c-1} + 1 \right\}$ then the following three useful inequalities can be derived:

$$x > x_1 + c + 1$$

$$\Rightarrow [x] > x_1 + c$$

$$\Rightarrow [x] - c > x_1 \text{ --- (a)}$$

On the other hand,

$$x > \frac{c^2}{c-1} + 1$$

$$\Rightarrow [x] > \frac{c^2}{c-1} + 1$$

$$\Rightarrow [x] > \frac{c^2}{c-1}$$

$$\Rightarrow [x](c-1) > c^2$$

$$\Rightarrow \frac{[x](c-1)}{c} > c$$

$$\Rightarrow \frac{c[x] - [x]}{c} > c$$

$$\Rightarrow [x] - \frac{[x]}{c} > c$$

$$\Rightarrow \lfloor x \rfloor - c > \frac{\lfloor x \rfloor}{c} \text{--- -- -- (b)}$$

Finally,

$$x > x_1 + c + 1$$

$$\Rightarrow \lfloor x \rfloor > x_1 \text{--- -- -- -- -- (c)}$$

Using above inequalities of (a), (b), (c), we conclude :

$$g\left(\left\lfloor \frac{x}{c} \right\rfloor\right) = g\left(\frac{\lfloor x \rfloor}{c}\right) \text{(using Part 1)}$$

$$\geq g\left(\frac{\lfloor x \rfloor}{c} - 1\right) \text{(since } g \text{ is a non - decreasing)}$$

$$= g\left(\frac{\lfloor x \rfloor - c}{c}\right)$$

$$\geq k_1 g(\lfloor x \rfloor - c) \text{(using Inequality (a))}$$

$$> k_1 g\left(\frac{\lfloor x \rfloor}{c}\right) \text{(using Inequality (b))}$$

$$> (k_1)^2 g(\lfloor x \rfloor) \text{(using Inequality (c))}$$

Therefore, for all $c > 1$, such that

$x_2 = \max\left(x_1 + c + 1, \frac{c^2}{c-1} + 1\right) > 0, k_2 = (k_1)^2 > 0$, such that

$g\left(\frac{\lfloor x \rfloor}{c}\right) \geq k_2 g(\lfloor x \rfloor)$, which completes the proof that $g(\lfloor x \rfloor) \in S$.

2. To prove that $f(x) \in S$, note that f is defined in terms of a finite sum of positive terms each of which is bounded and also positive. So, f is bounded and positive.

Also according to Theorem 1, $f(x) = f(\lfloor x \rfloor)$.

So, it is sufficient to prove that:

(A) $f(n)$ is non – decreasing for all $n > 0$,

(B) for all $c > 1$, there exists n_3, k_3 such that

$f\left(\frac{n}{c}\right) \geq k_3 f(n)$ for all $n > n_3$.

To prove $f(n)$ is non – decreasing, we use strong induction. Note that :

$f(0) = u_0$,

$$f(1) = \sum_{i=1}^k a_i u_0 + g(1) \geq u_0$$

$$\left(\text{since } \sum_{i=1}^k a_i \geq 1 \text{ and } g(1) \geq 0 \right)$$

Assume that for all $k < n$ we have $f(k) \geq f(k - 1)$, and let us prove that $f(n) \geq f(n - 1)$. Consider

$$f(n) = \sum_{i=1}^k a_i f\left(\frac{n}{b^i}\right) + g(n)$$

$$= \sum_{i=1}^k a_i f\left(\left\lfloor \frac{n}{b^i} \right\rfloor\right) + g(n)$$

$$\geq \sum_{i=1}^k a_i f\left(\left\lfloor \frac{n-1}{b^i} \right\rfloor\right) + g(n-1) \quad (g(n) \text{ is nondecreasing})$$

$$= \sum_{i=1}^k a_i f\left(\left\lfloor \frac{n-1}{b^i} \right\rfloor\right) + g(n-1)$$

$$= f(n-1)$$

So, for all $n \geq 1$ we have $f(n) \geq f(n - 1)$. Therefore, f is non – decreasing. To prove the regularity condition, we use strong induction again. Using the results of the previous Part, there exist k_2, x_2 , such that

$g\left(\left\lfloor \frac{x}{c} \right\rfloor\right) \geq k_2 g(\lfloor x \rfloor)$ for all $x \geq x_2, c > 1$. Consider:

$$n_0 = \lfloor x_2 \rfloor,$$

$$k_3 = \min \left\{ \frac{f(0)}{f(n_0)}, k_2 \right\}.$$

f is nondecreasing and strictly positive. So for all $m \in [0, n_0)$

$$f\left(\frac{m}{c}\right) \geq f(0)$$

$$\geq f(0) \frac{f(m)}{f(n_0)}$$

$$= f(m) \frac{f(0)}{f(n_0)}$$

$$\geq f(m) k_3$$

i. e., for all $m \in [0, n_0]$ we have $f\left(\frac{m}{c}\right) \geq k_3 f(m)$.

Now assume that for all $m \in [0, n)$ where $n > n_0$ we have

$f\left(\frac{m}{c}\right) \geq k_3 f(m)$, and let us prove that $f\left(\frac{n}{c}\right) \geq k_3 f(n)$.

Since $n > n_0$, we have $\frac{n}{c} > 1$. So,

$$f\left(\frac{n}{c}\right) = \sum_{i=1}^k f\left(\frac{n}{a_i}\right) + g\left(\left\lfloor \frac{n}{c} \right\rfloor\right)$$

$$= \sum_{i=1}^k f\left(\left\lfloor \frac{\left(\frac{n}{a_i}\right)}{c} \right\rfloor\right) + g\left(\left\lfloor \frac{n}{c} \right\rfloor\right) \quad (\text{since } f(n) = f(\lfloor n \rfloor))$$

$$= \sum_{i=1}^k f\left(\left\lfloor \frac{\left\lfloor \frac{n}{a_i} \right\rfloor}{c} \right\rfloor\right) + g\left(\left\lfloor \frac{n}{c} \right\rfloor\right) \quad \left(\text{since } \left\lfloor \frac{n}{c} \right\rfloor = \left\lfloor \frac{\lfloor n \rfloor}{c} \right\rfloor\right)$$

$$= \sum_{i=1}^k f\left(\frac{\left\lfloor \frac{n}{a_i} \right\rfloor}{c}\right) + g\left(\left\lfloor \frac{n}{c} \right\rfloor\right) \quad (\text{since } f(n) = f(\lfloor n \rfloor))$$

$$\geq \sum_{i=1}^k k_3 f\left(\frac{\left\lfloor \frac{n}{a_i} \right\rfloor}{c}\right) + g\left(\left\lfloor \frac{n}{c} \right\rfloor\right) \quad (\text{since } f(n) = f(\lfloor n \rfloor))$$

$$= k_3 f(n).$$

As a result ,

for all $c > 1$, there exist such that $n_3 = 0$,

$k_3 = \min \left\{ \frac{f(0)}{f(n_0)}, k_2 \right\}$ such that $f\left(\frac{n}{c}\right) \geq k_3 f(n)$ for all

$n \geq n_3$,

which completes the proof.

Theorem 3 : Let $f(x)$ be a function defined as in

Theorem 1. Let p_0 be the real solution of the equation

$\sum_{i=1}^k a_i b_i^{-p} = 1$. Then p_0 always exists and is unique and

positive. Furthermore,

$$f(x) = \Theta(x^{p_0}) + \Theta\left(x^{p_0} \int_{x_1}^x \frac{g(u)}{u^{p_0+1}} d(u)\right)$$

for x_1 large enough.

Proof:

Since $g(x) \in S$, then according Part 1 both $g(\lfloor x \rfloor)$ and $f(x)$ belong to S . Hence, both functions possess an Order transform . Rewriting the definition of $f(x)$,

$$f(x) = \sum_{i=1}^k a_i f\left(\frac{x}{b_i}\right) + g(\lfloor x \rfloor) \text{ for all } x > 1$$

$$P\{f(x)\} = P\left\{\sum_{i=1}^k a_i f\left(\frac{x}{b_i}\right) + g(\lfloor x \rfloor)\right\} \text{ for all } s > 1$$

$$P\{f(x)\} = \sum_{i=1}^k a_i P\left\{f\left(\frac{x}{b_i}\right)\right\} + \int_1^s g\left(\frac{\lfloor u \rfloor}{u^{p+1}}\right) du$$

$$F(s, p) = \sum_{i=1}^k a_i \left(b_i^{-p} F(s, p) - \Theta\left(\frac{f(s)}{s^p}\right) + \Theta_s(1) \right) + \int_1^s g\left(\frac{\lfloor u \rfloor}{u^{p+1}}\right) du$$

$$F(s, p) - \sum_{i=1}^k a_i \left(b_i^{-p} F(s, p) \right) + \Theta\left(\frac{f(s)}{s^p}\right) = \int_1^s g\left(\frac{\lfloor u \rfloor}{u^{p+1}}\right) du + \Theta_s(1)$$

Taking $F(s, p)$ common:

$$F(s, p) \left(1 - \sum_{i=1}^k a_i (b_i^{-p}) \right) + \Theta\left(\frac{f(s)}{s^p}\right) = \int_1^s g\left(\frac{\lfloor u \rfloor}{u^{p+1}}\right) du + \Theta_s(1)$$

Let $h(p) = 1 - \sum_{i=1}^k a_i(b_i^{-p})$.Then:

$$h(0) = 1 - \sum_{i=1}^k a_i(b_i^0) \Rightarrow 1 - \sum_{i=1}^k a_i \leq 0$$

$$\lim_{p \rightarrow \infty} h(p) = \lim_{p \rightarrow \infty} (1) - \sum_{i=1}^k a_i \lim_{p \rightarrow \infty} (b_i^{-p})$$

$$\Rightarrow 1 - \sum_{i=1}^k a_i \lim_{p \rightarrow \infty} \left(\frac{1}{b_i^p} \right)$$

$$\Rightarrow 1 - \sum_{i=1}^k a_i \left(\frac{1}{\infty_i} \right)$$

$$\Rightarrow 1 - \sum_{i=1}^k a_i \times 0$$

$$\Rightarrow 1 - 0$$

$$\Rightarrow 1$$

And

$$\lim_{p \rightarrow \infty} h(p) = 1 > 0$$

$$\frac{d}{dp} h(p) = \frac{d}{dp} \left(1 - \sum_{i=1}^k a_i (b_i^{-p}) \right)$$

$$\Rightarrow \frac{d}{dp} (1) - \frac{d}{dp} \left(\sum_{i=1}^k a_i (b_i^{-p}) \right)$$

$$\Rightarrow 0 - \frac{d}{dp} \left(\sum_{i=1}^k a_i (b_i^{-p}) \right) \left[\frac{d}{dx} (c) = 0, \text{ where } c \text{ is constant} \right]$$

Applying multiplication chain rule:

$$\frac{d}{dx} f(x)g(x) = f(x) \times \frac{d}{dx} g(x) + g(x) \times \frac{d}{dx} f(x), \text{ we get:}$$

$$= -1 \left(\sum_{i=1}^k a_i \times \frac{d}{dp} (b_i^{-p}) + (b_i^{-p}) \times \frac{d}{dp} \left(\sum_{i=1}^k a_i \right) \right)$$

Note: $\sum_{i=1}^k a_i$ is constant, hence:

$$= -1 \left(\sum_{i=1}^k a_i \times \frac{d}{dp} (b_i^{-p}) + (b_i^{-p}) \times 0 \right)$$

$$= -1 \left(\sum_{i=1}^k a_i \times \frac{d}{dp} (b_i^{-p}) \right)$$

Lets deduce $\frac{d}{dp} (b_i^{-p})$:

Applying exponent rule : $a^b = e^{b \log a}$

Hence , $b^{-p} = e^{(-p) \log b}$

$$= \frac{d}{dp} (e^{(-p) \log b})$$

Applying chain rule : $\frac{df(u)}{dx} = \frac{df}{du} \times \frac{du}{dx}$ we get:

$$f = e^u, u = (-p) \log b$$

$$= \frac{d}{du} (e^u) \frac{d}{dp} (-p) \log b$$

$$= e^u \frac{d}{dp} (-p) \log b \quad \left[as \frac{d}{du} e^u = e^u \right]$$

$$= e^u \frac{d}{dp} (-p) \log b \quad \left[as \frac{d}{du} e^u = e^u \right]$$

Substuting back $u = (-p) \log b$

$$= e^{(-p) \log b} \frac{d}{dp} (-p) \log b$$

$$= e^{(-p) \log b} \left[\log b \times \frac{d}{dp} (-p) \right]$$

$$= e^{(-p) \log b} \left[\log b \times -\frac{dp}{dp} \right]$$

$$= e^{(-p) \log b} [\log b \times -1]$$

$$= -e^{(-p) \log b} \log b$$

$$= -(e^{\log b})^{-p} \log b$$

This is actually $\log_e b$, hence $e^{\log_e b} = b$ as, $a^{\log_a b} = b$

$$= -(b)^{-p} \log b$$

$$= -b^{-p} \log b$$

Hence,

$$= -1 \left(\sum_{i=1}^k a_i \times \frac{d}{dp} (b_i^{-p}) \right)$$

$$= -1 \left(\sum_{i=1}^k a_i \times -b_i^{-p} \log b_i \right)$$

$$= \sum_{i=1}^k a_i (\log b_i) b_i^{-p}$$

Hence:

$$\frac{d}{dp} h(p) = \frac{d}{dp} \left(1 - \sum_{i=1}^k a_i (b_i^{-p}) \right) = \sum_{i=1}^k a_i (\log b_i) b_i^{-p} > 0$$

for all p ($b_i \geq 2, a_i > 0$).

So , $h(p) = 0$ has a unique positive solution p_0 .

We get,

$$F(s, p) \left(1 - \sum_{i=1}^k a_i (b_i^{-p}) \right) + \Theta \left(\frac{f(s)}{s^p} \right) = \int_1^s g \left(\frac{[u]}{u^{p+1}} \right) du + \Theta_s(1)$$

$$\Rightarrow F(s, p)h(p) + \Theta \left(\frac{f(s)}{s^p} \right) = \int_1^s g \left(\frac{[u]}{u^{p+1}} \right) du + \Theta_s(1)$$

$$\Rightarrow F(s, p) \times 0 + \Theta \left(\frac{f(s)}{s^p} \right) = \int_1^s g \left(\frac{[u]}{u^{p+1}} \right) du + \Theta_s(1)$$

$$\Rightarrow 0 + \Theta \left(\frac{f(s)}{s^p} \right) = \int_1^s g \left(\frac{[u]}{u^{p+1}} \right) du + \Theta_s(1)$$

$$\Rightarrow \Theta \left(\frac{f(s)}{s^p} \right) = \int_1^s g \left(\frac{[u]}{u^{p+1}} \right) du + \Theta_s(1)$$

Now replacing p with p_0 we get:

$$i. e., \Theta \left(\frac{f(s)}{s^{p_0}} \right) = \int_1^s g \left(\frac{[u]}{u^{p_0+1}} \right) du + \Theta_s(1)$$

$$i. e., \frac{\Theta(f(s))}{\Theta(s^{p_0})} = \int_1^s g \left(\frac{[u]}{u^{p_0+1}} \right) du + \Theta_s(1)$$

$$\Rightarrow \Theta(f(s)) = \Theta \left(s^{p_0} \int_1^s g \left(\frac{\lfloor u \rfloor}{u^{p_0+1}} \right) du \right) + \Theta_s(s^{p_0})$$

Replacing s with x , so that $\Theta(f(s)) = f(x)$, we get:

$$\Rightarrow f(x) = \Theta \left(x^{p_0} \int_1^x g \left(\frac{\lfloor u \rfloor}{u^{p_0+1}} \right) du \right) + \Theta(x^{p_0})$$

Since $g(x) \in S$ and $g(x)$ is non – decreasing , then:

for all $x \geq 1$, $g \left(\frac{x}{2} \right) \leq g(\lfloor x \rfloor)$ and

There exist $k_1, x_1 > 0$ such that $g \left(\frac{x}{2} \right) \geq k_1 g(x)$ for all $x > x_1$

In other words for all $x > x_1$, $k_1 g(x) < g(\lfloor x \rfloor) < g(x)$,

$$i. e., k_1 \int_{x_1}^x \frac{g(u)}{u^{p_0+1}} du \leq \int_{x_1}^x \frac{g(\lfloor u \rfloor)}{u^{p_0+1}} du \leq \int_{x_1}^x \frac{g(u)}{u^{p_0+1}} du$$

$$i. e., x^{p_0} \int_{x_1}^x \left(\frac{g(\lfloor u \rfloor)}{u^{p_0+1}} \right) du = \Theta \left(x^{p_0} \int_{x_1}^x \left(\frac{g(u)}{u^{p_0+1}} \right) du \right)$$

Replacing it in above equation

$$f(x) = \Theta \left(x^{p_0} \int_1^x \left(\frac{g[u]}{u^{p_0+1}} \right) du \right) + \Theta \left(x^{p_0} \int_{x_1}^x \left(\frac{g(u)}{u^{p_0+1}} \right) du \right) + \Theta(x^{p_0})$$

$$f(x) = \Theta \left(x^{p_0} \int_{x_1}^x \left(\frac{g(u)}{u^{p_0+1}} \right) du \right) + \Theta(x^{p_0})$$

which completes the proof .

Theorem 4 : Let $f(x)$ be a function defined in Theorem 1. Let p_0 be the unique solution of the characteristic equation:

1. if there exist $\varepsilon > 0$ such that $g(x) = O(x^{p_0-\varepsilon})$, then $f(x) = \Theta(x^{p_0})$.

2. If there exist $\varepsilon > 0$ such that $g(x) = \Omega(x^{p_0+\varepsilon})$ and $\frac{g(x)}{x^{p_0+\varepsilon}}$ is a non – decreasing function then $f(x) = \Theta(g(x))$.

3.. If $g(x) = \Theta(x^{p_0})$ then $f(x) = \Theta(x^{p_0} \log x)$

Proof

Suppose there exist $\varepsilon > 0$ such that $g(x) = O(x^{p_0-\varepsilon})$, i. e., there exists $x_0, k > 0$, such that $g(x) < kx^{p_0-\varepsilon}$, for all $x > x_0$.

Let $x_1 > x_0$, then for all $x > x_1$ we have :

$$\int_{x_1}^x \frac{g(u)}{u^{p_0+1}} d(u) \leq k \int_{x_1}^x \frac{ku^{p_0-\varepsilon}}{u^{p_0+1}} du$$

$$= k \int_{x_1}^x \frac{1}{u^{p_0+1}} du$$

$$= k \int_{x_1}^x \frac{1}{u^{p_0+1}} du$$

$$= k \int_{x_1}^x u^{-p_0-1} du$$

Lets make the power rule of integral: $\int x^p = \frac{x^{p+1}}{p+1}$

Hence:

$$= k \int_{x_1}^x \frac{u^{-p_0-1+1}}{-p_0-1+1} du$$

$$= k \int_{x_1}^x \frac{u^{-p_0}}{-p_0} du$$

$$= k \times -\frac{1}{p_0} \int_{x_1}^x u^{-p_0} du$$

Let $u = x$, and $p_0 = \varepsilon$, then:

$$= k \times -\frac{1}{\varepsilon} \int_{x_1}^x x^{-\varepsilon} dx$$

By computation of boundaries:

$$\int_a^b f(x) dx = F(b) - F(a) = \lim_{x \rightarrow b^-} F(x) - \lim_{x \rightarrow a^+} F(x)$$

$$= k \times -\frac{1}{\varepsilon} [x^{-\varepsilon}]_{x_1}^x$$

$$\lim_{x \rightarrow x^-} x^{-\varepsilon} = x^{-\varepsilon} \text{ and } \lim_{x \rightarrow x_1^+} x^{-\varepsilon} = x_1^{-\varepsilon}$$

$$= k \times -\frac{1}{\varepsilon} (x^{-\varepsilon} - x_1^{-\varepsilon})$$

$$= k \times -\frac{1}{\varepsilon} ((-1)x_1^{-\varepsilon} - x^{-\varepsilon})$$

$$= \frac{k}{\varepsilon} (x_1^{-\varepsilon} - x^{-\varepsilon})$$

$$= \frac{k}{\varepsilon} \left(\frac{1}{x_1^\varepsilon} - \frac{1}{x^{-\varepsilon}} \right)$$

$$< \frac{k}{\varepsilon} \times \frac{1}{x_1^\varepsilon}$$

$$= O(1)$$

But we proved that :

$$f(x) = O(x^{p_0}) + O\left(x^{p_0} \int_1^s \left(\frac{g(u)}{u^{p_0+1}}\right) du\right)$$

Therefore replacing we get:

$$\begin{aligned} f(x) &= O(x^{p_0}) + O(x^{p_0} \times O(1)) = O(x^{p_0}) + O(x^{p_0}) \\ &= O(x^{p_0}). \end{aligned}$$

2. Suppose there exists $\varepsilon > 0$ such that $g(x) = \Omega(x^{p_0+\varepsilon})$ and $\frac{g(x)}{x^{p_0+\varepsilon}}$ is a non – dereasing function for large x . Let $\phi(x) = \frac{g(x)}{x^{p_0+\varepsilon}}$. Then:

$$g(x) = \phi(x)x^{p_0+\varepsilon}$$

$$i. e., x^{p_0} \int_{x_1}^x \frac{g(u)}{u^{p_0+1}} du = x^{p_0} \int_{x_1}^x \frac{\phi(u)u^{p_0+\varepsilon}}{u^{p_0+1}} du$$

$$= x^{p_0} \int_{x_1}^x \phi(u)u^{p_0+\varepsilon-p_0-1} du$$

$$= x^{p_0} \int_{x_1}^x \phi(u)u^{\varepsilon-1} du$$

for x_1 large enough to make $\phi(x)$ non decreasing. Therefore, for all $x > x_1$:

$$x^{p_0} \int_{x_1}^x \frac{g(u)}{u^{p_0+1}} du < x^{p_0} \phi(x) \int_{x_1}^x u^{\varepsilon-1} du$$

Let $u = x$, we get:

$$x^{p_0} \phi(x) \int_{x_1}^x x^{\varepsilon-1} dx$$

$$Applying power rule: \int x^a dx = \frac{x^{a+1}}{a+1}, a \neq -1$$

$$= x^{p_0} \phi(x) \int_{x_1}^x \frac{x^{\varepsilon-1+1}}{\varepsilon-1+1} dx$$

$$= x^{p_0} \phi(x) \int_{x_1}^x \frac{x^\varepsilon}{\varepsilon} dx$$

Taking out $\frac{1}{\varepsilon}$ we get:

$$= x^{p_0} \phi(x) \times \frac{1}{\varepsilon} \int_{x_1}^x x^\varepsilon dx$$

$$= x^{p_0} \phi(x) \times \frac{1}{\varepsilon} \times [x^\varepsilon]_{x_1}^x$$

By computation of boundaries:

$$\int_a^b f(x) dx = F(b) - F(a) = \lim_{x \rightarrow b^-} F(x) - \lim_{x \rightarrow a^+} F(x)$$

$$\lim_{x \rightarrow x^-} (x^\varepsilon) = x^\varepsilon \text{ and } \lim_{x \rightarrow x_1^+} (x^\varepsilon) = x_1^\varepsilon$$

Hence:

$$= x^{p_0} \phi(x) \times \frac{1}{\varepsilon} \times (x^\varepsilon - x_1^\varepsilon)$$

$$< x^{p_0} \phi(x) \times \frac{1}{\varepsilon} \times (x^\varepsilon)$$

$$< \phi(x) x^{p_0+\varepsilon} \times \frac{1}{\varepsilon}$$

$$\textit{We know } \phi(x) x^{p_0+\varepsilon} = g(x)$$

$$< g(x) \times \frac{1}{\varepsilon}$$

$$= \mathbf{O}(g(x))$$

Further more,

$$x^{p_0} \int_{x_1}^x \left(\frac{g(u)}{u^{p_0+1}} \right) du = x^{p_0} \int_{\frac{x}{a}}^x \left(\frac{g(u)}{u^{p_0+1}} \right) du + x^{p_0} \int_{x_1}^{\frac{x}{a}} \left(\frac{g(u)}{u^{p_0+1}} \right) du$$

But $g(x) \in S$, so:

$$x^{p_0} \int_{\frac{x}{a}}^x \left(\frac{g(u)}{u^{p_0+1}} \right) du = \mathbf{O}(g(x))[\textit{we can prove like above}].$$

By scaling property:

$$\int_{\frac{x}{a}}^x \left(\frac{g(u)}{u^{p_0+1}} \right) du = \Theta \left(\frac{g(x)}{x^{p_0}} \right)$$

Therefore,

$$x^{p_0} \int_{x_1}^x \left(\frac{g(u)}{u^{p_0+1}} \right) du = x^{p_0} \int_{\frac{x}{a}}^x \left(\frac{g(u)}{u^{p_0+1}} \right) du + x^{p_0} \int_{x_1}^{\frac{x}{a}} \left(\frac{g(u)}{u^{p_0+1}} \right) du$$

$$\Rightarrow x^{p_0} \int_{x_1}^x \left(\frac{g(u)}{u^{p_0+1}} \right) du = x^{p_0} \int_{\frac{x}{a}}^x \left(\frac{g(u)}{u^{p_0+1}} \right) du + x^{p_0} \int_{x_1}^{\frac{x}{a}} \left(\frac{g(u)}{u^{p_0+1}} \right) du$$

$$\Rightarrow x^{p_0} \int_{x_1}^x \left(\frac{g(u)}{u^{p_0+1}} \right) du = \Theta(g(x)) + x^{p_0} \int_{x_1}^{\frac{x}{a}} \left(\frac{g(u)}{u^{p_0+1}} \right) du$$

$$\Rightarrow x^{p_0} \int_{x_1}^x \left(\frac{g(u)}{u^{p_0+1}} \right) du = \Omega(g(x))$$

As we got:

$$x^{p_0} \int_{x_1}^x \frac{g(u)}{u^{p_0+1}} du = o(g(x))$$

$$x^{p_0} \int_{x_1}^x \frac{g(u)}{u^{p_0+1}} du = \Omega(g(x))$$

Hence,

$$x^{p_0} \int_{x_1}^x \frac{g(u)}{u^{p_0+1}} du = \Theta(g(x))$$

But we proved in Theorem 3 that:

$$\Rightarrow f(x) = \Theta(x^{p_0}) + \Theta\left(x^{p_0} \int_{x_1}^x \frac{g(u)}{u^{p_0+1}} d(u)\right)$$

Hence by replacing we get:

$$f(x) = \Theta\left(\left(\Theta(g(x))\right)\right) + \Theta(x^{p_0})$$

$$\Rightarrow \Theta(g(x)) + \Theta(x^{p_0}) \text{ [Asymptotic Multiplication Property]}$$

$$\Rightarrow \Theta(g(x)) \text{ since } g(x) = \Omega(x^{p_0+\epsilon})$$

3. Suppose $g(x) = \Theta(x^{p_0})$. Replacing the result of theorem 3 we get:

$$f(x) = \Theta(x^{p_0}) + \Theta\left(x^{p_0} \int_{x_1}^x \frac{g(u)}{u^{p_0+1}} d(u)\right)$$

$$f(x) = \Theta\left(x^{p_0} \int_{x_1}^x \frac{1}{u} du\right) + \Theta_s(x^{p_0})$$

When $g(x)$ is asymptotically equivalent to x^{p_0} .

Then $\int_{x_1}^x \frac{1}{u}$ is asymptotically equivalent to x^{p_0+1} .

This is because :

$\int \frac{1}{u} du = \log u$ or $\log x$, And $\log x$ is asymptotically equivalent to x^{p_0+1} for large value of x .

We can check it , i. e.

$$f(x) = \Theta\left(x^{p_0} \int_{x_1}^s \left(\frac{g(u)}{u^{p_0+1}}\right) du\right) + \Theta(x^{p_0})$$

Hence if $u = x$ then:

$$= \Theta \left(x^{p_0} \int_{x_1}^x \left(\frac{g(x)}{x^{p_0+1}} \right) dx \right) + \Theta(x^{p_0})$$

$$= \Theta \left(x^{p_0} \int_{x_1}^x \left(\frac{x^{p_0}}{x^{p_0+1}} \right) dx \right) + \Theta(x^{p_0})$$

$$= \Theta \left(x^{p_0} \int_{x_1}^x (x^{p_0-p_0-1}) dx \right) + \Theta(x^{p_0})$$

$$= \Theta \left(x^{p_0} \int_{x_1}^x (x^{-1}) dx \right) + \Theta(x^{p_0})$$

$$= \Theta \left(x^{p_0} \int_{x_1}^x \left(\frac{1}{x} \right) dx \right) + \Theta(x^{p_0})$$

or,

$$= \Theta \left(x^{p_0} \int_{x_1}^x \left(\frac{1}{u} \right) du \right) + \Theta(x^{p_0})$$

$$\text{Continuing from : } \Theta \left(x^{p_0} \int_{x_1}^x \left(\frac{1}{x} \right) dx \right) + \Theta(x^{p_0})$$

$$f(x) = \Theta \left(x^{p_0} \int_{x_1}^x \frac{1}{x} dx \right) + \Theta_s(x^{p_0})$$

We know, $\int \frac{1}{x} dx = \log x$

Then:

$$= \Theta(x^{p_0} [\log x]_{x_1}^x) + \Theta_s(x^{p_0})$$

$$= \Theta(x^{p_0} [\log x]_{x_1}^x) + \Theta_s(x^{p_0})$$

By computation of boundaries:

$$\int_a^b f(x) dx = F(b) - F(a) = \lim_{x \rightarrow b^-} F(x) - \lim_{x \rightarrow a^+} F(x)$$

$$\lim_{x \rightarrow x^-} \log x = \log x \text{ and } \lim_{x \rightarrow x_1^+} \log x = \log x_1$$

Hence,

$$= \Theta(x^{p_0} (\log x - \log x_1)) + \Theta_s(x^{p_0})$$

$$= \Theta(x^{p_0} \log x - x^{p_0} \log x_1) + \Theta_s(x^{p_0})$$

$$= \Theta(x^{p_0} \log x - x^{p_0} \log x_1)$$

It doesnot matter of x or x_1 , hence the result will be :

$$= \Theta(x^{p_0} \log x)$$

Hence it completes the proof.

Corollary 1: *Let u_n be a sequence as in Equation 1. Then,*

$$u_n = \Theta(n^{p_0}) + \Theta\left(n^{p_0} \int_{n_1}^n \frac{g(u)}{u^{p_0+1}} du\right) \text{ for } n_1 \text{ large enough,}$$

where p_0 is the real solution of the $\sum_{i=1}^k a_i b_i^{-p} = 1$ which always exists and is unique and positive. Furthermore,

1. *if there exists $\varepsilon > 0$ such that $g(x) = O(x^{p_0-\varepsilon})$ then*

$$u_n = \Theta(n^{p_0}).$$

2. *if there exists $\varepsilon > 0$ such that $g(x) = \Omega(x^{p_0+\varepsilon})$ and*

$\frac{g(x)}{x^{p_0+\varepsilon}}$ is a non – decreasing function, then $u_n = \Theta(g(n))$.

3. *if $g(x) = \Theta(x^{p_0})$ then $u_n = \Theta(n^{p_0} \log n)$*

Proof: *We can prove these conditions by above process.*
