

1. Check that if each  $u$  satisfies the two-dimensional Poisson's equation or not, written below:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

Where,  $u$  is dependent on independent variable  $x$  and  $y$ .

- a •  $u = \frac{2y}{x}$
- b •  $u = x^4 + y^4$
- c •  $u = -2 \cos x \sin y$

a)

$$\frac{\partial}{\partial x^2} \left( \frac{2y}{x} \right) + \frac{\partial}{\partial y^2} \left( \frac{2y}{x} \right) = \frac{\partial}{\partial x} \left( \frac{-2y}{x^2} \right) + \frac{\partial}{\partial y} \left( \frac{2}{x} \right) = \frac{4y}{x^3} + 0 = \frac{4y}{x^3}$$

b)

$$\frac{\partial}{\partial x^2} (x^4 + y^4) + \frac{\partial}{\partial y^2} (x^4 + y^4) = \frac{\partial}{\partial x} (4x^3) + \frac{\partial}{\partial y} (4y^3) = 12x^2 + 12y^2 = 12(x^2 + y^2)$$

c)

$$\begin{aligned} & \frac{\partial}{\partial x^2} (-2 \cos(x) \cdot \sin(y)) + \frac{\partial}{\partial y^2} (-2 \cos(x) \cdot \sin(y)) \\ &= \frac{\partial}{\partial x} (2 \cdot \sin(y) \cdot \sin(x)) + \frac{\partial}{\partial y} (-2 \cos(x) \cdot \cos(y)) = 2 \cdot \sin(y) \cdot \cos(x) + 2 \cdot \cos(x) \cdot \sin(y) \\ &= 4 \cdot \cos(x) \cdot \sin(y) \end{aligned}$$

Yes, all these are functions of  $x$  and  $y$ , and are therefore solutions to the PDE.

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2. calculate **div F** and **curl F** for the vector field,

$$\mathbf{F} = \overset{f_1}{\cos x} \mathbf{i} - \overset{f_2}{\sin y} \mathbf{j} + \overset{f_3}{z} \mathbf{k}.$$

Finding divergence

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} (\cos(x)) + \frac{\partial}{\partial y} (-\sin(y)) + \frac{\partial}{\partial z} z = \underline{\underline{-\sin(x) - \cos(y) + 1}}$$

Finding curl

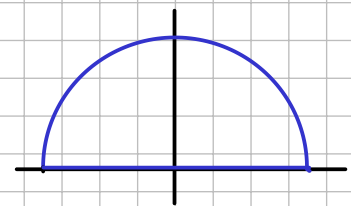
$$\left( \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \mathbf{i} + \left( \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \mathbf{j} + \left( \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \mathbf{k}$$

$$\text{curl } \vec{F} = (0-0)\mathbf{i} + (0-0)\mathbf{j} + (0-0)\mathbf{k} = \underline{\underline{\vec{0}}}$$

3. Using green's theorem evaluate  $\oint_C (x^2 y) dx + (xy^2) dy$ , clockwise boundary of the region.

$$0 \leq y \leq \sqrt{9-x^2}$$

$$\oint_C \vec{F}(x, y) \cdot d\vec{r} = \oint_C f_1(x, y) dx + f_2(x, y) dy = \iint_R \left( \frac{\partial f_1}{\partial y} - \frac{\partial f_2}{\partial x} \right) dA$$



$$-\oint_C \overset{f_1}{x^2 y} dx + \overset{f_2}{xy^2} dy = -\iint_R \frac{\partial}{\partial x} (x^2 y) - \frac{\partial}{\partial y} (xy^2) dA$$

$$= -\iint_R 2xy - 2xy dA = -\iint_R 0 dA = -0 = \underline{\underline{0}}$$

4. Use the Divergence Theorem to calculate the flux of the given vector field out of the sphere  $s$  with equation  $x^2 + y^2 + z^2 = a^2$ , where  $a > 0$ . and

$$\mathbf{F} = (x^2 + y^2)\mathbf{i} + (y^2 - z^2)\mathbf{j} + z\mathbf{k}$$

$\downarrow f_1$                        $\downarrow f_2$                        $\rightarrow f_3$

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iiint_V \operatorname{div} \vec{F} \, dV$$

Finding divergence of vector field

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 + y^2) + \frac{\partial}{\partial y}(y^2 - z^2) + \frac{\partial}{\partial z} z = 2x + 2y + 1$$

Convert to spherical coordinates

$$\iiint_V 2x + 2y + 1 \, dV$$

$$x = \rho \cdot \sin \phi \cdot \cos \theta$$

$$y = \rho \cdot \sin \phi \cdot \sin \theta$$

$$z = \rho \cdot \cos \phi$$

$$dV = \rho^2 \cdot \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= \iiint_V (2 \cdot \rho \cdot \sin(\phi) \cdot \cos(\theta) + 2 \rho \cdot \sin(\phi) \cdot \sin(\theta) + 1) \cdot \rho^2 \cdot \sin(\phi) \, d\rho \, d\phi \, d\theta$$

$$= \iiint_V \underbrace{2 \cdot \rho^3 \cdot \sin^2(\phi) \cdot \cos(\theta)} + \underbrace{2 \cdot \rho^3 \cdot \sin^2(\phi) \cdot \sin(\theta)} + \rho^2 \cdot \sin(\phi) \, d\rho \, d\phi \, d\theta$$

$$= \iiint_V 2 \cdot \rho^3 \cdot \sin^2(\phi) \cdot (\cos(\theta) + \sin(\theta)) + \rho^2 \cdot \sin(\phi) \, d\rho \, d\phi \, d\theta$$

Adding the boundaries

$$x^2 + y^2 + z^2 = a^2 \Rightarrow \rho = [0, a], \phi = [0, \pi], \theta = [0, 2\pi]$$

$$\Rightarrow \int_0^{2\pi} \int_0^{\pi} \int_0^a 2 \cdot \rho^3 \cdot \sin^2(\phi) \cdot (\cos(\theta) + \sin(\theta)) + \rho^2 \cdot \sin(\phi) \, d\rho \, d\phi \, d\theta$$

$$\Rightarrow \underbrace{2 \cdot \int_0^{2\pi} \int_0^{\pi} \int_0^a \rho^3 \cdot \sin^2(\phi) \cdot (\cos(\theta) + \sin(\theta)) \, d\rho \, d\phi \, d\theta}_{\text{first integral}} + \underbrace{\int_0^{2\pi} \int_0^{\pi} \int_0^a \rho^2 \cdot \sin(\phi) \, d\rho \, d\phi \, d\theta}_{\text{second integral}}$$

We start by evaluating the first integral

$$2 \cdot \int_0^{2\pi} \int_0^{\pi} \int_0^a \rho^3 \cdot \sin^2(\phi) \cdot (\cos(\theta) + \sin(\theta)) \, d\rho \, d\phi \, d\theta = 2 \cdot \int_0^{2\pi} \int_0^{\pi} \frac{a^4}{4} \cdot \sin^2(\phi) \cdot (\cos(\theta) + \sin(\theta)) \, d\phi \, d\theta$$

$$= 2 \cdot \int_0^{2\pi} \int_0^{\pi} \frac{a^4}{4} \cdot \sin^2(\phi) \cdot (\cos(\theta) + \sin(\theta)) \, d\phi \, d\theta$$

$$= \frac{a^4}{2} \int_0^{2\pi} \int_0^{\pi} \sin^2(\phi) \cdot (\cos(\theta) + \sin(\theta)) \, d\phi \, d\theta$$

$$= \frac{a^4}{2} \int_0^{2\pi} (\cos(\theta) + \sin(\theta)) \int_0^{\pi} \sin^2(\phi) \, d\phi \, d\theta$$

$$= \frac{a^4}{2} \int_0^{2\pi} (\cos(\theta) + \sin(\theta)) \cdot \left[ \frac{\phi}{2} - \frac{1}{4} \cdot \sin(2\phi) \right]_0^{\pi} \, d\theta$$

$$= \frac{a^4}{2} \int_0^{2\pi} (\cos(\theta) + \sin(\theta)) \cdot \frac{\pi}{2} - \frac{1}{4} \cdot \sin(2\pi) \, d\theta$$

$$= \frac{\pi a^4}{4} \int_0^{2\pi} \cos(\theta) + \sin(\theta) \, d\theta = \frac{\pi a^4}{4} \cdot [\sin(\theta) - \cos(\theta)]_0^{2\pi} = \frac{\pi a^4}{4} \cdot ((0-1) - (0-1)) = 0$$

Evaluating the second integral

$$\int_0^{2\pi} \int_0^{\pi} \int_0^a \rho^2 \cdot \sin(\phi) \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi} \left[ \frac{\rho^3}{3} \cdot \sin(\phi) \right]_0^a \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi} \frac{a^3}{3} \cdot \sin(\phi) \, d\phi \, d\theta$$

$$= \frac{a^3}{3} \cdot \int_0^{2\pi} \int_0^{\pi} \sin(\phi) \, d\phi \, d\theta = \frac{a^3}{3} \cdot \int_0^{2\pi} \sin(\phi) \, d\phi \, d\theta = \frac{a^3}{3} \cdot \int_0^{2\pi} [-\cos(\phi)]_0^{\pi} \, d\theta = \frac{a^3}{3} \cdot \int_0^{2\pi} (-\overset{0}{\cos(\pi)} + \overset{1}{\cos(0)}) \, d\theta$$

$$= \frac{a^3}{3} \cdot \int_0^{2\pi} 1 \, d\theta = \frac{a^3}{3} \cdot [\theta]_0^{2\pi} = \frac{2\pi a^3}{3}$$

Adding the integrals together

$$\Rightarrow 2 \cdot \int_0^{2\pi} \int_0^{\pi} \int_0^a \rho^3 \cdot \sin^2(\phi) \cdot (\cos(\theta) + \sin(\theta)) \, d\rho \, d\phi \, d\theta + \int_0^{2\pi} \int_0^{\pi} \int_0^a \rho^2 \cdot \sin(\phi) \, d\rho \, d\phi \, d\theta = 0 + \frac{2\pi a^3}{3}$$

$$\underline{\underline{= \frac{2\pi a^3}{3}}}$$

5. Determine whether the given vector field is conservative, and find a potential function if it is,

$$\mathbf{F}(x, y, z) = (2xy - z^2)\mathbf{i} + (2yz + x^2)\mathbf{j} - (2zx - y^2)\mathbf{k}$$

$$\left. \begin{array}{l} \frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}, \\ \frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x}, \\ \frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y}, \end{array} \right\} \text{ True for conservative fields. Let's test it!}$$

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} \Rightarrow 2x = 2x \quad \checkmark$$

$$\frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x} \Rightarrow -2z = -2z \quad \checkmark$$

$$\frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y} \Rightarrow 2y = 2y \quad \checkmark$$

The potential function must have this property

$$\nabla f = \vec{F}$$

$$\Rightarrow \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} = (2xy - z^2)\mathbf{i} + (2yz + x^2)\mathbf{j} - (2zx - y^2)\mathbf{k}$$

$$\Rightarrow \begin{cases} \frac{\partial f}{\partial x} = 2xy - z^2 \\ \frac{\partial f}{\partial y} = 2yz + x^2 \\ \frac{\partial f}{\partial z} = -(2zx - y^2) \end{cases} \Rightarrow \begin{cases} f = \int 2xy - z^2 dx \\ f = \int 2yz + x^2 dy \\ f = \int -2zx + y^2 dz \end{cases} \Rightarrow \begin{cases} f = x^2y - xz^2 + a(y, z) \\ f = y^2z + x^2y + b(x, z) \\ f = -z^2x + y^2z + c(x, y) \end{cases}$$

Summing up f's

$$f(x, y, z) = x^2y - xz^2 + a + y^2z + x^2y + b - z^2x + y^2z + c$$

$$\Rightarrow f(x, y, z) = \underline{x^2y} - \underline{xz^2} + \underline{y^2z} + \underline{x^2y} - \underline{z^2x} + \underline{y^2z} + a(y, z) + b(x, z) + c(x, y)$$

$$\Rightarrow \underline{f(x, y, z) = 2x^2y - 2xz^2 + 2y^2z + a(y, z) + b(x, z) + c(x, y)}$$

Finding a, b and c is left as an exercise for the reader;)