

Question 1:

Using the Laplace transform, find the solution for the following equation.

$$\frac{\partial^2}{\partial t^2} y(t) + 2 \frac{\partial}{\partial t} y(t) + 2y(t) = 0$$

with initial conditions $y(0) = 1$ and $Dy(0) = -1 \approx y'(t)$

$$y''(t) + 2 \cdot y'(t) + 2y(t) = 0$$

$$\Rightarrow (s^2 Y(s) - s \cdot y(0) - y'(0)) + 2 \cdot (s Y(s) - y(0)) + 2 \cdot Y(s) = 0$$

$$= (s^2 Y(s) - s \cdot 1 - (-1)) + 2 \cdot (s Y(s) - 1) + 2 \cdot Y(s) = 0$$

$$\Rightarrow s^2 Y(s) - \underline{s} + \underline{1} + 2s Y(s) - \underline{2} + 2 Y(s) = 0$$

$$\Rightarrow s^2 Y(s) + 2s Y(s) + 2 Y(s) = s + 1$$

$$\Rightarrow Y(s) \cdot (s^2 + 2s + 2) = s + 1$$

$$\Rightarrow Y(s) = \frac{s+1}{s^2+2s+2} \rightarrow \text{Transfer function}$$

Finding the poles

$$s^2 + 2s + 2 = 0 \Rightarrow \Delta = b^2 - 4ac = 4 - 4(1)(2) = 4 - 8 = -4$$

$$s = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-2 \pm 2i}{2} = \begin{cases} -1 - i \\ -1 + i \end{cases}$$

$$\Rightarrow Y(s) = \frac{s+1}{(s+1+i)(s+1-i)} = \frac{A}{s+1+i} + \frac{B}{s+1-i}$$

$$k_i = (s - p_i) \cdot Y(s) \Big|_{s=p_i}$$

$$A = k_1 \quad p_1 = -1 - i$$

$$A = \cancel{(s+1+i)} \cdot \frac{s+1}{\cancel{(s+1+i)}(s+1-i)} \Big|_{s=p_1} = \frac{s+1}{s+1-i} \Big|_{s=p_1} = \frac{(-1-i)+1}{(-1-i)+1-i} = \frac{-i}{-2i} = \frac{1}{2}$$

Complex Conjugate

$$B = \overline{A} = \frac{1}{2}$$

$$Y(s) = \frac{\frac{1}{2}}{s+1+i} + \frac{\frac{1}{2}}{s+1-i} = \frac{1}{2(s+1+i)} + \frac{1}{2(s+1-i)} = \frac{1}{2} \left(\frac{1}{s+1+i} + \frac{1}{s+1-i} \right)$$

$$\mathcal{L}\{e^{\alpha t}\} = \frac{1}{s-\alpha}$$

This is the result of the inverse laplace transformation

$$y(t) = \frac{1}{2} \left(e^{(1+i)t} + e^{(1-i)t} \right)$$

Using eulers formula:

$$\cos x = \operatorname{Re}(e^{ix}) = \frac{e^{ix} + e^{-ix}}{2},$$

$$\sin x = \operatorname{Im}(e^{ix}) = \frac{e^{ix} - e^{-ix}}{2i}.$$

$$= \frac{1}{2} \left(e^t \cdot e^{it} + e^t \cdot e^{-it} \right) = e^t \cdot \frac{1}{2} \cdot \left(e^{it} + e^{-it} \right)$$

$$= \underline{\underline{e^t \cdot \cos(t)}}$$

Question 2:

Find Fourier transform of

$$f(x) = \begin{cases} 1 - x^2 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

And hence evaluate:

$$\int_0^{\infty} \left(\frac{x \cos x - \sin x}{x^3} \right) \cos \frac{x}{2} dx$$

$$H(\omega) = \int_{-\infty}^{\infty} h(x) e^{-j\omega x} dx$$

$f(x)$ is only non-zero between $x=-1$ and $x=1$, therefore we can use these values as boundaries for our integral.

$$H(\omega) = \int_{-1}^1 (1 - x^2) \cdot e^{-j\omega x} dx$$

Partial Integration

$$\int u(x) \cdot v'(x) dx = u(x) \cdot v(x) - \int u'(x) \cdot v(x) dx$$

$$u = 1 - x^2 \quad v' = e^{-j\omega x}$$

$$u' = -2x \quad v = \frac{1}{-j\omega} \cdot e^{-j\omega x}$$

$$\Rightarrow H(\omega) = (1 - x^2) \cdot \frac{1}{-j\omega} \cdot e^{-j\omega x} - \int -2x \cdot \frac{1}{-j\omega} \cdot e^{-j\omega x} dx$$

$$\Rightarrow H(\omega) = (1 - x^2) \cdot \frac{1}{-j\omega} \cdot e^{-j\omega x} + \frac{2}{-j\omega} \int x \cdot e^{-j\omega x} dx$$

More partial integration

$$u = x \quad v' = e^{-j\omega x}$$

$$u' = 1 \quad v = \frac{1}{-j\omega} \cdot e^{-j\omega x}$$

$$\Rightarrow H(\omega) = (1-x^2) \cdot \frac{1}{j\omega} \cdot e^{-j\omega x} + \frac{2}{j\omega} \left(x \cdot \frac{1}{j\omega} \cdot e^{-j\omega x} - \int 1 \cdot \frac{1}{j\omega} \cdot e^{-j\omega x} dx \right)$$

$$\Rightarrow H(\omega) = (1-x^2) \cdot \frac{1}{j\omega} \cdot e^{-j\omega x} + \frac{2}{j\omega} \left(x \cdot \frac{1}{j\omega} \cdot e^{-j\omega x} - \frac{1}{j\omega} \cdot \frac{1}{j\omega} \cdot e^{-j\omega x} \right)$$

Simplifying

$$\Rightarrow H(\omega) = (1-x^2) \cdot \frac{1}{j\omega} \cdot e^{-j\omega x} + \frac{2x}{j^2\omega^2} \cdot e^{-j\omega x} - \frac{2}{j^3\omega^3} \cdot e^{-j\omega x}$$

$$\Rightarrow H(\omega) = (1-x^2) \cdot \frac{1}{j\omega} \cdot e^{-j\omega x} + \frac{2x}{j^2\omega^2} \cdot e^{-j\omega x} + \frac{2}{j^3\omega^3} \cdot e^{-j\omega x}$$

We can now evaluate with the limits

$$\left[\cancel{(1-x^2) \cdot \frac{1}{j\omega} \cdot e^{-j\omega x}} + \frac{2x}{j^2\omega^2} \cdot e^{-j\omega x} + \frac{2}{j^3\omega^3} \cdot e^{-j\omega x} \right]_{-1}^1$$

↪ In both limits this term becomes zero

$$= \left[\frac{2x}{j^2\omega^2} \cdot e^{-j\omega x} + \frac{2}{j^3\omega^3} \cdot e^{-j\omega x} \right]_{-1}^1$$

$$= \frac{2(1)}{j^2\omega^2} \cdot e^{-j\omega(1)} + \frac{2}{j^3\omega^3} \cdot e^{-j\omega(1)} - \frac{2(-1)}{j^2\omega^2} \cdot e^{-j\omega(-1)} - \frac{2}{j^3\omega^3} \cdot e^{-j\omega(-1)}$$

$$= \underbrace{\frac{2}{j^2\omega^2} \cdot e^{-j\omega}} + \underbrace{\frac{2}{j^3\omega^3} \cdot e^{-j\omega}} + \underbrace{\frac{2}{j^2\omega^2} \cdot e^{j\omega}} - \underbrace{\frac{2}{j^3\omega^3} \cdot e^{j\omega}}$$

Pull out constants to make it resemble eulers formula

$$= \left(\frac{2}{j^2\omega^2} \cdot (e^{-j\omega} + e^{j\omega}) + \frac{2}{j^3\omega^3} (e^{-j\omega} - e^{j\omega}) \right) \cdot \frac{2}{2}$$

$$= \frac{4}{j^2\omega^2} \cdot \left(\frac{e^{-j\omega} + e^{j\omega}}{2} \right) - \frac{4}{\omega^3} \cdot \left(\frac{e^{-j\omega} - e^{j\omega}}{2j} \right)$$

$$= \frac{4}{j^2\omega^2} \cdot \left(\frac{e^{-j\omega} + e^{j\omega}}{2} \right) + \frac{4}{\omega^3} \cdot \left(\frac{e^{-j\omega} - e^{j\omega}}{2j} \right)$$

$$\cos x = \operatorname{Re}(e^{ix}) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin x = \operatorname{Im}(e^{ix}) = \frac{e^{ix} - e^{-ix}}{2i}$$

$$= \frac{4}{j^2 \omega^2} \cdot \left(\frac{e^{j\omega} + e^{-j\omega}}{2} \right) + \frac{-4}{\omega^3} \cdot \left(\frac{e^{j\omega} - e^{-j\omega}}{2j} \right)$$

$$= \frac{-4}{\omega^2} \cdot \cos(\omega) + \frac{-4}{\omega^3} \cdot \sin(\omega)$$

$$= \frac{-4\omega \cdot \cos(\omega)}{\omega^3} + \frac{-4 \cdot \sin(\omega)}{\omega^3}$$

$$H(\omega) = \frac{-4\omega \cdot \cos(\omega) - 4 \cdot \sin(\omega)}{\omega^3} = -4 \cdot \frac{\omega \cdot \cos(\omega) - \sin(\omega)}{\omega^3}$$

And hence evaluate:

$$\int_0^\infty \left(\frac{x \cos x - \sin x}{x^3} \right) \cos \frac{x}{2} dx$$

This is the fourier transform

$$H(\omega) = \int_{-\infty}^{\infty} h(x) \cdot e^{-j\omega x} dx$$

This is the specific expression for our problem

$$H(\omega) = \int_{-1}^1 \underbrace{(1-x^2)}_{h(x)} \cdot e^{-j\omega x} dx = -4 \cdot \underbrace{\frac{\omega \cdot \cos(\omega) - \sin(\omega)}{\omega^3}}_{\downarrow}$$

This part seems to resemble the expression we are trying to evaluate...

Let's try to take the inverse fourier transform

$$h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{-4 \cdot \frac{\omega \cdot \cos(\omega) - \sin(\omega)}{\omega^3}}_{H(\omega)} \cdot e^{j\omega x} d\omega = \underbrace{1-x^2}_{\downarrow}$$

This part can NEVER be imaginary. Therefore, the integral on the left can also never be imaginary. This means that we can ignore the imaginary part.

$$\operatorname{Re}(e^{j\omega x}) = \cos(\omega x)$$

$$h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} -4 \cdot \frac{w \cdot \cos(w) - \sin(w)}{w^3} \cdot \cos(wx) dw = 1 - x^2$$

This inner expression now looks a lot like the expression we are trying to evaluate.

$$\int_0^{\infty} \left(\frac{x \cos x - \sin x}{x^3} \right) \cos \frac{x}{2} dx$$

$$h(x) = \frac{-4}{2\pi} \int_{-\infty}^{\infty} \frac{w \cdot \cos(w) - \sin(w)}{w^3} \cdot \cos(wx) dw = 1 - x^2$$

In fact, if we set $x=1/2$ they are equal

$$h\left(\frac{1}{2}\right) = \frac{-4}{2\pi} \int_{-\infty}^{\infty} \frac{w \cdot \cos(w) - \sin(w)}{w^3} \cdot \cos\left(\frac{w}{2}\right) dw = 1 - x^2$$

We can not isolate the integral to find its evaluation

$$\frac{-4}{2\pi} \int_{-\infty}^{\infty} \frac{w \cdot \cos(w) - \sin(w)}{w^3} \cdot \cos\left(\frac{w}{2}\right) dw = 1 - \left(\frac{1}{2}\right)^2$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{w \cdot \cos(w) - \sin(w)}{w^3} \cdot \cos\left(\frac{w}{2}\right) dw = \frac{2\pi \cdot (1 - \frac{1}{4})}{-4} = \frac{-2\pi + \frac{\pi}{2}}{4} = \frac{-4\pi + \pi}{8}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{w \cdot \cos(w) - \sin(w)}{w^3} \cdot \cos\left(\frac{w}{2}\right) dw = -\frac{3\pi}{8}$$

And there we have the answer!

$$\Rightarrow \int_{-\infty}^{\infty} \frac{w \cdot \cos(x) - \sin(x)}{x^3} \cdot \cos\left(\frac{x}{2}\right) dw = -\frac{3\pi}{8}$$

Question 3:

Find $\iint (\sin x + \cos y) dA$

$$R: \{x, y \mid 0 \leq x \leq \pi/2, 0 \leq y \leq \pi/2\}$$

$$\int_0^{\pi/2} \int_0^{\pi/2} (\sin(x) + \cos(y)) dx dy$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \sin(x) dx dy + \int_0^{\pi/2} \int_0^{\pi/2} \cos(y) dx dy$$

$$= \int_0^{\pi/2} [-\cos(x)]_0^{\pi/2} dy + \int_0^{\pi/2} [\cos(y) \cdot x]_0^{\pi/2} dy$$

$$= \int_0^{\pi/2} (-\cos(\frac{\pi}{2}) + \cos(0)) dy + \int_0^{\pi/2} \cos(y) \cdot \frac{\pi}{2} dy$$

$$= \int_0^{\pi/2} 1 dy + \int_0^{\pi/2} \cos(y) \cdot \frac{\pi}{2} dy = [y]_0^{\pi/2} + \frac{\pi}{2} [\sin(y)]_0^{\pi/2} = \frac{\pi}{2} + \frac{\pi}{2} \cdot 1$$

$$\underline{\underline{= \pi}}$$

Question 4

Find the $\iiint (x^2 + y^2 + z^2) dV$

where region is bounded by $z = c\sqrt{x^2 + y^2}$ and $(x^2 + y^2 + z^2 = a^2)$

Convert to spherical coordinates

$$\begin{aligned} x &= \rho \cdot \sin \phi \cdot \cos \theta \\ y &= \rho \cdot \sin \phi \cdot \sin \theta \\ z &= \rho \cdot \cos \phi \end{aligned}$$

$$dV = \rho^2 \cdot \sin(\phi) d\rho d\phi d\theta$$

Translate integral

$$\iiint x^2 + y^2 + z^2 dV = \iiint \rho^2 \cdot \rho^2 \cdot \sin(\phi) d\rho d\phi d\theta$$

$$= \iiint \rho^4 \cdot \sin(\phi) d\rho d\phi d\theta$$

Translate limits

$$z = c\sqrt{x^2 + y^2} \Rightarrow \rho \cdot \cos(\phi) = c \cdot \sqrt{(\rho \cdot \sin(\phi) \cdot \cos(\theta))^2 + (\rho \cdot \sin(\phi) \cdot \sin(\theta))^2}$$

$$\Rightarrow \rho \cdot \cos(\phi) = c \cdot \sqrt{\rho^2 \cdot \sin^2(\phi) \cdot \cos^2(\theta) + \rho^2 \cdot \sin^2(\phi) \cdot \sin^2(\theta)}$$

$$\Rightarrow \rho \cdot \cos(\phi) = c \cdot \sqrt{\rho^2 \cdot \sin^2(\phi) \cdot (\cos^2(\theta) + \sin^2(\theta))}$$

$$\Rightarrow p \cdot \cos(\phi) = c \cdot \sqrt{(p \cdot \sin(\phi))^2 \cdot (\cos(\theta)^2 + \sin(\theta)^2)} = c \cdot p \cdot \sin(\phi) \cdot 1$$

$$\Rightarrow p \cdot \cos(\phi) = c \cdot p \cdot \sin(\phi)$$

$$\Rightarrow \cos(\phi) = c \cdot \sin(\phi)$$

$$\Rightarrow c = \frac{\cos(\phi)}{\sin(\phi)} \Rightarrow \frac{1}{c} = \tan(\phi) \Rightarrow \phi = \tan^{-1}\left(\frac{1}{c}\right)$$

$$\text{Definition: } d = \tan^{-1}\left(\frac{1}{c}\right)$$

Second limit

$$\begin{aligned} x &= p \cdot \sin \phi \cdot \cos \theta \\ y &= p \cdot \sin \phi \cdot \sin \theta \\ z &= p \cdot \cos \phi \end{aligned}$$

This is a sphere with radius a.

$$x^2 + y^2 + z^2 = a^2 \Rightarrow p_{\min} = 0, p_{\max} = a$$

Inserting limits

$$\int_0^{2\pi} \int_0^d \int_0^a p^4 \cdot \sin(\phi) \, dp \, d\phi \, d\theta = \int_0^{2\pi} \int_0^d \sin(\phi) \int_0^a p^4 \, dp \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^d \sin(\phi) \left[\frac{p^5}{5} \right]_0^a \, d\phi \, d\theta = \int_0^{2\pi} \int_0^d \sin(\phi) \cdot \frac{a^5}{5} \, d\phi \, d\theta$$

$$= \frac{a^5}{5} \int_0^{2\pi} \int_0^d \sin(\phi) \, d\phi \, d\theta = \frac{a^5}{5} \int_0^{2\pi} \left[-\cos(\phi) \right]_0^d \, d\theta$$

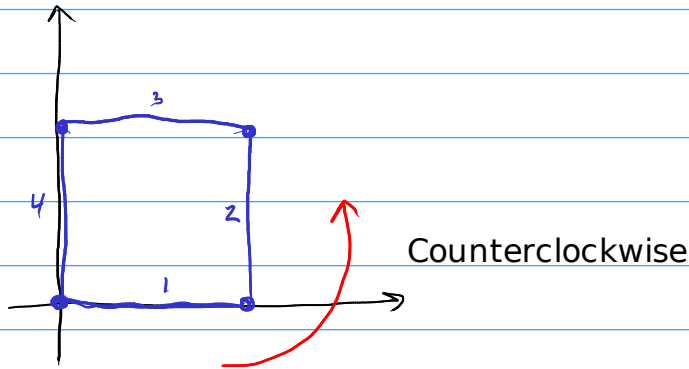
$$= \frac{a^5}{5} \cdot \int_0^{2\pi} (-\cos(d) - (-\cos(0))) \, d\theta = \frac{a^5}{5} \cdot \int_0^{2\pi} (1 - \cos(d)) \, d\theta$$

$$= \frac{a^5}{5} \cdot (1 - \cos(\alpha)) \int_0^{2\pi} 1 \, d\theta = \frac{a^5}{5} \cdot (1 - \cos(\alpha)) \cdot 2\pi$$

$$= \frac{a^5}{5} \cdot \cos\left(1 - \tan^{-1}\left(\frac{1}{2}\right)\right) \cdot 2\pi$$

Question 5

Evaluate $\oint_C x^2 y^2 dx + x^3 y dy$ counterclockwise around the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$



$$\textcircled{1}: y=0 \Rightarrow \int_{x=0}^{x=1} x^2 \cdot 0 dx + x^3 \cdot 0 \cdot 0 = 0$$

$$\textcircled{2}: x=1 \Rightarrow \int_{y=0}^{y=1} 1^2 \cdot y^2 \cdot 0 + 1^3 \cdot y dy = \int_0^1 y dy = \left[\frac{y^2}{2} \right]_0^1 = \frac{1}{2}$$

$$\textcircled{3}: y=1 \Rightarrow - \int_{x=0}^{x=1} x^2 \cdot 1^2 dx + x^3 \cdot 1 \cdot 0 = - \int_0^1 x^2 dx = \left[\frac{x^3}{3} \right]_0^1 = -\frac{1}{3}$$

$$\textcircled{4}: x=0 \Rightarrow - \int_{y=0}^{y=1} 0^2 \cdot y^2 \cdot 0 + 0^3 \cdot y dy = - \int_0^1 0 dy = 0$$

We can now add up the integrals

$$0 + \frac{1}{2} - \frac{1}{3} + 0 = \underline{\underline{\frac{1}{6}}}$$