430.457

Introduction to Intelligent Systems

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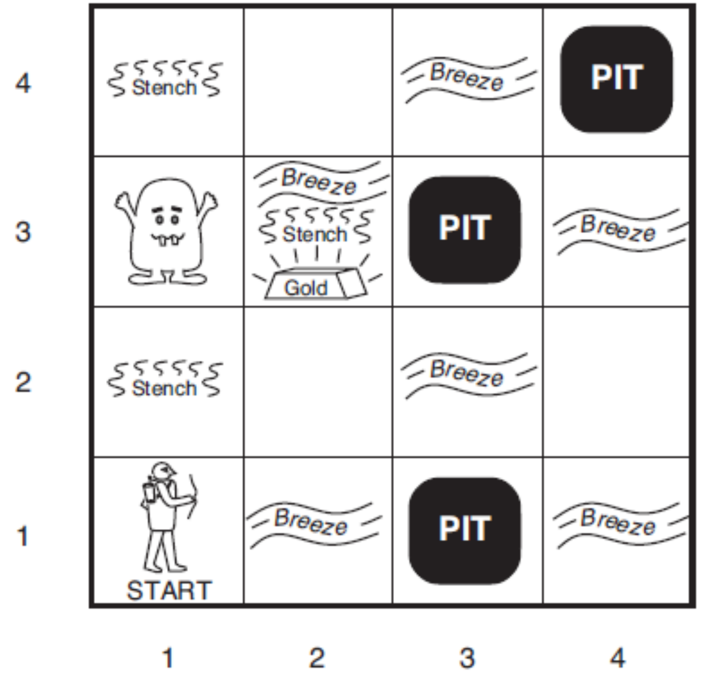
ECE, SNU

EXACT INFERENCE IN BAYESIAN NETWORKS

Wumpus World (Ch. 7)



Performance measure:
- +1000: climb out of the cave with the gold
- -1000: fall into a pit or eaten by the wumpus
- -1: each action taken
- -10: using up the arrow

Environment:
- Locations of gold and wumpus are chosen randomly
- A square has a pit with probability 0.2

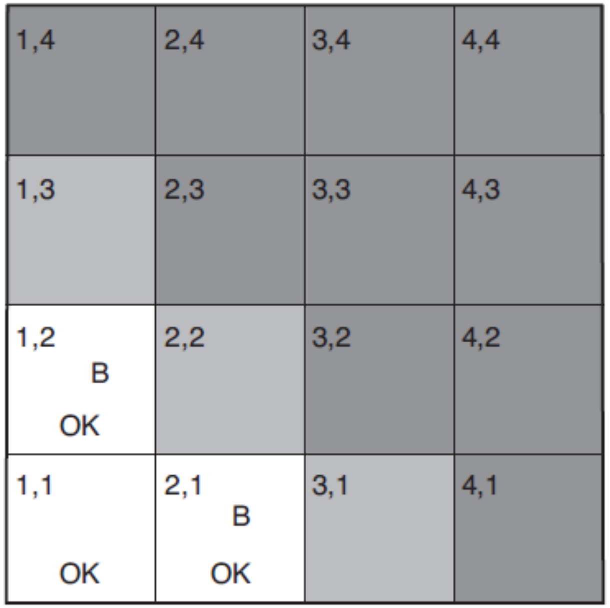
Actuators:
- Moves: *Forward, Turn Left, Turn Right*
- Agent dies if it enters a square with a pit or a live wumpus
- *Grab*: to pick up the gold
- *Shoot*: to fire an arrow (continues until it hits a wall), only one arrow is available
- *Climb*: to climb out of the cave

Sensors:
- *Stench*: in a square adjacent from the wumpus
- *Breeze*: in a square adjacent from a pit
- *Glitter*: in the square where the gold is
- *Bump*: when an agent walks into a wall
- *Scream*: when the wumpus is killed

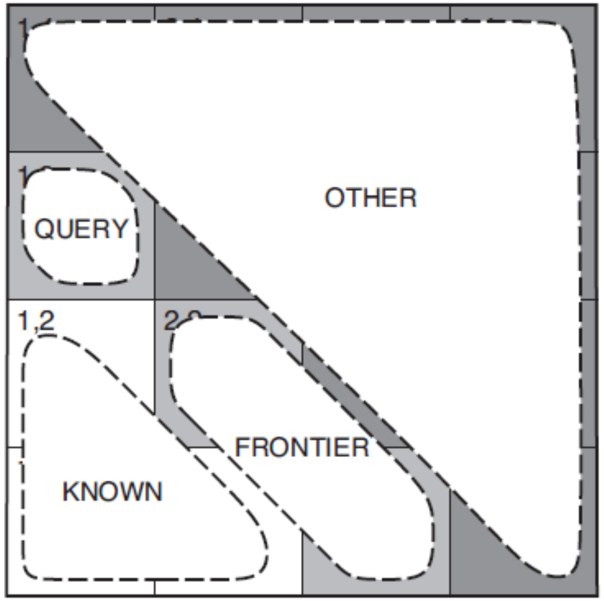
Wumpus World: Probabilistic Inference

For this example, we ignore the wumpus and the gold.

- $P_{i,j}$: True (or 1) if square $[i, j]$ contains a pit
- $B_{i,j}$: True if square $[i, j]$ is breezy
- Known facts and observations: $known = \neg p_{1,1} \wedge \neg p_{1,2} \wedge \neg p_{2,1}$, $b = \neg b_{1,1} \wedge b_{1,2} \wedge b_{2,1}$
- What is $P(P_{1,3}|known, b)$?

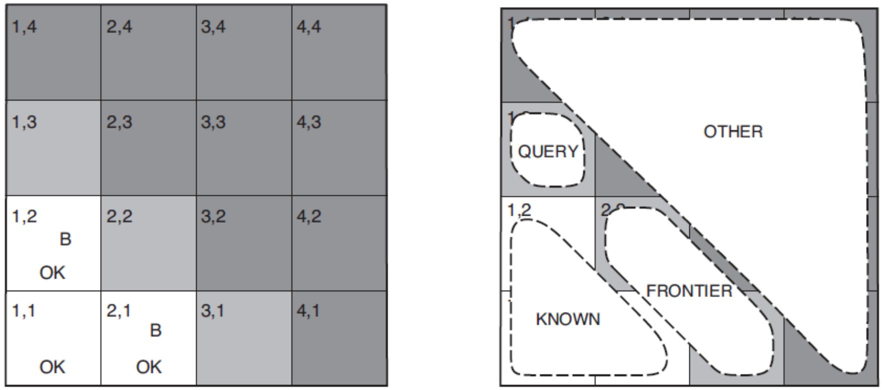


(a)



(b)

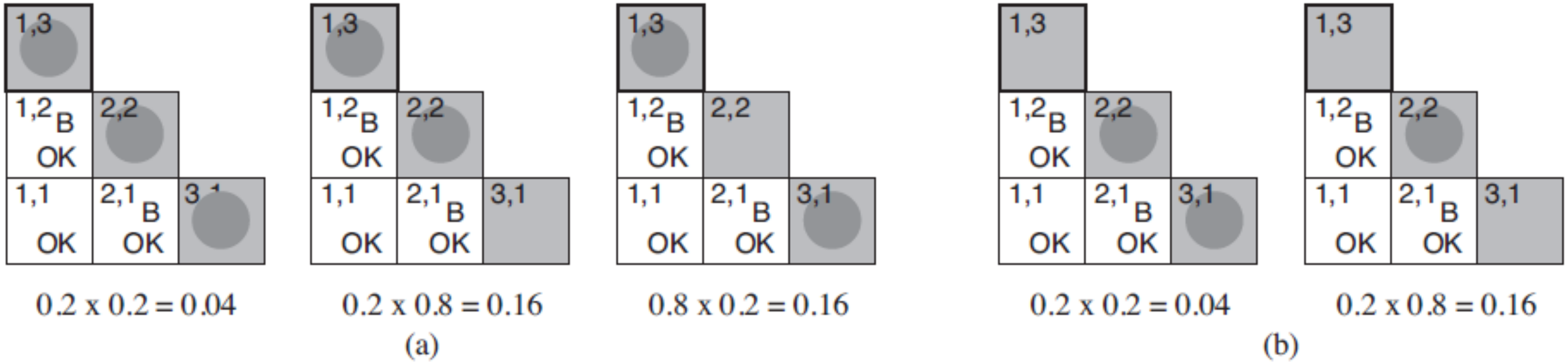
What is $P(P_{1,3}|known, b)$?



(a) (b)

$$
\begin{aligned}
P(P_{1,3}|known, b) &= \alpha \sum_{unknown} P(P_{1,3}, known, b, unknown) \\
&= \alpha \sum_{unknown} P(b|P_{1,3}, known, unknown) P(P_{1,3}, known, unknown) \\
&= \alpha \sum_{frontier} \sum_{other} P(b|P_{1,3}, known, frontier, other) P(P_{1,3}, known, frontier, other) \\
&= \alpha \sum_{frontier} \sum_{other} P(b|P_{1,3}, known, frontier) P(P_{1,3}, known, frontier, other) \\
&= \alpha \sum_{frontier} P(b|P_{1,3}, known, frontier) \sum_{other} P(P_{1,3}, known, frontier, other) \\
&= \alpha \sum_{frontier} P(b|P_{1,3}, known, frontier) \sum_{other} P(P_{1,3}) P(known) P(frontier) P(other) \\
&= \alpha P(P_{1,3}) P(known) \sum_{frontier} P(b|P_{1,3}, known, frontier) P(frontier) \sum_{other} P(other) \\
&= \alpha' P(P_{1,3}) \sum_{frontier} P(b|P_{1,3}, known, frontier) P(frontier)
\end{aligned}
$$

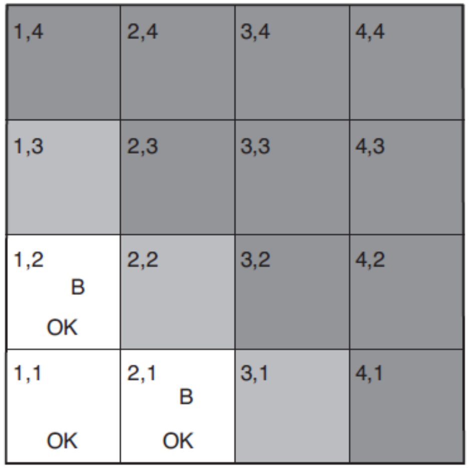
$$P(P_{1,3}|known, b) = \alpha' P(P_{1,3}) \sum_{frontier} P(b|P_{1,3}, known, frontier)P(frontier)$$



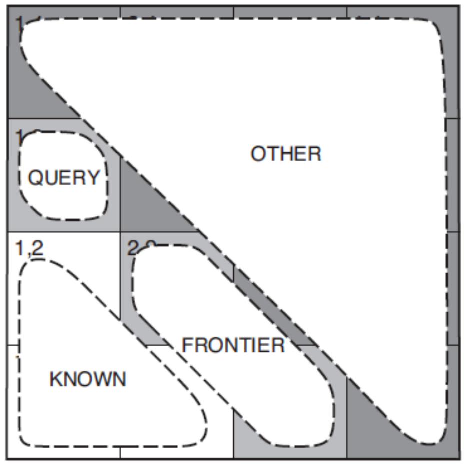
$$\begin{aligned} P(P_{1,3}|known, b) &= \alpha' \langle 0.2(0.04 + 0.16 + 0.16), 0.8(0.04 + 0.16) \rangle \\ &\approx \langle 0.31, 0.69 \rangle \end{aligned}$$

Probability Inference

- Partition variables in a Bayesian network as $\mathbf{X} = \{X\} \cup \mathbf{E} \cup \mathbf{Y}$
- X, **query variables**
- $\mathbf{E} = \{E_1, \ldots, E_m\}$, **evidence variables**, and $\mathbf{e}$ is a particular observed event
- $\mathbf{Y} = \{Y_1, \ldots, Y_l\}$, **hidden variables**
- Goal: $P(X|\mathbf{e})$, **posterior distribution**



(a)

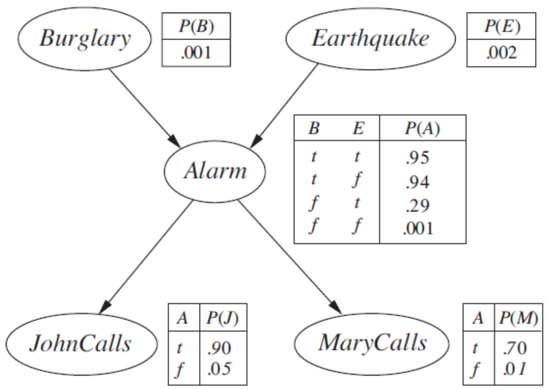


(b)

What is $P(P_{1,3}|known, b)$?

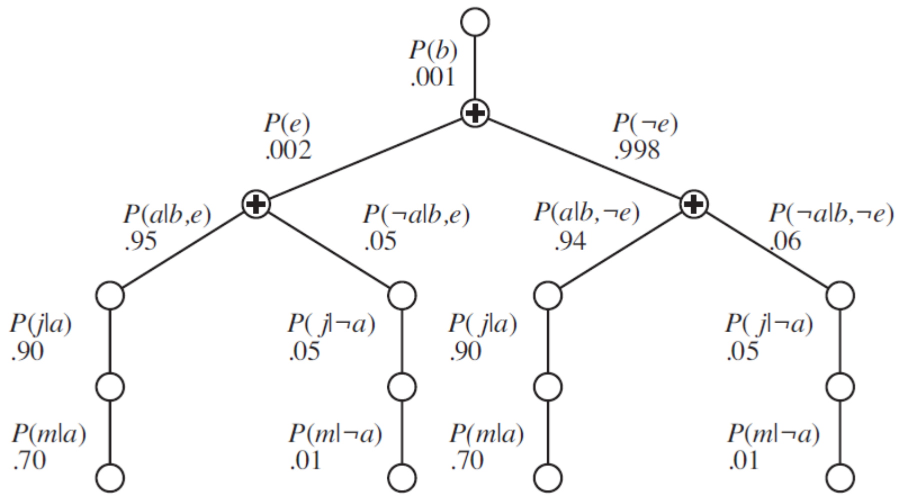
Inference by Enumeration

- $P(X|\mathbf{e}) = \alpha P(X, \mathbf{e}) = \alpha \sum_{\mathbf{y}} P(X, \mathbf{e}, \mathbf{y})$.
- A query can be answered using a Bayesian network by computing sums of products of conditional probabilities from the network.



$$
\begin{aligned}
P(B,E,A,J,M) &= P(B)P(E)P(A|B,E) \\
&\times P(J|A)P(M|A)
\end{aligned}
$$

$$
\begin{aligned}
P(b|j,m) &= \alpha P(b,j,m) = \alpha \sum_e \sum_a P(b,j,m,e,a) \\
&= \alpha \sum_e \sum_a P(b)P(e)P(a|b,e)P(j|a)P(m|a) \\
&= \alpha P(b) \sum_e P(e) \sum_a P(a|b,e)P(j|a)P(m|a)
\end{aligned}
$$



function ENUMERATION-ASK(X, $\mathbf{e}$, bn) **returns** a distribution over X
  inputs: X, the query variable
        $\mathbf{e}$, observed values for variables $\mathbf{E}$
        bn, a Bayes net with variables $\{X\} \cup \mathbf{E} \cup \mathbf{Y}$ /* $\mathbf{Y}$ = *hidden variables* */

  $\mathbf{Q}(X) \leftarrow$ a distribution over X, initially empty
  for each value x_i of X **do**
    $\mathbf{Q}(x_i) \leftarrow$ ENUMERATE-ALL(bn.VARS, $\mathbf{e}_{x_i}$)
      where $\mathbf{e}_{x_i}$ is $\mathbf{e}$ extended with $X = x_i$
  return NORMALIZE($\mathbf{Q}(X)$)

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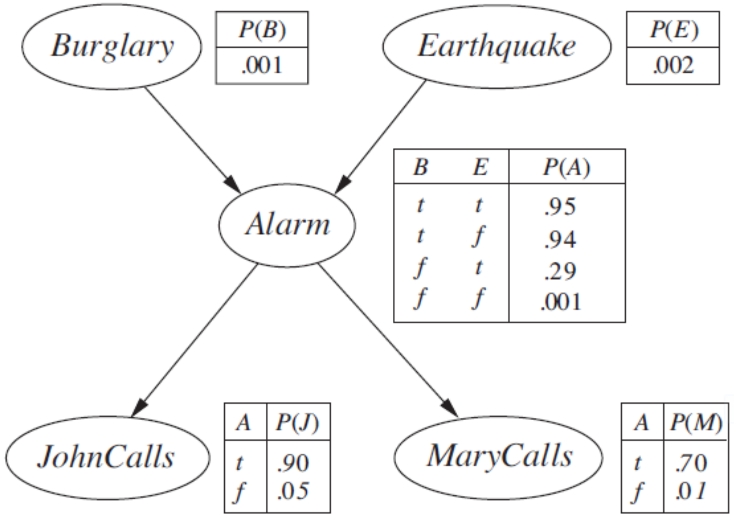
function ENUMERATE-ALL($vars$, $\mathbf{e}$) **returns** a real number
  if EMPTY?($vars$) **then return** 1.0
  $Y \leftarrow$ FIRST($vars$)
  if Y has value y in $\mathbf{e}$
    then return $P(y \mid parents(Y)) \times$ ENUMERATE-ALL(REST($vars$), $\mathbf{e}$)
    else return $\sum_y P(y \mid parents(Y)) \times$ ENUMERATE-ALL(REST($vars$), $\mathbf{e}_y$)
      where $\mathbf{e}_y$ is $\mathbf{e}$ extended with $Y = y$

For n Boolean variables, the time complexity is $O(2^n)$.

Variable Elimination Algorithm

$$\mathbf{P}(B \mid j, m) = \alpha \underbrace{\mathbf{P}(B)}_{\mathbf{f}_1(B)} \sum_e \underbrace{P(e)}_{\mathbf{f}_2(E)} \sum_a \underbrace{\mathbf{P}(a \mid B, e)}_{\mathbf{f}_3(A,B,E)} \underbrace{P(j \mid a)}_{\mathbf{f}_4(A)} \underbrace{P(m \mid a)}_{\mathbf{f}_5(A)}$$



$$\mathbf{f}_4(A) = \begin{pmatrix} P(j \mid a) \\ P(j \mid \neg a) \end{pmatrix} = \begin{pmatrix} 0.90 \\ 0.05 \end{pmatrix}$$

$$\mathbf{f}_5(A) = \begin{pmatrix} P(m \mid a) \\ P(m \mid \neg a) \end{pmatrix} = \begin{pmatrix} 0.70 \\ 0.01 \end{pmatrix}$$

$$\mathbf{P}(B \mid j, m) = \alpha \mathbf{f}_1(B) \times \sum_e \mathbf{f}_2(E) \times \sum_a \mathbf{f}_3(A, B, E) \times \mathbf{f}_4(A) \times \mathbf{f}_5(A)$$

point-wise product

$$\begin{aligned} \mathbf{f}_6(B, E) &= \sum_a \mathbf{f}_3(A, B, E) \times \mathbf{f}_4(A) \times \mathbf{f}_5(A) \\ &= (\mathbf{f}_3(a, B, E) \times \mathbf{f}_4(a) \times \mathbf{f}_5(a)) + (\mathbf{f}_3(\neg a, B, E) \times \mathbf{f}_4(\neg a) \times \mathbf{f}_5(\neg a)) \end{aligned}$$

$$\mathbf{P}(B \mid j, m) = \alpha \mathbf{f}_1(B) \times \sum_e \mathbf{f}_2(E) \times \mathbf{f}_6(B, E)$$

$$\begin{aligned} \mathbf{f}_7(B) &= \sum_e \mathbf{f}_2(E) \times \mathbf{f}_6(B, E) \\ &= \mathbf{f}_2(e) \times \mathbf{f}_6(B, e) + \mathbf{f}_2(\neg e) \times \mathbf{f}_6(B, \neg e) \end{aligned}$$

$$\mathbf{P}(B \mid j, m) = \alpha \mathbf{f}_1(B) \times \mathbf{f}_7(B)$$

function ELIMINATION-ASK(X, $\mathbf{e}$, bn) **returns** a distribution over X
  inputs: X, the query variable
    $\mathbf{e}$, observed values for variables $\mathbf{E}$
    bn, a Bayesian network specifying joint distribution $\mathbf{P}(X_1, \ldots, X_n)$

  $factors \leftarrow [\,]$
  for each var **in** ORDER(bn.VARS) **do**
    $factors \leftarrow$ [MAKE-FACTOR(var, $\mathbf{e}$)|$factors$]
    if var is a hidden variable **then** $factors \leftarrow$ SUM-OUT(var, $factors$)
  return NORMALIZE(POINTWISE-PRODUCT($factors$))

Example:

$$\mathbf{P}(B \mid j, m) = \alpha\, \mathbf{f}_1(B) \times \sum_e \mathbf{f}_2(E) \times \sum_a \mathbf{f}_3(A, B, E) \times \mathbf{f}_4(A) \times \mathbf{f}_5(A)$$

$$\mathbf{P}(B \mid j, m) = \alpha\, \mathbf{f}_1(B) \times \sum_e \mathbf{f}_2(E) \times \mathbf{f}_6(B, E)$$

$$\mathbf{P}(B \mid j, m) = \alpha\, \mathbf{f}_1(B) \times \mathbf{f}_7(B)$$

- The size of a factor is determined by the ordering of variables.
- Determining the optimal ordering is NP-hard