

Betragt følgende Butterworth 3. ordens lavpasfilter (frekvensnormeret filter)

$$\tilde{H}(s) = \frac{1}{s+1} \frac{1}{s^2 + s + 1}$$

og design et tilsvarende digitalt lavpasfilter med afskæringsfrekvens på 1 kHz og samplefrekvens på 8 kHz. Design filtret som følger

1. Design et digitalt lavpasfilter $H_1(z)$ ved brug af matched z -transformation.
2. Design et digitalt lavpasfilter $H_2(z)$ ved brug af impuls invariant z -transformation.
3. Benyt MATLAB til at sammenligne Bode plot for denormerede lavpasfilter $H(s)$, $H_1(z)$ og $H_2(z)$.
4. Benyt MATLAB til at sammenligne impulsresponserne for $H(s)$, $H_1(z)$ og $H_2(z)$.

$$f_s = 8 \text{ kHz}, f_a = 1 \text{ kHz}$$

①

$$\tilde{H}_1(s) = \frac{1}{s+1} \Rightarrow p_s = \{-1\}$$

$$\tilde{H}_2(s) = \frac{1}{s^2 + s + 1} \Rightarrow p_s = \left\{ \frac{-1 \pm i\sqrt{3}}{2} \right\}$$

$$p = \begin{bmatrix} -1 \\ \frac{-1+i\sqrt{3}}{2} \\ \frac{-1-i\sqrt{3}}{2} \end{bmatrix} \Rightarrow p_{\text{den}} = p \cdot 2\pi \cdot f_a = \begin{bmatrix} -2000\pi \\ -1000\pi + i \cdot 1000\sqrt{3}\pi \\ -1000\pi - i \cdot 1000\sqrt{3}\pi \end{bmatrix}$$

$$H_1(s) = \tilde{H}_1(s) \Big|_{s=\frac{z}{w_a}} = \frac{1}{s+1} \Big|_{s=\frac{z}{w_a}} = \frac{1}{\frac{z}{2000\pi} + 1} = \frac{2000\pi}{z + 2000\pi}$$

$$H_2(s) = \tilde{H}_2(s) \Big|_{s=\frac{z}{w_a}} = \frac{1}{s^2 + s + 1} \Big|_{s=\frac{z}{w_a}} = \frac{1}{\left(\frac{z}{2000\pi}\right)^2 + \frac{z}{2000\pi} + 1} = \frac{2000\pi}{\frac{1}{2000\pi} \cdot z^2 + z + 2000\pi}$$

Ryk til z domæne

$$z = e^{sT}, \quad T = \frac{1}{f_s} = \frac{1}{8000}, \quad b_n = -e^{sT}$$

$$b_1 = -e^{-2000\pi \cdot \frac{1}{8000}} = -e^{-\frac{\pi}{4}}$$

$$b_2 = -e^{(-1000\pi + i \cdot 1000\sqrt{3}) \cdot \frac{1}{8000}} = -e^{1000 \cdot (-\pi + i\sqrt{3}) \cdot \frac{1}{8000}} = -e^{\frac{\pi}{8} \cdot (-1 + i\sqrt{3})}$$

$$b_3 = -e^{\frac{\pi}{8} \cdot (-1 - i\sqrt{3})}$$

$$H(z) = \frac{a_0}{(z+b_1)(z+b_2)(z+b_3)} = \frac{a_1}{z+b_1} + \frac{a_2}{(z+b_2)(z+b_3)}$$

$$= \frac{a_1}{z - \underbrace{e^{-\frac{\pi}{4}}}_{b_1}} + \frac{a_2}{(z - \underbrace{e^{\frac{\pi}{8} \cdot (-1 - i\sqrt{3})}}_{b_2})(z - \underbrace{e^{\frac{\pi}{8} \cdot (-1 + i\sqrt{3})}}_{b_3})}$$

$$a_1: \frac{\underbrace{z000\pi}_{H_1(s)|_{s=0}}}{0 + z000\pi} = \frac{a_1}{1 - e^{-\frac{\pi}{4}}} \Rightarrow a_1 = 1 - e^{-\frac{\pi}{4}}$$

$$a_2: \frac{\underbrace{z000\pi}_{H_2(s)|_{s=0}}}{0 + 0 + z000\pi} = \frac{a_2}{(1 - e^{\frac{\pi}{8} \cdot (-1 - i\sqrt{3})})(1 - e^{\frac{\pi}{8} \cdot (-1 + i\sqrt{3})})}$$

$$\Rightarrow a_2 = (1 - e^{\frac{\pi}{8} \cdot (-1 - i\sqrt{3})})(1 - e^{\frac{\pi}{8} \cdot (-1 + i\sqrt{3})})$$

$$\Rightarrow \frac{1 - e^{-\frac{\pi}{4}}}{z - e^{-\frac{\pi}{4}}} + \frac{1 - e^{\frac{\pi}{8} \cdot (-1 + i\sqrt{3})} - e^{\frac{\pi}{8} \cdot (-1 - i\sqrt{3})}}{(z - e^{\frac{\pi}{8} \cdot (-1 - i\sqrt{3})})(z - e^{\frac{\pi}{8} \cdot (-1 + i\sqrt{3})})}$$

(2)

$$\tilde{H}(s) = \frac{1}{s+1} \frac{1}{s^2+s+1}$$

Partialbrøkopløs

$$\tilde{H}(s) = \left(\frac{1}{s+1} \cdot \frac{1}{s^2+s+1} = \frac{A}{s+1} + \frac{Bs+C}{s^2+s+1} \right) \cdot (s+1)(s^2+s+1)$$

$$\Rightarrow 1 = A(s^2+s+1) + (Bs+C)(s+1)$$

$$\Rightarrow 1 = As^2 + As + A + Bs^2 + Bs + Cs + C$$

Samle konstanter

$$\begin{array}{l} s^2: 0 = A+B \\ s: 0 = A+B+C \\ 1: 1 = A+C \end{array} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{array} \right] \xrightarrow{\substack{R_2 - R_1 \\ R_3 - R_1}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{array} \right] \xrightarrow{R_3 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{R_3 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array} \right] \cdot (-1) \\ \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{array} \right] \Rightarrow \begin{cases} A = 1 \\ B = -1 \\ C = 0 \end{cases}$$

$$\Rightarrow \tilde{H}(s) = \frac{A}{s+1} + \frac{Bs+C}{s^2+s+1} = \frac{1}{s+1} + \frac{-s}{s^2+s+1}$$

Denormer overføringsfunktionen

$$H(s) = \tilde{H}(s) \Big|_{s = \frac{s}{\omega_n}} = \frac{1}{\frac{s}{2000\pi} + 1} + \frac{\frac{-s}{2000\pi}}{\left(\frac{s}{2000\pi}\right)^2 + \frac{s}{2000\pi} + 1} = \underbrace{\frac{2000\pi}{s + 2000\pi}}_{H_1(s)} + \underbrace{\frac{-2000\pi \cdot s}{s^2 + 2000\pi \cdot s + (2000\pi)^2}}_{H_2(s)}$$

Find konstanter og z-transformer

$$H_1(s) = \frac{2000\pi}{s + 2000\pi} \Rightarrow \begin{cases} k_1 = 2000\pi \\ s_1 = -2000\pi \end{cases} \Rightarrow H_1(z) = \frac{1}{8000} \cdot 2000\pi \cdot \frac{1}{1 - e^{-2000\pi \cdot \frac{1}{8000}} \cdot z^{-1}} = \frac{\pi}{4} \cdot \frac{1}{1 - e^{-\frac{\pi}{4}} \cdot z^{-1}}$$

$$H_2(s) = \frac{-2000\pi \cdot s}{s^2 + 2000\pi \cdot s + (2000\pi)^2} = \frac{-2000\pi \cdot s}{(s - 1000\pi + 1000\pi\sqrt{3}i) \cdot (s - 1000\pi - 1000\pi\sqrt{3}i)}$$

Definition: $s_2 = 1000\pi - 1000\pi\sqrt{3}i$

Partialbrøkopløs

$$H_2(s) = \frac{-2000\pi \cdot s}{(s - s_1) \cdot (s - s_2^*)} = \frac{k_2}{s - s_2} + \frac{k_2^*}{s - s_1^*}$$

$$\Rightarrow k_2 = (s - s_1) \cdot H_2(s) \Big|_{s=s_2} = \frac{-2000\pi s}{s - s_1^*} \Big|_{s=s_2} = \frac{-2000\pi \cdot (1000\pi - 1000\pi\sqrt{3}i)}{(1000\pi - 1000\pi\sqrt{3}i) - (1000\pi + 1000\pi\sqrt{3}i)} \\ = \frac{-2000\pi \cdot (1000\pi - 1000\pi\sqrt{3}i)}{-2000\pi\sqrt{3}i} = \frac{1000\pi - 1000\pi\sqrt{3}i}{\sqrt{3}i} = \frac{1000\pi i + 1000\pi\sqrt{3}}{-\sqrt{3}} = -1000\pi - 1000\pi \frac{1}{\sqrt{3}}i$$

$$\begin{cases} k_2 = -1000\pi - 1000\pi \frac{1}{\sqrt{3}} i \\ s_2 = 1000\pi - 1000\pi \frac{1}{\sqrt{3}} i \end{cases} \Rightarrow \begin{cases} \alpha_2 = -1000\pi \\ \beta_2 = -1000\pi \frac{1}{\sqrt{3}} \\ \sigma_2 = 1000\pi \\ \omega_2 = -1000\pi \frac{1}{\sqrt{3}} \end{cases}$$

$$f_s = 8000 \Rightarrow T = \frac{1}{8000}$$

$$\sigma_2 T = 1000\pi \cdot \frac{1}{8000} = \frac{\pi}{8}$$

$$\omega_2 T = -1000\pi \frac{1}{\sqrt{3}} \cdot \frac{1}{8000} = \frac{-\pi\sqrt{3}}{8}$$

$$\Rightarrow \begin{cases} \alpha_0 = 2\alpha_2 \\ a_1 = -2e^{\sigma_2 T} (\alpha_2 \cos(\omega_2 T) - \beta_2 \sin(\omega_2 T)) \\ b_1 = -(2e^{\sigma_2 T} \cos(\omega_2 T)) \\ b_2 = e^{2\sigma_2 T} \end{cases} \Rightarrow \begin{cases} a_0 \rightarrow -2000\pi \\ a_1 \rightarrow -2e^{\pi/8} \left(-1000\pi \cos\left[\frac{\sqrt{3}\pi}{8}\right] - \frac{1000\pi \sin\left[\frac{\sqrt{3}\pi}{8}\right]}{\sqrt{3}} \right) \\ b_1 \rightarrow -2e^{\pi/8} \cos\left[\frac{\sqrt{3}\pi}{8}\right] \\ b_2 \rightarrow e^{\pi/4} \end{cases}$$

$$\Rightarrow \begin{cases} a_0 \rightarrow -6283.19 \\ a_1 \rightarrow 10613.3 \\ b_1 \rightarrow -2.3028 \\ b_2 \rightarrow 2.19328 \end{cases} \Rightarrow H_z(z) \approx \frac{-6283 + 10613z^{-1}}{1 - 2.3028z^{-1} + 2.1933z^{-2}}$$

Den samlede overføringsfunktion i z-domænet!

$$\Rightarrow H(z) = H_1(z) + H_2(z) = \frac{\pi}{4} \cdot \frac{1}{1 - e^{\frac{\pi}{4}} z^{-1}} + \frac{-6283 + 10613z^{-1}}{1 - 2.3028z^{-1} + 2.1933z^{-2}}$$



