

$$\iint_R \frac{x}{1+xy} dA$$

$$R: \{x, y \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

Answer: $\ln[4] - 1$

$$\int_0^1 \int_0^1 \frac{x}{1+xy} dy dx = \iint \frac{\cancel{x}}{u} \frac{1}{\cancel{x}} du dx = \int \left[\ln(u) \right]_{y=0}^{y=1} dx$$

$$u = 1+xy \Rightarrow \frac{du}{dy} = x \Rightarrow dy = \frac{1}{x} du$$

$$u_1 = 1+x(0)=1 \quad u_2 = 1+x(1)=x+1$$

$$= \int_0^1 \left[\ln(u) \right]_1^{x+1} dx = \int_0^1 \ln(x+1) - \cancel{\ln(1)} dx = \int_0^1 \ln(x+1) dx$$

$$= \int_0^1 \ln(u) du = \left[-u + u \cdot \ln(u) \right]_{u=1}^{u=x+1} = \left[-x-1 + (x+1) \cdot \ln(x+1) \right]_0^1$$

$u=x+1 \Rightarrow \frac{du}{dx} = 1 \Rightarrow du=dx$

$$= \left[-x-1 + x \cdot \ln(x+1) + \ln(x+1) \right]_0^1$$

$$= \left(-1-1+1 \cdot \ln(1+1) + \ln(1+1) \right) - \left(0-1+0 + \ln(1) \right)$$

$$= -2 + \ln(2) + \ln(2) + 1 = \ln(2 \cdot 2) - 1 = \underline{\underline{\ln(4) - 1}}$$

Problem 2:

$$\iint_R \left(\frac{\ln y}{y} \right) dA$$

$$R: \left\{ x, y \mid \begin{array}{l} 0 \leq x \leq \pi \\ e^{-2x} \leq y \leq e^{\cos x} \end{array} \right.$$

Answer: $\frac{\pi}{4} - \frac{2}{3}\pi^3$

$$\int_0^\pi \int_{e^{-2x}}^{e^{\cos(x)}} \ln(y) \cdot \frac{1}{y} dy dx = \int_0^\pi \int_{e^{-2x}}^{e^{\cos(x)}} u \cdot \frac{1}{y} \cdot \cancel{dy} dx = \int_0^\pi \int_{e^{-2x}}^{e^{\cos(x)}} u du dx$$

$$u = \ln(y) \Rightarrow \frac{du}{dy} = \frac{1}{y} \Rightarrow dy = y du$$

$$= \int_0^\pi \left[u \right]_{y=e^{-2x}}^{y=e^{\cos(x)}} dx = \int_0^\pi \left[\ln(y) \right]_{e^{-2x}}^{e^{\cos(x)}} dx = \int_0^\pi \cos(x) + 2x dx$$

$$= \left[\sin(x) + x^2 \right]_0^\pi = \cancel{\sin(\pi)} + \pi^2 - \cancel{\sin(0)} - 0 = \pi^2$$

Problem 3:

Calculate the given iterated integral

$$\int_0^1 dx \int_0^x (xy + y^2) dy$$

Answer: $\frac{5}{24}$

$$\int_0^1 \int_0^x xy + y^2 dy dx = \int_0^1 \left[x \cdot \frac{y^2}{2} + \frac{y^3}{3} \right]_0^x dx$$

$$= \int_0^1 x \cdot \frac{x^2}{2} + \frac{x^3}{3} - \cancel{x \cdot \frac{0^2}{2}} - \cancel{\frac{0^3}{3}} dx = \int_0^1 \frac{x^3}{2} + \frac{x^3}{3} dx = \int_0^1 \frac{5x^3}{6} dx$$

$$= \frac{5}{6} \cdot \int_0^1 x^3 dx = \frac{5}{6} \cdot \left[\frac{x^4}{4} \right]_0^1 = \frac{5}{6} \cdot \frac{1}{4} = \underline{\underline{\frac{5}{24}}}$$

Problem 4:

Calculate the given iterated integral

$$\int_0^{\pi} \int_{-x}^x \cos y \, dy \, dx$$

Answer: $-2 \cos x \Big|_0^{\pi} = 4$

$$\int_0^{\pi} \int_{-x}^x \cos(y) \, dy \, dx = \int_0^{\pi} [\sin(y)]_{-x}^x \, dx = \int_0^{\pi} \sin(x) - \sin(-x) \, dx$$

$$\int_0^{\pi} 2 \cdot \sin(x) \, dx = [-2 \cdot \cos(x)]_0^{\pi} = -2 \cdot \overset{-1}{\cos(\pi)} + 2 \cdot \overset{1}{\cos(0)} = 2 + 2 = \underline{4}$$

Problem 5:

Evaluate the double integral by iteration

$$\iint_R (x^2 + y^2) dA$$

where R is the rectangle $0 \leq x \leq a, 0 \leq y \leq b$

Answer: $\frac{1}{3}(a^3b + ab^3)$

$$\begin{aligned} \int_0^b \int_0^a x^2 + y^2 dx dy &= \int_0^b \left[\frac{x^3}{3} + y^2 x \right]_0^a dy = \int_0^b \frac{a^3}{3} + y^2 \cdot a dy \\ &= \left[\frac{a^3}{3} \cdot y + a \cdot \frac{y^3}{3} \right]_0^b = \frac{a^3}{3} \cdot b + a \cdot \frac{b^3}{3} = \underline{\underline{\frac{1}{3} (a^3b + ab^3)}} \end{aligned}$$

Problem 6:

Evaluate the double integral by iteration

$$\iint_R xy^2 dA$$

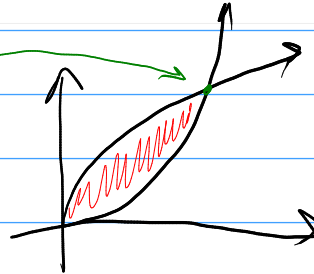
where R is the finite region in the first quadrant bounded by the curves $y = x^2$ and $x = y^2$

Answer: $\frac{1}{3} \left(\frac{2}{7} - \frac{1}{8} \right) = \frac{3}{56}$

Find the intercept

$$\begin{cases} y = x^2 \\ y^2 = x \Rightarrow y = \sqrt{x} \end{cases} \Rightarrow x^2 = \sqrt{x}$$

$$\Rightarrow x^4 = x \Rightarrow x^3 = 1 \Rightarrow x = 1$$

mister variable on last line?

$$\int_0^1 \int_{x^2}^{\sqrt{x}} xy^2 dy dx = \int_0^1 \left[x \cdot \frac{y^3}{3} \right]_{x^2}^{\sqrt{x}} dx = \int_0^1 x \cdot \frac{(\sqrt{x})^3}{3} - x \cdot \frac{x^6}{3} dx$$

$$= \frac{1}{3} \cdot \int_0^1 \underbrace{x \cdot (\sqrt{x})^3}_{x^1 \cdot (x^{\frac{1}{2}})^3 = x^1 \cdot x^{\frac{3}{2}} = x^{\frac{5}{2}}} - x^7 dx = \frac{1}{3} \cdot \int_0^1 x^{\frac{5}{2}} - x^7 dx = \frac{1}{3} \cdot \left[\frac{x^{\frac{7}{2}}}{\frac{7}{2}} - \frac{x^8}{8} \right]_0^1 = \frac{1}{3} \cdot \left[\frac{2(\sqrt{x})^7}{7} - \frac{x^8}{8} \right]_0^1$$

$$= \frac{1}{3} \cdot \left(\frac{2 \cdot 1}{7} - \frac{1}{8} - 0 \right) = \frac{1}{3} \cdot \left(\frac{2}{7} - \frac{1}{8} \right) = \frac{2}{21} - \frac{1}{24} = \underline{\underline{\frac{3}{56}}}$$

Problem 7:

Evaluate the double integral by iteration ??.

$$\iint_D x \cos y \, dA$$

where D is the finite region in the first quadrant bounded by the coordinate axes and the curve $y = 1 - x^2$

Answer: $\frac{1 - \cos(1)}{2}$

Problem 9:

Determine whether the given integral converges or diverges. Try to evaluate those that converge.

$$\iint_Q e^{-x-y} dA$$

where Q is the first quadrant of the xy -plane

Answer: $\left(\lim_{R \rightarrow \infty} (-e^{-x}) \Big|_0^R \right)^2 = 1$ (converges)

$$\int_0^{\infty} \int_0^{\infty} e^{-x-y} dx dy = - \int_0^{\infty} \int_0^{\infty} e^u du dy = - \int_0^{\infty} \left[e^{-x-y} \right]_0^{\infty} dy$$

$$u = -x-y \Rightarrow \frac{du}{dx} = -1 \Rightarrow dx = -1 du$$

$$= \lim_{a \rightarrow \infty} \left(- \int_0^{\infty} e^{-a-y} - e^{-0-y} dy \right) = \lim_{a \rightarrow \infty} \left(- \int_0^{\infty} e^{-a-y} - e^{-y} dy \right)$$

$$= \lim_{a \rightarrow \infty} \left(- \int_0^{\infty} e^{-a-y} dy + \int_0^{\infty} e^{-y} dy \right)$$

$$= \lim_{a \rightarrow \infty} \left(- \left[-e^{-a-y} \right]_0^{\infty} + \left[-e^{-y} \right]_0^{\infty} \right)$$

$$= \lim_{a \rightarrow \infty} \left(\left[e^{-a-y} \right]_0^{\infty} - \left[e^{-y} \right]_0^{\infty} \right)$$

$$= \lim_{\substack{a \rightarrow \infty \\ b \rightarrow \infty}} \left(e^{-a-b} - e^{-a} - (e^{-b} - e^{-0}) \right) = \lim_{\substack{a \rightarrow \infty \\ b \rightarrow \infty}} \left(\cancel{e^{-a-b}} - \cancel{e^{-a}} - \cancel{e^{-b}} + 1 \right)$$

$$= 1$$

Problem 10:

Determine whether the given integral converges or diverges. Try to evaluate those that converge.

$$\iint_T \frac{dA}{x^2 + y^2}$$

where T is the region satisfying $x \geq 1$ and $0 \leq y \leq x$

Answer: $\frac{\pi}{4} \int_1^\infty \frac{dx}{x} = \infty$ (The integral diverges to infinity.)

$$\int_1^\infty \int_0^x \frac{1}{x^2 + y^2} dy dx = \int_1^\infty \quad \text{find in table}$$

$$u = x^2 + y^2 \Rightarrow \frac{du}{dy} = 2y$$

Problem 11:

Evaluate $\iint_S (x + y) dA$, where S is the region in the first quadrant lying inside the disk $x^2 + y^2 \leq a^2$ and under the line $y = \sqrt{3}x$.

Answer: $\frac{(\sqrt{3}+1)a^3}{6}$

Translate to polar coordinates

$$x = r \cdot \cos(\theta), \quad y = r \cdot \sin(\theta)$$

$$dA = r d\theta dr$$

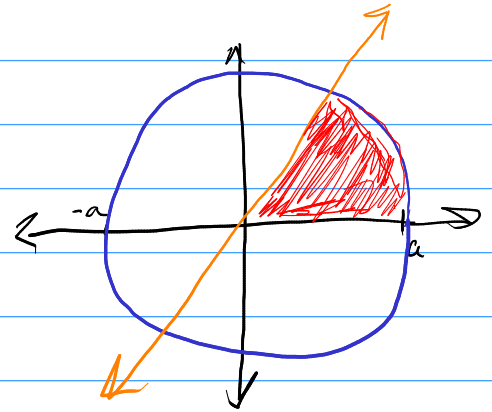
$$x^2 + y^2 \leq a^2 \Rightarrow 0 \leq r \leq a$$

$$y = \sqrt{3}x \Rightarrow r \cdot \sin(\theta) = \sqrt{3} \cdot r \cdot \cos(\theta)$$

$$\Rightarrow \sin(\theta) = \sqrt{3} \cdot \cos(\theta)$$

$$\Rightarrow \frac{\sin(\theta)}{\cos(\theta)} = \sqrt{3} \Rightarrow \tan(\theta) = \sqrt{3}$$

$$\Rightarrow \theta = \arctan(\sqrt{3}) = \frac{\pi}{3}$$



Integral in polar coordinates:

$$\int_0^a \int_0^{\pi/3} (r \cdot \cos(\theta) + r \cdot \sin(\theta)) \cdot r d\theta dr$$

$$= \int_0^a \int_0^{\pi/3} r^2 \cdot \cos(\theta) + r^2 \cdot \sin(\theta) d\theta dr = \int_0^a \left[r^2 \cdot \sin(\theta) - r^2 \cdot \cos(\theta) \right]_0^{\pi/3} dr$$

$$= \int_0^a r^2 \cdot \sin\left(\frac{\pi}{3}\right) - r^2 \cdot \cos\left(\frac{\pi}{3}\right) - \cancel{r^2 \cdot \sin(0)} + \cancel{r^2 \cdot \cos(0)} dr$$

$$= \int_0^a r^2 \cdot \sin\left(\frac{\pi}{3}\right) - r^2 \cdot \cos\left(\frac{\pi}{3}\right) + r^2 dr = \left[\frac{r^3}{3} \cdot \sin\left(\frac{\pi}{3}\right) - \frac{r^3}{3} \cdot \cos\left(\frac{\pi}{3}\right) + \frac{r^3}{3} \right]_0^a$$

$$= \left[\frac{r^3}{3} \cdot \left(\sin\left(\frac{\pi}{3}\right) - \cos\left(\frac{\pi}{3}\right) + 1 \right) \right]_0^a = \left[\frac{r^3}{3} \cdot \left(\frac{\sqrt{3}}{2} - \frac{1}{2} + 1 \right) \right]_0^a$$

$$= \frac{a^3}{3} \cdot \left(\frac{\sqrt{3}+1}{2} \right) = \underline{\underline{\frac{a^3(\sqrt{3}+1)}{3}}}$$

Problem 12:

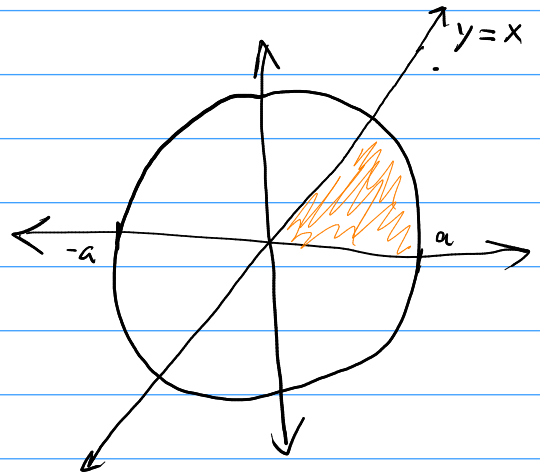
Evaluate $\iint_D xy \, dA$, where D is the region satisfying $x \geq 0$, $0 \leq y \leq x$, and $x^2 + y^2 \leq a^2$.

Answer: $\frac{a^4}{16}$

Convert to polar coordinates

$$x = r \cdot \cos(\theta), \quad y = r \cdot \sin(\theta)$$

$$dA = r \, d\theta \, dr$$



$$x^2 + y^2 \leq a^2 \Rightarrow 0 \leq r \leq a$$

$$0 \leq y \leq x \Rightarrow 0 \leq \cancel{r} \cdot \sin(\theta) \leq \cancel{r} \cdot \cos(\theta)$$

$$\Rightarrow 0 \leq \sin(\theta) \leq \cos(\theta) \Rightarrow 0 \leq \frac{\sin(\theta)}{\cos(\theta)} \leq 1 \Rightarrow 0 \leq \tan(\theta) \leq 1$$

$$\Rightarrow 0 \leq \theta \leq \arctan(1) \Rightarrow 0 \leq \theta \leq \frac{\pi}{4}$$

Integral in polar coordinates:

$$\int_0^a \int_0^{\frac{\pi}{4}} r \cdot \cos(\theta) \cdot r \cdot \sin(\theta) \cdot r \, d\theta \, dr = \int_0^a \int_0^{\frac{\pi}{4}} r^3 \cdot \cos(\theta) \cdot \sin(\theta) \, d\theta \, dr$$

$$= \int_0^a r^3 \int_0^{\frac{\pi}{4}} \cos(\theta) \cdot \sin(\theta) \, d\theta \, dr = \int_0^a r^3 \int_{\theta=0}^{\theta=\frac{\pi}{4}} u \cdot \sin(\theta) \cdot \frac{-1}{\sin(\theta)} \, du \, dr$$

$\hookrightarrow u = \cos(\theta) \Rightarrow \frac{du}{d\theta} = -\sin(\theta) \Rightarrow d\theta = \frac{-1}{\sin(\theta)} \, du$

$$\int_0^a r^3 \int_{\theta=0}^{\theta=\frac{\pi}{4}} -u \, du \, dr = \int_0^a r^3 \cdot \left[\frac{-u^2}{2} \right]_{\theta=0}^{\theta=\frac{\pi}{4}} \, dr = \int_0^a r^3 \cdot \left[\frac{-1}{2} \cos^2(\theta) \right]_0^{\frac{\pi}{4}} \, dr$$

$$= \int_0^a r^3 \cdot \left(\frac{-1}{2} \overset{\frac{\sqrt{2}}{2}}{\cos^2(\frac{\pi}{4})} + \frac{1}{2} \overset{1}{\cos^2(0)} \right) \, dr = \int_0^a r^3 \cdot \left(\frac{-1}{2} \cdot \left(\frac{\sqrt{2}}{2} \right)^2 + \frac{1}{2} \right) \, dr$$

$$\int_0^a r^3 \cdot \left(\frac{-1}{2} \cdot \frac{2}{4} + \frac{1}{2} \right) dr = \int_0^a r^3 \cdot \left(\frac{-1}{4} + \frac{2}{4} \right) dr = \int_0^a r^3 \cdot \frac{1}{4} dr$$

$$= \frac{1}{4} \cdot \int_0^a r^3 dr = \frac{1}{4} \cdot \left[\frac{r^4}{4} \right]_0^a = \frac{1}{4} \cdot \frac{a^4}{4} = \underline{\underline{\frac{a^4}{4}}}$$