**Problem 1:**

calculate **div F** and **curl F** for the vector field $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$

**Solution: div F** $= 0$, **curl F** $= -\mathbf{i} - \mathbf{j} - \mathbf{k}$

$$\textbf{div}\ \vec{v}(x,y) = \nabla \bullet \vec{v}(x,y) = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y}$$

$$\textbf{curl}\ \vec{v}(x,y,z) = \nabla \times \vec{v}(x,y,z) = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{bmatrix} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix}$$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \begin{bmatrix} \frac{d}{dx} \\ \frac{d}{dy} \\ \frac{d}{dz} \end{bmatrix} \cdot \begin{bmatrix} y \\ z \\ x \end{bmatrix} = \frac{d}{dx} y i + \frac{d}{dy} z j + \frac{d}{dz} x k = 0 + 0 + 0 = 0$$

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$det(M) = |M| = a \cdot \left|\begin{pmatrix} e & f \\ h & i \end{pmatrix}\right| - b \cdot \left|\begin{pmatrix} d & f \\ g & i \end{pmatrix}\right| + c \cdot \left|\begin{pmatrix} d & e \\ g & h \end{pmatrix}\right|$$

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F}(x,y,z) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ y & z & x \end{vmatrix}$$

$$= i \cdot \left(\frac{d}{dy} \cdot x - \frac{d}{dz} \cdot z\right) - j \cdot \left(\frac{d}{dx} \cdot x - \frac{d}{dz} \cdot y\right) + k \cdot \left(\frac{d}{dx} z - \frac{d}{dy} \cdot y\right)$$

$$= i \cdot (0 - 1) - j(1 - 0) + k \cdot (0 - 1) = \underline{\underline{-i - j - k}}$$

**Problem 2:**

calculate **div F** and **curl F** for the vector field $\mathbf{F} = x\mathbf{i} + x\mathbf{k}$

**Solution: div F** $= 1$, **curl F** $= -\mathbf{j}$

$$\textbf{curl}\ \vec{v}(x,y) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}, \textbf{ where } \vec{v}(x,y) = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$$

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$det(M) = |M| = a \cdot \left|\begin{pmatrix} e & f \\ h & i \end{pmatrix}\right| - b \cdot \left|\begin{pmatrix} d & f \\ g & i \end{pmatrix}\right| + c \cdot \left|\begin{pmatrix} d & e \\ g & h \end{pmatrix}\right|$$

$$\operatorname{div} \vec{F} = \nabla \vec{F} = \frac{d}{dx} x + \frac{d}{dy} 0 + \frac{d}{dz} x = 1$$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ x & 0 & x \end{vmatrix} = i \cdot \left(\frac{d}{dy} \cdot x - \frac{d}{dz} \cdot 0\right) - j\left(\frac{d}{dx} \cdot x - \frac{d}{dz} \cdot x\right) + k\left(\frac{d}{dx} \cdot 0 - \frac{d}{dy} x\right)$$

$$= i \cdot (0 - 0) - j(1 - 0) + k(0 - 0) = -j$$

**Problem 4:**

Evaluate $\oint_c (\sin x + 3y^2)\, dx + (2x - e^{-y^2})dy$, where $c$ is the boundary of the half-disk $x^2 + y^2 \leq a^2$, $y \geq 0$, oriented counterclockwise.

**Solution:** $\pi a^2 - 4a^3$

$$\oint_C \vec{F}(x,y)\, d\vec{r} = \oint_C P(x,y)dx + Q(x,y)dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

$P(x,y) = \sin(x) + 3y^2$

$\frac{dP}{dy} = 6y$

$Q(x,y) = 2x - e^{-y^2}$

$\frac{dQ}{dx} = 2$

$$\Rightarrow \iint_R 2 - 6y \; dA$$

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$$\iint_R (2 - 6 \cdot r \cdot \sin(\theta)) \cdot r \; dr \; d\theta$$

$r = 0..\pi$

$\theta = 0..a$

$$= \int_0^{\pi}\int_0^{a} 2r - 6 \cdot r^2 \cdot \sin(\theta) \; dr \; d\theta$$

$$= 2 \cdot \int_0^{\pi}\int_0^{a} r \; dr\, d\theta - 6 \int_0^{\pi}\int_0^{a} r^2 \cdot \sin(\theta) \; dr \; d\theta$$

$$= 2 \cdot \int_0^{\pi} \left[\frac{r^2}{2}\right]_0^a d\theta - 6 \cdot \int_0^{\pi} \sin(\theta) \cdot \left[\frac{r^3}{3}\right]_0^a \; d\theta = \cancel{2} \cdot \int_0^{\pi} \frac{a^2}{\cancel{2}} d\theta - 6 \cdot \int_0^{\pi} \sin(\theta) \cdot \frac{1}{3} \cdot a^3 \; d\theta$$

$$= a^2 \cdot \left[\theta\right]_0^{\pi} - 2a^3 \left[-\cos(\theta)\right]_0^{\pi} = \pi a^2 - 2a^3 (-0 - (-1)) = \underline{\underline{\pi a^2 - 2a^3}}$$

**Problem 5:**

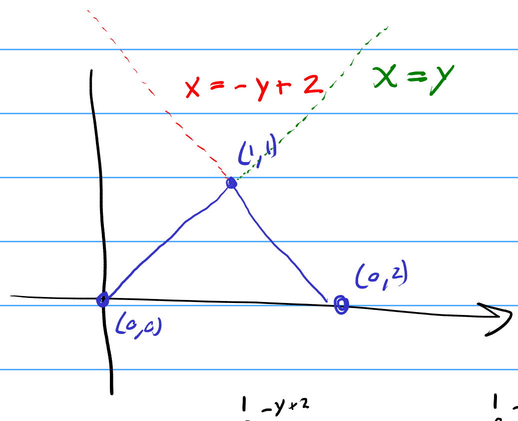
Evaluate $\oint_e (x^2 - xy)\,dx + (xy - y^2)dy$, clockwise around the triangle with vertices $(0,0)$, $(1,1)$, and $(2,0)$.

**Solution:** $-\frac{4}{3}$

$$\oint_C \vec{F}(x,y)\,\mathrm{d}\vec{r} = \oint_C P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{dA}$$

$$= -\iint_R \frac{d}{dx}(xy - y^2) - \frac{d}{dy}(x^2 - xy)\,dA$$

$$= -\iint_R y - (-x)\,dA = \iint y + x\,dA$$




$$-\int_0^1\int_y^{-y+2} y + x\,dx\,dy = -\int_0^1\int_y^{-y+2} y\,dxdy - \int_0^1\int_y^{-y+2} x\,dxdy$$

$$= -\int_0^1 \left[xy\right]_y^{-y+2} dy - \int_0^1 \left[\frac{x^2}{2}\right]_y^{-y+2} dy$$

$$= -\int_0^1 ((-y+2)y) - (y\cdot y)\,dy - \frac{1}{2}\int_0^1 (-y+2)^2 - y^2\,dy$$

$$= -\int_0^1 -y^2 + 2y - y^2\,dy - \frac{1}{2}\int_0^1 \cancel{y^2} - 4y + 4 - \cancel{y^2}\,dy$$

$$= -\int_0^1 2y - 2y^2\,dy - \frac{1}{2}\int_0^1 -4y + 4\,dy$$

$$= 2\int_0^1 y^2 + 2y\,dy + 2\int_0^1 y - 1\,dy$$

$$= 2\cdot\left[\frac{1}{3}\cdot y^3 + y^2\right]_0^1 - 2\left[\frac{y^2}{2} - y\right]_0^1 = 2\cdot\left(\frac{1}{3} + 1\right) - 2\cdot\left(\frac{1}{2} - 1\right) = \frac{2}{3} + 2 - 1 + 2$$

$$= \frac{6}{3}$$

**Problem 6:**

Use a line integral to find the plane area enclosed by the curve $r = a\cos^3 t\mathbf{i} + b\sin^3 t\mathbf{j}$, $(0 \le t \le 2\pi)$.

**Solution:** $\frac{3\pi ab}{8}$

$r = a \cdot \cos(t)^3 i + b \cdot \sin(t)^3 j$

**Problem 7:**

Use the Divergence Theorem to calculate the flux of the given vector field out of the sphere $\mathcal{s}$ with equation $x^2 + y^2 + z^2 = a^2$, where $a > 0$

$$\mathbf{F} = x\mathbf{i} - 2y\mathbf{j} + 4z\mathbf{k}$$

$$\iint_S \vec{F} \bullet \vec{n}\, dS = \iiint_V \mathbf{div}\, \vec{F}\, dV$$

$$\iiint_V div\, \vec{F}\, dV = \iiint_V \nabla \cdot \vec{F}\, dV = \iiint_V \frac{d}{dx} x + \frac{d}{dy} \cdot (-2y) + \frac{d}{dz} 4z\, dV$$

$$= \iiint_V 1 - 2 + 4\, dV = \iiint 3\, dV$$

Convert to spherical coordinates

$$x = \rho \cdot \sin\phi \cdot \cos\theta$$
$$y = \rho \cdot \sin\phi \cdot \sin\theta$$
$$z = \rho \cdot \cos\phi$$
$$dV = \rho \cdot \sin\phi\, d\rho\, d\phi\, d\theta$$

$$x^2 + y^2 + z^2 = a^2 \Rightarrow \rho = 0 \ldots a$$

$$\iiint_V 3\, dV \Rightarrow 3 \cdot \iiint \rho^2 \cdot \sin(\phi)\, d\theta\, d\rho\, d\phi$$

$$= 3 \cdot \iiint \rho^2 \cdot \sin(\phi)\, d\theta\, d\phi\, d\rho$$

We can now assign limits

$$= 3 \cdot \int_0^a \rho^2 \int_0^\pi \sin(\phi) \int_0^{2\pi} 1\, d\theta\, d\phi\, d\rho$$

$$= 3 \cdot \int_0^a \rho^2 \int_0^\pi \sin(\phi) \cdot 2\pi\, d\phi\, d\rho$$

$$= 6\pi \cdot \int_0^a \rho^2 \cdot \Big[ -\cos(\phi) \Big]_0^\pi\, d\rho = 6\pi \cdot \int_0^a \rho \cdot \big(-(-1) - (-1)\big)\, d\rho$$

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$$= 12\pi \int_0^a \rho^2\, d\rho = 12\pi \cdot \left[\frac{\rho^3}{3}\right]_0^a = 12\pi \cdot \frac{a^3}{3} = \underline{\underline{4\pi a^3}}$$