$Polynomial\ Reduction -$
$Solving\ Homogeneous\ Equation -$
$Example - 3$

$Example\ 3$: $Solve\ the\ following\ recurrence\ equation\ using\ the\ polynomial\ reduction\ method$:

$t_n - 5t_{n-1} + 8t_{n-2} - 4t_{n-3} = 0\ for\ n > 0$

$t_0 = 0$

$t_1 = 1$

$t_2 = 2$

$\underline{Solution}$:

$Let\ t_n = r^n\ be\ a\ solution\ of\ this\ second - order\ recurrence\ equation.$

$Then\ we\ have : r^n - 5r^{n-1} + 8r^{n-2} - 4^{n-3} = 0$

$If\ so, let\ us\ divide\ the\ entire\ equation\ by\ r^{n-3}\ seeing\ we\ have\ t_{n-3}.$

$$\Longrightarrow \frac{r^n}{r^{n-3}} - 5 \times \frac{r^{n-1}}{r^{n-3}} + 8 \times \frac{r^{n-2}}{r^{n-3}} - 4 \times \frac{r^{n-3}}{r^{n-3}} = 0$$

$$\Longrightarrow r^{n-(n-3)} - 5 \times r^{n-1-(n-3)} + 8 \times r^{n-2-(n-3)} - 4 \times r^{n-3-(n-3)}$$
$$= 0$$

$$\Longrightarrow r^3 - 5r^2 + 8r - 4 = 0$$

$\boldsymbol{Now\ using\ the\ rational\ root\ theorem\ on\ polynomials}$:

$\boldsymbol{factor\ of\ 4\ is\ 1, 2, and\ 4\ and\ factor\ of\ coefficient\ of}$
$\boldsymbol{r^3\ is\ 1 only\ , hence}$:

$$\pm \frac{1, 2, 4}{1}$$

$\therefore \frac{1}{1}$ $\boldsymbol{is\ the\ root\ of\ the\ expression, so\ factor\ out\ (r-1)}$:

$$\Longrightarrow (r-1) \times \frac{r^3 - 5r^2 + 8r - 4}{(r-1)}$$

$\boldsymbol{Now}$, $\frac{r^3}{r} = r^2$ $\boldsymbol{and}$ $r^2 \times (r-1) = r^3 - r^2$

$\boldsymbol{Next}$,

$$\boldsymbol{r^3 - 5r^2 + 8r - 4}$$

$\boldsymbol{r^3 \quad -r^2}$

$(-) \quad (+)$

$\boldsymbol{-4r^2 + 8r - 4}$

$\boldsymbol{Hence\ we\ can\ write}$:

$$\Rightarrow (\boldsymbol{r-1}) \times \left[\boldsymbol{r^2} + \frac{\boldsymbol{-4r^2 + 8r - 4}}{\boldsymbol{r-1}}\right]$$

$\boldsymbol{Again}$,

$\boldsymbol{Now}$, $\dfrac{\boldsymbol{-4r^2}}{\boldsymbol{r}} = \boldsymbol{-4r}\ \ \boldsymbol{and} - \boldsymbol{4r} \times (\boldsymbol{r-1}) = \boldsymbol{-4r^2 + 4r}$

$$\boldsymbol{-4r^2 \ + 8r - 4}$$

$\boldsymbol{-4r^2 \quad + 4r}$

$(+) \quad (-)$

$\boldsymbol{4r \quad - 4}$

$\boldsymbol{Hence}$,

$$\Rightarrow (\boldsymbol{r-1}) \times \left[\boldsymbol{r^2} - \boldsymbol{4r} + \frac{\boldsymbol{4r - 4}}{\boldsymbol{r-1}}\right]$$

$$\Longrightarrow (r-1)\times\left[r^2-4r+\frac{4(r-1)}{r-1}\right]$$

$$\Longrightarrow (r-1)\times[r^2-4r+4]$$

$$\Longrightarrow (r-1)\times[r^2-2r-2r+4]$$

$$\Longrightarrow (r-1)\times[r(r-2)-2(r-2)]$$

$$\Longrightarrow (r-1)\times(r-2)\times(r-2)$$

$Hence\ the\ roots\ are\ (r-1)(r-2)(r-2)\ , where$

$r-1=0\ or\ r=1,$

$r-2=0\ or\ r=2,$

$r-2=0\ \ or\ r=2,$

$i.e.\, r=1,2,2.$

$It\ can\ be\ observed\ that\ the\ roots\ are\ distinct. The\ root\ 2$

$has\ a\ multiplicity\ of\ 2; thus\ , Case\ 2\ of\ Theorem\ is\ applicable.$

$t_n=r^n\ , t_n=nr^n, t_n=n^2r^n, \dots, t_n=n^{m-1}r^n$

$Applying\ the\ theorem, one\ gets\ the\ following\ relation:$

$t_n=c_n1^n+c_22^n+c_3n2^n$

$i.e., if\ we\ consider\ the\ equation\ like\ this:$

$t_n = 1 + r^n + nr^n + n^2r^n + \cdots + n^{m-1}r^n$

$Therefore\ we\ will\ have:$

$t_n = c_n1^n + c_22^n + c_3n2^n$

$\underline{For\ n = 0:}$

$t_0 = c_11^0 + c_22^0 + c_3(0)2^0$

$0 \ = c_1 + c_2$

$\underline{For\ n = 1:}$

$t_1 = c_11^1 + c_22^1 + c_3(1)2^1$

$1 \ = c_1 + 2c_2 + 2c_3$

$\underline{For\ n = 2:}$

$t_2 = c_11^2 + c_22^2 + c_3(2)2^2$

$2 \ = c_1 + 4c_2 + 8c_3$

$\therefore The\ equations\ are\ as\ follows:$

$$c_1 + c_2 = 0 - - - (i)$$

$$c_1 + 2c_2 + 2c_3 = 1 - - - (ii)$$

$$c_1 + 4c_2 + 8c_3 = 2 - - - (iii)$$

Substracting (ii) ***from*** (i) ***we get***:

$$\begin{array}{l} c_1 + 2c_2 + 2c_3 = 1 \\ c_1 + c_2 \qquad\quad = 0 \\ (-)\ (-) \\ \hline c_2 + 2c_3 \qquad = 1 \\ \hline \end{array}$$

Now we get : $c_2 = 1 - 2c_3 - - - (iv)$

Applying (iv) ***in*** (iii) ***we get***:

$$c_1 + 4(1 - 2c_3) + 8c_3 = 2$$

$$\Rightarrow c_1 + 4 - 8c_3 + 8c_3 = 2$$

$$\Rightarrow c_1 = 2 - 4$$

$$\Rightarrow c_1 = -2 - - - (v)$$

Applying (v) ***in*** (i) ***we get***:

$$-2 + c_2 = 0$$
$$\Rightarrow c_2 = 2 - - - (iv)$$

$$Now\ c_1 = -2\ and\ c_2 = 2\ in\ (iii)\ we\ get:$$

$$c_1 + 4c_2 + 8c_3 = 2$$

$$\Rightarrow -2 + 4 \times 2 + 8c_3 = 2$$

$$\Rightarrow -2 + 8 + 8c_3 = 2$$

$$\Rightarrow 6 + 8c_3 = 2$$

$$\Rightarrow 8c_3 = 2 - 6$$

$$\Rightarrow 8c_3 = -4$$

$$\Rightarrow c_3 = -\frac{4}{8}$$

$$\Rightarrow c_3 = -\frac{1}{2}$$

$$Hence\ we\ get\ solutions: c_1 = -2, c_2 = 2\ and\ c_3 = -\frac{1}{2}.$$

$\boldsymbol{Therefore\ , the\ specific\ solution\ of\ the\ recurrence\ equation}$
$\boldsymbol{with\ respect\ to\ the\ initial\ conditions\ is\ as\ follows}$:

$$\boldsymbol{t_n = c_n 1^n + c_2 2^n + c_3 n 2^n}$$

$$\Longrightarrow\ (-\mathbf{2}) \times \mathbf{1}^{\boldsymbol{n}} + (\mathbf{2}) \times \mathbf{2}^{\boldsymbol{n}} + \left(-\frac{\mathbf{1}}{\mathbf{2}}(\boldsymbol{n})(\mathbf{2}^{\boldsymbol{n}})\right)$$

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