1. Check that if each u satisfies the two-dimensional Poisson's equation or not, written below:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

Where, u is dependent on independent variable x and y.

- a • $u = \frac{2y}{x}$
- b • $u = x^4 + y^4$
- c • $u = -2\cos x \sin y$

a)

$$\frac{\partial}{\partial x^2}\left(\frac{2y}{x}\right) + \frac{\partial}{\partial y^2}\left(\frac{2y}{x}\right) = \frac{\partial}{\partial x}\left(\frac{-2y}{x^2}\right) + \frac{\partial}{\partial y}\left(\frac{2}{x}\right) = \frac{4y}{x^3} + 0 = \frac{4y}{x^3}$$

b)

$$\frac{\partial}{\partial x^2}(x^4 + y^4) + \frac{\partial}{\partial y^2}(x^4 + y^4) = \frac{\partial}{\partial x}(4x^3) + \frac{\partial}{\partial y}(4y^3) = 12x^2 + 12y^2 = 12(x^2 + y^2)$$

c)

$$\frac{\partial}{\partial x^2}\left(-2\cos(x)\cdot\sin(y)\right) + \frac{\partial}{\partial y^2}\left(-2\cos(x)\cdot\sin(y)\right)$$

$$= \frac{\partial}{\partial x}\left(2\cdot\sin(y)\cdot\sin(x)\right) + \frac{\partial}{\partial y}\left(-2\cos(x)\cdot\cos(y)\right) = 2\cdot\sin(y)\cdot\cos(x) + 2\cdot\cos(x)\cdot\sin(y)$$

$$= 4\cdot\cos(x)\cdot\sin(y)$$

Yes, all these are functions of x and y, and are therefore solutions to the PDE.

2. calculate **div F** and **curl F** for the vector field,

$$\mathbf{F} = \overset{f_1}{\cos x}\,\mathbf{i} - \overset{f_2}{\sin y}\,\mathbf{j} + \overset{f_3}{z}\,\mathbf{k}.$$

Finding divergence

$$\nabla \cdot F = \frac{\partial}{\partial x}\left(\cos(x)\right) + \frac{\partial}{\partial y}\left(-\sin(y)\right) + \frac{\partial}{\partial z} z = \underline{\underline{-\sin(x) - \cos(y) + 1}}$$

Finding curl

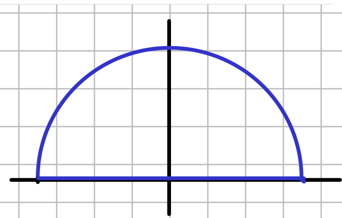
$$\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) i + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) j + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) k$$

$$\text{curl } \vec{F} = (0-0)\,i + (0-0)\,j + (0-0)\,k = \underline{\underline{\vec{0}}}$$

3. Using green's theorem evaluate $\oint_e (x^2y)\,dx + (xy^2)dy$, clockwise boundary of the region.

$$0 \le y \le \sqrt{9 - x^2}$$

$$\oint_C \vec{F}(x,y)\,\mathrm{d}\vec{r} = \oint_C f_1(x,y)\mathrm{dx} + f_2(x,y)\mathrm{dy} = \iint_R \frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y}\,\mathrm{dA}$$



$$-\oint_C \overset{f_1}{x^2y}\,dx + \overset{f_2}{xy^2}\,dy = -\iint_R \frac{\partial}{\partial x}(x^2y) - \frac{\partial}{\partial y}(xy^2)\,dA$$

$$= -\iint_R 2xy - 2xy\,dA = -\iint_R 0\,dA = -0 = \underline{\underline{0}}$$

4. Use the Divergence Theorem to calculate the flux of the given vector field out of the sphere $\mathcal{S}$ with equation $x^2 + y^2 + z^2 = a^2$, where $a > 0$. *and*

$$\mathbf{F} = \underbrace{(x^2 + y^2)}_{f_1}\mathbf{i} + \underbrace{(y^2 - z^2)}_{f_2}\mathbf{j} + \underbrace{z}_{f_3}\mathbf{k}$$

$$\iint_S \vec{F} \bullet \vec{n}\, \mathrm{dS} = \iiint_V \mathbf{div}\, \vec{F}\, \mathrm{dV}$$

Finding divergence of vector field

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2+y^2) + \frac{\partial}{\partial y}(y^2 - z^2) + \frac{d}{dz} z = 2x + 2y + 1$$

Convert to spherical coordinates

$$\begin{aligned} x &= \rho \cdot \sin\phi \cdot \cos\theta \\ y &= \rho \cdot \sin\phi \cdot \sin\theta \\ z &= \rho \cdot \cos\phi \\ \mathrm{dV} &= \rho^2 \cdot \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta \end{aligned}$$

$$\iiint_V 2x + 2y + 1 \, dV$$

$$= \iiint_V \left(2 \cdot \rho \cdot \sin(\phi) \cdot \cos(\theta) + 2\rho \cdot \sin(\phi) \cdot \sin(\theta) + 1\right) \cdot \rho^2 \cdot \sin(\phi) \; d\rho \, d\phi \, d\theta$$

$$= \iiint_V \underbrace{2 \cdot \rho^3 \cdot \sin^2(\phi)} \cdot \cos(\theta) + \underbrace{2 \cdot \rho^3 \cdot \sin^2(\phi)} \cdot \sin(\theta) + \rho^2 \cdot \sin(\phi) \; d\rho \, d\phi \, d\theta$$

$$= \iiint_V 2 \cdot \rho^3 \cdot \sin^2(\phi) \cdot \left(\cos(\theta) + \sin(\theta)\right) + \rho^2 \cdot \sin(\phi) \; d\rho \, d\phi \, d\theta$$

Adding the boundries

$$x^2 + y^2 + z^2 = a^2 \Rightarrow \rho = [0, a], \; \phi = [0, \pi], \; \theta = [0, 2\pi]$$

$$\Rightarrow \int_0^{2\pi} \int_0^{\pi} \int_0^{a} 2 \cdot \rho^3 \cdot \sin^2(\phi) \cdot \left(\cos(\theta) + \sin(\theta)\right) + \rho^2 \cdot \sin(\phi) \; d\rho \, d\phi \, d\theta$$

$$\Rightarrow \underbrace{2 \cdot \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^3 \cdot \sin^2(\phi) \cdot \left(\cos(\theta) + \sin(\theta)\right) d\rho \, d\phi \, d\theta} + \underbrace{\int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^2 \cdot \sin(\phi) \; d\rho \, d\phi \, d\theta}$$

We start by evaulating the first integral

$$2 \cdot \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^3 \cdot \sin^2(\phi) \cdot \left(\cos(\theta) + \sin(\theta)\right) d\rho \, d\phi \, d\theta = 2 \cdot \int_0^{2\pi} \int_0^{\pi} \frac{a^4}{4} \cdot \sin^2(\phi) \cdot \left(\cos(\theta) + \sin(\theta)\right) \; d\phi \, d\theta$$

$$= 2\cdot\int_0^{2\pi}\int_0^{\pi} \frac{a^4}{4}\cdot\sin^2(\phi)\cdot\left(\cos(\theta)+\sin(\theta)\right)\ d\phi\, d\theta$$

$$= \frac{a^4}{2}\int_0^{2\pi}\int_0^{\pi} \sin^2(\phi)\cdot\left(\cos(\theta)+\sin(\theta)\right)\ d\phi\, d\theta$$

$$= \frac{a^4}{2}\int_0^{2\pi}\left(\cos(\theta)+\sin(\theta)\right)\int_0^{\pi} \sin^2(\phi)\ d\phi\, d\theta$$

$$= \frac{a^4}{2}\int_0^{2\pi}\left(\cos(\theta)+\sin(\theta)\right)\cdot\left[\frac{\phi}{2}-\frac{1}{4}\cdot\sin(2\phi)\right]_0^{\pi}\ d\theta$$

$$= \frac{a^4}{2}\int_0^{2\pi}\left(\cos(\theta)+\sin(\theta)\right)\cdot\frac{\pi}{2}-\cancel{\frac{1}{4}\cdot\sin(2\pi)}\ d\theta$$

$$= \frac{\pi a^4}{4}\int_0^{2\pi}\cos(\theta)+\sin(\theta)\ d\theta = \frac{\pi a^4}{4}\cdot\left[\sin(\theta)-\cos(\theta)\right]_0^{2\pi} = \frac{\pi a^4}{4}\cdot\left(\left(\overset{0}{0}-1\right)-\left(0-1\right)\right)=0$$

Evaluating the second integral

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\rho^2\cdot\sin(\phi)\ d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi}\left[\frac{\rho^3}{3}\cdot\sin(\phi)\right]_0^{a} d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi}\frac{a^3}{3}\cdot\sin(\phi)\ d\phi\, d\theta$$

$$= \frac{a^3}{3}\cdot\int_0^{2\pi}\int_0^{\pi}\sin(\phi)\ d\phi\, d\theta = \frac{a^3}{3}\cdot\int_0^{2\pi}\int_0^{\pi}\sin(\phi)\ d\phi\, d\theta = \frac{a^3}{3}\cdot\int_0^{2\pi}\left[-\cos(\phi)\right]_0^{\pi} d\theta = \frac{a^3}{3}\cdot\int_0^{2\pi}\left(-\overset{0}{\cos(\pi)}+\overset{1}{\cos(0)}\right) d\theta$$

$$= \frac{a^3}{3}\cdot\int_0^{2\pi}1\ d\theta = \frac{a^3}{3}\cdot\left[\theta\right]_0^{2\pi} = \frac{2\pi a^3}{3}$$

Adding the integrals togeather

$$\Rightarrow 2\cdot\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\rho^3\cdot\sin^2(\phi)\cdot\left(\cos(\theta)+\sin(\theta)\right)\ d\rho\, d\phi\, d\theta + \int_0^{2\pi}\int_0^{\pi}\int_0^{a}\rho^2\cdot\sin(\phi)\ d\rho\, d\phi\, d\theta = 0 + \frac{2\pi a^3}{3}$$

$$= \frac{2\pi a^3}{3}$$

5. Determine whether the given vector field is conservative, and find a potential function if it is,

$$\mathbf{F}(x, y, z) = (2xy - z^2)\mathbf{i} + (2yz + x^2)\mathbf{j} - (2zx - y^2)\mathbf{k}$$

$$\left.\begin{array}{l} \dfrac{\partial f_1}{\partial y} = \dfrac{\partial f_2}{\partial x}, \\ \dfrac{\partial f_1}{\partial z} = \dfrac{\partial f_3}{\partial x}, \\ \dfrac{\partial f_2}{\partial z} = \dfrac{\partial f_3}{\partial y}, \end{array}\right\}$$

True for conservative fields. Let's test it!

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} \Rightarrow 2x = 2x \checkmark$$

$$\frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x} \Rightarrow -2z = -2z \checkmark$$

$$\frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y} \Rightarrow 2y = 2y \checkmark$$

The potential function must have this property

$$\nabla f = \vec{F}$$

$$\Rightarrow \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k = (2xy - z^2) i + (2yz + x^2) j - (2zx - y^2) k$$

$$\Rightarrow \begin{cases} \dfrac{\partial f}{\partial x} = 2xy - z^2 \\ \dfrac{\partial f}{\partial y} = 2yz + x^2 \\ \dfrac{\partial f}{\partial z} = -(2zx - y^2) \end{cases} \Rightarrow \begin{cases} f = \int 2xy - z^2 \, dx \\ f = \int 2yz + x^2 \, dy \\ f = \int -2zx + y^2 \, dz \end{cases} \Rightarrow \begin{cases} f = x^2y - xz^2 + a(y, z) \\ f = y^2z + x^2y + b(x, z) \\ f = -z^2x + y^2z + c(x, y) \end{cases}$$

Summing up f's

$$f(x, y, z) = x^2y - xz^2 + a + y^2z + x^2y + b - z^2x + y^2z + c$$

$$\Rightarrow f(x, y, z) = x^2y - xz^2 + y^2z + x^2y - z^2x + y^2z + a(y, z) + b(x, z) + c(x, y)$$

$$\Rightarrow f(x, y, z) = 2x^2y - 2xz^2 + 2y^2z + a(y, z) + b(x, z) + c(x, y)$$

Finding a, b and c is left as an exercise for the reader;)