Polynomial Reduction
−Solving Homogeneous Equations

To solve a recurrence equation using the polynomial reduction method, the first step is to formulate the characteristic equation:

$$T(n) = a_1 T(n-1) + a_2 T(n-2)$$

Let us re − arrange the equation as follows:

$$T(n) - a_1 T(n-1) - a_2 T(n-2) = 0$$

Let $T(n) = x^n$

$$\therefore\ T(n-1) = x^{n-1} \text{ and } T(n-2) = x^{n-2}$$

This gives the following equation:

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} = 0$$

As $x \neq 0$, divide the equation to get the following realtion:

$$\Rightarrow \frac{x^n}{x^{n-2}} + a_1 \times \frac{x^{n-1}}{x^{n-2}} + a_2 \times \frac{x^{n-2}}{x^{n-2}} = 0$$

$$\Rightarrow x^{n-(n-2)} + a_1 \times x^{n-1-(n-2)} + a_2 \times x^{n-2-(n-2)} = 0$$

$$\Rightarrow x^2 + a_1 \times x^1 + a_2 \times x^0 = 0$$

$$\Rightarrow x^2 + a_1 x + a_2 = 0$$

Therefore it can be observed that the recurrence equation is transformed into a characteristic equation, which is as follows:

$$T(n) - a_1 T(n-1) - a_2 T(n-2) = 0$$

$$\Longrightarrow T(n-0) - a_1 T(n-1) - a_2 T(n-2) = 0$$

$$\Longrightarrow x^2 + a_1 x^1 + a_2 x^0 = 0$$

$$\Longrightarrow x^2 + a_1 x + a_2 = 0$$

The order of the recurrence equation is 2; therefore, the characteristic equation is also of order 2.

$\boldsymbol{One\ can\ generalize\ this\ in\ the\ form\ a\ theorem\ \ for\ the\ conversion\ of\ a\ recurrence\ equation\ into\ the\ characteristic\ equation.}$

$\underline{\boldsymbol{Theorem}}$

$\boldsymbol{Theorem}$: $\boldsymbol{For\ a\ recurrence\ equation,}$
$\boldsymbol{a_0 t_n + a_1 t_{n-1} + \cdots + a_k t_{n-k} = 0}$, $\boldsymbol{having\ an\ order\ n,}$
$\boldsymbol{the\ characteristic\ equation\ will\ be\ of\ the\ following\ form}$:

$$\boldsymbol{a_0 r^k + a_1 r^{k-1} + \cdots + a_k r^0 = 0}$$

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