Problem 1:

Find $\iint_s x\, dS$ over the part of the parabolic cylinder $z = x^2/2$ that lies inside the first octant part of the cylinder $x^2 + y^2 = 1$.

Solution: $\pi/8$

$$z = \frac{x^2}{2}$$

$$\Rightarrow x = v, \quad z = \frac{v^2}{2}, \quad y = u \quad \Rightarrow \quad \vec{r}(u,v) = v i + u j + \frac{v^2 k}{2}$$

$$\Rightarrow \begin{cases} r_u = 1 j \\ r_v = 1 i + v k \end{cases}$$

$$dS = |r_u \times r_v| \, du\, dv$$

$$|r_u \times r_v| = \left\| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 1 & 0 & v \end{matrix} \right\| = |(1 \cdot v - 0 \cdot 0) i - (0 \cdot v - 1 \cdot 0) k + (0 \cdot 0 - 1 \cdot 0) k| = |v i - k| = \sqrt{v^2 + (-1)^2} = \sqrt{v^2 + 1}$$

$$\iint_S x\, dS = \iint_{v\,u} v \cdot \sqrt{v^2+1}\, du\, dv$$

Finding the bounds

$$0 \le x,\ 0 \le y \Rightarrow 0 \le v,\ 0 \le u$$

$$x^2 + y^2 \le 1 \Rightarrow v^2 + u^2 \le 1$$

Transform to polar coordinates

$x = r \cdot \cos(\theta) \qquad y = r \cdot \sin(\theta) \qquad \mathrm{dA} = r\, \mathrm{d}\theta\, \mathrm{dr} \qquad x^2 + y^2 = a^2 \Rightarrow a$

$$0 \le v,\ 0 \le u \Rightarrow 0 \le \theta \le \frac{\pi}{2}$$

$$v^2 + u^2 \le 1 \Rightarrow 0 \le r \le 1$$

$$\iint_{v\,u} v \cdot \sqrt{v^2+1}\, du\, dv \Rightarrow \int_0^{\frac{\pi}{2}} \int_0^1 r \cdot \cos(\theta) \cdot \sqrt{r^2 \cdot \cos^2(\theta) + 1} \cdot r\, dr\, d\theta = \underline{\underline{\frac{\pi}{8}}}$$

Problem 2:

Find the total charge on the surface

$$\mathbf{r} = e^u \cos v\mathbf{i} + e^u \sin v\mathbf{j} + u\mathbf{k}, \qquad (0 \le u \le 1, 0 \le v \le \pi),$$

if the charge density on the surface is $\delta = \sqrt{1 + e^{2u}}$.

Solution: $\frac{\pi}{3}(3e + e^3 - 4)$

$$\int_0^{\pi}\int_0^{1} \delta(u,v) \cdot \underbrace{|\vec{r}_u \times \vec{r}_v|} \, du \, dv$$

$$\begin{cases} \vec{r}_u = e^u \cdot \cos(v) i + e^u \cdot \sin(v) j + k \\ \vec{r}_v = -e^u \cdot \sin(v) i + e^u \cdot \cos(v) j \end{cases} \Rightarrow \vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ e^u \cdot \cos(v) & e^u \cdot \sin(v) & 1 \\ -e^u \cdot \sin(v) & e^u \cdot \cos(v) & 0 \end{vmatrix}$$

$$= \left(0 - e^u \cdot \cos(v)\right) i - \left(0 - (-e^u \cdot \sin(v))\right) j + \left(e^{u^2} \cdot \cos^2(v) - (-e^{u^2} \cdot \sin^2(v))\right) k$$

$$= -e^u \cdot \cos(v) i - e^u \cdot \sin(v) j + e^{u^2} \cdot \overbrace{\left(\cos^2(v) + \sin^2(v)\right)}^{1} k$$

$$= -e^u \cdot \cos(v) i - e^u \cdot \sin(v) j + e^{u^2} k$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{e^{u^2} \cdot \cos^2(v) + e^{u^2} \cdot \sin^2(v) + e^{u^4}} = \sqrt{e^{u^2} \cdot \underbrace{\left(\cos^2(v) + \sin^2(v)\right)}_{1} + e^{u^4}}$$

$$= \sqrt{e^{u^2} + e^{u^4}} = \sqrt{e^{u^2}\left(1 + e^{u^2}\right)} = e^u \sqrt{e^{u^2}+1} = e^u \sqrt{e^{2u}+1}$$

$$\Rightarrow \int_0^{\pi}\int_0^{1} \delta(u,v) \cdot |\vec{r}_u \times \vec{r}_v| \, du \, dv = \int_0^{\pi}\int_0^{1} \sqrt{e^{2u}+1} \cdot e^u \sqrt{e^{2u}+1} \, du \, dv = \int_0^{\pi}\int_0^{1} e^u \cdot (e^{2u}+1) \, du \, dv$$

$$= \int_0^{\pi}\int_0^{1} e^{3u} + e^u \, du \, dv = \int_0^{\pi} \left[\frac{1}{3} \cdot e^{3u} + e^u\right]_0^1 dv = \int_0^{\pi} \frac{1}{3} \cdot e^3 + e^1 - \frac{1}{3} \cdot e^0 - e^0 \, dv = \int_0^{\pi} \frac{1}{3} \cdot e^3 + e - \frac{1}{3} - 1 \, dv$$

$$= \left[\frac{1}{3} e^3 \cdot v + e \cdot v - \frac{v}{3} - v\right]_0^{\pi} = \frac{1}{3} e^3 \pi + e\pi - \frac{\pi}{3} - \pi = \frac{\pi}{3}\left(e^3 + 3e - 1 - 3\right) = \underline{\underline{\frac{\pi}{3}\left(e^3 + 3e - 4\right)}}$$

Problem 3:

Describe the parametric surface

$$x = au\cos v, \qquad y = au\sin v, \qquad z = bv$$

$(0 \le u \le 1, 0 \le v \le 2\pi)$, and find its area.

Solution: $\pi a\sqrt{a^2+b^2} + \pi b^2 \ln\left(\frac{a+\sqrt{a^2+b^2}}{b}\right)$ sq. units

$$\vec{r}(u,v) = au\cdot\cos(v)\hat{i} + au\cdot\sin(v)\hat{j} + bv\hat{k}$$

$$\Rightarrow \begin{cases} \vec{r}_u = a\cdot\cos(v)\hat{i} + a\cdot\sin(v)\hat{j} \\ \vec{r}_v = -au\cdot\sin(v)\hat{i} + au\cdot\cos(v)\hat{j} + b\hat{k} \end{cases}$$

$$\Rightarrow \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\cdot\cos(v) & a\cdot\sin(v) & 0 \\ -au\cdot\sin(v) & au\cdot\cos(v) & b \end{vmatrix} = (ab\cdot\sin(v) - \cancel{0})\hat{i} - (ab\cdot\cos(v) - \cancel{0})\hat{j} + (u\cdot a^2\cos^2(v) \underbrace{-(-}_{+} u\cdot a^2\sin^2(v))k$$

$$= ab\cdot\sin(v)\hat{i} - ab\cdot\cos(v)\hat{j} + (u\cdot a^2\cos^2(v) + u\cdot a^2\sin^2(v))k$$

$$= ab\cdot\sin(v)\hat{i} - ab\cdot\cos(v)\hat{j} + (u\cdot a^2\cdot\overbrace{(\cos^2(v) + \sin^2(v)}^{1})\,)k$$

$$= ab\cdot\sin(v)\hat{i} - ab\cdot\cos(v)\hat{j} + a^2uk$$

$$\Rightarrow |\vec{r}_u \times \vec{r}_v| = \sqrt{a^2b^2\sin^2(v) + a^2b^2\cos^2(v) + a^4u^2}$$

$$= \sqrt{a^2b^2\cdot\underbrace{(\sin^2(v)+\cos^2(v))}_{1} + a^4u^2}$$

$$= \sqrt{a^2b^2 + a^4u^2} = \sqrt{a^2\cdot(b^2+a^2u^2)}$$

$$= a\sqrt{b^2+a^2u^2}$$

$\lambda = b^2 + a^2u^2 \Rightarrow \frac{d\lambda}{du} = 2a^2u \Rightarrow du =$

$$\Rightarrow \int_0^{2\pi}\int_0^1 1\cdot a\sqrt{b^2+a^2u^2}\,du\,dv = a\cdot\int_0^{2\pi}\int_0^1 \sqrt{b^2+a^2u^2}\,du\,dv$$