

## ***B. B. 1. Simplified Master Theorem***

***The master theorem is used for solving divide – and – conquer recurrence relations.***

***While the master theorem doesnot solve all types of divide – and – conquer recurrences , it can solve the majority of recurrence equation.***

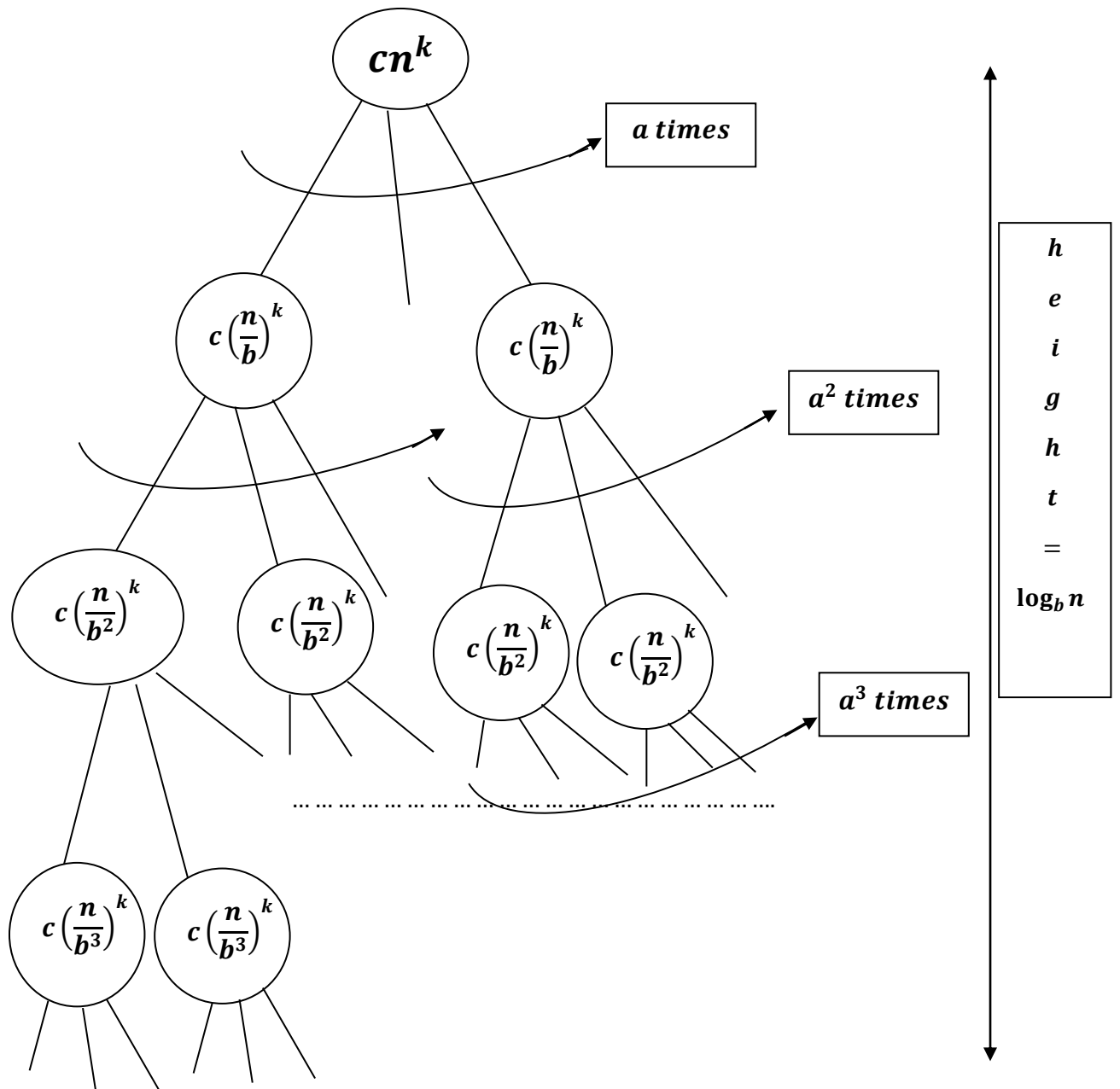
***At preliminary stage , we are provided with the formula:***

$$T(n) = aT\left(\frac{n}{b}\right) + cn^k.$$

***This implies that the problem is divided into `a` subproblems at every level and every subproblems is of the size***

$$\frac{cn^k}{b^k}.$$

***The total cost of the recurrence tree is calculated by adding the costs of all the levels .***



<i>Level</i>	<i>No. of problems</i>	<i>Problem Size</i>	<i>Work done = Problem Size × No. of Problems</i>
<b>0</b>	<b>1</b>	$cn^k$	$1 \times cn^k = cn^k$
<b>1</b>	$a$	$\frac{cn^k}{b^k}$	$a \times \left(\frac{cn^k}{b^k}\right)$
<b>2</b>	$a^2$	$c\left(\frac{n}{b^2}\right)^k$	$a^2 \times c\left(\frac{n}{b^2}\right)^k$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n - 1$	$a^k$	$c\left(\frac{n}{b^k}\right)^k$	$a^k \times c\left(\frac{n}{b^k}\right)^k$
$k = \log_b n$	$a^{\log_b n}$	$\left(c\left(\frac{n}{b^k}\right)\right)^{\log_b n}$	$a^{\log_b n} \times \left(c\left(\frac{n}{b^k}\right)\right)^{\log_b n}$

***Therefore , the total cost of the recurrence tree can be given as follows:***

$$T(n) = cn^k + a \times c \left(\frac{n}{b}\right)^k + a^2 \times c \left(\frac{n}{b^2}\right)^k + \dots$$

$$+ a^{\log_b n} \times \left(c \left(\frac{n}{b^k}\right)\right)^{\log_b n}$$

$$\Rightarrow cn^k \left( 1 + a \times \left(\frac{1}{b^k}\right) + a^2 \times \frac{1}{(b^2)^k} + \dots + a^{\log_b n} \times \left(\frac{1}{(b^k)^{\log_b n}}\right) \right)$$

$$\Rightarrow cn^k \left( 1 + \left(\frac{a}{b^k}\right) + \frac{a^2}{(b^2)^k} + \dots + \left(\frac{a^{\log_b n}}{(b^k)^{\log_b n}}\right) \right)$$

***The same expression can be written as:***

$$\Rightarrow cn^k \left( 1 + \left(\frac{a}{b^k}\right) + \left(\frac{a}{b^k}\right)^2 + \dots + \left(\frac{a}{b^k}\right)^{\log_b n} \right)$$

***if  $d = \left(\frac{a}{b^k}\right)$ , then we can rewrite the equation as:***

$$T(n) = cn^k(1 + (d) + (d)^2 + \dots + (d)^{\log_b n})$$

$$\text{Case 1: } \rightarrow \frac{a}{b^k} < 1$$

When  $d < 1$  i.e.  $\frac{a}{b^k} < 1$  hence,

To prove the geometric series is infinite :

$$\lim_{n \rightarrow \infty} (1 + d + d^2 + d^3 + \dots + d^{\log_b n})$$

As this is a geometric series which yields to:

$$= \lim_{n \rightarrow \infty} \left( \frac{d^{n+1} - 1}{d - 1} \right)$$

$$= \frac{1}{d - 1} \times \lim_{n \rightarrow \infty} d^{n+1} - \lim_{n \rightarrow \infty} 1$$

$$= \frac{1}{d - 1} \times (\infty - 1)$$

$$= \frac{1}{d - 1} \times (\infty - 1)$$

$$= \frac{1}{d-1} \times \infty$$

$$= \infty$$

*The series is infinite geometric series , as n goes infinite:*

*The series can be expressed as a common ratio `d` and a is 1 , Hence by formula of infinite geometric series we we get:*

$$S = \frac{a}{1-r} = \frac{1}{1-d}.$$

*To prove the series is convergent:*

$\lim_{n \rightarrow \infty} \left( \frac{1}{1-d} \right) = \frac{1}{1-0} = 1$  , hence series is convergent i. e. finite and unique.

*As the condition is  $d < 1$  , the series becomes convergent i. e. finite and unique i. e.:*

$$1 + d + d^2 + \dots = \frac{1}{1-d}$$

*Even if the sequence goes infinite , one can observe that the*

*sequence reduces to constant factors only, when  $d < 1$ .*

$$\begin{aligned} \text{Therefore : } cn^k \left[ \frac{1}{1-d} \right] &= \frac{cn^k}{1-d} = \Theta \left( \frac{cn^k}{1-d} \right) \\ &= \left( \frac{1}{1-d} \right) \times \Theta(cn^k) = \Theta(cn^k) = \Theta(n^k). \end{aligned}$$

$$\text{Case 2: } \rightarrow \frac{a}{b^k} = 1$$

*When  $d = 1$ , this implies  $a = b^k$ . Therefore we will have:*

$$T(n) = cn^k(1 + (d) + (d)^2 + \dots + (d)^{\log_b n})$$

*Substuting  $d$  with 1 we get:*

$$T(n) = cn^k(1 + (1) + (1)^2 + \dots + (1)^{\log_b n})$$

$$T(n) = cn^k(1 + (1 + 1 + \dots + \log_b n \text{ times}))$$

$$T(n) = cn^k(1 + 1 \times \log_b n)$$

$$T(n) = cn^k(1 + \log_b n)$$

$$T(n) = (cn^k + cn^k \times \log_b n)$$

*Here dominant term is  $cn^k \times \log_b n$ , hence result is:  
 $\Theta(n^k \log_b n)$  or  $\Theta(n^k \log n)$ .*

$$\text{Case 3: } \rightarrow \frac{a}{b^k} > 1$$

*When  $d \neq 1$ , this implies that  $\frac{a}{b^k} > 1$ , that is  $a > b^k$ , then:*

$$T(n) = cn^k(1 + (d) + (d)^2 + \dots + (d)^{\log_b n})$$

*We can write this as :*

$$T(n) = cn^k \times d^{\log_b n} \left( \underbrace{\frac{1}{d^{\log_b n}} + \dots + \frac{1}{d} + 1}_{\text{Constant}} \right)$$



*i. e.,  $\left(\frac{1}{d^{\log_b n}} + \dots + \frac{1}{d} + 1\right)$  will lead to constant, as the series converges to a finite, as  $n$  tends to infinity, each term in the summation tends to 0 when  $d > 1$ .*

*If  $n$  tends to infinity, the terms  $\frac{1}{d^{\log_b n}}, \frac{1}{d^{\log_b n-1}}, \dots$  etc. will become very small.*

*We can write it as:*

$$\begin{aligned} & \frac{1}{d^{\log_b n}} + \frac{1}{d^{\log_b n-1}} + \dots + \frac{1}{d} + 1 \\ &= \left(\frac{1}{d}\right)^{\log_b n} + \left(\frac{1}{d}\right)^{\log_b n-1} + \dots + \left(\frac{1}{d}\right)^1 + 1 \end{aligned}$$

*Now let's consider common ratio of the series  $r$ , which is equal to  $\frac{1}{d}$ .*

*Since  $0 < d < 1$  and  $n$  is infinite, we can apply the formula for the sum of a geometric series:*

$$S = \frac{a}{1-r}$$

*Substuting the values of  $a = \left(\frac{1}{d}\right)^{\log_b n}$  and  $r = \frac{1}{d}$  into the formula for the sum of an infinite geometric series:*

$$S = \frac{\left(\frac{1}{d}\right)^{\log_b n}}{1 - d}$$

$$S = \frac{\left(\frac{1}{d}\right)^{\log_b n}}{1 - \frac{1}{d}}$$

$$S = \frac{\left(\frac{1}{d}\right)^{\log_b n}}{\frac{d - 1}{d}}$$

$$S = \left(\frac{1}{d}\right)^{\log_b n} \times \frac{d}{(d - 1)}$$

*As  $n$  tends to infinity :  $\left(\frac{1}{d}\right)^{\log_b n}$  approaches to zero, since any positive base raised to a negative exponent*

*tends to zero. i. e. Suppose take an example:*

$$\left(\frac{1}{10}\right)^{\log_b 100}, \text{ then } \log_{10} 100 = 2$$

$$\left(\frac{1}{10}\right)^{\log_b 100} = \left(\frac{1}{10}\right)^2 = \frac{1}{100}, \text{ and the value is}$$

*less than one, hence exponent is negative i. e.  $10^{-2}$ .*

*\*\*\*\* Hence, the sum of series converges to a constant value. \*\*\**

*Note: The above expression can also be represented as:*

$$S = \frac{1}{d^{\log_b n}} \times \frac{d}{(d-1)} = \frac{1}{n^{\log_b d}} \times \frac{d}{(d-1)} \text{ also as:}$$

$$S = \frac{1}{n^{\log_b d} \times \frac{(d-1)}{d}}$$

*Also we can write:*

*As  $n$  tends to infinity, term  $\frac{1}{n^{\log_b d}}$  approaches to zero , since any positive base raised to a negative exponent tends to zero. Therefore, the sum of series converges to a constant value.*

*If we want an convergence test:*

## ***Convergence Test***

$$\frac{1}{d^{\log_b n}} + \frac{1}{d^{\log_b n-1}} + \dots + \frac{1}{d} + 1$$

*Lets consider the series:  $a_n = \frac{1}{d^{\log_b n}}$  , to apply the limit comparison test, we need to find another series  $b_n$  , whose convergence behaviour is known.*

*Lets choose  $b_n = \frac{1}{n^\epsilon}$  , where  $\epsilon$  is a positive constant.*

*Now we will compare the series  $a_n$  and  $b_n$ , by taking the limit as  $n$  tends to infinity:*

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\left( \frac{1}{d^{\log_b n}} \right)}{\frac{1}{n^\epsilon}}$$

*To simplify this expression, let's rewrite  $\frac{1}{d^{\log_b n}}$  as  $\frac{1}{n^{\log_b d}}$*

$$= \lim_{n \rightarrow \infty} \frac{1}{\frac{n^{\log_b d}}{1}} = \lim_{n \rightarrow \infty} \frac{1}{n^{\log_b d}}$$

$$= \lim_{n \rightarrow \infty} \frac{n^{\epsilon}}{n^{\log_b d}}$$

*Since  $b > 1$  and  $d > 1$ , we know that  $\log_b d > 0$ .*

*Therefore, as  $n$  tends to infinity,  $n^{\log_b d}$  grows faster than  $n^{\epsilon}$  for any positive  $\epsilon$ .*

*As a result the limit:  $\lim_{n \rightarrow \infty} \frac{n^{\epsilon}}{n^{\log_b d}}$  is equal to 0 for any positive  $\epsilon$ . This implies that  $a_n$  and  $b_n$  have the same convergence behaviour.*

*Since the series:  $\sum_{n=1}^{\infty} b_n$  with  $b_n = \frac{1}{n^{\epsilon}}$  converges  $\epsilon > 1$ ,*

*We can conclude that the series  $\sum_{n=1}^{\infty} a_n$  also converges.*

*Therefore the series :  $\frac{1}{d^{\log_b n}} + \frac{1}{d^{\log_b n-1}} + \dots + \frac{1}{d} + 1$  converges for  $d > 1$  and  $b > 1$ .*

*Therefore keeping out the constant:*

$$\left( \frac{1}{d^{\log_b n}} + \dots + \frac{1}{d} + 1 \right) \text{ generated from: } T(n) = cn^k \times d^{\log_b n},$$

*as it doesnot matter in time complexity, we continue:*

$$T(n) = cn^k \times d^{\log_b n}$$

$$\text{or, } T(n) = cn^k \times \left( \frac{a}{b^k} \right)^{\log_b n}$$

$$= cn^k \times \left( \frac{a^{\log_b n}}{b^{k \log_b n}} \right)$$

$$= cn^k \times \left( \frac{a^{\log_b n}}{n^k} \right) [b^{k \log_b n} = n^{k \log_b b}]$$

$$= cn^k \times \left( \frac{a^{\log_b n}}{n^k} \right) [b^{k \log_b n} = n^{k \log_b b}]$$

$$= c \times a^{\log_b n}$$

$\in \Theta(n^{\log_b a})$  [*as  $a^{\log_b n}$  can also be written as  $n^{\log_b a}$* ].

*Thus we get a simplified master theorem as follows:*

## ***Simplified Master Theorem:***

***Let the time complexity function  $T(n)$  be a positive and eventually a non – decreasing function of the following form:***

$$T(n) = aT\left(\frac{n}{b}\right) + cn^k$$

$$T(1) = d$$

***Here  $a, d, b, k$  are all constants .***

***Here  $b \geq 2, k \geq 0, a > 0, c > 0$  and  $d \geq 0$ . The solution for the recurrence equation is given as follows:***

***Case 1:  $T(n) \in \Theta(n^k)$  if  $a < b^k$***

***Case 2:  $T(n) \in \Theta(n^k \log n)$  if  $a = b^k$***

***Case 3:  $T(n) \in \Theta(n^{\log_b a})$  if  $a > b^k$***

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