

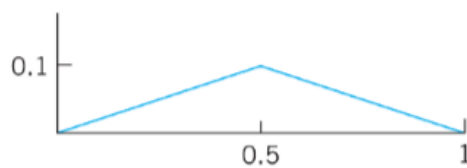
Exercise

Find $u(x, t)$ for the string of length $L=1$ and $c^2 = 1$ when the initial velocity is zero and the initial deflection with small k (say, 0.01) is as follows. Sketch or graph $u(x, t)$. Use the same information for problem 1, 2 and 3 and 4

Problem 1: $k \sin 2\pi x$

Problem 2: $k \left(\sin \pi x - \frac{1}{3} \sin 3\pi x \right)$

Problem 3:



$$u_{tt} = c^2 u_{xx} = c^2 \nabla^2 u$$

$$u(x, t) = k \cdot \sin(2\pi x) \Rightarrow \frac{\partial^2}{\partial t^2} (k \cdot \sin(2\pi x)) = \mathcal{L}^2 \cdot \frac{\partial^2}{\partial x^2} ($$

1. Find the solution of the one-dimensional wave equation

$$u_{tt}(x, t) = u_{xx}(x, t), \quad 0 \leq x \leq 1 \quad \boxed{c^2 = 1}$$

subject to the boundary conditions

$$u(0, t) = u(1, t) = 0 \quad \text{for all } t$$

and the initial condition

$$u(x, 0) = k_0 \sin(3\pi x), \quad u_t(x, 0) = 0, \quad 0 \leq x \leq 1,$$

where k_0 is a constant. Sketch the solution for $t = 0$, $t = 1/6$ and $t = 1/3$.

$$u_{tt}(x, t) = u_{xx}(x, t),$$

We assume that we can solve this with separation of variables

$$u(x, t) = F(x) G(t)$$

We now find the double derivatives

$$\frac{d^2}{dx^2} u(x, t) = \frac{d^2}{dx^2} (F(x) G(t)) = F''(x) G(t)$$

$$\frac{d^2}{dt^2} u(x, t) = \frac{d^2}{dt^2} (F(x) G(t)) = F(x) G''(t)$$

We now plug this into the wave equation

$$u_{tt}(x, t) = u_{xx}(x, t) \Rightarrow F(x) G''(t) = F''(x) G(t)$$

$$\Rightarrow \frac{F''(x)}{F(x)} = \frac{G''(t)}{G(t)} = \lambda$$

This must be a constant because the functions are functions of DIFFERENT variables. This also allows us to split up the equation to two less complicated ones.

$$\Rightarrow \begin{cases} F''(x) - \lambda \cdot F(x) = 0 \\ G''(t) - \lambda \cdot G(t) = 0 \end{cases}$$

We constrain our solutions to be 0 as $x=0$ and $x=1$.

We now take a look at the boundaries

$$u(0, t) = u(1, t) = 0$$

$u(0, t) = 0 \Rightarrow F(0) G(t) = 0 \Rightarrow$ Either $F(0) = 0$ or $G(t) = 0$. This last solution is however uninteresting because it simply leads to the conclusion that $u(x, t) = 0$ which is quite uninteresting.

$$\Rightarrow F(0) = 0$$

Now for the second boundary

$$u(1, t) = 0 \Rightarrow F(1) G(t) = 0 \Rightarrow F(1) = 0$$

We now have to determine λ , if the eigen value is equal to 0 we get the uninteresting that $\lambda=0$. Therefore, we ignore this case.

We assume that λ is positive.

$$\lambda > 0, \quad \lambda = \mu^2$$

This is a solvable diff. equation

$$\Rightarrow F''(x) - \lambda \cdot F(x) = 0 \Rightarrow F''(x) - \mu^2 \cdot F(x) = 0 \Rightarrow F(x) = A \cdot e^{\mu x} + B \cdot e^{-\mu x}$$

We can now combine this with our first boundary condition

$$F(0) = 0 \Rightarrow A \cdot e^0 + B \cdot e^0 = 0 \Rightarrow A + B = 0 \Rightarrow B = -A$$

And the second one

$$F(1) = 0 \Rightarrow A \cdot e^{\mu} + B \cdot e^{-\mu} = 0 \Rightarrow A \cdot e^{\mu} - A \cdot e^{-\mu} = 0 \Rightarrow A \cdot (e^{\mu} - e^{-\mu}) = 0$$

$$\Rightarrow A = 0 \Rightarrow B = 0 \Rightarrow F(x) = 0$$

This part is never negative.

This is also not an interesting result, therefore λ must be negative

We now assume that λ is negative.

$$\lambda < 0, \quad \lambda = -p^2$$

Solvable equation!

$$\Rightarrow F''(x) - \lambda \cdot F(x) = 0 \Rightarrow F''(x) + p^2 \cdot F(x) = 0 \Rightarrow F''(x) = -p^2 \cdot F(x)$$

$$\Rightarrow F(x) = A \cdot \cos(px) + B \cdot \sin(px)$$

Plugging in boundary conditions

$$F(0) = 0 \Rightarrow A \cdot \overset{n!}{\cos(0)} + B \cdot \overset{0}{\sin(0)} = 0 \Rightarrow A = 0$$

$$F(1) = 0 \Rightarrow B \cdot \sin(p) = 0 \Rightarrow p = n\pi, \quad n = 1, 2, 3, \dots$$

ANY B will satisfy this equation, we therefore choose 1

$$\Rightarrow B = 1$$

We now know a set of solutions

$$F(x) = F_n(x) = \sin(n\pi x), \quad n = 1, 2, 3, \dots$$

We now look at the function $G(t)$

Remember: $\lambda = -p^2$

$$G''(t) - \lambda \cdot G(t) = 0 \Rightarrow G''(t) + p^2 \cdot G(t) = 0 \Rightarrow G''(t) = -p^2 \cdot G(t)$$

$$\Rightarrow G(t) = A \cdot \cos(pt) + B \cdot \sin(pt)$$

Arbitrary constants

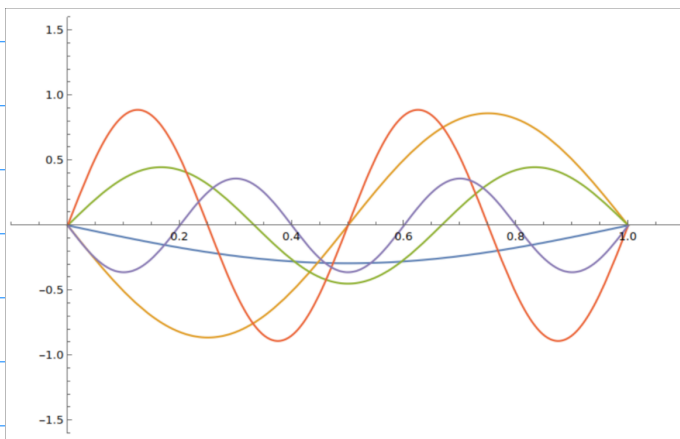
We can now write the set of solutions like this

$$u_n(x, t) = \sin(n\pi x) \cdot \left(A_n \cdot \cos(\pi n t) + B_n \sin(\pi n t) \right), \quad n = 1, 2, 3, \dots$$

And let's not forget

$$u(x, t) = 0$$

Here is a plot of some solutions at $t=0$



Bringing in initial conditions

$$u(x, 0) = k_0 \sin(3\pi x), \quad u_t(x, 0) = 0, \quad 0 \leq x \leq 1, \quad k_0 \text{ is constant}$$

$$u_n(x, t) = \sin(n\pi x) \cdot \left(A_n \cdot \cos(\pi n t) + B_n \sin(\pi n t) \right), \quad n = 1, 2, 3, \dots$$

Find the derivative of u

$$\begin{aligned} u_t &= \sin(n\pi x) \cdot \left(A_n \cdot n\pi \cdot (-\sin(\pi n t)) + B_n \cdot n\pi \cdot \cos(\pi n t) \right) \\ &= \sin(n\pi x) \cdot \left(B_n \cdot n\pi \cdot \cos(\pi n t) - A_n \cdot n\pi \cdot \sin(\pi n t) \right) \end{aligned}$$

$$u_t(x, 0) = 0$$

$$\Rightarrow \sin(n\pi x) \cdot \left(B_n \cdot n\pi \cdot \cos(0) - \cancel{A_n \cdot n\pi \cdot \sin(0)} \right) = 0$$

zero when $x = \frac{1}{n}$

never zero

$$\Rightarrow \sin(n\pi x) \cdot B_n \cdot n\pi = 0 \Rightarrow B_n = 0$$

$$\Rightarrow u_n(x, t) = \sin(n\pi x) \cdot A_n \cdot \cos(\pi n t), \quad n = 1, 2, 3, \dots$$

$$u(x, 0) = k_0 \cdot \sin(3\pi x)$$

$$\Rightarrow \sin(n\pi x) \cdot A_n \cdot \cos(\pi n(0)) = k_0 \cdot \sin(3\pi x)$$

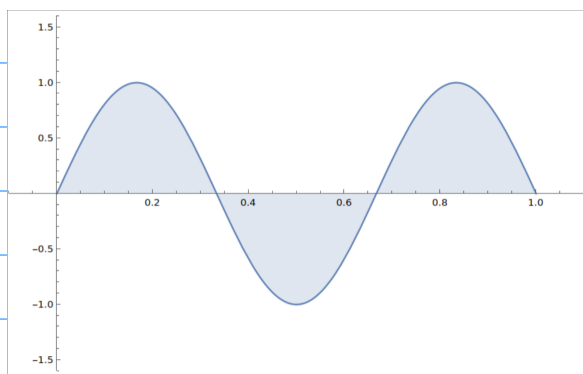
$$\Rightarrow \sin(n\pi x) \cdot A_n = k_0 \cdot \sin(3\pi x)$$

$$\Rightarrow n=3, A_n = k_0$$

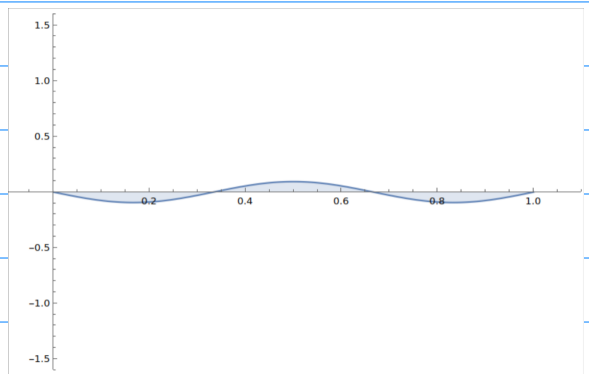
This is the final solution

$$\underline{\underline{u(x, t) = k_0 \cdot \sin(3\pi x) \cdot \cos(3\pi t)}}$$

Plot at $t=0$



Plot at $t=1.51$



2. Find the solution of the one-dimensional wave equation

$$u_{tt}(x, t) = u_{xx}(x, t), \quad 0 \leq x \leq 1$$

subject to the boundary conditions

$$u(0, t) = u(1, t) = 0 \quad \text{for all } t$$

and the initial condition

$$u(x, 0) = k_0 \sin(3\pi x), \quad u_t(x, 0) = 3\pi k_0 \sin(3\pi x), \quad 0 \leq x \leq 1,$$

where k_0 is a constant. Sketch the solution for $t = 0$, $t = 1/6$ and $t = 1/3$. Find t_0 at which $u(x, t_0) = 0$ for all x for the first time. Find t_1 at which $u(x, t_1) = 0$ for all x for the second time.

Boundary conditions are the same as before, so we can use the same general solution.

$$u_n(x, t) = \sin(n\pi x) \cdot (A_n \cdot \cos(n\pi t) + B_n \sin(n\pi t)), \quad n = 1, 2, 3, \dots$$

$$u_t(x, t) = \sin(n\pi x) \cdot (B_n \cdot n\pi \cdot \cos(n\pi t) - A_n \cdot n\pi \cdot \sin(n\pi t))$$

$$u(x, 0) = k_0 \cdot \sin(3\pi x)$$

$$\Rightarrow \sin(n\pi x) \cdot (A_n \cdot \overset{1}{\cos(0)} + B_n \cdot \overset{0}{\sin(0)}) = k_0 \cdot \sin(3\pi x)$$

$$\Rightarrow \sin(n\pi x) \cdot A_n = k_0 \cdot \sin(3\pi x) \Rightarrow n = 3, \quad A_n = k_0$$

$$u_t(x, 0) = \sin(n\pi x) \cdot (B_n \cdot n\pi \cdot \cos(n\pi t) - A_n \cdot n\pi \cdot \sin(n\pi t))$$

$$\Rightarrow \sin(3\pi x) \cdot (B_n \cdot 3\pi \cdot \overset{1}{\cos(0)} - A_n \cdot 3\pi \cdot \overset{0}{\sin(0)}) = 3\pi k_0 \cdot \sin(3\pi x)$$

$$\Rightarrow \sin(3\pi x) \cdot (B_n \cdot 3\pi) = 3\pi k_0 \cdot \sin(3\pi x)$$

$$\Rightarrow 3\pi \cdot B_n \cdot \sin(3\pi x) = 3\pi \cdot k_0 \cdot \sin(3\pi x) \Rightarrow k_0 = B_n$$

Putting it all together

$$u(x,t) = \sin(3\pi x) \cdot (k_0 \cdot 3\pi \cdot \cos(3\pi t) - k_0 \cdot 3\pi \cdot \sin(3\pi t))$$

$$\Rightarrow \underline{\underline{u(x,t) = k_0 \cdot 3\pi \cdot \sin(3\pi x) \cdot (\cos(3\pi t) - \sin(3\pi t))}}$$