Betragt følgende Butterworth 3. ordens lavpasfilter (frekvensnormeret filter)

$$\tilde{H}(s) = \frac{1}{s+1}\frac{1}{s^2+s+1}$$

og design et tilsvarende digitalt lavpasfilter med afskæringsfrekvens på 1 kHz og samplefrekvens på 8 kHz. Design filtret som følger

1. Design et digitalt lavpasfilter $H_1(z)$ ved brug af matched z-transformation.
2. Design et digitalt lavpasfilter $H_2(z)$ ved brug af impuls invariant z-transformation.
3. Benyt MATLAB til at sammenligne Bode plot for denormerede lavpasfilter $H(s)$, $H_1(z)$ og $H_2(z)$.
4. Benyt MATLAB til at sammenligne impulsresponserne for $H(s)$, $H_1(z)$ og $H_2(z)$.

Denormerer

$$\omega_a = 2\pi \cdot f_a = 2000\pi$$

$$H(s) = \tilde{H}(s)\Big|_{s=\frac{s}{\omega_a}} = \frac{1}{\frac{s}{2000\pi}+1}\cdot\frac{1}{\left(\frac{s}{2000\pi}\right)^2+\frac{s}{2000\pi}+1} = \frac{2000\pi}{s+2000\pi}\cdot\frac{(2000\pi)^2}{s^2+2000\pi s+(2000\pi)^2}$$

Finder $\underbrace{\text{poler}}_{p}$ og $\underbrace{\text{nulpunkter}}_{\text{ingen}}$

$$s+2000\pi = 0 \Rightarrow p_1 = -2000\pi$$

$$s^2+2000\pi s+(2000\pi)^2 = 0 \Rightarrow \begin{cases} p_2 = -3141-5441i \\ p_2 = -3141+5441i \end{cases}$$

Overfør poler til z-domæne

$$\boxed{z = e^{sT}} \qquad T = 8000^{-1}$$

$$\boxed{\text{Define: } a = 0{,}5249,\ b = 0{,}4247}$$

$$p_1' = e^{-2000\pi\cdot 8000^{-1}} = e^{-\frac{\pi}{4}} \approx 0{,}4559$$

$$p_2' = e^{(-3141-5441i)\cdot 8000^{-1}} \approx 0{,}5249-0{,}4247i = a-bi$$

$$p_3' = e^{(-3141+5441i)\cdot 8000^{-1}} \approx 0{,}5249+0{,}4247i = a+bi$$

Opskriv overføringsfunktion i z-domæne

$$H(z) = \frac{k}{(z-p_1')(z-p_2')(z-p_3')} = \frac{k}{(z-0{,}4559)\underbrace{(z-(a-bi))(z-(a+bi))}}$$

$$(z-(a-bi))(z-(a+bi)) = z^2-(a-bi)z-(a+bi)z+(a-bi)(a+bi)$$

$$= z^2-az\cancel{+bzi}-az\cancel{-bzi}+a^2+b^2 = z^2-2az+a^2+b^2$$

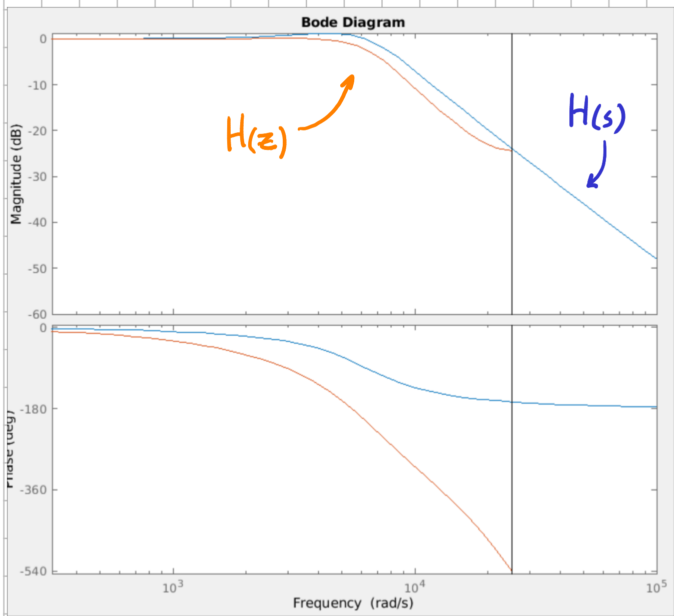
$$= \frac{k}{(z-0{,}4559)(z^2-2az+a^2+b^2)} = \frac{k}{(z-0{,}4559)(z^2-1{,}049z+0{,}4559)}$$

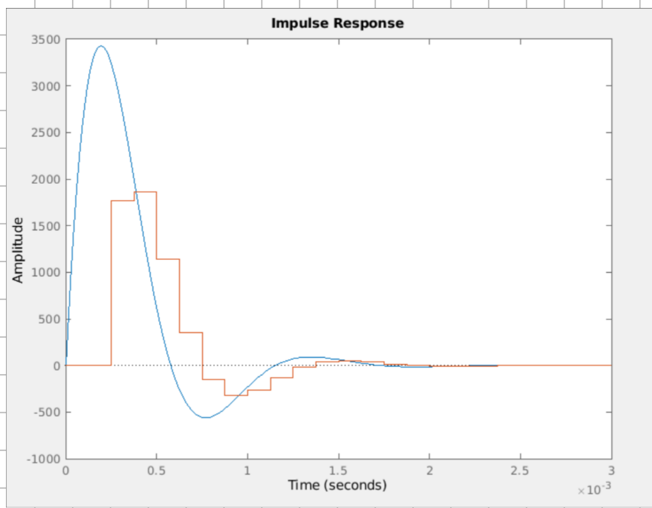
Lav konstant forstærkning

$$H(s)\big|_{s=0} = H(z)\big|_{z=1} \Rightarrow \frac{\cancel{2000\pi}}{0+\cancel{2000\pi}} \cdot \frac{\cancel{(2000\pi)^2}}{0+0+\cancel{(2000\pi)^2}} = \frac{k}{(1-0{,}4559)(1-1{,}049+0{,}4559)}$$

$$\Rightarrow 1 = \frac{k}{(1-0{,}4559)(1-1{,}049+0{,}4559)} = \frac{k}{0{,}2214} \Rightarrow k = 0{,}2214$$

$$H(z) = \frac{0{,}2214}{(z-0{,}4559)(z^2-1{,}049z+0{,}4559)}$$





(2) $$\tilde{H}(s) = \frac{1}{s+1}\frac{1}{s^2+s+1}$$

Partialbrøkopløs

$$\tilde{H}(s) = \left(\frac{1}{s+1}\cdot\frac{1}{s^2+s+1} = \frac{A}{s+1} + \frac{Bs+C}{s^2+s+1}\right)\cdot(s+1)(s^2+s+1)$$

$$\Rightarrow 1 = A(s^2+s+1) + (Bs+C)(s+1)$$

$$\Rightarrow 1 = As^2 + As + A + Bs^2 + Bs + Cs + C$$

Samle kostanter

$$\begin{aligned} s^2&: \quad 0 = A + B \\ s&: \quad 0 = A + B + C \\ 1&: \quad 1 = A + C \end{aligned} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right]\cdot(-1)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right] \Rightarrow \begin{cases} A = 1 \\ B = -1 \\ C = 0 \end{cases}$$

$$\Rightarrow \tilde{H}(s) = \frac{A}{s+1} + \frac{Bs+C}{s^2+s+1} = \frac{1}{s+1} + \frac{-s}{s^2+s+1}$$

Denormer overføringsfunktionen

$$H(s) = \tilde{H}(s)\Big|_{s=\frac{s}{\omega_a}} = \frac{1}{\frac{s}{2000\pi}+1} + \frac{\frac{-s}{2000\pi}}{\left(\frac{s}{2000\pi}\right)^2 + \frac{s}{2000\pi} + 1} = \overbrace{\frac{2000\pi}{s+2000\pi}}^{H_1(s)} + \overbrace{\frac{-2000\pi\cdot s}{s^2 + 2000\pi\cdot s + (2000\pi)^2}}^{H_2(s)}$$

Find konstanter og z-transformer

$$H_1(s) = \frac{2000\pi}{s+2000\pi} \Rightarrow \begin{cases} k_1 = 2000\pi \\ s_1 = -2000\pi \end{cases} \Rightarrow H_1(z) = \frac{1}{8000}\cdot 2000\pi\cdot\frac{1}{1-e^{-2000\pi\cdot\frac{1}{8000}}\cdot z^{-1}} = \frac{\pi}{4}\cdot\frac{1}{1-e^{\frac{-\pi}{4}}\cdot z^{-1}}$$

$$H_2(s) = \frac{-2000\pi\cdot s}{s^2+2000\pi\cdot s+(2000\pi)^2} = \frac{-2000\pi\cdot s}{(s-1000\pi+1000\pi\sqrt{3}\,i)\cdot(s-1000\pi-1000\pi\sqrt{3}\,i)}$$

Definition: $s_2 = 1000\pi - 1000\pi\sqrt{3}\,i$

Partialbrøkopløs

$$H_2(s) = \frac{-2000\pi\cdot s}{(s-s_2)\cdot(s-s_2^*)} = \frac{k_2}{s-s_2} + \frac{k_2^*}{s-s_2^*}$$

$$\Rightarrow k_2 = (s-s_2)\cdot H_2(s)\Big|_{s=s_2} = \frac{-2000\pi s}{s-s_2^*}\Big|_{s=s_2} = \frac{-2000\pi\cdot(1000\pi-1000\pi\sqrt{3}\,i)}{(1000\pi-1000\pi\sqrt{3}\,i)-(1000\pi+1000\pi\sqrt{3}\,i)}$$

$$= \frac{\cancel{-2000\pi}\cdot(1000\pi-1000\pi\sqrt{3}\,i)}{\cancel{-2000\pi}\sqrt{3}\,i} = \frac{1000\pi-1000\pi\sqrt{3}\,i}{\sqrt{3}i}\cdot\frac{i}{i} = \frac{1000\pi i+1000\pi\sqrt{3}}{-\sqrt{3}} = -1000\pi - 1000\pi\frac{1}{\sqrt{3}}i$$

$$\begin{cases} k_2 = -1000\pi - 1000\pi \frac{1}{\sqrt{3}} i \\ s_2 = 1000\pi - 1000\pi\sqrt{3}\, i \end{cases} \Rightarrow \begin{cases} \alpha_2 = -1000\pi \\ \beta_2 = -1000\pi \frac{1}{\sqrt{3}} \\ \sigma_2 = 1000\pi \\ \omega_2 = -1000\pi\sqrt{3} \end{cases}$$

$$\boxed{f_s = 8000 \Rightarrow T = \frac{1}{8000}}$$

$$\sigma_2 T = 1000\pi \cdot \frac{1}{8000} = \frac{\pi}{8}$$

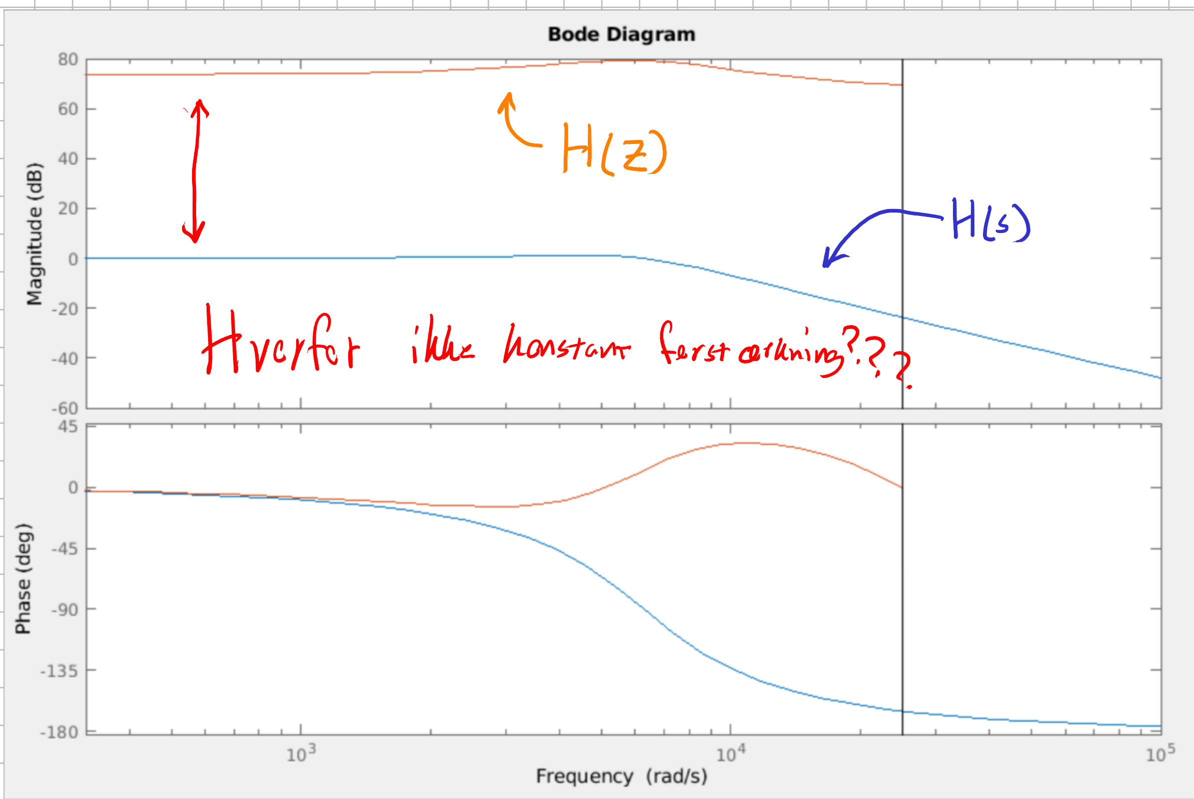
$$\omega_2 T = -1000\pi\sqrt{3} \cdot \frac{1}{8000} = \frac{-\pi\sqrt{3}}{8}$$

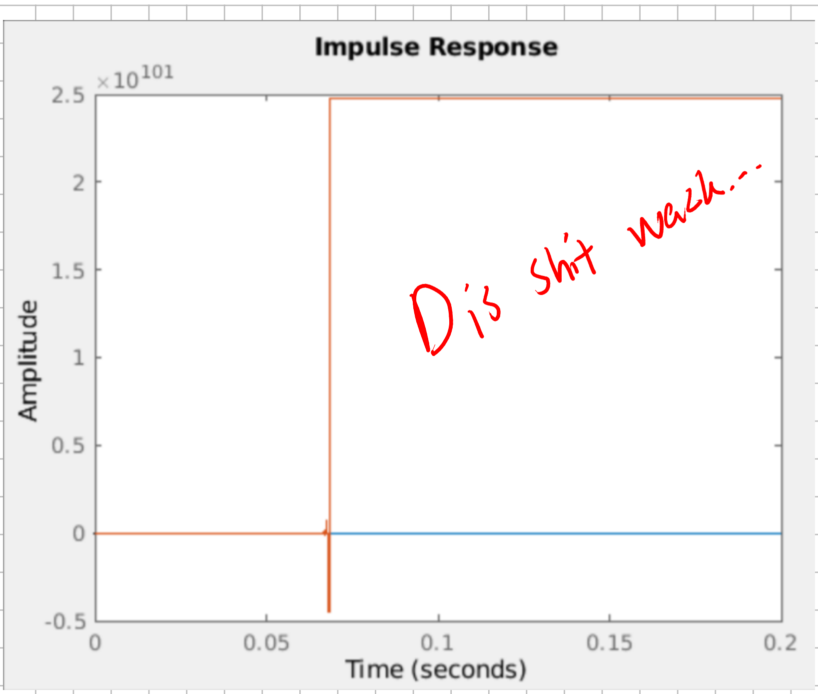
$$\Rightarrow \begin{cases} a_0 = 2\alpha_2 \\ a_1 = -2e^{\sigma_2 T}\left(\alpha_2 \cos(\omega_2 T) - \beta \cdot \sin(\omega T)\right) \\ b_1 = -\left(2e^{\sigma_2 T} \cdot \cos(\omega_2 T)\right) \\ b_2 = e^{2\sigma_2 T} \end{cases} \Rightarrow \begin{cases} a0 \to -2000\,\pi \\ a1 \to -2\, e^{\pi/8} \left(-1000\,\pi \operatorname{Cos}\left[\frac{\sqrt{3}\,\pi}{8}\right] - \frac{1000\,\pi \operatorname{Sin}\left[\frac{\sqrt{3}\,\pi}{8}\right]}{\sqrt{3}}\right) \\ b1 \to -2\, e^{\pi/8} \operatorname{Cos}\left[\frac{\sqrt{3}\,\pi}{8}\right] \\ b2 \to e^{\pi/4} \end{cases}$$

$$\approx \begin{cases} a0 \to -6283.19 \\ a1 \to 10613.3 \\ b1 \to -2.3028 \\ b2 \to 2.19328 \end{cases} \Rightarrow H_2(z) \approx \frac{-6283 + 10613 z^{-1}}{1 - 2{,}3028 z^{-1} + 2{,}1933 z^{-2}}$$

Den samplede overførinsfunktion i z-domænet!

$$\Rightarrow H(z) = H_1(z) + H_2(z) = \underline{\underline{\frac{\pi}{4} \cdot \frac{1}{1 - e^{-\frac{\pi}{4}} \cdot z^{-1}} + \frac{-6283 + 10613 z^{-1}}{1 - 2{,}3028 z^{-1} + 2{,}1933 z^{-2}}}}$$




Impulse Response
×10^101
2.5
2
1.5
1
0.5
0
-0.5
Amplitude
0
0.05
0.1
0.15
0.2
Time (seconds)
D is shit weak...