

**Problem 1:**calculate **div F** and **curl F** for the vector field  $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$ **Solution:**  $\text{div } \mathbf{F} = 0$ ,  $\text{curl } \mathbf{F} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$ 

$$\text{div } \vec{v}(x, y) = \nabla \cdot \vec{v}(x, y) = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y}$$

$$\text{curl } \vec{v}(x, y, z) = \nabla \times \vec{v}(x, y, z) = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} P(x, y, z) \\ Q(x, y, z) \\ R(x, y, z) \end{bmatrix} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix}$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} y \\ z \\ x \end{bmatrix} = \frac{\partial}{\partial x} y + \frac{\partial}{\partial y} z + \frac{\partial}{\partial z} x = 0 + 0 + 0 = 0$$

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\det(M) = |M| = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\text{curl } \vec{F} = \nabla \times \vec{F}(x, y, z) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= \hat{i} \cdot \left( \frac{\partial}{\partial y} x - \frac{\partial}{\partial z} z \right) - \hat{j} \cdot \left( \frac{\partial}{\partial x} x - \frac{\partial}{\partial z} y \right) + \hat{k} \cdot \left( \frac{\partial}{\partial x} z - \frac{\partial}{\partial y} y \right)$$

$$= \hat{i} \cdot (0 - 1) - \hat{j} \cdot (1 - 0) + \hat{k} \cdot (0 - 1) = \underline{\underline{-\hat{i} - \hat{j} - \hat{k}}}$$

**Problem 2:**calculate **div F** and **curl F** for the vector field  $\mathbf{F} = x\mathbf{i} + x\mathbf{k}$ **Solution:**  $\text{div } \mathbf{F} = 1$ ,  $\text{curl } \mathbf{F} = -\mathbf{j}$ 

$$\text{curl } \vec{v}(x, y) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}, \text{ where } \vec{v}(x, y) = \begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix}$$

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\det(M) = |M| = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} x = 1$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 0 & x \end{vmatrix} = \hat{i} \cdot \left( \frac{\partial}{\partial y} x - \frac{\partial}{\partial z} 0 \right) - \hat{j} \cdot \left( \frac{\partial}{\partial x} x - \frac{\partial}{\partial z} x \right) + \hat{k} \cdot \left( \frac{\partial}{\partial x} 0 - \frac{\partial}{\partial y} x \right)$$

$$= \hat{i} \cdot (0 - 0) - \hat{j} \cdot (1 - 0) + \hat{k} \cdot (0 - 0) = -\hat{j}$$

Evaluate  $\oint_c (\sin x + 3y^2) dx + (2x - e^{-y^2}) dy$ , where  $c$  is the boundary of the half-disk  $x^2 + y^2 \leq a^2, y \geq 0$ , oriented counterclockwise.

$$\oint_C \vec{F}(x, y) \, d\vec{r} = \oint_C P(x, y) dx + Q(x, y) dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA$$

$$Q(x, y) = 2x - e^{-y^2}$$

$$\frac{dQ}{dx} = 2$$

$$\Rightarrow \iint_R 2-6y \, dA$$

$$\iint_R (z - b \cdot r \cdot \sin(\varphi)) \cdot r \, dr \, d\varphi$$

$$0 = c..a$$

$$= \int_0^{\pi} \int_0^a 2r \cdot b \cdot r^2 \sin(\theta) \, dr \, d\theta$$

$$= 2 \int_0^\pi \int_0^a r \, dr \, d\theta - 6 \int_0^\pi \int_0^a r^3 \cdot \sin(\theta) \, dr \, d\theta$$

$$= 2 \cdot \int_0^{\pi} \left[ \frac{r^2}{2} \right]_0^a d\theta - b \cdot \int_0^{\pi} \sin(\theta) \cdot \left[ \frac{r^3}{3} \right]_0^a d\theta = 2 \cdot \int_0^{\pi} \frac{a^2}{2} d\theta - b \cdot \int_0^{\pi} \sin(\theta) \cdot \frac{1}{3} \cdot a^3 d\theta$$

$$= a^2 \left[ \theta \right]_0^\pi - 2a^3 \left[ -\cos(\theta) \right]_0^\pi = \pi a^2 - 2a^3 (-0 - (-1)) = \pi a^2 - 2a^3$$

**Problem 5:**

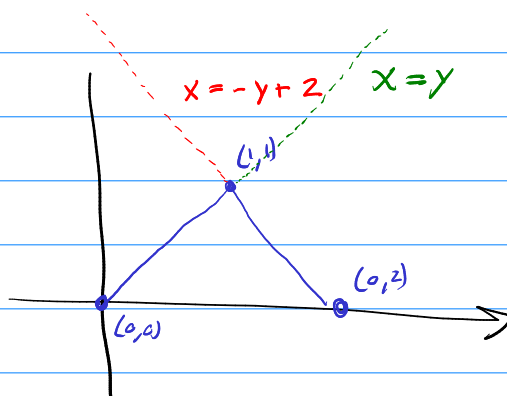
Evaluate  $\oint_C (x^2 - xy) dx + (xy - y^2) dy$ , clockwise around the triangle with vertices  $(0,0)$ ,  $(1,1)$ , and  $(2,0)$ .

**Solution:**  $-\frac{4}{3}$

$$\oint_C \vec{F}(x,y) d\vec{r} = \oint_C P(x,y) dx + Q(x,y) dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$$

$$= - \iint_R \frac{d}{dx} (xy - y^2) - \frac{d}{dy} (x^2 - xy) dA$$

$$= - \iint_R y - (-x) dA = \iint_R y + x dA$$



$$- \int_0^1 \int_y^{-y+2} y + x dx dy = - \int_0^1 \int_y^{-y+2} y dx dy - \int_0^1 \int_y^{-y+2} x dx dy$$

$$= - \int_0^1 [xy]_y^{-y+2} dy - \int_0^1 \left[ \frac{x^2}{2} \right]_y^{-y+2} dy$$

$$= - \int_0^1 ((-y+2)y) - (y \cdot y) dy - \frac{1}{2} \int_0^1 (-y+2)^2 - y^2 dy$$

$$= - \int_0^1 -y^2 + 2y - y^2 dy - \frac{1}{2} \int_0^1 \cancel{y^2} - 4y + 4 - \cancel{y^2} dy$$

$$= - \int_0^1 2y - 2y^2 dy - \frac{1}{2} \int_0^1 -4y + 4 dy$$

$$= 2 \int_0^1 y^2 + 2y dy + 2 \int_0^1 y - 1 dy$$

$$= 2 \cdot \left[ \frac{1}{3} y^3 + y^2 \right]_0^1 - 2 \left[ \frac{y^2}{2} - y \right]_0^1 = 2 \cdot \left( \frac{1}{3} + 1 \right) - 2 \cdot \left( \frac{1}{2} - 1 \right) = \frac{2}{3} + 2 - 1 + 2$$

$$= \frac{8}{3}$$

**Problem 6:**

Use a line integral to find the plane area enclosed by the curve  $r = a \cos^3 t \mathbf{i} + b \sin^3 t \mathbf{j}$ ,  $(0 \leq t \leq 2\pi)$ .

**Solution:**  $\frac{3\pi ab}{8}$

$$r = a \cdot \cos(t)^3 i + b \cdot \sin(t)^3 j$$

**Problem 7:**

Use the Divergence Theorem to calculate the flux of the given vector field out of the sphere  $\mathcal{S}$  with equation  $x^2 + y^2 + z^2 = a^2$ , where  $a > 0$

$$\mathbf{F} = x\mathbf{i} - 2y\mathbf{j} + 4z\mathbf{k}$$

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iiint_V \operatorname{div} \vec{F} \, dV$$

$$\begin{aligned} \iiint_V \operatorname{div} \vec{F} \, dV &= \iiint_V \nabla \cdot \vec{F} \, dV = \iiint_V \frac{d}{dx} x + \frac{d}{dy} (-2y) + \frac{d}{dz} 4z \, dV \\ &= \iiint_V 1 - 2 + 4 \, dV = \iiint_V 3 \, dV \end{aligned}$$

Convert to spherical coordinates

$$x^2 + y^2 + z^2 = a^2 \Rightarrow \rho = 0 \dots a$$

$$\begin{aligned} x &= \rho \cdot \sin \phi \cdot \cos \theta \\ y &= \rho \cdot \sin \phi \cdot \sin \theta \\ z &= \rho \cdot \cos \phi \\ dV &= \rho \cdot \sin \phi \, d\rho \, d\phi \, d\theta \end{aligned}$$

$$\begin{aligned} \iiint_V 3 \, dV &\Rightarrow 3 \cdot \iiint \rho^2 \sin(\phi) \, d\theta \, d\phi \, d\rho \\ &= 3 \cdot \iiint \rho^2 \sin(\phi) \, d\theta \, d\phi \, d\rho \end{aligned}$$

We can now assign limits

$$= 3 \cdot \int_0^a \rho^2 \int_0^\pi \sin(\phi) \int_0^{2\pi} 1 \, d\theta \, d\phi \, d\rho$$

$$= 3 \cdot \int_0^a \rho^2 \int_0^\pi \sin(\phi) \cdot 2\pi \, d\phi \, d\rho$$

$$= 6\pi \cdot \int_0^a \rho^2 \left[ -\cos(\phi) \right]_0^\pi d\rho = 6\pi \cdot \int_0^a \rho \cdot (-(-1) - (-1)) \, d\rho$$

$$= 12\pi \int_0^a \rho^2 \, d\rho = 12\pi \cdot \left[ \frac{\rho^3}{3} \right]_0^a = 12\pi \cdot \frac{a^3}{3} = \underline{\underline{4\pi a^3}}$$