State Feedback Control

Control Engineering (Reguleringsteknik)

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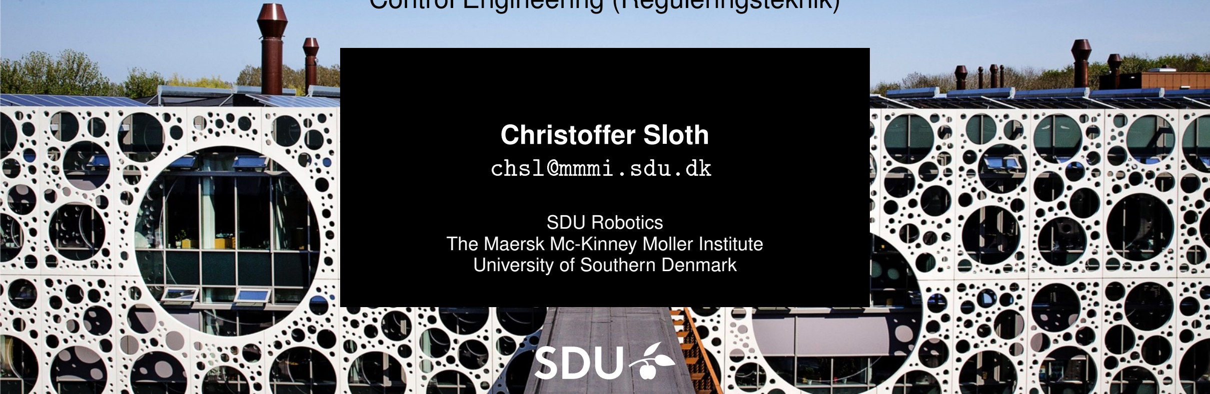

Agenda

Introduction

Controllability

Controllable Canonical Form

State Feedback and Pole Assignment

Summary

Introduction

Curriculum for Reguleringsteknik (REG)

Matematiske og grafiske metoder til syntese af lineære tidsinvariante systemer:[1]

- diskret og kontinuert tilstandsbeskrivelse
- analyse i tid og frekvens
- stabilitet, reguleringshastighed, følsomhed og fejl
- digitale PI, PID, LEAD og LAG regulatorer (serieregulatorer)
- tilstandsregulering, pole-placement og tilstands-estimering (observer)
- optimal regulering (least squares) og optimal tilstands-estimation (Kalman-filter)

Færdigheder:
Efter gennemførelse af kurset kan den succesfulde studerende:

- kunne analysere, dimensionere og implementere såvel kontinuert som tidsdiskret regulering af lineære tidsinvariante og stokastiske systemer

Kompetencer:
Efter gennemførelse af kurset kan den succesfulde studerende:

- anvende og implementere klassiske og moderne reguleringsteknikker for at kunne styre og regulere en robot hurtig og præcist

[1] Based on https://fagbesk.sam.sdu.dk/?fag_id=39673

The twelve lectures of the course are

- **Lecture 1**: Introduction to Linear Time-Invariant Systems
- **Lecture 2**: Stability and Performance Analysis
- **Lecture 3**: Introduction to Control
- **Lecture 4**: Design of PID Controllers
- **Lecture 5**: Root Locus
- **Lecture 6**: The Nyquist Plot
- **Lecture 7**: Dynamic Compensators and Stability Margins
- **Lecture 8**: Implementation
- **Lecture 9**: State Feedback
- **Lecture 10**: Observer Design
- **Lecture 11**: Optimal Control (Linear Quadratic Control)
- **Lecture 12**: The Kalman Filter

Contents

A continuous time system

$$\dot{x}(t) = Ax(t) + Bu(t)\,, \quad x(0) = 0$$

is said to be *controllable* iff for any $\xi \in \mathbb{R}^n$ there exists $u(t)$ such that for some $T > 0$, $x(T) = \xi$.

A discrete time system

$$x_{k+1} = \Phi x_k + \Gamma u_k\,, \quad x_0 = 0$$

is said to be *controllable* iff for any $\xi \in \mathbb{R}^n$ there exists $(u_0, u_1, \ldots)$ such that for some $N > 0$, $x_N = \xi$.

We consider the discrete time system

$$x_{k+1} = \Phi x_k + \Gamma u_k \,, \quad x_0 = 0$$

and iterate:

$$x_1 \quad = \Phi x_0 + \Gamma u_0$$

We consider the discrete time system

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x_1 &= \Gamma u_0 \\
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x_3 &= \Phi x_2 + \Gamma u_2
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We consider the discrete time system

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$$x_{k+1} = \Phi x_k + \Gamma u_k \, , \quad x_0 = 0$$

and iterate:

$$\begin{aligned} x_1 &= \Gamma u_0 \\ x_2 &= \Phi\Gamma u_0 + \Gamma u_1 \\ x_3 &= \Phi^2\Gamma u_0 + \Phi\Gamma u_1 + \Gamma u_2 \end{aligned}$$

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x_3 &= \Phi^2 \Gamma u_0 + \Phi \Gamma u_1 + \Gamma u_2 \\
&\vdots \\
x_n &= \Phi x_{n-1} + \Gamma u_{n-1}
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&\vdots \\
x_n &= \Phi^{n-1}\Gamma u_0 + \ldots + \Phi\Gamma u_{n-2} + \Gamma u_{n-1}
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Writing the equation

$$x_n = \Phi^{n-1}\Gamma u_0 + \ldots + \Phi\Gamma u_{n-2} + \Gamma u_{n-1}$$

in matrix form we obtain:

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$$x_n = \begin{bmatrix} \Gamma & \Phi\Gamma & \ldots & \Phi^{n-1}\Gamma \end{bmatrix} \begin{bmatrix} u_{n-1} \\ u_{n-2} \\ \vdots \\ u_0 \end{bmatrix}$$

Writing the equation

$$x_n = \Phi^{n-1}\Gamma u_0 + \ldots + \Phi\Gamma u_{n-2} + \Gamma u_{n-1}$$

in matrix form we obtain:

$$x_n = \underbrace{\begin{bmatrix}\Gamma & \Phi\Gamma & \ldots & \Phi^{n-1}\Gamma\end{bmatrix}}_{\text{Controllability matrix}} \begin{bmatrix} u_{n-1} \\ u_{n-2} \\ \vdots \\ u_0 \end{bmatrix}$$

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When is $x_n = \xi$ solvable for any $\xi \in \mathbb{R}^n$?

THEOREM. A system

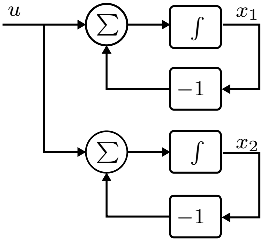
$$\Sigma\ :\ \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \text{ (continuous time)} \\ x_{k+1} &= \Phi x_k + \Gamma u_k \qquad \text{ (discrete time)} \end{cases}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, is controllable if and only if

$$\operatorname{rank} \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = n$$

For $m = 1$ this reduces to

$$\det \left(\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \right) \neq 0$$

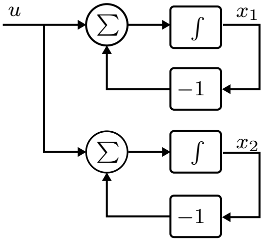


State space equations:

$$\begin{cases} \dot{x}_1 = -x_1 + u \\ \dot{x}_2 = -x_2 + u \end{cases}$$

State space equations in matrix form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$



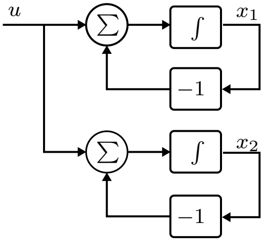
$$\dot{x} = Ax + Bu$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$



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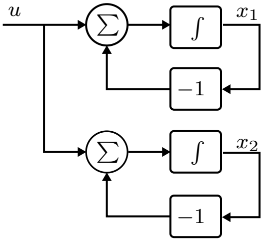

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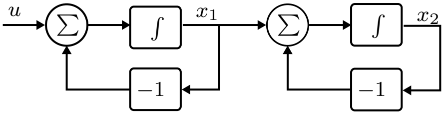
$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad \det(\mathcal{C}) = 0$$

$\mathrm{rank}(\mathcal{C}) = 1 < 2 \implies$ uncontrollable.

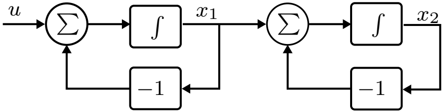


State equations:

$$\begin{cases} \dot{x}_1 &= -x_1 + u \\ \dot{x}_2 &= -x_2 + x_1 \end{cases}$$

State space model in matrix form:

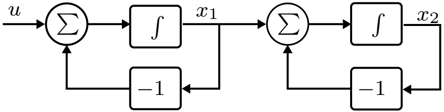
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$



$$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Controllability analysis

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$



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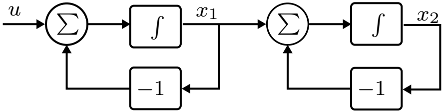
Controllability analysis

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \det(\mathcal{C}) = 1 \neq 0$$



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Controllability analysis

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \det(\mathcal{C}) = 1 \neq 0$$

$\operatorname{rank}(\mathcal{C}) = 2 \implies$ controllable

Controllable Canonical Form

The *single input* state space model

$$\dot{x}_c = A_c x_c + B_c u\,, \quad x_c \in \mathbb{R}^n\,,\, u \in \mathbb{R}$$

is said to be on ***controllable canonical form*** if

$$A_c = \left[\begin{array}{c|c} \multicolumn{2}{c}{a^T} \\ \hline I_{n-1} & 0_{(n-1)\times 1} \end{array}\right] \quad \text{and} \quad B_c = \left[\begin{array}{c} 1 \\ \hline 0_{(n-1)\times 1} \end{array}\right]$$

where $a \in \mathbb{R}^{n\times 1}$, $a^T = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$, I_{n-1} is an $(n-1)\times(n-1)$ identity matrix, and $0_{(n-1)\times 1}$ is an $(n-1)\times 1$ matrix of zeros.

Any controllable *single input* system can be written in the form:

$$\dot{x}_c = A_c x_c + B_c u\,, \quad x_c \in \mathbb{R}^n\,, u \in \mathbb{R}$$

where

$$A_c = \left[\begin{array}{c|c} \multicolumn{2}{c}{a^T} \\ \hline I_{n-1} & 0_{(n-1)\times 1} \end{array}\right], \quad B_c = \left[\begin{array}{c} 1 \\ \hline 0_{(n-1)\times 1} \end{array}\right]$$

and where $a \in \mathbb{R}^{n\times 1}$, $a^T = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$.
It can be shown that

$$\det(\lambda I - A_c) = \lambda^n - a_1\lambda^{n-1} - \ldots - a_n$$

For $n = 3$ the controllable canonical form becomes:

$$A_c = \left(\begin{array}{cc|c} a_1 & a_2 & a_3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right), \; B_c = \left(\begin{array}{c} 1 \\ \hline 0 \\ 0 \end{array}\right)$$

which is indeed controllable:

$$\mathcal{C}_c = \begin{bmatrix} B_c & A_c B_c & A_c^2 B_c \end{bmatrix} = \begin{bmatrix} 1 & a_1 & a_1^2 + a_2 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\det(\mathcal{C}) = 1 \neq 0 \implies$ system is controllable.

Given a state space model of a controllable system:

$$\dot{x} = Ax + Bu\,, \quad x \in \mathbb{R}^n\,,\, u \in \mathbb{R}$$

we wish to find a basis transformation $x = Tx_c$, such that:

$$\dot{x}_c = A_c x_c + B_c u\,, \quad x_c \in \mathbb{R}^n\,,\, u \in \mathbb{R}$$

where $A_c = T^{-1}AT$ and $B_c = T^{-1}B$, is in controllable canonical form. We can solve for T^{-1} by rewriting these equations as

$$A_c T^{-1} = T^{-1}A \quad \text{and} \quad B_c = T^{-1}B$$

We consider $n = 3$, and introduce the following notation for the rows of T^{-1}

$$T^{-1} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}, \quad s_1, s_2, s_3 \in \mathbb{R}^{1\times n}$$

Then we can rewrite the transformation equations $A_c T^{-1} = T^{-1} A$ and $T^{-1} B = B_c$ as:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} A \,, \quad \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Writing out these equations:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} A \,, \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

yields:

Controllable Canonical Form

Transformation to Controllable Canonical Form (3)

Writing out these equations:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} A \, , \quad \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

yields:

$$\left\{ \begin{array}{l} s_1 = s_2 A \\ \\ \end{array} \right\} , \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right\}$$

Writing out these equations:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ \boxed{0 \quad 1 \quad 0} \end{pmatrix} \boxed{\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}} = \begin{bmatrix} s_1 \\ s_2 \\ \boxed{s_3} \end{bmatrix} A \,, \quad \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

yields:

$$\left\{ \begin{array}{l} s_1 = s_2 A \\ \boxed{s_2 = s_3 A} \end{array} \right\} , \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right\}$$

Controllable Canonical Form

Transformation to Controllable Canonical Form (3)

Writing out these equations:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} A \,, \quad \begin{bmatrix} \boxed{s_1} \\ s_2 \\ s_3 \end{bmatrix} B = \begin{bmatrix} \boxed{1} \\ 0 \\ 0 \end{bmatrix}$$

yields:

$$\left\{ \begin{array}{l} s_1 = s_2 A \\ s_2 = s_3 A \end{array} \right\} , \left\{ \begin{array}{l} \boxed{s_1 B = 1} \\ \\ \\ \end{array} \right\}$$

Controllable Canonical Form

Transformation to Controllable Canonical Form (3)

Writing out these equations:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} A\,, \quad \begin{bmatrix} s_1 \\ \boxed{s_2} \\ s_3 \end{bmatrix} B = \begin{bmatrix} 1 \\ \boxed{0} \\ 0 \end{bmatrix}$$

yields:

$$\left\{ \begin{array}{l} s_1 = s_2 A \\ s_2 = s_3 A \end{array} \right\}, \quad \left\{ \begin{array}{l} s_1 B = 1 \\ \boxed{s_2 B = 0} \\ \end{array} \right\}$$

Controllable Canonical Form

Transformation to Controllable Canonical Form (3)

Writing out these equations:

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} A \, , \; \begin{bmatrix} s_1 \\ s_2 \\ \boxed{s_3} \end{bmatrix} B = \begin{bmatrix} 1 \\ 0 \\ \boxed{0} \end{bmatrix}$$

yields:

$$\left\{ \begin{array}{c} s_1 = s_2 A \\ s_2 = s_3 A \end{array} \right\} , \left\{ \begin{array}{c} s_1 B = 1 \\ s_2 B = 0 \\ \boxed{s_3 B = 0} \end{array} \right\}$$

Combining the equations

$$\left\{ \begin{array}{c} s_1 = s_2 A \\ s_2 = s_3 A \end{array} \right\}, \left\{ \begin{array}{c} s_1 B = 1 \\ s_2 B = 0 \\ s_3 B = 0 \end{array} \right\}$$

we obtain

$$s_3 \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

yielding the solution

$$s_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathcal{C}^{-1}, \, s_2 = s_3 A \, , \, s_1 = s_2 A$$

for nonsingular $\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$.

Controllable Canonical Form
Example: Companion Form (1)

We consider the system

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} x + \begin{bmatrix} 2 \\ 3 \end{bmatrix} u \\
y &= \begin{bmatrix} -3 & 2 \end{bmatrix} x
\end{aligned}
$$

having the controllability matrix

$$
\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ 3 & -7 \end{bmatrix}, \quad \det(\mathcal{C}) = 1 \neq 0
$$

We compute the rows of T^{-1} by

$$s_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathcal{C}^{-1} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -7 & 5 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 2 \end{bmatrix}$$

$$s_1 = s_2 A = \begin{bmatrix} -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 2 & -1 \end{bmatrix}$$

Thus,

$$T^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \implies T = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

Controllable Canonical Form

Example: Companion Form (3)

Eventually, we have

$$A_c = T^{-1}AT = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$
$$= \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}$$

and

Eventually, we have

$$
\begin{aligned}
A_c = T^{-1}AT &= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \\
&= \left(\begin{array}{c|c} -3 & -2 \\ \hline 1 & 0 \end{array} \right)
\end{aligned}
$$

and

Eventually, we have

$$A_c = T^{-1}AT = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \left(\begin{array}{c|c} -3 & -2 \\ \hline 1 & 0 \end{array} \right)$$

and

$$B_c = T^{-1}B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Eventually, we have

$$A_c = T^{-1}AT = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$
$$= \left(\begin{array}{c|c} -3 & -2 \\ \hline 1 & 0 \end{array} \right)$$

and

$$B_c = T^{-1}B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \left(\begin{array}{c} 1 \\ \hline 0 \end{array} \right)$$

Eventually, we have

$$A_c = T^{-1}AT = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$
$$= \left(\begin{array}{c|c} -3 & -2 \\ \hline 1 & 0 \end{array} \right) \Rightarrow \det(\lambda I - A) = \lambda^2 + 3\lambda + 2$$

and

$$B_c = T^{-1}B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \left(\frac{1}{0} \right)$$

Eventually, we have

$$A_c = T^{-1}AT = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \left(\begin{array}{cc} -3 & -2 \\ \hline 1 & \multicolumn{1}{|c}{0} \end{array} \right) \Rightarrow \det\left(\lambda I - A\right) = (\lambda + 1)(\lambda + 2)$$

and

$$B_c = T^{-1}B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \left(\begin{array}{c} 1 \\ \hline 0 \end{array} \right)$$

State Feedback and Pole Assignment

For a state space model

$$\dot{x} = Ax + Bu$$

a *state feedback* is a feedback of the form

$$u = Fx$$

Combining these two equations, we obtain:

$$\dot{x} = Ax + BFx = (A + BF)x$$

Thus, the result of a state feedback is a system with a modified system matrix, and thus with modified poles.

For a single input system in companion form, a state feedback takes a particular simple form:

$$A_c = \left(\begin{array}{cc|c} a_1 & a_2 & a_3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right), \quad B_c = \left(\begin{array}{c} 1 \\ \hline 0 \\ 0 \end{array}\right)$$

Applying the feedback $u = Fx$ with

$$F_c = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}$$

We obtain:

$$A_c + B_c F_c = \left(\begin{array}{cc|c} a_1 & a_2 & a_3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) + \left(\begin{array}{c} 1 \\ \hline 0 \\ 0 \end{array}\right) \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}$$

$$= \left(\begin{array}{cc|c} a_1 + f_1 & a_2 + f_2 & a_3 + f_3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

Thus, the characteristic polynomial has been changed from

$$\det(\lambda I - A_c) = \lambda^n - a_1\lambda^{n-1} - \ldots - a_n$$

to

$$\det(\lambda I - (A_c + B_c F_c)) = \lambda^n - (a_1 + f_1)\lambda^{n-1} - \ldots - (a_n + f_n)$$

By choosing $f_1, \ldots, f_n$ appropriately, *any* closed loop pole configuration can be obtained. This is known as *pole assignment*.

Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ be given.

1. Choose desired closed loop polynomial
   $\det(\lambda I - (A + BF)) = \lambda^n - a_{\text{cl},1}\lambda^{n-1} - \ldots - a_{\text{cl},n}$.

Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$ be given.

1. Choose desired closed loop polynomial $\det(\lambda I - (A + BF)) = \lambda^n - a_{\text{cl},1}\lambda^{n-1} - \ldots - a_{\text{cl},n}$.
2. Determine T, such that $A_c = T^{-1}AT$ and $B_c = T^{-1}B$ are in controllable canonical form.

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4. Define $F_c = \begin{bmatrix} a_{\text{cl},1} - a_1 & \ldots & a_{\text{cl},n} - a_n \end{bmatrix}$.
5. Compute resulting feedback gain $F = F_c T^{-1}$.

We consider again the system

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} x + \begin{bmatrix} 2 \\ 3 \end{bmatrix} u \\ y &= \begin{bmatrix} -3 & 2 \end{bmatrix} x\end{aligned}$$

for which we would like to move the poles to $\{-4, -5\}$.

1. Desired closed loop polynomial: $\lambda^2 + 9\lambda + 20$

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2. $T = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \Rightarrow A_c = \left(\begin{array}{c|c} -3 & -2 \\ \hline 1 & 0 \end{array}\right), B_c = \left(\begin{array}{c} 1 \\ \hline 0 \end{array}\right)$

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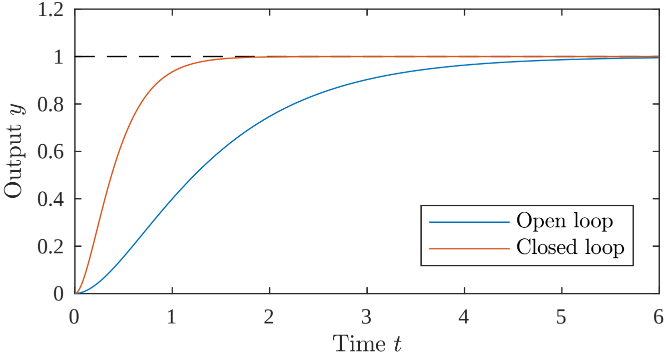
3. Open loop polynomial: $\lambda^2 + 3\lambda + 2$
4. $F_c = \begin{bmatrix} 3-9 & 2-20 \end{bmatrix} = \begin{bmatrix} -6 & -18 \end{bmatrix}$
5. $$\begin{aligned} F &= F_c T^{-1} = \begin{bmatrix} -6 & -18 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 42 & -30 \end{bmatrix} \end{aligned}$$



1.2
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6
Output y
Time t
Open loop
Closed loop

We consider a system with two inputs

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} x + \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} u \\ y &= \begin{bmatrix} -3 & 2 \end{bmatrix} x \end{aligned}$$

for which we would like to move the poles to $\{-4, -5\}$.

The system

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} x + \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} u \\
y &= \begin{bmatrix} -3 & 2 \end{bmatrix} x
\end{aligned}
$$

has controllability matrix

$$
\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 2 & 0 & -5 & -3 \\ 3 & 1 & -7 & -5 \end{bmatrix}, \quad \operatorname{rank}(\mathcal{C}) = 2 = n
$$

By use of the command `place` in MATLAB we obtain the controller gain

$$F = \begin{bmatrix} -31.5 \\ 5 - 4.5 \end{bmatrix}$$

Introduction

Controllability

Controllable Canonical Form

State Feedback and Pole Assignment

Summary

A continuous time system

$$\dot{x}(t) = Ax(t) + Bu(t)\,, \quad x(0) = 0$$

is said to be *controllable* if and only if for any $\xi \in \mathbb{R}^n$ there exists $u(t)$ such that for some $T > 0$, $x(T) = \xi$.

A discrete time system

$$x_{k+1} = \Phi x_k + \Gamma u_k\,, \quad x_0 = 0$$

is said to be *controllable* if and only if for any $\xi \in \mathbb{R}^n$ there exists $(u_0, u_1, \ldots)$ such that for some $N > 0$, $x_N = \xi$.

A system

$$\Sigma \; : \; \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) & \text{(continuous time)} \\ x_{k+1} = \Phi x_k + \Gamma u_k & \text{(discrete time)} \end{cases}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, is controllable if and only if

$$\operatorname{rank} \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = n$$

For a state space model

$$\dot{x} = Ax + Bu$$

a *state feedback* is a feedback of the form

$$u = Fx.$$

The closed-loop system dynamics is:

$$\dot{x} = (A + BF)x$$

If the system (A, B) is controllable then there exists a state feedback such that the eigenvalues of $A + BF$ can be placed anywhere in the complex plane.