Lecture 1: Euler-Lagrange Modelling

Underactuated Robotics

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Agenda

Knowledge:

- Derive dynamical state-space models of robots as control systems
- Analyze the stability of low dimensional linear and nonlinear systems
- Analyze the observability and controllability of linear control systems
- Use a variety of controllers for underactuated robots

Skills:

- Implement simulations of control systems in software
- Create concise technical reports presenting solutions to proposed problems

Competencies:

- Choose appropriate modern control techniques to solve control problems in robotics
- Apply modern control techniques to control simulated underactuated robots

- **Lesson 1**: Euler-Lagrange Modelling
- **Lesson 2**: Simulation of Robot Dynamics
- **Lesson 3**: Modelling and Simulation of BB8 Robot
- **Lesson 4**: Stability Analysis
- **Lesson 5**: Optimal Control
- **Lesson 6**: Feedback Linearisation
- **Lesson 7**: Energy Shaping Control
- **Lesson 8**: Simulation and Implementation of Control Systems
- **Lesson 9**: Sliding Mode Control
- **Lesson 10**: Help with hand-in
- **Lesson 11**: Help with hand-in
- **Lesson 12**: Help with hand-in

Euler-Lagrange modelling

Hamilton's Principle

The motion of a mechanical system from time a to b is such that the integral

$$I(t, q, \dot{q}) = \int_a^b \mathcal{L}(t, q, \dot{q})dt,$$

where $\mathcal{L} = E_{\text{kin}} - E_{\text{pot}}$ has a stationary value. The function $\mathcal{L}$ is called the **Lagrangian**.

Euler-Lagrange modelling can be used for finding the equations of motion of e.g. mechanical systems using the system's potential energy E_{pot} and kinetic energy E_{kin}.

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Generalized coordinates will most often be positions and/or angles of a mechanical system.

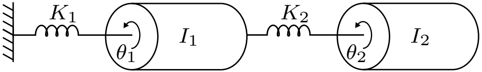
If q is a trajectory of a conservative mechanical system then

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = 0$$

where q is an n-dimensional vector of generalized coordinates and $\mathcal{L}$ is the *Lagrangian* given by

$$\mathcal{L} = E_{\text{kin}} - E_{\text{pot}} \quad \text{[J]}$$

where E_{pot} is the system's potential energy and E_{kin} is the system's kinetic energy.



The rotational mass-spring system has dynamics given by

$$I_1\ddot{\theta}_1 = -K_1\theta_1 - K_2(\theta_1 - \theta_2) \qquad \text{[Nm]}$$
$$I_2\ddot{\theta}_2 = -K_2(\theta_2 - \theta_1) \qquad \text{[Nm]}$$

where I_1, I_2 are moments of inertia [kgm^2] and K_1, K_2 are stiffnesses [N/rad].

The potential and kinetic energies are

$$E_{\text{pot}} = \frac{1}{2}K_1\theta_1^2 + \frac{1}{2}K_2(\theta_1 - \theta_2)^2$$
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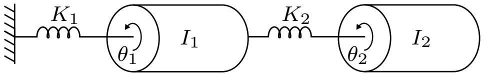
Euler-Lagrange modelling

Example: Rotational Mass-Spring System



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Euler-Lagrange modelling

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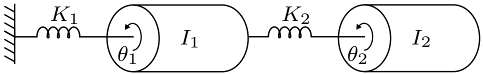

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This can be written as

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This extension is called ***Lagrange–D'Alembert's Principle***.

If q is a trajectory of a mechanical system that is affected by a generalized force Q then

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = \boldsymbol{Q}$$

where Q is an n-dimensional vector of generalized forces. ***Lagrange–D'Alembert's Principle*** can be written as (for $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)$)

$$\begin{aligned}
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} &- \frac{\partial \mathcal{L}}{\partial q_1} &= Q_1 \\
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2} &- \frac{\partial \mathcal{L}}{\partial q_2} &= Q_2 \\
\vdots & \quad\ \vdots & \vdots \\
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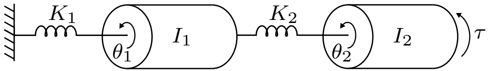
Euler-Lagrange Modelling

Example: Rotational Mass-Spring System with External Force



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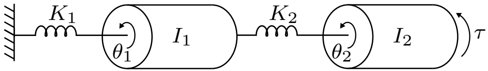

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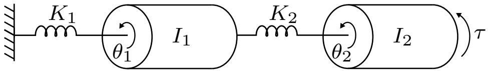

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If q is a trajectory of a mechanical system that is affected by a generalized force Q then

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = Q$$

where Q is an n-dimensional vector of generalized forces and $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)$ is the generalized coordinate and the Lagrangian is given by

$$\mathcal{L}(q, \dot{q}) = E_{\text{kin}}(q, \dot{q}) - E_{\text{pot}}(q).$$

Recall from Lecture 1 that

$$E_{\text{pot}}(\boldsymbol{q}) = -\sum_{i=1}^{n} m_{l_i} \boldsymbol{g}_0^T \boldsymbol{p}_{l_i}(\boldsymbol{q}) \qquad \text{[J]}$$

where m_{l_i} is the mass of Link i [kg], $\boldsymbol{g}_0$ is the gravitational acceleration in Base Frame [m/s^2] and $\boldsymbol{p}_{l_i}(\boldsymbol{q})$ is the position of the center of mass of Link i in Base Frame [m]; and

$$E_{\text{kin}}(q, \dot{q}) = \frac{1}{2}\dot{q}^T B(q)\dot{q} \qquad \text{[J]}$$

where $B(q)$ is the inertia tensor in Base Frame.

From Lagrange–D'Alembert's Principle, it is seen that

$$\frac{d}{dt}\frac{\partial E_{\text{kin}}}{\partial \dot{q}} - \frac{\partial E_{\text{kin}}}{\partial q} + \frac{\partial E_{\text{pot}}}{\partial q} = Q$$

where

$$\frac{\partial E_{\text{pot}}}{\partial q_i} = -\sum_{i=1}^{n} m_{l_i} \boldsymbol{g}_0^T \underbrace{\frac{\partial \boldsymbol{p}_{l_i}(\boldsymbol{q})}{\partial q_i}}_{=J_{P_i}^{l_i}}$$

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We define

$$g(q) = \begin{bmatrix} \frac{\partial E_{\text{pot}}}{\partial q_1} & \frac{\partial E_{\text{pot}}}{\partial q_2} & \cdots & \frac{\partial E_{\text{pot}}}{\partial q_n} \end{bmatrix}^T$$

The dynamical equation

$$\frac{d}{dt}\frac{\partial E_{\text{kin}}}{\partial \dot{q}} - \frac{\partial E_{\text{kin}}}{\partial q} + g(q) = Q$$

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This leads to

$$B(q)\ddot{q} + \dot{B}(q)\dot{q} - \frac{\partial E_{\mathsf{kin}}}{\partial q} + g(q) = Q$$

The final two terms of

$$B(q)\ddot{q} + \dot{B}(q)\dot{q} - \frac{\partial E_{\text{kin}}}{\partial q} + g(q) = Q$$

can be written as (the chain rule has been applied)

$$(\dot{B}(q)\dot{q})_i = \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial b_{ij}}{\partial q_k}\dot{q}_k\dot{q}_j$$

and

$$\frac{\partial E_{\text{kin}}}{\partial q_i} = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial b_{jk}}{\partial q_i}\dot{q}_k\dot{q}_j$$

The robot model given by the Euler-Lagrange equation can be formulated as

$$B(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau$$

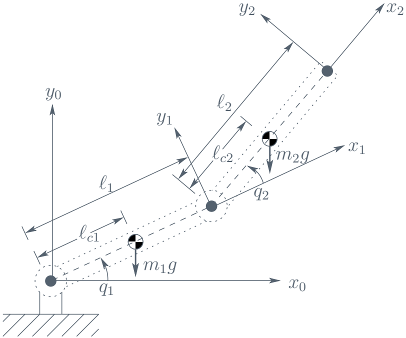
where $B(q)$ is the inertia tensor, $C(q, \dot{q})$ is a matrix containing Coriolis and centrifugal terms, $g(q)$ is the gravity vector, and τ is the actuator torque.

Robot with Two Joints

DH Parameters

The DH parameters for the robot are given in the following table.

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1
2	a_2	0	0	θ_2



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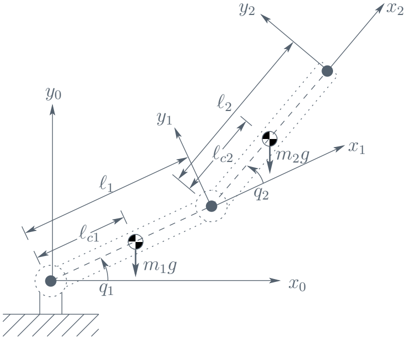
Each coordinate transformation is given by

$$A_i^{i-1}(\theta_i) = \begin{bmatrix} c_i & -s_i & 0 & a_i c_i \\ s_i & c_i & 0 & a_i s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where c_i (s_i) denotes $\cos(\theta_i)$ $(\sin(\theta_i))$ and c_{ij} (s_{ij}) denotes $\cos(\theta_i + \theta_j)$ $(\sin(\theta_i + \theta_j))$.

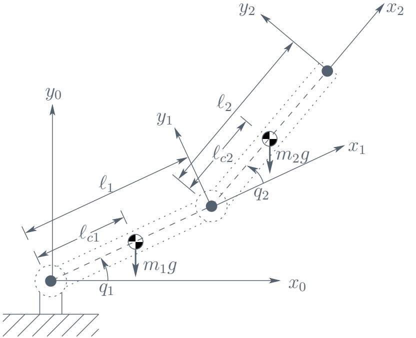


Robot with Two Joints
Center of Mass

The center of mass for Link 1 in Frame 0 is

$$\begin{bmatrix} \boldsymbol{p}_{l_1} \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} c_1 & -s_1 & 0 & l_1 c_1 \\ s_1 & c_1 & 0 & l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{=A_1^0} \begin{bmatrix} -l_1 + l_{c1} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{c1} c_1 \\ l_{c1} s_1 \\ 0 \\ 1 \end{bmatrix}$$



The center of mass for Link 2 in Frame 0 is

$$
\begin{bmatrix} \boldsymbol{p}_{l_2} \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} c_{12} & -s_{12} & 0 & l_1 c_1 + l_2 c_{12} \\ s_{12} & c_{12} & 0 & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{= A_1^0 A_2^1} \begin{bmatrix} -l_2 + l_{c2} \\ 0 \\ 0 \\ 1 \end{bmatrix}
$$

$$
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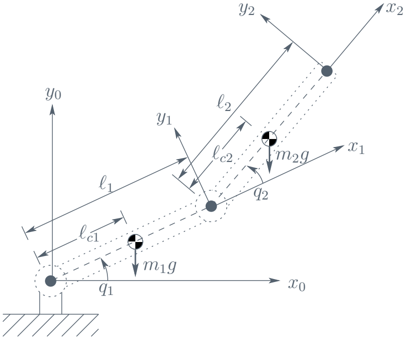

The Jacobian can be used for expressing the velocities of the center of mass of Link i as

$$\dot{\boldsymbol{p}}_{l_i} = J_P^{l_i}\dot{\boldsymbol{q}}$$
$$\boldsymbol{\omega}_i = J_O^{l_i}\dot{\boldsymbol{q}}$$

where

$$J_P^{l_i} = \begin{bmatrix}\boldsymbol{J}_{P1}^{l_i} & \boldsymbol{J}_{P2}^{l_i} & \dots & \boldsymbol{J}_{Pi}^{l_i} & \mathbf{0} & \dots & \mathbf{0}\end{bmatrix}$$
$$J_O^{l_i} = \begin{bmatrix}\boldsymbol{J}_{O1}^{l_i} & \boldsymbol{J}_{O2}^{l_i} & \dots & \boldsymbol{J}_{Oi}^{l_i} & \mathbf{0} & \dots & \mathbf{0}\end{bmatrix}$$

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For a revolute joint it is

$$\boldsymbol{J}_{Pj}^{l_i} = \boldsymbol{z}_{j-1} \times (\boldsymbol{p}_{l_i} - \boldsymbol{p}_{j-1}) \qquad \text{and} \qquad \boldsymbol{J}_{Oj}^{l_i} = \boldsymbol{z}_{j-1}$$

where $\boldsymbol{p}_{j-1}$ is the position vector to the origin of Frame $j-1$ and $\boldsymbol{z}_{j-1}$ is a unit vector in the direction of the z-axis of Frame $j-1$.

For Link 1 we obtain

$$\dot{\boldsymbol{p}}_{l_1} = J_P^{l_1}\dot{\boldsymbol{q}} \qquad \text{and} \qquad \boldsymbol{\omega}_1 = J_O^{l_1}\dot{\boldsymbol{q}}$$

where

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This implies that

$$J_P^{l_1} = \begin{bmatrix} \boldsymbol{J}_{P1}^{l_1} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} l_{c1}c_1 \\ l_{c1}s_1 \\ 0 \end{bmatrix} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} -l_{c1}s_1 & 0 \\ l_{c1}c_1 & 0 \\ 0 & 0 \end{bmatrix}$$

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For Link 2 haves

$$\dot{\boldsymbol{p}}_{l_2} = J_P^{l_2}\dot{\boldsymbol{q}} \qquad \text{and} \qquad \boldsymbol{\omega}_2 = J_O^{l_2}\dot{\boldsymbol{q}}$$

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$$\begin{aligned} J_P^{l_2} &= \begin{bmatrix} \boldsymbol{J}_{P1}^{l_2} & \boldsymbol{J}_{P2}^{l_2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{z}_0 \times (\boldsymbol{p}_{l_2} - \boldsymbol{p}_0) & \boldsymbol{z}_1 \times (\boldsymbol{p}_{l_2} - \boldsymbol{p}_1) \end{bmatrix} \\ J_O^{l_2} &= \begin{bmatrix} \boldsymbol{J}_{O1}^{l_2} & \boldsymbol{J}_{O2}^{l_2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{z}_0 & \boldsymbol{z}_1 \end{bmatrix} \end{aligned}$$

For Link 2 haves

$$\dot{\boldsymbol{p}}_{l_2} = J_P^{l_2}\dot{\boldsymbol{q}} \qquad \text{and} \qquad \boldsymbol{\omega}_2 = J_O^{l_2}\dot{\boldsymbol{q}}$$

where

$$J_P^{l_2} = \begin{bmatrix}\boldsymbol{J}_{P1}^{l_2} & \boldsymbol{J}_{P2}^{l_2}\end{bmatrix} = \begin{bmatrix}\boldsymbol{z}_0 \times (\boldsymbol{p}_{l_2} - \boldsymbol{p}_0) & \boldsymbol{z}_1 \times (\boldsymbol{p}_{l_2} - \boldsymbol{p}_1)\end{bmatrix}$$
$$J_O^{l_2} = \begin{bmatrix}\boldsymbol{J}_{O1}^{l_2} & \boldsymbol{J}_{O2}^{l_2}\end{bmatrix} = \begin{bmatrix}\boldsymbol{z}_0 & \boldsymbol{z}_1\end{bmatrix}$$

This implies that

$$J_P^{l_2} = \left[\begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}l_1c_1 + l_{c2}c_{12}\\ l_1s_1 + l_{c2}s_{12}\\ 0\end{bmatrix} \quad \begin{bmatrix}0\\0\\1\end{bmatrix} \times \left(\begin{bmatrix}l_1c_1 + l_{c2}c_{12}\\ l_1s_1 + l_{c2}s_{12}\\ 0\end{bmatrix} - \begin{bmatrix}l_1c_1\\ l_1s_1\\ 0\end{bmatrix}\right)\right]$$
$$= \begin{bmatrix}-l_1s_1 - l_{c2}s_{12} & -l_{c2}s_{12}\\ l_1c_1 + l_{c2}c_{12} & l_{c2}c_{12}\\ 0 & 0\end{bmatrix} \qquad \text{and } J_O^{l_2} = \begin{bmatrix}0 & 0\\ 0 & 0\\ 1 & 1\end{bmatrix}$$

The potential energy should be expressed in an inertial frame e.g. the base frame, which does not accelerate. Then the potential energy can be computed as

$$E_{\text{pot}}(\boldsymbol{q}) = \sum_{i=1}^{n} E_{\text{pot},l_i}(\boldsymbol{q}) \qquad \text{[J]}$$

where E_{pot,l_i} is the potential energy for Link i [J].

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The total potential energy becomes

$$E_{\text{pot}}(\boldsymbol{q}) = -\sum_{i=1}^{n} m_{l_i} \boldsymbol{g}_0^T \boldsymbol{p}_{l_i}(\boldsymbol{q}) \qquad \text{[J]}$$

where m_{l_i} is the mass of Link i [kg], $\boldsymbol{g}_0$ is the gravitational acceleration in Base Frame [m/s^2] and $\boldsymbol{p}_{l_i}(\boldsymbol{q})$ is the position of the center of mass of Link i in Base Frame [m].

The the considered robot manipulator's potential energy is

$$E_{\text{pot}}(\boldsymbol{q}) = -\sum_{i=1}^{2} m_{l_i} \boldsymbol{g}_0^T \boldsymbol{p}_{l_i}(\boldsymbol{q}) \qquad \text{[J]}$$

where

$$\boldsymbol{p}_{l_1} = \begin{bmatrix} l_{c1}c_1 \\ l_{c1}s_1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{p}_{l_2} = \begin{bmatrix} l_1c_1 + l_{c2}c_{12} \\ l_1s_1 + l_{c2}s_{12} \\ 0 \end{bmatrix}, \qquad \boldsymbol{g}_0 = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix}$$

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where

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This gives

$$E_{\text{pot}}(\boldsymbol{q}) = m_{l_1} g l_{c1} s_1 + m_{l_2} g (l_1 s_1 + l_{c2} s_{12}) \qquad \text{[J]}$$

The kinetic energy should be computed in an inertial frame, e.g., Base Frame that does not accelerate; thus, the kinetic energy can be computed as

$$E_{\mathsf{kin}}(\boldsymbol{q}) = \sum_{i=1}^{n} E_{\mathsf{kin},l_i}(\boldsymbol{q}) \qquad \text{[J]}$$

where E_{kin,l_i} is the kinetic energy of Link i [J].

The kinetic energy should be computed in an inertial frame, e.g., Base Frame that does not accelerate; thus, the kinetic energy can be computed as

$$E_{\text{kin}}(\boldsymbol{q}) = \sum_{i=1}^{n} E_{\text{kin},l_i}(\boldsymbol{q}) \qquad \text{[J]}$$

where E_{kin,l_i} is the kinetic energy of Link i [J].

The kinetic energy can be expressed as the sum of translational and rotational kinetic energy

$$E_{\text{kin},l_i}(\boldsymbol{q}) = \frac{1}{2} m_{l_i} \dot{\boldsymbol{p}}_{l_i}^T \dot{\boldsymbol{p}}_{l_i} + \frac{1}{2} \boldsymbol{\omega}_i^T I_{l_i}(\boldsymbol{q}) \boldsymbol{\omega}_i$$

where both $\dot{\boldsymbol{p}}_i$, $\boldsymbol{\omega}_i$ and I_{l_i} are given in Base Frame.

The inertia tensor I_{l_i} given in Base Frame can be computed by using an inertia tensor at the link's center of mass ($I^i_{l_i}$)

$$I_{l_i}(\boldsymbol{q}) = R^0_i(\boldsymbol{q}) I^i_{l_i} R^{0T}_i(\boldsymbol{q})$$

The inertia tensor I_{l_i} given in Base Frame can be computed by using an inertia tensor at the link's center of mass ($I_{l_i}^i$)

$$I_{l_i}(\boldsymbol{q}) = R_i^0(\boldsymbol{q}) I_{l_i}^i R_i^{0T}(\boldsymbol{q})$$

This gives the following expression for the kinetic energy

$$E_{\text{kin},l_i} = \frac{1}{2} m_{l_i} \dot{\boldsymbol{p}}_{l_i}^T \dot{\boldsymbol{p}}_{l_i} + \frac{1}{2} \boldsymbol{\omega}_i^T R_i^0 I_{l_i}^i R_i^{0T} \boldsymbol{\omega}_i$$

where both $\dot{\boldsymbol{p}}_i$ and $\boldsymbol{\omega}_i$ are given in Base Frame.

Robot with Two Joints

Kinetic Energy: Inertia Tensor

The inertia tensor I_{l_i} given in Base Frame can be computed by using an inertia tensor at the link's center of mass ($I^i_{l_i}$)

$$I_{l_i}(\boldsymbol{q}) = R^0_i(\boldsymbol{q}) I^i_{l_i} R^{0T}_i(\boldsymbol{q})$$

This gives the following expression for the kinetic energy

$$E_{\mathsf{kin},l_i} = \frac{1}{2} m_{l_i} \dot{\boldsymbol{p}}^T_{l_i} \dot{\boldsymbol{p}}_{l_i} + \frac{1}{2} \boldsymbol{\omega}^T_i R^0_i I^i_{l_i} R^{0T}_i \boldsymbol{\omega}_i$$

where both $\dot{\boldsymbol{p}}_i$ and $\boldsymbol{\omega}_i$ are given in Base Frame.

We intend to express $\dot{\boldsymbol{p}}_i$ and $\boldsymbol{\omega}_i$ by the use of generalized coordinates $\boldsymbol{q}$.

By using the Jacobian, the kinetic energy is expressed as

$$E_{\text{kin}}(\boldsymbol{q}) = \sum_{i=1}^{n} E_{\text{kin},l_i} \qquad [\text{J}]$$

where

$$E_{\text{kin},l_i} = \frac{1}{2} m_{l_i} \dot{\boldsymbol{q}}^T J_P^{l_i T} J_P^{l_i} \dot{\boldsymbol{q}} + \frac{1}{2} \dot{\boldsymbol{q}}^T J_O^{l_i T} R_i^0 I_{l_i}^i R_i^{0T} J_O^{l_i} \dot{\boldsymbol{q}}$$

Robot with Two Joints

Example: Kinetic Energy (I)

For Link 1 the kinetic energy is

$$
\begin{aligned}
E_{\text{kin},l_1} &= \frac{1}{2} m_{l_1} \dot{\boldsymbol{q}}^T J_P^{l_1 T} J_P^{l_1} \dot{\boldsymbol{q}} + \frac{1}{2} \dot{\boldsymbol{q}}^T J_O^{l_1 T} R_1^0 I_{l_1}^1 R_1^{0T} J_O^{l_1} \dot{\boldsymbol{q}} \\
&= \frac{1}{2} m_{l_1} \dot{\boldsymbol{q}}^T \begin{bmatrix} -l_{c1} s_1 & l_{c1} c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -l_{c1} s_1 & 0 \\ l_{c1} c_1 & 0 \\ 0 & 0 \end{bmatrix} \dot{\boldsymbol{q}} + \frac{1}{2} \begin{bmatrix} 0 & 0 & \dot{q}_1 \end{bmatrix} I_{l_1}^1 \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix} \\
&= \frac{1}{2} m_{l_1} l_{c1}^2 \dot{q}_1^2 + \frac{1}{2} \begin{bmatrix} 0 & 0 & \dot{q}_1 \end{bmatrix} I_{l_1}^1 \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix} \\
&= \frac{1}{2} \dot{\boldsymbol{q}}^T \begin{bmatrix} m_{l_1} l_{c1}^2 + I_{l_1,zz}^1 & 0 \\ 0 & 0 \end{bmatrix} \dot{\boldsymbol{q}}
\end{aligned}
$$

Robot with Two Joints

Example: Kinetic Energy (II)

For Link 2 the kinetic energy is

$$
\begin{aligned}
E_{\text{kin},l_2} &= \frac{1}{2} m_{l_2} \dot{\boldsymbol{q}}^T J_P^{l_2 T} J_P^{l_2} \dot{\boldsymbol{q}} + \frac{1}{2} \dot{\boldsymbol{q}}^T J_O^{l_2 T} R_2^0 I_{l_2}^2 R_2^{0T} J_O^{l_2} \dot{\boldsymbol{q}} \\
&= \frac{1}{2} m_{l_2} \dot{\boldsymbol{q}}^T \begin{bmatrix} -l_1 s_1 - l_{c2} s_{12} & l_1 c_1 + l_{c2} c_{12} & 0 \\ -l_{c2} s_{12} & l_{c2} c_{12} & 0 \end{bmatrix} \begin{bmatrix} -l_1 s_1 - l_{c2} s_{12} & -l_{c2} s_{12} \\ l_1 c_1 + l_{c2} c_{12} & l_{c2} c_{12} \\ 0 & 0 \end{bmatrix} \dot{\boldsymbol{q}} \\
&+ \frac{1}{2} \begin{bmatrix} 0 & 0 & \dot{q}_1 + \dot{q}_2 \end{bmatrix} I_{l_2}^2 \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 \end{bmatrix} \\
&= \frac{1}{2} m_{l_2} \dot{\boldsymbol{q}}^T \begin{bmatrix} l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} c_2 & l_{c2}^2 + l_1 l_{c2} c_2 \\ l_{c2}^2 + l_1 l_{c2} c_2 & l_{c2}^2 \end{bmatrix} \dot{\boldsymbol{q}} + \frac{1}{2} \begin{bmatrix} 0 & 0 & \dot{q}_1 + \dot{q}_2 \end{bmatrix} I_{l_2}^2 \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 \end{bmatrix} \\
&= \frac{1}{2} \dot{\boldsymbol{q}}^T \begin{bmatrix} m_{l_2}(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} c_2) + I_{l_2,zz}^2 & m_{l_2}(l_{c2}^2 + l_1 l_{c2} c_2) + I_{l_2,zz}^2 \\ m_{l_2}(l_{c2}^2 + l_1 l_{c2} c_2) + I_{l_2,zz}^2 & m_{l_2} l_{c2}^2 + I_{l_2,zz}^2 \end{bmatrix} \dot{\boldsymbol{q}}
\end{aligned}
$$

Robot with Two Joints

Example: Kinetic and Potential Energy

The potential and kinetic energy are

$$
\begin{aligned}
E_{\text{pot}} &= m_{l_1} g l_{c1} s_1 + m_{l_2} g (l_1 s_1 + l_{c2} s_{12}) \\
E_{\text{kin}} &= \frac{1}{2} \dot{\boldsymbol{q}}^T \underbrace{\begin{bmatrix} m_{l_1} l_{c1}^2 + I_{l_1,zz}^1 + m_{l_2}(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} c_2) + I_{l_2,zz}^2 & m_{l_2}(l_{c2}^2 + l_1 l_{c2} c_2) + I_{l_2,zz}^2 \\ m_{l_2}(l_{c2}^2 + l_1 l_{c2} c_2) + I_{l_2,zz}^2 & m_{l_2} l_{c2}^2 + I_{l_2,zz}^2 \end{bmatrix}}_{=B(\boldsymbol{q})} \dot{\boldsymbol{q}}
\end{aligned}
$$

where $B(\boldsymbol{q})$ is the inertia tensor expressed in Base Frame.

Lagrange–D'Alembert's Principle can be used for modelling the system, where $\boldsymbol{q}$ is a vector of the two joint angles, and τ_i is the torque applied at Joint i

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} - \frac{\partial \mathcal{L}}{\partial q_1} = \tau_1$$
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2} - \frac{\partial \mathcal{L}}{\partial q_2} = \tau_2$$

where

$$\mathcal{L} = E_{\text{kin}} - E_{\text{pot}}$$

The following procedure can be used for setting up a dynamical model of a robot with n degrees of freedom

0. Find the DH-parameters of the robot $(a_i, d_i, \alpha_i, \theta_i)$ for $i = 1, 2, \dots, n$.
1. Set up a kinematic model $T_n^0(\boldsymbol{q})$ of the robot.
2. Compute the coordinates $\boldsymbol{p}_{ci}^0(q)$ for center of mass for each link (given in Base frame).
3. Compute the angular velocities $\boldsymbol{\omega}_i^0(\boldsymbol{q}, \dot{\boldsymbol{q}})$ for each link (given in Base frame).
4. Compute velocities $\boldsymbol{v}_{ci}^0(\boldsymbol{q}, \dot{\boldsymbol{q}})$ for center of mass of each link (given in Base frame).
5. Compute the inertia-tensor $I_{l_i}^0(\boldsymbol{q})$ for each link (given in Base frame).
6. Compute the potential energy of the system $E_{\text{pot}}(\boldsymbol{q})$.
7. Compute the kinetic energy of the system $E_{\text{kin}}(\boldsymbol{q}, \dot{\boldsymbol{q}})$.
8. Set up the equations of motion for the system using Lagrange D'Alembert's principle.

Summary

If q is a trajectory of a conservative mechanical system then

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = 0$$

where q is an n-dimensional vector of generalized coordinates and $\mathcal{L}$ is the *Lagrangian* given by

$$\mathcal{L} = E_{\text{kin}} - E_{\text{pot}} \quad \text{[J]}$$

where E_{pot} is the system's potential energy and E_{kin} is the system's kinetic energy.

If q is a trajectory of a mechanical system that is affected by a generalized force Q then

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = \boldsymbol{Q}$$

where Q is an n-dimensional vector of generalized forces. This is called ***Lagrange–D'Alembert's Principle***.

The following procedure can be used for setting up a dynamical model of a serial robot manipulator with n degrees of freedom

0. Find the DH-parameters of the robot $(a_i, d_i, \alpha_i, \theta_i)$ for $i = 1, 2, \ldots, n$.
1. Set up a kinematic model $T_n^0(\boldsymbol{q})$ of the robot.
2. Compute the coordinates $\boldsymbol{p}_{ci}^0(q)$ for center of mass for each link (given in Base frame).
3. Compute the angular velocities $\boldsymbol{\omega}_i^0(\boldsymbol{q}, \dot{\boldsymbol{q}})$ for each link (given in Base frame).
4. Compute velocities $\boldsymbol{v}_{ci}^0(\boldsymbol{q}, \dot{\boldsymbol{q}})$ for center of mass of each link (given in Base frame).
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