

Opgave 1

Beregn volumen af det område, der er beskrevet ved ulighederne:

$$\left. \begin{aligned} z &\leq x + y + 10 \\ z &\geq x^2 + y^2 \\ x^2 + y^2 &\leq 4 \\ x &\geq 0 \\ y &\leq 0 \end{aligned} \right\} \text{Very cylinder}$$

$$x = r \cdot \cos \theta \quad y = r \sin \theta \quad z = z \quad dx dy dz = r \cdot d\theta dr dz \quad x^2 + y^2 = r^2$$

$$z \leq x + y + 10 \Rightarrow z \leq r \cdot \cos(\theta) + r \cdot \sin(\theta) + 10$$

$$z \geq r^2$$

$$r^2 \leq 4 \Rightarrow 0 \leq r \leq 2$$

$$\left. \begin{aligned} r \cdot \cos(\theta) &\geq 0 \Rightarrow \cos(\theta) \geq 0 \Rightarrow \theta \geq \frac{\pi}{2} + n\pi \\ r \cdot \sin(\theta) &\leq 0 \Rightarrow \sin(\theta) \leq 0 \Rightarrow \theta \leq 0 + n\pi \end{aligned} \right\} \frac{-\pi}{2} \leq \theta \leq 0$$

$$\int_0^2 \int_{-\frac{\pi}{2}}^0 \int_{r^2}^{r \cdot (\cos(\theta) + \sin(\theta)) + 10} r \, dz \, d\theta \, dr = \int_0^2 \int_{-\frac{\pi}{2}}^0 [r \cdot z]_{r^2}^{r \cdot (\cos(\theta) + \sin(\theta)) + 10} d\theta \, dr$$

$$\Rightarrow \int_0^2 \int_{-\frac{\pi}{2}}^0 r \cdot \cos(\theta) + r \cdot \sin(\theta) + 10 - r^3 \, d\theta \, dr = \int_0^2 \left[r \cdot \sin(\theta) - r \cdot \cos(\theta) + 10\theta - r^3\theta \right]_{-\frac{\pi}{2}}^0 dr$$

$$\Rightarrow \int_0^2 r \cdot \cancel{\sin(0)} - r \cdot \overset{A}{\cancel{\cos(0)}} + 10 \cdot 0 - r^3 \cdot 0 - r \cdot \overset{-1}{\sin(-\frac{\pi}{2})} + r \cdot \cancel{\cos(-\frac{\pi}{2})} - 10 \cdot \frac{\pi}{2} + r^3 \cdot \frac{\pi}{3} \, dr$$

$$= \int_0^2 -r + r + 5\pi - \frac{\pi r^3}{2} \, dr = \int_0^2 5\pi - \frac{\pi r^3}{2} \, dr = \left[5\pi r - \frac{\pi r^4}{8} \right]_0^2 = 10\pi - \frac{16\pi}{8} = 10\pi - 2\pi$$

$$\underline{\underline{= 8\pi}}$$

Opgave 2

Betragt vektorfeltet

$$\mathbf{F} = \begin{pmatrix} 4y + 2z \\ 4x + 2yz \\ 2x + y^2 \end{pmatrix}$$

a)

Beregn

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

hvor C er en ret linje fra $(1, 1, 1)$ til $(5, 7, 9)$.

b)

Beregn fluxen af \mathbf{F} op gennem disken D , der ligger i xy -planet med centrum i $(0, 0)$ og med radius 2.

$$\vec{r} = \begin{bmatrix} 5-1 \\ 7-1 \\ 9-1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}$$

The line from point $(1, 1, 1)$ to $(5, 7, 9)$ if $t=0..1$

$$\vec{r}(t) = \vec{r} \cdot t + \vec{p}_0 = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} \cdot t + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4t+1 \\ 6t+1 \\ 8t+1 \end{bmatrix} \Rightarrow \begin{cases} x = 4t+1 \\ y = 6t+1 \\ z = 8t+1 \end{cases} \quad \vec{r}'(t) = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}$$

$$\oint_C \mathbf{F}(x, y, z) \, d\vec{r} = \int_0^1 \begin{pmatrix} 4(6t+1) + 2(8t+1) \\ 4(4t+1) + 2(6t+1)(8t+1) \\ 2(4t+1) + (6t+1)^2 \end{pmatrix} \cdot \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} dt = \int_0^1 \begin{bmatrix} 16(6t+1) + 8(8t+1) \\ 24(4t+1) + 12(6t+1)(8t+1) \\ 16(4t+1) + 8(6t+1)^2 \end{bmatrix} dt$$

$$\oint_C \vec{F}(x, y) \, d\vec{r} = \int_a^b \vec{F}(x(t), y(t)) \cdot \vec{r}'(t) dt$$

Add them up (magic??)

$$\begin{aligned} &\Rightarrow \int_0^1 16(6t+1) + 8(8t+1) + \underline{24(4t+1)} + \underline{12(6t+1)(8t+1)} + \underline{16(4t+1)} + 8(6t+1)^2 \, dt \\ &= \int_0^1 40(4t+1) + 16(6t+1) + 8(8t+1) + 12(48t^2 + 14t + 1) + 8(36t^2 + 12t + 1) \, dt \\ &= \int_0^1 \underline{160t} + \underline{40} + \underline{96t} + \underline{16} + \underline{64t} + \underline{8} + \underline{576t^2} + \underline{168t} + \underline{12} + \underline{288t^2} + \underline{96t} + \underline{8} \, dt \\ &= \int_0^1 864t^2 + 584t + 84 \, dt = \left[288t^3 + 292t^2 + 84t \right]_0^1 = \underline{\underline{288 + 292 + 84 = 664}} \end{aligned}$$

b)

$$\vec{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Beregn fluxen af \mathbf{F} op gennem disken D , der ligger i xy -planet med centrum i $(0, 0)$ og med radius 2.

$$\mathbf{F} = \begin{pmatrix} 4y + 2z \\ 4x + 2yz \\ 2x + y^2 \end{pmatrix}$$

$$\hookrightarrow x^2 + y^2 = 4$$

$$\text{flux} = \iint_S \vec{F} \cdot \vec{n} \, dS$$

$$\Rightarrow \iint_D \begin{bmatrix} 4y + 2z \\ 4x + 2yz \\ 2x + y^2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \, dS = \iint_D 2x + y^2 \, dS$$

Convert to polar coordinates

$$x = r \cdot \cos(\theta) \quad y = r \cdot \sin(\theta) \quad dA = r \, d\theta \, dr \quad x^2 + y^2 = a^2 \Rightarrow a$$

$$\begin{aligned} \iint_D 2x + y^2 \, dx \, dy &\Rightarrow \int_0^2 \int_0^{2\pi} 2 \cdot r \cdot \cos(\theta) + r^2 \cdot \sin^2(\theta) \cdot r \, d\theta \, dr \\ &= \int_0^2 \left[2r^2 \cdot \sin(\theta) + r^3 \cdot \left(\frac{\theta}{2} - \frac{1}{4} \sin(2\theta) \right) \right]_0^{2\pi} \, dr \\ &= \int_0^2 r^3 \cdot \frac{2\pi}{2} \, dr = \int_0^2 \pi r^3 \, dr = \left[\frac{\pi r^4}{4} \right]_0^2 = \frac{16\pi}{4} = \underline{\underline{4\pi}} \end{aligned}$$

Opgave 3

Betrakt kurven C , der er parametriseret ved

$$\mathbf{r}(t) = \begin{pmatrix} \frac{1}{\sqrt{2}}t^2 \\ t \\ \frac{1}{3}t^3 \end{pmatrix}$$

hvor $0 \leq t \leq 2$.

a)

Beregn længden af kurven C .

$$\begin{aligned} \vec{r}(t) &= \begin{bmatrix} \frac{1}{\sqrt{2}}t^2 \\ t \\ \frac{1}{3}t^3 \end{bmatrix} \Rightarrow \vec{r}'(t) = \begin{bmatrix} \frac{2}{\sqrt{2}}t \\ 1 \\ t^2 \end{bmatrix} \Rightarrow |\vec{r}'(t)| = \sqrt{\frac{4}{2}t^2 + 1^2 + t^4} \\ &= \sqrt{t^4 + 2t^2 + 1} \\ &= \sqrt{(t^2 + 1)^2} \\ &= t^2 + 1 \end{aligned}$$

$$\Rightarrow \int_0^2 t^2 + 1 \, dt = \left[\frac{t^3}{3} + t \right]_0^2 = \frac{8}{3} + 2 = \underline{\underline{\frac{14}{3}}}$$

b)

Beregn

$$\int_C z \, ds \Rightarrow \vec{F}(x, y, z) = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{t^3}{3} \end{bmatrix}$$

$$\int_C \vec{F} \, d\vec{r} = \int_0^2 \begin{bmatrix} 0 \\ 0 \\ \frac{t^3}{3} \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{\sqrt{2}}t \\ 1 \\ t^2 \end{bmatrix} dt = \int_0^2 \frac{1}{3} \cdot t^5 \, dt = \left[\frac{1}{18} \cdot t^6 \right]_0^2 = \underline{\underline{\frac{64}{18}}}$$

why this not correct??

Opgave 4

Betragt differentialligningen

$$y''(t) + 4y(t) = \exp(-2t) = e^{-2t}$$

hvor $y(0) = 0$ og $y'(0) = 0$.

Brug nu Laplacetransformation til at løse ligningen og således bestemme $y(t)$.

$$\Rightarrow Y(s) \cdot s^2 + 4 \cdot Y(s) = \frac{1}{s+2} \Rightarrow Y(s) \cdot (s^2+4) = \frac{1}{s+2} \Rightarrow Y(s) = \frac{1}{s+2} \cdot \frac{1}{s^2+4}$$

$$\Rightarrow Y(s) = \frac{A}{s+2} + \frac{B \cdot s + C}{s^2+4}$$

$$\Rightarrow \frac{1}{s+2} \cdot \frac{1}{s^2+4} = \frac{A}{s+2} + \frac{B \cdot s + C}{s^2+4} \Rightarrow 1 \cdot 1 = A \cdot (s^2+4) + (B \cdot s + C)(s+2)$$

$$\Rightarrow 1 = A s^2 + 4A + B \cdot s^2 + 2Bs + Cs + 2C$$

$$\Rightarrow 1 = s^2(A+B) + s(2B+C) + 4A + 2C$$

$$\left\{ \begin{array}{l} A+B=0 \Rightarrow A=-B \\ 2B+C=0 \Rightarrow -2B=C \Rightarrow 2A=C \\ 4A+2C=1 \Rightarrow 2A+C=\frac{1}{2} \Rightarrow 2A=\frac{1}{2}-C \end{array} \right\} \Rightarrow \begin{array}{l} C=\frac{1}{2}-C \Rightarrow 2C=\frac{1}{2}=\frac{1}{4} \\ \Rightarrow 2A=\frac{1}{4} \Rightarrow A=\frac{1}{8} \\ \Rightarrow B=-\frac{1}{8} \end{array}$$

$$Y(s) = \frac{\frac{1}{8}}{s+2} + \frac{-\frac{1}{8}s + \frac{1}{4}}{s^2+4} = \frac{1}{8} \cdot \frac{1}{s+2} - \frac{1}{8} \cdot \frac{s}{s^2+4} + \frac{1}{8} \cdot \frac{2}{s^2+2^2}$$

$$y(t) = \frac{1}{8} \cdot e^{-2t} + \frac{1}{8} \cdot \cos(2t) + \frac{1}{8} \cdot \sin(2t)$$

$$\underline{\underline{y(t) = \frac{1}{8} (e^{-2t} + \cos(2t) + \sin(2t))}}$$