MECH201 ENGINEERING ANALYSIS

Lecture Notes
(Week 6)

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MATLAB example scripts

All lecture scripts uploaded weekly to:
https://github.com/AzdiarGazder/MECH201

Please follow the instructions posted there to download and run your local copy of the scripts.

Equation solving

Week 2 - Solving a **single variable nonlinear equation.**

$$5x^2 - x^3 + 7x^6 = 0$$

Week 3 - Solving **multiple variable linear algebraic equations.**

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$
$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$
$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

Week 5 - **Curve fitting and interpolation.**

Solving differential equations

Can we solve these differential equations analytically?

$$\frac{dy}{dx} = 5x + 3 \quad \xrightarrow{\text{Yes}} \quad y = \int (5x+3)dx = \frac{5}{2}x^2 + 3x + C$$

$$\frac{dy}{dx} = (1+x)^x \quad \xrightarrow{\text{NO}} \quad y = \int \left((1+x)^x\right)dx$$

No analytical solution. Numerical methods required to calculate the integral.

NO

$$\frac{d^2y}{dx^2} + y\frac{dy}{dx} = x + 2$$

No analytical solution. Numerical methods required to solve the 2nd order differential equation.

This week's topic: Numerical Integration.

Other applications requiring numerical integration

Average of a function f(x) over an interval

from x = a to x = b is:

$$\text{average} = \frac{\int_a^b f(x)dx}{b-a}$$

Average of a function f(x) over an area

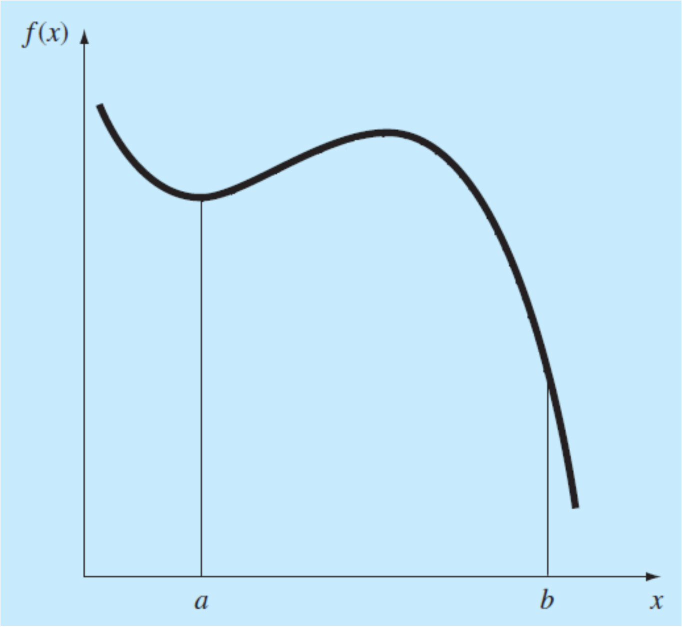
from x = [a, b], and y = [c, d] is

$$\textbf{average} \quad = \frac{\int_c^d \int_a^b f(x,y)dxdy}{(b-a)(d-c)}$$

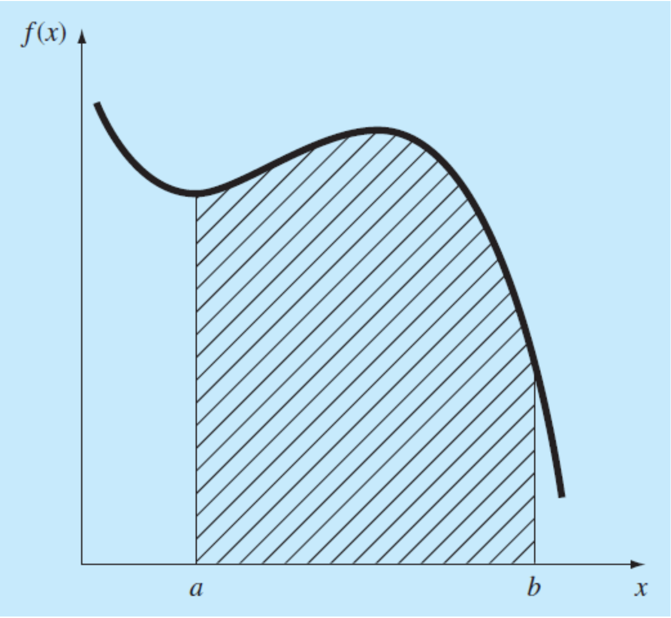
Numerical Integration

Integral of a function f(x)

Integration means **to bring together** parts to form a whole.



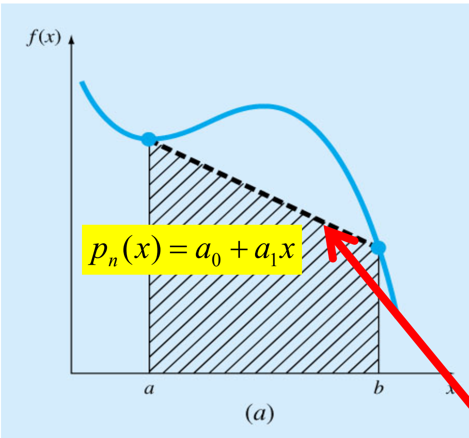
$$I = \int_a^b f(x)dx$$

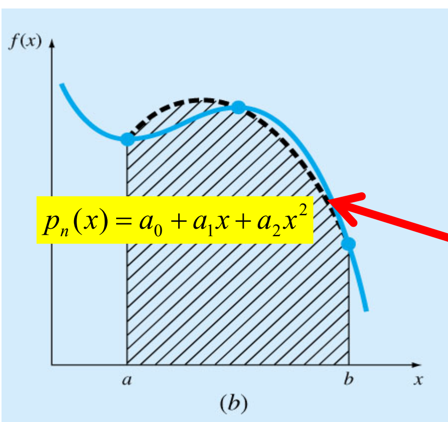


Integral of function f(x) between x = a and x = b
is the area under the curve of f(x)
above the x-axis between x = a and x = b.

L06E01_trapezoidalRule.m

Numerical Integration methods
Newton-Cotes formulae





- Newton-Cotes formulas are the most commonly used numerical integration methods.
- They are based on **replacing complicated functions** or tabular data **by approximating functions**.

$$I = \int_a^b f(x)dx \cong \int_a^b p_n(x)dx$$

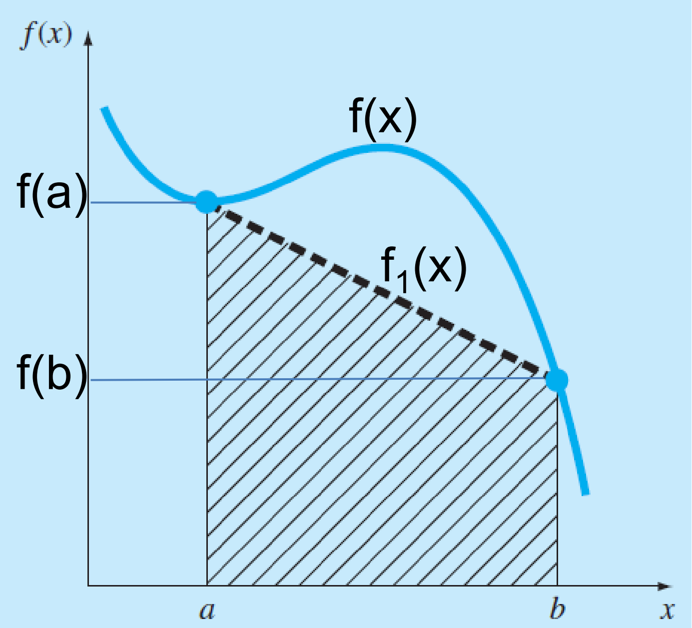
$$p_n(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n$$

Integrals – Trapezoidal rule
Approach 1

Principle: Use a straight line (i.e. – a **linear function $f_1(x)$**) to approximate the curve f(x) between the limits a and b.

At the limits, the straight line passes through the points:

$[x_1 = a, y_1 = f(a)]$ and $[x_2 = b, y_2 = f(b)]$.



The area below the straight line is a trapezoid.

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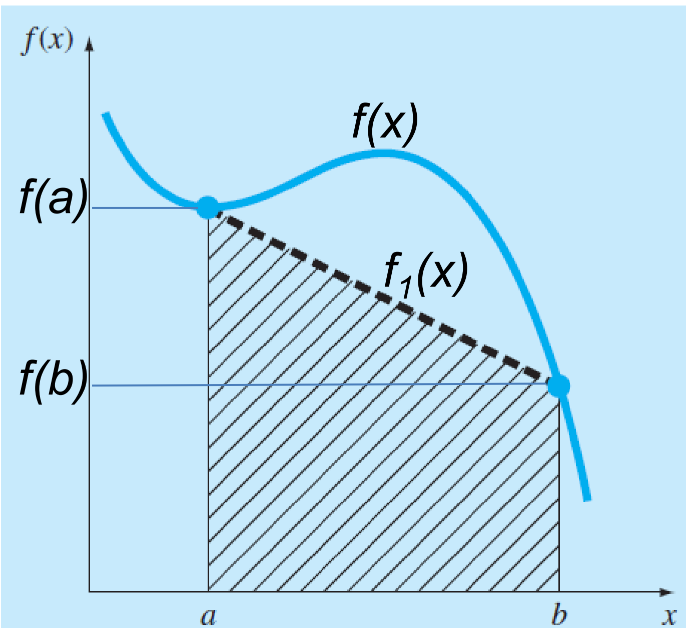
Integrals – Trapezoidal rule
Approach 1



Linear function $f_1(x)$:

$$f_1(x) = Ax + B$$

Linear function passes through [a, f(a)] and [b, f(b)]. So:

$$f_1(x = a) = A \times a + B = f(a)$$

$$f_1(x = b) = A \times b + B = f(b)$$

$$A = \frac{f(b) - f(a)}{b - a}$$

$$B = \frac{bf(a) - af(b)}{b - a}$$

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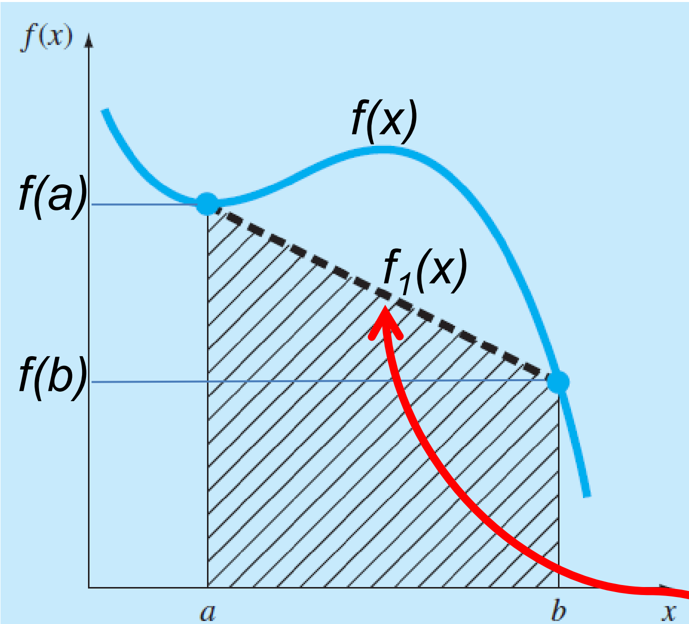
Integrals – Trapezoidal rule



Use linear function $f_1(x)$ to approx. f(x):

$$f(x) \cong f_1(x)$$

$$I = \int_a^b f(x)dx \cong \int_a^b f_1(x)dx$$

$$I \cong \int_a^b (Ax + B)dx$$

$$I \cong A\frac{x^2}{2}\bigg|_a^b + Bx\Big|_a^b$$

$$I \cong A\frac{b^2 - a^2}{2} + B(b - a)$$

Replacing A & B into the equation

$$A = \frac{f(b) - f(a)}{b - a}$$

$$B = \frac{bf(a) - af(b)}{b - a}$$

$$I = \int_a^b f(x)dx \cong (b - a)\left(\frac{f(a) + f(b)}{2}\right)$$

L06E01_trapezoidalRule.m

Example – Trapezoidal rule

Use the trapezoidal rule to numerically integrate the following function from **a = 0** to **b = 0.8**:

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

Solution:

$$a = 0 \rightarrow f(x = a) = 0.2$$

$$b = 0.8 \rightarrow f(x = b) = 0.232$$

$$I = \int_a^b f(x)dx \cong (b - a)\left(\frac{f(a) + f(b)}{2}\right)$$

$$\Rightarrow \quad I \cong (0.8 - 0)\left(\frac{0.2 + 0.232}{2}\right) = 0.1728$$

L06E01_trapezoidalRule.m

Relative error – Trapezoidal rule

True solution by the analytical method:

$$I = \int_a^b f(x)dx$$

$$= \int_0^{0.8} \left(0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5\right)dx$$

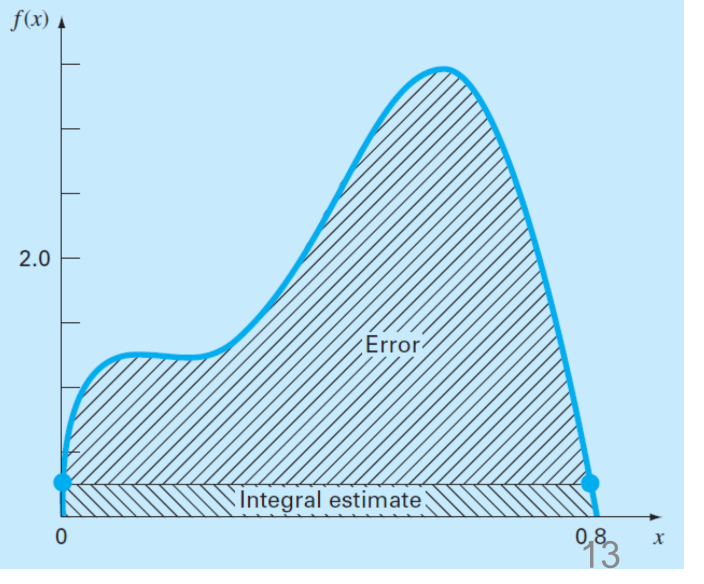
$$= \left(0.2x + 25\frac{x^2}{2} - 200\frac{x^3}{3} + 675\frac{x^4}{4} - 900\frac{x^5}{5} + 400\frac{x^6}{6}\right)_0^{0.8}$$

$$= 1.6405$$

Relative error:

$$|\varepsilon_t| = \left|\frac{1.6405 - 0.1728}{1.6405}\right| \times 100\% = 89.5\%$$

How do we improve accuracy?



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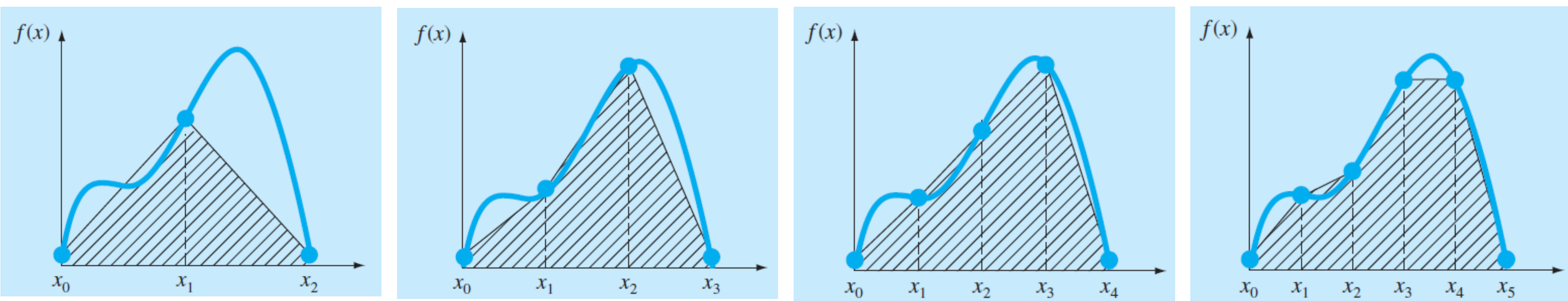
Multiple-application trapezoidal rule
Approach 2

To improve the accuracy of the trapezoidal rule:

Sub-divide the integration interval from a to b into a number of smaller segments and apply the method to each segment.



Increasing the segments reduces the error

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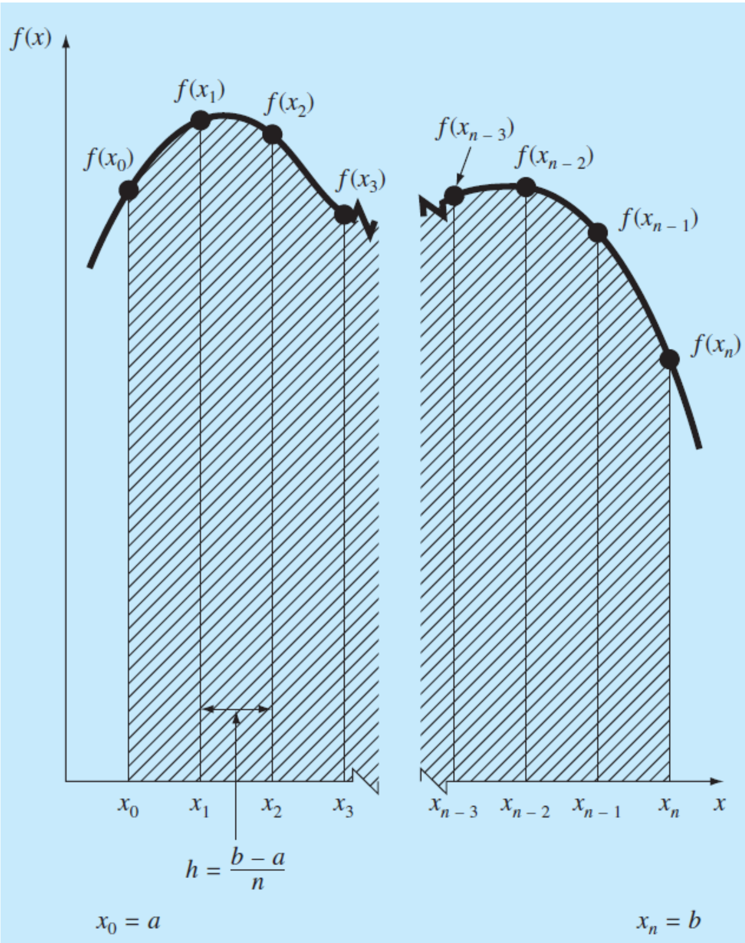
Multiple-application trapezoidal rule



1. There are n+1 equally spaced base points (x_0, x_1, x_2, …, x_n).
2. x_0 is a and x_n is b.
3. There are n segments of equal width.
4. Width of the segment (h) is

$$h = \frac{b-a}{n}$$

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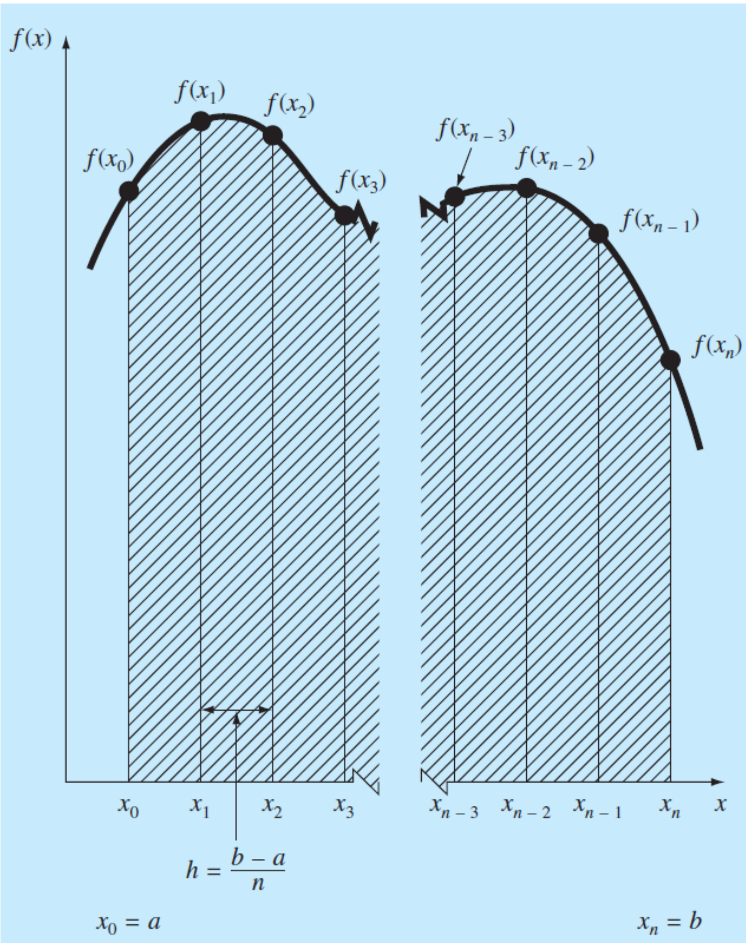
Multiple-application trapezoidal rule



Integration from a to b is the sum of integrations of n segments:

$$I = \int_a^b f(x)dx$$

$$= \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx$$

Applying the trapezoidal rule to each segment:

$$\int_{x_0}^{x_1} f(x)dx \cong (x_1 - x_0)\left(\frac{f(x_0) + f(x_1)}{2}\right) = h\left(\frac{f(x_0) + f(x_1)}{2}\right)$$

$$\int_{x_1}^{x_2} f(x)dx \cong (x_2 - x_1)\left(\frac{f(x_1) + f(x_2)}{2}\right) = h\left(\frac{f(x_1) + f(x_2)}{2}\right)$$

$$\vdots$$

$$\int_{x_{n-1}}^{x_n} f(x)dx \cong (x_n - x_{n-1})\left(\frac{f(x_{n-1}) + f(x_n)}{2}\right) = h\left(\frac{f(x_{n-1}) + f(x_n)}{2}\right)$$

L06E02_trapezoidalRule_multipleAppIN.m

Multiple-application trapezoidal rule

$$I = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx$$

$$I \cong h\left(\frac{f(x_0) + f(x_1)}{2}\right) + h\left(\frac{f(x_1) + f(x_2)}{2}\right) + \cdots + h\left(\frac{f(x_{n-1}) + f(x_n)}{2}\right)$$

$$\Rightarrow \quad \boxed{I \cong \frac{h}{2}\sum_{i=1}^{n}\left(f(x_{i-1}) + f(x_i)\right)}$$

or $$\boxed{I \cong \frac{h}{2}\left(f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right)}$$

L06E02_trapezoidalRule_multipleAppIN.m

Example: Multiple-application trapezoidal rule

Use the 2, 4,…,10 segments trapezoidal rule to numerically integrate the following function from

a = 0 to b = 0.8:

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

Solution:

For n = 2 segments:

$$h = (b - a)/n = (0.8 - 0)/2 = 0.4$$

$$x_0 = 0,\ f(x_0) = 0.2;\ x_1 = 0.4,\ f(x_1) = 2.456;$$

$$x_2 = 0.8,\ f(x_2) = 0.232$$

$$\Rightarrow \quad I \cong \frac{h}{2}\sum_{i=1}^{2}\left(f(x_{i-1}) + f(x_i)\right) = \frac{0.4}{2}\left(\left(f(x_0) + f(x_1)\right) + \left(f(x_1) + f(x_2)\right)\right)$$

$$= 1.0688$$

L06E02_trapezoidalRule_multipleAppIN.m

For n = 4 segments:

$$h = (b - a)/n = (0.8 - 0)/4 = 0.2$$

$$x_0 = 0,\ f(x_0) = 0.2;\ x_1 = 0.2,\ f(x_1) = 1.288;$$
$$x_2 = 0.4,\ f(x_2) = 2.456;\ x_3 = 0.6,\ f(x_3) = 3.464;$$
$$x_4 = 0.8,\ f(x_4) = 0.232$$

$$\Rightarrow \quad I \cong \frac{h}{2}\sum_{i=1}^{4}\left(f(x_{i-1}) + f(x_i)\right)$$
$$= \frac{0.2}{2}\left(\left(f(x_0) + f(x_1)\right) + \left(f(x_1) + f(x_2)\right) + \left(f(x_2) + f(x_3)\right) + \left(f(x_3) + f(x_4)\right)\right)$$
$$= 1.4848$$

L06E02_trapezoidalRule_multipleAppIN.m

Relative error - Multiple-application trapezoidal rule

True analytical solution = 1.6405

For n = 2 segments:

$$|\varepsilon_t| = \left|\frac{1.6405 - 1.0688}{1.6405}\right| \times 100\% = 34.9\%$$

For n = 4 segments:

$$|\varepsilon_t| = \left|\frac{1.6405 - 1.4848}{1.6405}\right| \times 100\% = 9.5\%$$

Many smaller segments = MORE ACCURATE RESULT!!

L06E02_trapezoidalRule_multipleAppIN.m

Results for different numbers (n) of segments

n	h	I	ε_t (%)
2	0.4	1.0688	34.9
3	0.2667	1.3695	16.5
4	0.2	1.4848	9.5
5	0.16	1.5399	6.1
6	0.1333	1.5703	4.3
7	0.1143	1.5887	3.2
8	0.1	1.6008	2.4
9	0.0889	1.6091	1.9
10	0.08	1.6150	1.6

L06E02_trapezoidalRule_multipleAppIN.m **

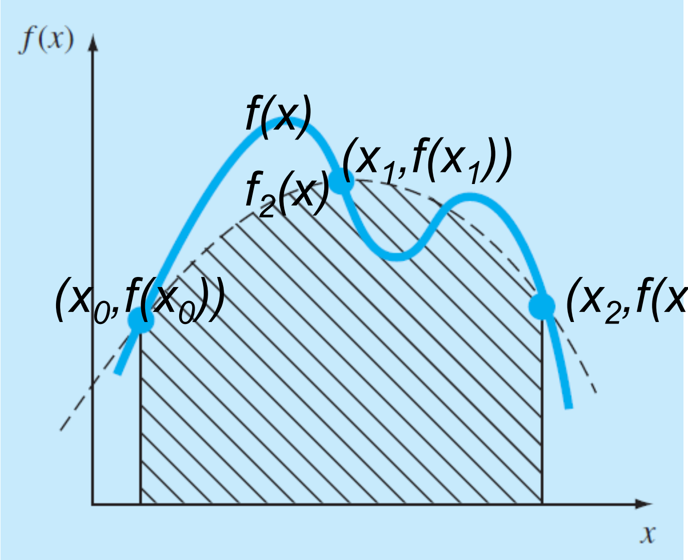
L06E03_trapezoidalRule_unequalWidths.m **

Integrals – Simpson's 1/3 rule
Approach 1

Principle:

Use a **second-order interpolating Lagrange polynomial** ($f_2(x)$)

$$f_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_0 - x_0)(x_0 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_0 - x_0)(x_0 - x_1)} f(x_2)$$



to approximate the curve (f(x)) between the limits a and b.

$f_2(x)$ passes through **three equally spaced points:**

$(x_0,f(x_0))$, $(x_1,f(x_1))$ and $(x_2, f(x_2))$. $x_1 - x_0 = x_2 - x_1 = h$.

L06E04_simpson13Rule.m

Integrals – Simpson's 1/3 rule

$$I = \int_a^b f(x)dx \cong \int_a^b f_2(x)dx$$

$$a = x_0 \quad b = x_2$$

$$I = \int_{x_0}^{x_2} \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \right] dx$$

$$\mathrm{I} \cong \frac{\mathrm{h}}{3}\left(\mathrm{f}(\mathrm{x}_0) + 4\mathrm{f}(\mathrm{x}_1) + \mathrm{f}(\mathrm{x}_2)\right)$$

where $\mathrm{h} = \dfrac{\mathrm{b}-\mathrm{a}}{2}$

L06E04_simpson13Rule.m

Example – Simpson's 1/3 rule

Use Simpson's 1/3 rule to numerically integrate the following function from a = 0 to b = 0.8.

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

Solution:

$$x_0 = 0,\ f(x_0) = 0.2;\ x_1 = 0.4,\ f(x_1) = 2.456;$$
$$x_2 = 0.8,\ f(x_2) = 0.232$$

$$h = \frac{b-a}{2} = 0.4$$

$$I \cong \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right) = \frac{0.4}{3}\left(0.2 + 4 \times 2.456 + 0.232\right)$$
$$= 1.3675$$

Relative error compared to analytical solution:

$$\left|\varepsilon_t\right| = \left|\frac{1.6405 - 1.3675}{1.6405}\right| \times 100\% = 16.6\%$$

How do we improve accuracy?

L06E04_simpson13Rule.m **

Integrals – Simpson's 1/3 Composite Rule Approach 2

- Sub-divide the integration interval from x_0 to x_n into several segments (n) of equal width:

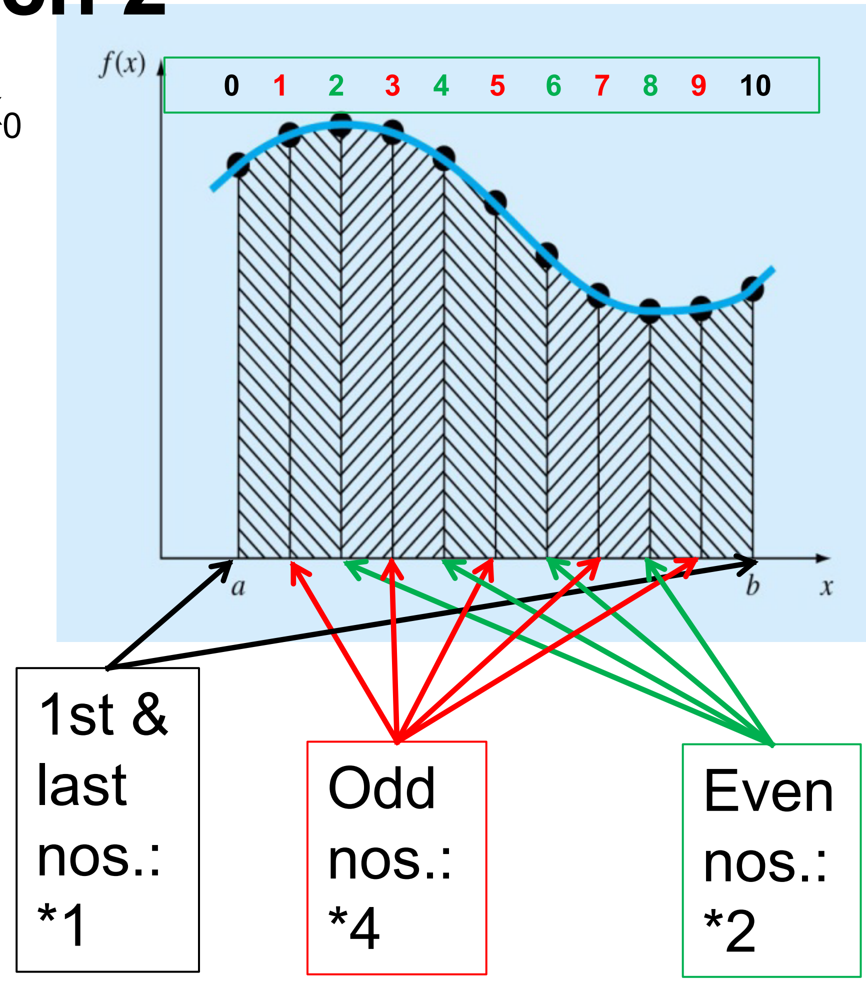
- Apply Simpson's 1/3 rule to **every two segments**:

$$I \approx \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)dx$$

$$= h\frac{f(x_0)+4f(x_1)+f(x_2)}{3} + h\frac{f(x_2)+4f(x_3)+f(x_4)}{3}$$

$$+\cdots+h\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{3}$$

$$I \approx \frac{h}{3}\left\{f(x_0)+4\sum_{i=1,3,5}^{n-1} f(x_i)+2\sum_{j=2,4,6}^{n-2} f(x_j)+f(x_n)\right\}$$



Note: The total number of segments for Simpson's 1/3 Composite Rule are EVEN

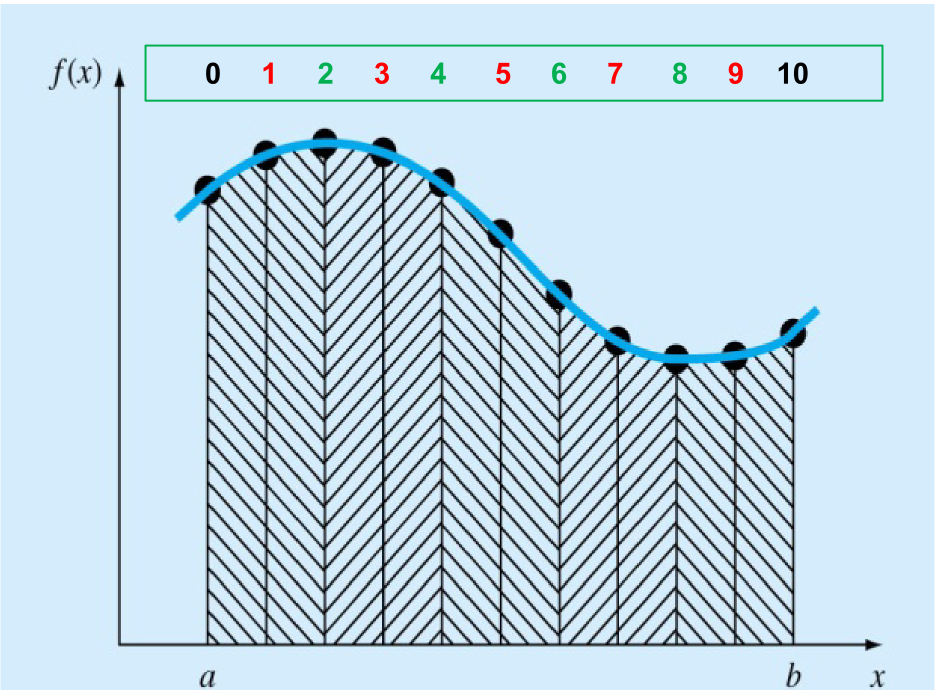
1+ 5 + 4 + 1 = 10

L06E05_simpson13CompositeRule_multipleAppIN.m

Integrals – Simpson's 1/3 Composite Rule

HOW to determine the weights for Simpson's 1/3 Composite Rule

$$I \approx \frac{h}{3}\left\{ f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n) \right\}$$



1 4 1 1 4 1 1 4 1

\+ 1 4 1 1 4 1

\- -

Weights = 1 4 2 4 2 4 2 4 2 4 1

L06E05_simpson13CompositeRule_multipleAppIN.m

Example – Simpson's 1/3 Composite Rule

Use Simpson's 1/3 Composite Rule with **n = 4** to numerically integrate the following function from a = 0 to b = 0.8.

(*exact value = 1.640533*)

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

Solution:

$$n = 4; \quad h = \frac{0.8-0}{4} = 0.2; \;\; x = 0; \quad 0.2; \quad 0.4; \quad 0.6; \quad 0.8$$

$$f(0) = 0.2 \quad f(0.2) = 1.288 \quad f(0.4) = 2.456 \quad f(0.6) = 3.464 \quad f(0.8) = 0.232$$

First Simpson's 1/3 ***Second Simpson's 1/3***

	1	4	1		
			1	4	1
Weights					
	1	4	2	4	1

$$I \cong \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right) + \frac{h}{3}\left(f(x_2) + 4f(x_3) + f(x_4)\right)$$

$$= \frac{0.2}{3}(0.2 + 4 \times 1.288 + 2.456) + \frac{0.2}{3}(2.456 + 4 \times 3.464 + 0.232)$$

$$= 1.6235$$

Error wrt true solution = $\varepsilon_t = \dfrac{1.640533 - 1.623467}{1.640533} 100\% = 1.04\%$

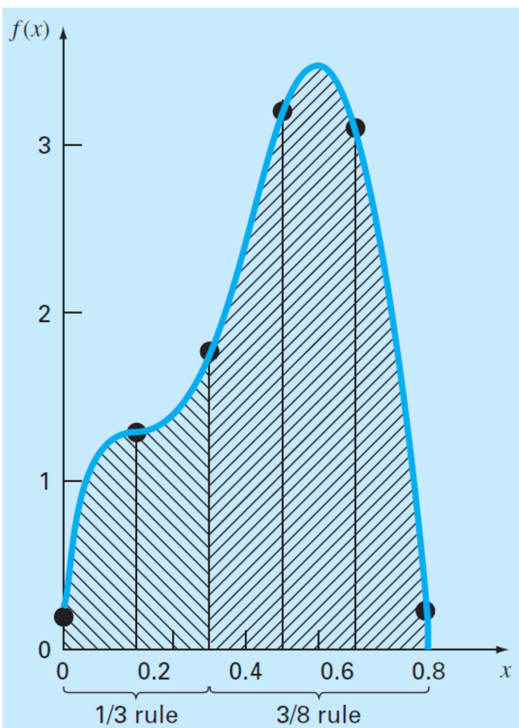
L06E05_simpson13CompositeRule_multipleAppIN.m **

Integrals – Simpson's 3/8 Rule
Approach 3

Principle: Use a **third-order polynomial curve** ($f_3(x)$) to approximate the curve f(x) between the limits a and b.

$f_3(x)$ passes through four **equally-spaced points** $(x_0,f(x_0))$, $(x_1,f(x_1))$, $(x_2, f(x_2))$ and $(x_3, f(x_3))$ such that

$x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = h$.



$$I = \int_a^b f(x)dx \cong \int_a^b f_3(x)dx$$

$$I \cong \frac{3h}{8}\left(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right)$$

L06E06_simpson38Rule.m **

Example – Simpson's 3/8 Composite Rule

Use Simpson's 3/8 Composite Rule with **n = 3** to numerically integrate the following function from a = 0 to b = 0.8.

(*exact value = 1.640533*)

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

Solution: n = 3 $\quad h = (b-a)/n = (0.8-0)/3 = 0.2667$

$$x = 0; \quad 0.2667; \quad 0.5333; \quad 0.8$$

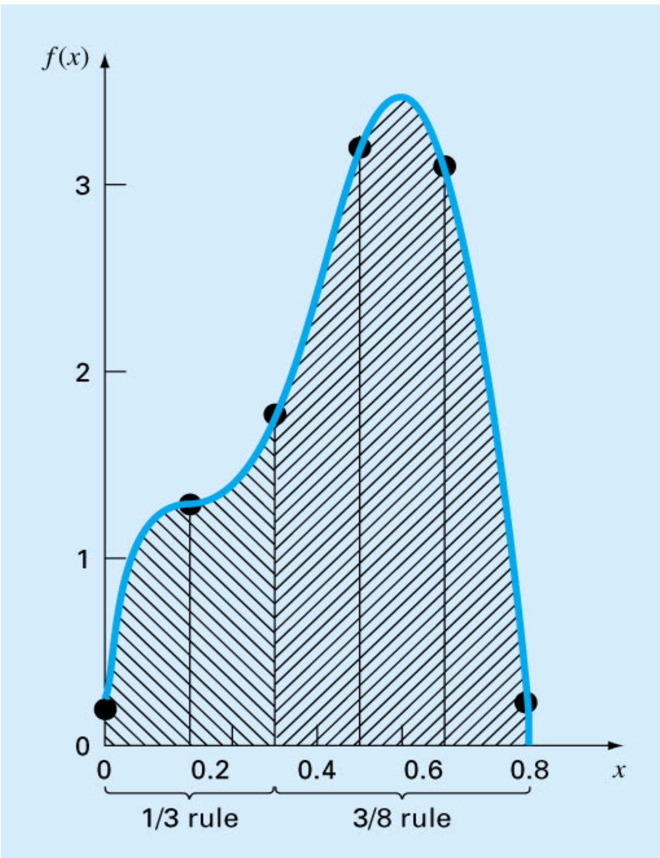
$$f(x_0) = 0.2; \quad f(x_1) = 1.4327; \quad f(x_2) = 3.4872; \quad f(x_3) = 0.232$$

$$I \cong \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right] = \frac{0.8}{8}\left(0.2 + 3\times 1.4327 + 3\times 3.4872 + 0.232\right) = 1.5192$$

Error wrt **true solution** = $\left|\varepsilon_t\right| = \left|\frac{1.6405 - 1.5192}{1.6405}\right| \times 100\% = 7.4\%$

L06E07_simpson38CompositeRule_multipleAppIN.m **

Integrals – Which Simpson's Composite Rule should be used?



- **Even** number of intervals (i.e., n = 4, 6,…)

  Use Simpson's 1/3 Composite Rule

- **Odd** number of intervals (i.e., n = 5, 7,…)

  Use Simpson's 3/8 Composite Rule
  for **max even** numbered intervals
  +
  Use Simpson's 3/8 rule
  for the **last three intervals**

MIXED RULE

For example, if the total intervals is 9.
Strategy = USE THE MIXED RULE
For intervals 1 to 6, use Simpson's 1/3 Composite Rule.
For intervals 7 to 9, use Simpson's 3/8 Composite Rule.

L06E08_compareError_13_38_CompositeRules.m **

Example – Simpson's Mixed Composite Rule Approach 4

Use Simpson's 3/8 Mixed Composite Rule with **n = 5** to numerically integrate the following function from a = 0 to b = 0.8.

(*exact value = 1.640533*)

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

Solution: n = 5

$$h = (b - a)/n = (0.8 - 0)/5 = 0.16$$

$$x_0 = 0;\ x_1 = 0.16;\ x_2 = 0.32;\ x_3 = 0.48;\ x_4 = 0.64;\ x_5 = 0.8,$$

$$\underbrace{f(x_0) = 0.2;\ f(x_1) = 1.2969;\ f(x_2) = 1.7434;}_{\text{Simpson's 1/3}}\ \underbrace{f(x_3) = 3.1860;\ f(x_4) = 3.1819;\ f(x_5) = 0.232}_{\text{Simpson's 3/8}}$$

$$I_1 \cong \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] = \frac{0.16}{3}\left(0.2 + 4 \times 1.2969 + 1.7434\right) = 0.3803$$

$$I_2 \cong \frac{3h}{8}\left[f(x_2) + 3f(x_3) + 3f(x_4) + f(x_5)\right] = \frac{0.48}{8}\left(1.7434 + 3 \times 3.1860 + 3 \times 3.1819 + 0.232\right) = 1.2648$$

$$I = I_1 + I_2 = 0.3803 + 1.2648 = 1.6451$$

$$|\varepsilon_t| = \left|\frac{1.6405 - 1.6451}{1.6405}\right| \times 100\% = 0.28\%$$

L06E09_simpson1338MixedRule.m **

Summary of Integrals (Trapezoidal and Simpson)

Trapezoidal rule (1 interval, 2 points)

$$I = (b-a)\frac{f(a)+f(b)}{2}$$

Composite Trapezoidal rule (any interval n >= 2)

$$I \cong \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) \right]$$

Simpson 1/3 rule (2 intervals, 3 points)

$$I \cong \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2) \right]$$

Composite Simpson 1/3 rule (even intervals)

$$I \approx \frac{h}{3}\left\{ f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n) \right\}$$

Simpson 3/8 rule (3 intervals, 4 points)

$$I \cong \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) \right]$$

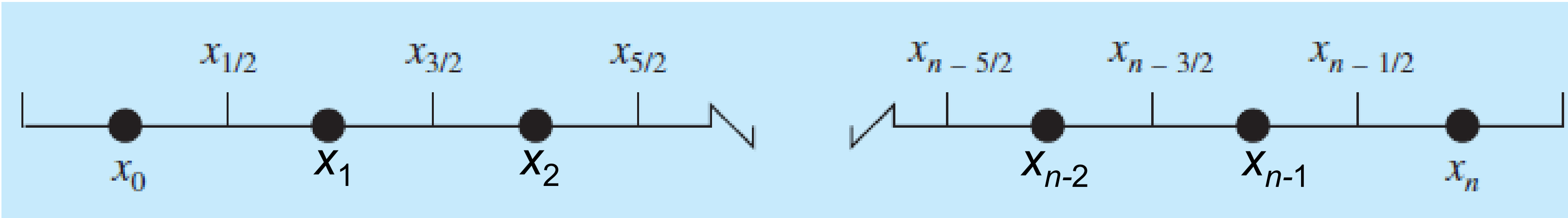
Composite Simpson 3/8 rule (multiples of 3 interval)

$$I \approx \frac{3h}{8} \sum_{i=0,3,6,\cdots}^{n-3} \left[f(x_i) + 3f(x_{i+1}) + 3f(x_{i+2}) + f(x_{i+3}) \right]$$

Mixed Simpson (any interval)

$$I \approx \frac{h}{3}\left\{ f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n) \right\} + I \cong \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) \right]$$

Integrals - Midpoint rule



$$\int_{x_0}^{x_n} f(x)dx = h\big(f(x_{1/2}) + f(x_{3/2}) + f(x_{5/2}) + \cdots + f(x_{n-3/2}) + f(x_{x-1/2})\big)$$

1. x_0 = a is the lower limit and x_n = b is the upper limit.
2. Equally spaced points (x_0, x_1, x_2, …, x_n). h = segment width.
3. Use the midpoints ($x_{1/2}$, $x_{3/2}$, $x_{5/2}$, …, $x_{n-3/2}$ and $x_{n-1/2}$).
4. Integral for each segment = the function of the midpoint multiplied by the segment width.
5. This method is widely used for improper integrals.

L06E10_midpointRule.m **

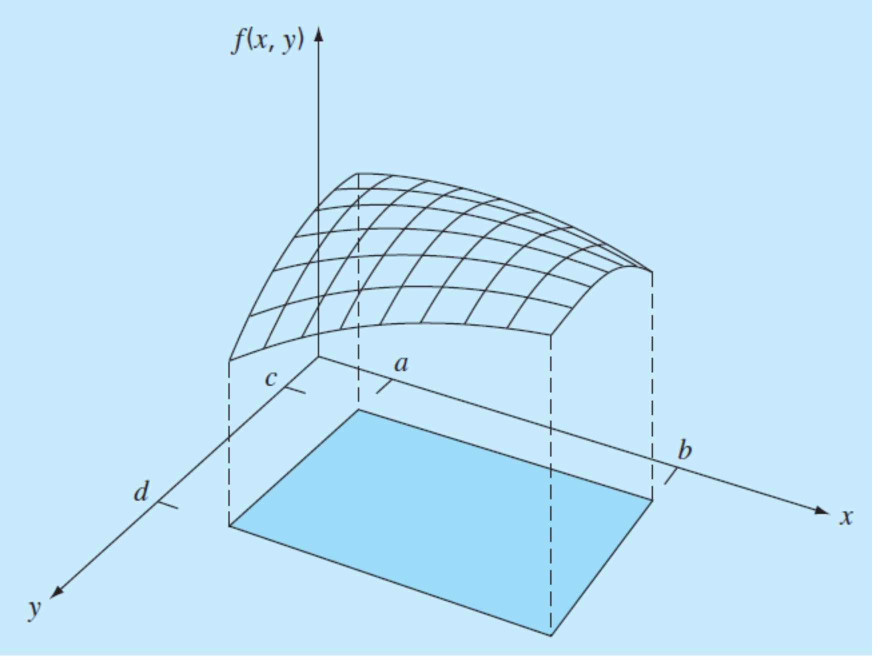
Double integrals

f(x,y) is a 2-D function representing a surface.

A double integral of f(x,y) over a rectangular area (a ≤ x ≤ b) and (c ≤ y ≤ d) is represented as:

$$\int_c^d \int_a^b f(x,y)\,dx\,dy$$

…and is the volume below the surface of f(x,y).



Numerical solution:

Step 1: Integrate along the 1st dimension (say, x) by trapezoidal or Simpson's rules.

Step 2: The result of the first integration is now integrated along the 2nd dimension (now y).

L06E11_trapezoidalRule_doubleIntegral.m

Example – Double integrals

If the temperature of a 8 m x 6 m rectangular heated plate is described by the following function:

$$T(x, y) = 2xy + 2x - x^2 - 2y^2 + 72$$

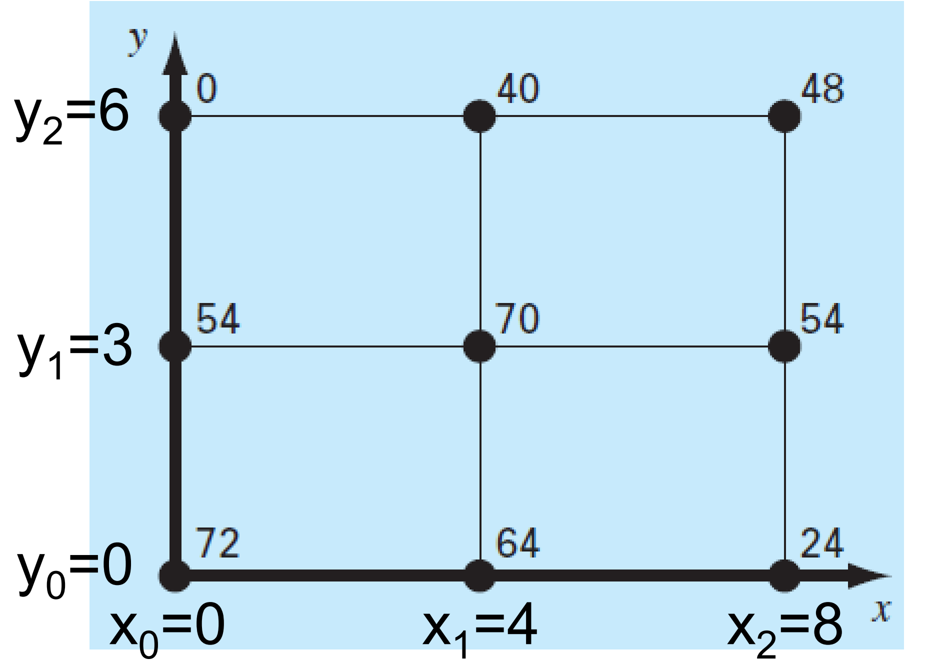
What is the average temperature?

Solution:

Use 2 segment trapezoidal rule along each direction.

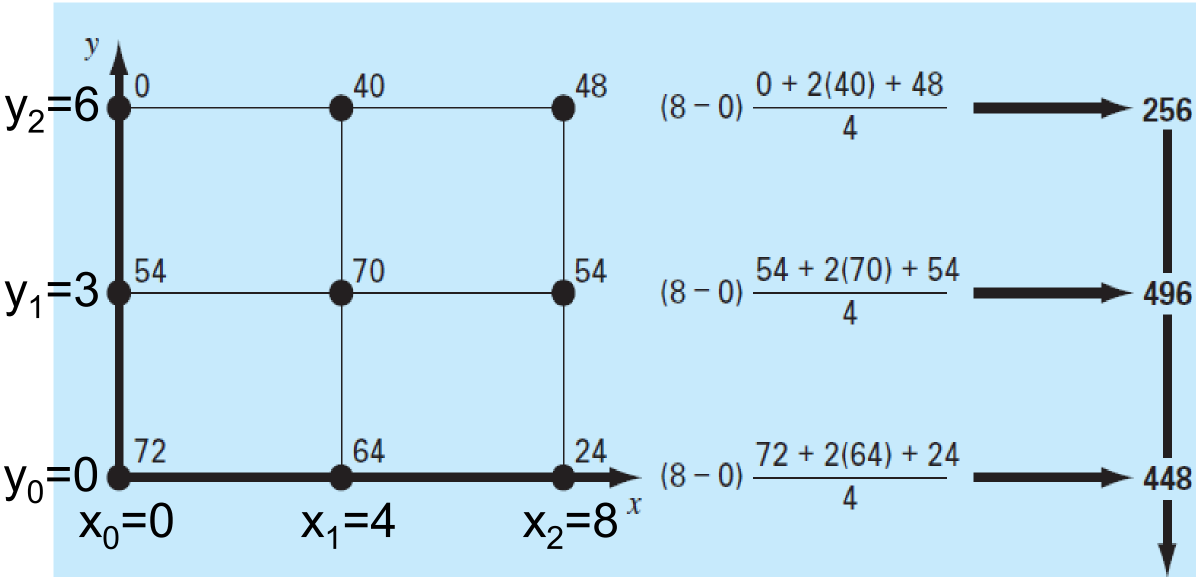
2 segments along the x axis = 3 points x_0, x_1 and x_2.

2 segments along the y axis = 3 points y_0, y_1 and y_2.



L06E11_trapezoidalRule_doubleIntegral.m

Step 1:
Applying the 2 segment trapezoidal rule along x for each y value



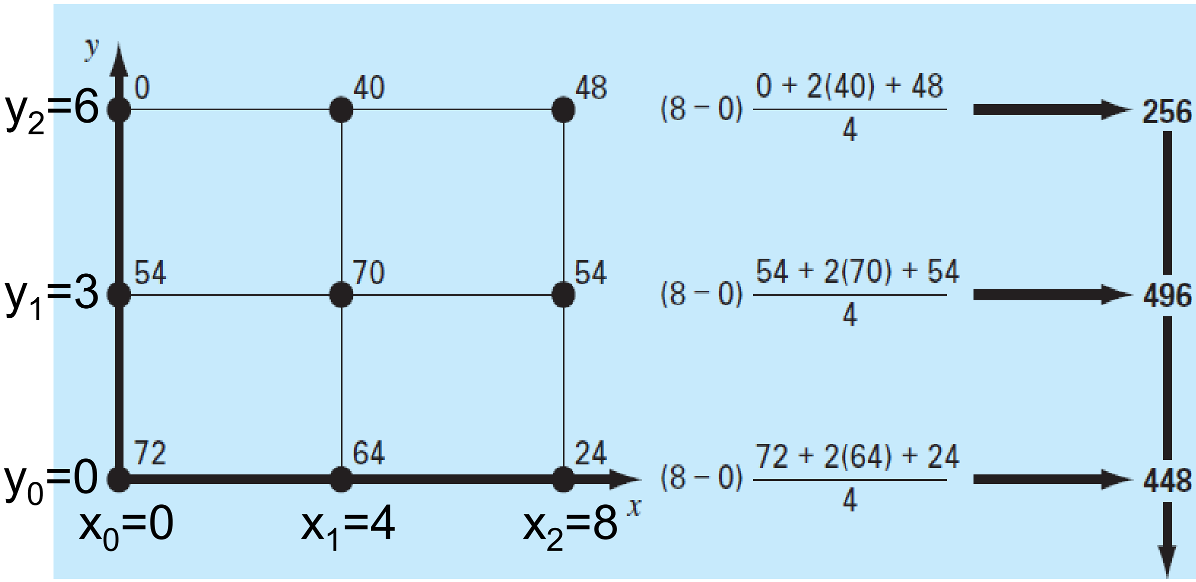
$$I_2 = \frac{(x_2 - x_0)}{4}\left(f(x_0, y_2) + 2f(x_1, y_2) + f(x_2, y_2)\right)$$

$$I_1 = \frac{(x_2 - x_0)}{4}\left(f(x_0, y_1) + 2f(x_1, y_1) + f(x_2, y_1)\right)$$

$$I_0 = \frac{(x_2 - x_0)}{4}\left(f(x_0, y_0) + 2f(x_1, y_0) + f(x_2, y_0)\right)$$

L06E11_trapezoidalRule_doubleIntegral.m

Step 2:
Applying the 2 segment trapezoidal rule along y



$$I_2 = \frac{(x_2 - x_0)}{4}\left(f(x_0, y_2) + 2f(x_1, y_2) + f(x_2, y_2)\right)$$

$$I_1 = \frac{(x_2 - x_0)}{4}\left(f(x_0, y_1) + 2f(x_1, y_1) + f(x_2, y_1)\right)$$

$$I_0 = \frac{(x_2 - x_0)}{4}\left(f(x_0, y_0) + 2f(x_1, y_0) + f(x_2, y_0)\right)$$

$$I = \frac{(y_2 - y_0)}{4}(I_0 + 2I_1 + I_2)$$

$$\text{average temperature} = \frac{\int_c^d \int_a^b T(x,y)\,dx\,dy}{(b-a)(d-c)}$$

$$= \frac{I}{(b-a)(d-c)} = 56$$

L06E11_trapezoidalRule_doubleIntegral.m **

Sequence of double integrals

Is the calculation sequence swappable?

x-direction first followed by y-direction

OR

y-direction first followed by x-direction

Answer: YES!

$$I=\int_a^b\left\{\int_c^d f(x,y)\,dy\right\}dx=\int_c^d\left\{\int_a^b f(x,y)\,dx\right\}dy$$

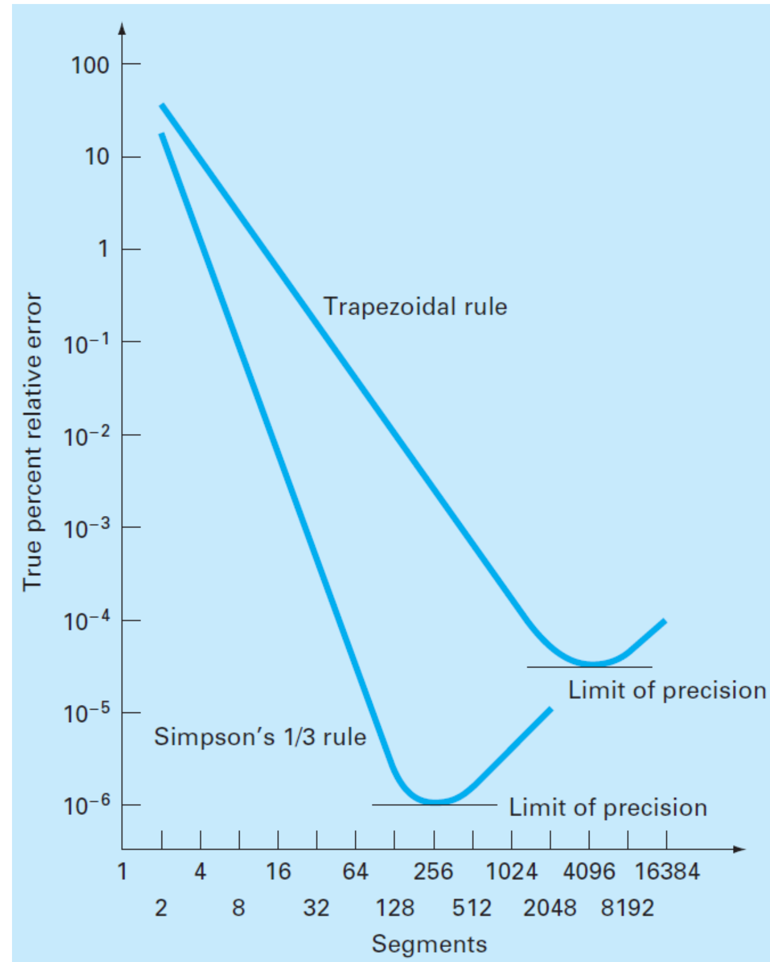
Techniques to improve accuracy

- High order approximations
  - Multi-application Trapezoidal Rule
  - Simpson's 1/3 composite Rule
  - Simpson's 3/8 composite Rule
  - Simpson's Mixed Rule
  - **Midpoint Rule**

- Multiple application
  (i.e. – using more segments).

- **Error correction techniques**
  - **Richardson's extrapolation**
  - **High order Romberg integration**

- **Gauss quadrature**



L06E12_midpointRule_doubleIntegral.m **

Richardson's Extrapolation

The exact value (I) of the integral can be expressed by:

$$I = I(h) + E(h)$$

where,

I(h) = the approximation from an *n*-segment application of trapezoidal rule with step size h =(b - a) / n

E(h) = truncation error which can be expressed as

$$E(h) \cong -\frac{b-a}{12} h^2 \bar{f}''$$

L06E13_trapezoidalRule_richardsonExtrapolation.m

Richardson's Extrapolation

If we make two separate estimates using step sizes (segment widths) of h_1 and h_2:

$$I = I(h_1) + E(h_1) = I(h_2) + E(h_2)$$

$$E(h_1) \cong -\frac{b-a}{12} h_1^2 \bar{f}''$$

$$E(h_2) \cong -\frac{b-a}{12} h_2^2 \bar{f}''$$

If we assume $\bar{f}''$ is constant regardless of step size, then:

$$\frac{E(h_1)}{E(h_2)} \cong \frac{h_1^2}{h_2^2} \Rightarrow E(h_2) = \frac{I(h_1) - I(h_2)}{1 - (h_1/h_2)^2}$$

$$\Downarrow$$

$$\boxed{I = I(h_2) + \frac{1}{(h_1/h_2)^2 - 1}\left(I(h_2) - I(h_1)\right)}$$

L06E13_trapezoidalRule_richardsonExtrapolation.m

Richardson's Extrapolation

If $h_2 = h_1/2$ then:

$$I = I(h_2) + \frac{1}{(2)^2 - 1}\left(I(h_2) - I(h_1)\right)$$

$$\Downarrow$$

$$I = \frac{4}{3} I(h_2) - \frac{1}{3} I(h_1)$$

I is the improved estimate of the integral.
It is better than both $I(h_1)$ and $I(h_2)$.

L06E13_trapezoidalRule_richardsonExtrapolation.m

Example – Richardson's Extrapolation

The single and multiple applications of the trapezoidal rule were used to estimate the integral of the following function from a = 0 to b = 0.8:

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

The results are:

Segments	h	Integral	ε_t, %
1	0.8	0.1728	89.5
2	0.4	1.0688	34.9
4	0.2	1.4848	9.5

Use Richardson's extrapolation method to improve the estimates of the integral.

L06E13_trapezoidalRule_richardsonExtrapolation.m

Segments	h	Integral	ε_t, %
1	0.8	0.1728	89.5
2	0.4	1.0688	34.9
4	0.2	1.4848	9.5

Combining the estimates for one (h_1 = 0.8) & two segments (h_2 = 0.4):

$$h_2 = h_1/2 \quad \Rightarrow \quad I = \frac{4}{3} I(h_2) - \frac{1}{3} I(h_1) = \frac{4}{3} \times 1.0688 - \frac{1}{3} \times 0.1728 = 1.367467$$

$$|\varepsilon_t| = 16.6\%$$

Combining the estimates for two (h_1 = 0.4) & four segments (h_2 = 0.2):

$$h_2 = h_1/2 \quad \Rightarrow \quad I = \frac{4}{3} I(h_2) - \frac{1}{3} I(h_1) = \frac{4}{3} \times 1.4848 - \frac{1}{3} \times 1.0688 = 1.623467$$

$$|\varepsilon_t| = 1.0\%$$

L06E13_trapezoidalRule_richardsonExtrapolation.m **

High order Romberg integration

- Apply **Richardson's Extrapolation** method twice for the two groups of step sizes:
  - For example, use $h_1 = 0.8$ and $h_2 = 0.4$ for the first time

  and

  - Then use $h_1 = 0.4$ and $h_2 = 0.2$ for the second time

- Using the previous example, we have 2 estimates of the integral:
  - **I_l = 1.367467** is the less accurate estimate from $h_1 = 0.8$ and $h_2 = 0.4$
  - and
  - **I_m = 1.623467** is the more accurate estimate from $h_1 = 0.4$ and $h_2 = 0.2$

- Now apply the error correction method again.

$$I = \frac{16}{15} I_m - \frac{1}{15} I_l$$

L06E13_trapezoidalRule_richardsonExtrapolation.m **
L06E14_trapezoidalRule_rombergQuadrature.m

General form of Romberg integration

$$I_{j,k} \cong \frac{4^{k-1} I_{j+1,k-1} - I_{j,k-1}}{4^{k-1} - 1}$$

k = level of integration

k = 1 returns estimates from the Trapezoidal rule

k = 2 returns estimates from Richardson's extrapolation

j = used to distinguish between more (j+1) and less (j) accurate estimates.

$I_{j,k}$ = the improved integral after integration level k.

$I_{j+1,k-1}$ = the more accurate estimate after integration level k-1.

$I_{j,k-1}$ = the less accurate estimate after integration level k-1.

L06E14_trapezoidalRule_rombergQuadrature.m

Example – Romberg Integration

Use Romberg integration to numerically integrate the following function from a = 0 to b = 0.8:

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

Solution:

	First level **k = 1**	**Second level** **k = 2**	**Third level** **k = 3**	**Fourth level** **k = 4**
j = 1 (**n = 1**, h = 0.8)	$I_{1,1}$ = 0.172800	$I_{1,2}$ = 1.367467	$I_{1,3}$ = 1.640533	$I_{1,4}$ = 1.640533
j = 2 (**n = 2**, h = 0.4)	$I_{2,1}$ = 1.068800	$I_{2,2}$ = 1.623467	$I_{2,3}$ = 1.640533	
j = 3 (**n = 4**, h = 0.2)	$I_{3,1}$ = 1.484800	$I_{3,2}$ = 1.639467		
j = 4 (**n = 8**, h = 0.1)	$I_{4,1}$ = 1.600800			

Trapezoidal rule

$$I = \frac{4}{3}I_m - \frac{1}{3}I_l$$

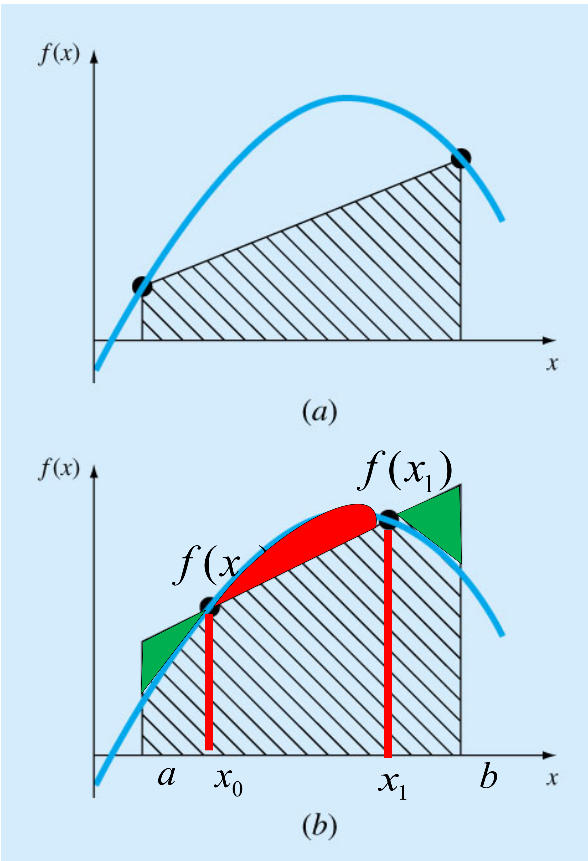
Richardson's extrapolation

$$I = \frac{16}{15}I_m - \frac{1}{15}I_l \qquad I = \frac{64}{63}I_m - \frac{1}{63}I_l$$

Romberg integration

L06E14_trapezoidalRule_rombergQuadrature.m **

Gauss quadrature



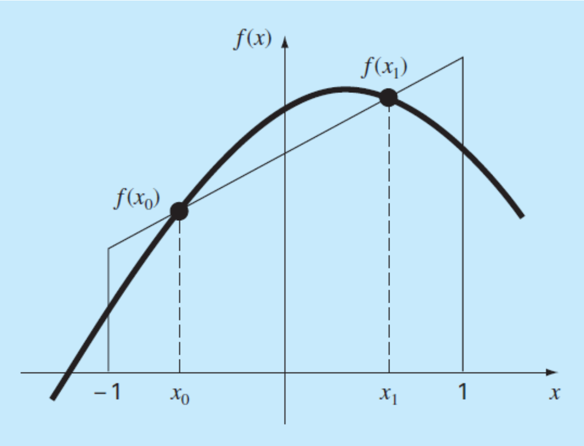
- **Gauss quadrature** implements a strategy of positioning **any two points** (rather than just the endpoints) on a curve to define a straight line that **balance out the positive and negative errors**.

- Hence the area evaluated under this straight line provides **an improved estimate of the integral**.

$$I = \int_a^b f(x)\,dx = \int_{-1}^{1} f(\lambda)d\lambda \cong c_0 f(\lambda_0 c_1 f(\lambda_1) + \cdots + +c_{n-1} f(\lambda_{n-1})$$

L06E15_gauss2PtQuadrature.m

General form of Gauss quadrature



For the integral estimate of the form:

$$I = \int_{-1}^{1} f(\lambda) d\lambda \cong c_0 f(\lambda_0) + c_1 f(\lambda_1) + \cdots + + c_{n-1} f(\lambda_{n-1})$$

The unknowns of the general Gauss-Legendre formula are:
weighting coefficients = c_0 …. c_{n-1}, and, λ_0 … λ_{n-1}

$$I = \sum c_i f(\lambda_i) = ?$$

L06E15_gauss2PtQuadrature.m

2 point Gauss-Legendre quadrature

For the integral:

$$I = \int_{-1}^{1} f(\lambda)d\lambda \cong c_0 f(\lambda_0) + c_1 f(\lambda_1)$$

There are **four** unknowns of the 2 point Gauss-Legendre formula:
$\mathbf{c_0}$ **,** $\mathbf{c_1}$**, and** $\mathbf{x_0}$ **,** $\mathbf{x_1}$

Assume:

- The integrals for 4 formulae:

y = constant,
y = λ,
y = λ²,
y = λ³,
are computed **exactly**

- Simplify the math by assuming integration intervals of -1 to +1

L06E15_gauss2PtQuadrature.m

2 point Gauss quadrature

Determining the constants:

$$c_0 f(\lambda_1) + c_1 f(\lambda_1) = \int_{-1}^{1} 1 d\lambda = \lambda |_{-1}^{+1} = (1) - (-1) = 2$$

$$c_0 f(\lambda_1) + c_1 f(\lambda_1) = \int_{-1}^{1} \lambda d\lambda = \frac{\lambda^2}{2} |_{-1}^{+1} = \left(\frac{1}{2}\right) - \left(\frac{(-1)^2}{2}\right) = 0$$

$$c_0 f(\lambda_1) + c_1 f(\lambda_1) = \int_{-1}^{1} \lambda^2 d\lambda = \frac{\lambda^3}{3} |_{-1}^{+1} = \left(\frac{1}{3}\right) - \left(\frac{(-1)^3}{3}\right) = \frac{2}{3}$$

$$c_0 f(\lambda_1) + c_1 f(\lambda_1) = \int_{-1}^{1} \lambda^3 d\lambda = \frac{\lambda^4}{4} |_{-1}^{+1} = \left(\frac{1}{4}\right) - \left(\frac{(-1)^4}{4}\right) = 0$$

Solve the 4 algebraic equations for the 4 unknowns c_0 , c_1, and x_0 , x_1

L06E15_gauss2PtQuadrature.m

Constants for the Gauss-Legendre formulae

# Points	Weighting Factors, c_i	Function Arguments, x_i
2	$c_0 = 1.0$ $c_1 = 1.0$	$x_0 = -1/\sqrt{3}$ $x_1 = 1/\sqrt{3}$
3	$c_0 = 5/9$ $c_1 = 8/9$ $c_2 = 5/9$	$x_0 = -\sqrt{3/5}$ $x_1 = 0.0$ $x_2 = \sqrt{3/5}$
4	$c_0 = \left(18 - \sqrt{30}\right)/36$ $c_1 = \left(18 + \sqrt{30}\right)/36$ $c_2 = \left(18 + \sqrt{30}\right)/36$ $c_3 = \left(18 - \sqrt{30}\right)/36$	$x_0 = -\sqrt{525 + 70\sqrt{30}}/35$ $x_1 = -\sqrt{525 - 70\sqrt{30}}/35$ $x_2 = \sqrt{525 - 70\sqrt{30}}/35$ $x_3 = \sqrt{525 + 70\sqrt{30}}/35$

L06E15_gauss2PtQuadrature.m

Simplified integration via 2 – point Gauss quadrature

$$I=\int_a^b f(x)dx$$

Substitute $$x=\frac{(b+a)+(b-a)x_d}{2}$$

$$dx=\frac{(b-a)}{2}dx_d$$

$$I=\int_{-1}^{1} f(\frac{(b+a)+(b-a)x_d}{2})\frac{(b-a)}{2}dx_d$$

$$I\cong f(\frac{(b+a)+(b-a)\left(-1/\sqrt{3}\right)}{2})\frac{(b-a)}{2}+f(\frac{(b+a)+(b-a)\left(1/\sqrt{3}\right)}{2})\frac{(b-a)}{2}$$

L06E15_gauss2PtQuadrature.m

Example – 2 pt Gauss quadrature

Use Gauss quadrature to numerically integrate the following function:

$$I = \int_2^6 \left(x^2 + 5x + 3\right) dx$$

Solution:

Step 1: Change the variables

$$\mathrm{x} = \frac{\mathrm{b}+\mathrm{a}}{2} + \frac{\mathrm{b}-\mathrm{a}}{2}\lambda = \frac{6+2}{2} + \frac{6-2}{2}\lambda = 4 + 2\lambda$$

$$\mathrm{dx} = \frac{(b-a)}{2} d\lambda = \frac{(6-2)}{2} d\lambda = 2d\lambda$$

Step 2: Substitute and change the integral

$$I = \int_2^6 (x^2 + 5x + 3) dx \qquad f(x) \qquad f(\lambda)$$

$$= \int_{-1}^{1} [(4 + 2\lambda)^2 + 5 \cdot (4 + 2\lambda) + 3] * 2d\lambda = \int_{-1}^{1} 2[(4 + 2\lambda)^2 + 5 \cdot (4 + 2\lambda) + 3] d\lambda$$

L06E15_gauss2PtQuadrature.m

Step 3: The transformed function

$$f(\lambda) = 2\left[(4+2\lambda)^2 + 5\cdot(4+2\lambda)+3\right]$$

Step 4: Values at the 2 Gauss quadrature points

$$f(\frac{1}{-\sqrt{3}}) = 2\left[(4+2\frac{1}{-\sqrt{3}})^2 + 5\cdot(4+2\frac{1}{-\sqrt{3}})+3\right] = 50.64445$$

$$f(\frac{1}{\sqrt{3}}) = 2\left[(4+2\frac{1}{\sqrt{3}})^2 + 5\cdot(4+2\frac{1}{\sqrt{3}})+3\right] = 111.68888$$

Step 5: The numerical solution to the integral

$$I = \int_a^b f(x)dx = \int_{-1}^{1} f(\lambda)d\lambda \cong f(\frac{1}{-\sqrt{3}}) + f(\frac{1}{\sqrt{3}}) = 50.64445 + 111.68888 = 161.3333$$

L06E15_gauss2PtQuadrature.m **

High order Gauss quadrature

$$I = \int_{-1}^{1} f(\lambda)d\lambda \cong c_0 f(\lambda_0) + c_1 f(\lambda_1) + \cdots + c_n f(\lambda_n)$$

Points	Weighting Factors	Function Arguments	Truncation Error
2	$c_0 = 1.0000000$	$x_0 = -0.577350269$	$\cong f^{(4)}(\xi)$
	$c_1 = 1.0000000$	$x_1 = 0.577350269$	
3	$c_0 = 0.5555556$	$x_0 = -0.774596669$	$\cong f^{(6)}(\xi)$
	$c_1 = 0.8888889$	$x_1 = 0.0$	
	$c_2 = 0.5555556$	$x_2 = 0.774596669$	
4	$c_0 = 0.3478548$	$x_0 = -0.861136312$	$\cong f^{(8)}(\xi)$
	$c_1 = 0.6521452$	$x_1 = -0.339981044$	
	$c_2 = 0.6521452$	$x_2 = 0.339981044$	
	$c_3 = 0.3478548$	$x_3 = 0.861136312$	
5	$c_0 = 0.2369269$	$x_0 = -0.906179846$	$\cong f^{(10)}(\xi)$
	$c_1 = 0.4786287$	$x_1 = -0.538469310$	
	$c_2 = 0.5688889$	$x_2 = 0.0$	
	$c_3 = 0.4786287$	$x_3 = 0.538469310$	
	$c_4 = 0.2369269$	$x_4 = 0.906179846$	
6	$c_0 = 0.1713245$	$x_0 = -0.932469514$	$\cong f^{(12)}(\xi)$
	$c_1 = 0.3607616$	$x_1 = -0.661209386$	
	$c_2 = 0.4679139$	$x_2 = -0.238619186$	
	$c_3 = 0.4679139$	$x_3 = 0.238619186$	
	$c_4 = 0.3607616$	$x_4 = 0.661209386$	
	$c_5 = 0.1713245$	$x_5 = 0.932469514$	

L06E16_gauss3PtQuadrature.m **

HOME READING
Improper integrals

For the case of integrals with a lower limit of -∞ and/or an upper limit of ∞:

$$I = \int_{-\infty}^{b} f(x)dx$$

$$I = \int_{-\infty}^{-A} f(x)dx + \int_{-A}^{b} f(x)dx$$

Choose a value for A.
Here A is a positive value.

$$x = \frac{1}{t}$$

$$I = \int_{-1/A}^{0} \frac{1}{t^2} f(\frac{1}{t})dt + \int_{-A}^{b} f(x)dx$$

Midpoint rule

Trapezoidal, Simpson's midpoint rules

L06E17_improperIntegrals.m

Example – Improper Integrals

Estimate the improper integral:

$$I = \int_{-\infty}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

Solution:

Choose A = 2.

$$I = \int_{-\infty}^{-2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx + \int_{-2}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$\int_{-\infty}^{-2} e^{-x^2/2} dx \xrightarrow{x = \frac{1}{t}} \int_{-1/2}^{0} \frac{1}{t^2} e^{-1/\left(2t^2\right)} dt$$

L06E17_improperIntegrals.m

Use the mid-point rule with h = 1/8:

$$\int_{-1/2}^{0} \frac{1}{t^2} e^{-1/\left(2t^2\right)} dt \cong \frac{1}{8}\left(f(x_{-7/16}) + f(x_{-5/16}) + f(x_{-3/16}) + f(x_{-1/16})\right)$$

$$= \frac{1}{8}\left(0.3833 + 0.0612 + 0 + 0\right) = 0.0556$$

Use Simpson's 1/3 rule with h = 0.5

$$\int_{-2}^{1} e^{-x^2/2} dx = (1-(-2))\frac{0.1353 + 4(0.3247 + 0.8825 + 0.8825) + 2(0.6065 + 1) + 0.6065}{3(6)}$$

$$= 2.0523$$

$$I = \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{-2} e^{-x^2/2} dx + \int_{-2}^{1} e^{-x^2/2} dx\right) = \frac{1}{\sqrt{2\pi}}(0.0556 + 2.0523)$$

$$= 0.8409$$

L06E17_improperIntegrals.m **

Mid-Semester Exam

- When = Monday, 4 September 2023 at 12:30 pm **(Week 7)**.
- Where = Bldg. 24, Room 201
- Duration = 1 hour 45 minutes.
- Exam syllabus = Lectures from Weeks 1 to 6.
- The exam question paper must not be removed from the exam venue. It must be returned along with the exam answer book.
- This is a restricted exam. You are allowed only one (1) double-sided A4 sheet of handwritten or typed notes in the form of a hardcopy.
- You are allowed the use of a UOW-approved calculator.
- Electrical devices, such as computers, mobile phones etc., are NOT allowed.
- Five multiple choice questions and four calculations need to be answered.