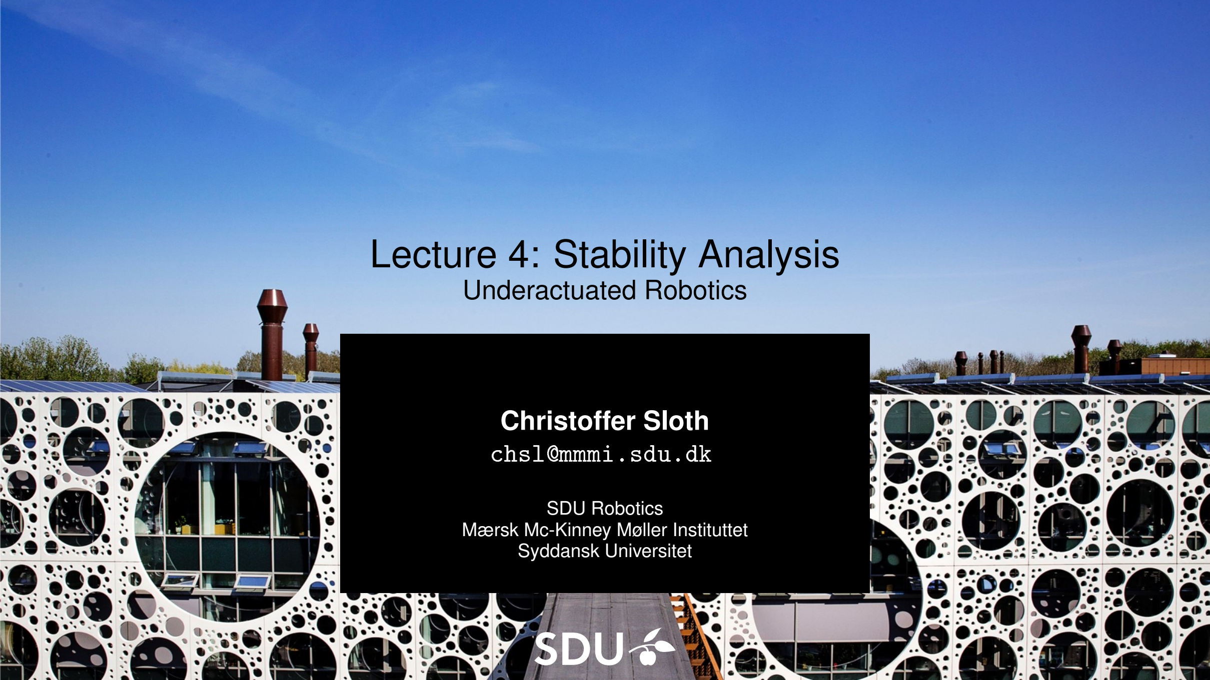

Lecture 4: Stability Analysis

Underactuated Robotics

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Agenda

Knowledge:

- Derive dynamical state-space models of robots as control systems
- Analyze the stability of low dimensional linear and nonlinear systems
- Analyze the observability and controllability of linear control systems
- Use a variety of controllers for underactuated robots

Skills:

- Implement simulations of control systems in software
- Create concise technical reports presenting solutions to proposed problems

Competencies:

- Choose appropriate modern control techniques to solve control problems in robotics
- Apply modern control techniques to control simulated underactuated robots

- **Lesson 1**: Newton-Euler Modelling
- **Lesson 2**: Euler-Lagrange Modelling
- **Lesson 3**: Simulation of Robot Dynamics
- **Lesson 4**: Stability Analysis
- **Lesson 5**: Optimal Control
- **Lesson 6**: Energy Shaping Control
- **Lesson 7**: Feedback Linearisation
- **Lesson 8**: Sliding Mode Control
- **Lesson 9**: Simulation and Implementation of Control Systems
- **Lesson 10**: Optimization-Based Control
- **Lesson 11**:
- **Lesson 12**:

If q is a trajectory of a mechanical system that is affected by a generalized force Q then

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = \boldsymbol{Q}$$

where Q is an n-dimensional vector of generalized forces.

The ***potential energy*** of the inverted pendulum is

$$E_{\text{pot}} = mgl(1 + \cos(\theta)) \qquad [\text{J}]$$

where m is the mass of the pendulum [kg], l is the length from the pivot point to the center of mass [m], and g is the gravitational acceleration [m/s^2].

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The ***kinetic energy*** is

$$E_{\text{kin}} = \frac{1}{2} m v^T v + \frac{1}{2} \omega^T I \omega$$

where v is the velocity of the center of mass [m/s], ω is the angular velocity of the center of mass [rad/s], and I is the moment of inertia of the pendulum with respect to the center of mass (given in Base frame) [kgm^2].

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$$E_{\text{kin}} = \frac{1}{2}mv^T v + \frac{1}{2}\omega^T I\omega$$

where v is the velocity of the center of mass [m/s], ω is the angular velocity of the center of mass [rad/s], and I is the moment of inertia of the pendulum with respect to the center of mass (given in Base frame) [kgm^2]. We assume that the pendulum is a point mass then ($I = 0$, $v = l\omega$)

$$E_{\text{kin}} = \frac{1}{2}ml^2\omega^2$$

By application of Lagrange–D'Alembert's Principle

$$ml^2\dot{\omega} - mgl\sin\theta = -b\omega + \tau$$

which leads to the following equation of motion

$$\dot{\omega} = \frac{g}{l}\sin\theta - \frac{b}{ml^2}\omega + \frac{1}{ml^2}\tau$$

We rewrite the equation of motion (with generalized coordinate $q = \theta$)

$$\dot{\omega} = \frac{g}{l}\sin\theta - \frac{b}{ml^2}\omega + \frac{1}{ml^2}\tau$$

on state space form

$$\dot{x} = f(x, u)$$

by defining the state and input

$$x = \begin{bmatrix} \theta \\ \omega \end{bmatrix}, \qquad u = \tau$$

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by defining the state and input

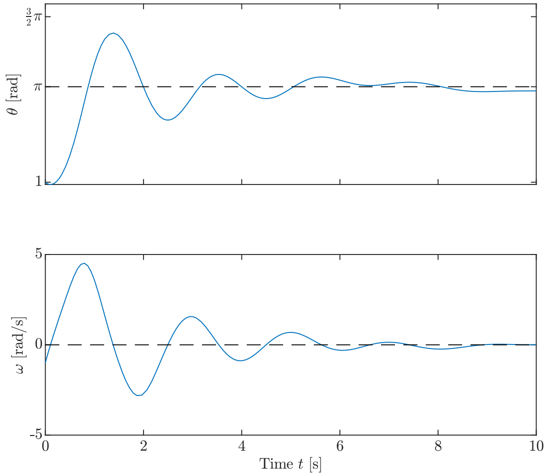
$$x = \begin{bmatrix} \theta \\ \omega \end{bmatrix}, \qquad u = \tau$$

This implies that

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \underbrace{\begin{bmatrix} \omega \\ \frac{g}{l}\sin\theta - \frac{b}{ml^2}\omega + \frac{1}{ml^2}\tau \end{bmatrix}}_{=f(x,u)}$$

Modelling
MATLAB Code for Simulation


```
%% Simulation of Pendulum
t0 = 0;            % [s] Start time for simulation
tEnd = 50;         % [s] End time for simulation
tu = (t0:0.05:tEnd)';
u = cos(tu);       % Specification of input
x0 = [1;-1];       % Specification of initial state
% Simulate the system
[t,x] = ode45(@invertedPendulum,[t0 tEnd],x0,[],[tu u]);
%% Function used for simulation via ode45
function dx = invertedPendulum(t,x,u)
     u = interp1(u(:,1),u(:,2),t); % determine u by linear interpolation
     theta = x(1); omega = x(2);tau = u;
     dx = [omega;
             g/l*sin(theta) - b/(l^2*m)*omega+1/(l^2*m)*tau];
end
```


$\frac{3}{2}\pi$
π
1
θ [rad]
5
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-5
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6
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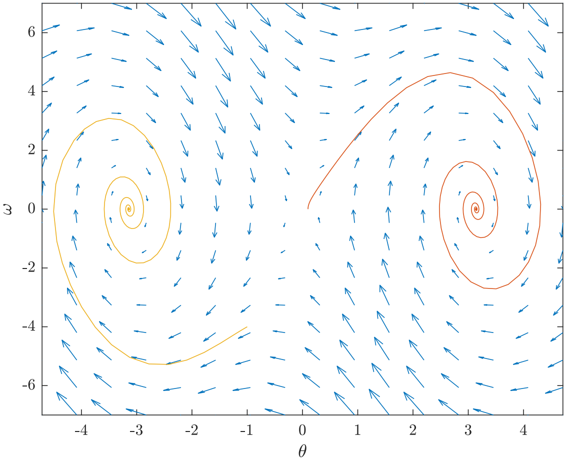
10
Time t [s]


6
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ω
-4
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-1
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3
4
θ

The system of 1st order differential equations

$$\dot{x} = f(x, u)$$

is approximated at $x = \bar{x}$ using first-order Taylor approximation by

$$\dot{x} \approx f(\bar{x}, \bar{u}) + \frac{\partial f}{\partial x}|_{x=\bar{x}, u=\bar{u}} \hat{x} + \frac{\partial f}{\partial u}|_{x=\bar{x}, u=\bar{u}} \hat{u}$$

where $\hat{x} = x - \bar{x}$ and all partial derivatives are evaluated at the operating point $p = (\bar{x}, \bar{u})$.

Linearization

First Order Taylor Approximation

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where $\hat{x} = x - \bar{x}$ and all partial derivatives are evaluated at the operating point $p = (\bar{x}, \bar{u})$.

We linearize systems at equilibrium points, i.e., where $\dot{x} = 0$. This implies that

$$\dot{\hat{x}} \approx \underbrace{\frac{\partial f}{\partial x}|_{x=\bar{x}, u=\bar{u}}}_{=A} \hat{x} + \underbrace{\frac{\partial f}{\partial u}|_{x=\bar{x}, u=\bar{u}}}_{B} \hat{u}$$

The dynamics of the inverted pendulum is given by

$$\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \omega \\ \frac{g}{l}\sin\theta - \frac{b}{ml^2}\omega + \frac{1}{ml^2}\tau \end{bmatrix}$$

This implies that

$$\frac{\partial f}{\partial(\theta, \omega)} = A = \begin{bmatrix} \frac{\partial f_1}{\partial \theta} & \frac{\partial f_1}{\partial \omega} \\ \frac{\partial f_2}{\partial \theta} & \frac{\partial f_2}{\partial \omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos\theta & -\frac{b}{ml^2} \end{bmatrix}$$

and

$$\frac{\partial f}{\partial \tau} = B = \begin{bmatrix} \frac{\partial f_1}{\partial \tau} \\ \frac{\partial f_2}{\partial \tau} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}$$

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and

$$\frac{\partial f}{\partial\tau} = B = \begin{bmatrix} \frac{\partial f_1}{\partial\tau} \\ \frac{\partial f_2}{\partial\tau} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}$$

At an equilibrium point $\dot{x} = f(\bar{x}, \bar{u}) = 0$, which means that

$$\bar{\omega} = 0, \qquad \bar{\tau} = mlg\sin(\bar{\theta})$$

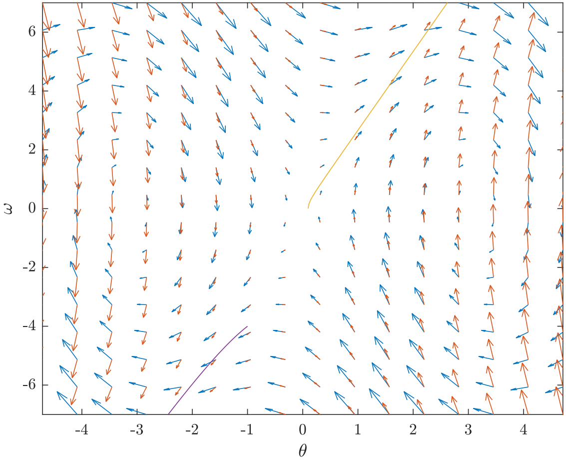
The linearized model of the inverted pendulum at equilibrium point $(\bar{x}, \bar{u}) = (\bar{\theta}, \bar{\omega}, \bar{\tau}) = (\bar{\theta}, 0, mlg\sin(\bar{\theta})$ is given by

$$\dot{\hat{x}} = \begin{bmatrix} \dot{\hat{\theta}} \\ \dot{\hat{\omega}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\cos\bar{\theta} & -\frac{b}{ml^2} \end{bmatrix} \underbrace{\begin{bmatrix} \hat{\theta} \\ \hat{\omega} \end{bmatrix}}_{=\hat{x}} + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} \hat{u}$$

where $\hat{x} = (x - \bar{x})$ and $\hat{u} = (u - \bar{u})$.

Linearization
MATLAB Code for Simulation


```
%% Simulation of Linearized Pendulum
pendul_lin = ss(A,B,C,D); % Definition of system model
t0 = 0;           % [s] Start time for simulation
tEnd = 50;        % [s] End time for simulation
t = (t0:0.05:tEnd)';
u = cos(t);       % Specification of input
x0 = [1;-1];      % Specification of initial state
% Simulate the system
[y,t,x] = lsim(pendul_lin,u,t,x0);
```



6
4
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-4
-6
ω
-4
-3
-2
-1
0
1
2
3
4
θ

A linear continuous-time system described by the state equation

$$\dot{x} = Ax$$

is asymptotically stable if and only if all eigenvalues have negative real part.

The eigenvalues of the system matrix (matrix A) of the linearized pendulum system are the values of λ that solves the equation (eigenvalues of A)

$$\det(\lambda I - A) = 0$$

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The eigenvalues are given by

$$\det\left(\begin{bmatrix} \lambda & -1 \\ -\frac{g}{l}\cos\bar{\theta} & \lambda + \frac{b}{ml^2} \end{bmatrix}\right) = \lambda\left(\lambda + \frac{b}{ml^2}\right) - \frac{g}{l}\cos\bar{\theta} = 0$$

The eigenvalues are therefore

$$\lambda = \frac{-\frac{b}{ml^2} \pm \sqrt{(\frac{b}{ml^2})^2 + 4\frac{g}{l}\cos\bar{\theta}}}{2}$$

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The eigenvalues are therefore

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In conclusion, one pole will be in the left half-plane, when $\cos(\bar{\theta}) > 0$; when $\cos(\bar{\theta}) < 0$ then the linearized system is stable.

By Hartman-Grobman Theorem, if the poles of a linearized system are not on the imaginary axis, then the nonlinear and linearized models behave similarly in a small neighborhood about the equilibrium point.

Stability Analysis

Relation between Stability of Linearized Pendulum and Nonlinear Pendulum

By Hartman-Grobman Theorem, if the poles of a linearized system are not on the imaginary axis, then the nonlinear and linearized models behave similarly in a small neighborhood about the equilibrium point.

For us: Design a stabilizing controller for the linearized system model, then the controller also stabilizes the nonlinear system around the equilibrium point.

A continuous time system

$$\dot{x}(t) = Ax(t) + Bu(t)\,, \quad x(0) = 0$$

is said to be *controllable* iff for any $\xi \in \mathbb{R}^n$ there exists $u(t)$ such that for some $T > 0$, $x(T) = \xi$.

A system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, is controllable if and only if

$$\text{rank}\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = n$$

For a state space model

$$\dot{x} = Ax + Bu$$

a *state feedback* is a feedback of the form

$$u = Fx$$

Combining these two equations, we obtain:

$$\dot{x} = Ax + BFx = (A + BF)x$$

Thus, the result of a state feedback is a system with a modified system matrix, and thus with modified poles.

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The poles of $A_{\text{cl}} = A + BF$ can be placed anywhere in the complex plane if the system is controllable.

The controllability matrix of the linearized pendulum is

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{ml^2} \\ \frac{1}{ml^2} & -\frac{b}{m^2 l^4} \end{bmatrix}$$

It is seen that the system is controllable, as $\mathcal{C}$ has full rank.

Stabilization using State Feedback

State Feedback of Linearized System

To stabilize the pendulum in upright, a feedback controller needs to be designed; it is possible since the linearized system is controllable. Poles must be placed in the open left-half plane, and the system will respond faster when the poles are more negative.

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A controller for the pendulum with poles $s = -1$ and $s = -2$ is

$$u = Fx$$

where

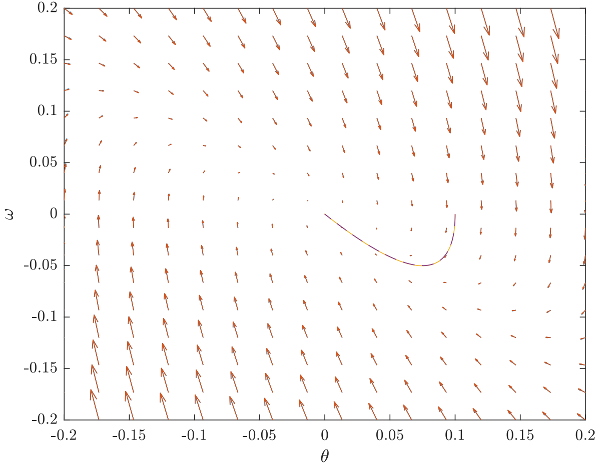
$$F = \begin{bmatrix} -11.82 & -2 \end{bmatrix}$$

Stabilization using State Feedback
MATLAB Code for Pole Placement


```
%% Pole Placement for Linearized Pendulum
pendul_lin = ss(A,B,C,D); % Definition of system model
P = [-1 -2];
F = -place(pendul_lin.A,pendul_lin.B,P);
```

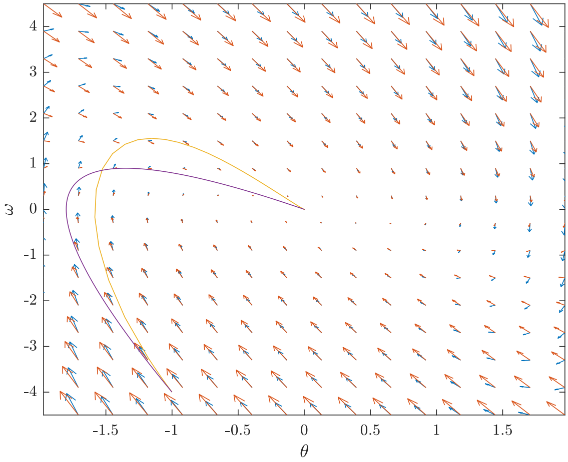
Stabilization using State Feedback

Phase Plot (Initial Condition Close to (0,0))



Stabilization using State Feedback

Phase Plot (Initial Condition Far from (0,0))



If q is a trajectory of a mechanical system that is affected by a generalized force Q then

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = \boldsymbol{Q}$$

where Q is an n-dimensional vector of generalized forces.

The ***potential energy*** of the inverted pendulum is

$$E_{\text{pot}} = g(m_1(-l_{c1}\cos\theta_1) + m_2(-l_1\cos\theta_1 - l_{c2}\cos(\theta_1+\theta_2))) \qquad \text{[J]}$$

where m is the mass of the pendulum [kg], l_{ci} is the length from the ith joint to the ith center of mass [m], and g is the gravitational acceleration [m/s^2].

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The ***kinetic energy*** is

$$E_{\text{kin}} = \frac{1}{2}\begin{bmatrix}\dot\theta_1 & \dot\theta_2\end{bmatrix}\begin{bmatrix} I_1 + I_2 + m_2 l_1^2 + 2m_2 l_1 l_{c2}\cos\theta_2 & I_2 + m_2 l_1 l_{c2}\cos\theta_2 \\ I_2 + m_2 l_1 l_{c2}\cos\theta_2 & I_2 \end{bmatrix}\begin{bmatrix}\dot\theta_1 \\ \dot\theta_2\end{bmatrix}$$

where v is the velocity of the center of mass [m/s], ω is the angular velocity of the center of mass [rad/s], and I is the moment of inertia of the pendulum with respect to the center of mass (given in Base frame) [kgm^2].

By application of Lagrange–D'Alembert's Principle ($q = (\theta_1, \theta_2)$)

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = B\tau$$

where

$$M(q) = \begin{bmatrix} I_1 + I_2 + m_2 l_1^2 + 2m_2 l_1 l_{c2}\cos\theta_2 & I_2 + m_2 l_1 l_{c2}\cos\theta_2 \\ I_2 + m_2 l_1 l_{c2}\cos\theta_2 & I_2 \end{bmatrix}$$

$$C(q,\dot{q}) = \begin{bmatrix} -2m_2 l_1 l_{c2}\sin\theta_2 \dot{q}_2 & -m_2 l_1 l_{c2}\sin\theta_2 \dot{q}_2 \\ m_2 l_1 l_{c2}\sin\theta_2 \dot{q}_1 & 0 \end{bmatrix}$$

$$g(q) = \begin{bmatrix} g(m_1(l_{c1}\sin\theta_1) + m_2(l_1\sin\theta_1 + l_{c2}\sin(\theta_1+\theta_2))) \\ g m_2 l_{c2}\sin(\theta_1+\theta_2) \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We rewrite the equation of motion (with generalized coordinate $q = \theta$)

$$\ddot{q} = \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = M^{-1}(q) \left(B\tau - C(q, \dot{q})\dot{q} - g(q) \right)$$

on state space form

$$\dot{x} = f(x, u)$$

by defining the state and input

$$x = \begin{bmatrix} \theta_1 & \theta_2 & \omega_1 & \omega_2 \end{bmatrix}^T, \qquad u = \tau$$

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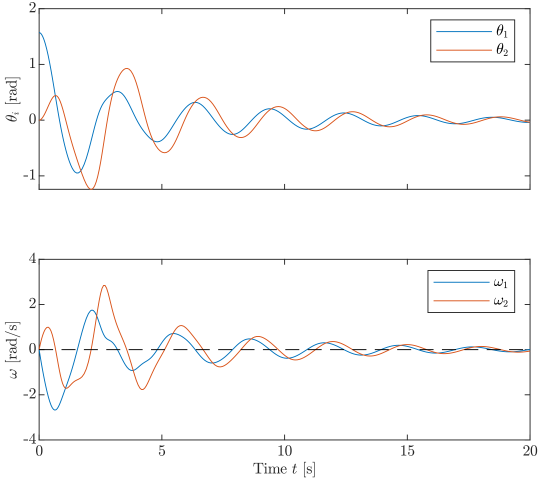

2
1
0
-1
θ_i [rad]
θ_1
θ_2
4
2
0
-2
-4
ω [rad/s]
ω_1
ω_2
0
5
10
15
20
Time t [s]

The system of 1st order differential equations

$$\dot{x} = f(x, u)$$

is approximated at $x = \bar{x}$ using first-order Taylor approximation by

$$\dot{x} \approx f(\bar{x}, \bar{u}) + \frac{\partial f}{\partial x}|_{x=\bar{x},u=\bar{u}}\hat{x} + \frac{\partial f}{\partial u}|_{x=\bar{x},u=\bar{u}}\hat{u}$$

where $\hat{x} = x - \bar{x}$ and all partial derivatives are evaluated at the operating point $p = (\bar{x}, \bar{u})$.

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We linearize systems at equilibrium points, i.e., where $\dot{x} = 0$. This implies that

$$\dot{\hat{x}} \approx \underbrace{\frac{\partial f}{\partial x}|_{x=\bar{x}, u=\bar{u}}}_{=A} \hat{x} + \underbrace{\frac{\partial f}{\partial u}|_{x=\bar{x}, u=\bar{u}}}_{B} \hat{u}$$

Linearization
Operating Point

The Acrobot is linearized at an equilibrium point $\dot{x} = f(\bar{x}, \bar{u}) = 0$; in particular, at its upright position which means that

$$\bar{\theta}_1 = \pi, \bar{\theta}_2 = 0, \bar{\omega}_1 = \bar{\omega}_2 = 0, \qquad \bar{\tau} = 0$$

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$$\bar{\theta}_1 = \pi, \bar{\theta}_2 = 0, \bar{\omega}_1 = \bar{\omega}_2 = 0, \qquad \bar{\tau} = 0$$

The linearized system has poles: $s = -4.6882, s = -2.2014, s = 2.2628, s = 1.7697$.

```
%% Linearization of Acrobot
% Nonlinear vector field
f = [omega1;omega2;
      M\(Bin*tau-diag([b1 b2])*[omega1;omega2]-C*[omega1;omega2]-grav)];
% Linearized model
Bsym = diff(f,tau);
Asym = [diff(f,theta1) diff(f,theta2) diff(f,omega1) diff(f,omega2)];
% Use the command 'subs' to insert values for parameters and state
```

The controllability of the Acrobot is determined by computing the controllability matrix $(n = 4)$

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$$

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$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$$

The linearized system is controllable; thus, we can design a state feedback controller by pole placement.

Stabilization using State Feedback

State Feedback of Linearized System

To stabilize the acrobot in upright, a feedback controller needs to be designed; it is possible since the linearized system is controllable. Poles must be placed in the open left-half plane, and the system will respond faster when the poles are more negative.

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A controller for the pendulum with poles $s = -1$, $s = -2$, $s = -3$, $s = -4$ is

$$u = Fx$$

where

$$F = \begin{bmatrix} 220.34 & 53.54 & 92.45 & 31.54 \end{bmatrix}$$

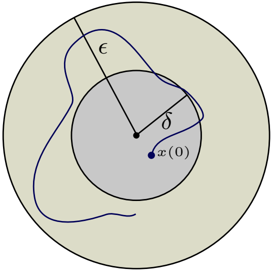
The equilibrium $x = 0$ of a dynamical system

$$\dot{x} = f(x)$$

is

- *stable* if, for each $\epsilon > 0$, there is δ such that

$$||x(0)|| < \delta \Rightarrow ||x(t)|| < \epsilon, \qquad \forall\, t \geq 0.$$



The equilibrium $x = 0$ of a dynamical system

$$\dot{x} = f(x)$$

is

- *stable* if, for each $\epsilon > 0$, there is δ such that

$$||x(0)|| < \delta \Rightarrow ||x(t)|| < \epsilon, \qquad \forall\, t \geq 0.$$

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- *unstable* if it is not stable.
- *asymptotically stable* if it is stable and δ can be chosen such that

$$||x(0)|| < \delta \Rightarrow \lim_{t\to\infty} x(t) = 0.$$

Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$, and let $V : \mathbb{R}^n \to \mathbb{R}$ be a differentiable. The equilibrium point $x = 0$ is

- *stable* if V satisfies

$$V(0) = 0 \tag{1a}$$

$$V(x) > 0 \quad \forall x \in \mathbb{R}^n \backslash \{0\} \tag{1b}$$

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If the function V satisfies the above conditions then it is called a ***Lyapunov function***.

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$$\dot{x} = -x^3$$

Determine if the system is globally asymptotically stable.

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This implies that V is a Lyapunov function and the system is globally asymptotically stable.

Let $D \subset \mathbb{R}^n$, let $x = 0 \in D$ be an equilibrium point of $\dot{x} = f(x)$, and let $V : D \to \mathbb{R}$ be a differentiable. The equilibrium point $x = 0$ is *locally*

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Any region $\Omega_\beta = \{x \in \mathbb{R}^n | V(x) \leq \beta\}$ such that $\Omega_\beta \subseteq D$ is a positively invariant set contained in the region of attraction of the equilibrium point.

A set $\Omega \subset \mathbb{R}^n$ is said to be an *invariant set* with respect to

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$$x(0) \in \Omega \Rightarrow x(t) \in \Omega, \quad \forall t \in \mathbb{R}.$$

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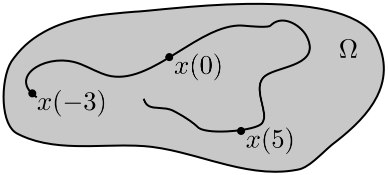
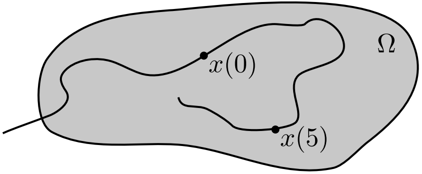


Ω
$x(0)$
$x(-3)$
$x(5)$



Ω
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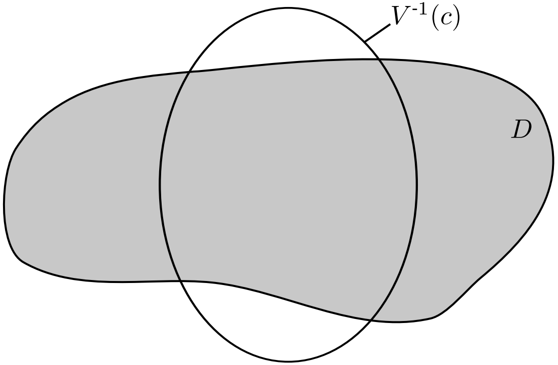
Why do we say: "*Any region* $\Omega_\beta = \{x \in \mathbb{R}^n | V(x) \leq \beta\}$ *such that* $\Omega_\beta \subseteq D$ *is a positively invariant set contained in the region of attraction of the equilibrium point.*"



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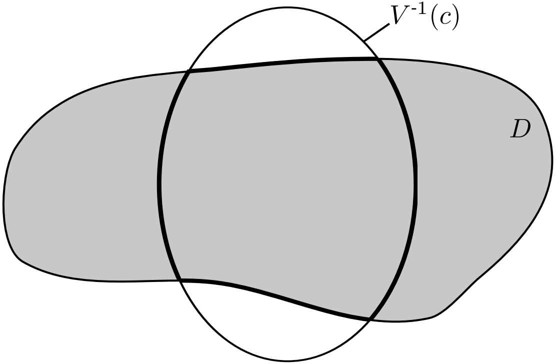

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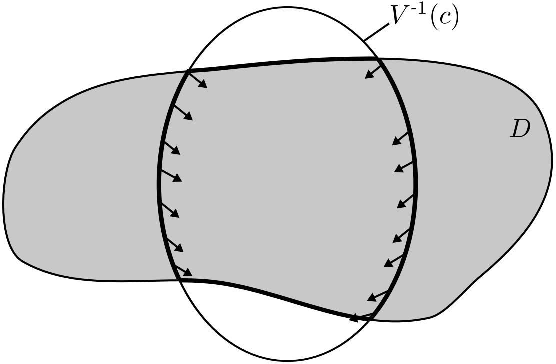

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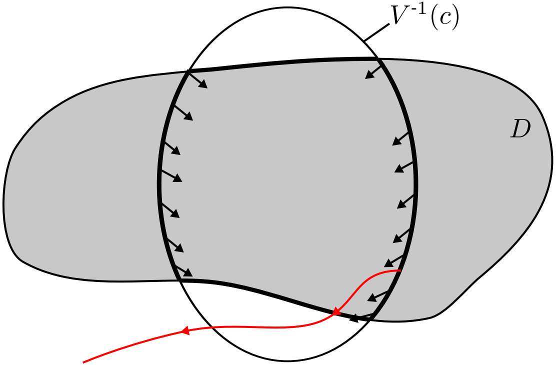

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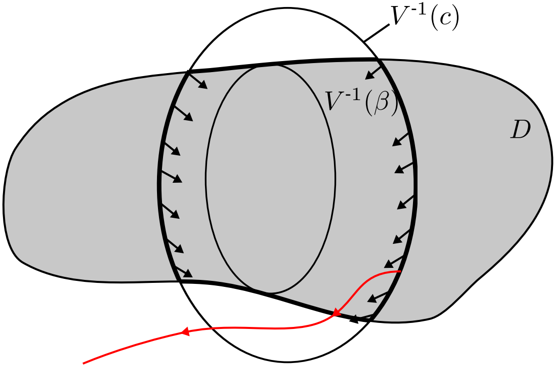

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