$Guess\ And\ Verify\ Method - Example - 4$

$Consider\ the\ recurrence\ T(n) = \sqrt{n}\ T(\sqrt{n}) + n$.

<u>$Solution$</u>:

$So, let\ us\ guess\ that\ T(n) = \Theta(nlogn), and\ then\ try\ to$

$prove\ that\ our\ guess\ is\ correct.$

<u>$1st\ we\ have\ to\ remember$</u>:

$Decreasing\ Rates\ of\ Growth$:

$$2^{2n} < n! < 4^{2n} < 2^n < n^3 < n^2 < nlogn < log(n!) < n$$
$$< 2^{logn} < log^2 n < \sqrt{logn} < loglogn < 1$$

2^{nd} $we\ have\ to\ remember\ that\ we\ are\ finding\ out\ the$

$average\ case\ \Theta\ i.e.both\ upper\ bound\ and\ lower\ bound.$

$Lets\ start\ by\ trying\ to\ prove\ an\ upper\ bound$,

$T(n) < cnlogn$:

$$T(n) = \sqrt{n}\ T(\sqrt{n}) + n$$
$$\leq \sqrt{n} \times c \times \sqrt{n}\, log\sqrt{n} + n$$
$$\leq c \times n \times log\sqrt{n} + n\ [\sqrt{n} \times \sqrt{n} = n]$$

$$\leq\ c \times n \times log n^{\frac{1}{2}} + n\ [\ log_a b^c = c \times log_a b]$$

$$\leq cn \times \frac{1}{2} log n + n$$

$$\leq cnlogn\ [\text{`}nlogn\text{`}\ is\ higher\ here\ comparing\ with\ \text{`}n\text{`}\ ]$$

$The\ last\ inequality\ assumes\ only\ that\ 1 \leq c \times \frac{1}{2} \times logn$.
$This\ is\ correct\ if\ n\ is\ sufficiently\ large\ and\ for\ any\ constant\ c, no\ matter\ how\ small. From\ the\ above\ proof, we\ can\ see\ that\ our\ guess\ is\ correct\ for\ the\ upper\ bound. Now, let\ us\ prove\ the\ lower\ bound\ for\ this\ recurrence.$

$$T(n) = \sqrt{n}\ T\left(\sqrt{n}\right) + n$$

$$\leq\ \sqrt{n}\ \times k \times \sqrt{n}\ log\sqrt{n} + n$$

$$\leq\ k \times n \times log\sqrt{n} + n\ \left[\sqrt{n}\ \times\ \sqrt{n} = n\right]$$

$$\leq\ k \times n \times log n^{\frac{1}{2}} + n\ [\ log_a b^c = c \times log_a b]$$

$$\leq kn \times \frac{1}{2} log n + n$$

$$\leq knlogn\ [\text{`}nlogn\text{`}\ is\ higher\ here\ comparing\ with\ \text{`}n\text{`}]$$

$The\ last\ inequality\ assumes\ only\ that\ 1 \geq k \times \frac{1}{2} \times logn.$

$This\ is\ incorrect\ if\ n\ is\ sufficiently\ large\ and\ for\ any\ constant\ k. From\ the\ above\ proof, we\ can\ see\ that\ our\ guess\ is\ incorrect\ for\ the\ lower\ bound.$

$From\ the\ above\ discussion, we\ understood\ that\ \Theta(nlogn)\ is\ too\ big. How\ about\ \Theta(n)\ ? The\ lower\ bound\ is\ easy\ to\ prove\ directly:$

$$T(n) = \sqrt{n}\ T(\sqrt{n}) + n \qquad \geq n$$

$Now, let\ us\ prove\ the\ upper\ bound\ for\ this\ \Theta(n).$

$$T(n) = \sqrt{n}\ T(\sqrt{n}) + n$$
$$\leq \sqrt{n} \times c \times \sqrt{n} + n$$
$$\leq c \times n + n\ (\sqrt{n} \times \sqrt{n} = n)$$
$$\leq cn + n$$
$$\leq n(c + 1)$$
$$\nleq cn$$

$From\ the\ above\ induction, \Theta(n)\ is\ too\ small\ and\ \Theta(nlogn)\ is\ too\ big\ . Therefore\ , we\ need\ something\ bigger\ than$ $`n`$ $and\ smaller\ than$ $`nlogn`$$. How\ about\ n\sqrt{logn}\ ?$

$Proving\ the\ upper\ bound\ for\ n\sqrt{logn}$:

$$T(n) = \sqrt{n}\ T(\sqrt{n}) + n$$

$$\leq \sqrt{n} \times c \times \sqrt{n} \times \sqrt{log\sqrt{n}} + n$$

$$\leq c \times n \times \sqrt{log\sqrt{n}} + n\ (\sqrt{n} \times \sqrt{n} = n)$$

$$\leq c \times n \times \log \sqrt{n}^{\frac{1}{2}} + n\ (\sqrt{n} \times \sqrt{n} = n)$$

$$\leq c \times n \times \frac{1}{2} log\sqrt{n} + n$$

$$\leq cnlog\sqrt{n}$$

$Proving\ the\ lower\ bound\ for\ n\sqrt{logn}$:

$$T(n) = \sqrt{n}\ T(\sqrt{n}) + n$$

$$\geq \sqrt{n} \times k \times \sqrt{n} \times \sqrt{log\sqrt{n}} + n$$

$$\geq k \times n \times \sqrt{log\sqrt{n}} + n\ (\sqrt{n} \times \sqrt{n} = n)$$

$$\geq k \times n \times \log \sqrt{n}^{\frac{1}{2}} + n\ (\sqrt{n} \times \sqrt{n} = n)$$

$$\geq k \times n \times \frac{1}{2} log\sqrt{n} + n$$

$$\ngeq knlog\sqrt{n}$$

$\boldsymbol{The\ last\ step\ doesn't\ work\ .What\ else\ is\ between}$
$\boldsymbol{n\ and\ nlogn?\ How\ about\ nloglogn?}$

$\boldsymbol{Proving\ upper\ bound\ for\ nloglogn:}$

$$
\begin{aligned}
T(n) &= \sqrt{n}\ T(\sqrt{n}) + n \\
&\le \sqrt{n} \times c \times \sqrt{n}\, loglog\sqrt{n} + n \ (\sqrt{n} \times \sqrt{n} = n) \\
&= c \times nloglog\sqrt{n} + n \\
&= c \times nloglogn - cn + n
\end{aligned}
$$

$\boldsymbol{i.e. we\ can\ say\ , nloglog\sqrt{n} = nloglogn - cn, where\ c\ is}$
$\boldsymbol{constant.}$

$$\le cnloglogn\ , if\ c \ge 1$$

$\boldsymbol{Proving\ lower\ bound\ for\ nloglogn:}$

$$
\begin{aligned}
T(n) &= \sqrt{n}\ T(\sqrt{n}) + n \\
&\ge \sqrt{n} \times k \times \sqrt{n}\, loglog\sqrt{n} + n \ (\sqrt{n} \times \sqrt{n} = n) \\
&\ge k \times nloglog\sqrt{n} + n \\
&\ge k \times nloglogn - kn + n
\end{aligned}
$$

$\boldsymbol{i.e. we\ can\ say\ , nloglog\sqrt{n} = nloglogn - cn, where\ c\ is}$
$\boldsymbol{constant.}$

$$\geq knloglogn\ , if\ k \leq 1$$

$From\ the\ above\ proofs, we\ can\ see\ that$

$T(n) \leq\ cnloglogn\ , if\ c \geq 1\ and\ T(n) \geq knloglogn, if\ k \leq 1.$

$$Hence\ it\ is\ fair : T(n) = \ \Theta(nloglogn).$$

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