$$\iint_R \frac{x}{1+xy}\,dA \qquad R:\left\{x,y \,\middle|\, \begin{matrix}0 \le x \le 1\\ 0 \le y \le 1\end{matrix}\right.$$

Answer: ln[4] − 1

$$\int_0^1\int_0^1 \frac{x}{1+xy}\,dy\,dx = \iint \frac{\cancel{x}}{u}\frac{1}{\cancel{x}}\,du\,dx = \int \Big[\ln(u)\Big]_{y=0}^{y=1} dx$$

$$u = 1+xy \Rightarrow \frac{du}{dy} = x \Rightarrow dy = \frac{1}{x}\,du$$

$$u_1 = 1+x(0) = 1 \quad u_2 = 1+x(1) = x+1$$

$$= \int_0^1 \Big[\ln(u)\Big]_1^{x+1} dx = \int_0^1 \ln(x+1) - \cancel{\ln(1)}\,dx = \int_0^1 \ln(x+1)\,dx$$

$$= \int_0^1 \ln(u)\,du = \Big[-u+u\cdot\ln(u)\Big]_{x=0}^{x=1} = \Big[-x-1+(x+1)\cdot\ln(x+1)\Big]_0^1$$

$$u = x+1 \Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx$$

$$= \Big[-x-1+x\cdot\ln(x+1)+\ln(x+1)\Big]_0^1$$

$$= \left(-1-1+1\cdot\ln(1+1)+\ln(1+1)\right) - \left(0-1+0+\underbrace{\ln(1)}_{0}\right)$$

$$= -2+\ln(2)+\ln(2)+1 = \ln(2\cdot 2)-1 = \underline{\underline{\ln(4)-1}}$$

Problem 2:

$$\iint_R \left(\frac{\ln y}{y}\right) dA \qquad R: \left\{x, y \,\middle|\, \begin{matrix} 0 \le x \le \pi \\ e^{-2x} \le y \le e^{\cos x} \end{matrix}\right.$$

Answer: $\frac{\pi}{4} - \frac{2}{3}\pi^3$

$$\int_0^{\pi}\int_{e^{-2x}}^{e^{\cos(x)}} \ln(y) \cdot \frac{1}{y} \, dy \, dx = \int_0^{\pi}\int_{e^{-2x}}^{e^{\cos(x)}} u \cdot \cancel{\frac{1}{y}} \cdot \cancel{y} \, du \, dx = \int_0^{\pi}\int_{e^{-u}}^{e^{\cos(x)}} u \, du \, dx$$

$$u = \ln(y) \Rightarrow \frac{du}{dy} = \frac{1}{y} \Rightarrow dy = y \, du$$

$$= \int_0^{\pi} [u]_{y=e^{-2x}}^{y=e^{\cos(x)}} dx = \int_0^{\pi} [\cancel{\ln}(y)]_{\cancel{e}^{-2x}}^{\cancel{e}^{\cos(x)}} dx = \int_0^{\pi} \cos(x) + 2x \, dx$$

$$= \left[\sin(x) + x^2\right]_0^{\pi} = \cancel{\sin(\pi)} + \pi^2 - \cancel{\sin(0)} - 0 = \pi^2$$

Problem 3:

Calculate the given iterated integral

$$\int_0^1 dx \int_0^x (xy + y^2)dy$$

Answer: $\frac{5}{24}$

$$\int_0^1 \int_0^x xy + y^2 \, dy \, dx = \int_0^1 \left[x \cdot \frac{y^2}{2} + \frac{y^3}{3} \right]_0^x dx$$

$$= \int_0^1 x \cdot \frac{x^2}{2} + \frac{x^3}{3} - \cancel{x \cdot \frac{0^2}{2}} - \cancel{\frac{0^3}{3}} \, dx = \int_0^1 \frac{x^3}{2} + \frac{x^3}{3} \, dx = \int_0^1 \frac{5x^3}{6} dx$$

$$= \frac{5}{6} \cdot \int_0^1 x^3 \, dx = \frac{5}{6} \cdot \left[\frac{x^4}{4} \right]_0^1 = \frac{5}{6} \cdot \frac{1}{4} = \frac{5}{24}$$

Problem 4:

Calculate the given iterated integral

$$\int_0^{\pi}\int_{-x}^{x} \cos y \, dy \, dx$$

Answer: $-2\cos x \,\Big|_0^{\pi} = 4$

$$\int_0^{\pi}\int_{-x}^{x} \cos(y)\, dy\, dx = \int_0^{\pi} \Big[\sin(y)\Big]_{-x}^{x} dx = \int_0^{\pi} \sin(x) - \sin(-x)\, dx$$

$$\int_0^{\pi} 2\cdot\sin(x)\, dx = \Big[-2\cdot\cos(x)\Big]_0^{\pi} = -2\cdot\overset{-1}{\cos(\pi)} + 2\cdot\overset{1}{\cos(0)} = 2+2 = \underline{\underline{4}}$$

Problem 5:

Evaluate the double integral by iteration

$$\iint_R (x^2 + y^2)\, dA$$

where R is the rectangle $0 \le x \le a$, $0 \le y \le b$

Answer: $\frac{1}{3}(a^3 b + ab^3)$

$$\int_0^b \int_0^a x^2 + y^2 \, dx\, dy = \int_0^b \left[\frac{x^3}{3} + y^2 x \right]_0^a dy = \int_0^b \frac{a^3}{3} + y^2 \cdot a \, dy$$

$$= \left[\frac{a^3}{3} \cdot y + a \cdot \frac{y^3}{3} \right]_0^b = \frac{a^3}{3} \cdot b + a \cdot \frac{b^3}{3} = \frac{1}{3}\left(a^3 b + ab^3\right)$$

Problem 6:

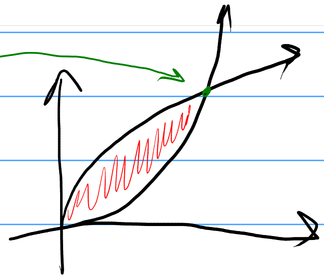
Evaluate the double integral by iteration

$$\iint_R xy^2\, dA$$

where R is the finite region in the first quadrant bounded by the curves $y = x^2$ and $x = y^2$

Answer: $\frac{1}{3}\left(\frac{2}{7} - \frac{1}{8}\right) = \frac{3}{56}$

Find the intercept



$$\begin{cases} y = x^2 \\ y^2 = x \Rightarrow y = \sqrt{x} \end{cases} \Rightarrow x^2 = \sqrt{x}$$

$$\Rightarrow x^4 = x \Rightarrow x^3 = 1 \Rightarrow x = 1$$

mister vi ikke en løsning?

$$\int_0^1 \int_{x^2}^{\sqrt{x}} xy^2\, dy\, dx = \int_0^1 \left[x \cdot \frac{y^3}{3} \right]_{x^2}^{\sqrt{x}} dx = \int_0^1 x \cdot \frac{(\sqrt{x})^3}{3} - x \cdot \frac{x^6}{3}\, dx$$

$$= \frac{1}{3} \cdot \int_0^1 \underbrace{x \cdot (\sqrt{x})^3}_{x^1 \cdot (x^{\frac{1}{2}})^3 = x^1 \cdot x^{\frac{3}{2}} = x^{\frac{5}{2}}} - x^7\, dx = \frac{1}{3} \int_0^1 x^{\frac{5}{2}} - x^7\, dx = \frac{1}{3} \left[\frac{x^{\frac{7}{2}}}{\frac{7}{2}} - \frac{x^8}{8} \right]_0^1 = \frac{1}{3} \left[\frac{2(\sqrt{x})^7}{7} - \frac{x^8}{8} \right]_0^1$$

$$= \frac{1}{3} \cdot \left(\frac{2 \cdot 1}{7} - \frac{1}{8} - 0 \right) = \frac{1}{3} \cdot \left(\frac{2}{7} - \frac{1}{8} \right) = \frac{2}{21} - \frac{1}{24} = \underline{\underline{\frac{3}{56}}}$$

Problem 7:

Evaluate the double integral by iteration ?.?.

$$\iint_D x \cos y \, dA$$

where D is the finite region in the first quadrant bounded by the coordinate axes and the curve $y = 1 - x^2$

Answer: $\frac{1-\cos(1)}{2}$

Problem 9:

Determine whether the given integral converges or diverges. Try to evaluate those that converge.

$$\iint_Q e^{-x-y}\, dA$$

where Q is the first quadrant of the xy-plane

Answer: $\left(\lim_{R\to\infty} (-e^{-x})\Big|_0^R\right)^2 = 1$ (converges)

$$\int_0^\infty\int_0^\infty e^{-x-y}\, dx\, dy = -\int_0^\infty\int_0^\infty e^{u}\, du\, dy = -\int_0^\infty \left[e^{-x-y}\right]_0^\infty dy$$

$u = -x-y \Rightarrow \frac{du}{dx} = -1 \Rightarrow dx = -1\, du$

$$= \lim_{a\to\infty}\left(-\int_0^\infty e^{-a-y} - e^{-0-y}\, dy\right) = \lim_{a\to\infty}\left(-\int_0^\infty e^{-a-y} - e^{-y}\, dy\right)$$

$$= \lim_{a\to\infty}\left(-\int_0^\infty e^{-a-y}\, dy + \int_0^\infty e^{-y}\, dy\right)$$

$$= \lim_{a\to\infty}\left(-\left[-e^{-a-y}\right]_0^\infty + \left[-e^{-y}\right]_0^\infty\right)$$

$$= \lim_{a\to\infty}\left(\left[e^{-a-y}\right]_0^\infty - \left[e^{-y}\right]_0^\infty\right)$$

$$= \lim_{\substack{a\to\infty\\ b\to\infty}}\left(e^{-a-b} - e^{-a} - \left(e^{-b} - e^{-0}\right)\right) = \lim_{\substack{a\to\infty\\ b\to\infty}}\left(\cancel{-e^{-a-b}} - \cancel{e^{-a}} - \cancel{e^{-b}} + 1\right)$$

$$= \underline{\underline{1}}$$

Problem 10:

Determine whether the given integral converges or diverges. Try to evaluate those that converge.

$$\iint_T \frac{dA}{x^2+y^2}$$

where T is the region satisfying $x \geq 1$ and $0 \leq y \leq x$

Answer: $\frac{\pi}{4}\int_1^\infty \frac{dx}{x} = \infty$ (The integral diverges to infinity.)

$$\int_1^\infty \int_0^x \frac{1}{x^2+y^2}\, dy\, dx = \int_1^\infty$$

find in table

$$u = x^2+y^2 \Rightarrow \frac{du}{dy} = 2y$$

Problem 11:

Evaluate $\iint_S (x+y)\,dA$, where S is the region in the first quadrant lying inside the disk $x^2 + y^2 \le a^2$ and under the line $y = \sqrt{3}x$.

Answer: $\frac{(\sqrt{3}+1)a^3}{6}$

Translate to polar coordinates

$$x = r\cdot\cos(\theta), \qquad y = r\cdot\sin(\theta)$$

$$\mathrm{dA} = r\,\mathrm{d}\theta\,\mathrm{dr}$$

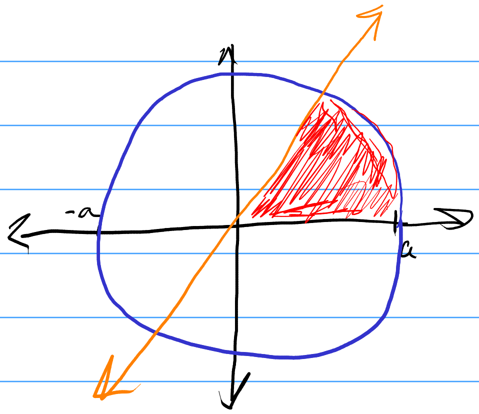
$$x^2+y^2 \le a^2 \Rightarrow 0 \le r \le a$$

$$y = \sqrt{3}\,x \Rightarrow \cancel{r}\cdot\sin(\theta) = \sqrt{3}\cdot\cancel{r}\cdot\cos(\theta)$$

$$\Rightarrow \sin(\theta) = \sqrt{3}\cdot\cos(\theta)$$

$$\Rightarrow \frac{\sin(\theta)}{\cos(\theta)} = \sqrt{3} \Rightarrow \tan(\theta) = \sqrt{3}$$

$$\Rightarrow \theta = \arctan(\sqrt{3}) = \frac{\pi}{3}$$



Integral in polar coordinates:

$$\int_0^a \int_0^{\frac{\pi}{3}} \left(r\cdot\cos(\theta) + r\cdot\sin(\theta)\right)\cdot r \;d\theta\; dr$$

$$= \int_0^a \int_0^{\frac{\pi}{3}} r^2\cdot\cos(\theta) + r^2\cdot\sin(\theta)\;d\theta\;dr = \int_0^a \Big[r^2\cdot\sin(\theta) - r^2\cdot\cos(\theta)\Big]_0^{\frac{\pi}{3}} dr$$

$$= \int_0^a r^2\cdot\sin\left(\tfrac{\pi}{3}\right) - r^2\cdot\cos\left(\tfrac{\pi}{3}\right) - \cancel{r^2\cdot\sin(0)} + r^2\cdot\cancel{\cos(0)}\;dr$$

$$= \int_0^a r^2\cdot\sin\left(\tfrac{\pi}{3}\right) - r^2\cdot\cos\left(\tfrac{\pi}{3}\right) + r^2\;dr = \left[\frac{r^3}{3}\cdot\sin\left(\tfrac{\pi}{3}\right) - \frac{r^3}{3}\cdot\cos\left(\tfrac{\pi}{3}\right) + \frac{r^3}{3}\right]_0^a$$

$$= \left[\frac{r^3}{3}\cdot\left(\sin\left(\tfrac{\pi}{3}\right) - \cos\left(\tfrac{\pi}{3}\right) + 1\right)\right]_0^a = \left[\frac{r^3}{3}\cdot\left(\frac{\sqrt{3}}{2} - \frac{1}{2} + 1\right)\right]_0^a$$

$$= \frac{a^3}{3} \cdot \left(\frac{\sqrt{3}+1}{2} \right) = \underline{\underline{\frac{a^3(\sqrt{3}+1)}{3}}}$$

Problem 12:

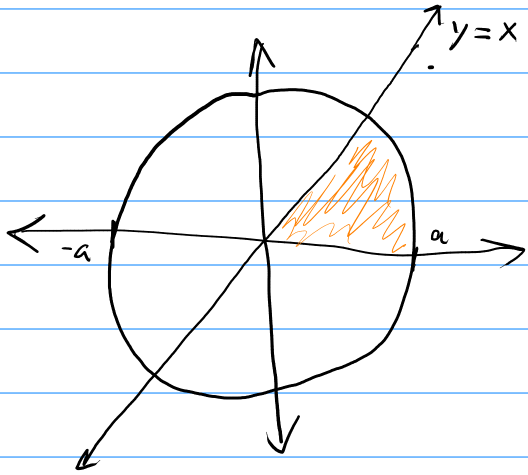
Evaluate $\iint_D xy\, dA$, where D is the region satisfying $x \geq 0$, $0 \leq y \leq x$, and $x^2 + y^2 \leq a^2$.

Answer: $\frac{a^4}{16}$

Convert to polar coordinates

$$x = r \cdot \cos(\theta), \qquad y = r \cdot \sin(\theta)$$

$$\mathrm{dA} = r\, \mathrm{d}\theta\, \mathrm{dr}$$



$$x^2 + y^2 \leq a^2 \Rightarrow 0 \leq r \leq a$$

$$0 \leq y \leq x \Rightarrow 0 \leq \cancel{r} \cdot \sin(\theta) \leq \cancel{r} \cdot \cos(\theta)$$

$$\Rightarrow 0 \leq \sin(\theta) \leq \cos(\theta) \Rightarrow 0 \leq \frac{\sin(\theta)}{\cos(\theta)} \leq 1 \Rightarrow 0 \leq \tan(\theta) \leq 1$$

$$\Rightarrow 0 \leq \theta \leq \arctan(1) \Rightarrow 0 \leq \theta \leq \frac{\pi}{4}$$

Integral in polar coordinates:

$$\int_0^a \int_0^{\frac{\pi}{4}} r \cdot \cos(\theta) \cdot r \cdot \sin(\theta) \cdot r \, d\theta \, dr = \int_0^a \int_0^{\frac{\pi}{4}} r^3 \cdot \cos(\theta) \cdot \sin(\theta) \, d\theta \, dr$$

$$= \int_0^a r^3 \int_0^{\frac{\pi}{4}} \cos(\theta) \cdot \sin(\theta) \, d\theta \, dr = \int_0^a r^3 \int_{\theta=0}^{\theta=\frac{\pi}{4}} u \cdot \cancel{\sin(\theta)} \cdot \frac{-1}{\cancel{\sin(\theta)}} \, du \, dr$$

$$\hookrightarrow u = \cos(\theta) \Rightarrow \frac{du}{dx} = -\sin(\theta) \Rightarrow dx = \frac{-1}{\sin(\theta)} du$$

$$\int_0^a r^3 \int_{\theta=0}^{\theta=\frac{\pi}{4}} -u \, du \, dr = \int_0^a r^3 \cdot \left[\frac{-u^2}{2} \right]_{\theta=0}^{\theta=\frac{\pi}{4}} dr = \int_0^a r^3 \cdot \left[\frac{-1}{2} \cos(\theta)^2 \right]_0^{\frac{\pi}{4}}$$

$$= \int_0^a r^3 \cdot \left(\frac{-1}{2} \overset{\frac{\sqrt{2}}{2}}{\cos\left(\frac{\pi}{4}\right)}^2 + \frac{1}{2} \overset{1}{\cos(0)}^2 \right) dr = \int_0^a r^3 \cdot \left(\frac{-1}{2} \cdot \left(\frac{\sqrt{2}}{2} \right)^2 + \frac{1}{2} \right) dr$$

$$\int_0^a r^3 \cdot \left(\frac{-1}{2} \cdot \frac{2}{4} + \frac{1}{2} \right) dr = \int_0^a r^3 \cdot \left(\frac{-1}{4} + \frac{2}{4} \right) dr = \int_0^a r^3 \cdot \frac{1}{4}\, dr$$

$$= \frac{1}{4} \cdot \int_0^a r^3\, dr = \frac{1}{4} \cdot \left[\frac{r^4}{4} \right]_0^a = \frac{1}{4} \cdot \frac{a^4}{4} = \underline{\underline{\frac{a^4}{4}}}$$