Lecture 9: Sliding Mode Control of Underactuated Systems

Underactuated Robotics

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Agenda

Introduction

Introduction to Sliding-Mode Control

System Model and Assumptions

Design of Sliding-Mode Control
- Case 1
- Case 2

Flight Control of a Quadrotor Helicopter

Knowledge:

- Derive dynamical state-space models of robots as control systems
- Analyze the stability of low dimensional linear and nonlinear systems
- Analyze the observability and controllability of linear control systems
- Use a variety of controllers for underactuated robots

Skills:

- Implement simulations of control systems in software
- Create concise technical reports presenting solutions to proposed problems

Competencies:

- Choose appropriate modern control techniques to solve control problems in robotics
- Apply modern control techniques to control simulated underactuated robots

- **Lesson 1**: Newton-Euler Modelling
- **Lesson 2**: Euler-Lagrange Modelling
- **Lesson 3**: Simulation of Robot Dynamics
- **Lesson 4**: Stability Analysis
- **Lesson 5**: Optimal Control
- **Lesson 6**: Feedback Linearisation
- **Lesson 7**: Energy Shaping Control
- **Lesson 8**: Simulation and Implementation of Control Systems
- **Lesson 9**: Sliding Mode Control
- **Lesson 10**: Help with hand-in
- **Lesson 11**: Help with hand-in
- **Lesson 12**: Help with hand-in

Sliding-mode control is a favorable control method when

- The parameters of the considered system are unknown
- Finite time convergence is needed

Introduction

Introduction to Sliding-Mode Control

System Model and Assumptions

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Flight Control of a Quadrotor Helicopter

Consider the second order system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = h(x) + g(x)u$$

where h and g are unknown functions and $g(x) \geq g_0 > 0$ for all x.

How can a controller be designed to stabilize the origin despite the unknown parameters?

Suppose that we can find a control law that constrains the motion of the system to the surface

$$s = a_1 x_1 + x_2 = 0$$

On the surface, the system dynamics are

$$\dot{x}_1 = -a_1 x_1$$

If $a_1 > 0$ then $x(t)$ converges to the origin when t goes to infinite, and the motion on s is independent of g and h.

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If $a_1 > 0$ then $x(t)$ converges to the origin when t goes to infinite, and the motion on s is independent of g and h.

To reach the sliding surface (in finite time) the following control can be applied

$$u = -\beta \texttt{sign}(s)$$

System Model and Assumptions

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System Model and Assumptions

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Flight Control of a Quadrotor Helicopter

The equation of motion for an underactuated system can be written as

$$m_{11}(\boldsymbol{q})\ddot{\boldsymbol{q}}_1 + m_{12}(\boldsymbol{q})\ddot{\boldsymbol{q}}_2 + \boldsymbol{h}_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0$$
$$m_{21}(\boldsymbol{q})\ddot{\boldsymbol{q}}_1 + m_{22}(\boldsymbol{q})\ddot{\boldsymbol{q}}_2 + \boldsymbol{h}_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{\tau}$$

where $\boldsymbol{q} = (\boldsymbol{q}_1, \boldsymbol{q}_2)$ is the generalized coordinate, $\boldsymbol{\tau}$ is the control input, and h_i contains Coriolis, centrifugal and gravity terms.

System Model and Assumptions

Underactuated System Model in Cascade Normal Form

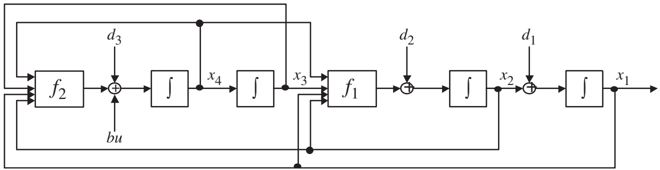
The equation of motion for a class of underactuated systems can be rewritten into cascade normal form

$$\begin{aligned}
\dot{x}_1 &= x_2 + d_1 \\
\dot{x}_2 &= f_1(x_1, x_2, x_3, x_4) + d_2 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_2(x_1, x_2, x_3, x_4) + b(x_1, x_2, x_3, x_4)u + d_3
\end{aligned}$$

where $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ is the system state, $\boldsymbol{u} \in \mathbb{R}^n$ is the control input, and $f_1, f_2 : \mathbb{R}^{4n} \to \mathbb{R}^n$ and $b : \mathbb{R}^{4n} \to \mathbb{R}^{n\times n}$ are nonlinear smooth sunctions, b is invertible, and $d_1, d_2, d_3 \in \mathbb{R}^n$ are disturbances.

System Model and Assumptions
Structure of Cascade Normal Form

The following diagram shows the structure of the cascade normal form.



We assume the following about the system model

1. $f_1(0,0,0,0) = 0$
2. $\partial f_1/\partial x_3$ or $\partial f_1/\partial x_4$ is invertible.
3. $f_1(0,0,x_3,x_4) = 0$ is asymptotically stable.

System Model and Assumptions

Assumptions for System

We assume the following about the system model

1. $f_1(0,0,0,0) = 0$
2. $\partial f_1/\partial x_3$ or $\partial f_1/\partial x_4$ is invertible.
3. $f_1(0,0,x_3,x_4) = 0$ is asymptotically stable.

The system does not necessarily satisfy Assumption 2 and 3, but then d_2 can be chosen to ensure this.

Cascade normal form

$$
\begin{aligned}
\dot{x}_1 &= x_2 + d_1\\
\dot{x}_2 &= f_1(x_1, x_2, x_3, x_4) + d_2\\
\dot{x}_3 &= x_4\\
\dot{x}_4 &= f_2(x_1, x_2, x_3, x_4) + b(x_1, x_2, x_3, x_4)u + d_3
\end{aligned}
$$

Considered system

$$
\begin{aligned}
\dot{x}_1 &= x_2\\
\dot{x}_2 &= \frac{g}{l}\sin(x_1) + x_3^3\\
\dot{x}_3 &= x_4\\
\dot{x}_4 &= u
\end{aligned}
$$

System Model and Assumptions

Example (System)

Cascade normal form

$$\begin{aligned}
\dot{x}_1 &= x_2 + d_1\\
\dot{x}_2 &= f_1(x_1, x_2, x_3, x_4) + d_2\\
\dot{x}_3 &= x_4\\
\dot{x}_4 &= f_2(x_1, x_2, x_3, x_4) + b(x_1, x_2, x_3, x_4)u + d_3
\end{aligned}$$

Considered system

$$\begin{aligned}
\dot{x}_1 &= x_2\\
\dot{x}_2 &= \frac{g}{l}\sin(x_1) + x_3^3 + x_3 - x_3\\
\dot{x}_3 &= x_4\\
\dot{x}_4 &= u
\end{aligned}$$

System Model and Assumptions
Example (System)

Cascade normal form

$$
\begin{aligned}
\dot{x}_1 &= x_2 + d_1 \\
\dot{x}_2 &= f_1(x_1, x_2, x_3, x_4) + d_2 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_2(x_1, x_2, x_3, x_4) + b(x_1, x_2, x_3, x_4)u + d_3
\end{aligned}
$$

Considered system

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \underbrace{\frac{g}{l}\sin(x_1) + x_3^3 + x_3}_{=f_1} + \underbrace{-x_3}_{=d_2} \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= u
\end{aligned}
$$

System Model and Assumptions
Example (Assumptions)

For

$$f_1 = \frac{g}{l}\sin(x_1) + x_3^3 + x_3$$

1. $f_1(0,0,0,0) = 0$
2. $\partial f_1/\partial x_3$ is invertible.

Design of Sliding-Mode Control

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Introduction to Sliding-Mode Control

System Model and Assumptions

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Flight Control of a Quadrotor Helicopter

The controller that will be developed will be on the form

$$u = u_{eq} + u_{sw}$$

where u_{eq} is the control on the sliding manifold and u_{sw} will ensure finite time convergence to the sliding manifold.

We consider two cases

1. $\partial f_1/\partial x_4 = 0$ and $\partial f_1/\partial x_3$ is invertible.
2. $\partial f_1/\partial x_4$ is invertible.

Design of Sliding-Mode Control
Error Variable

The following error variables are used in the controller design

$$
\begin{aligned}
e_1 &= x_1 \\
e_2 &= x_2 \\
e_3 &= f_1(x_1, x_2, x_3, x_4) \\
e_4 &= \frac{\partial f_1}{\partial x_1} x_2 + \frac{\partial f_1}{\partial x_2} f_1 + \frac{\partial f_1}{\partial x_3} x_4 \\
E_1 &= [e_1, e_2, e_3]^T \\
E_2 &= [e_1, e_2]^T
\end{aligned}
$$

When $\partial f_1/\partial x_4 = 0$ and $\partial f_1/\partial x_3$ is invertible then the switching surface is defined as

$$s = c_1 e_1 + c_2 e_2 + c_3 e_3 + e_4$$

When $\partial f_1/\partial x_4 = 0$ and $\partial f_1/\partial x_3$ is invertible then the switching surface is defined as

$$s = c_1 e_1 + c_2 e_2 + c_3 e_3 + e_4$$

The constants c_i for $i = 1, 2, 3$ should be chosen such that all eigenvalues of

$$A_{n1} = \begin{bmatrix} 0 & I_n & 0 \\ 0 & 0 & I_n \\ -c_1 I_n & -c_2 I_n & -c_3 I_n \end{bmatrix}$$

have negative real part and

$$\max\{\bar{d}_1, \bar{d}_2, \beta_1 \bar{d}_1 + \beta_2 \bar{d}_2\} < \gamma$$

where γ is the real part of the smallest eigenvalue of $-A_{n1}$ and $\beta_1, \beta_2 \geq 0$ are related to row sums, see Assumptions 4 and 5.

If $\partial f_1/\partial x_4 = 0$ and $\partial f_1/\partial x_3$ is invertible, then the control

$$u = u_{eq} + u_{sw}$$

with $\rho, \lambda > 0$ and

$$u_{eq} = -\left[\frac{\partial f_1}{\partial x_3} b\right]^{-1} \left(c_1 x_2 + c_2 f_1 + c_3 \frac{\partial f_1}{\partial x_1} x_2 + c_3 \frac{\partial f_1}{\partial x_2} f_1 + c_3 \frac{\partial f_1}{\partial x_3} x_4 + \frac{d}{dt}\left[\frac{\partial f_1}{\partial x_1} x_2\right] + \frac{d}{dt}\left[\frac{\partial f_1}{\partial x_2} f_1\right] + \frac{d}{dt}\left[\frac{\partial f_1}{\partial x_3}\right] x_4 + \frac{\partial f_1}{\partial x_3} f_2\right)$$

$$u_{sw} = -\left[\frac{\partial f_1}{\partial x_3} b\right]^{-1} (M \,\mathsf{sign}(s) + \lambda s)$$

$$M = (c_1 \bar{d}_1 + c_2 \bar{d}_2 + c_3 \beta_1 \bar{d}_1 + c_3 \beta_2 \bar{d}_2)||E_1||_2 + \beta_3(\bar{d}_3 + \bar{d}_2||\zeta(x)||_2) + \rho$$

will asymptotically stabilize the following system to 0

$$\begin{aligned}
\dot{x}_1 &= x_2 + d_1 \\
\dot{x}_2 &= f_1(x_1, x_2, x_3, x_4) + d_2 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_2(x_1, x_2, x_3, x_4) + b(x_1, x_2, x_3, x_4)u + d_3
\end{aligned}$$

When $\partial f_1/\partial x_4$ is invertible then the switching surface is defined as

$$s = c_1 e_1 + c_2 e_2 + e_3$$

Design of Sliding-Mode Control

Switching Surface

When $\partial f_1/\partial x_4$ is invertible then the switching surface is defined as

$$s = c_1 e_1 + c_2 e_2 + e_3$$

The constants c_i for $i = 1, 2, 3$ should be chosen such that all eigenvalues of

$$A_{n1} = \begin{bmatrix} 0 & I_n \\ -c_1 I_n & -c_2 I_n \end{bmatrix}$$

have negative real part and

$$\max\{\bar{d}_1, \bar{d}_2\} < \gamma$$

where γ is the real part of the smallest eigenvalue of $-A_{n2}$.

If $\partial f_1/\partial x_4$ is invertible, then the control

$$u = u_{eq} + u_{sw}$$

with $\rho, \lambda > 0$ and

$$\begin{aligned}
u_{eq} &= -\left[\frac{\partial f_1}{\partial x_4} b\right]^{-1}\left(c_1 x_2 + c_2 f_1 + \frac{\partial f_1}{\partial x_1} x_2 + \frac{\partial f_1}{\partial x_2} f_1 + \frac{\partial f_1}{\partial x_3} x_4 + \frac{\partial f_1}{\partial x_4} f_2\right)\\
u_{sw} &= -\left[\frac{\partial f_1}{\partial x_4} b\right]^{-1}(M \text{sign}(s) + \lambda s)\\
M &= (c_1 \bar{d}_1 + c_2 \bar{d}_2 + \beta_1 \bar{d}_1 + \beta_2 \bar{d}_2)||E_2||_2 + \beta_4(\bar{d}_3 + \bar{d}_2||\zeta(x)||_2) + \rho
\end{aligned}$$

will asymptotically stabilize the following system to 0

$$\begin{aligned}
\dot{x}_1 &= x_2 + d_1\\
\dot{x}_2 &= f_1(x_1, x_2, x_3, x_4) + d_2\\
\dot{x}_3 &= x_4\\
\dot{x}_4 &= f_2(x_1, x_2, x_3, x_4) + b(x_1, x_2, x_3, x_4)u + d_3
\end{aligned}$$

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Flight Control of a Quadrotor Helicopter

The dynamic model of the quadrotor helicopter is given by

$$
\begin{aligned}
\ddot{x} &= u_1(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi) - K_1\dot{x}/m \\
\ddot{y} &= u_1(\sin\phi\sin\theta\cos\psi + \cos\phi\sin\psi) - K_2\dot{y}/m \\
\ddot{z} &= u_1(\cos\phi\cos\psi) - g - K_3\dot{z}/m \\
\ddot{\theta} &= u_2 - lK_4\dot{\theta}/I_1 \\
\ddot{\psi} &= u_3 - lK_5\dot{\psi}/I_2 \\
\ddot{\phi} &= u_4 - lK_4\dot{\phi}/I_3
\end{aligned}
$$

where (x, y, z) is the position and (θ, ψ, ϕ) is the orientation given by Euler angles (pitch, roll, yaw).

The dynamic model of the quadrotor helicopter can be reformulated as two sub-systems

Fully actuated subsystem:

$$\begin{bmatrix} \ddot{z} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} u_1(\cos\phi\cos\psi) - g \\ u_4 \end{bmatrix} + \begin{bmatrix} -K_3\dot{z}/m \\ -lK_4\dot{\phi}/I_3 \end{bmatrix}$$

Underactuated subsystem:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} u_1(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi) \\ u_1(\sin\phi\sin\theta\cos\psi + \cos\phi\sin\psi) \end{bmatrix} + \begin{bmatrix} -K_1\dot{x}/m \\ -K_2\dot{y}/m \end{bmatrix}$$

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} -lK_4\dot{\theta}/I_1 \\ -lK_5\dot{\psi}/I_2 \end{bmatrix}$$

The dynamic model of the quadrotor helicopter can be reformulated as two sub-systems

Fully actuated subsystem:

$$\begin{bmatrix} \ddot{z} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} u_1(\cos\phi\cos\psi) - g \\ u_4 \end{bmatrix} + \begin{bmatrix} -K_3\dot{z}/m \\ -lK_4\dot{\phi}/I_3 \end{bmatrix}$$

Underactuated subsystem:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} u_1(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi) \\ u_1(\sin\phi\sin\theta\cos\psi + \cos\phi\sin\psi) \end{bmatrix} + \begin{bmatrix} -K_1\dot{x}/m \\ -K_2\dot{y}/m \end{bmatrix}$$

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} -lK_4\dot{\theta}/I_1 \\ -lK_5\dot{\psi}/I_2 \end{bmatrix}$$

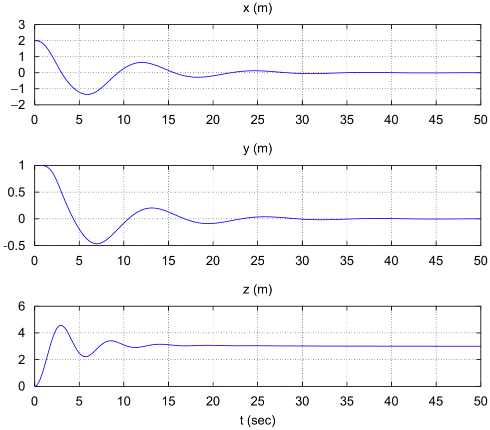
Only the underactuated subsystem is of interest in this course.

The underactuated subsystem has dynamics given by

$$
\begin{aligned}
u &= \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} \\
f_1 &= \begin{bmatrix} \sin\theta\cos\psi & \sin\psi \end{bmatrix}^T \\
f_2 &= \begin{bmatrix} 0 & 0 \end{bmatrix}^T \\
b &= I \\
d_1 &= -\frac{\dot{u}_1}{u_1} x_1 \\
d_2 &= -\frac{\dot{u}_1}{u_1} x_2 - \hat{T}^{-1} \begin{bmatrix} \frac{K_1}{m} & 0 \\ 0 & \frac{K_2}{m} \end{bmatrix} \hat{T} x_2 \\
d_3 &= \begin{bmatrix} -\frac{lK_4}{I_1} & 0 \\ 0 & -\frac{lK_5}{I_2} \end{bmatrix}
\end{aligned}
$$

Flight Control of a Quadrotor Helicopter

Simulation Results (I)



Flight Control of a Quadrotor Helicopter

Simulation Results (II)