Question 1.1: Find the Fourier transform of Delta function $\delta(x-a)$

The forier transform of an impule is can be found by table

$$\mathcal{F}\left[\delta(t-t_0)\right] = e^{-j\omega t_0} \Rightarrow \mathcal{F}\left[\delta(t-a)\right] = e^{j\omega a}$$

Question 1.2: Find Inverse Laplace of $\left[\frac{1}{S}+\frac{2}{S^2}+\frac{3}{S^2+4}+\frac{S}{S^2+16}\right]$

$a=1$ $a=2$ $a=4$

Hint: Use Laplace table

From the table:

$$\mathcal{L}[1] = \frac{1}{s}, \quad \mathcal{L}\left[\sin(at)\right] = \frac{a}{s^2+a^2}, \quad \mathcal{L}\left[\cos(at)\right] = \frac{s}{s^2+a^2}$$

$$f(t) = 1 + 2\cdot\sin(at) + \frac{2}{3}\cdot\sin(2t) + \cos(4t)$$

Question 1.3: Provide the definition for a function to be homogenous.

Check if $f(x,y,z) = x^2+y^2-z^2$ is homogenous or not.

Homogenous functions can be scaled by their inputs like this:

$$f(tx, ty) = t\cdot f(x,y)$$

Check

$$f(tx, ty, tz) = t\cdot f(x,y,z)$$

$$\Rightarrow (tx)^2 + (ty^2) - (tz)^2 = t\cdot(x^2+y^2-z^2)$$

$$\Rightarrow t^2x^2 + t^2y^2 - t^2z^2 \neq tx^2 + ty^2 - tz^2$$

f(x, y ,z) is NOT homogenous

Question 1.4: Find partial derivative $f_{122}(x,y,z)$ when $f = e^{(3x+4y)}\cos(5z)$

$$f_x(x,y,z) = \frac{d}{dx}\left(e^{(3x+4y)}\cdot\cos(5z)\right) = e^{(3x+4y)}\cdot 3\cdot\cos(5z)$$

$$f_{xy}(x,y,z) = \frac{d}{dy}\left(e^{(3x+4y)}\cdot 3\cdot\cos(5z)\right) = 3\cdot 4\cdot e^{(3x+4y)}\cdot\cos(5z)$$

$$f_{xyy}(x,y,z) = \frac{d}{dy}\left(12\cdot e^{(3x+4y)}\cdot\cos(5z)\right) = 12\cdot 4\cdot e^{(3x+4y)}\cdot\cos(5z)$$

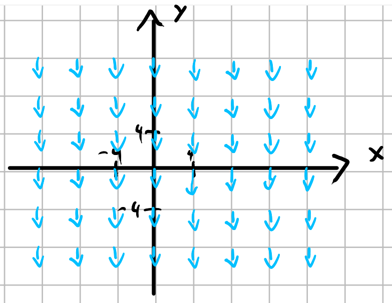
$$= 48\cdot e^{(3x+4y)}\cdot\cos(5z)$$

Question 1.5: Evaluate the limit for the expression $\lim_{(x,y)\to(2,-1)} xy + x^2$

Inserting directly

$$(2\cdot(-1)) + 2^2 = -2 + 4 = \underline{\underline{2}}$$

Question 1.6: Sketch the vector field for $\mathbf{F}(x, y) = -2\mathbf{j}$



Question 1.7: Find divergence of vector field $\mathbf{F}(x, y) = -\sin x\mathbf{i} + \sin y\mathbf{j}$

$$\operatorname{div} \vec{F}(x,y) = \nabla \cdot \vec{F}(x,y) = \frac{d}{dx}(-\sin(x)) + \frac{d}{dy}(\sin(y))$$

$$\underline{\underline{= -\cos(x) + \cos(y)}}$$

Question 1.8: Consider one to one transformation (x, y) from domain S in the uv plane on to a domain D in the xy plane. If $f(x, y)$ is integrable on D and has continuous partial derivatives of (x, y) with respect to (u, v) in S domain, then which of the following statement is correct:

a) The function g is not necessarily integrable on S.

b) If $\frac{\partial(x,y)}{\partial(u,v)}$ are continuous in S, the $g(u, v)$ is integrable on S, and $\iint_D f(x, y)\, dxdy = \iint_S g(u, v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$

c) The continuity of $\frac{\partial(x,y)}{\partial(u,v)}$ in S-domain does not guarantee the integrability of function g on S.

d) The integrability of function g on S is independent of the continuity of $\frac{\partial(x,y)}{\partial(u,v)}$ in S.

Question 1.9: Consider a material with a thermal conductivity (k), density (ρ), and specific heat (q). The heat conduction equation for this material is given by $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ where, $u(x, t)$ is temperature in space, and $c^2 = k/\rho q$ is thermal diffusivity.

Which of the following statement is correct about k, ρ, and q

a) Increasing thermal conductivity leads to faster temperature.

b) Increasing density results in a higher thermal diffusivity.

c) Higher specific heat increases the rate of temperature change.

d) The heat conduction equation is independent of the material properties.

Question 1.10: In the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ where $u(x,t)$ is displacement of wave, if the initial conditions are $u(x,0) = \sin(\pi x)$ and $\frac{\partial u}{\partial t}(x,0) = 0$, what is the solution for $u(x,t)$

a) $u(x,t) = \sin(\pi x)\cos(\pi ct)$ ✓
b) $u(x,t) = \sin(\pi x)\sin(\pi ct)$ ✗
c) $u(x,t) = \sin(\pi x)\, e^{-\lambda_n^2 t}$ ✓
d) $u(x,t) = \cos(\pi x)\, e^{-\lambda_n^2 t}$ ✗

Only a and c satisfy initial condition

$$t=0 \Rightarrow \begin{cases} \text{a) } u(x,0) = \sin(\pi x)\cos(\pi c 0) = \sin(\pi x) \checkmark \\ \text{b) } u(x,0) = \sin(\pi x)\sin(\pi c 0) = 0 \times \\ \text{c) } u(x,0) = \sin(\pi x)\, e^{-\lambda_n^2 0} = \sin(\pi x) \checkmark \\ \text{d) } u(x,0) = \cos(\pi x)\, e^{-\lambda_n^2 0} = \cos(\pi x) \times \end{cases}$$

Finding their derivatives

a) $u_t = \frac{\partial}{\partial t}\left(\sin(\pi x)\cdot\cos(\pi ct)\right) = -\pi c\cdot\sin(\pi x)\cdot\sin(\pi ct)$

$\Rightarrow u_t(x,0) = -\pi c\cdot\sin(\pi x)\cdot\sin(\pi c\cdot 0) = 0$

The answer is a

Question2.1: Calculate the Fourier transform of the function $\Delta(t) = \begin{cases} 1-2|x| & |x| \le \frac{1}{2} \\ 0 & \text{Otherwise} \end{cases}$

$$f(t) = \begin{cases} 1-2|x| , & \frac{-1}{2} \le x \le \frac{1}{2} \\ 0 , & \text{Otherwise} \end{cases}$$

This is the formula for the fourier transform

$$X(w) = \int_{-\infty}^{\infty} x(t) \cdot e^{-jwt} \, dt$$

In our case we can split up the integral like this

$$F(w) = \underbrace{\int_{-\infty}^{-\frac{1}{2}} f(t) \cdot e^{-jwt} \, dt}_{0} + \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) \cdot e^{-jwt} \, dt + \underbrace{\int_{\frac{1}{2}}^{\infty} f(t) \cdot e^{-jwt} \, dt}_{0}$$

The integral of the funtion is ZERO outside its first condition

$$F(w) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) \cdot e^{-jwt} \, dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} (1-2|x|) \cdot e^{-jwt} \, dt$$

Instead of integrating over the negative part of the domain,
we integrate twice over the positive part, to get rid of the
absolute value. This is possible because the function is symetric.

$$F(w) = 2 \cdot \int_{0}^{\frac{1}{2}} (1-2x) \cdot e^{-jwt} \, dt = 2 \cdot \int_{0}^{\frac{1}{2}} e^{-jwt} \, dt + 2 \cdot \int_{0}^{\frac{1}{2}} -2x \cdot e^{-jwt}$$

$$= 2 \cdot \left[e^{-jwt} \cdot (-jw) \right]_0^{\frac{1}{2}} + 2 \cdot [$$

Finish evaluation of integral

Question 2.2: Use Stokes theorem to show that $\oint_C ydx + zdy + xdz = \sqrt{3}\pi a^2$, where C is the suitably oriented intersection of the surfaces $x^2 + y^2 + z^2 = a^2$, and $x + y + z = 0$.

$$\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$$

$$\iint_R \text{curl}\,\vec{F}\cdot\vec{n}\,dS$$

$$\vec{n} = \hat{i} + \hat{j} + \hat{k}$$

$$\text{curl}\,\vec{F} = -\hat{i} + \hat{j} - \hat{k}$$

$$\Rightarrow \iint_R (-\hat{i} + \hat{j} - \hat{k})\cdot(\hat{i} + \hat{j} + \hat{k})\,dS = \iint_R -1\,dS$$

Convert to cylindrical coordinates to find limits

Insert limits

Evaluate integral

Question3.1: Evaluate the double integral $\iint_R y\,dA$ where R is the region bounded by a shifted circle $x^2 + y^2 = 2x$ and a line $y = x$.

Convert to polar coordinates

$$r \cdot \sin(\theta) = r \cdot \cos(\theta) \Rightarrow \sin(\theta) = \cos(\theta) \Rightarrow \begin{cases} \theta = \frac{\pi}{4} \\ \theta = \frac{5\pi}{4} \end{cases}$$

$$r^2 = 2 \cdot r \cdot \cos(\theta) \Rightarrow r = 2 \cdot \cos(\theta)$$

Writing the integral

$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \int_0^{2\cdot\cos(\theta)} 1 \, dr\, d\theta = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left[r\right]_0^{2\cdot\cos(\theta)} d\theta = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} 2\cdot\cos(\theta)\, d\theta = 2 \cdot \left[\sin(\theta)\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = 2 \cdot \left(\sin\left(\frac{5\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right)$$

$$= 2 \cdot \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = 2 \cdot \left(-\sqrt{2}\right) = \underline{\underline{-2\sqrt{2}}}$$

Question3.2: Find the type of PDE, transform to normal form, and solve it. Show your work in detail. $u_{xx} + 5u_{xy} + 4u_{yy} = 0$

Obtain values

$$Au_{xx} + Bu_{xy} + C_{yy} + Cu_x + Eu_y + Fu = G$$

$$\Rightarrow A = 1,\ B = 5,\ C = 4$$

Classify

$$d = B^2 - 4ac = 5^2 - 4\cdot1\cdot4 = 25 - 16 = 9$$

$$d > 0 \Rightarrow \text{Hyperbolic!}$$

Solve equation to obtain new variables

$$A\lambda^2 - B\lambda + C = 0 \Rightarrow \lambda^2 - 5\lambda + 4 = 0$$

$$\lambda = \frac{-b \pm \sqrt{d}}{2a} = \frac{5 \pm \sqrt{9}}{2} = \frac{5 \pm 3}{2} = \begin{cases} \lambda_1 = \frac{2}{2} = 1 \\ \lambda_2 = \frac{8}{2} = 4 \end{cases}$$

$$\begin{cases} \frac{dy}{dx} = \lambda_1 = 1 \Rightarrow \int dy = \int 1 dx \Rightarrow y = x + c_1 \Rightarrow c_1 = y - x \\ \frac{dy}{dx} = \lambda_2 = 4 \Rightarrow \int dy = \int 4 dx \Rightarrow y = 4x + c_2 \Rightarrow c_2 = y - 4x \end{cases}$$

$$\xi = y - x,\quad \eta = y - 4x$$

Find the needed derivatives

$$\xi_x = -1,\ \xi_y = 1,\ \xi_{xx} = \xi_{xy} = \xi_{yy} = 0$$

$$\eta_x = -4,\ \eta_y = 1,\ \eta_{xx} = \eta_{xy} = \eta_{yy} = 0$$

$$u_{xx} = u_{\xi\xi}\cdot(\xi_x)^2 + 2u_{\xi\eta}\cdot\xi_x\eta_x + u_{\eta\eta}(\eta_x)^2 + \cancel{u_\xi \xi_{xx}} + \cancel{u_\eta \eta_{xx}}$$

$$= u_{\xi\xi}\cdot(-1)^2 + 2u_{\xi\eta}\cdot(-1)(-4) + u_{\eta\eta}\cdot(-4)^2$$

$$= u_{\xi\xi} + 8u_{\xi\eta} + 16\cdot u_{\eta\eta}$$

$$u_{yy} = u_{\xi\xi}(\xi_y)^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\cdot(\eta_y)^2 + \cancel{u_\xi \xi_{yy}} + \cancel{u_\eta \eta_{yy}}$$

$$= u_{\xi\xi}\cdot 1^2 + 2u_{\xi\eta}\cdot 1\cdot 1 + u_{\eta\eta}\cdot 1^2$$

$$= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

$$u_{xy} = u_{\xi\xi}\cdot\xi_x\xi_y + u_{\eta\eta}\eta_x\eta_y + \cancel{u_\xi \xi_{xy}} + \cancel{u_\eta \eta_{xy}} + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x)$$

$$= u_{\xi\xi}\cdot(-1)\cdot(1) + u_{\eta\eta}(-4)(1) + u_{\xi\eta}((-1)\cdot 1) + 1\cdot(-4))$$

$$= -u_{\xi\xi} - 4u_{\eta\eta} - 5u_{\xi\eta}$$

Putting in the original equation

$$u_{xx} + 5u_{xy} + 4u_{yy} = 0$$

$$\Rightarrow u_{\varepsilon\varepsilon} + 8u_{\varepsilon\eta} + 16 \cdot u_{\eta\eta} + 5\left(-u_{\varepsilon\varepsilon} - 4u_{\eta\eta} - 5u_{\varepsilon\eta}\right) + 4\left(u_{\varepsilon\varepsilon} + 2u_{\varepsilon\eta} + u_{\eta\eta}\right)$$

$$\Rightarrow 0 \cdot u_{\varepsilon\varepsilon} - 9u_{\varepsilon\eta} + 0_{\eta\eta} = 0 \Rightarrow -9u_{\varepsilon\eta} = 0 \Rightarrow u_{\varepsilon\eta} = 0$$

$$u_{\varepsilon} = \int 0 \, d\eta = f_1(\varepsilon)$$

$$u = \int c_1(\varepsilon) \, d\varepsilon = f_{\varepsilon}(\varepsilon) + f_2(\eta)$$