

## Opgave 1 (20 point)

Beregn volumen af det område, der er beskrevet ved ulighederne:

$$z \leq -x^2 - y^2 + 10$$

$$z \geq x - 2y - 5$$

$$x^2 + y^2 \leq 1$$

$$x \leq 0$$

$$x = r \cdot \cos \theta \quad y = r \sin \theta \quad z = z \quad dx dy dz = r \cdot d\theta dr dz$$

$$x^2 + y^2 = a^2 \Rightarrow r = a$$

$$\Rightarrow \begin{cases} z \leq -r^2 \cdot \cos^2(\theta) - r^2 \cdot \sin^2(\theta) + 10 \\ z \geq r \cdot \cos(\theta) - 2r \cdot \sin(\theta) - 5 \\ r \leq 1 \\ r \cdot \cos(\theta) \leq 0 \Rightarrow \cos(\theta) \leq 0 \Rightarrow \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \end{cases}$$

$$\Rightarrow \int_0^1 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{r \cdot \cos(\theta) - 2r \cdot \sin(\theta) - 5}^{-r^2 \cdot \cos^2(\theta) - r^2 \cdot \sin^2(\theta) + 10} r \, dz \, d\theta \, dr$$

$$= \int_0^1 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} -r^3 \cdot \cos(\theta)^2 - r^3 \cdot \sin(\theta)^2 + \underline{10r} - r^2 \cdot \cos(\theta) + 2r^2 \cdot \sin(\theta) + \underline{5r} \, d\theta \, dr$$

$$= \int_0^1 \left[ -r^3 \cdot \left( \frac{\theta}{2} + \frac{1}{4} \sin(2\theta) \right) - r^3 \cdot \left( \frac{\theta}{2} - \frac{1}{4} \sin(2\theta) \right) - r^2 \cdot \sin(\theta) - 2r^2 \cdot \cos(\theta) + 15r\theta \right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} dr$$

$$= \int_0^1 \left[ -r^3 \cdot \left( \frac{\theta}{2} + \frac{1}{4} \sin(2\theta) \right) - r^3 \cdot \left( \frac{\theta}{2} - \frac{1}{4} \sin(2\theta) \right) - r^2 \cdot \sin(\theta) + 15r\theta \right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} dr$$

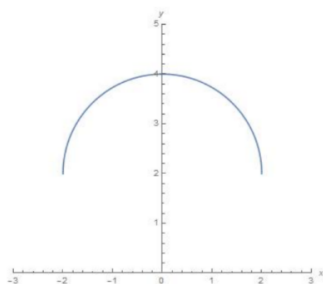
$$= \int_0^1 \left[ -r^3 \cdot \left( \frac{\frac{\pi}{2}}{2} + \frac{1}{4} \sin\left(2 \cdot \frac{\pi}{2}\right) \right) - r^3 \cdot \left( \frac{\frac{\pi}{2}}{2} - \frac{1}{4} \sin\left(2 \cdot \frac{\pi}{2}\right) \right) - r^2 \cdot \sin\left(\frac{\pi}{2}\right) + 15r \left( \frac{\pi}{2} \right) \right. \\ \left. + r^3 \cdot \left( \frac{\frac{3\pi}{2}}{2} + \frac{1}{4} \sin\left(2 \cdot \frac{3\pi}{2}\right) \right) + r^3 \cdot \left( \frac{\frac{3\pi}{2}}{2} - \frac{1}{4} \sin\left(2 \cdot \frac{3\pi}{2}\right) \right) + r^2 \cdot \sin\left(\frac{3\pi}{2}\right) - 15r \left( \frac{3\pi}{2} \right) \right]$$

$$= \int_0^1 \left[ -r^3 \cdot \frac{\pi}{4} - r^3 \cdot \frac{\pi}{4} - r^3 + \frac{15\pi r}{2} - r^3 \cdot \frac{\pi}{4} - r^3 \cdot \frac{\pi}{4} + r^3 + \frac{15\pi r}{2} \right]$$

$$= \int_0^1$$

## Opgave 2 (30 point)

Betragt kurven  $C$ , der ligger i planen  $z = 0$  og er den øvre del af cirklen med centrum i  $(0, 2, 0)$  og har radius 2. Kurven starter i  $(2, 2, 0)$  og slutter i  $(-2, 2, 0)$ . Se skitse herunder.



a)

Beregn

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

hvor vektorfeltet  $\mathbf{F}$  er givet ved

$$\mathbf{F} = \begin{pmatrix} y+z \\ x \\ x \end{pmatrix} \begin{matrix} f_1 \\ f_2 \\ f_3 \end{matrix}$$

Hint: undersøg om  $\mathbf{F}$  er konservativt.

$$\begin{aligned} \frac{\partial f_1}{\partial y} &= \frac{\partial f_2}{\partial x}, \\ \frac{\partial f_1}{\partial z} &= \frac{\partial f_3}{\partial x}, \\ \frac{\partial f_2}{\partial z} &= \frac{\partial f_3}{\partial y}, \end{aligned}$$

$$\left. \begin{aligned} \frac{\partial f_1}{\partial y} &= \frac{\partial f_2}{\partial x} \Rightarrow 1+0=1 \quad \checkmark \\ \frac{\partial f_1}{\partial z} &= \frac{\partial f_3}{\partial x} \Rightarrow 0+1=1 \quad \checkmark \\ \frac{\partial f_2}{\partial z} &= \frac{\partial f_3}{\partial y} \Rightarrow 0=0 \quad \checkmark \end{aligned} \right\} \Rightarrow \text{Conservative!}$$

$$\oint_C \mathbf{F} \cdot d\vec{r} = f(B) - f(A)$$

Finding potential function

$$\nabla f = \vec{F} \Rightarrow \begin{cases} \frac{\partial f}{\partial x} = y+z \\ \frac{\partial f}{\partial y} = x \\ \frac{\partial f}{\partial z} = x \end{cases} \Rightarrow \begin{cases} f = xy + xz + A(y, z) \\ f = xy + B(x, z) \\ f = xz + C(x, y) \end{cases} \Rightarrow \begin{cases} A(y, z) = 0 \\ B(x, z) = xz \\ C(x, y) = xy \end{cases}$$

$$\Rightarrow f(x, y, z) = xy + xz$$

$$\oint_C \vec{F} \cdot d\vec{r} = f(B) - f(A) = f(-2, 2, 0) - f(2, 2, 0) = (-2) \cdot 2 + (-2) \cdot 0 - 2 \cdot 2 + 2 \cdot 0 = -4 - 4 = \underline{\underline{-8}}$$

**b)**

Beregn

$$\int_C f(x, y, z) ds$$

hvor  $f(x, y, z) = x + y + z$ .

??

### Opgave 3 (30 point)

Betragt overfladen  $S$ , der er parametriseret ved

$$\mathbf{r}(t) = \begin{pmatrix} u \cos(v) \\ u \sin(v) \\ u \end{pmatrix}$$

hvor  $0 \leq u \leq 2$  og  $0 \leq v \leq \pi$ .

Betragt også vektorfeltet

$$\mathbf{F} = \begin{pmatrix} -xz \\ -yz \\ z \end{pmatrix}$$

a)

Beregn arealet af  $S$ .

b)

Beregn fluxen af  $\mathbf{F}$  opad gennem  $S$ .

$$\text{flux} = \iint_S \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) du dv$$

$$\begin{bmatrix} u \cos(v) \\ u \sin(v) \\ u \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \vec{r}_u \times \vec{r}_v = \begin{bmatrix} -u \cos(v) \\ -u \sin(v) \\ u \end{bmatrix}$$

$$\iint_0^\pi \int_0^2 \begin{bmatrix} -u \cos(v) \cdot u \\ -u \sin(v) \cdot u \\ u \end{bmatrix} \cdot \begin{bmatrix} -u \cos(v) \\ -u \sin(v) \\ u \end{bmatrix} du dv$$

$$= \int_0^\pi \int_0^2 (u^2 \cos^2(v) + u^2 \sin^2(v) + u^2) du dv$$

$$= \int_0^\pi \int_0^2 u^2 \cdot (\underbrace{\cos^2(v) + \sin^2(v)}_1) + u^2 du dv$$

$$= \int_0^\pi \int_0^2 (u^3 + u^2) du dv = \int_0^\pi \left[ \frac{u^4}{4} + \frac{u^3}{3} \right]_0^2 dv = \int_0^\pi \left( \frac{16}{4} + \frac{8}{3} \right) dv = \int_0^\pi \left( 4 + \frac{8}{3} \right) dv = \left[ 4v + \frac{8v}{3} \right]_0^\pi = 4\pi + \frac{8\pi}{3} = \underline{\underline{\frac{20\pi}{3}}}$$

a)

$$\iint_S |\vec{r}_u \times \vec{r}_v| du dv$$

$$\vec{r}_u = \begin{bmatrix} \cos(v) \\ \sin(v) \\ 1 \end{bmatrix}, \quad \vec{r}_v = \begin{bmatrix} -u \sin(v) \\ u \cos(v) \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ -(a_1 b_3 - a_3 b_1) \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

$$\vec{r}_u \times \vec{r}_v = \begin{bmatrix} \cancel{\cos(v)} \cdot 0 - 1 \cdot u \cos(v) \\ -(\cancel{\cos(v)} \cdot 0 - 1 \cdot (-u \sin(v))) \\ \cos(v) \cdot u \cos(v) - \sin(v) \cdot (-u \sin(v)) \end{bmatrix}$$

$$= \begin{bmatrix} -u \cos(v) \\ -u \sin(v) \\ u(\cos^2(v) + \sin^2(v)) \end{bmatrix} = \begin{bmatrix} -u \cos(v) \\ -u \sin(v) \\ u \end{bmatrix}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{u^2 \cos^2(v) + u^2 \sin^2(v) + u^2}$$

$$= \sqrt{u^2 (\underbrace{\cos^2(v) + \sin^2(v)}_1) + u^2}$$

$$= \sqrt{u^2 + u^2} = \sqrt{2u} = u\sqrt{2}$$

$$\iint_0^\pi \int_0^2 u\sqrt{2} du dv = \sqrt{2} \cdot \int_0^\pi \int_0^2 u du dv$$

$$= \sqrt{2} \cdot \int_0^\pi \int_0^2 u du dv = \sqrt{2} \cdot \int_0^\pi \left[ \frac{u^2}{2} \right]_0^2 dv$$

$$= \sqrt{2} \cdot \int_0^\pi \frac{4}{2} dv = \sqrt{2} \cdot \int_0^\pi 2 dv = \sqrt{2} \cdot [2v]_0^\pi$$

$$= \underline{\underline{2\pi\sqrt{2}}}$$

## Opgave 4 (20 point)

Betragt den partielle differentialligning

$$u_{yy} - u_{xx} = 0$$

Klassificer ligningen, omskriv den til normal form ved at gennemføre et passende variabelskift.

Løs den fremkomne ligning, løsningen skal skrives som funktion af  $x$  og  $y$ .

NOOOOOOOOOOOOOOOPE