$B.B.1. Simplified\ Master\ Theorem$

The master theorem is used for solving divide − and − conquer recurrence relations.

While the master theorem doesnot solve all types of divide − and − conquer recurrences , it can solve the majority of recurrene equation.
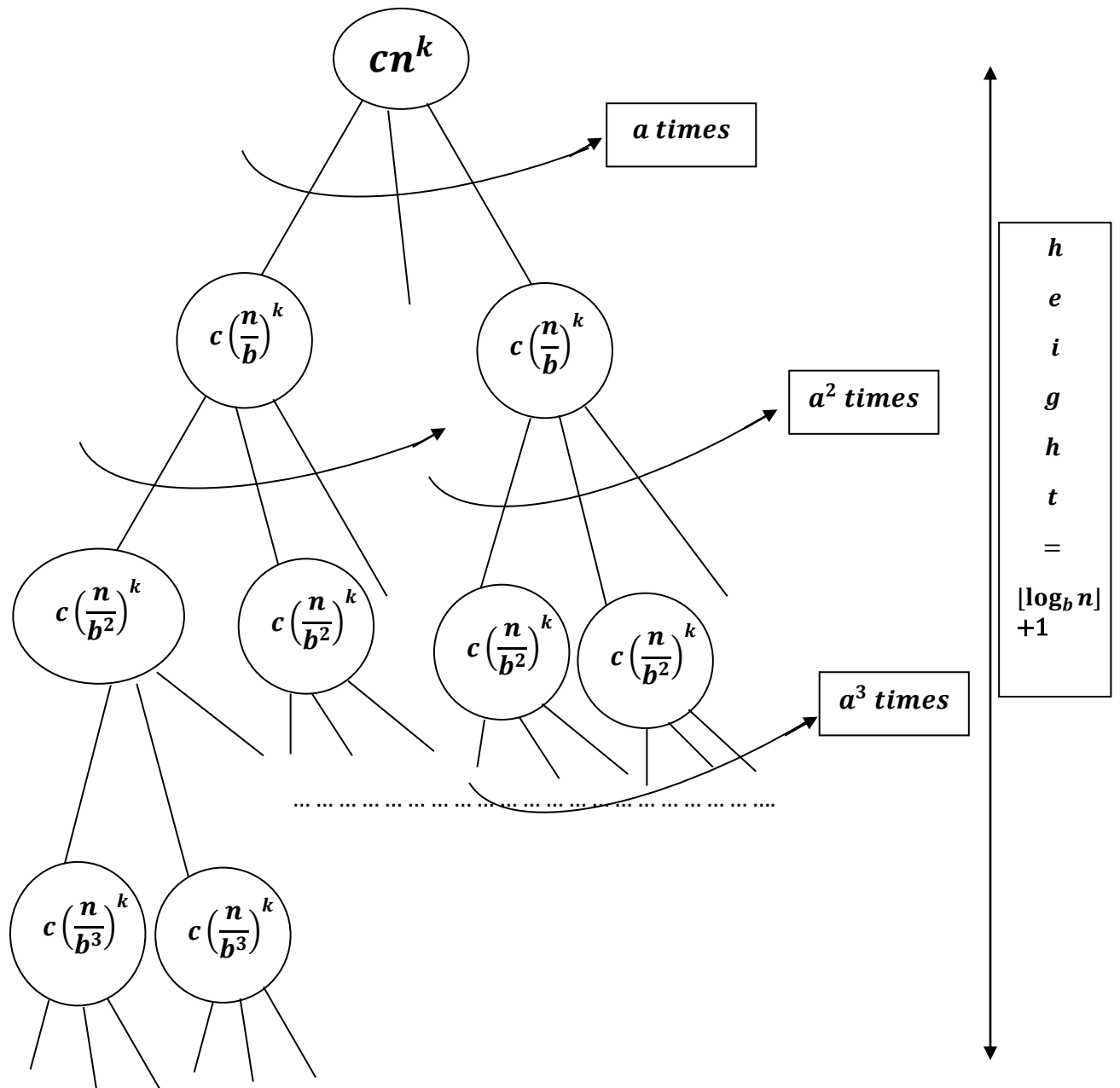
At preliminary stage , we are provided with the formula:

$$T(n) = aT\left(\frac{n}{b}\right) + cn^k.$$

This implies that the problem is divided into `a` subproblems at every level and every subproblems is of the size
$$\frac{cn^k}{b^k}.$$

The total cost of the recurrence tree is calculated by adding the costs of all the levels .

cn^k

$c\left(\dfrac{n}{b}\right)^k$

$c\left(\dfrac{n}{b}\right)^k$

a times

$c\left(\dfrac{n}{b^2}\right)^k$

$c\left(\dfrac{n}{b^2}\right)^k$

$c\left(\dfrac{n}{b^2}\right)^k$

$c\left(\dfrac{n}{b^2}\right)^k$

a^2 times

a^3 times

……………………………………………………………………

$c\left(\dfrac{n}{b^3}\right)^k$

$c\left(\dfrac{n}{b^3}\right)^k$

h
e
i
g
h
t
=
$\lfloor\log_b n\rfloor$
+1

Level	No. of problems	Problem Size	Work done = Problem Size × No. of Problems
0	1	cn^k	$1 \times cn^k = cn^k$
1	a	$\dfrac{cn^k}{b^k}$	$a \times \left(\dfrac{cn^k}{b^k}\right)$
2	a^2	$c\left(\dfrac{n}{b^2}\right)^k$	$a^2 \times c\left(\dfrac{n}{b^2}\right)^k$
. .	. .	. .	. .
$n-1$	a^k	$c\left(\dfrac{n}{b^k}\right)^k$	$a^k \times c\left(\dfrac{n}{b^k}\right)^k$
$k = \log_b n$	$a^{\log_b n}$	$\left(c\left(\dfrac{n}{b^k}\right)\right)^{\log_b n}$	$a^{\log_b n} \times \left(c\left(\dfrac{n}{b^k}\right)\right)^{\log_b n}$

Therefore , the total cost of the recurrence tree can be given as follows:

$$T(n) = cn^k + a \times c \left(\frac{n}{b}\right)^k + a^2 \times c \left(\frac{n}{b^2}\right)^k + \cdots$$

$$+ a^{\log_b n} \times \left(c \left(\frac{n}{b^k}\right) \right)^{\log_b n}$$

$$\Rightarrow cn^k \left(1 + a \times \left(\frac{1}{b^k}\right) + a^2 \times \frac{1}{(b^2)^k} + \cdots + a^{\log_b n} \times \left(\frac{1}{(b^k)^{\log_b n}} \right) \right)$$

$$\Rightarrow cn^k \left(1 + \left(\frac{a}{b^k}\right) + \frac{a^2}{(b^2)^k} + \cdots + \left(\frac{a^{\log_b n}}{(b^k)^{\log_b n}} \right) \right)$$

The same expression can be written as:

$$\Rightarrow cn^k \left(1 + \left(\frac{a}{b^k}\right) + \left(\frac{a}{b^k}\right)^2 + \cdots + \left(\frac{a}{b^k}\right)^{\log_b n} \right)$$

if $d = \left(\frac{a}{b^k}\right)$, *then we can rewrite the equation as*:

$$T(n) = cn^k\left(1 + (d) + (d)^2 + \cdots + (d)^{\log_b n}\right)$$

$$\textit{Case 1:} \rightarrow \frac{a}{b^k} < 1$$

When $d < 1$ *i.e.* $\frac{a}{b^k} < 1$ *hence,*

<u>To prove the geometric series is infinite</u> :

$$\lim_{n \to \infty}\left(1 + d + d^2 + d^3 + \cdots + d^{\log_b n}\right)$$

As this is a geometric series which yields to:

$$= \lim_{n \to \infty}\left(\frac{d^{n+1} - 1}{d - 1}\right)$$

$$= \frac{1}{d - 1} \times \lim_{n \to \infty} d^{n+1} - \lim_{n \to \infty} 1$$

$$= \frac{1}{d - 1} \times (\infty - 1)$$

$$= \frac{1}{d - 1} \times (\infty - 1)$$

$$= \frac{1}{d-1} \times \infty$$

$$= \infty$$

The series is infinite geometric series , as n goes infinite:

*The series can be expressed as a common ratio `d` and a is **1** , Hence by formula of infinite geometric series we we get*:

$$S = \frac{a}{1-r} = \frac{1}{1-d}.$$

<u>***To prove the series is convergent***</u>:

$$\lim_{n \to \infty} \left(\frac{1}{1-d} \right) = \frac{1}{1-0} = 1 , hence \ series \ is \ covergent \ i.e.$$

finite and unique.

*As the condition is d < **1** , the series becomes convergent i.e. finite and unique i.e.*:

$$1 + d + d^2 + \cdots = \frac{1}{1-d}$$

Even if the sequence goes infinite , one can observe that the

sequence reduces to constant factors only, when $d < 1$.

$$Therefore: cn^k \left[\frac{1}{1-d}\right] = \frac{cn^k}{1-d} = \Theta\left(\frac{cn^k}{1-d}\right)$$

$$= \left(\frac{1}{1-d}\right) \times \Theta(cn^k) = \Theta(cn^k) = \Theta(n^k).$$

$$Case\ 2: \rightarrow \frac{a}{b^k} = 1$$

When $d = 1$, this implies $a = b^k$. Therefore we will have:

$$T(n) = cn^k(1 + (d) + (d)^2 + \cdots + (d)^{\log_b n})$$

Substuting d with 1 we get:

$$T(n) = cn^k(1 + (1) + (1)^2 + \cdots + (1)^{\log_b n})$$

$$T(n) = cn^k(1 + (1 + 1 + \cdots + \log_b n\ times))$$

$$T(n) = cn^k(1 + 1 \times \log_b n)$$

$$T(n) = cn^k(1 + \log_b n)$$

$$T(n) = \left(cn^k + \ cn^k \times \log_b n \right)$$

Here dominant term is $cn^k \times \log_b n$, *hence result is*:

$\Theta\left(n^k \log_b n\right)$ *or* $\Theta\left(n^k \log n\right)$.

$$Case\ 3: \rightarrow \ \frac{a}{b^k} > 1$$

When $d \neq 1,$ *this implies that* $\dfrac{a}{b^k} > 1$, *that is* $a > b^k,$ *then*:

$$T(n) = \ cn^k(1 + \ (d) + (d)^2 + \cdots + (d)^{\log_b n})$$

We can write this as :

$$T(n) = \ cn^k \times d^{\log_b n} \left(\frac{1}{d^{\log_b n}} + \cdots + \frac{1}{d} + 1 \right)$$

$$\boxed{Constant}$$

i.e., $\left(\dfrac{1}{d^{\log_b n}} + \cdots + \dfrac{1}{d} + 1\right)$ *will lead to constant , as the series converges to a finite , as n tends to inifinity , each term in the summation tends to 0 when d > 1.*

If n tends to infinity , the terms , $\dfrac{1}{d^{\log_b n}}, \dfrac{1}{d^{\log_b n-1}}$ *, ... etc. will become very small .*

We can write it as:

$$\frac{1}{d^{\log_b n}} + \frac{1}{d^{\log_b n-1}} + \cdots + \frac{1}{d} + 1$$

$$= \left(\frac{1}{d}\right)^{\log_b n} + \left(\frac{1}{d}\right)^{\log_b n-1} + \cdots + \left(\frac{1}{d}\right)^{1} + 1$$

Now lets consider common ratio of the series r, which is equal to $\dfrac{1}{d}$.

Since 0 < d < 1 and n is infinite, we can apply the formula for the sum of an geometric series:

$$S = \frac{a}{1-r}$$

Substuting the values of $a = \left(\dfrac{1}{d}\right)^{\log_b n}$ *and* $r = \dfrac{1}{d}$ *into the formula for the sum of an infinite geometric series:*

$$S = \frac{\left(\dfrac{1}{d}\right)^{\log_b n}}{1 - d}$$

$$S = \frac{\left(\dfrac{1}{d}\right)^{\log_b n}}{1 - \dfrac{1}{d}}$$

$$S = \frac{\left(\dfrac{1}{d}\right)^{\log_b n}}{\dfrac{d - 1}{d}}$$

$$S = \left(\frac{1}{d}\right)^{\log_b n} \times \frac{d}{(d - 1)}$$

As n tends to inifinity : $\left(\dfrac{1}{d}\right)^{\log_b n}$ *approaches to zero, since any positive base raised to a negative exponent*

tends to zero. i.e. Suppose take an example:

$$\left(\frac{1}{10}\right)^{\log_b 100}, then \ \log_{10} 100 = 2$$

$$\left(\frac{1}{10}\right)^{\log_b 100} = \left(\frac{1}{10}\right)^2 = \frac{1}{100}, and \ the \ value \ is$$

less than one, hence exponent is negative i.e. 10^{-2}.

**** *Hence, the sum of series converges to a constant value.* ***

Note: The above expression can also be represented as:

$$S = \frac{1}{d^{\log_b n}} \times \frac{d}{(d-1)} = \frac{1}{n^{\log_b d}} \times \frac{d}{(d-1)} \ also \ as:$$

$$S = \frac{1}{n^{\log_b d} \times \frac{(d-1)}{d}}$$

Also we can write:

As n tends to infinity, term $\dfrac{1}{n^{\log_b d}}$ approaches to zero , since any positive base raised to a negative exponent tends to zero. Therefore, the sum of series converges to a constant value.

If we want an convergence test:

Convergence Test

$$\frac{1}{d^{\log_b n}} + \frac{1}{d^{\log_b n-1}} + \cdots + \frac{1}{d} + 1$$

Lets consider the series: $a_n = \dfrac{1}{d^{\log_b n}}$,to apply the limit comparison test, we need to find another series b_n , whose convergence behaviour is known.

Lets choose $b_n = \dfrac{1}{n^\epsilon}$,where ϵ is a positive constant. Now we will compare the series a_n and b_n, by taking the limit as n tends to infinity:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\dfrac{1}{d^{\log_b n}}\right)}{\dfrac{1}{n^\epsilon}}$$

To simplify this expression, lets rewrite $\dfrac{1}{d^{\log_b n}}$ *as* $\dfrac{1}{n^{\log_b d}}$

$$= \lim_{n \to \infty} \frac{\dfrac{1}{n^{\log_b d}}}{\dfrac{1}{n^{\in}}}$$

$$= \lim_{n \to \infty} \frac{n^{\in}}{n^{\log_b d}}$$

Since $b > 1$ *and* $d > 1$ *, we know that* $\log_b d > 0$ *.*
Therefore , as `n` tends to inifinity , $n^{\log_b d}$ *grows faster*
than $n^{\in}$ *for any positive* $\in$*.*

As a result the limit: $\lim_{n \to \infty} \dfrac{n^{\in}}{n^{\log_b d}}$ *is equal to* 0 *for any*
positive $\in$*. This implies that* a_n *and* b_n *have the same*
convergence behaviour .

Since the series: $\displaystyle\sum_{n=1}^{\infty} b_n$ *with* $b_n = \dfrac{1}{n^{\in}}$ *converges* $\in > 1,$

We can conclude that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ *also converges.*

Therefore the series $: \dfrac{1}{d^{\log_b n}} + \dfrac{1}{d^{\log_b n-1}} + \cdots + \dfrac{1}{d} + 1$

converges for $d > 1$ *and* $b > 1.$

Therefore keeping out the constant:

$\left(\dfrac{1}{d^{\log_b n}} + \cdots + \dfrac{1}{d} + 1 \right)$ *generated from*$: T(n) = cn^k \times d^{\log_b n},$

as it doesnot matter in time complexity, we continue:

$T(n) = cn^k \times d^{\log_b n}$

$or, T(n) = cn^k \times \left(\dfrac{a}{b^k} \right)^{\log_b n}$

$= cn^k \times \left(\dfrac{a^{\log_b n}}{b^{k \log_b n}} \right)$

$= cn^k \times \left(\dfrac{a^{\log_b n}}{n^k} \right) \left[b^{k \log_b n} = n^{k \log_b b} \right]$

$= cn^k \times \left(\dfrac{a^{\log_b n}}{n^k} \right) \left[b^{k \log_b n} = n^{k \log_b b} \right]$

$= c \times a^{\log_b n}$

$\in \Theta\left(n^{\log_b a}\right)$ [as $a^{\log_b n}$ can also be written as $n^{\log_b a}$].

Thus we get a simplified master theorem as follows:

Simplified Master Theorem:

Let the time complexity function $T(n)$ be a positive and eventually a non − decreasing function of the following form:

$$T(n) = aT\left(\frac{n}{b}\right) + cn^k$$

$$T(1) = d$$

Here a, d, b, k are all constants .

Here $b \geq 2, k \geq 0, a > 0, c > 0$ and $d \geq 0.$ The solution for the recurrence equation is given as follows:

Case 1: $T(n) \in \Theta\left(n^k\right)$ if $a < b^k$

Case 2: $T(n) \in \Theta\left(n^k \log n\right)$ if $a = b^k$

Case 3: $T(n) \in \Theta\left(n^{\log_b a}\right)$ if $a > b^k$

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