

**Problem 1:** Find the type, transform to normal form, and solve. (Show the details of your work)

$$u_{xy} - u_{yy} = 0$$

$$Au_{xx} + Bu_{xy} + C_{yy} + Du_x + Eu_y + Fu = G$$

$$\Rightarrow B=1, C=-1, A=D=E=F=G=0$$

Calculate discriminant

$$\Delta = B^2 - 4AC = 1^2 - 4 \cdot 0 \cdot (-1) = 1$$

$$\Delta > 0 \Rightarrow \text{Hyperbolic!}$$

Get  $\xi$  and  $\eta$

$$A\lambda^2 - B\lambda + C = 0 \Rightarrow -\lambda - 1 = 0 \Rightarrow \lambda = -1$$

$$\frac{d\eta}{d\xi} = \lambda \Rightarrow \int d\eta = \int \lambda d\xi = \eta = -\xi$$

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Characteristic Equations

$$Ay'^2 + By' + C = 0 \Rightarrow 0 \cdot y'^2 + y' - 1 = 0 \Rightarrow y' = 1$$

$$\Rightarrow y = x + C_1 \Rightarrow C_1 = y - x$$

New variable w

$$w = y - x \Rightarrow w_x = -1, w_y = 1$$

Describe u in terms of w

$$u_x = u_w \overset{-1}{w_x} = -u_w$$

$$u_{xy} = (-u_w)_w \overset{1}{w_y} = -u_{ww} \cdot 1 = -u_{ww}$$

$$u_y = u_w \overset{1}{w_y} = u_w$$

$$u_{yy} = (u_w)_w \overset{1}{w_y} = u_{ww}$$

Substitute variables

$$u_{xy} - u_{yy} = 0 \Rightarrow -u_{ww} - u_{ww} = 0 \Rightarrow -2u_{ww} = 0$$

$$\Rightarrow u_{ww} = 0 \quad \text{Normal form !!}$$

Solution

$$u_{ww} = f_1(w) \Rightarrow \iint f_1(w) dw dw = f_2(w)$$

$$\underline{\underline{u(x, y) = f_2(y - x)}}$$

**Problem 1:** Find the type, transform to normal form, and solve. (Show the details of your work)

$$u_{xy} - u_{yy} = 0$$

This is a linear PDE as nothing is a function of  $u$

$$u(x, y) = F(x) G(y)$$

We find the double derivatives

$$u_{yy} = F(x) G''(y)$$

$$u_{xy} = F'(x) G'(y)$$

Substitute in the original equation

$$F'(x) G'(y) - F(x) G''(y) = 0 \Rightarrow F'(x) G'(y) = F(x) G''(y)$$

$$\Rightarrow \frac{F'(x)}{F(x)} = \frac{G''(y)}{G'(y)} = \lambda$$

$$\begin{cases} \textcircled{1} F'(x) - \lambda \cdot F(x) = 0 \\ \textcircled{2} G''(y) - \lambda \cdot G'(y) = 0 \end{cases}$$

Solving the first ODE:

$$f'(x) - \lambda \cdot f(x) = 0 \Rightarrow a=0, b=1, c=-\lambda$$

$$(a \cdot r^2 + b \cdot r + c) \cdot e^{rt} = 0 \Rightarrow a \cdot r^2 + b \cdot r + c = 0$$

$$\Rightarrow r - \lambda = 0 \Rightarrow r = \lambda$$

We have one root. Therefore we use this solution:

$$f(x) = A \cdot e^{rx} + B \cdot x \cdot e^{rx} = A \cdot e^{\lambda x} + B \cdot x \cdot e^{\lambda x}$$

Solving the second ODE:

$$f''(x) - \lambda \cdot f'(x) = 0 \Rightarrow a=1, b=-\lambda, c=0$$

$$\Rightarrow r^2 - \lambda b = 0$$

Solving with quadratic equation

$$\Delta = b^2 - 4ac = \lambda^2$$

$$r = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{\lambda \pm \sqrt{\lambda^2}}{2} = \frac{\lambda \pm \lambda}{2} \Rightarrow \begin{cases} 0 \\ \lambda \end{cases}$$

Here we have two solutions

$$f(x) = A \cdot e^{r_1 x} + B \cdot e^{r_2 x} = A \cdot e^0 + B \cdot e^{\lambda x} = A + B \cdot e^{\lambda x}$$

We now have solutions to the ODEs

$$F(x) = A \cdot e^{\lambda x} + B \cdot x \cdot e^{\lambda x} \quad G(y) = C + D \cdot e^{\lambda y}$$

Combining them to get PDE solution

$$u(x, y) = F(x) G(y) = (A \cdot e^{\lambda x} + B \cdot x \cdot e^{\lambda x}) (C + D \cdot e^{\lambda y})$$

**Problem 2:** Find the type, transform to normal form, and solve. (Show the details of your work)

$$u_{xx} + 2u_{xy} + u_{yy} = 0$$

$$Au_{xx} + Bu_{xy} + C_{yy} + Du_x + Eu_y + Fu = G$$

$$\Rightarrow A=1, B=2, C=1, E=F=G=0$$

$$B^2 - 4AC = 4 - 4 \cdot 1 \cdot 1 = 0 \Rightarrow \text{Parabolic!}$$

Characteristic Equation

$$\frac{dy}{dx} = \frac{B}{2A} = \frac{2}{2 \cdot 1} = 1 \Rightarrow y = \int 1 dx \Rightarrow y = x + C_1$$

$$C_1 = y - x \Rightarrow \xi = y - x$$

I choose  $\eta = x$

Check jacobian

$$\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0 \quad \checkmark$$

Get needed derivatives (from chain rule)

$$\xi_x = -1, \xi_y = 1, \xi_{xx} = 0, \xi_{yx} = 0, \xi_{xy} = 0$$

$$\eta_x = 1, \eta_y = 0, \eta_{xx} = 0, \eta_{yy} = 0, \eta_{xy} = 0$$

$$\begin{aligned} u_x &= u_\xi \xi_x + u_\eta \eta_x \\ u_y &= u_\xi \xi_y + u_\eta \eta_y \\ u_{xx} &= u_{\xi\xi} (\xi_x)^2 + 2u_{\xi\eta} (\xi_x \eta_x) + u_{\eta\eta} (\eta_x)^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx} \\ u_{yy} &= u_{\xi\xi} (\xi_y)^2 + 2u_{\xi\eta} (\xi_y \eta_y) + u_{\eta\eta} (\eta_y)^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy} \\ u_{xy} &= u_{\xi\xi} (\xi_x \xi_y) + u_{\eta\eta} (\eta_x \eta_y) + u_\xi \xi_{xy} + u_\eta \eta_{xy} + u_{\xi\eta} (\xi_x \eta_y + \xi_y \eta_x) \end{aligned}$$

$$u_{xx} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$$

$$u_{yy} = u_{\xi\xi}$$

$$u_{xy} = -u_{\xi\xi} + u_{\xi\eta}$$

Replace in original PDE

$$u_{xx} + 2u_{xy} + u_{yy} = 0$$

$$\Rightarrow u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} - 2u_{\xi\xi} + 2u_{\xi\eta} + u_{\xi\xi} = 0$$

$$\Rightarrow u_{\eta\eta} = 0$$

Solve dat shit!

$$u_\eta = \int 0 d\eta = f_1(\xi)$$

$$u = \int f_1(\xi) d\eta = f_1(\xi) \cdot \int 1 d\eta = f_1(\xi) \cdot \eta + f_2(\xi)$$

Remeber when we were young and naive

$$\xi = y - x, \eta = x$$

$$\underline{\underline{u = f_1(y-x) \cdot x + f_2(y-x)}}$$

**Problem 3:** Find the type, transform to normal form, and solve. (Show the details of your work)

$$u_{xx} - 4u_{xy} + 3u_{yy} = 0$$

$$Au_{xx} + Bu_{xy} + C_{yy} + Du_x + Eu_y + Fu = G$$

$$A=1, B=-4, C=3$$

$$\Delta = B^2 - 4AC = (-4)^2 - 4(1)(3) = 16 - 12 = 4$$

$\Delta > 0 \Rightarrow$  Hyperbolic!

Find new variables

$$A\lambda^2 - B\lambda + C = 0 \Rightarrow \lambda^2 + 4\lambda + 3 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{4}}{2} = \frac{-4 \pm 2}{2} = -2 \pm 1 \Rightarrow \begin{cases} -1 \\ -3 \end{cases}$$

$$\frac{dy}{dx} = -1 \Rightarrow \int dy = \int -1 dx \Rightarrow y = -x + C_1 \Rightarrow C_1 = x + y$$

$$\frac{dy}{dx} = -3 \Rightarrow \int dy = \int -3 dx \Rightarrow y = -3x + C_2 \Rightarrow C_2 = 3x + y$$

$$\Rightarrow \begin{cases} \xi = x + y \\ \eta = 3x + y \end{cases}$$

Find the needed derivatives

$$\xi_x = 1, \xi_y = 1, \xi_{xx} = \xi_{yy} = \xi_{xy} = 0$$

$$\eta_x = 3, \eta_y = 1, \eta_{xx} = \eta_{yy} = \eta_{xy} = 0$$

$$\begin{aligned} u_x &= u_\xi \xi_x + u_\eta \eta_x \\ u_y &= u_\xi \xi_y + u_\eta \eta_y \\ u_{xx} &= u_{\xi\xi}(\xi_x)^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}(\eta_x)^2 + u_{\xi\xi\xi} + u_{\eta\xi\xi} \\ u_{yy} &= u_{\xi\xi}(\xi_y)^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}(\eta_y)^2 + u_{\xi\xi\xi} + u_{\eta\xi\xi} \\ u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}\xi_x\eta_y + u_{\xi\eta}\xi_y\eta_x + u_{\eta\eta}\eta_x\eta_y + u_{\xi\xi\xi}(\xi_x\xi_y + \xi_y\xi_x) \end{aligned}$$

$$u_{xx} = u_{\xi\xi} + 6u_{\xi\eta} + 9u_{\eta\eta}$$

$$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

$$u_{xy} = u_{\xi\xi} + 3u_{\xi\eta} + 4u_{\eta\eta}$$

Substitute in original equation

$$u_{xx} - 4u_{xy} + 3u_{yy} = 0$$

$$\Rightarrow u_{\xi\xi} + 6u_{\xi\eta} + 9u_{\eta\eta} - 4u_{\xi\xi} - 12u_{\xi\eta} - 16u_{\eta\eta} + 3u_{\xi\xi} + 6u_{\xi\eta} + 3u_{\eta\eta} = 0$$

$$\Rightarrow -4u_{\xi\eta} = 0 \Rightarrow u_{\xi\eta} = 0$$

A laterally insulated bar of length 10cm and constant cross-sectional area  $1\text{cm}^2$ , of density  $10.6\text{ gm/cm}^3$ , thermal conductivity  $1.04\text{ cal/(cm sec }^\circ\text{C)}$ , and specific heat  $0.056\text{ cal/(gm }^\circ\text{C)}$  (this corresponds to silver, a good heat conductor) has initial temperature  $f(x)$  and is kept at  $0^\circ\text{C}$  at the ends  $x = 0$  and  $x = 10$ . Find the temperature  $u(x, t)$  at later times. Here  $f(x)$  equals:

**Problem 4:**  $f(x) = \sin 0.4 \pi x$

$$C = 0,056$$

$$k = 1,04$$

$$\rho = 10.6$$

Calculate alpha

$$\alpha = \frac{k}{C\rho} \approx 1,752$$

$$u_t = \alpha \cdot u_{xx}$$

Initial conditions

$$u(0, t) = 0 \quad u(x, 0) = f(x)$$

$$u(10, t) = 0$$

Assume that we can solve with separation of variables

$$u(x, t) = F(x) G(t)$$

$$\Rightarrow u_t = F(x) G'(t), \quad u_{xx} = F''(x) G(t)$$

Rewrite equation

$$F(x) G'(t) = \alpha \cdot F''(x) G(t) \Rightarrow \frac{G'(t)}{G(t)} = \alpha \cdot \frac{F''(x)}{F(x)} = \lambda$$

Convert to ODEs

$$\begin{cases} \alpha \cdot F''(x) - \lambda \cdot F(x) = 0 \\ G'(t) - \lambda \cdot G(t) = 0 \end{cases}$$

$$\frac{\partial u}{\partial t} = \alpha \cdot \nabla^2 u, \quad \alpha = \frac{k}{c\rho}$$

$t$ : Time

$u$ : Temperature as a function of position and time.

$k$ : Thermal conductivity

$c$ : Specific Heat Capacity

$\rho$ : Density



Solving the first ODE

$$\alpha \cdot F''(x) - \lambda \cdot F(x) = 0 \Rightarrow a = \alpha, b = -\lambda, c = 0$$

$$\Rightarrow e^{rt} (a \cdot r^2 + b \cdot r + c) = 0 \Rightarrow a \cdot r^2 + b \cdot r + c = 0$$

$$\Rightarrow \Delta = b^2 - 4ac = \lambda^2$$

$$\Rightarrow r = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{\lambda \pm \sqrt{\lambda^2}}{2\alpha} = \frac{\lambda \pm \lambda}{2\alpha} = \begin{cases} r_1 = 0 \\ r_2 = \frac{\lambda}{\alpha} \end{cases}$$

Because we find two roots we use the following solution

$$F(x) = A \cdot e^{r_1 x} + B \cdot e^{r_2 x} = A \cdot e^0 + B \cdot e^{\frac{\lambda}{\alpha} x} = A + B \cdot e^{\frac{\lambda}{\alpha} x}$$

Solving the second one

$$G'(t) - \lambda \cdot G(t) = 0 \Rightarrow a = 0, b = 1, c = -\lambda$$

$$e^{rt} (a \cdot r^2 + b \cdot r + c) = 0 \Rightarrow a \cdot r^2 + b \cdot r + c = 0 \Rightarrow r - \lambda = 0$$

$$\Rightarrow r = \lambda$$

Here we only get one solution

$$G(t) = C \cdot e^{\lambda t} + D \cdot x \cdot e^{\lambda t}$$

We combine the functions to get a general solution

$$u(x, t) = F(x) G(t) = \left( A + B \cdot e^{\frac{\lambda}{\alpha} x} \right) \left( C \cdot e^{\lambda t} + D \cdot x \cdot e^{\lambda t} \right)$$

Plugging in initial conditions

$$u(0, t) = F(0) G(t) = 0 \Rightarrow F(0) = 0$$

$G(t) = 0$  is also a solution, but it is uninteresting.

$$u(10, t) = F(10) G(t) = 0 \Rightarrow F(10) = 0$$

We use this to find A and B

$$F(0) = 0 \Rightarrow A + B \cdot e^{\frac{\lambda}{\alpha} \cdot 0} = 0 \Rightarrow A + B = 0 \Rightarrow B = -A$$

$$F(10) = 0 \Rightarrow A - A \cdot e^{\frac{\lambda}{\alpha} \cdot 10} = 0 \Rightarrow A \cdot (1 - e^{\frac{\lambda}{\alpha} \cdot 10}) = 0$$

This is not great. Here either A or  $\lambda$  must be 0 leading to  $u = 0$

$$\lambda = 0 \Rightarrow u(x, t) = 0$$

$$A = 0 \Rightarrow u(x, t) = 0$$

Instead we now assume that  $\lambda < 0$

And that we can express it like this:  $\lambda = -p^2$

We now solve the first ODE again

$$\alpha \cdot F''(x) - \lambda \cdot F(x) = 0 \Rightarrow \alpha \cdot F''(x) + p^2 \cdot F(x) = 0$$

$$\Rightarrow a = \alpha, b = 0, c = p^2, \Delta = b^2 - 4ac = -4\alpha p^2$$

$$\Rightarrow a \cdot r^2 + b \cdot r + c = 0 \Rightarrow r = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{\pm \sqrt{-4\alpha p^2}}{2\alpha} = \frac{\pm \sqrt{-1} \sqrt{4} \sqrt{\alpha} \sqrt{p^2}}{2\alpha}$$

$$= \frac{\pm i \cdot 2 \cdot \sqrt{\alpha} \cdot p}{2\alpha} \cdot \frac{\sqrt{\alpha}}{\sqrt{\alpha}} = \frac{\pm i \cdot \alpha \cdot p}{\alpha \cdot \sqrt{\alpha}} = \pm i \cdot \frac{p}{\sqrt{\alpha}}$$

$$r = k \pm wi \Rightarrow k = 0, w = \frac{p}{\sqrt{\alpha}}$$

The solution can now be expressed like this

$$\begin{aligned} F(x) &= A \cdot e^{kx} \cdot \cos(wx) + B \cdot e^{kx} \cdot \sin(wx) = A \cdot \cancel{e^0} \cdot \cos\left(\frac{p}{\sqrt{\alpha}}x\right) + B \cdot \cancel{e^0} \cdot \sin\left(\frac{p}{\sqrt{\alpha}}x\right) \\ &= A \cdot \cos\left(\frac{p}{\sqrt{\alpha}}x\right) + B \cdot \sin\left(\frac{p}{\sqrt{\alpha}}x\right) \end{aligned}$$

Plugging in initial conditions

$$u(0, t) = 0 \Rightarrow F(0) = 0 \Rightarrow A \cdot \overset{\uparrow}{\cos(0)} + B \cdot \cancel{\sin(0)} = 0 \Rightarrow A = 0 \Rightarrow F(x) = B \cdot \sin\left(\frac{p}{\sqrt{\alpha}}x\right)$$

We assume that  $B \neq 0$  because otherwise  $u(x, t) = 0$

$$u(10, t) = 0 \Rightarrow F(10) = 0 \Rightarrow B \cdot \sin\left(\frac{p}{\sqrt{\alpha}} \cdot 10\right) = 0 \Rightarrow \sin\left(p \cdot \frac{10}{\sqrt{\alpha}}\right) = 0$$

$$\Rightarrow p = \frac{n\pi\sqrt{\alpha}}{10}, \quad n = \{1, 2, 3, \dots\}$$

Inserting  $p$

$$F(x) = B \cdot \sin\left(\frac{n\pi\sqrt{\alpha}}{10} \cdot \frac{x}{\sqrt{\alpha}}\right) = B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right), \quad n = \{1, 2, 3, \dots\}$$

Let's now take a look at  $G(y)$

$$G'(t) - \lambda \cdot G(t) = 0 \Rightarrow G'(t) + p^2 \cdot G(t) = 0$$

$$\Rightarrow a=0, \quad b=1, \quad c=p^2 \Rightarrow a \cdot r^2 + b \cdot r + c = 0 \Rightarrow r + p^2 = 0 \Rightarrow r = -p^2$$

$$\Rightarrow r = -\left(\frac{n\pi\sqrt{\alpha}}{10}\right)^2 = -\frac{n^2\pi^2\alpha}{100}$$

We still get one double root

$$G(t) = C \cdot e^{r_1 t} + D \cdot x \cdot e^{r_2 t} = C \cdot e^{-\frac{n^2\pi^2\alpha}{100} \cdot t} + D \cdot t \cdot e^{-\frac{n^2\pi^2\alpha}{100} \cdot t}$$

Combining solutions

$$u(x, t) = F(x) G(t) = B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right) \cdot \left(C \cdot e^{-\frac{n^2\pi^2\alpha}{100} \cdot t} + D \cdot t \cdot e^{-\frac{n^2\pi^2\alpha}{100} \cdot t}\right)$$

Plugging in initial condition

$$u(x, 0) = f(x) \Rightarrow \sin(0, 4\pi x) = B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right) \cdot (C \cdot e^0 + D \cdot 0 \cdot e^0)$$

$$\Rightarrow \sin(0, 4\pi x) = B \cdot \sin\left(\frac{n\pi}{10}\right) \cdot (C) = C \cdot B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right)$$

$$\Rightarrow \sin(0, 4\pi x) = C \cdot B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right) \Rightarrow \begin{cases} C=1 \\ B=1 \end{cases}$$

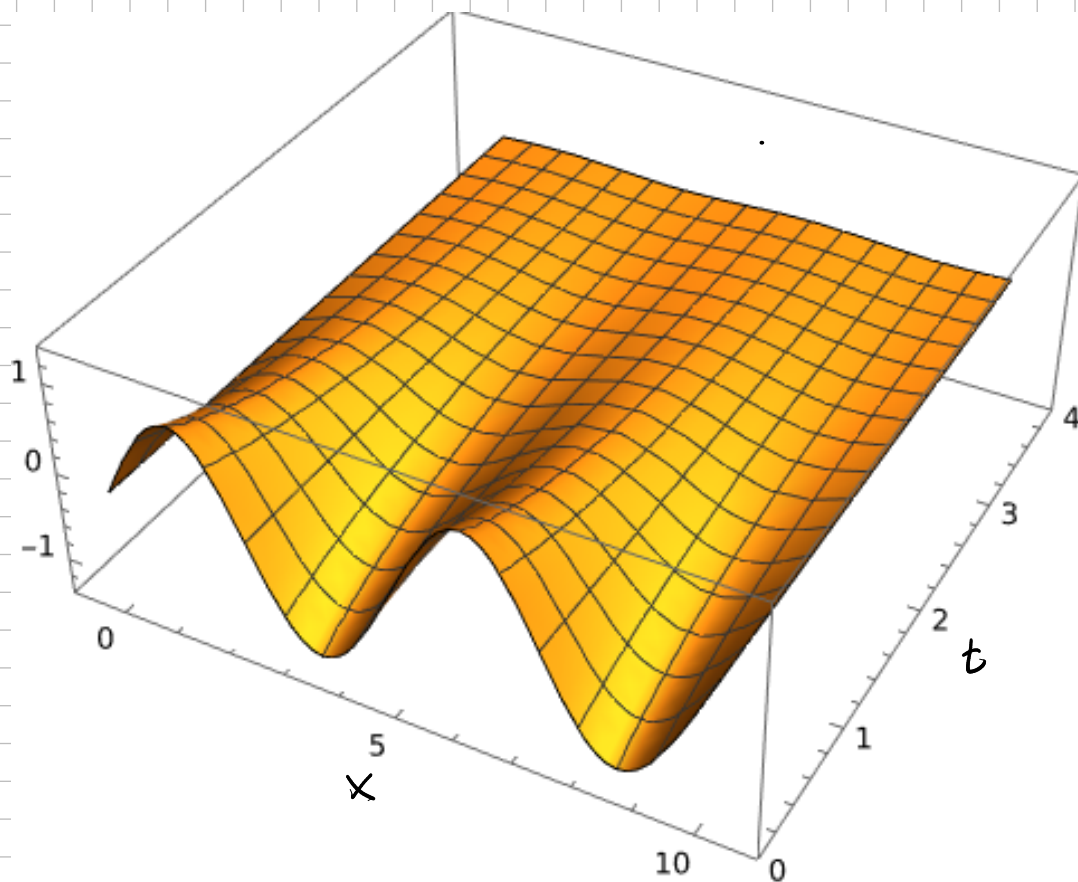
$$\Rightarrow \sin(0, 4\pi x) = \sin\left(\frac{n}{10} \pi x\right) \Rightarrow n=4$$

We now have all our constants and can write the specific solution

$$u(x,t) = B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right) \cdot \left(C \cdot e^{\frac{n^2 \pi^2 \alpha}{100} \cdot t} + D \cdot t \cdot e^{\frac{n^2 \pi^2 \alpha}{100} \cdot t}\right)$$

$$= \cancel{1} \cdot \sin\left(\frac{4}{10} \pi x\right) \cdot \left(\cancel{1} \cdot e^{\frac{16 \pi^2 \alpha}{100} \cdot t} + 0 \cdot \cancel{t} \cdot e^{\frac{16 \pi^2 \alpha}{100} \cdot t}\right)$$

$$\Rightarrow \underline{\underline{u(x,t) = \sin\left(\frac{4}{10} \pi x\right) \cdot e^{\frac{16 \pi^2 \alpha}{100} \cdot t}}}$$



**Problem 5:**  $f(x) = \sin 0.1 \pi x + \frac{1}{2} \sin 0.2 \pi x$

We start with our general solution:

$$u(x, t) = B \cdot \sin\left(\frac{n\pi}{10} x\right) \cdot \left(C \cdot e^{\frac{n^2 \pi^2 \alpha}{100} t} + D \cdot t \cdot e^{\frac{n^2 \pi^2 \alpha}{100} t}\right)$$

Plugging in initial condition

$$\begin{aligned} u(x, 0) = f(x) &= B \cdot \sin\left(\frac{n\pi}{10} x\right) \cdot \left(C \cdot \cancel{e^0} + D \cdot \cancel{0} \cdot \cancel{e^0}\right) \\ &= B \cdot \sin\left(\frac{n\pi}{10} x\right) \cdot C \end{aligned}$$

Because this is a linear PDE the sum of two solutions will also be a solution. We can therefore split this up into two solutions and add them together afterwards.

$$u(x, 0) = f_1(x) + f_2(x) = u_1(x, t) + u_2(x, t)$$

$$f_1(x) = \sin(0.1 \pi x), \quad f_2(x) = \frac{1}{2} \cdot \sin(0.2 \pi x)$$

Get first solution

$$\sin(0.1 \pi x) = B \cdot C \cdot \sin\left(\frac{n\pi}{10} x\right) \Rightarrow \begin{cases} n=1 \\ B \cdot C = 1 \Rightarrow \underbrace{B=1, C=1} \end{cases}$$

I could also have chosen values like  $B=2$  and  $C=0.5$  but this is a more elegant solution

$$u_1(x, t) = 1 \cdot \sin\left(\frac{\pi}{10} x\right) \cdot \left(1 \cdot e^{\frac{1 \cdot \pi^2 \alpha}{100} t} + D \cdot t \cdot e^{\frac{1 \cdot \pi^2 \alpha}{100} t}\right)$$

I also set  $D=0$  to simplify

$$u_1(x, t) = \sin\left(\frac{\pi}{10} x\right) \cdot e^{\frac{-\pi^2 \alpha}{100} t}$$

Second solution

$$\frac{1}{2} \cdot \sin(0.2 \pi x) = B \cdot C \cdot \sin\left(\frac{n\pi}{10} x\right) \Rightarrow \begin{cases} n=2 \\ B=\frac{1}{2} \\ C=1 \end{cases}$$

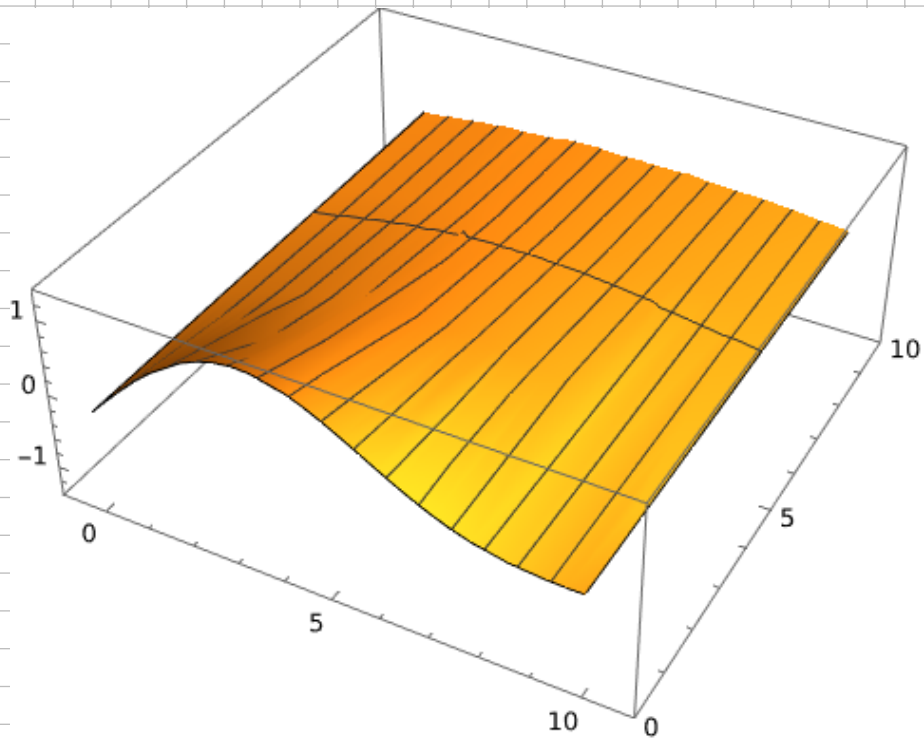
$$u_2(x, t) = \frac{1}{2} \sin\left(\frac{2\pi}{10} x\right) \cdot \left(1 \cdot e^{\frac{4 \cdot \pi^2 \alpha}{100} t} + 0 \cdot t \cdot e^{\frac{4 \cdot \pi^2 \alpha}{100} t}\right)$$

again i set  $D=0$

$$= \frac{1}{2} \cdot \sin\left(\frac{2\pi}{10} x\right) \cdot e^{\frac{-4\pi^2 \alpha}{100} t}$$

We can now add these solutions together to get the final solution

$$\underline{u(x,t) = u_1(x,t) + u_2(x,t) = \sin\left(\frac{\pi}{10}x\right) \cdot e^{-\frac{\pi^2\alpha}{100}t} + \frac{1}{2} \cdot \sin\left(\frac{2\pi}{10}x\right) \cdot e^{-\frac{4\pi^2\alpha}{100}t}}$$



## Problem 6: $f(x) = 1 - 0.2|x - 5|$

We start by approximating the function as a fourier-series

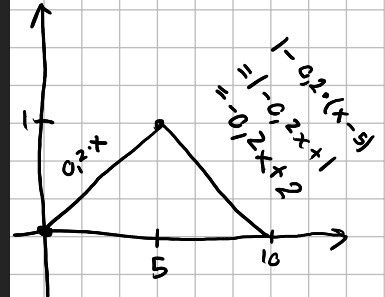
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

$n$ : A how many times of the base frequency.

$L$ : The half period.

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 0$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n > 0$$



We know that we can only use

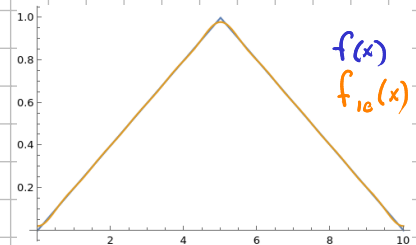
$$f(x) = 1 - 0.2|x - 5| \Rightarrow \int_0^{10} f(x) dx = \int_0^5 0.2x dx + \int_5^{10} -0.2x + 2 dx$$

$$2L = 10 \Rightarrow L = 5$$

$$N = 10$$

Calculated with mathematica:

$$f_a(x) = \underbrace{-0.405285 \cos\left(\frac{\pi x}{5}\right)}_{f_1} - \underbrace{0.0450316 \cos\left(\frac{3\pi x}{5}\right)}_{f_2} - \underbrace{0.0162114 \cos(\pi x)}_{f_3} - \underbrace{0.00827112 \cos\left(\frac{7\pi x}{5}\right)}_{f_4} - \underbrace{0.00500352 \cos\left(\frac{9\pi x}{5}\right)}_{f_5} + \underbrace{0.5}_{\text{offset?}}$$



Find general solution to subfunctions on the following form:

$$f(x) = a \cdot \cos\left(\frac{b\pi x}{5}\right)$$

We start with the general solution

$$u(x, t) = B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right) \cdot \left( C \cdot e^{\frac{n^2 \pi^2 \alpha}{100} \cdot t} + D \cdot t \cdot e^{\frac{n^2 \pi^2 \alpha}{100} \cdot t} \right)$$

Insert the initial condition

$$u(x, 0) = f(x) = a \cdot \cos\left(\frac{b\pi x}{5}\right) = B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right) \cdot \left( C \cdot e^0 + D \cdot 0 \cdot e^0 \right)$$

$$\Rightarrow a \cdot \cos\left(\frac{b\pi x}{5}\right) = B \cdot C \cdot \sin\left(\frac{n\pi}{10} \cdot x\right)$$

$$\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$$

I will set  $D=0$  and  $C=1$  for simplicity

$$\Rightarrow a \cdot \cos\left(\frac{b\pi x}{5}\right) = B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right) \Rightarrow a \cdot \sin\left(\frac{2b\pi}{10} \cdot x + \frac{\pi}{2}\right) = B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right)$$

$$a = B \Rightarrow \sin\left(\frac{2b\pi}{10} \cdot x + \frac{\pi}{2}\right) = \sin\left(\frac{n\pi}{10} \cdot x\right)$$

Idk what to do here, i will assume that  $n = 2b$

$$\begin{aligned} u(x, t) &= a \cdot \sin\left(\frac{2b\pi}{10} \cdot x\right) \cdot \left( e^{-\frac{4b^2\pi^2\alpha}{100} \cdot t} + 0 \cdot t \cdot e^{-\frac{4b^2\pi^2\alpha}{100} \cdot t} \right) \\ &= a \cdot \sin\left(\frac{2b\pi}{10} \cdot x\right) \cdot e^{-\frac{4b^2\pi^2\alpha}{100} \cdot t} \end{aligned}$$



**Problem 7: Arbitrary temperatures at ends.** If the ends  $x = 0$  and  $x = L$  of the bar in the text are kept at constant temperatures  $U_1$  and  $U_2$  respectively, what is the temperature  $u_1(x)$  in the bar after a long time (theoretically, as  $t \rightarrow \infty$ )? First guess, then calculate.