

B. B. 1. Simplified Master Theorem

The master theorem is used for solving divide – and – conquer recurrence relations.

While the master theorem doesnot solve all types of divide – and – conquer recurrences , it can solve the majority of recurrence equation.

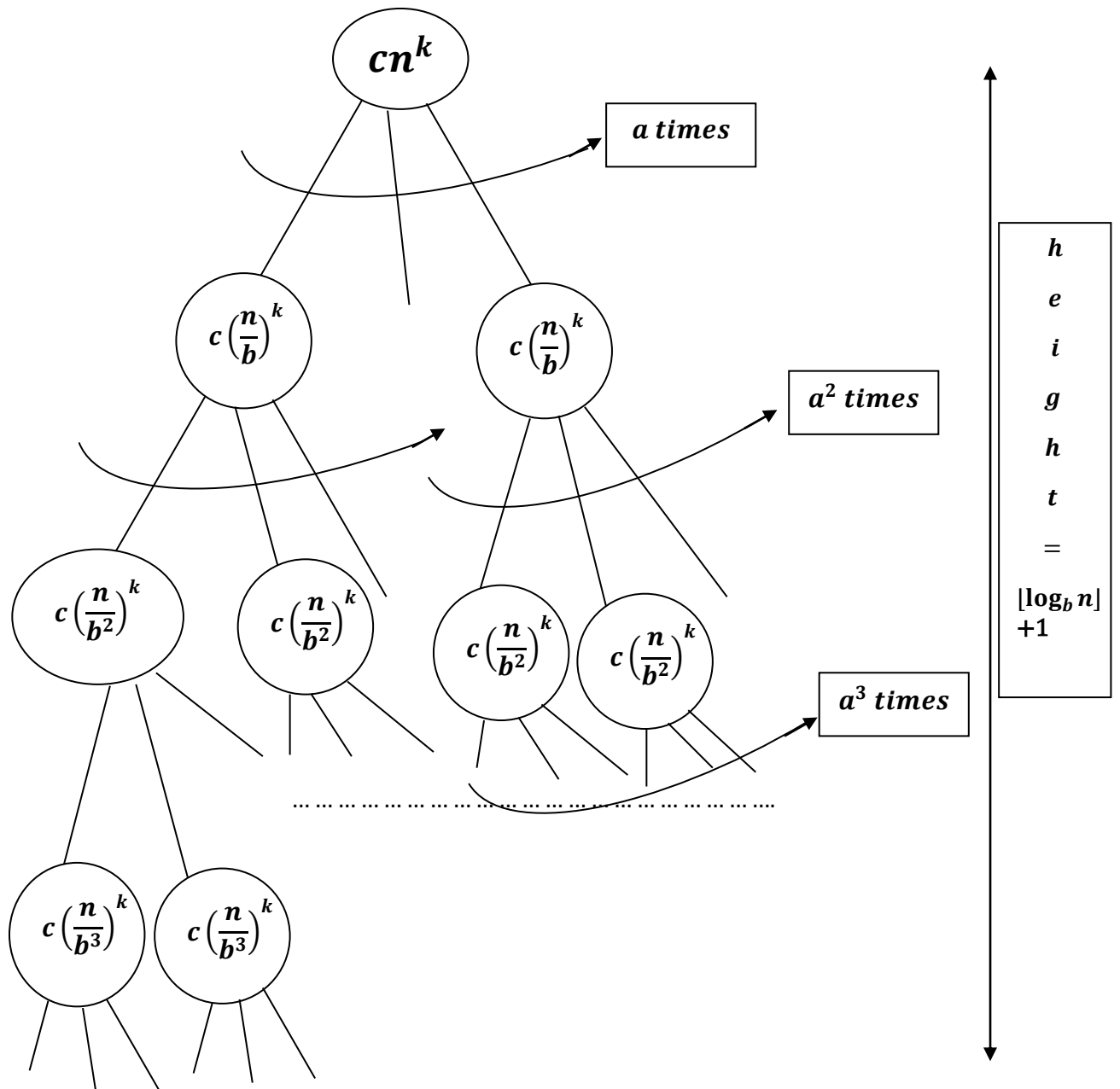
At preliminary stage , we are provided with the formula:

$$T(n) = aT\left(\frac{n}{b}\right) + cn^k.$$

This implies that the problem is divided into `a` subproblems at every level and every subproblems is of the size

$$\frac{cn^k}{b^k}.$$

The total cost of the recurrence tree is calculated by adding the costs of all the levels .



<i>Level</i>	<i>No. of problems</i>	<i>Problem Size</i>	<i>Work done = Problem Size × No. of Problems</i>
0	1	cn^k	$1 \times cn^k = cn^k$
1	a	$\frac{cn^k}{b^k}$	$a \times \left(\frac{cn^k}{b^k}\right)$
2	a^2	$c\left(\frac{n}{b^2}\right)^k$	$a^2 \times c\left(\frac{n}{b^2}\right)^k$
\vdots	\vdots	\vdots	\vdots
$n - 1$	a^k	$c\left(\frac{n}{b^k}\right)^k$	$a^k \times c\left(\frac{n}{b^k}\right)^k$
$k = \log_b n$	$a^{\log_b n}$	$\left(c\left(\frac{n}{b^k}\right)\right)^{\log_b n}$	$a^{\log_b n} \times \left(c\left(\frac{n}{b^k}\right)\right)^{\log_b n}$

Therefore , the total cost of the recurrence tree can be given as follows:

$$T(n) = cn^k + a \times c \left(\frac{n}{b}\right)^k + a^2 \times c \left(\frac{n}{b^2}\right)^k + \dots$$

$$+ a^{\log_b n} \times \left(c \left(\frac{n}{b^k}\right)\right)^{\log_b n}$$

$$\Rightarrow cn^k \left(1 + a \times \left(\frac{1}{b^k}\right) + a^2 \times \frac{1}{(b^2)^k} + \dots + a^{\log_b n} \times \left(\frac{1}{(b^k)^{\log_b n}}\right) \right)$$

$$\Rightarrow cn^k \left(1 + \left(\frac{a}{b^k}\right) + \frac{a^2}{(b^2)^k} + \dots + \left(\frac{a^{\log_b n}}{(b^k)^{\log_b n}}\right) \right)$$

The same expression can be written as:

$$\Rightarrow cn^k \left(1 + \left(\frac{a}{b^k}\right) + \left(\frac{a}{b^k}\right)^2 + \dots + \left(\frac{a}{b^k}\right)^{\log_b n} \right)$$

if $d = \left(\frac{a}{b^k}\right)$, then we can rewrite the equation as:

$$T(n) = cn^k(1 + (d) + (d)^2 + \dots + (d)^{\log_b n})$$

$$\text{Case 1: } \rightarrow \frac{a}{b^k} < 1$$

When $d < 1$ i.e. $\frac{a}{b^k} < 1$ hence,

To prove the geometric series is infinite :

$$\lim_{n \rightarrow \infty} (1 + d + d^2 + d^3 + \dots + d^{\log_b n})$$

As this is a geometric series which yields to:

$$= \lim_{n \rightarrow \infty} \left(\frac{d^{n+1} - 1}{d - 1} \right)$$

$$= \frac{1}{d - 1} \times \lim_{n \rightarrow \infty} d^{n+1} - \lim_{n \rightarrow \infty} 1$$

$$= \frac{1}{d - 1} \times (\infty - 1)$$

$$= \frac{1}{d - 1} \times (\infty - 1)$$

$$= \frac{1}{d-1} \times \infty$$

$$= \infty$$

The series is infinite geometric series , as n goes infinite:

The series can be expressed as a common ratio `d` and a is 1 , Hence by formula of infinite geometric series we we get:

$$S = \frac{a}{1-r} = \frac{1}{1-d}.$$

To prove the series is convergent:

$\lim_{n \rightarrow \infty} \left(\frac{1}{1-d} \right) = \frac{1}{1-0} = 1$, hence series is convergent i. e. finite and unique.

As the condition is $d < 1$, the series becomes convergent i. e. finite and unique i. e.:

$$1 + d + d^2 + \dots = \frac{1}{1-d}$$

Even if the sequence goes infinite , one can observe that the

sequence reduces to constant factors only, when $d < 1$.

$$\begin{aligned} \text{Therefore : } cn^k \left[\frac{1}{1-d} \right] &= \frac{cn^k}{1-d} = \Theta \left(\frac{cn^k}{1-d} \right) \\ &= \left(\frac{1}{1-d} \right) \times \Theta(cn^k) = \Theta(cn^k) = \Theta(n^k). \end{aligned}$$

$$\text{Case 2: } \rightarrow \frac{a}{b^k} = 1$$

When $d = 1$, this implies $a = b^k$. Therefore we will have:

$$T(n) = cn^k(1 + (d) + (d)^2 + \dots + (d)^{\log_b n})$$

Substituting d with 1 we get:

$$T(n) = cn^k(1 + (1) + (1)^2 + \dots + (1)^{\log_b n})$$

$$T(n) = cn^k(1 + (1 + 1 + \dots + \log_b n \text{ times}))$$

$$T(n) = cn^k(1 + 1 \times \log_b n)$$

$$T(n) = cn^k(1 + \log_b n)$$

$$T(n) = (cn^k + cn^k \times \log_b n)$$

Here dominant term is $cn^k \times \log_b n$, hence result is:

$\Theta(n^k \log_b n)$ or $\Theta(n^k \log n)$.

$$\text{Case 3: } \rightarrow \frac{a}{b^k} > 1$$

When $d \neq 1$, this implies that $\frac{a}{b^k} > 1$, that is $a > b^k$, then:

$$T(n) = cn^k(1 + (d) + (d)^2 + \dots + (d)^{\log_b n})$$

We can write this as :

$$T(n) = cn^k \times d^{\log_b n} \left(\underbrace{\frac{1}{d^{\log_b n}} + \dots + \frac{1}{d} + 1}_{\text{Constant}} \right)$$

i. e., $\left(\frac{1}{d^{\log_b n}} + \dots + \frac{1}{d} + 1\right)$ will lead to constant, as the series converges to a finite, as n tends to infinity, each term in the summation tends to 0 when $d > 1$.

If n tends to infinity, the terms $\frac{1}{d^{\log_b n}}, \frac{1}{d^{\log_b n-1}}, \dots$ etc. will become very small.

We can write it as:

$$\begin{aligned} & \frac{1}{d^{\log_b n}} + \frac{1}{d^{\log_b n-1}} + \dots + \frac{1}{d} + 1 \\ &= \left(\frac{1}{d}\right)^{\log_b n} + \left(\frac{1}{d}\right)^{\log_b n-1} + \dots + \left(\frac{1}{d}\right)^1 + 1 \end{aligned}$$

Now let's consider common ratio of the series r , which is equal to $\frac{1}{d}$.

Since $0 < d < 1$ and n is infinite, we can apply the formula for the sum of an geometric series:

$$S = \frac{a}{1-r}$$

Substuting the values of $a = \left(\frac{1}{d}\right)^{\log_b n}$ and $r = \frac{1}{d}$ into the formula for the sum of an infinite geometric series:

$$S = \frac{\left(\frac{1}{d}\right)^{\log_b n}}{1 - d}$$

$$S = \frac{\left(\frac{1}{d}\right)^{\log_b n}}{1 - \frac{1}{d}}$$

$$S = \frac{\left(\frac{1}{d}\right)^{\log_b n}}{\frac{d - 1}{d}}$$

$$S = \left(\frac{1}{d}\right)^{\log_b n} \times \frac{d}{(d - 1)}$$

As n tends to infinity : $\left(\frac{1}{d}\right)^{\log_b n}$ approaches to zero, since any positive base raised to a negative exponent

tends to zero. i. e. Suppose take an example:

$$\left(\frac{1}{10}\right)^{\log_b 100}, \text{ then } \log_{10} 100 = 2$$

$$\left(\frac{1}{10}\right)^{\log_b 100} = \left(\frac{1}{10}\right)^2 = \frac{1}{100}, \text{ and the value is}$$

less than one, hence exponent is negative i. e. 10^{-2} .

***** Hence, the sum of series converges to a constant value. ****

Note: The above expression can also be represented as:

$$S = \frac{1}{d^{\log_b n}} \times \frac{d}{(d-1)} = \frac{1}{n^{\log_b d}} \times \frac{d}{(d-1)} \text{ also as:}$$

$$S = \frac{1}{n^{\log_b d} \times \frac{(d-1)}{d}}$$

Also we can write:

As n tends to infinity, term $\frac{1}{n^{\log_b d}}$ approaches to zero , since any positive base raised to a negative exponent tends to zero. Therefore, the sum of series converges to a constant value.

If we want an convergence test:

Convergence Test

$$\frac{1}{d^{\log_b n}} + \frac{1}{d^{\log_b n-1}} + \dots + \frac{1}{d} + 1$$

Lets consider the series: $a_n = \frac{1}{d^{\log_b n}}$, to apply the limit comparison test, we need to find another series b_n , whose convergence behaviour is known.

Lets choose $b_n = \frac{1}{n^\epsilon}$, where ϵ is a positive constant.

Now we will compare the series a_n and b_n , by taking the limit as n tends to infinity:

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{d^{\log_b n}} \right)}{\frac{1}{n^\epsilon}}$$

To simplify this expression, let's rewrite $\frac{1}{d^{\log_b n}}$ as $\frac{1}{n^{\log_b d}}$

$$= \lim_{n \rightarrow \infty} \frac{1}{\frac{n^{\log_b d}}{1}} = \lim_{n \rightarrow \infty} \frac{1}{n^{\log_b d}}$$

$$= \lim_{n \rightarrow \infty} \frac{n^{\epsilon}}{n^{\log_b d}}$$

Since $b > 1$ and $d > 1$, we know that $\log_b d > 0$.

Therefore, as n tends to infinity, $n^{\log_b d}$ grows faster than n^{ϵ} for any positive ϵ .

As a result the limit: $\lim_{n \rightarrow \infty} \frac{n^{\epsilon}}{n^{\log_b d}}$ is equal to 0 for any positive ϵ . This implies that a_n and b_n have the same convergence behaviour.

Since the series: $\sum_{n=1}^{\infty} b_n$ with $b_n = \frac{1}{n^{\epsilon}}$ converges $\epsilon > 1$,

We can conclude that the series $\sum_{n=1}^{\infty} a_n$ also converges.

Therefore the series : $\frac{1}{d^{\log_b n}} + \frac{1}{d^{\log_b n-1}} + \dots + \frac{1}{d} + 1$ converges for $d > 1$ and $b > 1$.

Therefore keeping out the constant:

$$\left(\frac{1}{d^{\log_b n}} + \dots + \frac{1}{d} + 1 \right) \text{ generated from: } T(n) = cn^k \times d^{\log_b n},$$

as it doesnot matter in time complexity, we continue:

$$T(n) = cn^k \times d^{\log_b n}$$

$$\text{or, } T(n) = cn^k \times \left(\frac{a}{b^k} \right)^{\log_b n}$$

$$= cn^k \times \left(\frac{a^{\log_b n}}{b^{k \log_b n}} \right)$$

$$= cn^k \times \left(\frac{a^{\log_b n}}{n^k} \right) [b^{k \log_b n} = n^{k \log_b b}]$$

$$= cn^k \times \left(\frac{a^{\log_b n}}{n^k} \right) [b^{k \log_b n} = n^{k \log_b b}]$$

$$= c \times a^{\log_b n}$$

$\in \Theta(n^{\log_b a})$ [*as $a^{\log_b n}$ can also be written as $n^{\log_b a}$*].

Thus we get a simplified master theorem as follows:

Simplified Master Theorem:

Let the time complexity function $T(n)$ be a positive and eventually a non – decreasing function of the following form:

$$T(n) = aT\left(\frac{n}{b}\right) + cn^k$$

$$T(1) = d$$

Here a, d, b, k are all constants .

Here $b \geq 2, k \geq 0, a > 0, c > 0$ and $d \geq 0$. The solution for the recurrence equation is given as follows:

Case 1: $T(n) \in \Theta(n^k)$ if $a < b^k$

Case 2: $T(n) \in \Theta(n^k \log n)$ if $a = b^k$

Case 3: $T(n) \in \Theta(n^{\log_b a})$ if $a > b^k$
