Stability and Performance Analysis

Control Engineering (Reguleringsteknik)

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Agenda

Introduction
Curriculum for Reguleringsteknik (REG)

Matematiske og grafiske metoder til syntese af lineære tidsinvariante systemer:[1]

- diskret og kontinuert tilstandsbeskrivelse
- analyse i tid og frekvens
- stabilitet, reguleringshastighed, følsomhed og fejl
- digitale PI, PID, LEAD og LAG regulatorer (serieregulatorer)
- tilstandsregulering, pole-placement og tilstands-estimering (observer)
- optimal regulering (least squares) og optimal tilstands-estimation (Kalman-filter)

Færdigheder:
Efter gennemførelse af kurset kan den succesfulde studerende:

- kunne analysere, dimensionere og implementere såvel kontinuert som tidsdiskret regulering af lineære tidsinvariante og stokastiske systemer

Kompetencer:
Efter gennemførelse af kurset kan den succesfulde studerende:

- anvende og implementere klassiske og moderne reguleringsteknikker for at kunne styre og regulere en robot hurtig og præcist

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[1] Based on https://fagbesk.sam.sdu.dk/?fag_id=39673

The twelve lectures of the course are

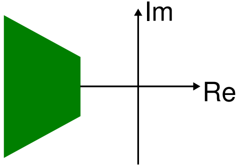
- **Lecture 1**: Introduction to Linear Time-Invariant Systems
- **Lecture 2**: Stability and Performance Analysis
- **Lecture 3**: Introduction to Control
- **Lecture 4**: Design of PID Controllers
- **Lecture 5**: Root Locus
- **Lecture 6**: The Nyquist Plot
- **Lecture 7**: Dynamic Compensators and Stability Margins
- **Lecture 8**: Implementation
- **Lecture 9**: State Feedback
- **Lecture 10**: Observer Design
- **Lecture 11**: Optimal Control (Linear Quadratic Control)
- **Lecture 12**: The Kalman Filter

The performance of the system can be specified in both time and frequency.

Time Domain

- Rise time
- Settling time
- Overshoot

Frequency domain



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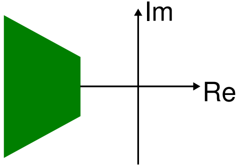


The stability of the dynamical system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

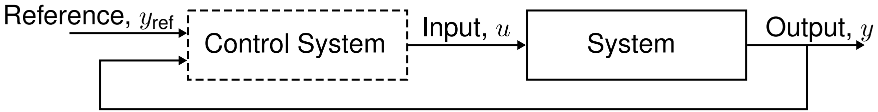
can be determined from the eigenvalues of A.

The stability of the transfer function

$$G(s) = \frac{Q(s)}{P(s)}$$

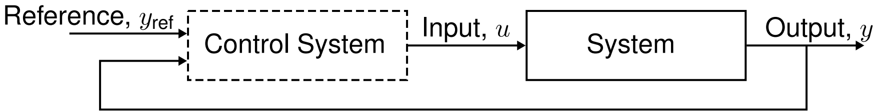
can be determined from the poles of $G(s)$.

Task: Design a cruise control for a car.



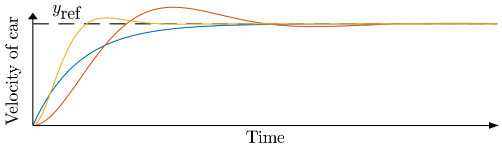
- **Control Input**: Throttle position u
- **Measured Output**: Velocity of the car y
- **Reference Input**: Desired velocity of the car y_{ref}

Task: Design a cruise control for a car.



Today, we answers to the following question:

- What is "good" performance?



Basic System Classes

A first-order system has one pole and is described by

$$H(s) = \frac{k}{\tau s + 1}$$

$$\begin{cases} \dot{x} = -\frac{1}{\tau}x + \frac{k}{\tau}u \\ y = x \end{cases}$$

where k is the **DC-gain** and τ is the **time-constant**.

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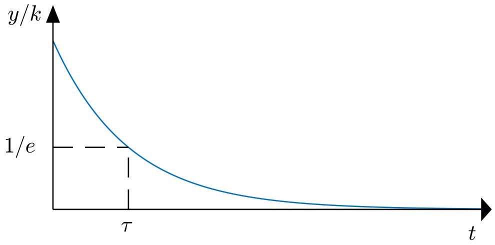
where k is the **DC-gain** and τ is the **time-constant**.

The system has a pole in $s = -\frac{1}{\tau}$, i.e., the smaller time-constant, the faster system response.

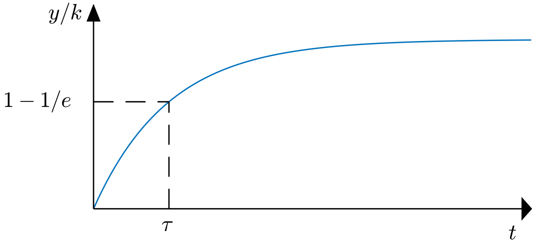
The impulse response of a first-order system is

$$y(t) = ke^{-\frac{1}{\tau}t}$$



The step response of a first-order system is

$$y(t) = k(1 - e^{-\frac{1}{\tau}t})$$



The transfer function of a second-order system is

$$H(s) = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

and is described by the gain k and two parameters: $\zeta > 0$ and $\omega_n > 0$.

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The system has two poles, which are $s \in \mathbb{C}$ where

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

The poles of

$$H(s) = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

are values of s given by

$$s = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4 \cdot 1 \cdot \omega_n^2}}{2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}.$$

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Terminology

- The parameter ζ is called the **damping ratio**.
- The parameter ω_n is called the **undamped natural frequency**.

The damping ratio ζ of an underdamped second-order system is smaller than one; thus, it has a complex pole pair with poles at

$$-\underbrace{\zeta\omega_n}_{\sigma} \pm j\,\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}.$$

Second-Order Systems

Underdamped Second-Order System (1)

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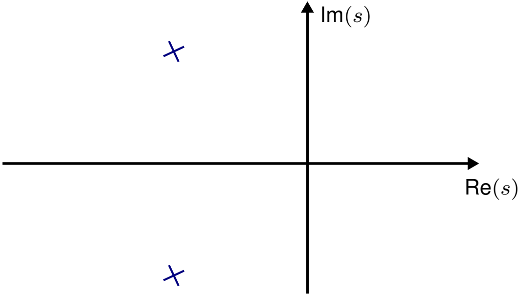

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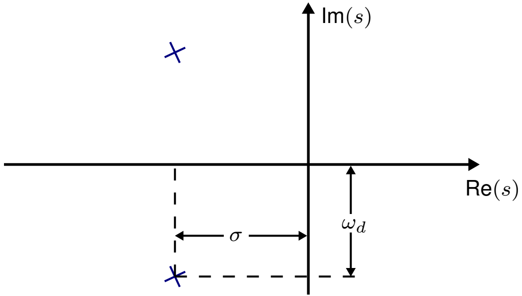

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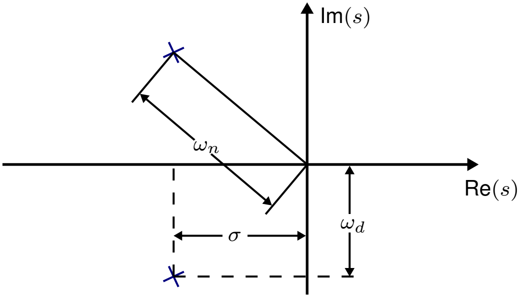

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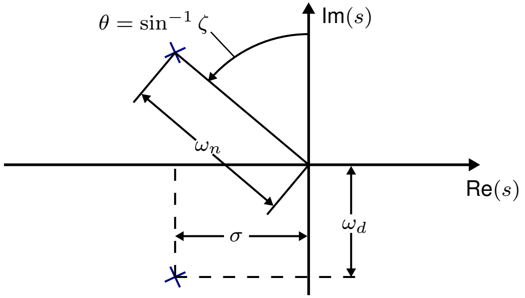

The impulse response of the system is

$$h(t) = k\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\sigma t}\sin(\omega_d t)1(t).$$

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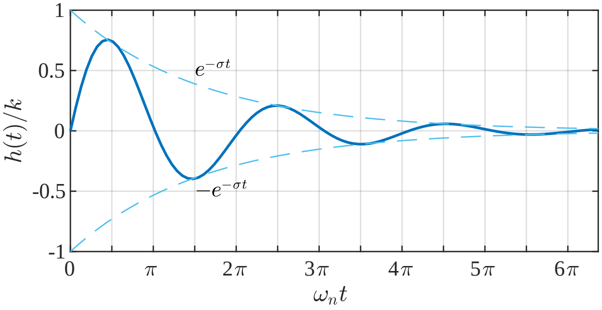

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$$y(t) = k\left(1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)\right)$$

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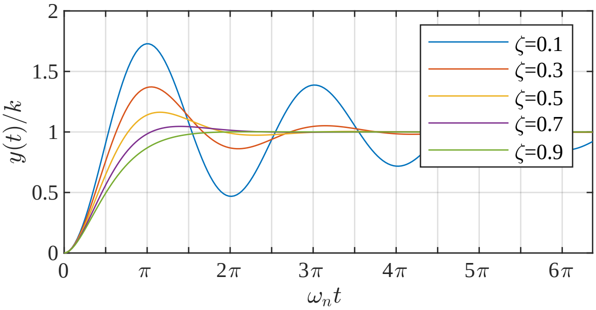
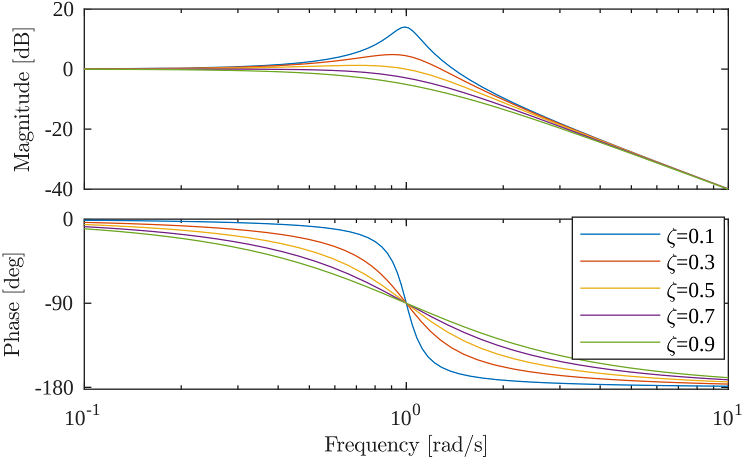


The Bode plot of a second-order system depends on the damping ratio ζ.



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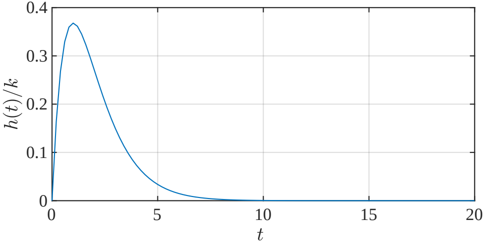

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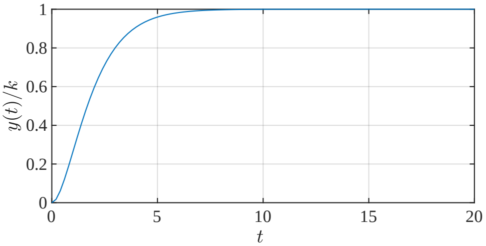

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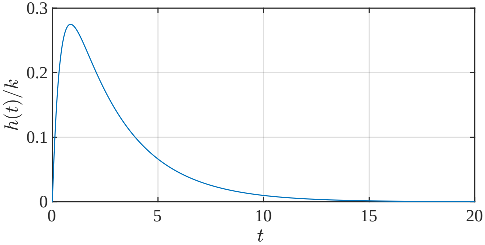

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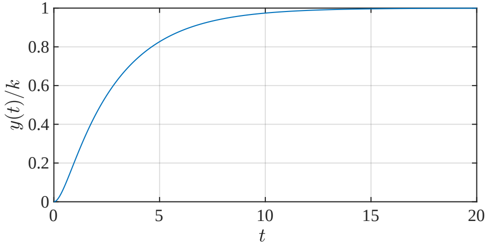
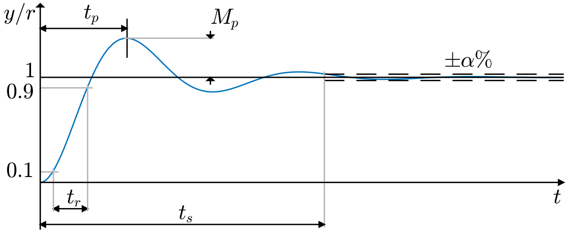


We consider three different performance measures of dynamical systems

1. The **rise time** t_r.
2. The **settling time** t_s.
3. The **overshoot** M_p.

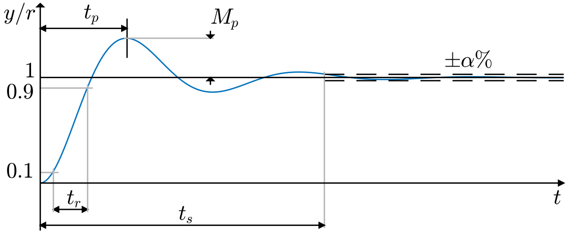


The **peak time** is denoted t_p.

Performance Specification

Rise Time

The rise time of a dynamical system is the time it takes to come from 0.1 times the setpoint value to reach 0.9 times the setpoint value.



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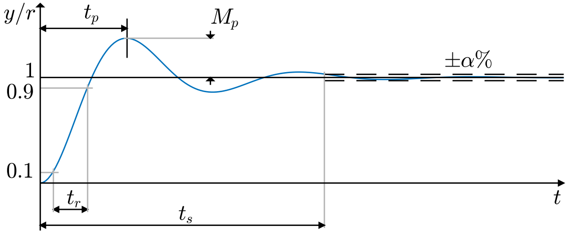
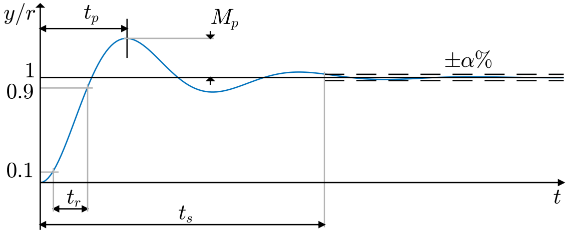


For a second-order system the rise time can be approximated as

$$t_r = \frac{1.8}{\omega_n}$$

The $\alpha\%$-settling time is the time it takes for the step response to reach $\pm\alpha\%$ of the setpoint value without leaving it again.



Performance Specification

Settling Time

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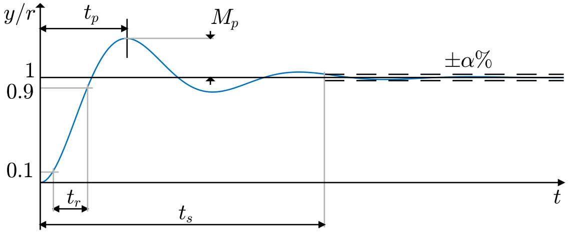

For a second-order system the settling time can be approximated as

$$t_s = \frac{-\log(\alpha/100)}{\omega_n \zeta}$$

The peak time t_p can be found from the step response of the system

$$y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

At the maximum of $y(t)$, we have $\dot{y}(t) = 0$

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$$\dot{y}(t) = e^{-\sigma t}\left(\omega_d \sin(\omega_d t) + \frac{\sigma^2}{\omega_d}\sin(\omega_d t)\right)$$

Thus, $\dot{y}(t) = 0$ for $\sin(\omega_d t)$, i.e.,

$$t_p = \frac{\pi}{\omega_d}.$$

The overshoot is computed from the step response at the peak time

$$
\begin{aligned}
y(t_p) &= 1 - e^{-\sigma\pi/\omega_d}\left(\cos(\pi) + \frac{\sigma}{\omega_d}\sin(\pi)\right) \\
&= 1 + M_p
\end{aligned}
$$

The overshoot is computed from the step response at the peak time

$$\begin{aligned} y(t_p) &= 1 - e^{-\sigma\pi/\omega_d}\left(\cos(\pi) + \frac{\sigma}{\omega_d}\sin(\pi)\right) \\ &= 1 + M_p \end{aligned}$$

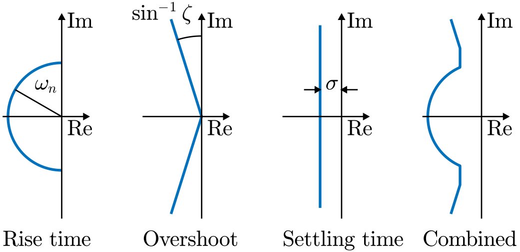
This gives an expression for the overshoot

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

for $0 \leq \zeta \leq 1$.

Performance Specification

Frequency and Time Specifications



To obtain a rise rise shorter than t_r

$$\omega_n \geq \frac{1.8}{t_r}$$

To obtain an overshoot that is smaller than M_p

$$\zeta \geq \sqrt{\frac{\left(\frac{\log(M_p)}{-\pi}\right)^2}{1+\left(\frac{\log(M_p)}{-\pi}\right)^2}}$$

To obtain an α%-settling time shorter than t_s

$$\sigma \geq \frac{-\log(\alpha/100)}{t_s}$$

With

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

we have that:

$$G(s) \to \infty \text{ for } s \to p \quad \Rightarrow \quad \det\,(pI - A) = 0$$

Hence,

$$p \text{ is a pole for } G(s) \;\Rightarrow$$

Poles of State Space Models

Definition

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Hence,

$$p \text{ is a pole for } G(s) \;\Rightarrow\; p \text{ is an eigenvalue for } A$$

For the mass-spring-damper system, the A matrix was:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

which has the characteristic polynomial:

$$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{vmatrix} \\ &= \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) \end{aligned}$$

Thus, the system has poles in $\{-1, -2\}$.

With

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

we have that:

$$\begin{aligned}
G(z)u = 0 \quad &\Rightarrow \quad C\,(zI - A)^{-1}\,Bu + Du = 0 \\
\Rightarrow \quad & C\xi + Du = 0\,,\ \xi = (zI - A)^{-1}\,Bu \\
\Rightarrow \quad & C\xi + Du = 0\,,\ (A - zI)\,\xi + Bu = 0 \\
\Rightarrow \quad & \begin{bmatrix} A - zI & B \\ C & D \end{bmatrix} \begin{bmatrix} \xi \\ u \end{bmatrix} = 0
\end{aligned}$$

Zeros of State Space Models

Definition

$$G(z)u = 0 \quad \Rightarrow \quad C\,(zI - A)^{-1}\,Bu + Du = 0$$

$$\Rightarrow \quad C\xi + Du = 0\,,\ \xi = (zI - A)^{-1}\,Bu$$

$$\Rightarrow \quad C\xi + Du = 0\,,\ (A - zI)\,\xi + Bu = 0$$

$$\Rightarrow \quad \begin{bmatrix} A - zI & B \\ C & D \end{bmatrix} \begin{bmatrix} \xi \\ u \end{bmatrix} = 0$$

Thus, z is a zero for $G(s)$ $\Rightarrow$

$$\begin{bmatrix} A - zI & B \\ C & D \end{bmatrix}$$ does not have full column rank

Zeros of State Space Models

Example: Mass-Spring-Damper

For the mass-spring-damper system, zeros must satisfy:

$$\begin{vmatrix} A - zI & B \\ C & D \end{vmatrix} = 0$$

or

$$\begin{vmatrix} -z & 1 & 0 \\ -2 & -3-z & 1 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} -z & 1 \\ 0 & 1 \end{vmatrix} \cdot (-1) = z = 0$$

Hence, the system has a zero in the origin.

A linear discrete-time system

$$x_{k+1} = \Phi x_k$$

where $x_k \in \mathbb{R}^n$ and $\Phi \in \mathbb{R}^{n \times n}$ is ***asymptotically stable*** if

$$\lim_{k \to \infty} x_k = 0$$

for any $x_0 \in \mathbb{R}^n$.

A linear discrete-time system described by the state equation

$$x_{k+1} = \Phi x_k$$

is asymptotically stable if and only if all eigenvalues have magnitude smaller than one (they are within the unit circle).

Assume that Φ is diagonalizable then by spectral decomposition Φ can be written as

$$\Phi = Q \Lambda Q^{-1}$$

where $\Lambda \in \mathbb{C}^{n \times n}$ is a diagonal matrix with the eigenvalues of Φ on the diagonal, and each column of Q is a (right) eigenvector of Φ.

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The state at sample k can be written as

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The state at sample k can be written as

$$x_k = \Phi^k x_0 = Q\Lambda^k Q^{-1} x_0.$$

Thus, the system is stable if and only if the magnitude of all diagonal elements of Λ is smaller than one.

A linear continuous-time system

$$\dot{x} = Ax$$

where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ is ***asymptotically stable*** if

$$\lim_{t \to \infty} x(t) = 0$$

for any $x(0) = x_0 \in \mathbb{R}^n$.

A linear continuous-time system described by the state equation

$$\dot{x} = Ax$$

is asymptotically stable if and only if all eigenvalues have negative real part.

Recall that the solution to the differential equation is

$$x(t) = e^{At}x(0)$$

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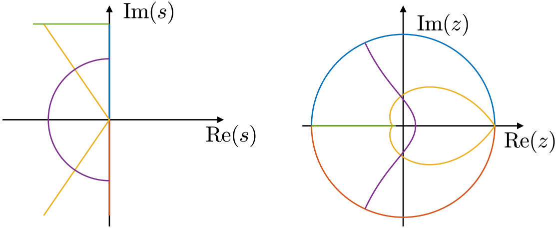
Thus, the system is stable if and only if the real parts of all diagonal elements of Λ are negative.

The equivalent characteristics in the z-plane are related to those in the s-plane by the expression

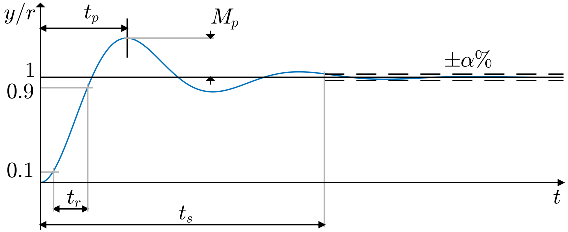
$$z = e^{sT}$$

where T is the sample period.



We consider three different performance measures of dynamical systems

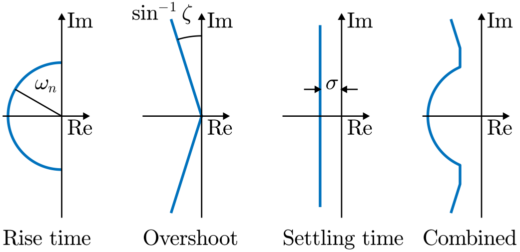
1. The **rise time** t_r.
2. The **settling time** t_s.
3. The **overshoot** M_p.



The **peak time** is denoted t_p.

Summary

Frequency-Domain Performance Specifications



To obtain a rise rise shorter than t_r

$$\omega_n \geq \frac{1.8}{t_r}$$

To obtain an overshoot that is smaller than M_p

$$\zeta \geq \sqrt{\frac{\left(\frac{\log(M_p)}{-\pi}\right)^2}{1+\left(\frac{\log(M_p)}{-\pi}\right)^2}}$$

To obtain an α%-settling time shorter than t_s

$$\sigma \geq \frac{-\log(\alpha/100)}{t_s}$$

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