

Problem 1:

Find $\iint_S x \, dS$ over the part of the parabolic cylinder $z = x^2/2$ that lies inside the first octant part of the cylinder $x^2 + y^2 = 1$.

Solution: $\pi/8$

$$z = \frac{x^2}{2}$$

$$\Rightarrow x = v, \quad z = \frac{v^2}{2}, \quad y = u \Rightarrow \vec{r}(u, v) = v\vec{i} + u\vec{j} + \frac{v^2}{2}\vec{k}$$

$$\Rightarrow \begin{cases} r_u = \vec{j} \\ r_v = \vec{i} + v\vec{k} \end{cases}$$

$$dS = |r_u \times r_v| \, du \, dv$$

$$|r_u \times r_v| = \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 1 & 0 & v \end{vmatrix} \right\| = |(1 \cdot v - 0 \cdot 0)\vec{i} - (0 \cdot v - 1 \cdot 0)\vec{k} + (0 \cdot 0 - 1 \cdot 0)\vec{j}| = |v\vec{i} - \vec{k}| = \sqrt{v^2 + (-1)^2} = \sqrt{v^2 + 1}$$

$$\iint_S x \, dS = \iint_{v,u} v \cdot \sqrt{v^2 + 1} \, du \, dv$$

Finding the bounds

$$0 \leq x, \quad 0 \leq y \Rightarrow 0 \leq v, \quad 0 \leq u$$

$$x^2 + y^2 \leq 1 \Rightarrow v^2 + u^2 \leq 1$$

Transform to polar coordinates

$$x = r \cdot \cos(\theta) \quad y = r \cdot \sin(\theta) \quad dA = r \, d\theta \, dr \quad x^2 + y^2 = a^2 \Rightarrow a$$

$$0 \leq v, \quad 0 \leq u \Rightarrow 0 \leq \theta \leq \frac{\pi}{2}$$

$$v^2 + u^2 \leq 1 \Rightarrow 0 \leq r \leq 1$$

$$\iint_{v,u} v \cdot \sqrt{v^2 + 1} \, du \, dv \Rightarrow \int_0^{\frac{\pi}{2}} \int_0^1 r \cos(\theta) \cdot \sqrt{r^2 \cos^2(\theta) + 1} \cdot r \, dr \, d\theta = \frac{\pi}{8}$$

Problem 2:

Find the total charge on the surface

$$\mathbf{r} = e^u \cos v \mathbf{i} + e^u \sin v \mathbf{j} + u \mathbf{k}, \quad (0 \leq u \leq 1, 0 \leq v \leq \pi),$$

if the charge density on the surface is $\delta = \sqrt{1 + e^{2u}}$.

Solution: $\frac{\pi}{3}(3e + e^3 - 4)$

$$\int_0^\pi \int_0^1 \delta(u, v) \cdot |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

$$\begin{cases} \vec{r}_u = e^u \cdot \cos(v) \mathbf{i} + e^u \cdot \sin(v) \mathbf{j} + \mathbf{k} \\ \vec{r}_v = -e^u \cdot \sin(v) \mathbf{i} + e^u \cdot \cos(v) \mathbf{j} \end{cases} \Rightarrow \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^u \cdot \cos(v) & e^u \cdot \sin(v) & 1 \\ -e^u \cdot \sin(v) & e^u \cdot \cos(v) & 0 \end{vmatrix}$$

$$= (0 - e^u \cdot \cos(v)) \mathbf{i} - (0 - (-e^u \cdot \sin(v))) \mathbf{j} + (e^{2u} \cdot \cos^2(v) - (-e^{2u} \cdot \sin^2(v))) \mathbf{k}$$

$$= -e^u \cdot \cos(v) \mathbf{i} - e^u \cdot \sin(v) \mathbf{j} + e^{2u} \cdot (\cos^2(v) + \sin^2(v)) \mathbf{k}$$

$$= -e^u \cdot \cos(v) \mathbf{i} - e^u \cdot \sin(v) \mathbf{j} + e^{2u} \mathbf{k}$$

$$\begin{aligned} |\vec{r}_u \times \vec{r}_v| &= \sqrt{e^{2u} \cdot \cos^2(v) + e^{2u} \cdot \sin^2(v) + e^{4u}} = \sqrt{e^{2u} \cdot (\cos^2(v) + \sin^2(v) + e^{2u})} \\ &= \sqrt{e^{2u} + e^{4u}} = \sqrt{e^{2u}(1 + e^{2u})} = e^u \sqrt{e^{2u} + 1} = e^u \sqrt{e^{2u} + 1} \end{aligned}$$

$$\Rightarrow \int_0^\pi \int_0^1 \delta(u, v) \cdot |\vec{r}_u \times \vec{r}_v| \, du \, dv = \int_0^\pi \int_0^1 \sqrt{e^{2u} + 1} \cdot e^u \sqrt{e^{2u} + 1} \, du \, dv = \int_0^\pi \int_0^1 e^{2u} (e^{2u} + 1) \, du \, dv$$

$$= \int_0^\pi \int_0^1 (e^{4u} + e^{2u}) \, du \, dv = \int_0^\pi \left[\frac{1}{3} e^{3u} + e^u \right]_0^1 \, dv = \int_0^\pi \left(\frac{1}{3} e^3 + e - \frac{1}{3} e^0 - e^0 \right) \, dv = \int_0^\pi \left(\frac{1}{3} e^3 + e - \frac{1}{3} - 1 \right) \, dv$$

$$= \left[\frac{1}{3} e^3 \cdot v + e \cdot v - \frac{v}{3} - v \right]_0^\pi = \frac{1}{3} e^3 \pi + e\pi - \frac{\pi}{3} - \pi = \frac{\pi}{3} (e^3 + 3e - 1 - 3) = \underline{\underline{\frac{\pi}{3} (e^3 + 3e - 4)}}$$

Problem 3:

Describe the parametric surface

$$x = au \cos v, \quad y = au \sin v, \quad z = bv$$

($0 \leq u \leq 1, 0 \leq v \leq 2\pi$), and find its area.

Solution: $\pi a \sqrt{a^2 + b^2} + \pi b^2 \ln \left(\frac{a + \sqrt{a^2 + b^2}}{b} \right)$ sq. units

$$\vec{r}(u, v) = au \cos(v) \hat{i} + au \sin(v) \hat{j} + bv \hat{k}$$

$$\Rightarrow \begin{cases} \vec{r}_u = a \cos(v) \hat{i} + a \sin(v) \hat{j} \\ \vec{r}_v = -au \sin(v) \hat{i} + au \cos(v) \hat{j} + b \hat{k} \end{cases}$$

$$\Rightarrow \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a \cos(v) & a \sin(v) & 0 \\ -au \sin(v) & au \cos(v) & b \end{vmatrix} = (ab \sin(v) - 0) \hat{i} - (ab \cos(v) - 0) \hat{j} + (u \cdot a^2 \cos^2(v) - (-u \cdot a^2 \sin^2(v))) \hat{k}$$

$$= ab \sin(v) \hat{i} - ab \cos(v) \hat{j} + (u \cdot a^2 \cos^2(v) + u \cdot a^2 \sin^2(v)) \hat{k}$$

$$= ab \sin(v) \hat{i} - ab \cos(v) \hat{j} + (u \cdot a^2 \cdot (\cos^2(v) + \sin^2(v))) \hat{k}$$

$$= ab \sin(v) \hat{i} - ab \cos(v) \hat{j} + a^2 u \hat{k}$$

$$\begin{aligned} \Rightarrow |\vec{r}_u \times \vec{r}_v| &= \sqrt{a^2 b^2 \sin^2(v) + a^2 b^2 \cos^2(v) + a^4 u^2} \\ &= \sqrt{a^2 b^2 (\sin^2(v) + \cos^2(v)) + a^4 u^2} \\ &= \sqrt{a^2 b^2 + a^4 u^2} = \sqrt{a^2 (b^2 + a^2 u^2)} \\ &= a \sqrt{b^2 + a^2 u^2} \end{aligned}$$

Not good...

$$\lambda = b^2 + a^2 u^2 \Rightarrow \frac{d\lambda}{du} = 2a^2 u \Rightarrow du = \frac{1}{2a^2} d\lambda$$

$$\Rightarrow \int_0^{2\pi} \int_0^1 a \sqrt{b^2 + a^2 u^2} du dv = a \int_0^{2\pi} \int_0^1 \sqrt{b^2 + a^2 u^2} du dv =$$

Problem 4:

Find the flux of $\mathbf{F} = x\mathbf{i} + x\mathbf{j} + \mathbf{k}$ upward through the part of the surface $z = x^2 - y^2$ inside the cylinder $x^2 + y^2 = a^2$.

Solution: $\frac{\pi}{2} a^2 (2 - a^2)$

Convert to cylindrical coordinates

$$f(\theta, r, z) = (\underbrace{r \cdot \cos \theta}_x, \underbrace{r \cdot \sin \theta}_y, \underbrace{z}_z)$$

$$||J|| = r$$

$$\begin{aligned} z = x^2 - y^2 &\Rightarrow z(\theta, r) = r^2 \cdot \cos^2(\theta) - r^2 \cdot \sin^2(\theta) \\ &= r^2 \cdot (\cos^2(\theta) - \sin^2(\theta)) \\ &= r^2 \cdot \cos(2\theta) \end{aligned}$$

We only need the upward component of F

$$f(x, y, z) = x \Rightarrow f(\theta, r, z(\theta, r)) = r \cdot \cos(\theta)$$

Finding limits

$$\begin{aligned} x^2 + y^2 \leq a^2 &\Rightarrow 0 \leq r \leq a \\ 0 \leq \theta &\leq 2\pi \end{aligned}$$

Writing the integral

$$\int_0^a \int_0^{2\pi} f(\theta, r, z(\theta, r)) \cdot \sqrt{\left(\frac{\partial z}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 + 1} \, d\theta \, dr$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial}{\partial \theta} r^2 \cdot \cos(2\theta) = -2 r^2 \cdot \sin(2\theta)$$

$$\frac{\partial z}{\partial r} = \frac{\partial}{\partial r} r^2 \cdot \cos(2\theta) = 2r \cdot \cos(2\theta)$$

Spørg Jacob!

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$$\begin{aligned} z = x^2 - y^2 &\Rightarrow z = r^2 \cdot \cos^2(\theta) - r^2 \cdot \sin^2(\theta) \\ &= r^2 \cdot (\cos^2(\theta) - \sin^2(\theta)) \\ &= r^2 \cdot \cos(2\theta) \end{aligned}$$

Parameterization

$$u = r, \quad v = \theta, \quad z = u \cdot \cos(2v)$$

$$\vec{r} = u\hat{i} + v\hat{j} + u \cdot \cos(2v)\hat{k}$$

$$\begin{cases} \vec{r}_u = \hat{i} + \cos(2v)\hat{k} \\ \vec{r}_v = \hat{j} - u \cdot \sin(2v) \cdot 2 \end{cases}$$

$$\begin{aligned} \vec{r}_u \times \vec{r}_v &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \cos(2v) \\ 0 & 1 & -2u \sin(2v) \end{vmatrix} = (0 - \cos(2v))\hat{i} - (-2u \sin(2v) - 0)\hat{j} + (1 - 0)\hat{k} \\ &= -\cos(2v)\hat{i} + 2u \sin(2v)\hat{j} + \hat{k} \end{aligned}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{\cos^2(2v) + 4u^2 \sin^2(2v) + 1}$$

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} = \frac{-\cos(2v)\hat{i} + 2u \sin(2v)\hat{j} + \hat{k}}{\sqrt{\cos^2(2v) + 4u^2 \sin^2(2v) + 1}}$$

Getting the limits

$$x^2 + y^2 \leq a^2 \Rightarrow 0 \leq r \leq a \Rightarrow 0 \leq u \leq a$$

$$0 \leq \theta < 2\pi \Rightarrow 0 \leq v < 2\pi$$

$$dS = r \cdot d\theta dr \Rightarrow dS = u \, dv \, du$$

Parameterization of vector field

$$\vec{F}(x(u,v), y(u,v), z(u,v)) = u\hat{i} + u\hat{j} + \mathbf{1}$$

Writing the integral

$$\iint_S \vec{F} \times \vec{n} \, dS = \int_0^a \int_0^{2\pi} (u\hat{i} + u\hat{j} + \mathbf{1}) \cdot \left(\frac{-\cos(2v)\hat{i} + 2u \sin(2v)\hat{j} + \hat{k}}{\sqrt{\cos^2(2v) + 4u^2 \sin^2(2v) + 1}} \right) \cdot u \, dv \, du$$