$B.B.1.$ *Simplified Master Theorem*

The master theorem is used for solving divide – and – conquer recurrence relations.

While the master theorem doesnot solve all types of divide – and – conquer recurrences , it can solve the majority of recurrene equation.

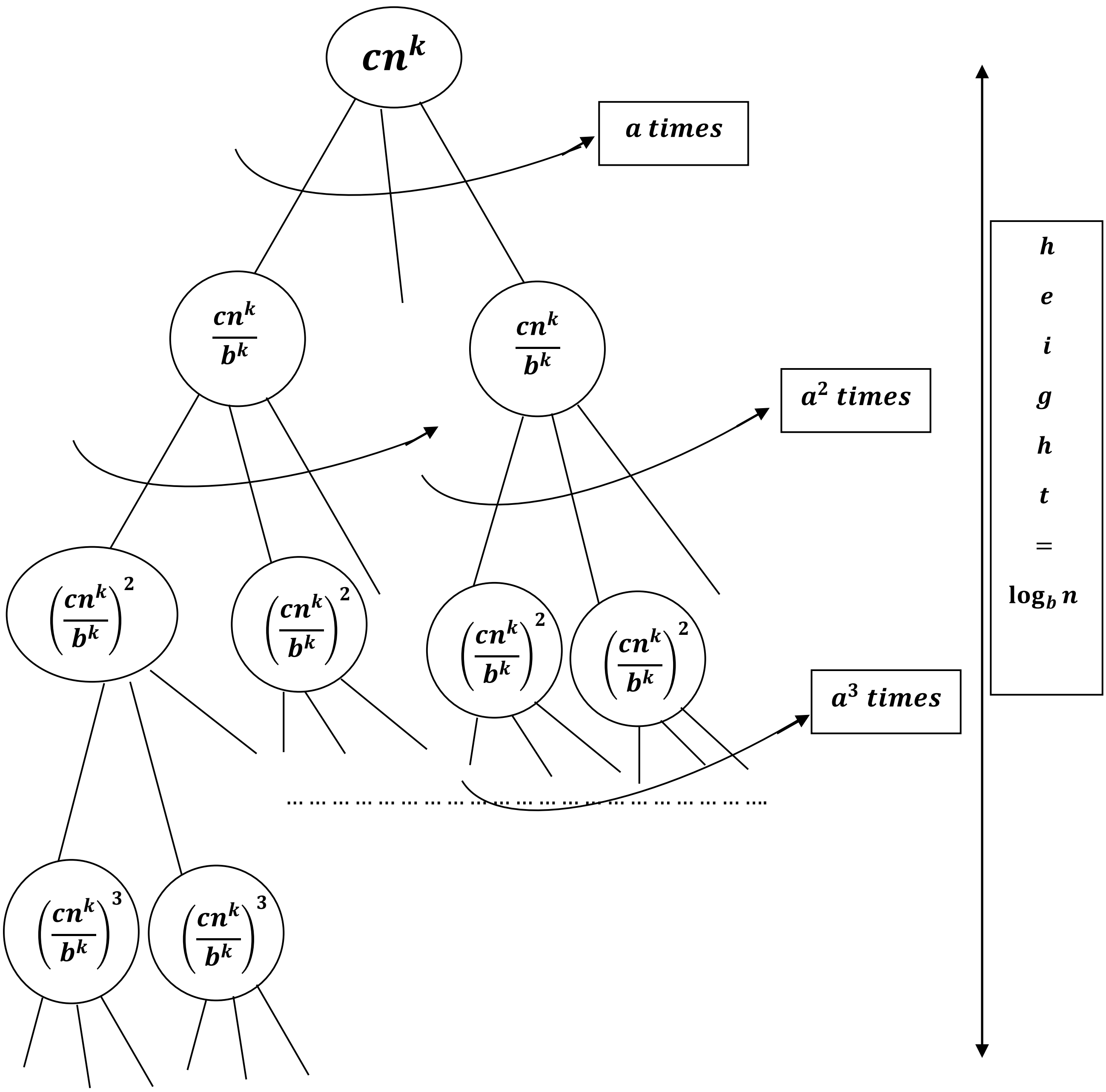
At preliminary stage , we are provided with the formula:

$$T(n) = aT\left(\frac{n}{b}\right) + \mathrm{cn^k}.$$

This implies that the problem is divided into `a` subproblems at every level and every subproblems is of the size

$$\frac{cn^k}{b^k}.$$

The total cost of the recurrence tree is calculated by adding the costs of all the levels .



cn^k
$a\ times$
$\frac{cn^k}{b^k}$
$\frac{cn^k}{b^k}$
$a^2\ times$
$\left(\frac{cn^k}{b^k}\right)^2$
$\left(\frac{cn^k}{b^k}\right)^2$
$\left(\frac{cn^k}{b^k}\right)^2$
$\left(\frac{cn^k}{b^k}\right)^2$
$a^3\ times$
$\left(\frac{cn^k}{b^k}\right)^3$
$\left(\frac{cn^k}{b^k}\right)^3$
$height = \log_b n$

$\boldsymbol{Level}$	$\boldsymbol{No. of\ problems}$	$\boldsymbol{Problem\ Size}$	$\boldsymbol{Work\ done = Problem\ Size \times No. of\ Problems}$
$\mathbf{0}$	$\mathbf{1}$	$\boldsymbol{cn^k}$	$\boldsymbol{1 \times cn^k = cn^k}$
$\mathbf{1}$	$\boldsymbol{a}$	$\boldsymbol{\frac{cn^k}{b^k}}$	$\boldsymbol{a \times \left(\frac{cn^k}{b^k}\right)}$
$\mathbf{2}$	$\boldsymbol{a^2}$	$\boldsymbol{\left(\frac{cn^k}{b^k}\right)^2}$	$\boldsymbol{a^2 \times \left(\frac{cn^k}{b^k}\right)^2}$
. .	. .	. .	. .
$\boldsymbol{n-1}$	$\boldsymbol{a^k}$	$\boldsymbol{\left(\frac{cn^k}{b^k}\right)^k}$	$\boldsymbol{a^k \times \left(\frac{cn^k}{b^k}\right)^k}$
$\boldsymbol{k = \log_b n}$	$\boldsymbol{a^{\log_b n}}$	$\boldsymbol{\left(\frac{cn^k}{b^k}\right)^{\log_b n}}$	$\boldsymbol{a^{\log_b n} \times \left(\frac{cn^k}{b^k}\right)^{\log_b n}}$

$\boldsymbol{Therefore\ ,the\ total\ cost\ of\ the\ \ recurrence\ tree\ can\ be\ given\ as\ follows}$:

$$T(n) = cn^k + \ a \times \left(\frac{cn^k}{b^k}\right) + a^2 \times \left(\frac{cn^k}{b^k}\right)^2 + \cdots + a^{\log_b n} \times \left(\frac{cn^k}{b^k}\right)^{\log_b n}$$

$$\Rightarrow cn^k \left(1 + \ a \times \left(\frac{1}{b^k}\right) + a^2 \times \left(\frac{1}{b^k}\right)^2 + \cdots + a^{\log_b n} \times \left(\frac{1}{b^k}\right)^{\log_b n}\right)$$

$$\Rightarrow cn^k \left(1 + \ \left(\frac{a}{b^k}\right) + \left(\frac{a}{b^k}\right)^2 + \cdots + \left(\frac{a}{b^k}\right)^{\log_b n}\right)$$

$\boldsymbol{if\ d = \left(\frac{a}{b^k}\right), then\ we\ can\ rewrite\ the\ equation\ as}$:

$$T(n) = \ cn^k\left(1 + \ (d) + (d)^2 + \cdots + (d)^{\log_b n}\right)$$

$$\boldsymbol{Case\ 1: \rightarrow\ \frac{a}{b^k} < 1}$$

$\boldsymbol{When\ d < 1\ i.e. \frac{a}{b^k} < 1\ \ hence,}$

$$\lim_{n\to\infty}\left(1 + d + d^2 + d^3 + \cdots + d^{\log_b n}\right)$$

$\boldsymbol{As\ this\ is\ a\ geometric\ series\ which\ yields\ to}$:

$$= \lim_{n\to\infty}\left(\frac{d^{n+1} - 1}{d - 1}\right)$$

$$= \frac{1}{d-1} \times \lim_{n\to\infty} d^{n+1} - \lim_{n\to\infty} 1$$

$$= \frac{1}{d-1} \times (\infty - 1)$$

$$= \frac{1}{d-1} \times (\infty - 1)$$

$$= \frac{1}{d-1} \times \infty$$

$\boldsymbol{Now\ the\ condition\ is\ d < 1\ , the\ series\ becomes\ convergent\ i.e. finite\ and\ unique\ then\ it\ becomes}$:

$$1 + d + d^2 + \cdots = \frac{1}{d-1}$$

$\boldsymbol{Even\ if\ the\ sequence\ goes\ infinite\ , one\ can\ observe\ that\ the\ sequence\ reduces\ to\ constant\ factors\ only, when\ d < 1}$.

$$Therefore : cn^k \left[\frac{1}{d-1}\right] = \frac{cn^k}{d-1} = \Theta\left(\frac{cn^k}{d-1}\right)$$

$$= \left(\frac{1}{d-1}\right) \times \Theta\left(\mathrm{cn}^{\mathrm{k}}\right) = \Theta\left(\mathrm{cn}^{\mathrm{k}}\right) = \Theta\left(\mathrm{n}^{\mathrm{k}}\right).$$

$$\boldsymbol{Case\ 2:\rightarrow\ \frac{a}{b^k} = 1}$$

$\boldsymbol{When\ d = 1\ , this\ implies\ a = b^k.\ Therefore\ we\ will\ have}$:

$$\boldsymbol{T(n) = \ cn^k(1 + \ (d) + (d)^2 + \cdots + (d)^{\log_b n})}$$

$\boldsymbol{Substuting\ d\ with\ 1\ we\ get}$:

$$\boldsymbol{T(n) = \ cn^k\left(1 + \ (1) + (1)^2 + \cdots + (1)^{\log_b n}\right)}$$

$$\boldsymbol{T(n) = \ cn^k(1 + \ \ (1 + 1 + \cdots + \ \log_b n\ times))}$$

$$\boldsymbol{T(n) = \ cn^k(1 + \ \ 1 \times \log_b n)}$$

$$\boldsymbol{T(n) = \ cn^k(1 + \ \ \log_b n)}$$

$$\boldsymbol{T(n) = \ \left(cn^k + \ \ cn^k \times \log_b n\right)}$$

$\boldsymbol{Here\ dominant\ term\ is\ cn^k \times \log_b n\ , hence\ result\ is}$:

$\boldsymbol{\Theta\left(n^k \log_b n\right)\ or\ \Theta\left(n^k \log n\right)}$.

$$Case\ 3 :\rightarrow\ \frac{a}{b^k} > 1$$

$When\ d\ \neq 1, this\ implies\ that\ \frac{a}{b^k} > 1\ , that\ is\ a > b^k, then:$

$$T(n) =\ cn^k(1 +\ (d) + (d)^2 + \cdots + (d)^{\log_b n})$$

$We\ can\ write\ this\ as\ :$

$$T(n) =\ cn^k\ \times d^{\log_b n}\underbrace{\left(\frac{1}{d^{\log_b n}} + \cdots + \frac{1}{d} + 1\right)}_{\boxed{Constant}}$$

$i.e., \left(\frac{1}{d^{\log_b n}} + \cdots + \frac{1}{d} + 1\right)\ will\ lead\ to\ constant\ , as\ the$

$series\ converges\ to\ a\ finite\ , as\ n\ tends\ to\ inifinity\ , each$

$term\ in\ the\ summation\ tends\ to\ 0\ when\ d > 1.$

$$i.e.\lim_{n\to\infty}\left(\frac{1}{d^{\log_b n}} + \cdots + \frac{1}{d} + 1\right)$$

$$\Rightarrow \boldsymbol{We\ know}: \lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

$$\Rightarrow \left(\frac{1}{\infty} + \cdots + 0 + 1\right)$$

$\boldsymbol{As\ we\ know}\ \frac{1}{\infty}\ \boldsymbol{is}\ 0, \boldsymbol{hence}\ 1\ \boldsymbol{is\ the\ answer}\ , \boldsymbol{therefore}$

$\boldsymbol{convergent}.$

$\boldsymbol{Hence},$

$$T(n) = cn^k \times d^{\log_b n}$$

$$or, T(n) = cn^k \times \left(\frac{a}{b^k}\right)^{\log_b n}$$

$$= cn^k \times \left(\frac{a^{\log_b n}}{b^{k \log_b n}}\right)$$

$$= cn^k \times \left(\frac{a^{\log_b n}}{n^k}\right) \left[\, b^{k \log_b n} = n^{k \log_b b}\right]$$

$$= cn^k \times \left(\frac{a^{\log_b n}}{n^k}\right) \left[\, b^{k \log_b n} = n^{k \log_b b}\right]$$

$$= \boldsymbol{c} \times \boldsymbol{a}^{\log_b n}$$

$$\in \ \boldsymbol{\Theta}\left(\boldsymbol{n}^{\log_b a}\right).$$

$\boldsymbol{Thus\ we\ get\ a\ simplified\ master\ theorem\ as\ follows}$:

Simplified Master Theorem:

Let the time complexity function $T(n)$ *be a positive and eventually a non-decreasing function of the following form*:

$$T(n) = aT\left(\frac{n}{b}\right) + cn^k$$

$$T(1) = d$$

Here a, d, b, k *are all constants* .

Here $b \geq 2, k \geq 0, a > 0, c > 0$ *and* $d \geq 0$. *The solution for the recurrence equation is given as follows*:

Case 1: $T(n) \in \Theta(n^k)$ *if* $a < b^k$

Case 2: $T(n) \in \Theta(n^k \log n)$ *if* $a = b^k$

Case 3: $T(n) \in \Theta(n^{\log_b a})$ *if* $a > b^k$

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