Opgave 1 (20 point)

Beregn volumen af det område, der er beskrevet ved ulighederne:

$$
\begin{aligned}
&z \leq -x^2 - y^2 + 10\\
&z \geq x - 2y - 5\\
&x^2 + y^2 \leq 1\\
&x \leq 0
\end{aligned}
$$

$x = r \cdot \cos\theta \qquad y = r\sin\theta \qquad z = z \qquad \mathrm{dxdydz} = r \cdot \mathrm{d}\theta\mathrm{drdz}$

$x^2 + y^2 = a^2 \Rightarrow r = a$

$$
\Rightarrow \begin{cases}
z \leq -r^2 \cdot \cos^2(\theta) - r^2 \cdot \sin^2(\theta) + 10\\
z \geq r \cdot \cos(\theta) - 2r \cdot \sin(\theta) - 5\\
r \leq 1\\
r \cdot \cos(\theta) \leq 0 \Rightarrow \cos(\theta) \leq 0 \Rightarrow \frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}
\end{cases}
$$

$$
\Rightarrow \int_0^1 \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_{r\cdot\cos(\theta) - 2r\cdot\sin(\theta) - 5}^{-r^2\cdot\cos^2(\theta) - r^2\cdot\sin^2(\theta) + 10} r \, dz \, d\theta \, dr
$$

$$
= \int_0^1 \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} -r^3\cdot\cos(\theta)^2 - r^3\cdot\sin(\theta)^2 + \underline{10r} - r^2\cos(\theta) + 2r^2\sin(\theta) + 5\underline{r} \; d\theta \, dr
$$

$$
= \int_0^1 \left[-r^3\cdot\left(\frac{\theta}{2} + \frac{1}{4}\sin(2\theta)\right) - r^3\cdot\left(\frac{\theta}{2} - \frac{1}{4}\sin(2\theta)\right) - r^2\cdot\sin(\theta) \cancel{- 2r^2\cdot\cos(\theta)} + 15r\theta\right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}} dr
$$

$$
= \int_0^1 \left[-r^3\cdot\left(\frac{\theta}{2} + \frac{1}{4}\sin(2\theta)\right) - r^3\cdot\left(\frac{\theta}{2} - \frac{1}{4}\sin(2\theta)\right) - r^2\cdot\sin(\theta) + 15r\theta\right]_{\frac{-\pi}{2}}^{\frac{\pi}{2}} dr
$$

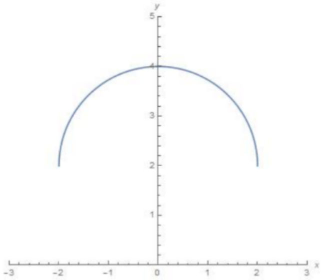
$$
\begin{aligned}
= \int_0^1 \Bigg[& -r^3\cdot\left(\frac{\frac{\pi}{2}}{2} + \cancel{\frac{1}{4}\overset{0}{\sin}\left(2\frac{\pi}{2}\right)}\right) - r^3\cdot\frac{\frac{\pi}{2}}{2} - \cancel{\frac{1}{4}\overset{0}{\sin}\left(2\cdot\frac{\pi}{2}\right)} - r^3\cdot\overset{1}{\sin}\left(\frac{\pi}{2}\right) + 15r\left(\frac{\pi}{2}\right)\\
& + r^3\cdot\left(\frac{\frac{-\pi}{2}}{2} + \cancel{\frac{1}{4}\overset{0}{\sin}\left(2\frac{-\pi}{2}\right)}\right) + r^3\cdot\left(\frac{\frac{-\pi}{2}}{2} - \cancel{\frac{1}{4}\overset{0}{\sin}\left(2\frac{-\pi}{2}\right)}\right) + r^3\cdot\overset{1}{\sin}\left(\frac{\pi}{2}\right) - 15r\left(\frac{-\pi}{2}\right)
\end{aligned}
$$

$$
= \int_0^1 \underline{-r^3\cdot\frac{\pi}{4}} - \underline{r^3\cdot\frac{\pi}{4}} \cancel{- r^3} + \frac{15\pi r}{2} \underline{- r^3\cdot\frac{\pi}{4}} - \underline{r^3\cdot\frac{\pi}{4}} + \cancel{r^3} + \frac{15\pi r}{2}
$$

$$
= \int_0^1
$$

Opgave 2 (30 point)

Betragt kurven C, der ligger i planen $z = 0$ og er den øvre del af cirklen med centrum i $(0, 2, 0)$ og har radius 2. Kurven starter i $(2, 2, 0)$ og slutter i $(-2, 2.0)$. Se skitse herunder.



a)

Beregn

$$\int_C \mathbf{F} \cdot \mathbf{dr}$$

hvor vektorfeltet $\mathbf{F}$ er givet ved

$$\mathbf{F} = \begin{pmatrix} y+z \\ x \\ x \end{pmatrix} \begin{matrix} f_1 \\ f_2 \\ f_3 \end{matrix}$$

Hint: undersøg om $\mathbf{F}$ er konservativt.

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}, \quad \frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x}, \quad \frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y},$$

$$\left.\begin{array}{l} \frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} \Rightarrow 1 + 0 = 1 \quad \checkmark \\ \frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x} \Rightarrow 0 + 1 = 1 \quad \checkmark \\ \frac{\partial f_2}{\partial z} = \frac{\partial f_3}{\partial y} \Rightarrow 0 = 0 \quad \checkmark \end{array}\right\} \Rightarrow \text{Conservative!}$$

$$\oint_C \mathbf{F} \cdot d\vec{r} = f(B) - f(A)$$

Finding potential function

$$\nabla f = \vec{F} \Rightarrow \begin{cases} \frac{\partial f}{\partial x} = y + z \\ \frac{\partial f}{\partial y} = x \\ \frac{\partial f}{\partial z} = x \end{cases} \Rightarrow \begin{cases} f = xy + xz + A(y,z) \\ f = xy + B(x,z) \\ f = xz + C(x,y) \end{cases} \Rightarrow \begin{cases} A(y,z) = 0 \\ B(x,z) = xz \\ C(x,y) = xy \end{cases}$$

$$\Rightarrow f(x,y,z) = xy + xz$$

$$\oint_C \vec{F}\, d\vec{r} = f(B) - f(A) = f(-2,2,0) - f(2,2,0) = (-2)\cdot 2 + (-2)\cdot 0 - 2\cdot 2 + 2\cdot 0 = -4 - 4 = \underline{\underline{-8}}$$

b)

Beregn

$$\int_C f(x, y, z)ds$$

hvor $f(x, y, z) = x + y + z$.

??

Opgave 3 (30 point)

Betragt overfladen S, der er parametriseret ved

$$\mathbf{r}(t) = \begin{pmatrix} u\cos(v) \\ u\sin(v) \\ u \end{pmatrix}$$

hvor $0 \leq u \leq 2$ og $0 \leq v \leq \pi$.

Betragt også vektorfeltet

$$\mathbf{F} = \begin{pmatrix} -xz \\ -yz \\ z \end{pmatrix}$$

a)

Beregn arealet af S.

b)

Beregn fluxen af $\mathbf{F}$ opad gennem S.

a) $\iint_S |\vec{r}_u \times \vec{r}_v| \text{ du dv}$

$$\vec{r}_u = \begin{bmatrix} \cos(v) \\ \sin(v) \\ 1 \end{bmatrix} \qquad \vec{r}_v = \begin{bmatrix} -u\cdot\sin(v) \\ u\cdot\cos(v) \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ -(a_1b_3 - a_3b_1) \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

$$\vec{r}_u \times \vec{r}_v = \begin{bmatrix} \cancel{\cos(v)\cdot 0} - 1\cdot u\cdot\cos(v) \\ -(\cancel{\cos(v)\cdot 0} - 1\cdot(-u\cdot\sin(v))) \\ \cos(v)\cdot u\cdot\cos(v) - \sin(v)\cdot(-u\cdot\sin(v)) \end{bmatrix}$$

$$= \begin{bmatrix} -u\cdot\cos(v) \\ -u\cdot\sin(v) \\ u\cdot\underbrace{(\cos^2(v)+\sin^2(v))}_{1} \end{bmatrix} = \begin{bmatrix} -u\cdot\cos(v) \\ -u\cdot\sin(v) \\ u \end{bmatrix}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{u^2\cdot\cos(v)^2 + u^2\cdot\sin(v)^2 + u^2}$$

$$= \sqrt{u^2\cdot\underbrace{(\cos(v)^2 + \sin(v)^2)}_{1} + u^2}$$

$$= \sqrt{u^2+u^2} = \sqrt{2u^2} = u\sqrt{2}$$

$$\int_0^\pi\int_0^2 u\sqrt{2}\, du\, dv = \sqrt{2}\cdot\int_0^\pi\int_0^2 u\, du\, dv$$

$$= \sqrt{2}\cdot\int_0^\pi\int_0^2 u\, du\, dv = \sqrt{2}\cdot\int_0^\pi \left[\frac{u^2}{2}\right]_0^2 dv$$

$$= \sqrt{2}\cdot\int_0^\pi \frac{4}{2}\, dv = \sqrt{2}\cdot\int_0^\pi 2\, dv = \sqrt{2}\cdot[2v]_0^\pi$$

$$= 2\pi\sqrt{2}$$

$$\text{flux} = \iint_S \vec{F}\cdot(\vec{r}_u \times \vec{r}_v) \text{ du dv}$$

$$\begin{bmatrix} u\cdot\cos(v) \\ u\cdot\sin(v) \\ u \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad \vec{r}_u \times \vec{r}_v = \begin{bmatrix} -u\cdot\cos(v) \\ -u\cdot\sin(v) \\ u \end{bmatrix}$$

$$\int_0^\pi\int_0^2 \begin{bmatrix} -u\cdot\cos(v)\cdot u \\ -u\cdot\sin(v)\cdot u \\ u \end{bmatrix} \cdot \begin{bmatrix} -u\cdot\cos(v) \\ -u\cdot\sin(v) \\ u \end{bmatrix} du\, dv$$

$$= \int_0^\pi\int_0^2 u^3\cdot\cos^2(v) + u^3\cdot\sin^2(v) + u^2\, du\, dv$$

$$= \int_0^\pi\int_0^2 u^3\cdot\overbrace{(\cos^2(v)+\sin^2(v))}^{1} + u^2\, du\, dv$$

$$= \int_0^\pi\int_0^2 u^3 + u^2\, du\, dv = \int_0^\pi \left[\frac{u^4}{4} + \frac{u^3}{3}\right]_0^2 dv = \int_0^\pi \frac{16}{4} + \frac{8}{3}\, dv = \left[4v + \frac{8v}{3}\right]_0^\pi = 4\pi + \frac{8\pi}{3} = \frac{20\pi}{3}$$

Opgave 4 (20 point)

Betragt den partielle differentialligning

$$u_{yy} - u_{xx} = 0$$

Klassificer ligningen, omskriv den til normal form ved at gennemføre et passende variabelskift.
Løs den fremkomne ligning, løsningen skal skrives som funktion af x og y.

NOOOOOOOOOOOOOOOPE