Chapter 2. Ring Theory

Hu Zheng
Department of Mathematics, Zhejiang University

April 22, 2015

1 Basic Notions

Problem 1.1 (Kaplansky)**.** *If some element in a unital ring R has more than one right inverse, then it has infinitely many inverses.*

Solution: Maybe we need first to find some examples to get a good handle of it...

- Suppose $R = M_n(k)$ for some field k.

Note that if $rq = 1$, then $r = rqr, r(1 - qr) = 0$.

Suppose $rq_1 = 1, rq_2 = 1$, then $r(q_1 - q_2) = 0$, and $r(q_1 + (q_1 - q_2)u) = 0$ for any u. We only need to find infinite us such that $(q_1 - q_2)u$ are all distinctive.

◀

Problem 1.2. *(1) Suppose L is a noncommutative field, and a is out of the center of L, then L is generated by all the congruent elements of a.*

(2) Suppose L is a field and K its proper subfield, and $K^* = K - \{0\}$ is a normal subgroup of L^*, then K is contained in the center of L.

Solution: ◀

2 Homeomorphism Theorems for Rings

3 Applications of Homomorphisms

4 Various Kinds of Integral Domains

Problem 4.1. *Is $x+1$ a unit in $\mathbb{Z}[x]$ and $\mathbb{Z}[[x]]$? Is x^2+3x+2 irreducible in $\mathbb{Z}[x]$ and $\mathbb{Z}[[x]]$.*

Solution: $x+1$ is not a unit in $\mathbb{Z}[x]$. But it's a unit in $\mathbb{Z}[[x]]$, and its inverse is $1-x+x^2-x^3+\cdots$.

x^2+3x+2 is not a unit in $\mathbb{Z}[x]$. But it is a unit in $\mathbb{Z}[[x]]$ as it equals $(x+1)(x+2)$, and its inverse is $(1-x+x^2-\cdots)(1-2x+4x^2-\cdots)$. ◀

Problem 4.2. *Suppose $f(x)$ is an irreducible polynomial of odd degree in $\mathbb{Q}$. α and β are different roots of $f(x)$ in some extension of $\mathbb{Q}$. Prove that $\alpha+\beta \notin \mathbb{Q}$.*

Solution: First let's ask why odd degree? Because in even degrees α and β can be congruent. ◀