Observability, Observers, and Observer Based Control

Control Engineering (Reguleringsteknik)

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Agenda

Introduction

Observability

Full Order Observer

Observer Based Control

Integral Control

- Example: Integral Control

Matematiske og grafiske metoder til syntese af lineære tidsinvariante systemer:[1]

- diskret og kontinuert tilstandsbeskrivelse
- analyse i tid og frekvens
- stabilitet, reguleringshastighed, følsomhed og fejl
- digitale PI, PID, LEAD og LAG regulatorer (serieregulatorer)
- tilstandsregulering, pole-placement og tilstands-estimering (observer)
- optimal regulering (least squares) og optimal tilstands-estimation (Kalman-filter)

Færdigheder:
Efter gennemførelse af kurset kan den succesfulde studerende:

- kunne analysere, dimensionere og implementere såvel kontinuert som tidsdiskret regulering af lineære tidsinvariante og stokastiske systemer

Kompetencer:
Efter gennemførelse af kurset kan den succesfulde studerende:

- anvende og implementere klassiske og moderne reguleringsteknikker for at kunne styre og regulere en robot hurtig og præcist

[1] Based on https://fagbesk.sam.sdu.dk/?fag_id=39673

The twelve lectures of the course are

- **Lecture 1**: Introduction to Linear Time-Invariant Systems
- **Lecture 2**: Stability and Performance Analysis
- **Lecture 3**: Introduction to Control
- **Lecture 4**: Design of PID Controllers
- **Lecture 5**: Root Locus
- **Lecture 6**: The Nyquist Plot
- **Lecture 7**: Dynamic Compensators and Stability Margins
- **Lecture 8**: Implementation
- **Lecture 9**: State Feedback
- **Lecture 10**: Observer Design
- **Lecture 11**: Optimal Control (Linear Quadratic Control)
- **Lecture 12**: The Kalman Filter

A continuous time system

$$\dot{x}(t) = Ax(t)\,, \quad y(t) = Cx(t)$$

is said to be *observable* iff $y(t) \equiv 0 \Rightarrow x(t) \equiv 0$.

A discrete time system

$$x_{k+1} = \Phi x_k\,, \quad y_k = Cx_k$$

is said to be *observable* iff $y_k \equiv 0 \Rightarrow x_k \equiv 0$.

We consider the discrete time system

$$x_{k+1} = \Phi x_k \,, \quad y_k = C x_k \,, \quad x_0 = x_0$$

and iterate:

$$x_0 = x_0 \qquad y_0 = C x_0$$

We consider the discrete time system

$$x_{k+1} = \Phi x_k \,, \quad y_k = C x_k \,, \quad x_0 = x_0$$

and iterate:

$$\begin{aligned} x_0 &= x_0 & \qquad y_0 &= C x_0 \\ x_1 &= \Phi x_0 \end{aligned}$$

Observability

Condition for Observability (1)

We consider the discrete time system

$$x_{k+1} = \Phi x_k \,, \quad y_k = C x_k \,, \quad x_0 = x_0$$

and iterate:

$$\begin{aligned} x_0 &= x_0 & \qquad y_0 &= C x_0 \\ x_1 &= \Phi x_0 \end{aligned}$$

We consider the discrete time system

$$x_{k+1} = \Phi x_k\,, \quad y_k = C x_k\,, \quad x_0 = x_0$$

and iterate:

$$\begin{aligned} x_0 &= x_0 & y_0 &= C x_0 \\ x_1 &= \Phi x_0 & y_1 &= C\Phi x_0 \end{aligned}$$

We consider the discrete time system

$$x_{k+1} = \Phi x_k \,, \quad y_k = C x_k \,, \quad x_0 = x_0$$

and iterate:

$$\begin{aligned} x_0 &= x_0 & \qquad y_0 &= C x_0 \\ x_1 &= \Phi x_0 & \qquad y_1 &= C \Phi x_0 \\ x_2 &= \Phi x_1 \end{aligned}$$

We consider the discrete time system

$$x_{k+1} = \Phi x_k \,, \quad y_k = C x_k \,, \quad x_0 = x_0$$

and iterate:

$$\begin{aligned} x_0 &= x_0 & \qquad y_0 &= C x_0 \\ x_1 &= \Phi x_0 & \qquad y_1 &= C \Phi x_0 \\ x_2 &= \Phi x_1 \end{aligned}$$

We consider the discrete time system

$$x_{k+1} = \Phi x_k \,, \quad y_k = C x_k \,, \quad x_0 = x_0$$

and iterate:

$$\begin{aligned} x_0 &= x_0 & y_0 &= C x_0 \\ x_1 &= \Phi x_0 & y_1 &= C\Phi x_0 \\ x_2 &= \Phi^2 x_0 & y_2 &= C\Phi^2 x_0 \end{aligned}$$

We consider the discrete time system

$$x_{k+1} = \Phi x_k\,, \quad y_k = C x_k\,, \quad x_0 = x_0$$

and iterate:

$$\begin{aligned}
x_0 &= x_0 & \quad y_0 &= C x_0 \\
x_1 &= \Phi x_0 & \quad y_1 &= C\Phi x_0 \\
x_2 &= \Phi^2 x_0 & \quad y_2 &= C\Phi^2 x_0 \\
&\vdots & & \\
x_{n-1} &= \Phi x_{n-2} & &
\end{aligned}$$

We consider the discrete time system

$$x_{k+1} = \Phi x_k\,, \quad y_k = C x_k\,, \quad x_0 = x_0$$

and iterate:

$$\begin{aligned}
x_0 &= x_0 & y_0 &= C x_0 \\
x_1 &= \Phi x_0 & y_1 &= C\Phi x_0 \\
x_2 &= \Phi^2 x_0 & y_2 &= C\Phi^2 x_0 \\
&\vdots \\
x_{n-1} &= \Phi x_{n-2}
\end{aligned}$$

We consider the discrete time system

$$x_{k+1} = \Phi x_k \,, \quad y_k = C x_k \,, \quad x_0 = x_0$$

and iterate:

$$\begin{array}{ccccccc} x_0 & = & x_0 & & y_0 & = & C x_0 \\ x_1 & = & \Phi x_0 & & y_1 & = & C \Phi x_0 \\ x_2 & = & \Phi^2 x_0 & & y_2 & = & C \Phi^2 x_0 \\ & \vdots & & & & & \\ x_{n-1} & = & \Phi^{n-1} x_0 & & y_{n-1} & = & C \Phi^{n-1} x_0 \end{array}$$

Writing the equations

$$y_k = C\Phi^k x_0\,,\; k = 0, \ldots, n-1$$

in matrix form we obtain:

Writing the equations

$$y_k = C\Phi^k x_0\,,\; k = 0, \ldots, n-1$$

in matrix form we obtain:

$$\underbrace{\begin{bmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{n-1} \end{bmatrix}}_{\text{Observability matrix}} x_0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Writing the equations

$$y_k = C\Phi^k x_0 \, , \, k = 0, \ldots, n-1$$

in matrix form we obtain:

$$\begin{bmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{n-1} \end{bmatrix} x_0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

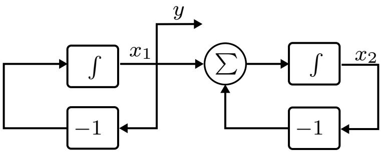
When is this equation solvable for some $x_0 \neq 0$?

THEOREM. A system

continuous time	discrete time
$\Sigma : \begin{cases} \dot{x}(t) &= Ax(t) \\ y(t) &= Cx(t) \end{cases}$	$\Sigma : \begin{cases} x_{k+1} &= \Phi x_k \\ y_k &= Cx_k \end{cases}$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, is observable if and only if

$$\operatorname{rank} \mathcal{O} = \operatorname{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$



State and output equations:

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & -x_1 \\ \dot{x}_2 & = & -x_2 + x_1 \\ y & = & x_1 \end{array}\right\}$$

State space model:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For the state space matrices:

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

the observability matrix $\mathcal{O}$ becomes:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

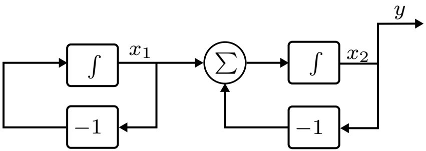
For the state space matrices:

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

the observability matrix $\mathcal{O}$ becomes:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

$\det \mathcal{O} = 0 \implies$ system is unobservable.



State and output equations:

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & -x_1 \\ \dot{x}_2 & = & -x_2 + x_1 \\ y & = & x_2 \end{array} \right\}$$

State space model:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For the state space matrices:

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

the observability matrix $\mathcal{O}$ becomes:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

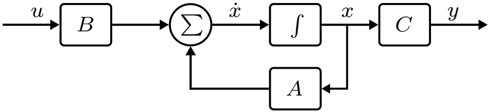
For the state space matrices:

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

the observability matrix $\mathcal{O}$ becomes:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$\det \mathcal{O} = -1 \neq 0 \implies$ system is observable.



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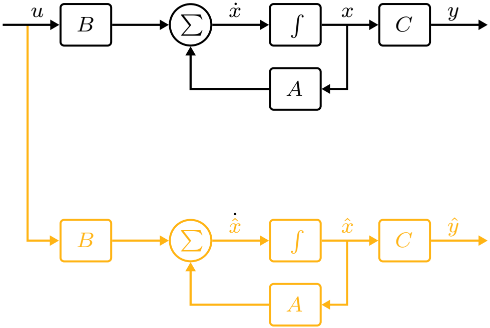

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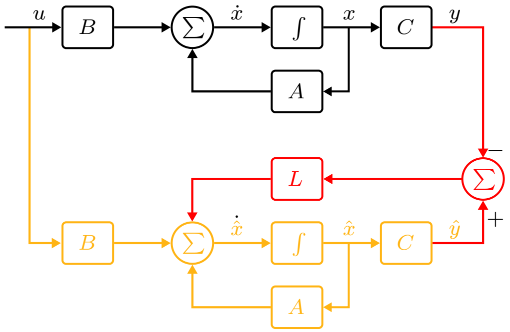
Full Order Observer

The Full Order Observer (1)



System:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

Observer:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\ \hat{y} &= C\hat{x} \end{aligned}$$

System:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

Observer:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\ \hat{y} &= C\hat{x} \end{aligned}$$

Error, $e = \hat{x} - x$:

$$\dot{e} = \dot{\hat{x}} - \dot{x} = A\hat{x} + Bu + L(C\hat{x} - y) - (Ax + Bu)$$

Full Order Observer

The Full Order Observer (2)

System:
$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

Observer:
$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\ \hat{y} &= C\hat{x} \end{aligned}$$

Error, $e = \hat{x} - x$:

$$\begin{aligned} \dot{e} = \dot{\hat{x}} - \dot{x} &= A\hat{x} + Bu + L(C\hat{x} - y) - (Ax + Bu) \\ &= A(\hat{x} - x) + L(C\hat{x} - Cx) \end{aligned}$$

System:
$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

Observer:
$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\ \hat{y} &= C\hat{x} \end{aligned}$$

Error, $e = \hat{x} - x$:

$$\begin{aligned} \dot{e} = \dot{\hat{x}} - \dot{x} &= A\hat{x} + Bu + L(C\hat{x} - y) - (Ax + Bu) \\ &= A(\hat{x} - x) + L(C\hat{x} - Cx) \\ &= (A + LC)(\hat{x} - x) = (A + LC)e \end{aligned}$$

THEOREM. A full order observer for the system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

with observer gain L is stable, if and only if the eigenvalues of the matrix $A + LC$ all have negative real part.

Moreover, such an L always exists, if (A, C) is observable.

Observer Design

Example: Observer Pole Assignment

We consider again the system

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} x + \begin{bmatrix} 2 \\ 3 \end{bmatrix} u \\ y &= \begin{bmatrix} -3 & 2 \end{bmatrix} x\end{aligned}$$

for which we would like to assign observer poles to $\{-4, -5\}$, i.e. to design L such that $A + LC$ has eigenvalues in $\{-4, -5\}$.

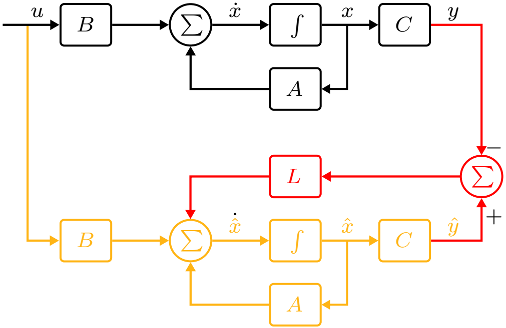
We consider again the system

$$\dot{x} = \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} x + \begin{bmatrix} 2 \\ 3 \end{bmatrix} u$$
$$y = \begin{bmatrix} -3 & 2 \end{bmatrix} x$$

for which we would like to assign observer poles to $\{-4, -5\}$, i.e. to design L such that $A + LC$ has eigenvalues in $\{-4, -5\}$.

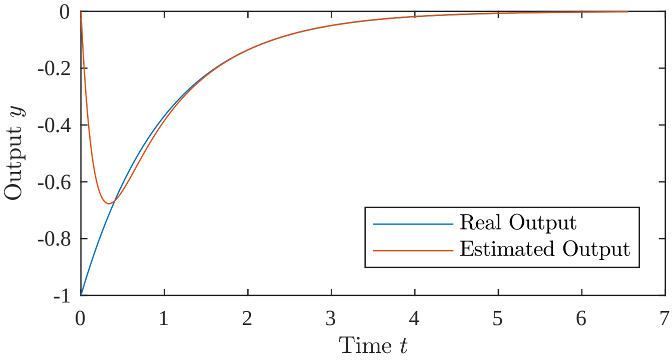
We can use `place` in MATLAB to compute L.

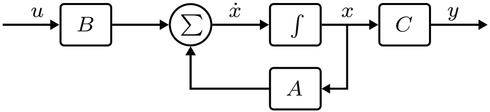
Observer Design
The Full Order Observer



Observer Design

Example: Observer Pole Assignment

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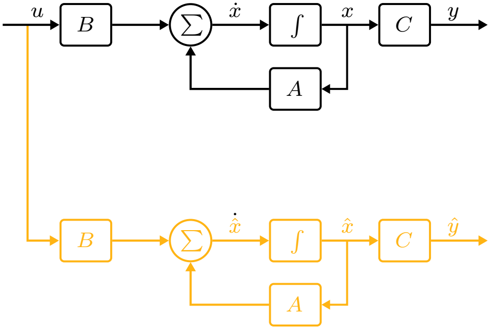
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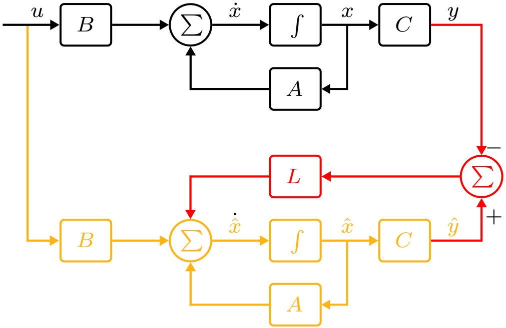
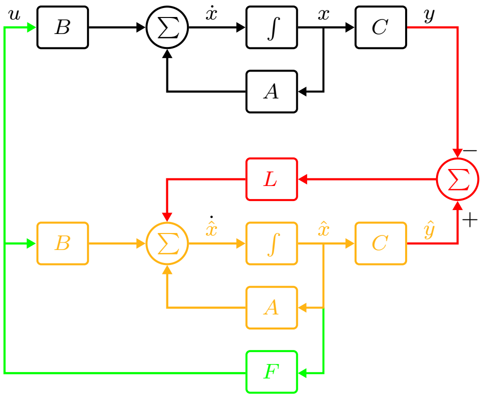
Observer Based Control

Observer Based Control (1)



Observer Based Control

Observer Based Control (1)



System:
$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

Observer:
$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\ \hat{y} &= C\hat{x} \end{aligned}$$

Feedback:
$$u = F\hat{x}$$

System:
$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

Observer:
$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\ \hat{y} &= C\hat{x} \end{aligned}$$

Feedback: $u = F\hat{x}$

Error, $e = \hat{x} - x$:

System: $\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$

Observer: $\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\ \hat{y} &= C\hat{x} \end{aligned}$

Feedback: $u = F\hat{x}$

Error, $e = \hat{x} - x$:

$$\begin{aligned} \dot{e} &= \dot{\hat{x}} - \dot{x} \\ &= A\hat{x} + BF\hat{x} + L(C\hat{x} - y) - (Ax + BF\hat{x}) \end{aligned}$$

System:
$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx\end{aligned}$$

Observer:
$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y)\\ \hat{y} &= C\hat{x}\end{aligned}$$

Feedback: $u = F\hat{x}$

Error, $e = \hat{x} - x$:

$$\begin{aligned}\dot{e} &= \dot{\hat{x}} - \dot{x}\\ &= A\hat{x} + BF\hat{x} + L(C\hat{x} - y) - (Ax + BF\hat{x})\\ &= A(\hat{x} - x) + L(C\hat{x} - Cx)\end{aligned}$$

System:
$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

Observer:
$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\ \hat{y} &= C\hat{x} \end{aligned}$$

Feedback: $u = F\hat{x}$

Error, $e = \hat{x} - x$:

$$\begin{aligned} \dot{e} &= \dot{\hat{x}} - \dot{x} \\ &= A\hat{x} + BF\hat{x} + L(C\hat{x} - y) - (Ax + BF\hat{x}) \\ &= A(\hat{x} - x) + L(C\hat{x} - Cx) \\ &= (A + LC)(\hat{x} - x) = (A + LC)e \end{aligned}$$

Combining the two equations:

$$\begin{aligned}\dot{x} &= Ax + Bu = Ax + BF\hat{x} = Ax + BF(e + x)\\ &= (A + BF)x + BFe\end{aligned}$$

and

$$\dot{e} = (A + LC)e$$

gives:

$$\begin{bmatrix}\dot{x}\\ \dot{e}\end{bmatrix} = \begin{bmatrix}A + BF & BF\\ 0 & A + LC\end{bmatrix}\begin{bmatrix}x\\ e\end{bmatrix}$$

THEOREM. An observer based controller for the system

$$
\begin{aligned}
\dot{x} &= Ax + Bu \quad , \quad x \in \mathbb{R}^n \\
y &= Cx
\end{aligned}
$$

with observer gain L and feedback gain F results in $2n$ closed loop poles, coinciding with the eigenvalues of the two matrices:

$$A + BF \quad \text{and} \quad A + LC$$

We consider again the system

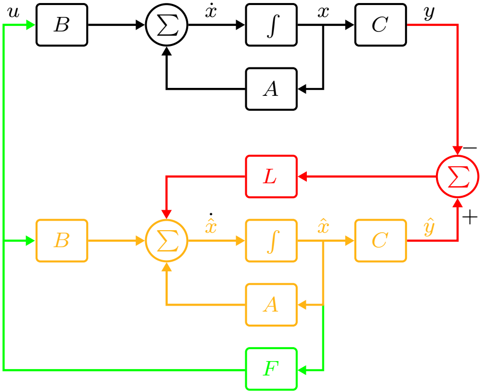
$$\begin{aligned} \dot{x} &= \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} x + \begin{bmatrix} 2 \\ 3 \end{bmatrix} u \\ y &= \begin{bmatrix} -3 & 2 \end{bmatrix} x \end{aligned}$$

for which we apply an observer based controller with

$$L = \begin{bmatrix} -6 \\ -12 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 42 & -30 \end{bmatrix}$$

Observer Based Control

Observer Based Control



The transfer function of the controller becomes:

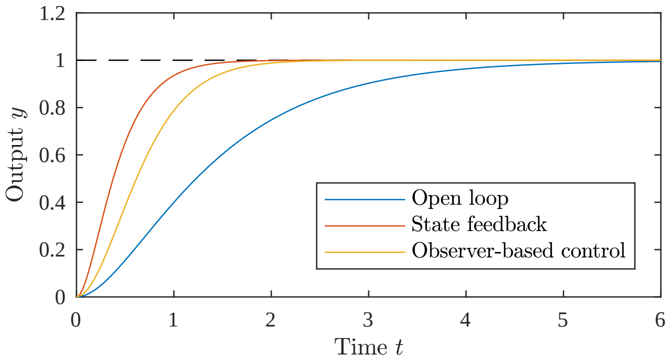
$$
\begin{aligned}
K(s) &= -F\,(sI - A - BF - LC)^{-1}\,L \\
&= -108\frac{s+\frac{7}{3}}{s^2+15s+74}
\end{aligned}
$$

The closed loop transfer function becomes:

$$
G(s)\,(I - K(s)G(s))^{-1} = \frac{s^2+15s+74}{(s+5)^2(s+4)^2}
$$

Observer Based Control

Example: Observer Based Control (3)



Integral Control

Integral Control

Augmented System Description (1)

We consider a state space system of the form:

$$\begin{array}{rcl} \dot{x} & = & Ax + Bu \\ y & = & Cx \end{array}$$

for which we wish to design a feedback law:

$$u(t) = F x(t) + F_I x_I(t)$$

where

$$x_I(t) = \int_0^t y(\tau) - r(\tau)\, d\tau$$

or

$$\dot{x}_I(t) = y(t) - r(t)$$

The equations:

$$\begin{array}{rclclcl} \dot{x} & = & Ax & + & Bu & & \\ \dot{x}_I & = & y & & & - & r \\ y & = & Cx & & & & \end{array}$$

can be combined into an extended state model:

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{x}_I \end{bmatrix} &= \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_I \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -I \end{bmatrix} r \\ y &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_I \end{bmatrix} \end{aligned}$$

Integral Control
Augmented System Description (2)

can be combined into an extended state model:

$$\begin{bmatrix} \dot{x} \\ \dot{x}_I \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_I \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -I \end{bmatrix} r$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_I \end{bmatrix}$$

for which the feedback law becomes:

$$u = Fx + F_I x_I = \begin{bmatrix} F & F_I \end{bmatrix} \begin{bmatrix} x \\ x_I \end{bmatrix}$$

Thus, the integral control problem has been reduced to a conventional state feedback problem:

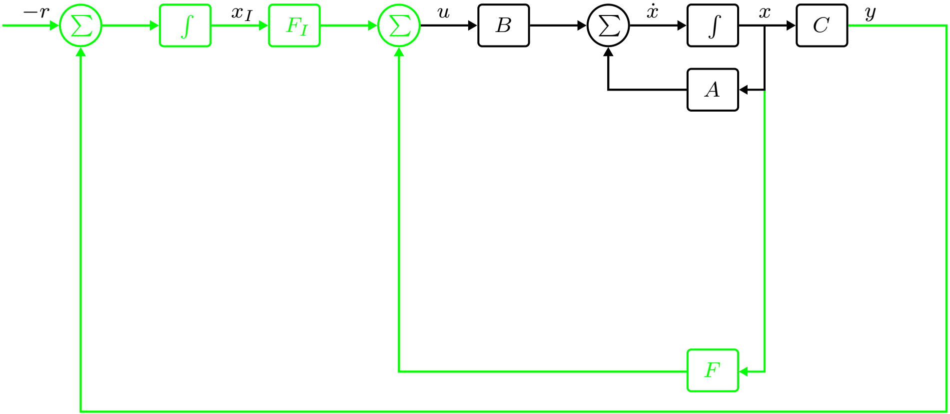
$$\begin{aligned} \dot{x}_e &= A_e x_e + B_e u \\ y &= C_e x_e \end{aligned}$$

for which we have to design a state feedback $u = F_e x_e$, where:

$$F_e = \begin{bmatrix} F & F_I \end{bmatrix}, \quad x_e = \begin{bmatrix} x \\ x_I \end{bmatrix}$$

$$A_e = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, B_e = \begin{bmatrix} B \\ 0 \end{bmatrix}, C_e = \begin{bmatrix} C & 0 \end{bmatrix}$$

Integral Control
Block Diagram



If the states are unavailable for feedback, they can be estimated by e.g. a full order observer:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\ \hat{y} &= C\hat{x} \end{aligned}$$

where L is chosen such that $A + LC$ is stable with desirable eigenvalues.

If the states are unavailable for feedback, they can be estimated by e.g. a full order observer:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\ \hat{y} &= C\hat{x} \end{aligned}$$

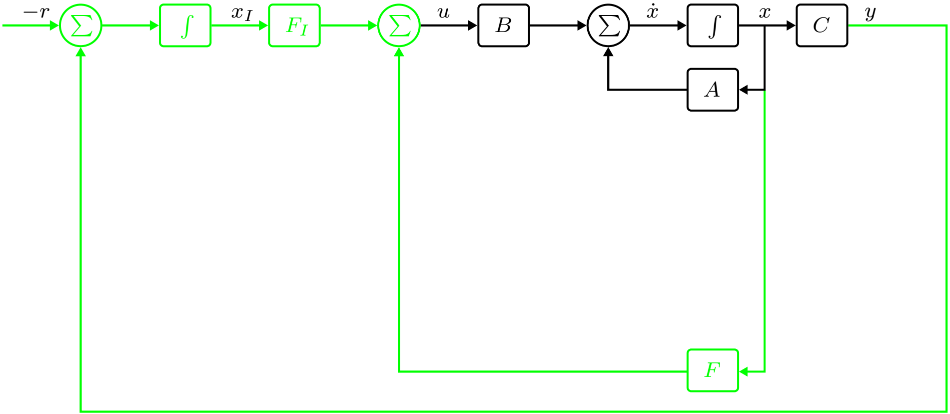
where L is chosen such that $A + LC$ is stable with desirable eigenvalues.

Separation result: The closed loop poles of such an observer based integral control scheme consist of the eigenvalues of

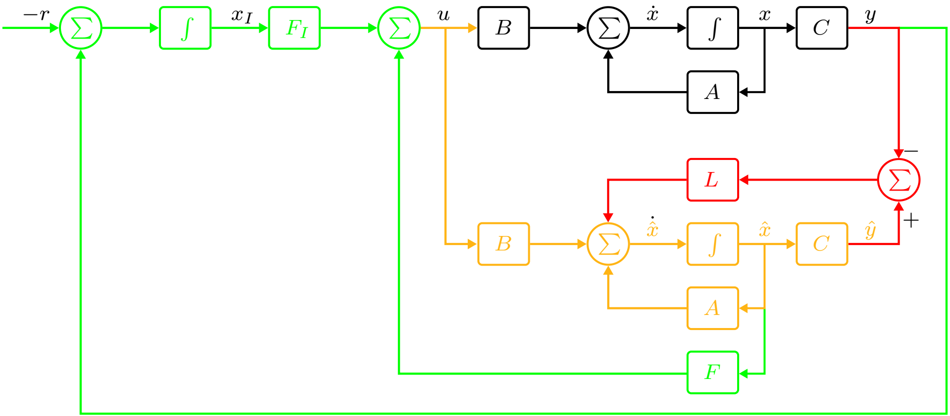
$$A_e + B_e F_e \quad \text{and of} \quad A + LC$$

Integral Control

Block Diagram of Observer-Based Integral Control



Integral Control
Block Diagram of Observer-Based Integral Control



We consider again the system

$$\begin{aligned}\dot{x} &= \begin{bmatrix}2 & -3\\ 4 & -5\end{bmatrix} x + \begin{bmatrix}2\\ 3\end{bmatrix} u\\ y &= \begin{bmatrix}-3 & 2\end{bmatrix} x\end{aligned}$$

for which we have already computed an observer gain assigning poles in $\{-4, -5\}$:

$$L = \begin{bmatrix}-6\\ -12\end{bmatrix}$$

The extended system becomes:

$$A_e = \left[\begin{array}{c|c} A & 0 \\ \hline C & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 4 & -5 & 0 \\ \hline -3 & 2 & 0 \end{array}\right]$$

$$B_e = \left[\begin{array}{c} B \\ \hline 0 \end{array}\right] = \left[\begin{array}{c} 2 \\ 3 \\ \hline 0 \end{array}\right]$$

$$C_e = \left[\begin{array}{c|c} C & 0 \end{array}\right] = \left[\begin{array}{cc|c} -3 & 2 & 0 \end{array}\right]$$

Using e.g. controllable canonical form, an extended state feedback can be found, which assigns poles in $\{-3, -4, -5\}$:

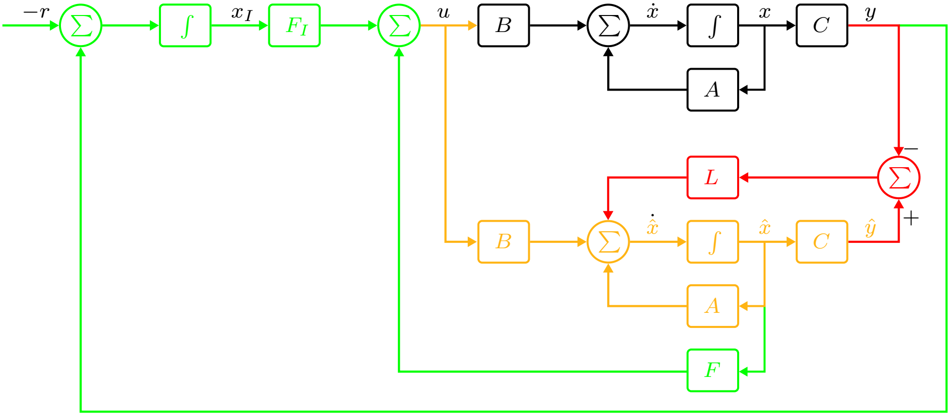
$$F_e = \begin{bmatrix} 117 & -81 & -60 \end{bmatrix}$$
$$\Rightarrow \; F = \begin{bmatrix} 117 & -81 \end{bmatrix}, \quad F_I = -60$$

The resulting controller can be shown to have the transfer function:

$$-\frac{1}{6s} \cdot \frac{55s^2 + 207s + 200}{s^2 + 18s + 119}$$

Integral Control

Example (4)



Integral Control
Example (5)