Question 1:

Using the Laplace transform, find the solution for the following equation.

$$\frac{\partial^2}{\partial t^2} y(t) + 2\frac{\partial}{\partial t} y(t) + 2y(t) = 0$$

with initial conditions $y(0) = 1$and $Dy(0) = -1$ $= y'(t)$

$$y''(t) + 2 \cdot y'(t) + 2y(t) = 0$$

$$\Rightarrow \left(s^2 Y(s) - s\,y(0) - y'(0)\right) + 2 \cdot \left(s \cdot Y(s) - y(0)\right) + 2 \cdot Y(s) = 0$$

$$= \left(s^2 Y(s) - s \cdot 1 - (-1)\right) + 2 \cdot \left(s\,Y(s) - 1\right) + 2 \cdot Y(s) = 0$$

$$\Rightarrow s^2 Y(s) - s + 1 + 2s\,Y(s) - 2 + 2\,Y(s) = 0$$

$$\Rightarrow s^2 Y(s) + 2s\,Y(s) + 2\,Y(s) = s + 1$$

$$\Rightarrow Y(s) \cdot \left(s^2 + 2s + 2\right) = s + 1$$

$$\Rightarrow Y(s) = \frac{s+1}{s^2 + 2s + 2} \longrightarrow \text{Transfor function}$$

Finding the poles

$$s^2 + 2s + 2 = 0 \Rightarrow d = b^2 - 4ac = 4 - 4(1)(2) = 4 - 8 = -4$$

$$s = \frac{-b \pm \sqrt{d}}{2a} = \frac{-2 \pm 2i}{2} = \begin{cases} -1 - i \\ -1 + i \end{cases}$$

$$\Rightarrow Y(s) = \frac{s+1}{(s+1+i)(s+1-i)} = \frac{A}{s+1+i} + \frac{B}{s+1-i}$$

$$k_i = (s-p_i)\cdot Y(s)\Big|_{s=p_i}$$

$A = k_1 \qquad p_1 = -1-i$

$$A = \cancel{(s+1+i)} \cdot \frac{s+1}{\cancel{(s+1+i)}(s+1-i)}\Bigg|_{s=p_1} = \frac{s+1}{s+1-i}\Bigg|_{s=p_1} = \frac{(-1-i)+1}{(-1-i)+1-i} = \frac{-i}{-2i} = \frac{1}{2}$$

Complex Conjugate

$$B = \bar{A} = \frac{1}{2}$$

$$Y(s) = \frac{\frac{1}{2}}{s+1+i} + \frac{\frac{1}{2}}{s+1-i} = \frac{1}{2(s+1+i)} + \frac{1}{2(s+1-i)} = \frac{1}{2}\left(\frac{1}{s+1+i} + \frac{1}{s+1-i}\right)$$

$$\mathcal{L}\{e^{\alpha t}\} = \frac{1}{s-\alpha}$$

This is the result of the inverse laplace transformation

$$y(t) = \frac{1}{2}\left(e^{(1+i)t} + e^{(1-i)t}\right)$$

Using eulers formula:

$$\cos x = \operatorname{Re}(e^{ix}) = \frac{e^{ix}+e^{-ix}}{2},$$
$$\sin x = \operatorname{Im}(e^{ix}) = \frac{e^{ix}-e^{-ix}}{2i}.$$

$$= \frac{1}{2}\left(e^{t}\cdot e^{it} + e^{t}\cdot e^{-it}\right) = e^{t}\cdot\frac{1}{2}\cdot\left(e^{it} + e^{-it}\right)$$

$$\underline{\underline{= e^{t}\cdot\cos(t)}}$$

Question 2:

Find Fourier transform of

$$f(x) = \begin{cases} 1 - x^2 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

And hence evaluate:

$$\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right) \cos\frac{x}{2}\, dx$$

$$H(\omega) = \int_{-\infty}^{\infty} h(\cancel{t}\, x) e^{-j\omega \cancel{t}\, x} d\cancel{t}\, x$$

f(x) is only non-zero between x=-1 and x=1, therefore we can use these values as boundries for our integral.

$$H(w) = \int_{-1}^{1} (1-x^2) \cdot e^{-jwx} \, dx$$

Partial Integration

$$\int u(x) \cdot v'(x) dx = u(x) \cdot v(x) - \int u'(x) \cdot v(x) dx$$

$$u = 1 - x^3 \qquad v' = e^{-jwx}$$

$$u' = -2x \qquad v = \frac{1}{-jw} \cdot e^{-jwx}$$

$$\Rightarrow \quad H(w) = (1-x^2) \cdot \frac{1}{-jwx} \cdot e^{-jwx} - \int -2x \cdot \frac{1}{-jwx} \cdot e^{-jwx} \, dx$$

$$\Rightarrow \quad H(w) = (1-x^2) \cdot \frac{1}{-jw} \cdot e^{-jwx} + \frac{2}{-jw} \int x \cdot e^{-jwx} \, dx$$

More parital integration

$$u = x \qquad v' = e^{-jwx}$$

$$u' = 1 \qquad v = \frac{1}{-jw} \cdot e^{-jwx}$$

$$\Rightarrow \quad H(w) = (1-x^2)\cdot\frac{1}{-jw}\cdot e^{-jwx} + \frac{2}{-jw}\left(x\cdot\frac{1}{-jw}\cdot e^{-jwx} - \int 1\cdot\frac{1}{-jw}\cdot e^{-jwx}\,dx\right)$$

$$\Rightarrow H(w) = (1-x^2)\cdot\frac{1}{-jw}\cdot e^{-jwx} + \frac{2}{-jw}\left(x\cdot\frac{1}{-jw}\cdot e^{-jwx} - \frac{1}{-jw}\cdot\frac{1}{-jw}\cdot e^{-jwx}\right)$$

Simplifying

$$\Rightarrow H(w) = (1-x^2)\cdot\frac{1}{-jw}\cdot e^{-jwx} + \frac{2x}{j^2w^2}\cdot e^{-jwx} - \frac{2}{-j^3w^3}\cdot e^{-jwx}$$

$$\Rightarrow H(w) = (1-x^2)\cdot\frac{1}{-jw}\cdot e^{-jwx} + \frac{2x}{j^2w^2}\cdot e^{-jwx} + \frac{2}{j^3w^3}\cdot e^{-jwx}$$

We can now evaluate with the limits

$$\left[\cancel{(1-x^2)\cdot\frac{1}{-jw}\cdot e^{-jwx}} + \frac{2x}{j^2w^2}\cdot e^{-jwx} + \frac{2}{j^3w^3}\cdot e^{-jwx}\right]_{-1}^{1}$$

In both limits this term becomes zero

$$= \left[\frac{2x}{j^2w^2}\cdot e^{-jwx} + \frac{2}{j^3w^3}\cdot e^{-jwx}\right]_{-1}^{1}$$

$$= \frac{2(1)}{j^2w^2}\cdot e^{-jw(1)} + \frac{2}{j^3w^3}\cdot e^{-jw(1)} - \frac{2(-1)}{j^2w^2}\cdot e^{-jw(-1)} - \frac{2}{j^3w^3}\cdot e^{-jw(-1)}$$

$$= \frac{2}{j^2w^2}\cdot e^{-jw} + \frac{2}{j^3w^3}\cdot e^{-jw} + \frac{2}{j^2w^2}\cdot e^{jw} - \frac{2}{j^3w^3}\cdot e^{jw}$$

Pull out costants to make it resemble eulers formula

$$= \left(\frac{2}{j^2w^2}\cdot\left(e^{-jw} + e^{jw}\right) + \frac{2}{j^3w^3}\left(e^{-jw} - e^{jw}\right)\right)\cdot\frac{2}{2}$$

$$= \frac{4}{j^2w^2}\cdot\left(\frac{e^{-jw}+e^{jw}}{2}\right) - \frac{4}{w^3}\cdot\left(\frac{e^{-jw}-e^{jw}}{2j}\right)$$

$$= \frac{4}{j^2w^2}\cdot\left(\frac{e^{-jw}+e^{jw}}{2}\right) + \frac{4}{w^3}\cdot\left(\frac{e^{-jw}-e^{jw}}{2j}\right)$$

$$\cos x = Re(e^{ix}) = \frac{e^{ix}+e^{-ix}}{2}$$

$$\sin x = Im(e^{ix}) = \frac{e^{ix}-e^{-ix}}{2i}$$

$$= \frac{4}{\underbrace{j^2}_{-1} w^2}\cdot\left(\frac{e^{jw}+e^{-jw}}{2}\right) + \frac{-4}{w^3}\cdot\left(\frac{e^{jw}-e^{-jw}}{2j}\right)$$

$$= \frac{-4}{w^2}\cdot\cos(w) + \frac{-4}{w^3}\cdot\sin(w) \qquad w/w$$

$$= \frac{-4w\cdot\cos(w)}{w^3} + \frac{-4\cdot\sin(w)}{w^3}$$

$$H(w) = \frac{-4w\cdot\cos(w) - 4\cdot\sin(w)}{w^3} = -4\cdot\frac{w\cdot\cos(w)-\sin(w)}{w^3}$$

And hence evaluate:

$$\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\,dx$$

This is the fourier transform

$$H(w) = \int_{-\infty}^{\infty} h(x)\cdot e^{-jwx}$$

This is the specific expression for our problem

$$H(w) = \int_{-1}^{1} \underbrace{(1-x^2)}_{h(x)}\cdot e^{-jwx}\,dx = \underbrace{-4\cdot\frac{w\cdot\cos(w)-\sin(w)}{w^3}}$$

This part seems to resemple the expression we are trying to evaluate...

Let's try to take the inverse fourier transform

$$h(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{-4\cdot\frac{w\cdot\cos(w)-\sin(w)}{w^3}}_{H(w)}\cdot e^{jwx}\,dw = \underbrace{1-x^2}$$

This part can NEVER be imaginary. Therefore, the integral on the left can also never be imaginary. This means that we can ignore the imaginary part.

$$Re(e^{jwx}) = \cos(wx)$$

$$h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} -4 \cdot \frac{w \cdot \cos(w) - \sin(w)}{w^3} \cdot \cos(wx)\, dw = 1 - x^2$$

This inner expression now looks a lot like the expression we are trying to evaluate.

$$\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right) \cos\frac{x}{2}\, dx$$

$$h(x) = \frac{-4}{2\pi} \int_{-\infty}^{\infty} \frac{w \cdot \cos(w) - \sin(w)}{w^3} \cdot \cos(wx)\, dw = 1 - x^2$$

In fact, if we set x=1/2 they are equal

$$h\left(\frac{1}{2}\right) = \frac{-4}{2\pi} \int_{-\infty}^{\infty} \frac{w \cdot \cos(w) - \sin(w)}{w^3} \cdot \cos\left(\frac{w}{2}\right) dw = 1 - x^2$$

We can not isolate the integral to find its evaluation

$$\frac{-4}{2\pi} \int_{-\infty}^{\infty} \frac{w \cdot \cos(w) - \sin(w)}{w^3} \cdot \cos\left(\frac{w}{2}\right) dw = 1 - \left(\frac{1}{2}\right)^2$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{w \cdot \cos(w) - \sin(w)}{w^3} \cdot \cos\left(\frac{w}{2}\right) dw = \frac{2\pi \cdot \left(1 - \frac{1}{4}\right)}{-4} = \frac{-2\pi + \frac{\pi}{2}}{4} = \frac{-4\pi + \pi}{8}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{w \cdot \cos(w) - \sin(w)}{w^3} \cdot \cos\left(\frac{w}{2}\right) dw = \frac{-3\pi}{8}$$

And there we have the answer!

$$\Rightarrow \int_{-\infty}^{\infty} \frac{w \cdot \cos(x) - \sin(x)}{x^3} \cdot \cos\left(\frac{x}{2}\right) dw = \frac{-3\pi}{8}$$

Question 3:

Find $\iint (\sin x + \cos y)dA$ $\quad R:\left\{x,y \,\middle|\, \begin{matrix} 0 \le x \le \pi/2 \\ 0 \le y \le \pi/2 \end{matrix}\right.$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin(x) + \cos(y)\, dx\, dy$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin(x)\, dx\, dy + \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \cos(y)\, dx\, dy$$

$$= \int_0^{\frac{\pi}{2}} \left[-\cos(x)\right]_0^{\frac{\pi}{2}} dy + \int_0^{\frac{\pi}{2}} \left[\cos(y) \cdot x\right]_0^{\frac{\pi}{2}} \cdot dy$$

$$= \int_0^{\frac{\pi}{2}} -\underset{0}{\cos\left(\tfrac{\pi}{2}\right)} + \underset{1}{\cos(0)}\, dy + \int_0^{\frac{\pi}{2}} \cos(y) \cdot \frac{\pi}{2}\, dy$$

$$= \int_0^{\pi/2} 1\, dy + \int_0^{\frac{\pi}{2}} \cos(y) \cdot \frac{\pi}{2}\, dy = \left[y\right]_0^{\frac{\pi}{2}} + \frac{\pi}{2} \cdot \left[\sin(y)\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2} + \frac{\pi}{2} \cdot 1$$

$$= \underline{\underline{\pi}}$$

Question 4

Find the $\iiint (x^2 + y^2 + z^2)dV$

where region is bounded by $z = c\sqrt{(x^2 + y^2)}$ and $(x^2 + y^2 + z^2 = a^2)$

Convert to spherical coordinates

$$x = \cancel{R}\rho \cdot \sin\phi \cdot \cos\theta$$
$$y = \cancel{R}\rho \cdot \sin\phi \cdot \sin\theta$$
$$z = \cancel{R}\rho \cdot \cos\phi$$

$$dV = \rho^2 \cdot \sin(\phi)\, d\rho\, d\phi\, d\theta$$

Translate integral

$$\iiint x^2 + y^2 + z^2\, dV = \iiint \rho^2 \cdot \rho^2 \cdot \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= \iiint \rho^4 \cdot \sin(\phi)\, d\rho\, d\phi\, d\theta$$

Translate limits

$$z = c \cdot \sqrt{x^2 + y^2} \Rightarrow \rho \cdot \cos(\phi) = c \cdot \sqrt{(\rho \cdot \sin(\phi) \cdot \cos(\theta))^2 + (\rho \cdot \sin(\phi) \cdot \sin(\theta))^2}$$

$$\Rightarrow \rho \cdot \cos(\phi) = c \cdot \sqrt{\rho^2 \cdot \sin(\phi)^2 \cdot \cos(\theta)^2 + \rho^2 \cdot \sin(\phi)^2 \cdot \sin(\theta))^2}$$

$$\Rightarrow \rho \cdot \cos(\phi) = c \cdot \sqrt{\rho^2 \cdot \sin(\phi)^2 \cdot (\cos(\theta)^2 + \sin(\theta)^2)}$$

$$\Rightarrow \rho\cdot\cos(\phi) = c\cdot\sqrt{(\rho\cdot\sin(\phi))^2}\cdot\sqrt{(\cos(\theta)^2+\sin(\theta)^2)} = c\cdot\rho\cdot\sin(\phi)\cdot 1$$

$$\Rightarrow \rho\cdot\cos(\phi) = c\cdot\rho\cdot\sin(\phi)$$

$$\Rightarrow \cos(\phi) = c\cdot\sin(\phi)$$

$$\Rightarrow c = \frac{\cos(\phi)}{\sin(\phi)} \Rightarrow \frac{1}{c} = \tan(\phi) \Rightarrow \phi = \tan^{-1}\left(\frac{1}{c}\right)$$

Definition: $\alpha = \tan^{-1}\left(\frac{1}{c}\right)$

Second limit

$$x = \rho\cdot\sin\phi\cdot\cos\theta$$
$$y = \rho\cdot\sin\phi\cdot\sin\theta$$
$$z = \rho\cdot\cos\phi$$

This is a sphere with radius a.

$$x^2+y^2+z^2 = a^2 \Rightarrow \rho_{min} = 0,\ \rho_{max} = a$$

Inserting limits

$$\int_0^{2\pi}\int_0^{\alpha}\int_0^{a} \rho^4\cdot\sin(\phi)\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\alpha} \sin(\phi)\int_0^{a} \rho^4\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\alpha} \sin(\phi)\left[\frac{\rho^5}{5}\right]_0^a d\phi\, d\theta = \int_0^{2\pi}\int_0^{\alpha} \sin(\phi)\cdot\frac{a^5}{5}\, d\phi\, d\theta$$

$$= \frac{a^5}{5}\int_0^{2\pi}\int_0^{\alpha} \sin(\phi)\, d\phi\, d\theta = \frac{a^5}{5}\int_0^{2\pi}\left[-\cos(\phi)\right]_0^{\alpha} d\theta$$

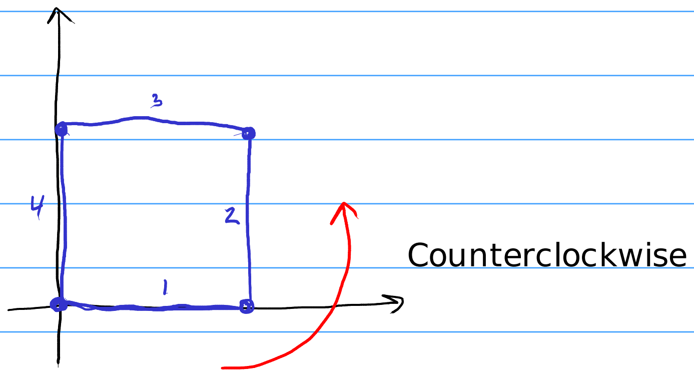
$$= \frac{a^5}{5}\cdot\int_0^{2\pi}(-\cos(0)-(-\cos(\alpha))\, d\theta = \frac{a^5}{5}\cdot\int_0^{2\pi}(1-\cos(\alpha))\, d\theta$$

$$= \frac{a^5}{5} \cdot \left(1 - \cos(\alpha)\right) \int_0^{2\pi} 1 \, d\theta = \frac{a^5}{5} \cdot \left(1 - \cos(\alpha)\right) \cdot 2\pi$$

$$= \frac{a^5}{5} \cdot \cos\left(1 - \tan^{-1}\left(\frac{1}{c}\right)\right) \cdot 2\pi$$

Question 5

Evaluate $\oint_C x^2y^2dx + x^3y\,dy$ counterclockwise around the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$



$$① : y=0 \Rightarrow \int_{x=0}^{x=1} x^2\cdot 0\,dx + x^3\cdot 0\cdot 0 = 0$$

$$② \quad x=1 \Rightarrow \int_{y=0}^{y=1} 1^2\cdot y^2\cdot 0 + 1^3\cdot y\,dy = \int_0^1 y\,dy = \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}$$

$$③ \quad y=1 \Rightarrow -\int_{x=0}^{x=1} x^2\cdot 1^2\,dx + x^3\cdot 1\cdot 0 = -\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{-1}{3}$$

$$④ \quad x=0 \Rightarrow -\int_{y=0}^{y=1} 0^2\cdot y^2\cdot 0 + 0^3\cdot y\,dy = -\int_0^1 0\,dy = 0$$

We can now add up the integrals

$$0 + \frac{1}{2} + \frac{-1}{3} + 0 = \frac{1}{6}$$