Opgave 1.1 (Fourierrækker)

Find Fourierkoefficienterne og Fourierrækken for firkant-signalet defineret som

$$f(x) = \begin{cases} 0 & \text{hvis } -1 \le x < 0 \\ 1 & \text{hvis } 0 \le x < 1 \end{cases}$$

og

$$f(x+2) = f(x)$$

Løsning:

$$f(x) = \frac{1}{2} + \sum_{n=1,3,5,\dots} \frac{2}{n\pi}\sin(n\pi x)$$

$$-1 \le x < 1 \Rightarrow T = 2 \Rightarrow L = 1$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cdot\cos\left(\frac{n\pi x}{L}\right)dx = \int_{-1}^{0} \underbrace{f(x)}_{0}\cdot\cos(n\pi x)\,dx + \int_{0}^{1} \underbrace{f(x)}_{1}\cdot\cos(n\pi x)\,dx$$

$$= \int_0^1 \cos(n\pi x)\,dx = \int_{x=0}^{x=1} \cos(u)\,\frac{1}{n\pi}\,du = \left[\frac{1}{n\pi}\cdot\sin(u)\right]_{x=0}^{x=1} = \left[\frac{1}{n\pi}\cdot\sin(n\pi x)\right]_0^1$$

$$u = n\pi x \Rightarrow \frac{du}{dx} = n\pi \Rightarrow dx = \frac{1}{n\pi}du$$

$$= \frac{2}{n\pi}\cdot\underbrace{\sin(n\pi)}_{0} - \frac{2}{n\pi}\cdot\underbrace{\sin(0)}_{0} = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\cdot\sin\left(\frac{n\pi x}{L}\right)dx = \int_{-1}^{1} f(x)\cdot\sin(n\pi x)\,dx$$

$$= \cancel{\int_{-1}^{0} \underbrace{f(x)}_{0}\cdot\sin(n\pi x)\,dx} + \int_0^1 \underbrace{f(x)}_{1}\cdot\sin(n\pi x)\,dx = \int_0^1 \sin(\underbrace{n\pi x}_{u})\,dx$$

$$\frac{du}{dx} = n\pi \Rightarrow dx = \frac{1}{n\pi}$$

$$= \int_{x=0}^{x=1} \sin(u)\cdot\frac{1}{n\pi}\,du = \frac{1}{n\pi}\cdot\left[-\cos(u)\right]_{x=0}^{x=1} = \frac{1}{\pi n}\cdot\left[\cos(n\pi)\right]_0^1 = \frac{1}{\pi n}\cdot\left(\underbrace{\cos(\pi)}_{1} + \underbrace{\cos(0)}_{1}\right) = \frac{2}{\pi n}$$

$$a_0 = \int_0^1 f(x)\cdot\cos(0)\,dx = \int_0^1 f(x)\,dx = \int_0^1 1\,dx = [x]_0^1 = 1$$

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cdot\cos\left(\frac{n\pi x}{L}\right) + b_n\cdot\sin\left(\frac{n\pi x}{L}\right)\right) = \underline{\underline{\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{\pi n}\cdot\sin(n\pi x)\right)}}$$

Opgave 1.2 (Fouriertransformation)

Enhedstrinfunktionen defineret som

$$u(t-a) = \begin{cases} 1 \text{ for } t-a > 0 \\ 0 \text{ for } t-a < 0 \end{cases}$$

benyttes til at definere en firkantimpuls

$$x(t) = u(t-a) - u(t-b)$$

hvor $a < b$.

- Tegn grafen for firkantimpulsen $x(t)$
- Udregn den Fouriertransformerede af $x(t)$

Løsning:

$$X(\omega) = \frac{e^{-j\omega a} - e^{-j\omega b}}{j\omega}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \mathrm{dt}$$

$$X(w) = \int_{-\infty}^{\infty} x(t) \cdot e^{-jwt}\, dt = \int_{-\infty}^{a} \cancel{\underbrace{x(t)}_{0} \cdot e^{-jwt}}\, dt + \int_{a}^{b} \underbrace{x(t)}_{1} \cdot e^{-jwt}\, dt + \int_{b}^{\infty} \cancel{\underbrace{x(t)}_{0} \cdot e^{-jwt}}\, dt$$

$$= \int_{a}^{b} e^{-jwt}\, dt = \int_{t=a}^{t=b} e^{u} \cdot \frac{-1}{jw}\, du = \left[e^{u} \cdot \frac{-1}{jw}\right]_{t=a}^{t=b} = \left[e^{-jwt} \cdot \frac{-1}{jw}\right]_{a}^{b}$$

$$u = -jwt \Rightarrow \frac{du}{dt} = -jw \Rightarrow dt = \frac{-1}{jw}\, du$$

$$\int_{a} \underbrace{x(t)}_{1} \cdot e^{-jwt}\, dt$$

$$= \frac{-e^{-jwb} + e^{-jwa}}{jw} = \underline{\underline{\frac{e^{-jwa} - e^{-jwb}}{jw}}}$$

Opgave 1.3 (Invers Fouriertransformation)

Betragt signalet (i frekvensdomæne) $X(\omega) = \delta(\omega - \omega_0)$ hvor ω_0 er en konstant. Benyt invers Fouriertransformation til at finde $x(t)$. **Løsning**:

$$x(t) = \frac{1}{2\pi} e^{j\omega_0 t}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \mathrm{d}\omega$$

$$x(t) = \frac{1}{2\pi} \left(\int_{-\infty}^{\omega_0^-} \underbrace{X(\omega)}_{0} \cdot e^{j\omega t} \, d\omega + \int_{\omega_0^-}^{\omega_0^+} X(\omega) \cdot e^{j\omega t} \, d\omega + \int_{\omega_0^+}^{\infty} \underbrace{X(\omega)}_{0} \cdot e^{j\omega t} \, d\omega \right)$$

$$= \frac{1}{2\pi} \int_{\omega_0^-}^{\omega_0^+} X(\omega) \cdot e^{j\omega t} \, d\omega = \frac{1}{2\pi} \cdot \int_{\omega_0^-}^{\omega_0^+} \underbrace{X(\omega)}_{1} \cdot e^{j\omega_0 t} = \underline{\underline{\frac{1}{2\pi} \cdot e^{j\omega_0 t}}}$$

Opgave 1.4 (Fouriertransformation)

Betragt signalet (i tidsdomæne) $x(t) = \sin(\omega_0 t)$ hvor ω_0 er en konstant. Benyt Fouriertransformation til at finde $X(\omega)$. **Løsning**:

$$X(\omega) = -j\pi(\delta(\omega - \omega_0) - \delta(\omega + \omega_0))$$

Signal	Fouriertransform
$\delta(t)$	1
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$
$\delta(t - t_0)$	$e^{-j\omega t_0}$
$\sin(\omega_0 t)$	$-j\pi(\delta(\omega - \omega_0) - \delta(\omega + \omega_0))$
$\cos(\omega_0 t)$	$\pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$
1	$2\pi\delta(\omega)$

$$\underline{X(\omega) = \mathcal{F}\left[\sin(\omega_0 t)\right] = -j\pi\left(\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right)}$$