① If A is an invertible matrix, then so is A^T.

X 참 True

○ 거짓 False

② $(A+B)^{-1} = A^{-1} + B^{-1}$

○ 참

X 거짓

③ If A and B are symmetric matrix, then AB is also symmetric.

X 참

○ 거짓

④ Every vector space possesses a basis

X 참

○ 거짓

⑤ Let a $= \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$ and b $= \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$. Then {a, b} forms a orthonormal basis of $\mathbb{R}^2$

X 참

○ 거짓

Length of vectors should be 1

$$\|a\| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1 \quad \checkmark$$

$$\|b\| = \sqrt{\left(\frac{-4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = 1 \quad \checkmark$$

Check if they are perpendicular with the dot product.

$$a \cdot b = \frac{3}{5} \cdot \frac{-4}{5} + \frac{4}{5} \cdot \frac{3}{5} = 0 \quad \checkmark$$

=> a and b are orthonormal!

(6) If $\text{rank}(A) = \text{rank}(B)$, then $\text{rank}(A^2) = \text{rank}(B^2)$.

X 참

○ 거짓

(7) $\text{rank}(B^T A^T) = \text{rank}(AB)$.

$\text{rank}(A^T B^T) = \text{rank}((AB)^T) = \text{rank}(AB)$

X 참

○ 거짓

(8) If A is symmetric and invertible, then A^{-1} is also symmetric and invertible.

X 참

○ 거짓

(9) Solve the following system using the augmented matrix $[A \| b]$ (write down your answers on the PDF file).

$$\begin{array}{rrrrrcr} -2x_1 & +4x_2 & -2x_3 & -x_4 & +4x_5 & = & -3 \\ 4x_1 & -8x_2 & +3x_3 & -3x_4 & +x_5 & = & 2 \\ x_1 & -2x_2 & +x_3 & -x_4 & +x_5 & = & 0 \\ x_1 & -2x_2 & & -3x_4 & +4x_5 & = & -1 \end{array}$$

The form of the solution would be as follows. Please find the right values for a to o.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} + \lambda_1 \begin{bmatrix} f \\ g \\ h \\ i \\ j \end{bmatrix} + \lambda_2 \begin{bmatrix} k \\ l \\ m \\ n \\ o \end{bmatrix}$$

$$A\vec{x} = \vec{b} \Rightarrow \begin{bmatrix} -2 & 4 & -2 & -1 & 4 \\ 4 & -8 & 3 & -3 & 1 \\ 1 & -2 & 1 & -1 & 1 \\ 1 & -2 & 0 & -3 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} -3 \\ 2 \\ 0 \\ -1 \end{bmatrix} \Rightarrow \left[\begin{array}{ccccc|c} -2 & 4 & -2 & -1 & 4 & -3 \\ 4 & -8 & 3 & -3 & 1 & 2 \\ 1 & -2 & 1 & -1 & 1 & 0 \\ 1 & -2 & 0 & -3 & 4 & -1 \end{array}\right] \begin{array}{l} 2 \\ \\ -1 \\ \end{array} \sim \left[\begin{array}{ccccc|c} -2 & 4 & -2 & -1 & 4 & -3 \\ 0 & 0 & -1 & -5 & 9 & -4 \\ 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & -2 & 3 & -1 \end{array}\right] \begin{array}{l} \frac{1}{2} \\ \\ -1 \\ \end{array}$$

$$\sim \left[\begin{array}{ccccc|c} -2 & 4 & -2 & -1 & 4 & -3 \\ 0 & 0 & -1 & -5 & 9 & -4 \\ 0 & 0 & 0 & -1.5 & 3 & -1.5 \\ 0 & 0 & 0 & 3 & -6 & 3 \end{array}\right] 2 \sim \left[\begin{array}{ccccc|c} -2 & 4 & -2 & -1 & 4 & -3 \\ 0 & 0 & -1 & -5 & 9 & -4 \\ 0 & 0 & 0 & -1.5 & 3 & -1.5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \cdot \frac{-2}{3} \sim \left[\begin{array}{ccccc|c} -2 & 4 & -2 & -1 & 4 & -3 \\ 0 & 0 & -1 & -5 & 9 & -4 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

let $x_5 = \lambda_1$, then

$x_4 - 2\lambda_1 = 1 \Rightarrow x_4 = 1 + 2\lambda_1$

$-x_3 - 5(1 + 2\lambda_1) + 9\lambda_1 = -4 \Rightarrow x_3 = -5 - 10\lambda_1 + 9\lambda_1 + 4 \Rightarrow x_3 = -\lambda_1 - 1$

$-2x_1 + 4x_2 - 2(-\lambda_1 - 1) - (1 + 2\lambda_1) + 4\lambda_1 = -2x_1 + 4x_2 + \cancel{2\lambda_1} + 2 - 1 - \cancel{2\lambda_1} + 4\lambda_1 = -2x_1 + 4x_2 + 4\lambda_1 + 1 = -3$

$\Rightarrow -2x_1 + 4x_2 + 4\lambda_1 + 4 = 0 \Rightarrow -x_1 + 2x_2 + 2\lambda_1 + 2 = 0$

let $x_2 = \lambda_2$, then

$x_1 = 2\lambda_2 + 2\lambda_1 + 2$

$$\begin{cases} x_1 = 2\lambda_2 + 2\lambda_1 + 2 \\ x_2 = \lambda_2 \\ x_3 = -\lambda_1 - 1 \\ x_4 = 1 + 2\lambda_1 \\ x_5 = \lambda_1 \end{cases} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 2 \\ 0 \\ -1 \\ 2 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(10) Are the following vectors linearly independent?

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 5 \\ 7 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 6 \end{bmatrix}$$

X 참

○ 거짓

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 5 & 7 & 0 \\ 2 & 0 & 1 & 6 \end{bmatrix} \begin{matrix} \\ -1 \\ -2 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & -4 \\ 0 & -4 & -5 & -2 \end{bmatrix} \begin{matrix} \\ 4 \\ \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 1 & -18 \end{bmatrix} \Rightarrow \text{INDEPENDENT}$$

(11) Compute the inverse of the following matrices. Find the right values for a to m.

(a) $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

(b) $B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & -1 \\ 4 & 2 & 1 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B^{-1} = \begin{bmatrix} e & f & g \\ h & i & j \\ k & l & m \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] -\tfrac{3}{2} \sim \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right] \cdot 2 \sim \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 1 & -3 & 2 \end{array}\right] -1 \sim \left[\begin{array}{cc|cc} 2 & 0 & 4 & -2 \\ 0 & 1 & -3 & 2 \end{array}\right] \cdot \tfrac{1}{2} \sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -3 & 2 \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 3 & 2 & -1 & 0 & 1 & 0 \\ 4 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ -\frac{3}{2} \\ -2 \end{matrix} \sim \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \begin{matrix} \\ \\ 1 \end{matrix} \sim \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{7}{2} & 1 & 1 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \begin{matrix} \cdot \frac{1}{2} \\ \cdot 2 \\ \end{matrix}$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -7 & 2 & 2 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] -1 \sim \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 8 & -2 & -2 \\ 0 & 1 & 0 & -7 & 2 & 2 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \cdot \tfrac{1}{2} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -1 & -1 \\ 0 & 1 & 0 & -7 & 2 & 2 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \Rightarrow B^{-1} = \begin{bmatrix} 4 & -1 & -1 \\ -7 & 2 & 2 \\ -2 & 0 & 1 \end{bmatrix}$$

(12) Find the rank of the following matrix

$$M = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ -1 & 1 & 3 & -5 \\ -1 & 2 & 5 & -6 \\ -1 & -2 & -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ -1 & 1 & 3 & -5 \\ -1 & 2 & 5 & -6 \\ -1 & -2 & -3 & 1 \end{bmatrix} \begin{matrix} 1 \\ -1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ -1 & 1 & 3 & -5 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} 1 \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ 0 & 3 & 6 & -6 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \cdot\frac{1}{3} \\ -\frac{1}{3} \end{matrix}$$

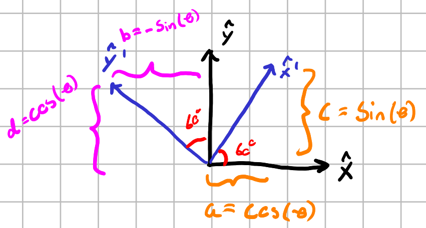
$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} -2 \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -5 & -10 & 10 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} 5 \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank}(M) = 3$$

(13) Find the linear transformation matrix that rotates a point in a 2-dimensional space counterclockwise by 60 degrees.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\hat{x}' = R\hat{x} \Rightarrow \hat{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$

$$\hat{y}' = R\hat{y} \Rightarrow \hat{y}' = \begin{bmatrix} b \\ d \end{bmatrix}$$



$$\Rightarrow R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$R(60°) = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

(14) Find the basis of the kernel of matrix A, the dimension of the image of matrix A.

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & -3 & 2 & 0 \end{bmatrix}$$

$$\text{Dim(image)} = k$$

$$\text{Basis for the Kernel} = \left\{ \begin{bmatrix} a \\ b \\ 1 \\ c \end{bmatrix} \begin{bmatrix} d \\ e \\ f \\ 1 \end{bmatrix} \right\}.$$

$$\ker(A) = \underbrace{\{X \in \mathbb{R}^4 \mid Ax = \vec{0}\}}_{\text{We want to find the vectors that map to } \vec{0}.} \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 3 & 1 & 0 & 2 & 0 \\ 1 & -3 & 2 & 0 & 0 \end{array}\right] \begin{matrix} -3 \\ -1 \end{matrix} \sim \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 0 & -5 & 3 & -1 & 0 \\ 0 & -5 & 3 & -1 & 0 \end{array}\right] -1$$

$$\sim \begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 0 & -5 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 + 2x_2 - x_3 + x_4 = 0 \\ -5x_2 + 3x_3 - x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -2x_2 + x_3 - x_4 \\ x_2 = \frac{3}{5}x_3 - \frac{1}{5}x_4 \end{cases}$$

let $\lambda_1 = x_3, \lambda_2 = x_4$

$$\Rightarrow \begin{cases} x_1 = -2x_2 + \lambda_1 - \lambda_2 \\ x_2 = \frac{3}{5}\lambda_1 - \frac{1}{5}\lambda_2 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{-6}{5}\lambda_1 + \frac{2}{5}\lambda_2 + \lambda_1 - \lambda_2 \\ x_2 = \frac{3}{5}\lambda_1 - \frac{1}{5}\lambda_2 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{-1}{5}\lambda_1 + \frac{-3}{5}\lambda_2 \\ x_2 = \frac{3}{5}\lambda_1 - \frac{1}{5}\lambda_2 \end{cases}$$

$$\Rightarrow \ker(A) = \lambda_1 \overset{b_1}{\begin{bmatrix} \frac{-1}{5} \\ \frac{-3}{5} \\ 1 \\ 0 \end{bmatrix}} + \lambda_2 \cdot \overset{b_2}{\begin{bmatrix} \frac{3}{5} \\ \frac{-1}{5} \\ 0 \\ 1 \end{bmatrix}}$$

b_1 and b_2 are the basis of the kernel

$$B = \left\{ \begin{bmatrix} \frac{-1}{5} \\ \frac{-3}{5} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{3}{5} \\ \frac{-1}{5} \\ 0 \\ 1 \end{bmatrix} \right\} \Rightarrow \dim(\ker(A)) = 2$$

The dimensions of ker(A) and Im(A) should add up to 4 (dimensions of original space).

$$\underbrace{\dim(\ker(A))}_{=2} + \dim(\mathrm{Im}(A)) = 4$$

$$\Rightarrow \dim(\mathrm{Im}(A)) = 2$$

15

Compute the distance between

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad y = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

using

$$\langle x, y \rangle := x^T A y, \quad A = \begin{bmatrix} 2 & 2 & -1 \\ 2 & 5 & 4 \\ -1 & 4 & 8 \end{bmatrix}$$

$$x^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 2 & -1 \\ 2 & 5 & 4 \\ -1 & 4 & 8 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -2+0-2 \\ -2+0+8 \\ 1+0+16 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -4 \\ 6 \\ 17 \end{bmatrix} = -4 + 12 + 51 = 59$$

(16)

Consider the Euclidean vector space $\mathbb{R}^4$ with the dot product. A subspace $U \subseteq \mathbb{R}^4$ and $x \in \mathbb{R}^4$ are given by

Basis Vectors

$$U = \operatorname{span}\left\{ \begin{bmatrix} 3 \\ 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \end{bmatrix} \right\}, \quad x = \begin{bmatrix} -1 \\ -7 \\ 1 \\ 5 \end{bmatrix}.$$

Determine the orthogonal projection $\pi_U(x)$ of x onto U.

$$\pi_U(x) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

Mapping

$$A = \begin{bmatrix} 3 & 0 \\ 3 & 0 \\ 0 & 4 \\ 3 & 0 \end{bmatrix} \Rightarrow P = A(A^T A)^{-1} A^T$$

Projection matrix

$$A^T A = \begin{bmatrix} 3 & 3 & 0 & 3 \\ 0 & 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 3 & 0 \\ 0 & 4 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 27 & 0 \\ 0 & 16 \end{bmatrix}$$

$$(A^T A)^{-1} = \begin{bmatrix} 27 & 0 \\ 0 & 16 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{27} & 0 \\ 0 & \frac{1}{16} \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} 3 & 0 \\ 3 & 0 \\ 0 & 4 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{27} & 0 \\ 0 & \frac{1}{16} \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 & 3 \\ 0 & 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{9} & 0 \\ \frac{1}{9} & 0 \\ 0 & \frac{1}{4} \\ \frac{1}{9} & 0 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 & 3 \\ 0 & 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

The projection into subspace U can thus be written as

$$\mu_U(x) = P\vec{x} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -1 \\ -7 \\ 1 \\ 5 \end{bmatrix} = \underline{\underline{\begin{bmatrix} -1 \\ -1 \\ 1 \\ -1 \end{bmatrix}}}$$