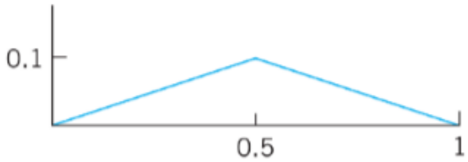
Exercise

Find $u(x,t)$ for the string of length L=1 and $c^2 = 1$ when the initial velocity is zero and the initial deflection with small k (say, 0.01) is as follows. Sketch or graph $u(x,t)$. Use the same information for problem 1,2 and 3 and 4

Problem 1: $k \sin 2\pi x$

Problem 2: $k\left(\sin \pi x - \frac{1}{3}\sin 3\pi x\right)$

Problem 3:



$$u_{tt} = c^2 u_{xx} = c^2 \nabla^2 u$$

$u(x,t) = k \cdot \sin(2\pi x) \Rightarrow \frac{d^2}{dt^2}\left(k \cdot \sin(2\pi x)\right) = c^2 \cdot \frac{d^2}{dx^2}\Big($

1. Find the solution of the one-dimensional wave equation

$$u_{tt}(x,t) = u_{xx}(x,t), \quad 0 \le x \le 1 \qquad \boxed{c^2 = 1}$$

subject to the boundary conditions

$$u(0,t) = u(1,t) = 0 \ \text{ for all t}$$

and the initial condition

$$u(x,0) = k_0 \sin(3\pi x), \quad u_t(x,0) = 0, \quad 0 \le x \le 1,$$

where k_0 is a constant. Sketch the solution for $t = 0$, $t = 1/6$ and $t = 1/3$.

$$u_{tt}(x,t) = u_{xx}(x,t),$$

We assume that we can solve this with seperation of variables

$$u(x,t) = F(x)\,G(t)$$

We now find the double derivatives

$$\frac{d^2}{dx^2} u(x,t) = \frac{d^2}{dx^2}\left(F(x)\,G(t)\right) = F''(x)\,G(t)$$

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We now plug this into the wave equation

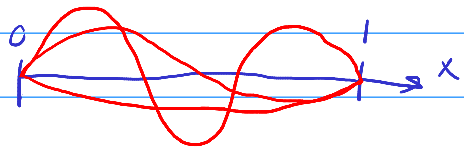
$$u_{tt}(x,t) = u_{xx}(x,t) \Rightarrow F(x)\,G''(t) = F''(x)\,G(t)$$

$$\Rightarrow \frac{F''(x)}{F(x)} = \frac{G''(x)}{G(x)} = \lambda$$

This must be a constant because the functions are functions of DIFFERENT variables. This also allows us to split op the equation to two less complicated ones.

$$\Rightarrow \begin{cases} F''(x) - \lambda \cdot F(x) = 0 \\ G''(t) - \lambda \cdot G(t) = 0 \end{cases}$$

We constain our solutions to be 0 as x=0 and x=1.



We now take a look at the boundries

$$u(0,t) = u(1,t) = 0$$

$$u(0,t) = 0 \Rightarrow F(0)\,G(t) = 0 \Rightarrow$$

Either F(0) = 0 or G(t)=0. This last solution is however uninteresting because it simply leads to the conclusion that u(x, t) = 0 which is quite uninteresting.

$$\Rightarrow F(0) = 0$$

Now for the second boundary

$$u(1,t) = 0 \Rightarrow F(1)\,G(t) = 0 \Rightarrow F(1) = 0$$

We now have to determine λ, if the eigen value is equal to 0 we get the uninteresting that λ=0. Therefore, we ignore this case.

We assume that λ is positive.

$$\lambda > 0, \quad \lambda = \mu^2$$

This is a solvable diff. equation

$$\Rightarrow F''(x) - k \cdot F(x) = 0 \Rightarrow F''(x) - \mu^2 \cdot F(x) = 0 \Rightarrow F(x) = A \cdot e^{\mu x} + B \cdot e^{-\mu x}$$

We can now combine this with out first boundary condition

$$F(0) = 0 \Rightarrow A \cdot e^0 + B \cdot e^0 = 0 \Rightarrow A + B = 0 \Rightarrow B = -A$$

And the second one

$$F(1) = 0 \Rightarrow A \cdot e^{\mu} + B \cdot e^{-\mu} = 0 \Rightarrow A \cdot e^{\mu} - A \cdot e^{-\mu} = 0 \Rightarrow A \cdot \underbrace{(e^{\mu} - e^{-\mu})}_{\text{This part is never negative.}} = 0$$

$$\Rightarrow A = 0 \Rightarrow B = 0 \Rightarrow F(x) = 0$$

This is also not an interesting result, therefore λ must be negative

We now assume that λ is negative.

$$\lambda < 0, \quad \lambda = -p^2$$

$$\Rightarrow F''(x) - \lambda \cdot F(x) = 0 \Rightarrow F''(x) + p^2 \cdot F(x) = 0 \Rightarrow \overbrace{F''(x) = -p^2 \cdot F(x)}^{\text{Solvable equation!}}$$

$$\Rightarrow F(x) = A \cdot \cos(px) + B \cdot \sin(px)$$

Plugging in boundary conditions

$$F(0) = 0 \Rightarrow A \cdot \overset{1}{\cos(0)} + B \cdot \overset{0}{\sin(0)} = 0 \Rightarrow A = 0$$

$$F(1) = 0 \Rightarrow B \cdot \sin(p) = 0 \Rightarrow p = n\pi, \quad n = 1, 2, 3 \ldots$$

ANY B will satisfy this equation, we therefore choose 1

$$\Rightarrow B = 1$$

We now know a set of solutions

$$F(x) = F_n(x) = \sin(n\pi x), \quad n = 1, 2, 3, \ldots$$

We now look at the function G(t)

Remember: $\lambda = -p^2$

$$G''(t) - \lambda \cdot G(t) = 0 \Rightarrow G''(t) + p^2 \cdot G(t) = 0 \Rightarrow G''(t) = -p^2 \cdot G(t)$$

$$\Rightarrow G(t) = A \cdot \cos(pt) + B \cdot \sin(pt)$$

Arbitrery constants

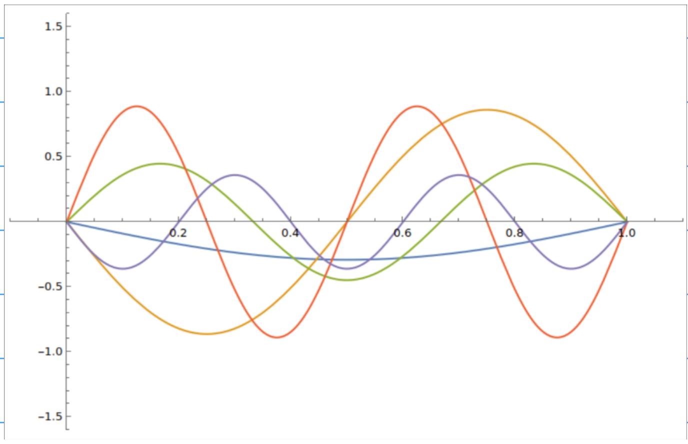
We can now write the set of solutions like this

$$u_n(x,t) = \sin(n\pi x)\cdot\left(A_n\cdot\cos(\pi n t) + B_n\sin(\pi n t)\right), \quad n = 1, 2, 3, \ldots$$

And let's not forget

$u(x,t) = 0$

Here is a plot of some solutions at t=0



Bringing in initial conditions

$$u(x,0) = k_0 \sin(3\pi x), \quad u_t(x,0) = 0, \quad 0 \le x \le 1,$$ k_0 is constant

$$u_n(x,t) = \sin(n\pi x)\cdot\left(A_n\cdot\cos(\pi n t) + B_n\sin(\pi n t)\right), \quad n = 1, 2, 3, \ldots$$

Find the derivative of u

$$u_t = \sin(n\pi x)\cdot\left(A_n\cdot n\pi\cdot\left(-\sin(n\pi t)\right) + B_n\cdot n\pi\cdot\cos(n\pi t)\right)$$

$$= \sin(n\pi x)\cdot\left(B_n\cdot n\pi\cdot\cos(n\pi t) - A_n\cdot n\pi\cdot\sin(n\pi t)\right)$$

$u_t(x,0) = 0$

$$\Rightarrow \sin(n\pi x)\cdot\left(B_n\cdot n\pi\cdot\cos(0) - \cancel{A_n\cdot n\pi\cdot\sin(0)}\right) = 0$$

$$\Rightarrow \underbrace{\sin(n\pi x)}_{\text{zero when } x=\frac{1}{n}} \cdot B_n \cdot \underbrace{n\pi}_{\text{never zero}} = 0 \Rightarrow B_n = 0$$

$$\Rightarrow \quad u_n(x,t) = \sin(n\pi x)\cdot A_n \cdot \cos(\pi n t) \quad , \quad n = 1, 2, 3, \ldots$$

$$u(x,0) = k_0 \cdot \sin(3\pi x)$$

$$\Rightarrow \quad \sin(n\pi x)\cdot A_n \cdot \cos(\pi n(0)) = k_0 \cdot \sin(3\pi x)$$

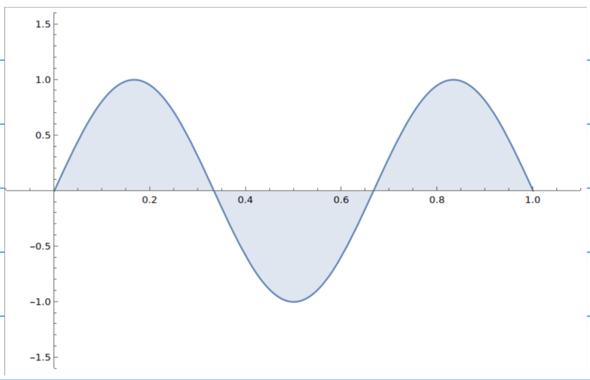
$$\Rightarrow \quad \sin(n\pi x)\cdot A_n = k_0 \cdot \sin(3\pi x)$$

$$\Rightarrow \quad n = 3, \quad A_n = k_0$$

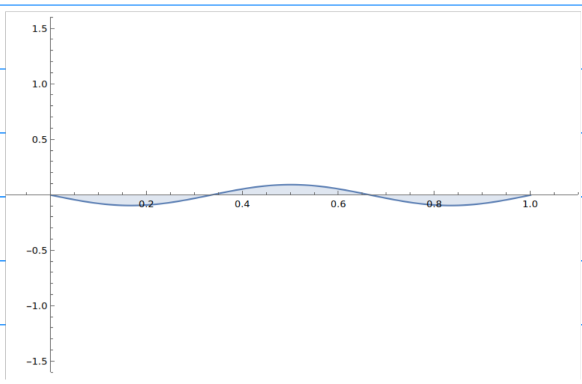
This is the final solution

$$\underline{\underline{u(x,t) = k_0 \cdot \sin(3\pi x)\cdot\cos(3\pi t)}}$$

Plot at t=0



Plot at t=1.51



2. Find the solution of the one-dimensional wave equation

$$u_{tt}(x,t) = u_{xx}(x,t), \quad 0 \le x \le 1$$

subject to the boundary conditions

$$u(0,t) = u(1,t) = 0 \quad \text{for all t}$$

and the initial condition

$$u(x,0) = k_0 \sin(3\pi x), \quad u_t(x,0) = 3\pi k_0 \sin(3\pi x), \quad 0 \le x \le 1,$$

where k_0 is a constant. Sketch the solution for $t = 0$, $t = 1/6$ and $t = 1/3$. Find t_0 at which $u(x,t_0) = 0$ for all x for the first time. Find t_1 at which $u(x,t_1) = 0$ for all x for the second time.

Boundary conditions are the same as before, so we can use the same general solution.

$$u_n(x,t) = \sin(n\pi x)\cdot\left(A_n\cdot\cos(\pi n t) + B_n \sin(\pi n t)\right), \quad n = 1, 2, 3, \dots$$

$$u_t(x,t) = \sin(n\pi x)\cdot\left(B_n\cdot n\pi\cdot\cos(n\pi t) - A_n\cdot n\pi\cdot\sin(n\pi t)\right)$$

$$u(x,0) = k_0\cdot\sin(3\pi x)$$

$$\Rightarrow \sin(n\pi x)\cdot\left(A_n\cdot\overset{1}{\cos(0)} + B_n \overset{0}{\sin(0)}\right) = k_0\cdot\sin(3\pi x)$$

$$\Rightarrow \sin(n\pi x)\cdot A_n = k_0\cdot\sin(3\pi x) \Rightarrow n = 3, \quad A_n = k_0$$

$$u_t(x,0) = \sin(n\pi x)\cdot\left(B_n\cdot n\pi\cdot\cos(n\pi t) - A_n\cdot n\pi\cdot\sin(n\pi t)\right)$$

$$\Rightarrow \sin(3\pi x)\cdot\left(B_n\cdot 3\pi\cdot\overset{1}{\cos(0)} - A_n\cdot 3\pi\cdot\overset{0}{\sin(0)}\right) = 3\pi k_0\cdot\sin(3\pi x)$$

$$\Rightarrow \sin(3\pi x)\cdot\left(B_n\cdot 3\pi\right) = 3\pi k_0\cdot\sin(3\pi x)$$

$$\Rightarrow 3\pi\cdot B_n\cdot\sin(3\pi x) = 3\pi\cdot k_0\cdot\sin(3\pi x) \Rightarrow k_0 = B_n$$

Putting it all togeather

$$u(x,t) = \sin(3\pi x) \cdot \left(k_0 \cdot 3\pi \cdot \cos(3\pi t) - k_0 \cdot 3\pi \cdot \sin(3\pi t) \right)$$

$$\Rightarrow \underline{\underline{u(x,t) = k_0 \cdot 3\pi \cdot \sin(3\pi x) \cdot \left(\cos(3\pi t) - \sin(3\pi t) \right)}}$$