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1 题

1.1 题

$$\begin{aligned} \max \quad & 3w_1 + 6w_2 + 2w_3 + 2w_4 \\ s.t. \quad & w_1 + 3w_2 + w_4 \le 8 \\ & 2w_1 + w_2 \le 6 \\ & w_2 + w_3 + w_4 \le 3 \\ & w_1 + w_2 + w_3 \le 6 \\ & w_1, w_2, w_3, w_4 \ge 0 \end{aligned}$$

1.2 题

$$Ax^{*T} - b = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 3 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 6 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore w_4^* = 0$$

$$\because x_1^*, x_2^*, x_3^* \ne 0$$

$$\therefore \begin{cases} w_1 + 3w_2 + w_4 = 8 \\ 2w_1 + w_2 = 6 \\ w_2 + w_3 + w_4 = 3 \end{cases} \Rightarrow \begin{cases} w_1 = 2 \\ w_2 = 2 \\ w_3 = 1 \end{cases}$$

$$\therefore w^* = (2, 2, 1, 0)$$

2 题

2.1 题

$$\begin{cases} x_1 - 1 - \alpha_1 - \alpha_2 = 0 \\ x_2 - 2 - \alpha_1 - \alpha_3 = 0 \\ (x_1 + x_2 - \kappa)\alpha_1 = 0 \\ x_1\alpha_2 = 0 \\ x_2\alpha_3 = 0 \\ \alpha_1, \alpha_2, \alpha_3 \ge 0 \\ x_1 + x_2 - \kappa \ge 0 \\ x_1, x_2 \ge 0 \end{cases}$$

代入$\kappa = 4, x_1 = 1.5, x_2 = 2.5$得：

$\alpha_1 = 0.5, \alpha_2 = 0, \alpha_3 = 0$

$\therefore x^*$为KKT点

显然, 所给问题为凸规划

$\therefore x^*$是最优解

2.2 题

$$\begin{cases} x_1 - 1 = 0 \\ x_2 - 2 = 0 \\ x_1 + x_2 - \kappa > 0 \\ x_1, x_2 > 0 \end{cases}$$
$$\therefore \kappa < 3$$
$$\text{最优解为}(1,2), \text{目标函数最优值为} -\frac{5}{2}$$

2.3 题

由(2)得$\kappa < 3$时, 最优解不在边界上

假设最优解落在边界$x_1 + x_2 - \kappa = 0$, 进而$x_1 + x_2 = \kappa$

$$\begin{cases} x_1 = \alpha_1 + \alpha_2 + 1 \\ x_2 = \alpha_1 + \alpha_3 + 2 \Rightarrow x_1, x_2 > 0 \\ \alpha_1, \alpha_2, \alpha_3 \geq 0 \end{cases}$$
$$\begin{cases} x_1, x_2 > 0 \\ x_1\alpha_2 = 0 \quad \Rightarrow \alpha_2 = \alpha_3 = 0 \\ x_2\alpha_3 = 0 \end{cases}$$
$$\begin{cases} x_1 = \alpha_1 + 1 \\ x_2 = \alpha_1 + 2 \\ x_1 + x_2 = \kappa \\ \alpha_1 \geq 0 \end{cases} \Rightarrow x_1 = \frac{\kappa - 1}{2}, x_2 = \frac{\kappa + 1}{2}, \kappa \geq 3$$

综上, 当$\kappa \geq 3$时, 最优解为可行区域边界点, 最优解为$(\frac{\kappa-1}{2}, \frac{\kappa+1}{2})$, 目标函数最优值为$\frac{1}{4}\kappa^2 - \frac{3}{2}\kappa - \frac{1}{4}$

2.4 题

$$\begin{cases} x_1 - 1 - w = 0 \\ x_2 - 2 - w = 0 \\ x_1, x_2 > 0 \\ w \geq 0 \end{cases} \Rightarrow \begin{cases} x_1 = w + 1 \\ x_2 = w + 2 \end{cases}$$
$$\therefore \text{对偶问题为}$$
$$\begin{aligned} \max \quad & \theta(w) = -w^2 + (\kappa - 3)w - \frac{5}{2} \\ s.t. \quad & w \geq 0 \end{aligned}$$

3 题

3.1 题

$$\begin{aligned} \min \quad & x_1 + x_2 - 3x_3 + M(x_6 + x_7) \\ s.t. \quad & x_1 - 2x_2 + x_3 + x_4 = 11 \\ & 2x_1 + x_2 - 4x_3 - x_5 + x_6 = 3 \\ & x_1 - 2x_3 + x_7 = 1 \\ & x_i \geq 0, i = 1, \ldots, 7 \end{aligned}$$

	x_1	**x_2**	**x_3**	**x_4**	**x_5**	**x_6**	**x_7**	
x_4	1	-2	1	1	0	0	0	11
x_6	2	1	-4	0	-1	1	0	3
x_7	1	0	-2	0	0	0	1	1
	3M-1	M-1	-6M+3	0	-M	0	0	4M

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_4	0	-2	3	1	0	0	-1	10
x_6	0	1	0	0	-1	1	-2	1
x_1	1	0	-2	0	0	0	1	1
	0	M-1	1	0	-M	0	1-3M	M+1

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_4	0	0	3	1	-2	2	-5	12
x_2	0	1	0	0	-1	1	-2	1
x_1	1	0	-2	0	0	0	1	1
	0	0	1	0	-1	1-M	-1-M	2

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_3	0	0	1	1/3	-2/3	2/3	-5/3	4
x_2	0	1	0	0	-1	1	-2	1
x_1	1	0	0	2/3	-4/3	4/3	-7/3	9
	0	0	0	-1/3	-1/3	1/3-M	2/3-M	-2

$$\text{最优解：}(9,1,4)^T$$
$$\text{最优值：}-2$$

3.2 题

$$\text{新的右端向量：}\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{5}{3} \\ 0 & 1 & -2 \\ \frac{2}{3} & \frac{4}{3} & -\frac{7}{3} \end{pmatrix}\begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ 1 \\ \frac{1}{3} \end{pmatrix}$$
$$\text{新的目标函数值：}\frac{1}{3}+1+1=\frac{7}{3}$$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_3	0	0	1	1/3	-2/3	2/3	-5/3	-1/3
x_2	0	1	0	0	-1	1	-2	1
x_1	1	0	0	2/3	-4/3	4/3	-7/3	1/3
	0	0	0	-1/3	-1/3	1/3-M	2/3-M	7/3

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_5	0	0	-3/2	-1/2	1	-1	5/2	1/2
x_2	0	1	-3/2	-1/2	0	0	1/2	3/2
x_1	1	0	-2	0	0	0	1	1
	0	0	-1/2	-1/2	0	-M	3/2-M	5/2

$$\text{新问题最优解：}(1,\frac{3}{2},0)^T$$

$$\text{新问题最优值：}\frac{5}{2}$$

4 题

$L=(x_1-1)^2+x_2^2-v(x_1-\frac{x_2^2}{\beta})$

$KKT\text{条件：}\begin{cases}2(x_1-1)-v=0\\2x_2+\frac{2v}{\beta}x_2=0\\x_1-\frac{x_2^2}{\beta}=0\end{cases}$

将$x_1=x_2=0$代入得$v=-2$

$x_1-\dfrac{x_2^2}{\beta}$在$(0,0)$处的梯度为$(1,0)^T$

$G=\{d=(d_1,d_2)^T|(1,0)\times(d_1,d_2)^T=0\}=\{d=(d_1,d_2)^T|d_1=0\}$

$\nabla^2L=\begin{pmatrix}2&0\\0&2+\frac{2v}{\beta}\end{pmatrix}$

代入$x_1=x_2=0,v=-2$得$\nabla^2L=\begin{pmatrix}2&0\\0&2-\frac{4}{\beta}\end{pmatrix}$

$2-\dfrac{4}{\beta}>0\Rightarrow\beta>2$时$,\overline{x}$一定是局部最优解

$2-\dfrac{4}{\beta}<0\Rightarrow 0<\beta<2$时$,\overline{x}$一定不是局部最优解

$2-\dfrac{4}{\beta}=0\Rightarrow\beta=2$时$,x_2^2=2x_1,$目标函数变为$(x_1-1)^2+2x_1(x_1\geq 0)$

即目标函数为$x_1^2+1(x_1\geq 0),$显然$x_1=0$时达到最优值$,$此时$x_2=0,$故$\overline{x}$是局部最优解

综上$,\beta\geq 2$时$,\overline{x}$是局部最优解

$\beta<2$时$,\overline{x}$不是局部最优解

5 题

5.1 题

$Ad = Bd_B + Nd_N = B(-B^{-1}Nd_N) + Nd_N = 0$
$\therefore \forall\lambda > 0, A(x+\lambda d) = Ax + \lambda Ad = Ax = b$
$\therefore d$是x处的可行方向
$\nabla f(x)^T d = \nabla f_{x_B}(x)^T d_B + \nabla f_{x_N}(x)^T d_N$
$= \nabla f_{x_B}(x)^T(-B^{-1}Nd_N) + \nabla f_{x_N}(x)^T d_N$
$= [-\nabla f_{x_B}(x)^T B^{-1}N + \nabla f_{x_N}(x)^T]d_N$
$= r(x_N)^T d_N$
$= -d_N^T d_N$
若$d_N = 0$, 则$d_B = -B^{-1}Nd_N = 0$, 于是$d = 0$, 矛盾
$\therefore d_N \neq 0$
$\therefore \nabla f(x)^T d = -d_N^T d_N < 0$
$\therefore d$是x处的下降方向
综上, d是x处的可行下降方向

5.2 题

$L = f(x) - v^T(Ax - b)$
$\nabla L = \nabla f(x) - A^T v = \begin{pmatrix} \nabla_{x_B} f(x) \\ \nabla_{x_N} f(x) \end{pmatrix} - \begin{pmatrix} B^T \\ N^T \end{pmatrix} v$

必要性：
$d = 0 \Rightarrow d_N = 0 \Rightarrow r(x_N) = 0 \Rightarrow \nabla_{x_N} f(x) - (B^{-1}N)^T \nabla_{x_B} f(x) = 0$
令$v = (B^{-1})^T \nabla_{x_B} f(x)$
则$\nabla_{x_B} f(x) - B^T v = 0$
$\nabla_{x_N} f(x) - N^T v = \nabla_{x_N} f(x) - N^T[(B^{-1})^T \nabla_{x_B} f(x)]$
$= \nabla_{x_N} f(x) - N^T (B^{-1})^T \nabla_{x_B} f(x)$
$= \nabla_{x_N} f(x) - (B^{-1}N)^T \nabla_{x_B} f(x) = 0$
又x是可行解, 满足$Ax = b$
$\therefore x$是KKT点

充分性：
$\because x$是KKT点
$\therefore \begin{cases} \nabla_{x_B} f(x) - B^T v = 0 & ① \\ \nabla_{x_N} f(x) - N^T v = 0 & ② \end{cases}$
由①得$v = (B^{-1})^T \nabla_{x_B} f(x)$, 代入②得
$\nabla_{x_N} f(x) - N^T (B^{-1})^T \nabla_{x_B} f(x) = 0$
$\therefore \nabla_{x_N} f(x) - (B^{-1}N)^T \nabla_{x_B} f(x) = 0$
$\therefore r(x_N) = 0$
$\therefore d_N = -r(x_N) = 0$
$\therefore d_B = -B^{-1}Nd_N = 0$
$d_N = 0, d_B = 0 \Rightarrow d = 0$

证毕