$C.1.$ *Recurrence Tree Method − Example 3*

Solve the following recurrence equation:

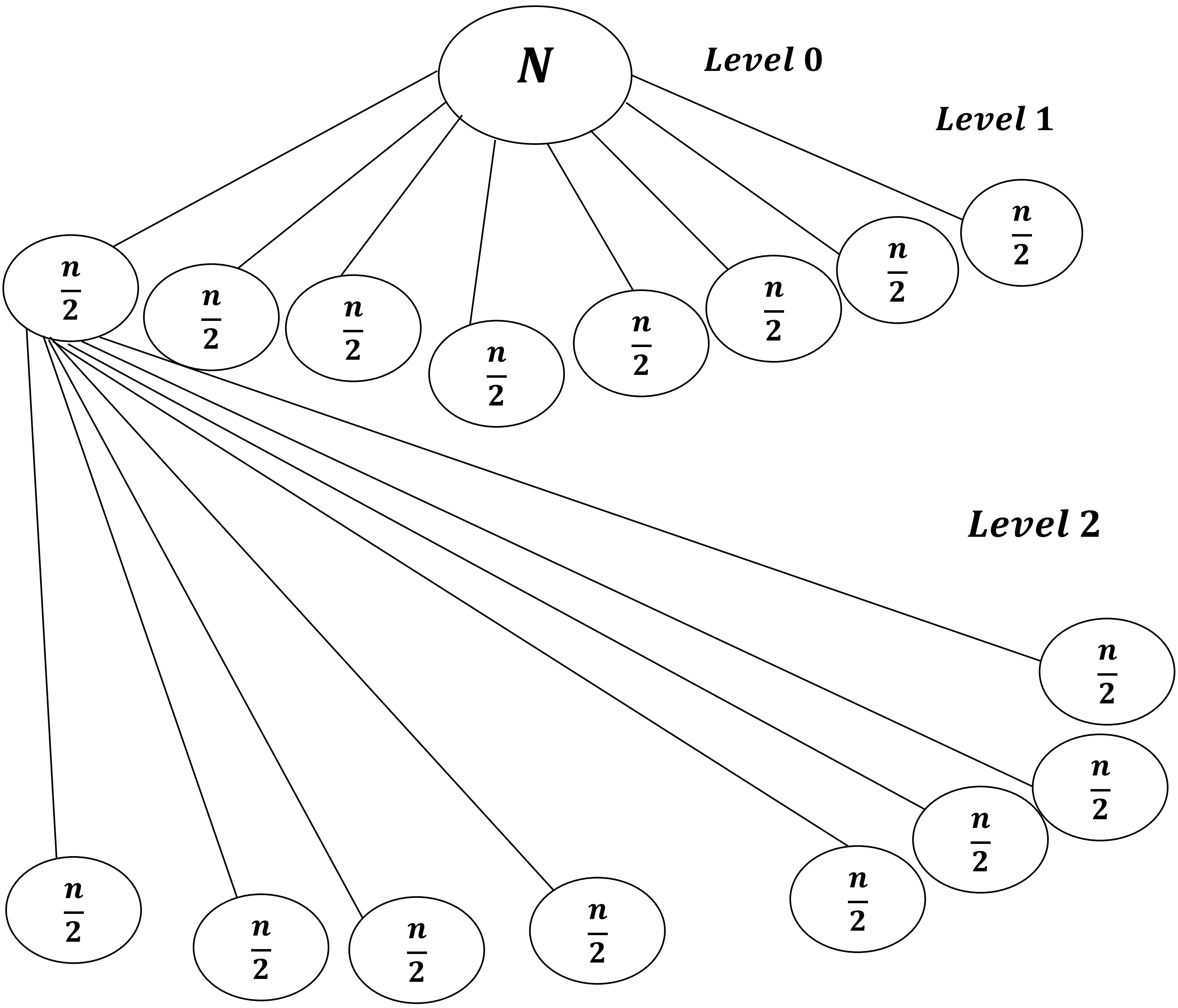
$$t_n = \begin{cases} \mathbf{1} & \text{for } n = 1 \\ \mathbf{8}T\left(\frac{n}{2}\right) & \text{for } n > 1 \end{cases}$$

<u>Solution</u>:

Initially the input is n. It is sub − divided into eight sub − problems. In the next level, each of these eight sub − problems is again divided into eight sub problems(so a total of $8 \times 8 = 64$ sub problems).

Here $8T$ represents 8 subdivision and $\frac{n}{2}$ represents $\frac{n}{2}$ increase of problem size.

This process is continued till a pattern is obtained.



$Recurrence\ tree\ at\ levels\ 1\ and\ 2$ ($and\ only\ one\ problem\ division\ is\ shown$)

$Level$	$No. of\ problems$	$Problem\ Size$	$Work\ done = No. of\ problems \times problem\ size$
$\mathbf{0}$	$\mathbf{1}$	$\boldsymbol{n}$	$\mathbf{1} \times \boldsymbol{n} = \boldsymbol{n}$
$\mathbf{1}$	$\mathbf{8}$	$\dfrac{\boldsymbol{n}}{\mathbf{8}}$	$\mathbf{8} \times \dfrac{\boldsymbol{n}}{\mathbf{8}} = \boldsymbol{n}$
$\mathbf{2}$	$\mathbf{8^2}$	$\dfrac{\boldsymbol{n}}{\mathbf{8^2}}$	$\mathbf{8^2} \times \dfrac{\boldsymbol{n}}{\mathbf{8^2}} = \boldsymbol{n}$
.	.	.	.
.	.	.	.
$\boldsymbol{k}$	$\mathbf{8}^{\boldsymbol{k}}$	$\dfrac{\boldsymbol{n}}{\mathbf{8}^{\boldsymbol{k}}}$	$\mathbf{8}^{\boldsymbol{k}} \times \dfrac{\boldsymbol{n}}{\mathbf{8}^{\boldsymbol{k}}} = \boldsymbol{n}$
$\log_2 \boldsymbol{n}$	$\mathbf{8}^{\log_2 \boldsymbol{n}}$	$\mathbf{1}$	$\mathbf{8}^{\log_2 \boldsymbol{n}} \times \mathbf{1} = \boldsymbol{n}^3$

$\boldsymbol{The\ problem\ size\ reduces\ to\ 1\ as\ T(n) = 1}$

$\boldsymbol{for\ n = 1\ i.e. T(1) = 1\ or\ t_1 = 1.}$

$\boldsymbol{It\ can\ be\ observed\ that\ at\ every\ level\ a\ problem\ is\ divided}$

$\boldsymbol{into\ eight\ subproblems\ or\ nodes\ is\ increasing\ in\ the}$

$\boldsymbol{following\ pattern: 1, 8, 64, \ldots \left(8^0, 8^1, 8^2, \ldots., 8^i\right).}$

$The\ problem\ size\ is\ decreasing\ in\ a\ geometric\ series\ as\ follows: \left(n, \frac{n}{8}, \frac{n}{8^2}, \dots, \frac{n}{8^k}, \dots, 1\right). Based\ on\ the\ table, the\ amount\ of\ work\ done\ at\ the\ \log_2 n\ can\ be\ calculated\ as\ follows:$

$8^{\log_2 n} \times T(1)$

$= (2^3)^{\log_2 n} \times 1$

$= \left(2^{\log_2 n}\right)^3$

$= (n)^3\ \left[a^{\log_a b} = b\right]$

$= n^3$

$Alternatively,$

$8^{\log_2 n} \times T(1)$

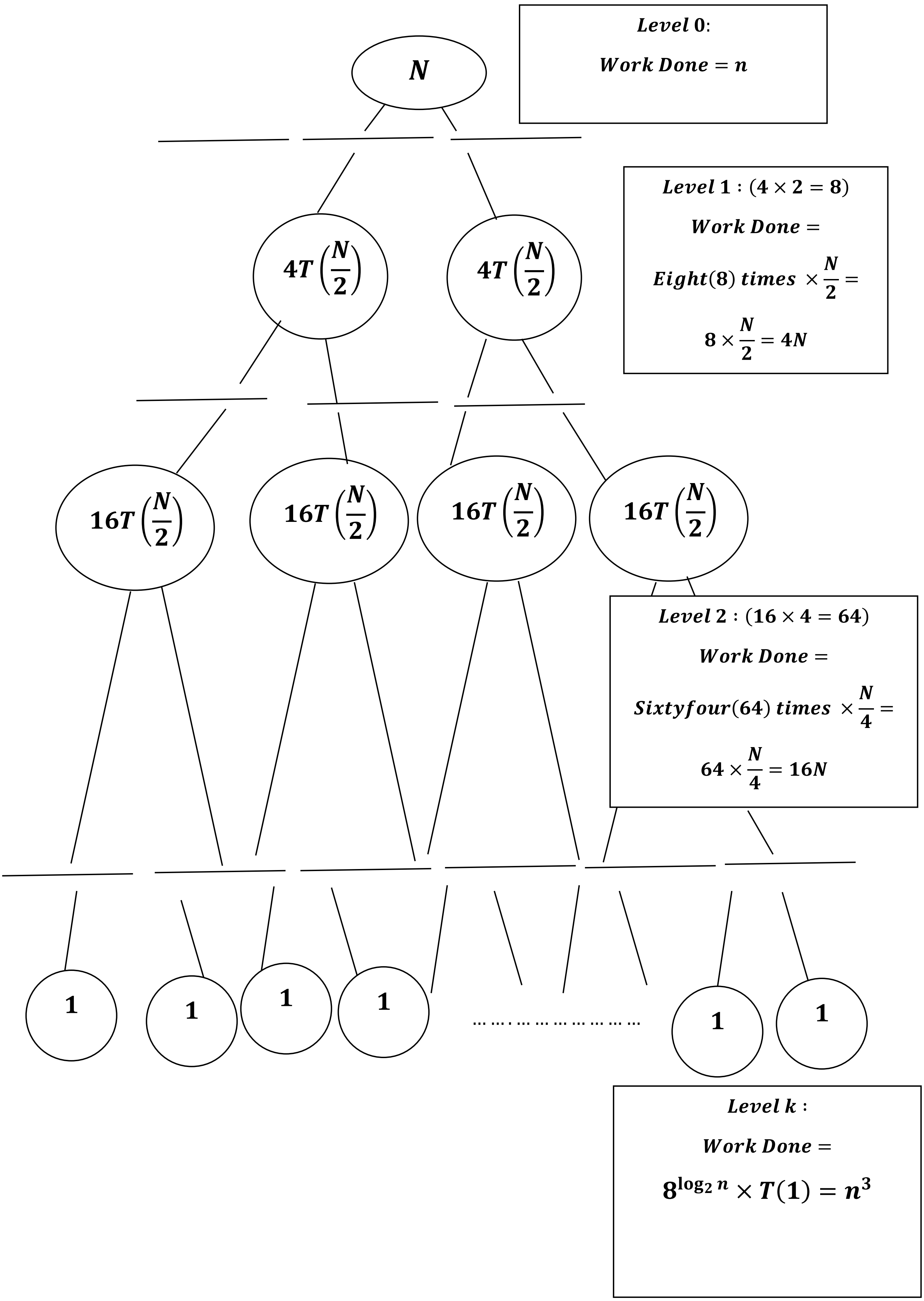
$= (2^3)^{\log_2 n} \times 1$

$= \left(2^{\log_2 n}\right)^3$

$= \left(n^{\log_2 2}\right)^3\ \left[a^{\log_a b} = b^{\log_a a} = b\right]$

$= n^3$

$Now\ to\ calculate\ amount\ of\ work\ done\ at\ other\ level\ lets\ divide\ the\ above\ again:$


N
$Level\ 0$:
$Work\ Done = n$
$4T\left(\frac{N}{2}\right)$
$4T\left(\frac{N}{2}\right)$
$Level\ 1 : (4 \times 2 = 8)$
$Work\ Done =$
$Eight(8)\ times\ \times \frac{N}{2} =$
$8 \times \frac{N}{2} = 4N$
$16T\left(\frac{N}{2}\right)$
$16T\left(\frac{N}{2}\right)$
$16T\left(\frac{N}{2}\right)$
$16T\left(\frac{N}{2}\right)$
$Level\ 2 : (16 \times 4 = 64)$
$Work\ Done =$
$Sixtyfour(64)\ times\ \times \frac{N}{4} =$
$64 \times \frac{N}{4} = 16N$
1
1
1
1
… … . … … … … … … … …
1
1
$Level\ k$:
$Work\ Done =$
$8^{\log_2 n} \times T(1) = n^3$

$\boldsymbol{Re - writing\ the\ above\ table\ as}$:

$\boldsymbol{Level}$	$\boldsymbol{No.of\ problems}$	$\boldsymbol{Problem\ Size}$	$\boldsymbol{Work\ done = No.of\ problems \times problem\ size}$
$\mathbf{0}$	$\mathbf{1}$	$\boldsymbol{n}$	$\mathbf{1} \times \boldsymbol{n} = \boldsymbol{n}$
$\mathbf{1}$	$\mathbf{8}$	$\boldsymbol{\frac{n}{2}}$	$\mathbf{8} \times \boldsymbol{\frac{n}{2}} = \mathbf{4}\boldsymbol{n}$
$\mathbf{2}$	$\mathbf{8^2}$	$\boldsymbol{\frac{n}{4}}$	$\mathbf{8^2} \times \boldsymbol{\frac{n}{4}} = \mathbf{16}\boldsymbol{n}$
.	.	.	.
.	.	.	.
$\boldsymbol{k}$	$\mathbf{8}^{\boldsymbol{k}}$	$\boldsymbol{\frac{n}{2^k}}$	$\mathbf{8}^{\boldsymbol{k}} \times \boldsymbol{\frac{n}{2^k}} = \boldsymbol{n} \times \mathbf{2}^{\mathbf{2}\boldsymbol{k}}$
$\mathbf{\log_2} \boldsymbol{n}$	$\mathbf{8}^{\mathbf{\log_2} \boldsymbol{n}}$	$\mathbf{1}$	$\mathbf{8}^{\mathbf{\log_2} \boldsymbol{n}} \times \mathbf{1} = \boldsymbol{n}^{\mathbf{3}}$

$\boldsymbol{Hence\ at\ level\ 0\ , work\ done\ is\ N. At\ the\ next\ level\ , the\ cost}$
$\boldsymbol{is\ 8\ times\ \left(\frac{N}{2}\right) = 4N\ ; at\ level\ 2\ , the\ cost\ is\ 64\ times\ \left(\frac{N}{4}\right)}$
$\boldsymbol{= 16N\ and\ so\ on.}$

$\boldsymbol{The\ work\ done\ is\ \ increasing\ in\ the\ following\ pattern}$:

$\mathbf{1, 4, 16, \ldots\ldots}$ $\boldsymbol{Therefore\ the\ total\ cost\ is\ the\ work\ done}$
$\boldsymbol{at\ the\ last\ level\ and\ work\ done\ at\ all\ other\ level}(\mathbf{0, 1, 2, 3, \ldots},$
$(\log_2 n - 1)$.

$\boldsymbol{Thus}$, $\boldsymbol{the\ total\ cost\ of\ the\ tree\ can\ be\ esitmated\ as}$
$\boldsymbol{follows}$:

$$\Rightarrow \sum_{i=0}^{\log_2 n-1} 4^i n + 8^{\log_2 n} \times T(1)$$

$\boldsymbol{Where}$, $8^{log_2 n} \times T(1)$ $\boldsymbol{is\ the\ last\ level\ for}\ \log_2 n$.

$\boldsymbol{and\ for}\ \ \log_2 n - 1\ \boldsymbol{i.e.till}\ \ \log_2 n - 1\ \boldsymbol{we\ have}$: $\displaystyle\sum_{i=0}^{\log_2 n-1} 4^i n$

$\boldsymbol{Hence\ we\ got}$: $\displaystyle\sum_{i=0}^{\log_2 n-1} 4^i n + 8^{\log_2 n} \times T(1)$

$\boldsymbol{And\ we\ know}$: $8^{\log_2 n} \times T(1) = n^3$, $\boldsymbol{hence}$:

$$\Rightarrow \sum_{i=0}^{\log_2 n-1} 4^i n + n^3$$

$$\Rightarrow n \times \sum_{i=0}^{\log_2 n-1} 4^i + n^3$$

And we know the geometric series :

$$\sum_{k=1}^{n} k = 1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

And hence for $\sum_{i=0}^{\log_2 n-1} 4^i$ ***, we get***

$$\Rightarrow n \times \left(\frac{4^{(\log_2 n-1)+1} - 1}{4 - 1}\right) + n^3$$

$$\Rightarrow n \times \left(\frac{4^{\log_2 n} - 1}{4 - 1}\right) + n^3$$

$$\Rightarrow n \times \left(\frac{(n)^2 - 1}{4 - 1}\right) + n^3 \left[\because 4^{\log_2 n} = \left(2^{\log_2 n}\right)^2 = n^2 \right]$$

$$\Rightarrow n \times \left(\frac{n^2 - 1}{3}\right) + n^3$$

$$\Rightarrow \frac{n^3 - n}{3} + n^3$$

$$\Rightarrow \frac{n^3 - n + 3n^3}{3}$$

$$\Rightarrow \frac{4n^3 - n}{3}$$

$$Hence\ total\ cost\ of\ the\ tree\ as: \Theta\left(\frac{4n^3 - n}{3}\right)$$

$$\Rightarrow \Theta\left(\frac{4n^3}{3} - \frac{n}{3}\right)$$

$$\Rightarrow \Theta\left(\frac{4n^3}{3}\right)$$

$$\Rightarrow \frac{4}{3} \times \Theta(n^3)$$

$$\Rightarrow \Theta(n^3)$$

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