Problem 1: Find the type, transform to normal form, and solve. (Show the details of your work)

$$u_{xy} - u_{yy} = 0$$

$$Au_{xx} + Bu_{xy} + C_{yy} + Du_x + Eu_y + Fu = G$$

$\Rightarrow B = 1, C = -1, A = D = E = F = G = 0$

Calculate discriminant

$d = B^2 - 4AC = 1^2 - 4 \cdot 0 \cdot (-1) = 1$

$d > 0 \Rightarrow$ Hyperbolic!!

Get xi and eta

$A\lambda^2 - B\lambda + C = 0 \Rightarrow -\lambda - 1 = 0 \Rightarrow \lambda = 1$

$\frac{dy}{dx} = \lambda \Rightarrow \int dy = \int \lambda \, dx = y = x$

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$\Rightarrow B = 1,\ C = -1,\ A = D = E = F = G = 0$

Calculate discriminant

$B^2 - 4AC = 1^2 - 4 \cdot 0 \cdot (-1) = 1 \Rightarrow$ Hyperbolic!!

Characteristic Equations

$Ay'^2 + By' + C = 0 \Rightarrow 0 \cdot y'^2 + y' - 1 = 0 \Rightarrow y' = 1$

$\Rightarrow y = x + C_1 \Rightarrow C_1 = y - x$

New variable w

$w = y - x \Rightarrow w_x = -1,\ w_y = 1$

Describe u in terms of w

$u_x = u_w \overbrace{w_x}^{-1} = -u_w$

$u_{xy} = (-u_w)_w \overbrace{w_y}^{1} = -u_{ww} \cdot 1 = -u_{ww}$

$u_y = u_w \overbrace{w_y}^{1} = u_w$

$u_{yy} = (u_w)_w \cdot \overbrace{w_y}^{1} = u_{ww}$

Substitute variables

$u_{xy} - u_{yy} = 0 \Rightarrow -u_{ww} - u_{ww} = 0 \Rightarrow -2u_{ww} = 0$

$\Rightarrow u_{ww} = 0$ Normal form !!

Solution

$u_{ww} = f_1(w) \Rightarrow \iint f_1(w)\,dw\,dw = f_2(w)$

$\underline{u(x, y) = f_2(y - x)}$

Problem 1: Find the type, transform to normal form, and solve. (Show the details of your work)

$$u_{xy} - u_{yy} = 0$$

This is a linear PDE as nothing is a function of u

$$u(x,y) = F(x)G(y)$$

We find the double derivatives

$$u_{yy} = F(x)G''(y)$$

$$u_{xy} = F'(x)G'(y)$$

Substitute in the original equation

$$F'(x)G'(y) - F(x)G''(y) = 0 \Rightarrow F'(x)G'(y) = F(x)G''(y)$$

$$\Rightarrow \frac{F'(x)}{F(x)} = \frac{G''(y)}{G'(y)} = \lambda$$

$$\begin{matrix} 1 \\ 2 \end{matrix} \begin{cases} F'(x) - \lambda \cdot F(x) = 0 \\ G''(y) - \lambda \cdot G'(y) = 0 \end{cases}$$

Solving the first ODE:

$$f'(x) - \lambda \cdot f(x) = 0 \Rightarrow a = 0,\ b = 1,\ c = -\lambda$$

$$(a \cdot r^2 + b \cdot r + c) \cdot e^{rt} = 0 \Rightarrow a \cdot r^2 + b \cdot r + c = 0$$

$$\Rightarrow r - \lambda = 0 \Rightarrow r = \lambda$$

We have one root. Therefore we use this solution:

$$f(x) = A \cdot e^{rx} + B \cdot x \cdot e^{rx} = A \cdot e^{\lambda x} + B \cdot x \cdot e^{rx}$$

Solving the second ODE:

$$f''(x) - \lambda \cdot f'(x) = 0 \Rightarrow a = 1, \; b = -\lambda, \; c = 0$$

$$\Rightarrow r^2 - \lambda b = 0$$

Solving with quadratic equation

$$d = b^2 - 4ac = \lambda^2$$

$$r = \frac{-b \pm \sqrt{d}}{2a} = \frac{\lambda \pm \sqrt{\lambda^2}}{2} = \frac{\lambda \pm \lambda}{2} \Rightarrow \begin{cases} 0 \\ \lambda \end{cases}$$

Here we have two solutions

$$f(x) = A \cdot e^{r_1 x} + B \cdot e^{r_2 x} = A \cdot e^{0} + B \cdot e^{\lambda x} = A + B \cdot e^{\lambda x}$$

We now have solutions to the ODEs

$$F(x) = A \cdot e^{\lambda x} + B \cdot x \cdot e^{rx} \qquad G(y) = C + D \cdot e^{\lambda y}$$

Combining them to get PDE solution

$$u(x,y) = F(x)\, G(y) = \left(A \cdot e^{\lambda x} + B \cdot x \cdot e^{rx}\right)\left(C + D \cdot e^{\lambda y}\right)$$

Problem 2: Find the type, transform to normal form, and solve. (Show the details of your work)

$$u_{xx} + 2u_{xy} + u_{yy} = 0$$

$$Au_{xx} + Bu_{xy} + C_{yy} + Du_x + Eu_y + Fu = G$$

$\Rightarrow A=1,\ B=2,\ C=1,\ E=F=G=0$

$B^2-4AC = 4-4\cdot1\cdot1 = 0 \Rightarrow$ Parabolic!

Characteristic Equation

$$\frac{dy}{dx} = \frac{B}{2A} = \frac{2}{2\cdot1} = 1 \Rightarrow y = \int 1\,dx \Rightarrow y = x + c_1$$

$$c_1 = y - x \Rightarrow \xi = y - x$$

I choose $\eta = x$

Check jacobian

$$\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0 \ \checkmark$$

Get needed derivatives (from chain rule)

$$\xi_x = -1,\ \xi_y = 1,\ \xi_{xx} = 0,\ \xi_{yy} = 0,\ \xi_{xy} = 0$$

$$\eta_x = 1,\ \eta_y = 0,\ \eta_{xx} = 0,\ \eta_{yy} = 0,\ \xi_{xy} = 0$$

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$
$$u_y = u_\xi \xi_y + u_\eta \eta_y$$
$$u_{xx} = u_{\xi\xi}(\xi_x)^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}(\eta_x)^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$$
$$u_{yy} = u_{\xi\xi}(\xi_y)^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}(\eta_y)^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}$$
$$u_{xy} = u_{\xi\xi}\xi_x\xi_y + u_{\eta\eta}\eta_x\eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy} + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x)$$

$$u_{xx} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$$

$$u_{yy} = u_{\xi\xi}$$

$$u_{xy} = -u_{\xi\xi} + u_{\xi\eta}$$

Replace in original PDE

$$u_{xx} + 2u_{xy} + u_{yy} = 0$$

$$\Rightarrow \underbrace{u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}} \underbrace{- 2u_{\xi\xi} + 2u_{\xi\eta}} + \underbrace{u_{\xi\xi}} = 0$$

$$\Rightarrow u_{\eta\eta} = 0$$

Solve dat shit!

$$u_\eta = \int 0 \, d\eta = f_1(\xi)$$

$$u = \int f_1(\xi) \, d\eta = f_1(\xi) \cdot \int 1 \, d\eta = f_1(\xi) \cdot \eta + f_2(\xi)$$

Remeber when we were young and naive

$$\xi = y - x \, , \quad \eta = x$$

$$\underline{\underline{u = f_1(y-x) \cdot x + f_2(y-x)}}$$

Problem 2: Find the type, transform to normal form, and solve. (Show the details of your work)

$$u_{xx} + 2u_{xy} + u_{yy} = 0$$

$$Au_{xx} + Bu_{xy} + C_{yy} + Du_x + Eu_y + Fu = G$$

$\Rightarrow A=1,\ B=2,\ C=1,\ E=F=G=0$

$B^2-4AC = 4-4\cdot1\cdot1=0 \Rightarrow$ Parabolic!

Characteristic Equation

$$\frac{dy}{dx} = \frac{B}{2A} = \frac{2}{2\cdot1} = 1 \Rightarrow y = \int 1\,dx \Rightarrow y = x + c_1$$

$c_1 = y-x \Rightarrow \xi = y-x$

Eta must not be parallel to xi

$\eta = y$

$$\boxed{\begin{array}{l} \xi = y-x,\ \xi_x = -1,\ \xi_y = 1 \\ \eta = y,\ \eta_x = 0,\ \eta_y = 1 \end{array}}$$

$$\bar{A}u_{\xi\xi} + \bar{B}u_{\xi\eta} + \bar{C}_{\eta\eta} + \bar{D}u_\xi + \bar{E}u_\eta + \bar{F}u = \bar{G}$$

where

$$\begin{cases} \bar{A} &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\ \bar{B} &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_x \\ \bar{C} &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\ \bar{D} &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\ \bar{E} &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\ \bar{F} &= F \\ \bar{G} &= G \end{cases}$$

$\bar{A} = 1\cdot(-1)^2 + 2\cdot(-1)\cdot1 + 1\cdot1^2 = 1-2+1=0$

$\bar{B} = 2\cdot1\cdot(-1)\cdot0 + 2\cdot(-1)\cdot1$

Problem 3: Find the type, transform to normal form, and solve. (Show the details of your work)

$$u_{xx} - 4u_{xy} + 3u_{yy} = 0$$

$$Au_{xx} + Bu_{xy} + C_{yy} + Du_x + Eu_y + Fu = G$$

$A=1$, $B=-4$, $C=3$

$d = B^2 - 4ac = (-4)^2 - 4(1)(3) = 16-12=4$

$d>0 \Rightarrow$ Hypobolic!

Find new variables

$A\lambda^2 - B\lambda + C = 0 \Rightarrow \underset{a}{\lambda^2} + \underset{b}{4\lambda} + \underset{c}{3} = 0$

$\lambda = \frac{-4 \pm \sqrt{4}}{2} = \frac{-4\pm 2}{2} = -2 \pm 1 \Rightarrow \begin{cases} -1 \\ -3 \end{cases}$

$\frac{dy}{dx} = -1 \Rightarrow \int dy = \int -1 dx \Rightarrow y = -x + C_1 \Rightarrow C_1 = x+y$

$\frac{dy}{dx} = -3 \Rightarrow \int dy = \int -3\, dx \Rightarrow y = -3x + C_2 \Rightarrow C_2 = 3x+y$

$\Rightarrow \begin{cases} \xi = x+y \\ \eta = 3x+y \end{cases}$

Find the needed derivatives

$\xi_x = 1$, $\xi_y = 1$, $\xi_{xx} = \xi_{yy} = \xi_{xy} = 0$

$\eta_x = 3$, $\eta_y = 1$, $\eta_{xx} = \eta_{yy} = \eta_{xy} = 0$

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$
$$u_y = u_\xi \xi_y + u_\eta \eta_y$$
$$u_{xx} = u_{\xi\xi}(1)^2 + 2u_{\xi\eta} 1\cdot 3 + u_{\eta\eta}(3)^2 + \cancel{u_\xi \xi_{xx}} + \cancel{u_\eta \eta_{xx}}$$
$$u_{yy} = u_{\xi\xi}(1)^2 + 2u_{\xi\eta} 1\cdot 1 + u_{\eta\eta}(1)^2 + \cancel{u_\xi \xi_{yy}} + \cancel{u_\eta \eta_{yy}}$$
$$u_{xy} = u_{\xi\xi} 1\cdot 1 + u_{\eta\eta} 3\cdot 1 + \cancel{u_\xi \xi_{xy}} + \cancel{u_\eta \eta_{xy}} + u_{\xi\eta}(1\cdot 1 + 1\cdot 3)$$

$u_{xx} = u_{\xi\xi} + 6u_{\xi\eta} + 9u_{\eta\eta}$

$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

$u_{xy} = u_{\xi\xi} + 3u_{\eta\eta} + 4u_{\xi\eta}$

Substitute in original equation

$u_{xx} - 4u_{xy} + 3u_{yy} = 0$

$\Rightarrow u_{\xi\xi} + 6u_{\xi\eta} + 9u_{\eta\eta} - 4u_{\xi\xi} - 12u_{\eta\eta} - 16u_{\xi\eta} + 3u_{\xi\xi} + 6u_{\xi\eta} + 3_{\eta\eta} = 0$

$\Rightarrow -4u_{\xi\eta} = 0 \Rightarrow u_{\xi\eta} = 0$

A laterally insulated bar of length 10cm and constant cross-sectional area $1cm^2$, of density $10.6\,gm/cm^3$, thermal conductivity $1.04\,cal/(cm\sec{}^\circ C)$, and specific heat $0.056\,cal/(gm\,{}^\circ C)$ (this corresponds to silver, a good heat conductor) has initial temperature $f(x)$ and is kept at $0°C$ at the ends $x = 0$ and $x = 10$. Find the temperature $u(x,t)$ at later times. Here $f(x)$ equals:

Problem 4: $f(x) = \sin 0.4\,\pi x$

$$\frac{\partial u}{\partial t} = \alpha \cdot \nabla^2 u, \qquad \alpha = \frac{k}{c\rho}$$

t: Time

u: Temperature as a function of position and time.

k: Thermal conductivity

c: Specific Heat Capacity

ρ: Density

$$c = 0{,}056$$
$$k = 1{,}04$$
$$\rho = 10.6$$

Calculate alpha

$$\alpha = \frac{k}{c\rho} \approx 1{,}752 \qquad u_t = \alpha \cdot u_{xx}$$

Initial contitions

$$u(0,t) = 0 \qquad u(x,0) = f(x)$$
$$u(10,t) = 0$$

Assume that we can solve with seperation of variables

$$u(x,t) = F(x)\,G(t)$$

$$\Rightarrow u_t = F(x)\,G'(t), \quad u_{xx} = F''(x)\,G(t)$$

Rewrite equation

$$F(x)\,G'(t) = \alpha \cdot F''(x)\,G(t) \Rightarrow \frac{G'(t)}{G(t)} = \alpha \cdot \frac{F''(x)}{F(x)} = \lambda$$

Convert to ODEs

$$\begin{cases} \alpha \cdot F''(x) - \lambda \cdot F(x) = 0 \\ G'(t) - \lambda \cdot G(t) = 0 \end{cases}$$

Solving the first ODE

$$\alpha \cdot F''(x) - \lambda \cdot F(x) = 0 \Rightarrow a = \alpha, \; b = -\lambda, \; c = 0$$

$$\Rightarrow e^{rt} \cdot (a \cdot r^2 + b \cdot r + c) = 0 \Rightarrow a \cdot r^2 + b \cdot r + c = 0$$

$$\Rightarrow d = b^2 - 4ac = \lambda^2$$

$$\Rightarrow r = \frac{-b \pm \sqrt{d}}{2a} = \frac{\lambda \pm \sqrt{\lambda^2}}{2\alpha} = \frac{\lambda \pm \lambda}{2\alpha} = \begin{cases} r_1 = 0 \\ r_2 = \frac{\lambda}{\alpha} \end{cases}$$

Because we find two root we use the following solution

$$F(x) = A \cdot e^{r_1 x} + B \cdot e^{r_2 x} = A \cdot e^0 + B \cdot e^{\frac{\lambda}{\alpha} \cdot x} = A + B \cdot e^{\frac{\lambda}{\alpha} \cdot x}$$

Solving the second one

$$G'(t) - \lambda \cdot G(t) = 0 \Rightarrow a = 0, \; b = 1, \; c = -\lambda$$

$$e^{rt} \cdot (a \cdot r^2 + b \cdot r + c) = 0 \Rightarrow a \cdot r^2 + b \cdot r + c = 0 \Rightarrow r - \lambda = 0$$

$$\Rightarrow r = \lambda$$

Here we only get one solution

$$G(t) = C \cdot e^{\lambda t} + D \cdot x \cdot e^{\lambda t}$$

We combine the functions to get a general solution

$$u(x,t) = F(x) G(t) = \left(A + B \cdot e^{\frac{\lambda}{\alpha} \cdot x}\right)\left(C \cdot e^{\lambda t} + D \cdot x \cdot e^{\lambda t}\right)$$

Plugging in initial conditions

$$u(0,t) = F(0) G(t) = 0 \Rightarrow F(0) = 0$$

G(t) = 0 is also a solution, but it is uninteresting.

$$u(10,t) = F(10) G(t) = 0 \Rightarrow F(10) = 0$$

We use this to find A and B

$$F(0)=0 \Rightarrow A+B\cdot e^{\frac{\lambda}{\alpha}\cdot 0}=0 \Rightarrow A+B=0 \Rightarrow B=-A$$

$$F(10)=0 \Rightarrow A-A\cdot e^{\frac{\lambda}{\alpha}\cdot 10}=0 \Rightarrow A\cdot\left(1-e^{\frac{\lambda}{\alpha}\cdot 10}\right)=0$$

This is not great. Here either A or λ must be 0 leading to u = 0

$$\lambda=0 \Rightarrow u(x,t)=0$$

$$A=0 \Rightarrow u(x,t)=0$$

Instead we now assume that $\lambda < 0$

And that we can express i like this: $\lambda = -p^2$

We now solve the first ODE again

$$\alpha\cdot F''(x)-\lambda\cdot F(x)=0 \Rightarrow \alpha\cdot F''(x)+p^2\cdot F(x)=0$$

$$\Rightarrow a=\alpha,\ b=0,\ c=p^2,\ d=b^2-4ac=-4\alpha p^2$$

$$\Rightarrow a\cdot r^2+b\cdot r+c=0 \Rightarrow r=\frac{-b\pm\sqrt{d}}{2a}=\frac{\pm\sqrt{-4\alpha p^2}}{2\alpha}=\frac{\pm\sqrt{-1}\sqrt{4}\sqrt{\alpha}\sqrt{p^2}}{2\alpha}$$

$$=\frac{\pm i\cdot\cancel{2}\cdot\sqrt{\alpha}\cdot p}{\cancel{2}\alpha}\cdot\frac{\sqrt{\alpha}}{\sqrt{\alpha}}=\frac{\pm i\cdot\cancel{\alpha}\cdot p}{\cancel{\alpha}\cdot\sqrt{\alpha}}=\pm i\cdot\frac{p}{\sqrt{\alpha}}$$

$$r=k\pm wi \Rightarrow k=0,\ w=\frac{p}{\sqrt{\alpha}}$$

The solution can now be expressed like this

$$F(x)=A\cdot e^{kx}\cdot\cos(wx)+B\cdot e^{kx}\cdot\sin(wx)=A\cdot\cancel{e^0}\cdot\cos\left(\frac{p}{\sqrt{\alpha}}x\right)+B\cdot\cancel{e^0}\cdot\sin\left(\frac{p}{\sqrt{\alpha}}x\right)$$

$$=A\cdot\cos\left(\frac{p}{\sqrt{\alpha}}x\right)+B\cdot\sin\left(\frac{p}{\sqrt{\alpha}}\right)$$

Plugging in initial conditions

$$u(0,t)=0 \Rightarrow F(0)=0 \Rightarrow A\cdot\overset{1}{\cos(0)}+\cancel{B\cdot\sin(0)}=0 \Rightarrow A=0 \Rightarrow F(x)=B\cdot\sin\left(\frac{p}{\sqrt{\alpha}}x\right)$$

We assume that B != 0 because otherwise u(x, t) = 0

$$u(10,t) = 0 \Rightarrow F(10) = 0 \Rightarrow B \cdot \sin\left(\frac{p}{\sqrt{\alpha}} \cdot 10\right) = 0 \Rightarrow \sin\left(p \cdot \frac{10}{\sqrt{\alpha}}\right) = 0$$

$$\Rightarrow p = \frac{n\pi\sqrt{\alpha}}{10}, \quad n = \{1, 2, 3 \ldots\}$$

Inserting p

$$F(x) = B \cdot \sin\left(\frac{n\pi\cancel{\sqrt{\alpha}}}{10} \cdot \frac{x}{\cancel{\sqrt{\alpha}}}\right) = B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right), \quad n = \{1, 2, 3 \ldots\}$$

Let's now take a look at G(y)

$$G'(t) - \lambda \cdot G(t) = 0 \Rightarrow G'(t) + p^2 \cdot G(t) = 0$$

$$\Rightarrow a = 0, \ b = 1, \ c = p^2 \Rightarrow a \cdot r^2 + b \cdot r + c = 0 \Rightarrow r + p^2 = 0 \Rightarrow r = -p^2$$

$$\Rightarrow r = -\left(\frac{n\pi\sqrt{\alpha}}{10}\right)^2 = -\frac{n^2\pi^2\alpha}{100}$$

We still get one double root

$$G(t) = C \cdot e^{r_1 t} + D \cdot x \cdot e^{r_2 t} = C \cdot e^{-\frac{n^2\pi^2\alpha}{100} \cdot t} + D \cdot t \cdot e^{-\frac{n^2\pi^2\alpha}{100} \cdot t}$$

Kombining solutions

$$u(x,t) = F(x)\,G(t) = B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right) \cdot \left(C \cdot e^{-\frac{n^2\pi^2\alpha}{100} \cdot t} + D \cdot t \cdot e^{-\frac{n^2\pi^2\alpha}{100} \cdot t}\right)$$

Plugging in initial condition

$$u(x,0) = f(x) \Rightarrow \sin(0{,}4\pi x) = B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right) \cdot \left(C \cdot \cancel{e^0} + \cancel{D \cdot 0 \cdot e^0}\right)$$

$$\Rightarrow \sin(0{,}4\pi x) = B \cdot \sin\left(\frac{n\pi}{10}\right) \cdot (C + 0 \cdot t) = C \cdot B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right)$$

$$\Rightarrow \sin(0{,}4\pi x) = C \cdot B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right) \Rightarrow \begin{cases} C = 1 \\ B = 1 \end{cases}$$

$$\Rightarrow \sin(0{,}4\pi x) = \sin\left(\frac{n}{10}\pi x\right) \Rightarrow n = 4$$

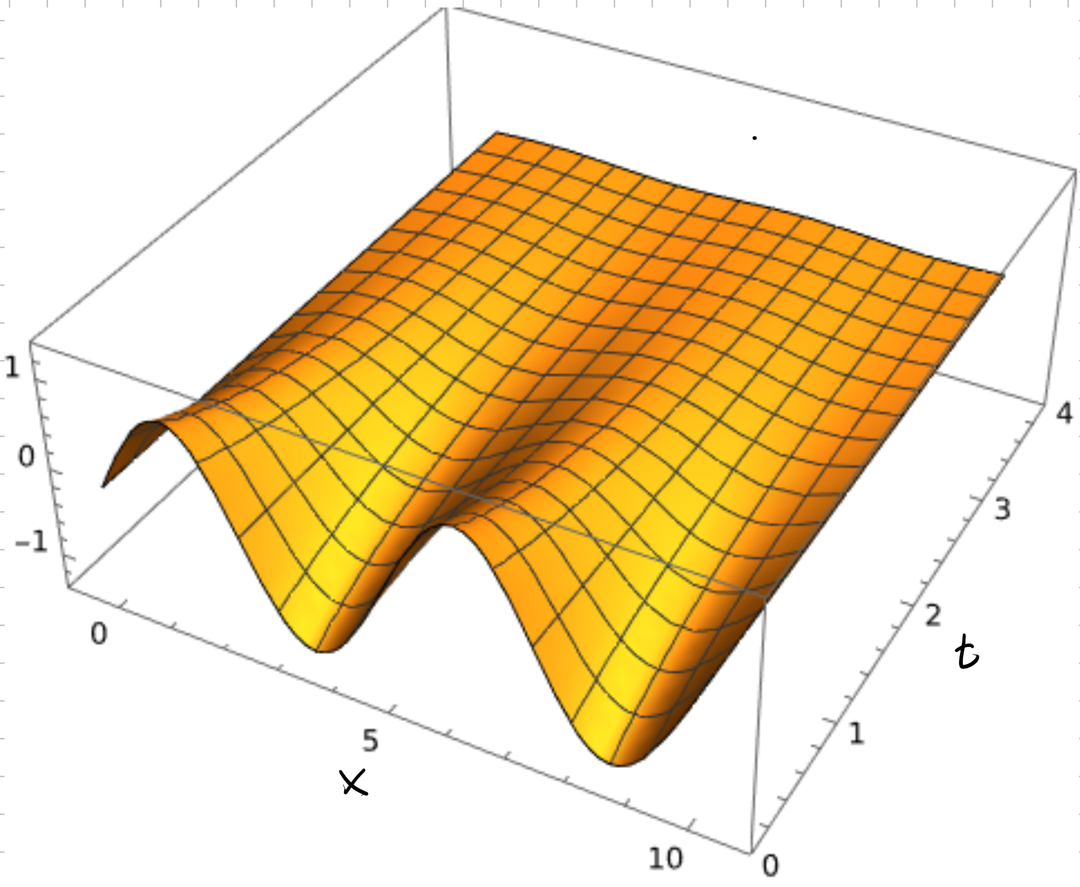
I will set D=0 as this provides a more simple solution, but D does not have to be zero.

We now have all our constants and can write the specific solution

$$u(x,t) = B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right) \cdot \left(C \cdot e^{-\frac{n^2\pi^2\alpha}{100} \cdot t} + D \cdot t \cdot e^{-\frac{n^2\pi^2\alpha}{100} \cdot t}\right)$$

$$= \cancel{1} \cdot \sin\left(\frac{4}{10}\pi x\right) \cdot \left(\cancel{1} \cdot e^{-\frac{n^2\pi^2\alpha}{100} \cdot t} + \cancel{0 \cdot t \cdot e^{-\frac{n^2\pi^2\alpha}{100} \cdot t}}\right)$$

$$\Rightarrow \underline{\underline{u(x,t) = \sin\left(\frac{4}{10}\pi x\right) \cdot e^{-\frac{16 \cdot \pi^2 \alpha}{100} \cdot t}}}$$



Problem 5: $f(x) = \sin 0.1\,\pi x + \frac{1}{2}\sin 0.2\,\pi x$

We start with our general solution:

$$u(x,t) = B\cdot\sin\left(\frac{n\pi}{10}\cdot x\right)\cdot\left(C\cdot e^{-\frac{n^2\pi^2\alpha}{100}\cdot t} + D\cdot t\cdot e^{-\frac{n^2\pi^2\alpha}{100}\cdot t}\right)$$

Plugging in initial condition

$$u(x,0) = f(x) = B\cdot\sin\left(\frac{n\pi}{10}\cdot x\right)\cdot\left(C\cdot \cancel{e^0} + \cancel{D\cdot 0\cdot e^0}\right)$$

$$= B\cdot\sin\left(\frac{n\pi}{10}x\right)\cdot C$$

Because this is a linear PDE the sum of two solutions will also be a solution. We can therefore split this up into two solutions and add them togeather afterwards.

$$u(x,0) = f_1(x) + f_2(x) = u_1(x,t) + u_2(x,t)$$

$$f_1(x) = \sin(0{,}1\pi x),\quad f_2(x) = \frac{1}{2}\cdot\sin(0{,}2\pi x)$$

Get first solution

$$\sin(0{,}1\pi x) = B\cdot C\cdot\sin\left(\frac{n\pi}{10}x\right) \Rightarrow \begin{cases} n=1 \\ B\cdot C = 1 \Rightarrow \underbrace{B=1,\ C=1} \end{cases}$$

I could also have chosen values like B=2 and C=0.5 but this is a more elegant solution

$$u_1(x,t) = 1\cdot\sin\left(\frac{\pi}{10}\cdot x\right)\cdot\left(1\cdot e^{-\frac{1\cdot\pi^2\alpha}{100}\cdot t} + D\cdot t\cdot e^{-\frac{1\cdot\pi^2\alpha}{100}\cdot t}\right)$$

I also set D=0 to simplify

$$u_1(x,t) = \sin\left(\frac{\pi}{10}\cdot x\right)\cdot e^{-\frac{\pi^2\alpha}{100}}$$

Second solution

$$\frac{1}{2}\cdot\sin(0{,}2\pi x) = B\cdot C\cdot\sin\left(\frac{n\pi}{10}x\right) \Rightarrow \begin{cases} n=2 \\ B=\frac{1}{2} \\ C=1 \end{cases}$$

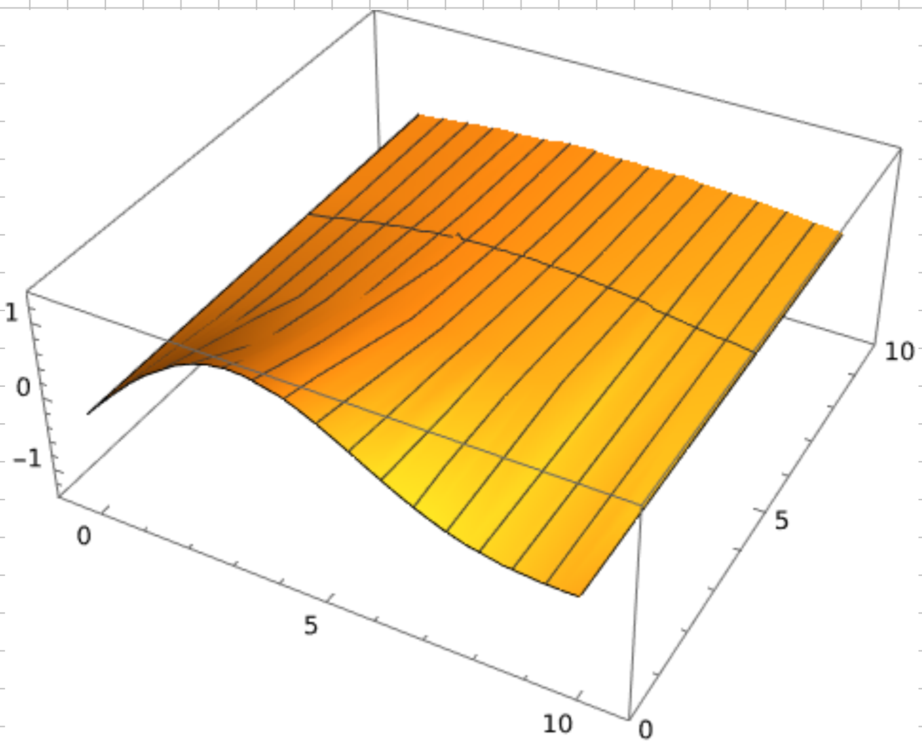
again i set D=0

$$u_2(x,t) = \frac{1}{2}\sin\left(\frac{2\pi}{10}\cdot x\right)\cdot\left(1\cdot e^{-\frac{4\cdot\pi^2\alpha}{100}\cdot t} + 0\cdot t\cdot e^{-\frac{4\cdot\pi^2\alpha}{100}\cdot t}\right)$$

$$= \frac{1}{2}\cdot\sin\left(\frac{2\pi}{10}\cdot x\right)\cdot e^{-\frac{4\pi^2\alpha}{100}\cdot t}$$

We can now add these solutions togeather to get the final solution

$$u(x,t) = u_1(x,t) + u_2(x,t) = \sin\left(\frac{\pi}{10}\cdot x\right)\cdot e^{\frac{-\pi^2\alpha}{100}\cdot t} + \frac{1}{2}\cdot\sin\left(\frac{2\pi}{10}\cdot x\right)\cdot e^{\frac{-4\pi^2\alpha}{100}\cdot t}$$



Problem 6: $f(x) = 1 - 0.2|x - 5|$

We start by approximating the function as a fourier-series

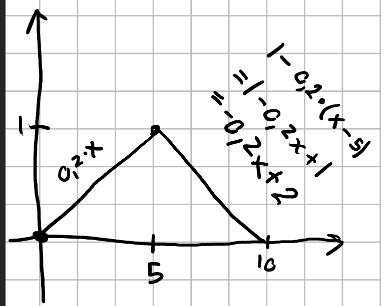
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

n: A how many times of the base frequency.

L: The half period.

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \qquad n \geq 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \qquad n > 0$$



We know that we can only use

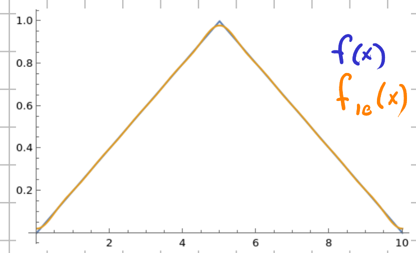
$$f(x) = 1 - 0{,}2\cdot|x-5| \Rightarrow \int_0^{10} f(x)\,dx = \int_0^5 0{,}2x\,dx + \int_5^{10} -0{,}2x+2\,dx$$

$$2L = 10 \Rightarrow L = 5$$

$$N = 10$$

Calculated with mathematica:

$$f_a(x) = \overbrace{-0.405285\cos\left(\frac{\pi x}{5}\right)}^{f_1} \overbrace{-0.0450316\cos\left(\frac{3\pi x}{5}\right)}^{f_2} \overbrace{-0.0162114\cos(\pi x)}^{f_3} - \underbrace{0.00827112\cos\left(\frac{7\pi x}{5}\right)}_{f_4} \underbrace{-0.00500352\cos\left(\frac{9\pi x}{5}\right)}_{f_5} + \underbrace{0.5}_{\text{offset?}}$$



Find general solution to subfunctions on the following form:

$$f(x) = a\cdot\cos\left(\frac{b\pi x}{5}\right)$$

We start with the general solution

$$u(x,t) = B\cdot\sin\left(\frac{n\pi}{10}\cdot x\right)\cdot\left(C\cdot e^{-\frac{n^2\pi^2\alpha}{100}\cdot t} + D\cdot t\cdot e^{-\frac{n^2\pi^2\alpha}{100}\cdot t}\right)$$

Insert the initial condition

$$u(x,0) = f(x) = a\cdot\cos\left(\frac{b\pi x}{5}\right) = B\cdot\sin\left(\frac{n\pi}{10}\cdot x\right)\cdot\left(C\cdot e^0 + D\cdot 0\cdot e^0\right)$$

$$\Rightarrow \quad a \cdot \cos\left(\frac{b\pi x}{5}\right) = B \cdot C \cdot \sin\left(\frac{n\pi}{10} \cdot x\right)$$

$$\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$$

I will set D=0 and C=1 for simplicity

$$\Rightarrow a \cdot \cos\left(\frac{b\pi x}{5}\right) = B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right) \Rightarrow a \cdot \sin\left(\frac{2b\pi}{10} \cdot x + \frac{\pi}{2}\right) = B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right)$$

$$a = B \Rightarrow \sin\left(\frac{2b\pi}{10} \cdot x + \frac{\pi}{2}\right) = \sin\left(\frac{n\pi}{10} \cdot x\right)$$

Idk what to do here, i will assume that n = 2b

$$u(x,t) = a \cdot \sin\left(\frac{2b\pi}{10} \cdot x\right) \cdot \left(e^{-\frac{4b^2\pi^2\alpha}{100} \cdot t} + 0 \cdot t \cdot e^{-\frac{4b^2\pi^2\alpha}{100} \cdot t}\right)$$

$$= a \cdot \sin\left(\frac{2b\pi}{10} \cdot x\right) \cdot e^{-\frac{4b^2\pi^2\alpha}{100} \cdot t}$$

Problem 7: Arbitrary temperatures at ends. If the ends $x = 0$ and $x = L$ of the bar in the text are kept at constant temperatures U_1 and U_2 respectively, what is the temperature $u_1(x)$ in the bar after a long time (theoretically, as $t \to \infty$)? First guess, then calculate.