# Dynamic Compensators and Stability Margins

## Control Engineering (Reguleringsteknik)


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SDU

# Agenda



Introduction
- Curriculum

Stability Margins

Dynamic Compensation
- Lead Compensation
- Lag Compensation

# Introduction
## Curriculum for Reguleringsteknik (REG)



Matematiske og grafiske metoder til syntese af lineære tidsinvariante systemer:[1]

- diskret og kontinuert tilstandsbeskrivelse
- analyse i tid og frekvens
- stabilitet, reguleringshastighed, følsomhed og fejl
- digitale PI, PID, LEAD og LAG regulatorer (serieregulatorer)
- tilstandsregulering, pole-placement og tilstands-estimering (observer)
- optimal regulering (least squares) og optimal tilstands-estimation (Kalman-filter)

**Færdigheder**:
Efter gennemførelse af kurset kan den succesfulde studerende:

- kunne analysere, dimensionere og implementere såvel kontinuert som tidsdiskret regulering af lineære tidsinvariante og stokastiske systemer

**Kompetencer**:
Efter gennemførelse af kurset kan den succesfulde studerende:

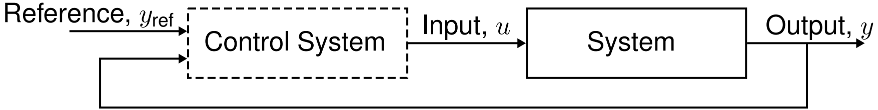
- anvende og implementere klassiske og moderne reguleringsteknikker for at kunne styre og regulere en robot hurtig og præcist

[1] Based on https://fagbesk.sam.sdu.dk/?fag_id=39673

The twelve lectures of the course are

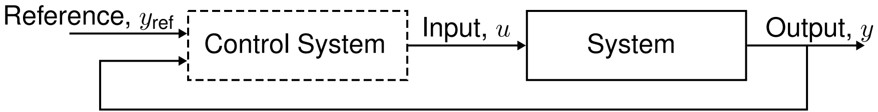
- **Lecture 1**: Introduction to Linear Time-Invariant Systems
- **Lecture 2**: Stability and Performance Analysis
- **Lecture 3**: Introduction to Control
- **Lecture 4**: Design of PID Controllers
- **Lecture 5**: Root Locus
- **Lecture 6**: The Nyquist Plot
- **Lecture 7**: Dynamic Compensators and Stability Margins
- **Lecture 8**: Implementation
- **Lecture 9**: State Feedback
- **Lecture 10**: Observer Design
- **Lecture 11**: Optimal Control (Linear Quadratic Control)
- **Lecture 12**: The Kalman Filter

**Task**: Design a cruise control for a car.




- **Control Input**: Throttle position $u$
- **Measured Output**: Velocity of the car $y$
- **Reference Input**: Desired velocity of the car $y_{\text{ref}}$

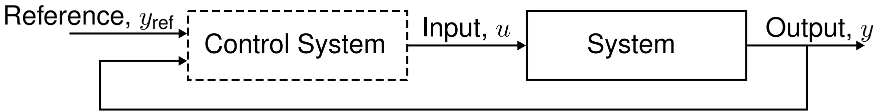
**Task**: Design a cruise control for a car.




Today, an answer to the following question is provided:

1. How can one ensure that a control is stable despite uncertainties in the system?

**Task**: Design a cruise control for a car.




Today, an answer to the following question is provided:

1. How can one ensure that a control is stable despite uncertainties in the system?
   - Uncertainties may affect the *gain* of the system (e.g. the inertia).
   - Uncertainties may affect the *phase* of the system (e.g. communication delays between controller and sensor).

The ***gain margin*** is the factor by which the gain can be rained before a system becomes unstable.

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Recall that a system is ***neutrally stable*** for

$$|L(j\omega)| = 1 \quad \text{and} \quad \angle L(j\omega) = 180^\circ$$

where $L(s) = K(s)G(s)$.

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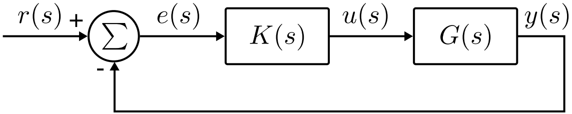


# Stability Margins
## Phase Margin: Definition



The ***phase margin*** is the amount by which the phase can be rained before a system becomes unstable (before it exceeds -$180°$).

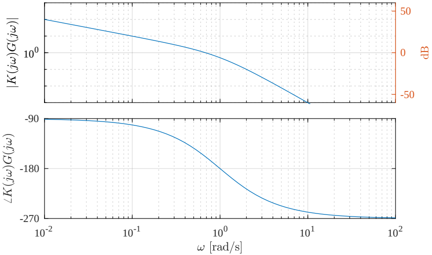
A system is ***neutrally stable*** for

$$|L(j\omega)| = 1 \quad \text{and} \quad \angle L(j\omega) = 180^\circ$$

where $L(s) = K(s)G(s)$.

The stability margins can be determined from a Bode plot as (use `margin` in MATLAB)

- **Gain Margin**: Find the value of $\omega$ where $\angle K(j\omega)G(j\omega) = -180°$, and denote it by $\omega_{GM}$. The gain margin (in dB) is
$$GM = -|K(j\omega_{GM})G(j\omega_{GM})|$$
- **Phase Margin**: Find the value of $\omega$ where $|K(j\omega)G(j\omega)| = 0$ dB, and denote it by $\omega_c$ (this frequency is called the ***crossover frequency***). The phase margin is
$$PM = \angle K(j\omega_c)G(j\omega_c) + 180°$$



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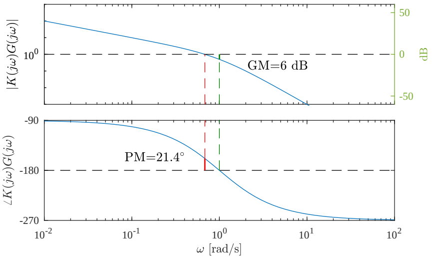
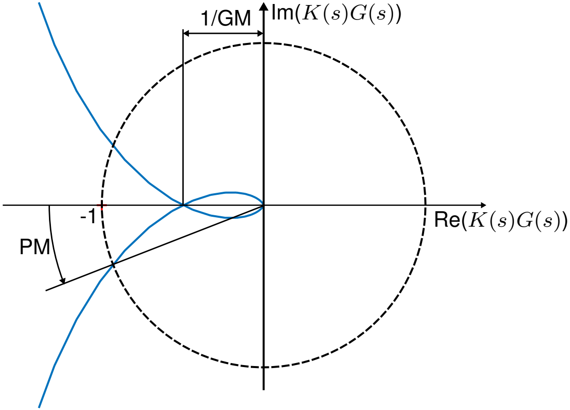


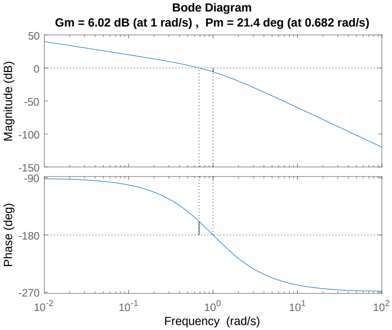
The stability margins can also be determined from a Nyquist plot, where the margins are determined by the closeness of the Nyquist plot to the point -1.



Consider the loop gain

$$L(s) = K(s)G(s) = \frac{1}{s(s+1)^2}$$

with the following Bode plot.



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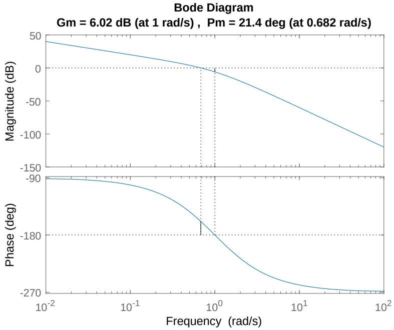


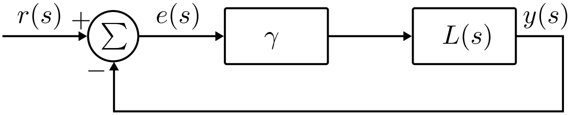


The gain $\gamma$ can be increased to 2 before the system becomes unstable.

Consider a loop gain

$$L(s) = \frac{1.3(s+2)}{s^3 + s^2 + bs + 1}$$

where $b$ is some parameter.

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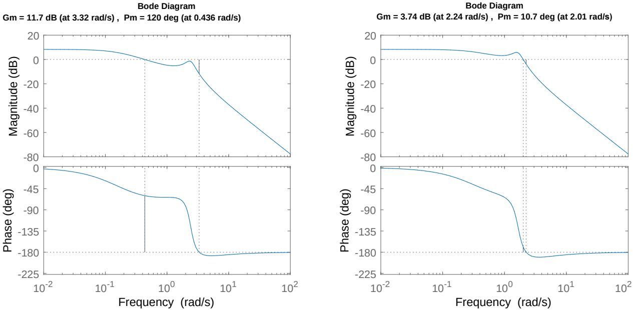
where $b$ is some parameter. A parameter change from $b = 6$ (left) to $b = 3$ (right) changes the margins significantly.



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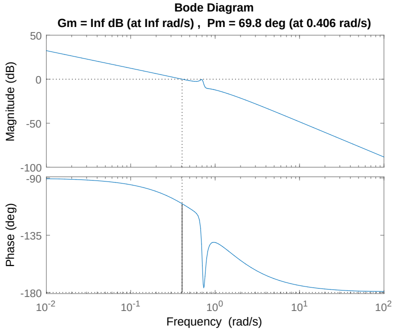
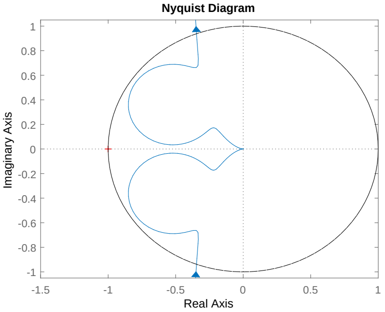
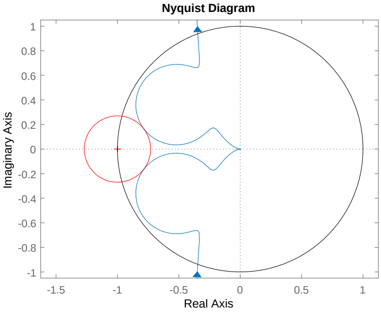
If the gain and phase is changed simultaneously, issues can occur.



The phase and gain margins determine stability properties, when only the gain OR phase is changed.

This is the motivation for looking at the vector margin that gives the shortest distance between the Nyquist plot and -1.



# Dynamic Compensation

## Overview

Two types of compensators are considered

- **Lead Compensation**: Approximates the PD control, i.e., it lowers the rise time and decreases the overshoot.
- **Lag Compensation**: Approximates the PI control, i.e., it improves the steady state tracking.

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$$D(s) = K\frac{s+z}{s+p}.$$

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Both compensators are given by the transfer function

$$D(s) = K\frac{s+z}{s+p}.$$

- If $z < p$, then $D(s)$ is called a lead compensation.
- If $z > p$, then $D(s)$ is called a lag compensation

A lead compensator is given by

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where $z < p \in \mathbb{R}$.

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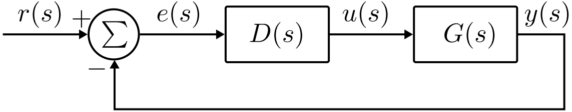
The transfer function $D(s)$ has a zero followed by a pole, and acts as a filtered PD controller, when used as a feedback controller.



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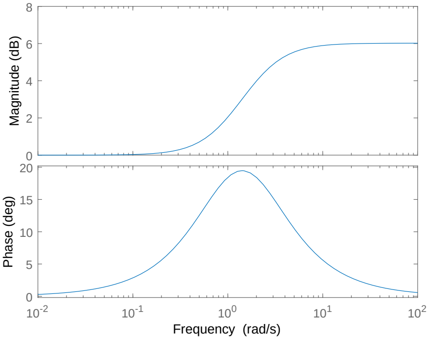

A lead compensation is given by

$$D(s) = \frac{Ts+1}{\alpha Ts+1} \qquad \text{where } \alpha < 1$$

and $1/\alpha$ is called the ***lead ratio***.

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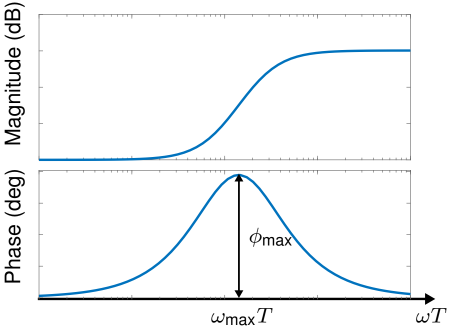


We have the following

$$\omega_{\text{max}} = \frac{1}{T\sqrt{\alpha}} = \sqrt{|z||p|}$$

and

$$\sin\phi_{\text{max}} = \frac{1-\alpha}{1+\alpha}$$

1. Determine open-loop gain $K$ to satisfy error or bandwidth requirements:
   1.1 To meet error requirement, pick $K$ to satisfy error constants ($K_p$, $K_V$, $K_a$) so that $e_ss$ error specification is met.
   1.2 Alternatively, to meet bandwidth requirement, pick $K$ so that the open-loop crossover frequency is a factor of two below the desired closed-loop bandwidth.
2. Evaluate the phase margin of the uncompensated system using the value $K$ obtained from Step 1.
3. Allow for extra margin (about $10°$), and determine the needed phase lead $\phi_{\text{max}}$ (one lead compensation should contribute a maximum of $60°$ to the phase).
4. Determine $\alpha$ from

$$\sin \phi_{\text{max}} = \frac{1-\alpha}{1+\alpha}$$

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$$\sin \phi_{\text{max}} = \frac{1-\alpha}{1+\alpha}$$
5. Pick $\omega_{\text{max}}$ to be the crossover frequency; thus, the zero is at $1/T = \omega_{\text{max}}\sqrt{\alpha}$ and the pole is at $1/\alpha T = \omega_{\text{max}}/\sqrt{\alpha}$.
6. Draw the compensated frequency response and check the phase margin.
7. Iterate on the design. Adjust compensator parameters (poles, zeros, and gain) until all specifications are met.

Consider the system

$$KG(s) = \frac{K}{(s/0.5+1)(s+1)(s/2+1)}$$

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Design a lead compensator such that the steady state error to a step input is

$$\frac{1}{1+K_p}$$

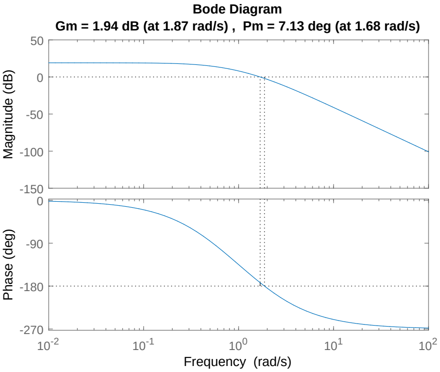
where $K_p = 9$ and the phase margin is at least $25°$.

Step 1: Meet steady state error requirement.

We use the Final Value Theorem to find $K$

$$K_p = \lim_{s \to 0} KG(s) = K = 9$$

Step 2: The Bode plot of $KG(s)$ with $K = 9$ is shown below.




The phase margin is $7^\circ$ at $\omega = 1.68$ rad/s.

Step 3: To allow a phase margin of $25^\circ$ (requirement) plus $10^\circ$ (extra margin), the needed phase lead is

$$\phi_{\mathsf{max}} = 25^\circ + 10^\circ - 7^\circ = 28^\circ$$

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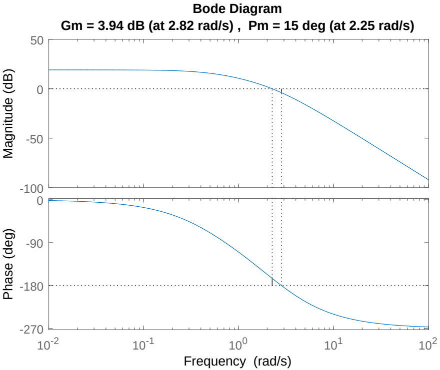
Step 5: Pick $\omega_{\text{max}}$ to be at the crossover frequency, i.e., $\omega_{\text{max}} = 1.68$ rad/s. Thereby,

$$T = \frac{1}{\omega_{\text{max}} \sqrt{\alpha}} = 0.9906$$

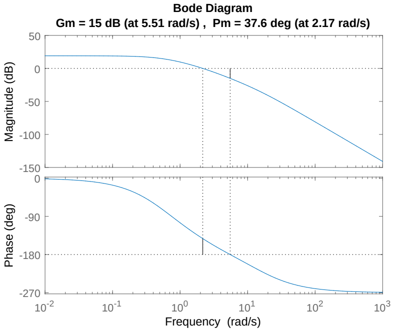
Step 6: Verify that the response satisfies the requirements. Since the crossover frequency has changed to 2.25 rad/s, the design does not work.



Step 7: The compensator is redesigned, as

$$D(s) = \frac{s/1.5 + 1}{s/15 + 1}$$

This gives the following frequency response



A lag compensator is given by

$$D(s) = K\frac{s+z}{s+p}$$

where $z > p \in \mathbb{R}$

# Dynamic Compensation
## Lag Compensation: Definition

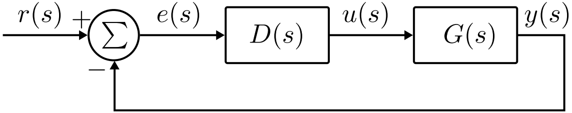


A lag compensator is given by

$$D(s) = K\frac{s+z}{s+p} \qquad or \qquad D(s) = K_0\alpha\frac{Ts+1}{\alpha Ts+1}$$

where $z > p \in \mathbb{R}$ and $\alpha > 1$.

The transfer function $D(s)$ has a pole followed by a zero, and approximates a PI controller, when used as a feedback controller.



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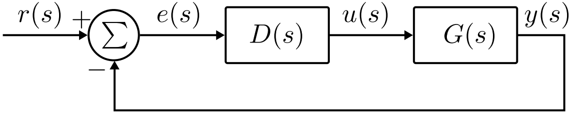


**Idea**: Improve the steady-state performance without affecting the other dynamics.

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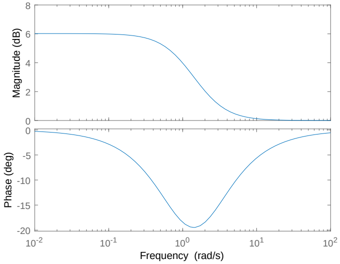

1. Determine open-loop gain $K$ that will meet the phase margin requirement without compensation.
2. Draw the Bode plot of the uncompensated system with crossover frequency from Step 1, and evaluate the low-frequency gain.
3. Determine $\alpha$ to meet the low-frequency gain error requirement.
4. Choose the corner frequency $\omega = 1/T$ (the zero of the lag compensator) to be one decade below the crossover frequency $\omega_c$.
5. The other corner frequency (the pole location of the lag compensator) is $\omega = 1/\alpha T$
6. Iterate on the design. Adjust compensator parameters (poles, zeros, and gain) until all specifications are met.

Design a lag compensator so that the phase margin is at least $40^\circ$ and $K_p = 9$ for the system

$$KG(s) = \frac{K}{(s/0.5+1)(s+1)(s/2+1)}$$

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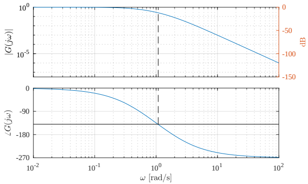

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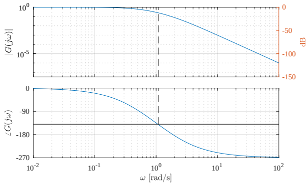
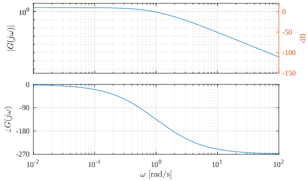



We choose $K = 3$.

Step 2: Draw Bode plot, and evaluate low-frequency gain of $KG(s)$.



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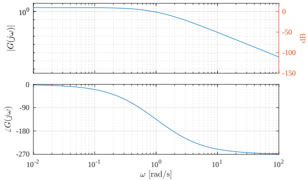


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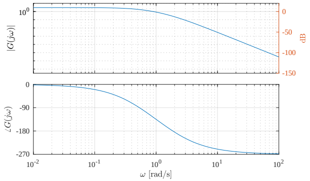
Step 2: Draw Bode plot, and evaluate low-frequency gain of $KG(s)$.

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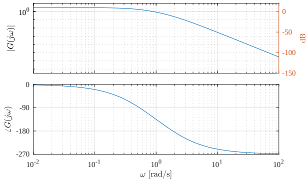


The phase margin is $53.5°$ and the low-frequency gain is $10$ dB (or 3.16 times). Step 3: Determine $\alpha$ such that desired low-frequency gain is attained.

To get $K_p = 9$, $\alpha$ must be 3.

Step 4: Choose the zero of the lag compensator (corner frequency $\omega = 1/T$) to be one octave slower than the crossover frequency, i.e., $\omega = 0.2$ rad/s (and $T = 5$).

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Step 5: The corner frequency of the pole of the lag compensator should be chosen to be $\omega = 1/\alpha T$. Thereby, the lag compensator is given by

$$D(s) = 3\frac{5s+1}{15s+1}$$

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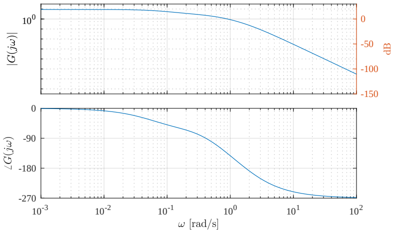


The system complies with the specification.