Problem 1: $u_{yy} + 16u = 0$ $\Rightarrow$ $u = f(x)$

Problem 2: $u_{yy} = 0$ $\Rightarrow$ $u = f(x)$

Problem 3: $u_y + 2yu = 0$ $\Rightarrow$ $u = f(x)$

Problem 4: $u_y + u = e^{xy}$

Problem 5: $u_{xx} = 4y^2u$

Problem 6: $u_y = 2xyu$

(4) $u_y + u = e^{xy} \Rightarrow u_y + u - e^{xy} = 0$

$\Rightarrow f(x),$

Verify (by substitution) that the given function is a solution of the indicated PDE (wave equation).

Problem 7: $u = \sin kx \cos kct$

Wave equation: $u_{tt} = c^2\nabla^2 u$

$u = \sin(kx)\cdot\cos(kct)$

$u_t = \sin(kx)\cdot\left(-\sin(kct)\cdot kc\right) = -kc\cdot\sin(kx)\cdot\sin(kct)$

$u_{tt} = -kc\cdot\sin(kx)\cdot\cos(kct)\cdot kc = -k^2c^2\cdot\sin(kx)\cdot\cos(kct)$

$c^3\nabla^2 u = c^2\frac{d^2u}{dx^2} = c^2\cdot\frac{d}{dx}\left(\cos(kx)\cdot k\cdot\cos(kct)\right)$

$= c^2\cdot\left(-k^2\cdot\sin(kx)\cdot\cos(kct)\right)$

Equal!

$= -k^2c^2\cdot\sin(kx)\cdot\cos(kct)$

Verify (by substitution) that the given function is a solution of the indicated PDE (heat equation).

Problem 8: $u = e^{-\omega^2 c^2 t} \cos \omega x$

Heat equation: $\underbrace{\frac{\partial u}{\partial t}} = \underbrace{\alpha \cdot \nabla^2 u}$

$$u_t = \frac{d}{dt}\left(e^{-w^2c^2t} \cdot \cos(wx)\right)$$

$$= -w^2c^2 \cdot e^{-w^2c^2t} \cdot \cos(wx)$$

$$\alpha \cdot \nabla^2 u = \alpha \cdot \frac{d^2}{dx^2}\left(e^{-w^2c^2t} \cdot \cos(wx)\right)$$

$$= \alpha \cdot \frac{d}{dx}\left(e^{-w^2c^2t} \cdot (-\sin(wx) \cdot w)\right)$$

$$= -\alpha \cdot e^{-w^2c^2t} \cdot \cos(wx) \cdot w^2$$

$$= -w^2 \cdot \alpha \cdot e^{-w^2c^2t} \cdot \cos(wx)$$

Putting them together

$$-w^2c^2 \cdot e^{-w^2c^2t} \cdot \cos(wx) = -w^2 \cdot \alpha \cdot e^{-w^2c^2t} \cdot \cos(wx)$$

$\Rightarrow u$ is a solution if $\alpha = c^2$

Problem 10: Boundary value problem: Verify that the function $u(x, y) = a\ln(x^2 + y^2) + b$ satisfies Laplace's equation and determine a and b so that u satisfies the boundary conditions $u = 110$ on the circle $x^2 + y^2 = 1$ and $u = 0$ on the circle $x^2 + y^2 = 100$.

$$u(x,y) = a \cdot \ln(x^2+y^2) + b$$

Check: $\nabla^2 u = 0$

$$\frac{d^2}{dx^2}\left(a \cdot \ln(x^2+y^2)+b\right) + \frac{d^2}{dy^2}\left(a \cdot \ln(x^2+y^2)+b\right) = 0$$

$$\Rightarrow \underbrace{\frac{d}{dx}\left(a \cdot \frac{1}{x^2+y^2} \cdot 2x\right)} + \frac{d}{dy}\left(a \cdot \frac{1}{x^3+y^2} \cdot 2y\right) = 0$$

$$a \cdot \left(\frac{d}{dx}\left(\frac{1}{x^2+y^2}\right) \cdot 2x + \frac{d}{dx}(2x) \cdot \frac{1}{x^2+y^2} \right)$$

$$= a \cdot \left(\frac{-1}{(x^2+y^2)^2} \cdot 2x \cdot 2x + 2 \cdot \frac{1}{x^2+y^2} \right)$$

$$= \frac{-4ax^2}{(x^2+y^2)^2} + \frac{2a}{x^2+y^2} = \frac{-4ax^2}{(x^2+y^2)^2} + \frac{2a(x^2+y^2)}{(x^3+y^2)^2}$$

$$= \frac{-4ax^2 + 2a(x^2+y^2)}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{-4ax^2+2a(x^2+y^2)}{(x^2+y^2)^2} + \overbrace{\frac{-4ay^2+2a(x^2+y^2)}{(x^2+y^2)^2}}^{\text{Inforced}} = 0$$

$$\Rightarrow \frac{-4ax^2-4ay^2+4a(x^2+y^2)}{(x^2+y^2)^2} = 0$$

$$\Rightarrow \frac{\cancel{-4ax^2}-\cancel{4ay^2}+\cancel{4ax^2}+\cancel{4ay^2}}{(x^2+y^2)^2} = 0$$

$$\Rightarrow \frac{0}{(x^2+y^2)^2} = 0 \qquad \text{True!}$$

Problem 10: Boundary value problem: Verify that the function $u(x,y) = a\ln(x^2+y^2)+b$ satisfies Laplace's equation and determine a and b so that u satisfies the boundary conditions $u = 110$ on the circle $x^2+y^2=1$ and $u=0$ on the circle $x^2+y^2=100$.

$$u(x,y) = a\cdot\ln(x^2+y^2)+b$$

$x^2+y^2=1 \Rightarrow y^2=1-x^2$	$x^2+y^2=100 \Rightarrow y^2=100-x^2$
$\Rightarrow a\cdot\ln(x+1-x^2)+b=110$	$\Rightarrow a\cdot\ln(x+100-x^2)=0$
$\Rightarrow a\cdot\ln(-x^2+x+1)+b=110$	$\Rightarrow a\cdot\ln(-x^2+x+100)=0$

$$\Rightarrow \begin{cases} a\cdot\ln(-x^2+x+1)+b=110 \\ a\cdot\ln(-x^2+x+100)+b=0 \end{cases}$$

$$\Rightarrow a \cdot \ln(-x^2+x+1) \cancel{+b} - a \cdot \ln(-x^2+x+100) \cancel{-b} = 110 - 0$$

$$\Rightarrow a \cdot \left(\ln(-x^2+x+1) - \ln(-x^2+x+100)\right) = 110$$

$$\Rightarrow a = \frac{110}{\ln\left(\frac{-x^2+x+1}{-x^2+x+100}\right)}$$