

Opgave 1.1 (Fourierrækker)

Find Fourierkoefficienterne og Fourierrækken for firkant-signalet defineret som

$$f(x) = \begin{cases} 0 & \text{hvis } -1 \leq x < 0 \\ 1 & \text{hvis } 0 \leq x < 1 \end{cases}$$

og

$$f(x+2) = f(x)$$

Løsning:

$$f(x) = \frac{1}{2} + \sum_{n=1,3,5,\dots} \frac{2}{n\pi} \sin(n\pi x)$$

$$-1 \leq x < 1 \Rightarrow T=2 \Rightarrow L=1$$

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^L f(x) \cdot \cos\left(\frac{n\pi x}{L}\right) dx = \int_{-1}^0 \underbrace{f(x)}_0 \cdot \cos(n\pi x) dx + \int_0^1 \underbrace{f(x)}_1 \cdot \cos(n\pi x) dx \\ &= \int_0^1 \cos(n\pi x) dx = \int_{x=0}^{x=1} \cos(u) \cdot \frac{1}{n\pi} du = \left[\frac{1}{n\pi} \cdot \sin(u) \right]_{x=0}^{x=1} = \left[\frac{1}{n\pi} \cdot \sin(n\pi x) \right]_0^1 \\ &\quad u = n\pi x \Rightarrow \frac{du}{dx} = n\pi \Rightarrow dx = \frac{1}{n\pi} du \\ &= \frac{2}{n\pi} \cdot \underbrace{\sin(n\pi)}_0 - \frac{2}{n\pi} \cdot \underbrace{\sin(0)}_0 = 0 \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{L} \int_{-L}^L f(x) \cdot \sin\left(\frac{n\pi x}{L}\right) dx = \int_{-1}^1 f(x) \cdot \sin(n\pi x) dx \\ &= \int_{-1}^0 \underbrace{f(x)}_0 \cdot \sin(n\pi x) dx + \int_0^1 \underbrace{f(x)}_1 \cdot \sin(n\pi x) dx = \int_0^1 \sin(n\pi x) dx \\ &\quad \frac{du}{dx} = n\pi \Rightarrow dx = \frac{1}{n\pi} \\ &= \int_{x=0}^{x=1} \sin(u) \cdot \frac{1}{n\pi} du = \frac{1}{n\pi} \cdot \left[-\cos(u) \right]_{x=0}^{x=1} = \frac{1}{n\pi} \cdot \left[\cos(n\pi) \right]_0^1 = \frac{1}{n\pi} \cdot (\underbrace{\cos(\pi)}_{-1} + \underbrace{\cos(0)}_1) = \frac{2}{n\pi} \end{aligned}$$

$$a_0 = \int_{-1}^1 f(x) \cdot \cos(0) dx = \int_{-1}^1 f(x) dx = \int_0^1 1 dx = \left[x \right]_0^1 = 1$$

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \cos\left(\frac{n\pi x}{L}\right) + b_n \cdot \sin\left(\frac{n\pi x}{L}\right) \right) = \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{2}{n\pi} \cdot \sin(n\pi x) \right)$$

Opgave 1.2 (Fouriertransformation)

Enhedstrinfunktionen defineret som

$$u(t-a) = \begin{cases} 1 & \text{for } t-a > 0 \\ 0 & \text{for } t-a < 0 \end{cases}$$

benyttes til at definere en firkantimpuls

$$x(t) = u(t-a) - u(t-b)$$

hvor $a < b$.

- Tegn grafen for firkantimpulsen $x(t)$
- Udregn den Fouriertransformerede af $x(t)$

Løsning:

$$X(\omega) = \frac{e^{-j\omega a} - e^{-j\omega b}}{j\omega}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt = \int_{-\infty}^a \underbrace{x(t)}_0 \cdot e^{-j\omega t} dt + \int_a^b \underbrace{x(t)}_1 \cdot e^{-j\omega t} dt + \int_b^{\infty} \underbrace{x(t)}_0 \cdot e^{-j\omega t} dt$$

$$= \int_a^b e^{-j\omega t} dt = \int_{t=a}^{t=b} e^{u \cdot \frac{-1}{j\omega}} du = \left[e^{u \cdot \frac{-1}{j\omega}} \right]_{t=a}^{t=b} = \left[e^{-j\omega t \cdot \frac{-1}{j\omega}} \right]_a^b$$

$u = -j\omega t \Rightarrow \frac{du}{dt} = -j\omega \Rightarrow dt = \frac{-1}{j\omega} du$

$$= \frac{-e^{-j\omega b} + e^{-j\omega a}}{j\omega} = \underline{\underline{\frac{e^{-j\omega a} - e^{-j\omega b}}{j\omega}}}$$

Opgave 1.3 (Invers Fouriertransformation)

Betragt signalet (i frekvensdomæne) $X(\omega) = \delta(\omega - \omega_0)$ hvor ω_0 er en konstant. Benyt invers Fouriertransformation til at finde $x(t)$. **Løsning:**

$$x(t) = \frac{1}{2\pi} e^{j\omega_0 t}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \left(\int_{-\infty}^{\omega_0^-} \underbrace{X(\omega)}_0 \cdot e^{j\omega t} d\omega + \int_{\omega_0^-}^{\omega_0^+} X(\omega) \cdot e^{j\omega t} d\omega + \int_{\omega_0^+}^{\infty} \underbrace{X(\omega)}_0 \cdot e^{j\omega t} d\omega \right) \\ &= \frac{1}{2\pi} \int_{\omega_0^-}^{\omega_0^+} X(\omega) \cdot e^{j\omega t} d\omega = \frac{1}{2\pi} \cdot \int_{\omega_0^-}^{\omega_0^+} \underbrace{X(\omega)}_1 \cdot e^{j\omega_0 t} = \underline{\underline{\frac{1}{2\pi} \cdot e^{j\omega_0 t}}} \end{aligned}$$

Opgave 1.4 (Fouriertransformation)

Betragt signalet (i tidsdomæne) $x(t) = \sin(\omega_0 t)$ hvor ω_0 er en konstant. Benyt Fouriertransformation til at finde $X(\omega)$. **Løsning:**

$$X(\omega) = -j\pi(\delta(\omega - \omega_0) - \delta(\omega + \omega_0))$$

Signal	Fouriertransform
$\delta(t)$	1
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$
$\delta(t - t_0)$	$e^{-j\omega t_0}$
$\sin(\omega_0 t)$	$-j\pi(\delta(\omega - \omega_0) - \delta(\omega + \omega_0))$
$\cos(\omega_0 t)$	$\pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$
1	$2\pi\delta(\omega)$

$$\underline{X(\omega) = \mathcal{F}[\sin(\omega_0 t)] = -j\pi(\delta(\omega - \omega_0) - \delta(\omega + \omega_0))}$$