Problem 1:

Calculate $f_{223}(x,y,z)$, $f_{232}(x,y,z)$, and $f_{322}(x,y,z)$ for the function $f(x,y,z) = e^{x-2y+3z}$.

Solution:

$$f_{223}(x,y,z) = \frac{\partial}{\partial z}\frac{\partial}{\partial y}\frac{\partial}{\partial y}e^{x-2y+3z} = 12e^{x-2y+3z}$$

$$f_{232}(x,y,z) = \frac{\partial}{\partial y}\frac{\partial}{\partial z}\frac{\partial}{\partial y}e^{x-2y+3z} = 12e^{x-2y+3z}$$

$$f_{322}(x,y,z) = \frac{\partial}{\partial y}\frac{\partial}{\partial y}\frac{\partial}{\partial z}e^{x-2y+3z} = 12e^{x-2y+3z}$$

It is the same!

Problem 2:

If $z = \sin(x^2 y)$, where $x = st^2$ and $y = s^2 + \frac{1}{t}$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$

(a) by direct substitution and the single-variable form of the chain rule, and

(b) by using (two-variable) chain rule.

Solution:

(a) By direct substitution:

$$z = \sin\left((st^2)^2\left(s^2 + \frac{1}{t}\right)\right)$$

$$\frac{\partial z}{\partial s} = (4s^3t^4 + 2st^3)\cos(s^4t^4 + s^2t^3)$$

$$\frac{\partial z}{\partial t} = (4s^4t^3 + 3s^2t^2)\cos(s^4t^4 + s^2t^3)$$

(b) Using the chain rule:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (4s^3t^4 + 2st^3)\cos(s^4t^4 + s^2t^3)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (4s^4t^3 + 3s^2t^2)\cos(s^4t^4 + s^2t^3)$$

I beleive the solution to be correct:)

Problem 3:

Find the rate of change of $f(x, y) = y^4 + 2xy^3 + x^2 + y^2$ at $(0,1)$ measured in each of the following directions:

(a) $\mathbf{i} + 2\mathbf{j}$, (b) $\mathbf{j} - 2\mathbf{i}$, (c) $3\mathbf{i}$, (d) $\mathbf{i} + \mathbf{j}$.

Solution:

We calculate.

$$\nabla f(x, y) = (2y^3 + 2xy^2)\mathbf{i} + (4y^3 + 6xy^2 + 2x^2y)\mathbf{j},$$

$$\nabla f(0,1) = 2\mathbf{i} + 4\mathbf{j}$$

(a) The unit vector in the direction of $\mathbf{i} + 2\mathbf{j}$ is $\frac{\mathbf{i}+2\mathbf{j}}{\sqrt{5}}$.

$$\frac{\mathbf{i} + 2\mathbf{j}}{\sqrt{5}} \cdot (2\mathbf{i} + 4\mathbf{j}) = 2\sqrt{5}$$

(b) $\frac{-2\mathbf{i}+\mathbf{j}}{\sqrt{5}} \cdot (2\mathbf{i} + 4\mathbf{j}) = 0$

(c) The unit vector in the direction of $3\mathbf{i}$ is just $\mathbf{i}$.

$$\mathbf{i} \cdot (2i + 4j) = 2$$

(d) The unit vector in the direction of $\mathbf{i} + \mathbf{j}$ is $\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}$.

$$\frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}} \cdot (2\mathbf{i} + 4\mathbf{j}) = 3\sqrt{2}$$

Problem 4:

Find the Jacobian matrix $D\mathbf{f}(1,0)$ for the transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ given by

$$\mathbf{f}(\underset{x_1}{x},\underset{x_2}{y}) = (\underbrace{xe^y + \cos(\pi y)}_{f_1}, \underbrace{x^2}_{f_2}, \underbrace{x - e^y}_{f_3})$$

And use it to find an approximate value for $\mathbf{f}(1.02, 0.01)$.

Solution:

$$d\mathbf{f} = D\mathbf{f}(1,0)d\mathbf{x} = \begin{pmatrix} 0.03 \\ 0.04 \\ 0.01 \end{pmatrix}$$

Therefore, $\mathbf{f}(1.02, 0.01) \approx (2.03, 2.04, 0.01)$.

$$J(a,b) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}\Big|_{(a,b)} & \frac{\partial f_1}{\partial x_2}\Big|_{(a,b)} & \cdots & \frac{\partial f_1}{\partial x_n}\Big|_{(a,b)} \\ \frac{\partial f_2}{\partial x_1}\Big|_{(a,b)} & \frac{\partial f_2}{\partial x_2}\Big|_{(a,b)} & \cdots & \frac{\partial f_2}{\partial x_n}\Big|_{(a,b)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}\Big|_{(a,b)} & \frac{\partial f_m}{\partial x_2}\Big|_{(a,b)} & \cdots & \frac{\partial f_m}{\partial x_n}\Big|_{(a,b)} \end{bmatrix}$$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{bmatrix} = \begin{bmatrix} e^y & xe^y - \sin(\pi y)\pi \\ 2x & 0 \\ 1 & -e^y \end{bmatrix}$$

$$J\Big|_{(1,0)} = \begin{bmatrix} e^0 & 1\cdot e^0 - \sin(\pi)\cdot\pi \\ 2\cdot 1 & 0 \\ 1 & -e^0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{bmatrix}$$

$$\left(J\Big|_{(1,0)}\right) \times \begin{bmatrix} 1.02 \\ 0.01 \end{bmatrix} = \begin{pmatrix} 1.03 \\ 2.04 \\ 1.01 \end{pmatrix}$$