Problem 1:

Find $\iint_S x\, dS$ over the part of the parabolic cylinder $z = x^2/2$ that lies inside the first octant part of the cylinder $x^2 + y^2 = 1$.

Solution: $\pi/8$

$$z = \frac{x^2}{2}$$

$$\Rightarrow x = v, \quad z = \frac{v^2}{2}, \quad y = u \quad \Rightarrow \quad \vec{r}(u,v) = v\,i + u\,j + \frac{v^2 k}{2}$$

$$\Rightarrow \begin{cases} r_u = 1j \\ r_v = 1i + vk \end{cases}$$

$$dS = |r_u \times r_v| \; du\,dv$$

$$|r_u \times r_v| = \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 1 & 0 & v \end{vmatrix} \right\| = |(1\cdot v - 0\cdot 0)i - (0\cdot v - 1\cdot 0)k + (0\cdot 0 - 1\cdot 0)k| = |vi - k| = \sqrt{v^2 + (-1)^2} = \sqrt{v^2+1}$$

$$\iint_S x\, dS = \iint_{v\,u} v \cdot \sqrt{v^2+1}\; du\,dv$$

Finding the bounds

$$0 \le x, \; 0 \le y \Rightarrow 0 \le v, \; 0 \le u$$

$$x^2 + y^2 \le 1 \Rightarrow v^2 + u^2 \le 1$$

Transform to polar coordinates

$x = r \cdot \cos(\theta) \qquad y = r \cdot \sin(\theta) \qquad \mathrm{dA} = r\, \mathrm{d}\theta\, \mathrm{dr} \qquad x^2 + y^2 = a^2 \Rightarrow a$

$$0 \le v, \; 0 \le u \Rightarrow 0 \le \theta \le \frac{\pi}{2}$$

$$v^2 + u^2 \le 1 \Rightarrow 0 \le r \le 1$$

$$\iint_{v\,u} v \cdot \sqrt{v^2+1}\; du\,dv \Rightarrow \int_0^{\frac{\pi}{2}} \int_0^1 r \cdot \cos(\theta) \cdot \sqrt{r^2 \cdot \cos^2(\theta) + 1} \cdot r \; dr\, d\theta = \underline{\underline{\frac{\pi}{8}}}$$

Problem 2:

Find the total charge on the surface

$$\mathbf{r} = e^u \cos v\mathbf{i} + e^u \sin v\mathbf{j} + u\mathbf{k}, \qquad (0 \le u \le 1, 0 \le v \le \pi),$$

if the charge density on the surface is $\delta = \sqrt{1 + e^{2u}}$.

Solution: $\frac{\pi}{3}(3e + e^3 - 4)$

$$\int_0^{\pi}\int_0^{1} \delta(u,v)\cdot\underbrace{|\vec{r}_u \times \vec{r}_v|}\,du\,dv$$

$$\begin{cases} \vec{r}_u = e^u\cdot\cos(v)i + e^u\cdot\sin(v)j + k \\ \vec{r}_v = -e^u\cdot\sin(v)i + e^u\cdot\cos(v)j \end{cases} \Rightarrow \vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ e^u\cdot\cos(v) & e^u\cdot\sin(v) & 1 \\ -e^u\cdot\sin(v) & e^u\cdot\cos(v) & 0 \end{vmatrix}$$

$$= (0 - e^u\cdot\cos(v))\,i - (0 - (-e^u\cdot\sin(v)))\,j + (e^{u^2}\cdot\cos^2(v) - (-e^{u^2}\cdot\sin^2(v)))\,k$$

$$= -e^u\cdot\cos(v)\,i - e^u\cdot\sin(v)\,j + e^{u^2}\cdot\overbrace{(\cos^2(v)+\sin^2(v))}^{1}\,k$$

$$= -e^u\cdot\cos(v)\,i - e^u\cdot\sin(v)\,j + e^{u^2}\,k$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{e^{u^2}\cdot\cos^2(v) + e^{u^2}\cdot\sin^2(v) + e^{u^4}} = \sqrt{e^{u^2}\cdot\underbrace{(\cos^2(v)+\sin^2(v))}_{1} + e^{u^4}}$$

$$= \sqrt{e^{u^2}+e^{u^4}} = \sqrt{e^{u^2}(1+e^{u^2})} = e^u\sqrt{e^{u^2}+1} = e^u\sqrt{e^{2u}+1}$$

$$\Rightarrow \int_0^{\pi}\int_0^{1} \delta(u,v)\cdot|\vec{r}_u \times \vec{r}_v|\,du\,dv = \int_0^{\pi}\int_0^{1}\sqrt{e^{2u}+1}\cdot e^u\sqrt{e^{2u}+1}\,du\,dv = \int_0^{\pi}\int_0^{1} e^u\cdot(e^{2u}+1)\,du\,dv$$

$$= \int_0^{\pi}\int_0^{1} e^{3u} + e^u\,du\,dv = \int_0^{\pi}\left[\frac{1}{3}\cdot e^{3u} + e^u\right]_0^1 dv = \int_0^{\pi}\frac{1}{3}\cdot e^3 + e^1 - \frac{1}{3}\cdot e^0 - e^0\,dv = \int_0^{\pi}\frac{1}{3}\cdot e^3 + e - \frac{1}{3} - 1\,dv$$

$$= \left[\frac{1}{3}\cdot e^3\cdot v + e\cdot v - \frac{v}{3} - v\right]_0^{\pi} = \frac{1}{3}e^3\pi + e\pi - \frac{\pi}{3} - \pi = \frac{\pi}{3}(e^3 + 3e - 1 - 3) = \underline{\underline{\frac{\pi}{3}(e^3 + 3e - 4)}}$$

Problem 3:

Describe the parametric surface

$$x = au\cos v, \quad y = au\sin v, \quad z = bv$$

$(0 \le u \le 1, 0 \le v \le 2\pi)$, and find its area.

Solution: $\pi a\sqrt{a^2+b^2} + \pi b^2 \ln\left(\frac{a+\sqrt{a^2+b^2}}{b}\right)$ sq. units

$$\vec{r}(u,v) = au\cdot\cos(v)\,\hat{i} + au\cdot\sin(v)\,\hat{j} + bv\,\hat{k}$$

$$\Rightarrow \begin{cases} \vec{r}_u = a\cdot\cos(v)\,\hat{i} + a\cdot\sin(v)\,\hat{j} \\ \vec{r}_v = -au\cdot\sin(v)\,\hat{i} + au\cdot\cos(v)\,\hat{j} + b\,\hat{k} \end{cases}$$

$$\Rightarrow \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cdot\cos(v) & a\cdot\sin(v) & 0 \\ -au\cdot\sin(v) & au\cdot\cos(v) & b \end{vmatrix} = (ab\cdot\sin(v) - \cancel{0})\,\hat{i} - (ab\cdot\cos(v) - \cancel{0})\,\hat{j} + (u\cdot a^2\cos^2(v) \underbrace{-(-}_{+} u\cdot a^2\sin^2(v)))k$$

$$= ab\cdot\sin(v)\,\hat{i} - ab\cdot\cos(v)\,\hat{j} + (u\cdot a^2\cos^2(v) + u\cdot a^2\sin^2(v))k$$

$$= ab\cdot\sin(v)\,\hat{i} - ab\cdot\cos(v)\,\hat{j} + (u\cdot a^2\cdot\overbrace{(\cos^2(v)+\sin^2(v))}^{1})k$$

$$= ab\cdot\sin(v)\,\hat{i} - ab\cdot\cos(v)\,\hat{j} + a^2uk$$

$$\Rightarrow |\vec{r}_u \times \vec{r}_v| = \sqrt{a^2b^2\sin^2(v) + a^2b^2\cos^2(v) + a^4u^2}$$

$$= \sqrt{a^2b^2\cdot\underbrace{(\sin^2(v)+\cos^2(v))}_{1} + a^4u^2}$$

$$= \sqrt{a^2b^2 + a^4u^2} = \sqrt{a^2\cdot(b^2+a^2u^2)}$$

$$= a\sqrt{b^2+a^2u^2}$$

Not good...

$\lambda = b^2 + a^2u^2 \Rightarrow \frac{d\lambda}{du} = 2a^2u \Rightarrow du = \frac{1}{2a^2u}\,d\lambda$

$$\Rightarrow \int_0^{2\pi}\int_0^1 1\cdot a\sqrt{b^2+a^2u^2}\,du\,dv = a\cdot\int_0^{2\pi}\int_0^1 \sqrt{b^2+a^2u^2}\,du\,dv =$$

Problem 4:

Find the flux of $\mathbf{F} = x\mathbf{i} + x\mathbf{j} + \mathbf{k}$ upward through the part of the surface $z = x^2 - y^2$ inside the cylinder $x^2 + y^2 = a^2$.

Solution: $\frac{\pi}{2}a^2(2 - a^2)$

Convert to cylindrical coordinates

$$f(\theta, r, z) = (\underbrace{r \cdot \cos\theta}_{x}, \underbrace{r \cdot \sin\theta}_{y}, \underbrace{z}_{z})$$

$$||J|| = r$$

$$z = x^2 - y^2 \Rightarrow z(\theta, r) = r^2 \cdot \cos^2(\theta) - r^2 \cdot \sin^2(\theta)$$
$$= r^2 \cdot (\cos^2(\theta) - \sin^2(\theta))$$
$$= r^2 \cdot \cos(2\theta)$$

We only need the upward component of F

$$f(x, y, z) = x \Rightarrow f(\theta, r, z(\theta, r)) = r \cdot \cos(\theta)$$

Finding limits

$$x^2 + y^2 \leq a^2 \Rightarrow 0 \leq r \leq a$$
$$0 \leq \theta \leq 2\pi$$

Writing the integral

$$\int_0^a \int_0^{2\pi} f(\theta, r, z(\theta, r)) \cdot \sqrt{\left(\frac{\partial z}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 + 1} \; d\theta \, dr$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial}{\partial \theta} r^2 \cdot \cos(2\theta) = -2r^2 \cdot \sin(2\theta)$$

$$\frac{\partial z}{\partial r} = \frac{\partial}{\partial \theta} r^2 \cdot \cos(2\theta) = 2r \cdot \cos(2\theta)$$

Spørg Jacob!

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$$= r^2 \cdot (\cos^2(\theta) - \sin^2(\theta))$$
$$= r^2 \cdot \cos(2\theta)$$

Parameterization

$u = r, \quad v = \theta, \quad z = u \cdot \cos(2v)$

$$\vec{r} = u\hat{i} + v\hat{j} + u \cdot \cos(2v)\hat{k}$$

$$\begin{cases} \vec{r}_u = \hat{i} + \cos(2v)\hat{k} \\ \vec{r}_v = \hat{j} - u \cdot \sin(2v) \cdot 2 \end{cases}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \cos(2v) \\ 0 & 1 & -2u\sin(2v) \end{vmatrix} = (0 - \cos(2v))\hat{i} - (-2u \cdot \sin(2v) - 0)\hat{j} + (1 - 0)\hat{k}$$
$$= -\cos(2v)\hat{i} + 2u \cdot \sin(2v)\hat{j} + \hat{k}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{\cos^2(2v) + 4u^2 \cdot \sin^2(2v) + 1}$$

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} = \frac{-\cos(2v)\hat{i} + 2u \cdot \sin(2v)\hat{j} + \hat{k}}{\sqrt{\cos^2(2v) + 4u^2 \cdot \sin^2(2v) + 1}}$$

Getting the limits

$$x^2 + y^2 \le a^2 \Rightarrow 0 \le r \le a \Rightarrow 0 \le u \le a$$
$$0 \le \theta < 2\pi \Rightarrow 0 \le v \le 2\pi$$
$$dS = r \cdot d\theta dr \Rightarrow dS = u \, dv du$$

Parameterization of vector field

$$\vec{F}(x(u,v), y(u,v), z(u,v)) = u\hat{i} + u\hat{j} + 1$$

Writing the integral

$$\iint_S \vec{F} \times \vec{n} \, dS = \int_0^a \int_0^{2\pi} (u\hat{i} + u\hat{j} + 1) \cdot \left(\frac{-\cos(2v)\hat{i} + 2u \cdot \sin(2v)\hat{j} + \hat{k}}{\sqrt{\cos^2(2v) + 4u^2 \cdot \sin^2(2v) + 1}} \right) \cdot v \, dv \, d\theta$$