

①

If A is an invertible matrix, then so is  $A^T$ .

☒ 참 True

☐ 거짓 False

②

$$(A + B)^{-1} = A^{-1} + B^{-1}$$

☐ 참

☒ 거짓

③

If A and B are symmetric matrix, then AB is also symmetric.

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④

Every vector space possesses a basis

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⑤

Let  $a = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$  and  $b = \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$ . Then  $\{a, b\}$  forms a orthonormal basis of  $\mathbb{R}^2$

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Length of vectors should be 1

$$\|a\| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1 \quad \checkmark$$

$$\|b\| = \sqrt{\left(\frac{-4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = 1 \quad \checkmark$$

Check if they are perpendicular with the dot product.

$$a \cdot b = \frac{3}{5} \cdot \frac{-4}{5} + \frac{4}{5} \cdot \frac{3}{5} = 0 \quad \checkmark$$

=> a and b are orthonormal!

6 If  $\text{rank}(A) = \text{rank}(B)$ , then  $\text{rank}(A^2) = \text{rank}(B^2)$ .

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7  $\text{rank}(B^T A^T) = \text{rank}(AB)$ .

$$\text{rank}(A^T B^T) = \text{rank}((AB)^T) = \text{rank}(AB)$$

☒ 참

☐ 거짓

8 If  $A$  is symmetric and invertible, then  $A^{-1}$  is also symmetric and invertible.

☒ 참

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9 Solve the following system using the augmented matrix  $[A|b]$  (write down your answers on the PDF file).

$$\begin{aligned} -2x_1 + 4x_2 - 2x_3 - x_4 + 4x_5 &= -3 \\ 4x_1 - 8x_2 + 3x_3 - 3x_4 + x_5 &= 2 \\ x_1 - 2x_2 + x_3 - x_4 + x_5 &= 0 \\ x_1 - 2x_2 - 3x_4 + 4x_5 &= -1 \end{aligned}$$

The form of the solution would be as follows. Please find the right values for  $a$  to  $o$ .

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} + \lambda_1 \begin{bmatrix} f \\ g \\ h \\ i \\ j \end{bmatrix} + \lambda_2 \begin{bmatrix} k \\ l \\ m \\ n \\ o \end{bmatrix}$$

$$A\vec{x} = \vec{b} \Rightarrow \left[ \begin{array}{ccccc|c} -2 & 4 & -2 & -1 & 4 & -3 \\ 4 & -8 & 3 & -3 & 1 & 2 \\ 1 & -2 & 1 & -1 & 1 & 0 \\ 1 & -2 & 0 & -3 & 4 & -1 \end{array} \right] \vec{x} = \begin{bmatrix} -3 \\ 2 \\ 0 \\ -1 \end{bmatrix} \Rightarrow \left[ \begin{array}{ccccc|c} -2 & 4 & -2 & -1 & 4 & -3 \\ 4 & -8 & 3 & -3 & 1 & 2 \\ 1 & -2 & 1 & -1 & 1 & 0 \\ 1 & -2 & 0 & -3 & 4 & -1 \end{array} \right] \begin{matrix} \leftarrow 2 \\ \leftarrow -1 \\ \leftarrow -1 \end{matrix} \sim \left[ \begin{array}{ccccc|c} -2 & 4 & -2 & -1 & 4 & -3 \\ 0 & 0 & -1 & -5 & 9 & -4 \\ 1 & -2 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & -2 & 3 & -1 \end{array} \right] \begin{matrix} \\ \leftarrow \frac{1}{2} \\ \\ \leftarrow -1 \end{matrix}$$

$$\sim \left[ \begin{array}{ccccc|c} -2 & 4 & -2 & -1 & 4 & -3 \\ 0 & 0 & -1 & -5 & 9 & -4 \\ 0 & 0 & 0 & -1.5 & 3 & -1.5 \\ 0 & 0 & 0 & 3 & -6 & 3 \end{array} \right] \begin{matrix} \\ \\ \leftarrow 2 \\ \leftarrow \frac{2}{3} \end{matrix} \sim \left[ \begin{array}{ccccc|c} -2 & 4 & -2 & -1 & 4 & -3 \\ 0 & 0 & -1 & -5 & 9 & -4 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Let  $x_5 = \lambda_1$ , then

$$x_4 - 2\lambda_1 = 1 \Rightarrow x_4 = 1 + 2\lambda_1$$

$$-x_3 - 5(1 + 2\lambda_1) + 9\lambda_1 = -4 \Rightarrow x_3 = -5 - 10\lambda_1 + 9\lambda_1 + 4 \Rightarrow x_3 = -\lambda_1 - 1$$

$$-2x_1 + 4x_2 - 2(-\lambda_1 - 1) - (1 + 2\lambda_1) + 4\lambda_1 = -2x_1 + 4x_2 + 2\lambda_1 + 2 - 1 - 2\lambda_1 + 4\lambda_1 = -2x_1 + 4x_2 + 4\lambda_1 + 1 = -3$$

$$\Rightarrow -2x_1 + 4x_2 + 4\lambda_1 + 4 = 0 \Rightarrow -x_1 + 2x_2 + 2\lambda_1 + 2 = 0$$

let  $x_2 = \lambda_2$ , then

$$x_1 = 2\lambda_2 + 2\lambda_1 + 2$$

$$\begin{cases} x_1 = 2\lambda_2 + 2\lambda_1 + 2 \\ x_2 = \lambda_2 \\ x_3 = -\lambda_1 - 1 \\ x_4 = 1 + 2\lambda_1 \\ x_5 = \lambda_1 \end{cases} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 2 \\ 0 \\ -1 \\ 2 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Are the following vectors linearly independent?

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 5 \\ 7 \\ 0 \end{bmatrix}, x_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 6 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 5 & 7 & 0 \\ 2 & 0 & 1 & 6 \end{bmatrix} \xrightarrow{-2} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & -4 \\ 0 & -4 & -5 & -2 \end{bmatrix} \xrightarrow{4} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 1 & -18 \end{bmatrix} \Rightarrow \text{INDEPENDENT}$$

–Compute the inverse of the following matrices. Find the right values for  $a$  to  $m$ .

(a)  $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

(b)  $B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & -1 \\ 4 & 2 & 1 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B^{-1} = \begin{bmatrix} e & f & g \\ h & i & j \\ k & l & m \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{bmatrix} \xrightarrow{\cdot \frac{1}{2}} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{bmatrix} \xrightarrow{\cdot 2} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & -3 & 2 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 2 & 0 & 4 & -2 \\ 0 & 1 & -3 & 2 \end{bmatrix} \xrightarrow{\cdot \frac{1}{2}} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -3 & 2 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

$$\left[ \begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 3 & 2 & -1 & 0 & 1 & 0 \\ 4 & 2 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow[\leftarrow]{\begin{array}{l} -\frac{3}{2} \\ 2 \end{array}} \sim \left[ \begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array} \right] \xrightarrow{1} \sim \left[ \begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{3}{2} & 1 & 1 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array} \right] \cdot \begin{array}{l} \frac{1}{2} \\ 2 \end{array}$$

$$\sim \left[ \begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -7 & 2 & 2 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array} \right] \xrightarrow{-1} \sim \left[ \begin{array}{ccc|ccc} 2 & 0 & 0 & 8 & -2 & -2 \\ 0 & 1 & 0 & -7 & 2 & 2 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array} \right] \xrightarrow{\cdot \frac{1}{2}} \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -1 & -1 \\ 0 & 1 & 0 & -7 & 2 & 2 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array} \right] \Rightarrow B^{-1} = \left[ \begin{array}{ccc|ccc} 4 & -1 & -1 \\ -7 & 2 & 2 \\ -2 & 0 & 1 \end{array} \right]$$

Find the rank of the following matrix

$$M = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ -1 & 1 & 3 & -5 \\ -1 & 2 & 5 & -6 \\ -1 & -2 & -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ -1 & 1 & 3 & -5 \\ -1 & 2 & 5 & -6 \\ -1 & -2 & -3 & 1 \end{bmatrix} \xrightarrow{\sim} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ -1 & 1 & 3 & -5 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\sim} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ 0 & 3 & 6 & -6 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\cdot \frac{1}{3}}$$

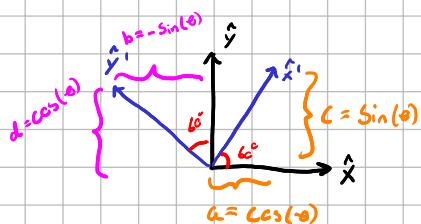
$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-2} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -5 & -10 & 10 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{5} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank}(M) = 3$$

Find the linear transformation matrix that rotates a point in a 2-dimensional space counterclockwise by 60 degrees.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\hat{x}' = R \hat{x} \Rightarrow \hat{x}' = \begin{bmatrix} a & b \\ c & a \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$

$$\hat{y}' = R \hat{y} \Rightarrow \hat{y}' = \begin{bmatrix} b \\ a \end{bmatrix}$$



$$\Rightarrow R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$R(60^\circ) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

-Find the basis of the kernel of matrix  $A$ , the dimension of the image of matrix  $A$ .

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & -3 & 2 & 0 \end{bmatrix}$$

$$\text{Dim}(\text{image}) = k$$

$$\text{Basis for the Kernel} = \left\{ \begin{bmatrix} a \\ b \\ 1 \\ c \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \\ 1 \end{bmatrix} \right\}.$$

$$\ker(A) = \{ \underbrace{x \in \mathbb{R}^4 \mid Ax = \vec{0}} \Rightarrow \left[ \begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 3 & 1 & 0 & 2 & 0 \\ 1 & -3 & 2 & 0 & 0 \end{array} \right] \begin{array}{l} \leftarrow -3 \\ \leftarrow -1 \end{array} \sim \left[ \begin{array}{cccc|c} 1 & 2 & -1 & 1 & 0 \\ 0 & -5 & 3 & -1 & 0 \\ 0 & -5 & 3 & -1 & 0 \end{array} \right] \begin{array}{l} \\ \leftarrow -1 \end{array}$$

We want to find the vectors that map to  $\vec{0}$ .

$$\sim \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 & 2 & -1 & 1 & 0 \\ 0 & -5 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 + 2x_2 - x_3 + x_4 = 0 \\ -5x_2 + 3x_3 - x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -2x_2 + x_3 - x_4 \\ x_2 = \frac{3}{5}x_3 - \frac{1}{5}x_4 \end{cases}$$

let  $\lambda_1 = x_3, \lambda_2 = x_4$

$$\Rightarrow \begin{cases} x_1 = -2x_2 + \lambda_1 - \lambda_2 \\ x_2 = \frac{3}{5}\lambda_1 - \frac{1}{5}\lambda_2 \end{cases} \Rightarrow \begin{cases} x_1 = -\frac{6}{5}\lambda_1 + \frac{2}{5}\lambda_2 + \lambda_1 - \lambda_2 \\ x_2 = \frac{3}{5}\lambda_1 - \frac{1}{5}\lambda_2 \end{cases} \Rightarrow \begin{cases} x_1 = -\frac{1}{5}\lambda_1 + \frac{3}{5}\lambda_2 \\ x_2 = \frac{3}{5}\lambda_1 - \frac{1}{5}\lambda_2 \end{cases}$$

$$\Rightarrow \ker(A) = \lambda_1 \begin{bmatrix} -\frac{1}{5} \\ \frac{3}{5} \\ 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} \frac{3}{5} \\ -\frac{1}{5} \\ 0 \\ 1 \end{bmatrix}$$

$b_1$  and  $b_2$  are the basis of the kernel

$$B = \left\{ \begin{bmatrix} -\frac{1}{5} \\ \frac{3}{5} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{3}{5} \\ -\frac{1}{5} \\ 0 \\ 1 \end{bmatrix} \right\} \Rightarrow \dim(\ker(A)) = 2$$

The dimensions of  $\ker(A)$  and  $\text{Im}(A)$  should add up to 4 (dimensions of original space).

$$\dim(\ker(A)) + \dim(\text{Im}(A)) = 4$$

$$\Rightarrow \underline{\underline{\dim(\text{Im}(A)) = 2}}$$

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Compute the distance between

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad y = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

using

$$\langle x, y \rangle := x^T A y, \quad A = \begin{bmatrix} 2 & 2 & -1 \\ 2 & 5 & 4 \\ -1 & 4 & 8 \end{bmatrix}$$

$$x^T = [1 \ 2 \ 3]$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = [1 \ 2 \ 3] \begin{bmatrix} 2 & 2 & -1 \\ 2 & 5 & 4 \\ -1 & 4 & 8 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

$$= [1 \ 2 \ 3] \begin{bmatrix} -2 + 0 - 2 \\ -2 + 0 + 8 \\ 1 + 0 + 16 \end{bmatrix}$$

$$= [1 \ 2 \ 3] \begin{bmatrix} -4 \\ 6 \\ 17 \end{bmatrix} = -4 + 12 + 51 = \underline{\underline{59}}$$

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Consider the Euclidean vector space  $\mathbb{R}^4$  with the dot product. A subspace  $U \subseteq \mathbb{R}^4$  and  $x \in \mathbb{R}^4$  are given by

$$U = \text{span} \left\{ \begin{bmatrix} 3 \\ 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \end{bmatrix} \right\}, \quad x = \begin{bmatrix} -1 \\ -7 \\ 1 \\ 5 \end{bmatrix}.$$

Basis Vectors

Determine the orthogonal projection  $\pi_U(x)$  of  $x$  onto  $U$ .

$$\pi_U(x) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

Mapping ↗

$$A = \begin{bmatrix} 3 & 0 \\ 3 & 0 \\ 0 & 4 \\ 3 & 0 \end{bmatrix} \Rightarrow P = A(A^T A)^{-1} A^T$$

Projection matrix

$$A^T A = \begin{bmatrix} 3 & 3 & 0 & 3 \\ 0 & 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 3 & 0 \\ 0 & 4 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 27 & 0 \\ 0 & 16 \end{bmatrix}$$

$$(A^T A)^{-1} = \begin{bmatrix} 27 & 0 \\ 0 & 16 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{27} & 0 \\ 0 & \frac{1}{16} \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} 3 & 0 \\ 3 & 0 \\ 0 & 4 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{27} & 0 \\ 0 & \frac{1}{16} \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 & 3 \\ 0 & 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{9} & 0 \\ \frac{1}{9} & 0 \\ 0 & \frac{1}{4} \\ \frac{1}{9} & 0 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 & 3 \\ 0 & 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

The projection into subspace  $U$  can thus be written as

$$\pi_U(x) = P\vec{x} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -1 \\ -7 \\ 1 \\ 5 \end{bmatrix} = \underline{\underline{\begin{bmatrix} -1 \\ -1 \\ 1 \\ -1 \end{bmatrix}}}$$