Problem 1:

$$\iiint xy^2\cos(z)\;dydzdx \quad R:\left\{x,y,z\,\middle|\,\begin{matrix}0\le x\le 2\\ 0\le y\le 3\;and\;0\le z\le \pi/2\end{matrix}\right.$$

Answer: 18

$$\int_0^2\int_0^{\frac{\pi}{2}}\int_0^3 xy^2\cdot\cos(z)\;dy\,dz\,dx = \int_0^2\int_0^{\frac{\pi}{2}}\left[x\cdot\cos(z)\cdot\frac{y^3}{3}\right]_0^3 dz\,dx$$

$$= \int_0^2\int_0^{\frac{\pi}{2}} x\cdot\cos(z)\cdot\frac{3^3}{3}\;dz\,dx = 9\cdot\int_0^2\int_0^{\frac{\pi}{2}} x\cdot\cos(z)\;dz\,dx$$

$$= 9\cdot\int_0^2\left[x\cdot\sin(z)\right]_0^{\frac{\pi}{2}} dx = 9\cdot\int_0^2 x\;dx = 9\cdot\left[\frac{x^2}{2}\right]_0^2 = 9\cdot\frac{4}{2} = \underline{\underline{18}}$$

Problem 2:

$$\iiint 2\sqrt{y}e^{-x^2}\,dz\,dx\,dy \quad R:\left\{x,y,z \,\middle|\, \begin{matrix} 0\le x\le 1 \\ 0\le y\le 4 \text{ and } 0\le z\le x\end{matrix}\right.$$

Answer: $\frac{16}{3}(1-e^{-1})$

$$\int_0^4\int_0^1\int_0^x 2\sqrt{y}\,e^{-x^2}\,dz\,dx\,dy = 2\cdot\int_0^4\sqrt{y}\int_0^1 e^{-x^2}\int_0^x 1\,dz\,dx\,dy$$

$$= 2\cdot\int_0^4\sqrt{y}\int_0^1 e^{-x^2}\left[z\right]_0^x\,dx\,dy = 2\cdot\int_0^4\sqrt{y}\int_0^1 e^{-x^2}\cdot x\,dx\,dy$$

$$u=-x^2 \Rightarrow \frac{du}{dx} = -2x \Rightarrow dx = \frac{-1}{2x}\,du$$

$$= 2\cdot\int_0^4\sqrt{y}\int_{x=0}^{x=1} e^u\cdot\cancel{x}\cdot\frac{-1}{2\cancel{x}}\,du\,dy = -\int_0^4\sqrt{y}\int_{x=0}^{x=1} e^u\,du\,dy$$

$$= -\int_0^4\sqrt{y}\cdot\left[e^u\right]_{x=0}^{x=1}\,dy = -\int_0^4\sqrt{y}\cdot\left[e^{-x^2}\right]_0^1\,dy = -\int_0^4\sqrt{y}\cdot\left(e^{-1}-e^{-0}\right)dy$$

$$= -\left(e^{-1}-1\right)\cdot\int_0^4\sqrt{y}\,dy = \left(1-e^{-1}\right)\cdot\int_0^4 y^{\frac{1}{2}}dy = \left(1-e^{-1}\right)\cdot\int_0^4 y^{\frac{1}{2}}dy$$

$$= \left(1-e^{-1}\right)\cdot\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4 = \left(1-e^{-1}\right)\cdot\left[\frac{2}{3}\cdot y^{\frac{3}{2}}\right]_0^4 = \left(1-e^{-1}\right)\cdot\frac{2}{3}\left[\left(\sqrt{y}\right)^3\right]_0^4$$

$$= \left(1-e^{-1}\right)\cdot\frac{2}{3}\cdot\left(\left(\sqrt{4}\right)^3-\cancel{\left(\sqrt{0}\right)^3}\right) = \left(1-e^{-1}\right)\cdot\frac{2}{3}\cdot 2^3 = \underline{\left(1-e^{-1}\right)\cdot\frac{16}{3}}$$

Problem 3:

$$\iiint 1 + 2x - 3y \, dxdydz \qquad R:\left\{x,y,z \,\middle|\, \begin{matrix} -a \le x \le a \\ -b \le y \le b \text{ and } -c \le z \le c \end{matrix}\right.$$

Answer: 8abc

$$\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a} 1+2x-3y \, dx\,dy\,dz = \int_{-c}^{c}\int_{-b}^{b}\left[x+x^2-3yx\right]_{-a}^{a} dy\,dz$$

$$= \int_{-c}^{c}\int_{-b}^{b} a+a^2-3ya-\left(-a+(-a)^2-3y(-a)\right) dy\,dz$$

$$= \int_{-c}^{c}\int_{-b}^{b} a+\cancel{a^2}-3ya+a-\cancel{a^2}-3ay \, dy\,dz = \int_{-c}^{c}\int_{-b}^{b} 2a-6ay \, dy\,dz$$

$$= \int_{-c}^{c}\left[2ay-3ay^2\right]_{-b}^{b} dz = \int_{-c}^{c} 2ab-3ab^2-\left(2a(-b)-3a(-b)^2\right) dz$$

$$= \int_{-c}^{c} 2ab-\cancel{3ab^2}+2ab+\cancel{3ab^2} \, dz = \int_{-c}^{c} 4ab \, dz = \left[4abz\right]_{-c}^{c}$$

$$= 4abc - 4ab(-c) = \underline{\underline{8abc}}$$

Problem 4:

Find triple integral of function $\frac{1}{(x+y+z)3} dxdydz$ when R is bounded by 6 planes $z = 1, z = 2, y = 0, y = z, x = 0$, and $x = y + z$.

Answer: $\frac{3}{16}\ln 2$.

$$\int_1^2\int_0^z\int_0^{y+z} \frac{1}{(x+y+z)^3}\,dx\,dy\,dz = \int_1^2\int_0^z\int_0^{y+z} u\cdot\frac{1}{(x+y+z)^2}\cdot\left(-(x+y+z)^2\right)du\,dy\,dx$$

$$u=\frac{1}{x+y+z} \Rightarrow \frac{du}{dx} = \frac{-1}{(x+y+z)^2}\cdot 1 \Rightarrow dx = -(x+y+z)^2\,du$$

$$= \int_1^2\int_0^z\int_0^{y+z} u\cdot(-1)\,du\,dy\,dx = -\int_1^2\int_0^z\int_0^{y+z} u\,du\,dy\,dx = -\int_1^2\int_0^z\left[\frac{u^2}{2}\right]_0^{y+z} dy\,dx$$

$$= \frac{-1}{2}\int_1^2\int_0^z\left[u^2\right]_0^{y+z} dy\,dx = \frac{-1}{2}\int_1^2\int_0^z\left[\left(\frac{1}{x+y+z}\right)^2\right]_0^{y+z} dy\,dz = \frac{-1}{2}\int_1^2\int_0^z\left[\frac{1}{(x+y+z)^2}\right]_0^{y+z} dy\,dz$$

$$= \frac{-1}{2}\int_1^2\int_0^z \frac{1}{(y+z+y+z)^2} - \frac{1}{(y+z)^2}\,dy\,dz = \frac{-1}{2}\int_1^2\int_0^z\frac{1}{(2y+2z)^2} - \frac{1}{(y+z)^2}\,dy\,dz$$

$$\frac{1}{(2y+2z)^2} = \frac{1}{(2\cdot(y+z))^2} = \frac{1}{4\cdot(y+z)^2} = \frac{1}{4}\cdot\frac{1}{(y+z)^2}$$

$$= \frac{-1}{2}\int_1^2\int_0^z \frac{1}{4}\cdot\frac{1}{(y+z)^2} - \frac{1}{(y+z)^2}\,dy\,dz$$

$$= \frac{-1}{2}\int_1^2\int_0^z \frac{-3}{4}\cdot\frac{1}{(y+z)^2}\,dy\,dz = \frac{3}{8}\int_1^2\int_0^z\frac{1}{(y+z)^2}\,dy\,dz = \frac{3}{8}\int_1^2\int_{y=0}^{y=z} u^{-2}\,du\,dz$$

$$u = y+z \Rightarrow du = dy$$

$$= \frac{3}{8}\int_1^2\left[-u^{-1}\right]_{y=0}^{y=z} dz = \frac{-3}{8}\int_1^2\left[(y+z)^{-1}\right]_0^z dz = \frac{-3}{8}\int_1^2 (z+z)^{-1} - (0+z)^{-1}\,dz = \frac{-3}{8}\int_1^2\frac{1}{2z} - \frac{1}{z}\,dz$$

$$= \frac{\cancel{-}3}{8}\int_1^2\frac{\cancel{-}1}{z}\cdot\left(\frac{-1}{2}+1\right)dz = \frac{3}{8}\int_1^2\frac{1}{z}\cdot\frac{1}{2}\,dz = \frac{3}{16}\int_1^2\frac{1}{z}\,dz = \frac{3}{16}\cdot\left[\ln(z)\right]_1^2 = \frac{3}{16}\cdot\left(\ln(2)\overbrace{-\ln(1)}^{0}\right) = \underline{\underline{\frac{3}{16}\cdot\ln(2)}}$$

Problem 5:

Find determinant of the Jacobian for x=2u+w y=$-v^2$ *and* $z = u + v^2 - 2w^2$

Answer: 16vw+~~4u~~v+2v

$$f(u, v, w) = (\overset{f_1}{2u+w}, \overset{f_2}{-v^2}, \overset{f_3}{u+v^2-2w^2})$$

$$J = \begin{bmatrix} \frac{f_1}{\partial u} & \frac{f_1}{\partial v} & \frac{f_1}{\partial w} \\ \frac{f_2}{\partial u} & \frac{f_2}{\partial v} & \frac{f_2}{\partial w} \\ \frac{f_3}{\partial u} & \frac{f_3}{\partial v} & \frac{f_3}{\partial w} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -2v & 0 \\ 1 & 2v & -4w \end{bmatrix}$$

$$|J| = 2(8vw - 0) - 0 + 1 \cdot (0 - (-2v)) = \underline{\underline{16vw + 2v}}$$

I choose to believe that this is the correct answer!

Problem 6:

Find the volume of solid bounded by $\underbrace{x^2+y^2+z^2=9}$ *and* $\underbrace{x^2+y^2=8z}$

Answer: $\frac{40}{3}\pi$

Convert to cylindrical coordinates

$$\begin{aligned} x &\to r\cdot\cos\theta \\ y &\to r\sin\theta \\ z &\to z \\ \mathrm{dxdydz} &\to r\cdot \mathrm{d\theta drdz} \\ x^2+y^2=a^2 &\to r=a \end{aligned}$$

$$x^2+y^2=8z \Rightarrow \underbrace{r^2\cdot\cos^2(\theta)+r^2\cdot\sin^2(\theta)}_{r^2(\cos^2(\theta)+\sin^2(\theta))}=8z \Rightarrow r^2=8z \Rightarrow z=\frac{r^2}{8}$$

$$x^2+y^2+z^2=9 \Rightarrow r^2+z^2=9 \Rightarrow z^2=9-r^2 \Rightarrow z=\sqrt{9-r^2}$$

$$\Rightarrow \int_0^{2\pi}\int_?^?\int_{\frac{r^2}{8}}^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta$$

Try to find the limits for r

$$\begin{cases} z=\frac{r^2}{8} \\ z=\sqrt{9-r^2}\end{cases} \Rightarrow \left(\frac{r^2}{8}=\sqrt{9-r^2}\right)^2 \Rightarrow \frac{1}{64}r^4=9-r^2 \Rightarrow \frac{1}{64}r^4+r^2=9 \Rightarrow r^4+64r^2-9\cdot64=0$$

$$9\cdot64=8\cdot72$$

$$\Rightarrow r^4+64r^2-9\cdot64=0 \Rightarrow (r^2-8)(r^2+72)=0$$

$$\Rightarrow \begin{cases} r^2-8=0 \Rightarrow r^2=8 \Rightarrow r_1=\sqrt{8} \\ r^2+72=0 \Rightarrow r^2=-72 \Rightarrow \underbrace{r_2=\sqrt{-72}}_{\text{Imaginary part is ignored}}\end{cases} \Rightarrow r=\sqrt{8}$$

This is the boundary for r.

Rewriting the integral

$$\int_0^{2\pi}\int_0^{\sqrt{8}}\int_{\frac{r^2}{8}}^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{8}}\Big[rz\Big]_{\frac{r^2}{8}}^{\sqrt{9-r^2}}dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{8}} r\sqrt{9-r^2}-\frac{1}{8}r^3\,dr\,d\theta$$

From table:

$$\int x\sqrt{-x^2\pm a^2}\,\mathrm{dx}=-\frac{1}{3}(x^2\pm a^2)^{3/2}$$

$$\int_0^{2\pi}\left[-\frac{1}{3}(9-r^2)^{\frac{3}{2}}-\frac{1}{32}r^4\right]_0^{\sqrt{8}}d\theta = \int_0^{2\pi}-\frac{1}{3}(9-(\sqrt{8})^2)^{\frac{3}{2}}-\frac{1}{32}(\sqrt{8})^4+\frac{1}{3}(9-0^2)^{\frac{3}{2}}+\cancel{\frac{1}{32}\cdot0^4}\,d\theta$$

$$=\int_0^{2\pi}-\frac{1}{3}(9-8)^{\frac{3}{2}}-\frac{1}{32}\cdot8^2+\frac{1}{3}\cdot9^{\frac{3}{2}}\,d\theta = \int_0^{2\pi}-\frac{1}{3}\cdot(\sqrt{1})^3-\frac{64}{32}+\frac{1}{3}\cdot(\sqrt{9})^3\,d\theta$$

$$=\int_0^{2\pi}-\frac{1}{3}-2+(\sqrt{9})^2\,d\theta = \int_0^{2\pi}-\frac{1}{3}-2+9\,d\theta = \int_0^{2\pi}-\frac{1}{3}+7\,d\theta$$

$$=\int_0^{2\pi}\frac{20}{3}\,d\theta = \left[\frac{20\theta}{3}\right]_0^{2\pi} = \underline{\underline{\frac{40\pi}{3}}}$$