Problem 1:

Find $\iint_S x\,dS$ over the part of the parabolic cylinder $z = x^2/2$ that lies inside the first octant part of the cylinder $x^2 + y^2 = 1$.

Solution: $\pi/8$

$$z = \frac{x^2}{2}$$

$$\Rightarrow x = v, \quad z = \frac{v^2}{2}, \quad y = u \quad \Rightarrow \quad \vec{r}(u,v) = v\,i + u\,j + \frac{v^2 k}{2}$$

$$\Rightarrow \begin{cases} r_u = 1j \\ r_v = 1i + vk \end{cases}$$

$$dS = |r_u \times r_v| \; du\,dv$$

$$|r_u \times r_v| = \left\| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 1 & 0 & v \end{matrix} \right\| = |(1\cdot v - 0\cdot 0)i - (0\cdot v - 1\cdot 0)k + (0\cdot 0 - 1\cdot 0)k| = |vi - k| = \sqrt{v^2 + (-1)^2} = \sqrt{v^2+1}$$

$$\iint_S x\,dS = \int_v\int_u v\cdot\sqrt{v^2+1}\;du\,dv$$

Finding the bounds

$$0 \le x,\; 0 \le y \Rightarrow 0 \le v,\; 0 \le u$$

$$x^2 + y^2 \le 1 \Rightarrow v^2 + u^2 \le 1$$

Transform to polar coordinates

$x = r\cdot\cos(\theta) \quad y = r\cdot\sin(\theta) \quad \mathrm{dA} = r\,\mathrm{d}\theta\,\mathrm{dr} \quad x^2 + y^2 = a^2 \Rightarrow a$

$$0 \le v,\; 0 \le u \Rightarrow 0 \le \theta \le \frac{\pi}{2}$$

$$v^2 + u^2 \le 1 \Rightarrow 0 \le r \le 1$$

$$\int_v\int_u v\cdot\sqrt{v^2+1}\;du\,dv \Rightarrow \int_0^{\frac{\pi}{2}}\int_0^1 r\cdot\cos(\theta)\cdot\sqrt{r^2\cdot\cos^2(\theta)+1}\cdot r\;dr\,d\theta = \underline{\underline{\frac{\pi}{8}}}$$

Problem 2:

Find the total charge on the surface

$$\mathbf{r} = e^u \cos v\mathbf{i} + e^u \sin v\mathbf{j} + u\mathbf{k}, \qquad (0 \le u \le 1, 0 \le v \le \pi),$$

if the charge density on the surface is $\delta = \sqrt{1+e^{2u}}$.

Solution: $\frac{\pi}{3}(3e + e^3 - 4)$

$$\int_0^\pi\int_0^1 \delta(u,v)\cdot\underbrace{|\vec{r}_u \times \vec{r}_v|}\,du\,dv$$

$$\begin{cases} \vec{r}_u = e^u\cdot\cos(v)i + e^u\cdot\sin(v)j + k \\ \vec{r}_v = -e^u\cdot\sin(v)i + e^u\cdot\cos(v)j \end{cases} \Rightarrow \vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ e^u\cdot\cos(v) & e^u\cdot\sin(v) & 1 \\ -e^u\cdot\sin(v) & e^u\cdot\cos(v) & 0 \end{vmatrix}$$

$$= (0 - e^u\cdot\cos(v))\,i - (0-(-e^u\cdot\sin(v)))\,j + (e^{u^2}\cdot\cos^2(v) - (-e^{u^2}\cdot\sin^2(v)))\,k$$

$$= -e^u\cdot\cos(v)\,i - e^u\cdot\sin(v)\,j + e^{u^2}\cdot\overbrace{(\cos^2(v)+\sin^2(v))}^{1}\,k$$

$$= -e^u\cdot\cos(v)\,i - e^u\cdot\sin(v)\,j + e^{u^2}\,k$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{e^{u^2}\cdot\cos^2(v) + e^{u^2}\cdot\sin^2(v) + e^{u^4}} = \sqrt{e^{u^2}\cdot\underbrace{(\cos^2(v)+\sin^2(v))}_{1} + e^{u^4}}$$

$$= \sqrt{e^{u^2}+e^{u^4}} = \sqrt{e^{u^2}(1+e^{u^2})} = e^u\sqrt{e^{u^2}+1} = e^u\sqrt{e^{2u}+1}$$

$$\Rightarrow \int_0^\pi\int_0^1 \delta(u,v)\cdot|\vec{r}_u \times \vec{r}_v|\,du\,dv = \int_0^\pi\int_0^1 \sqrt{e^{2u}+1}\cdot e^u\sqrt{e^{2u}+1}\,du\,dv = \int_0^\pi\int_0^1 e^u\cdot(e^{2u}+1)\,du\,dv$$

$$= \int_0^\pi\int_0^1 e^{3u} + e^u\,du\,dv = \int_0^\pi \left[\frac{1}{3}\cdot e^{3u} + e^u\right]_0^1 dv = \int_0^\pi \frac{1}{3}\cdot e^3 + e^1 - \frac{1}{3}\cdot e^0 - e^0\,dv = \int_0^\pi \frac{1}{3}\cdot e^3 + e - \frac{1}{3} - 1\,dv$$

$$= \left[\frac{1}{3}\cdot e^3\cdot v + e\cdot v - \frac{v}{3} - v\right]_0^\pi = \frac{1}{3}e^3\pi + e\pi - \frac{\pi}{3} - \pi = \frac{\pi}{3}(e^3 + 3e - 1 - 3) = \underline{\underline{\frac{\pi}{3}(e^3+3e-4)}}$$

Problem 3:

Describe the parametric surface

$$x = au\cos v, \quad y = au\sin v, \quad z = bv$$

$(0 \le u \le 1, 0 \le v \le 2\pi)$, and find its area.

Solution: $\pi a\sqrt{a^2+b^2} + \pi b^2 \ln\left(\frac{a+\sqrt{a^2+b^2}}{b}\right)$ sq. units

$$\vec{r}(u,v) = au\cdot\cos(v)\hat{i} + au\cdot\sin(v)\hat{j} + bv\hat{k}$$

$$\Rightarrow \begin{cases} \vec{r}_u = a\cdot\cos(v)\hat{i} + a\cdot\sin(v)\hat{j} \\ \vec{r}_v = -au\cdot\sin(v)\hat{i} + au\cdot\cos(v)\hat{j} + b\hat{k} \end{cases}$$

$$\Rightarrow \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cdot\cos(v) & a\cdot\sin(v) & 0 \\ -au\cdot\sin(v) & au\cdot\cos(v) & b \end{vmatrix} = (ab\cdot\sin(v) - \cancel{0})\hat{i} - (ab\cdot\cos(v) - \cancel{0})\hat{j} + (u\cdot a^2\cos^2(v) \underbrace{-(-}_{+} u\cdot a^2\sin^2(v))k$$

$$= ab\cdot\sin(v)\hat{i} - ab\cdot\cos(v)\hat{j} + (u\cdot a^2\cos^2(v) + u\cdot a^2\sin^2(v))k$$

$$= ab\cdot\sin(v)\hat{i} - ab\cdot\cos(v)\hat{j} + (u\cdot a^2\cdot\overbrace{(\cos^2(v)+\sin^2(v)}^{1})k$$

$$= ab\cdot\sin(v)\hat{i} - ab\cdot\cos(v)\hat{j} + a^2uk$$

$$\Rightarrow |\vec{r}_u \times \vec{r}_v| = \sqrt{a^2b^2\sin^2(v) + a^2b^2\cos^2(v) + a^4u^2}$$

$$= \sqrt{a^2b^2\cdot\underbrace{(\sin^2(v)+\cos^2(v))}_{1} + a^4u^2}$$

$$= \sqrt{a^2b^2 + a^4u^2} = \sqrt{a^2\cdot(b^2+a^2u^2)}$$

$$= a\sqrt{b^2+a^2u^2}$$

Not good...

$$\lambda = b^2 + a^2u^2 \Rightarrow \frac{d\lambda}{du} = 2a^2u \Rightarrow du = \frac{1}{2a^2u}d\lambda$$

$$\Rightarrow \int_0^{2\pi}\int_0^1 1\cdot a\sqrt{b^2+a^2u^2}\,du\,dv = a\cdot\int_0^{2\pi}\int_0^1 \sqrt{b^2+a^2u^2}\,du\,dv =$$