Problem 3:

Determine whether the given vector field is conservative, and find a potential if it is

$$\mathbf{F}(x, y, z) = x\mathbf{i} - 2y\mathbf{j} + 3z\mathbf{k}$$

Solution:

$$\phi(x, y, z) = \frac{x^2}{2} - y^2 + \frac{3z^2}{2}$$

Check if conservative

$f_1 = x, \; f_2 = -2y, \; f_3 = 3z$

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} \Rightarrow 0 = 0 \quad \checkmark$$

$$\frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x} \Rightarrow 0 = 0 \quad \checkmark$$

$$\frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y} \Rightarrow 0 = 0 \quad \checkmark$$

Conservative!

$$\nabla f = \vec{F}(x, y, z) \Rightarrow \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} x \\ -2y \\ 3z \end{bmatrix} \Rightarrow \begin{bmatrix} f \\ f \\ f \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x^2 + A(y,z) \\ -y^2 + B(x,z) \\ \frac{3}{2}z^2 + C(x,y) \end{bmatrix}$$

$$\Rightarrow f = \frac{1}{2}x^2 + A(y,z) = -y^2 + B(x,z) = \frac{3}{2}z^2 + C(x,y)$$

$$\Rightarrow A(y,z) = -y^2 + \frac{3}{2}z^2, \; B(x,z) = \frac{1}{2}x^2 + \frac{3}{2}z^2, \; C(x,y) = \frac{1}{2}x^2 - y^2$$

$$\Rightarrow \underline{\underline{f = \frac{1}{2}x^2 - y^2 + \frac{3}{2}z^2}}$$

Problem 4:

Determine whether the given vector field is conservative, and find a potential if it is

$$\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$$

Solution:

$$\phi(x, y, z) = xy + \frac{z^3}{3}$$

$f_1 = y, \; f_2 = x, \; f_3 = z^2$

$$\left.\begin{aligned} \frac{\partial f_1}{\partial y} &= \frac{\partial f_2}{\partial x} \Rightarrow 1 = 1 \quad \checkmark \\ \frac{\partial f_1}{\partial z} &= \frac{\partial f_3}{\partial x} \Rightarrow 0 = 0 \quad \checkmark \\ \frac{\partial f_2}{\partial z} &= \frac{\partial f_3}{\partial y} \Rightarrow 0 = 0 \quad \checkmark \end{aligned}\right\} \text{CONSERVATIVE!}$$

$$\Rightarrow \nabla f = \vec{F}(x, y, z) \Rightarrow \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} y \\ x \\ z^2 \end{bmatrix} \Rightarrow \begin{cases} f = xy + A(y, z) \\ f = xy + B(x, z) \\ f = \frac{1}{3}z^3 + C(x, y) \end{cases}$$

All three equations are true if: $A(y, z) = B(x, z) = \frac{1}{3}z^3, \; C(x, y) = xy$

$$\Rightarrow \underline{\underline{f = xy + \frac{1}{3}z^3}}$$

Problem 5:

Determine whether the given vector field is conservative, and find a potential if it is

$$F(x,y) = \frac{x\mathbf{i} - y\mathbf{j}}{x^2 + y^2}$$

Solution:

$$\frac{\partial F_1}{\partial y} = -\frac{2xy}{(x^2+y^2)^2}, \frac{\partial F_2}{\partial x} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{F_1(x,y)}{\partial y} = \frac{d}{dy} \cdot \frac{x}{x^2+y^2} = x \cdot \frac{-1}{(x^2+y^2)^2} \cdot 2y = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{F_2(x,y)}{\partial x} = \frac{d}{dx} \cdot \frac{-y}{x^2+y^2} = -y \cdot \frac{-1}{(x^2+y^2)^2} \cdot 2x = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{F_1(x,y)}{\partial y} \neq \frac{F_2(x,y)}{\partial x} \Rightarrow \text{NOT Conservative}$$

Problem 6:

Evaluate the given line integral over the specified curve C

$$\int_C (x+y)\,ds, \quad \mathbf{r} = at\mathbf{i} + bt\mathbf{j} + ct\mathbf{k}, \; 0 \le t \le m$$

Solution:

$$\int_C (x+y)\,ds = \frac{(a+b)\sqrt{a^2+b^2+c^2}}{2} m^2$$

$$\int_{t=a}^{t=b} f(x(t), y(t))\mathrm{dS}$$

$$\mathrm{dS} = \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2}\,\mathrm{dt}$$

$f(x, y, z) = x + y$

$x(t) = at, \quad y(t) = bt, \quad z(t) = ct$

$f(x(t), y(t), z(t)) = at + bt$

$$\int_{t=0}^{t=m} f(x(t), y(t), z(t))\, dS = \int_{t=0}^{t=m} at + bt \, dS$$

$$dS = \sqrt{x_t^2 + y_t^2 + z_t^2}\; dt = \sqrt{a^2+b^2+c^2}\; dt$$

$$\Rightarrow \int_{t=0}^{t=m} (at+bt)\sqrt{a^2+b^2+c^2}\; dt = \sqrt{a^2+b^2+c^2} \cdot \int_0^m at + bt\; dt$$

$$= \sqrt{a^2+b^2+c^2} \cdot \left[\frac{a}{2}t^2 + \frac{b}{2}t^2\right]_0^m = \sqrt{a^2+b^2+c^2} \cdot \left(\frac{am^2}{2} + \frac{bm^2}{2}\right)$$

$$= \sqrt{a^2+b^2+c^2} \cdot \frac{1}{2} \cdot (a+b) \cdot m^2 = \underline{\underline{\frac{\sqrt{a^2+b^2+c^2} \cdot (a+b)m^2}{2}}}$$

Problem 7:

Find the mass of a wire along the curve

$$\mathbf{r} = 3t\mathbf{i} + 3t^2\mathbf{j} + 2t^3\mathbf{k}, \ (0 \le t \le 1)$$

if the density at $\mathbf{r}(t)$ is $1 + t$ g/unit length.

Solution:

$$m = 8\ g$$

$$\int_0^1 f(\vec{r}(t))\,ds = \int_0^1 1+t\,ds$$

$$ds = \sqrt{x_t^2+y_t^2+z_t^2} = \sqrt{3^2+(6t)^2+(6t^2)^2} = \sqrt{\underbrace{9+36t^2+36t^4}} = \sqrt{9\cdot(2t^2+1)^2} = \sqrt{9}\sqrt{(2t^2+1)^2} = 3\cdot(2t^2+1)\,dt$$

$$= 9\cdot(4t^4+4t^2+1) = 9\cdot(2t^2+1)^2$$

$$\Rightarrow \int_0^1 (t+1)(6t^2+3)\,dt = \int_0^1 6t^3+3t+6t^2+3\,dt = \int_0^1 6t^3+6t^2+3t+3\,dt = \left[\frac{6}{4}t^4+2t+\frac{3}{2}t+3t\right]_0^1$$

$$= \frac{3}{2}+2+\frac{3}{2}+3 = \underline{\underline{8}}$$

Problem 8:

Evaluate the line integral of the tangential component of the given vector field along the given curve

$$\mathbf{F}(x,y) = xy\mathbf{i} - x^2\mathbf{j} \text{ along } y = x^2 \text{ from } (0,0) \text{ to } (1,1)$$

Solution:

$$-\frac{1}{4}$$

$$x(t) = t, \quad y(t) = t^2 \Rightarrow \vec{r}(t) = t\hat{i} + t^2\hat{j}, \quad 0 \le t \le 1$$

Project the vector field onto r(t) to get the tangential component

Projection af $\vec{u}$ på $\vec{v}$. Dvs. $\vec{u}$ i $\vec{v}$'s retning.

$$s = \vec{u} \bullet \frac{\vec{v}}{|\vec{v}|} = |\vec{u}| \cdot \cos\theta$$

$$s = F(x(t), y(t)) \frac{\vec{r}(t)}{\underbrace{|\vec{r}(t)|}}$$

$$|\vec{r}(t)| = |t\hat{i} + t^2\hat{j}| = \sqrt{t^2 + t^4} = \sqrt{t^2 \cdot (t^2+1)} = t\sqrt{t^2+1}$$

dot

$$\Rightarrow s = (t^3\hat{i} - t^2\hat{j}) \cdot \frac{\cancel{t} i + t^{\cancel{2}} j}{\cancel{t}\sqrt{t^2+1}} = (t^3\hat{i} - t^2\hat{j}) \cdot \left(\frac{1}{\sqrt{t^2+1}}\hat{i} + \frac{t}{\sqrt{t^2+1}}\hat{j}\right) = \frac{t^3}{\sqrt{t^2+1}} - \frac{t^3}{\sqrt{t^2+1}} = 0$$

???

Makes no sense...