Betragt følgende Butterworth 3. ordens lavpasfilter (frekvensnormeret filter)

$$\tilde{H}(s) = \frac{1}{s+1}\frac{1}{s^2+s+1}$$

og design et tilsvarende digitalt lavpasfilter med afskæringsfrekvens på 1 kHz og samplefrekvens på 8 kHz. Design filtret som følger

1. Design et digitalt lavpasfilter $H_1(z)$ ved brug af matched z-transformation.
2. Design et digitalt lavpasfilter $H_2(z)$ ved brug af impuls invariant z-transformation.
3. Benyt MATLAB til at sammenligne Bode plot for denormerede lavpasfilter $H(s)$, $H_1(z)$ og $H_2(z)$.
4. Benyt MATLAB til at sammenligne impulsresponserne for $H(s)$, $H_1(z)$ og $H_2(z)$.

$f_s = 8kHz, \quad f_a = 1kHz$

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$$\tilde{H}_1(s) = \frac{1}{s+1} \Rightarrow p_s = \{-1\}$$

$$\tilde{H}_2(s) = \frac{1}{s^2+s+1} \Rightarrow p_s = \left\{\frac{-1 \pm i\sqrt{3}}{2}\right\}$$

$$p = \begin{bmatrix} -1 \\ \frac{-1+i\sqrt{3}}{2} \\ \frac{-1-\sqrt{3}}{2} \end{bmatrix} \Rightarrow p_{den} = p \cdot 2\pi \cdot f_a = \begin{bmatrix} -2000\pi \\ -1000\pi + i \cdot 1000\sqrt{3}\pi \\ -1000\pi - i \cdot 1000\sqrt{3}\pi \end{bmatrix}$$

$$H_1(s) = \tilde{H}_1(s)\Big|_{s=\frac{s}{w_a}} = \frac{1}{s+1}\Big|_{s=\frac{s}{w_a}} = \frac{1}{\frac{s}{2000\pi}+1} = \frac{2000\pi}{s+2000\pi}$$

$$H_2(s) = \tilde{H}_2(s)\Big|_{s=\frac{s}{w_a}} = \frac{1}{s^2+s+1}\Big|_{s=\frac{s}{w_a}} = \frac{1}{\left(\frac{s}{2000\pi}\right)^2+\frac{s}{2000\pi}+1} = \frac{2000\pi}{\frac{1}{2000\pi}\cdot s^2+s+2000\pi}$$

Ryk til z domæne

$$z = e^{sT}, \quad T = \frac{1}{f_s} = \frac{1}{8000}, \quad b_n = -e^{sT}$$

$$b_1 = -e^{-2000\pi \cdot \frac{1}{8000}} = -e^{\frac{-\pi}{4}}$$

$$b_2 = -e^{(-1000\pi + i \cdot 1000\sqrt{3}) \cdot \frac{1}{8000}} = -e^{1000 \cdot (-\pi + i\sqrt{3}) \cdot \frac{1}{8000}} = -e^{\frac{\pi}{8} \cdot (-1 + i\sqrt{3})}$$

$$b_3 = -e^{\frac{\pi}{8} \cdot (-1 - i\sqrt{3})}$$

$$H(z) = \frac{a_0}{(z+b_1)(z+b_2)(z+b_3)} = \frac{a_1}{z+b_1} \cdot \frac{a_2}{(z+b_2)(z+b_3)}$$

$$= \frac{a_1}{z \underbrace{- e^{\frac{-\pi}{4}}}_{b_1}} \cdot \frac{a_2}{\left(z \underbrace{- e^{\frac{\pi}{8}\cdot(-1-i\sqrt{3})}}_{b_2}\right)\left(z \underbrace{- e^{\frac{\pi}{8}\cdot(-1+i\sqrt{3})}}_{b_3}\right)}$$

$$a_1: \quad \underbrace{\frac{2000\pi}{0+2000\pi}}_{H_1(s)|_{s=0}} = \frac{a_1}{1-e^{\frac{-\pi}{4}}} \Rightarrow a_1 = 1 - e^{\frac{-\pi}{4}}$$

$$a_2: \quad \underbrace{\frac{2000\pi}{0+0+2000\pi}}_{H_2(s)|_{s=0}} = \frac{a_2}{\left(1-e^{\frac{\pi}{8}\cdot(-1-i\sqrt{3})}\right)\left(1-e^{\frac{\pi}{8}\cdot(-1+i\sqrt{3})}\right)}$$

$$\Rightarrow a_2 = \left(1-e^{\frac{\pi}{8}\cdot(-1-i\sqrt{3})}\right)\left(1-e^{\frac{\pi}{8}\cdot(-1+i\sqrt{3})}\right)$$

$$\Rightarrow \frac{1-e^{\frac{-\pi}{4}}}{z-e^{\frac{-\pi}{4}}} \cdot \frac{1-e^{\frac{\pi}{8}\cdot(-1+i\sqrt{3})} - e^{\frac{\pi}{8}\cdot(-1-i\sqrt{3})}}{\left(z-e^{\frac{\pi}{8}\cdot(-1-i\sqrt{3})}\right)\left(z-e^{\frac{\pi}{8}\cdot(-1+i\sqrt{3})}\right)}$$

(2) $$\tilde{H}(s) = \frac{1}{s+1}\frac{1}{s^2+s+1}$$

Partialbrøkopløs

$$\tilde{H}(s) = \left(\frac{1}{s+1}\cdot\frac{1}{s^2+s+1} = \frac{A}{s+1} + \frac{Bs+C}{s^2+s+1}\right)\cdot(s+1)(s^2+s+1)$$

$$\Rightarrow 1 = A(s^2+s+1) + (Bs+C)(s+1)$$

$$\Rightarrow 1 = As^2 + As + A + Bs^2 + Bs + Cs + C$$

Samle kostanter

$$\begin{aligned} s^2: &\quad 0 = A+B \\ s: &\quad 0 = A+B+C \\ 1: &\quad 1 = A+C \end{aligned} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\cdot\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right]\cdot(-1)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right] \Rightarrow \begin{cases} A = 1 \\ B = -1 \\ C = 0 \end{cases}$$

$$\Rightarrow \tilde{H}(s) = \frac{A}{s+1} + \frac{Bs+C}{s^2+s+1} = \frac{1}{s+1} + \frac{-s}{s^2+s+1}$$

Denormer overføringsfunktionen

$$H(s) = \tilde{H}(s)\Big|_{s=\frac{s}{\omega_a}} = \frac{1}{\frac{s}{2000\pi}+1} + \frac{\frac{-s}{2000\pi}}{\left(\frac{s}{2000\pi}\right)^2 + \frac{s}{2000\pi} + 1} = \overbrace{\frac{2000\pi}{s+2000\pi}}^{H_1(s)} + \overbrace{\frac{-2000\pi\cdot s}{s^2+2000\pi\cdot s+(2000\pi)^2}}^{H_2(s)}$$

Find konstanter og z-transformer

$$H_1(s) = \frac{2000\pi}{s+2000\pi} \Rightarrow \begin{cases} k_1 = 2000\pi \\ s_1 = -2000\pi \end{cases} \Rightarrow H_1(z) = \frac{1}{8000}\cdot 2000\pi\cdot\frac{1}{1-e^{-2000\pi\cdot\frac{1}{8000}}\cdot z^{-1}} = \frac{\pi}{4}\cdot\frac{1}{1-e^{\frac{-\pi}{4}}\cdot z^{-1}}$$

$$H_2(s) = \frac{-2000\pi\cdot s}{s^2+2000\pi\cdot s+(2000\pi)^2} = \frac{-2000\pi\cdot s}{(s-1000\pi+1000\pi\sqrt{3}\,i)\cdot(s-1000\pi-1000\pi\sqrt{3}\,i)}$$

Definition: $s_2 = 1000\pi - 1000\pi\sqrt{3}\,i$

Partialbrøkopløs

$$H_2(s) = \frac{-2000\pi\cdot s}{(s-s_2)\cdot(s-s_2^*)} = \frac{k_2}{s-s_2} + \frac{k_2^*}{s-s_2^*}$$

$$\Rightarrow k_2 = (s-s_2)\cdot H_2(s)\Big|_{s=s_2} = \frac{-2000\pi s}{s-s_2^*}\Bigg|_{s=s_2} = \frac{-2000\pi\cdot(1000\pi-1000\pi\sqrt{3}\,i)}{(1000\pi-1000\pi\sqrt{3}\,i)-(1000\pi+1000\pi\sqrt{3}\,i)}$$

$$= \frac{\cancel{-2000\pi}\cdot(1000\pi-1000\pi\sqrt{3}\,i)}{\cancel{-2000\pi}\sqrt{3}\,i} = \frac{1000\pi-1000\pi\sqrt{3}\,i}{\sqrt{3}i}\cdot\frac{i}{i} = \frac{1000\pi i+1000\pi\sqrt{3}}{-\sqrt{3}} = -1000\pi - 1000\pi\frac{1}{\sqrt{3}}i$$

$$\begin{cases} k_2 = -1000\pi - 1000\pi \frac{1}{\sqrt{3}} i \\ s_2 = 1000\pi - 1000\pi\sqrt{3}\, i \end{cases} \Rightarrow \begin{cases} \alpha_2 = -1000\pi \\ \beta_2 = -1000\pi \frac{1}{\sqrt{3}} \\ \sigma_2 = 1000\pi \\ \omega_2 = -1000\pi\sqrt{3} \end{cases}$$

$$\boxed{f_s = 8000 \Rightarrow T = \frac{1}{8000}}$$

$$\sigma_2 T = 1000\pi \cdot \frac{1}{8000} = \frac{\pi}{8}$$

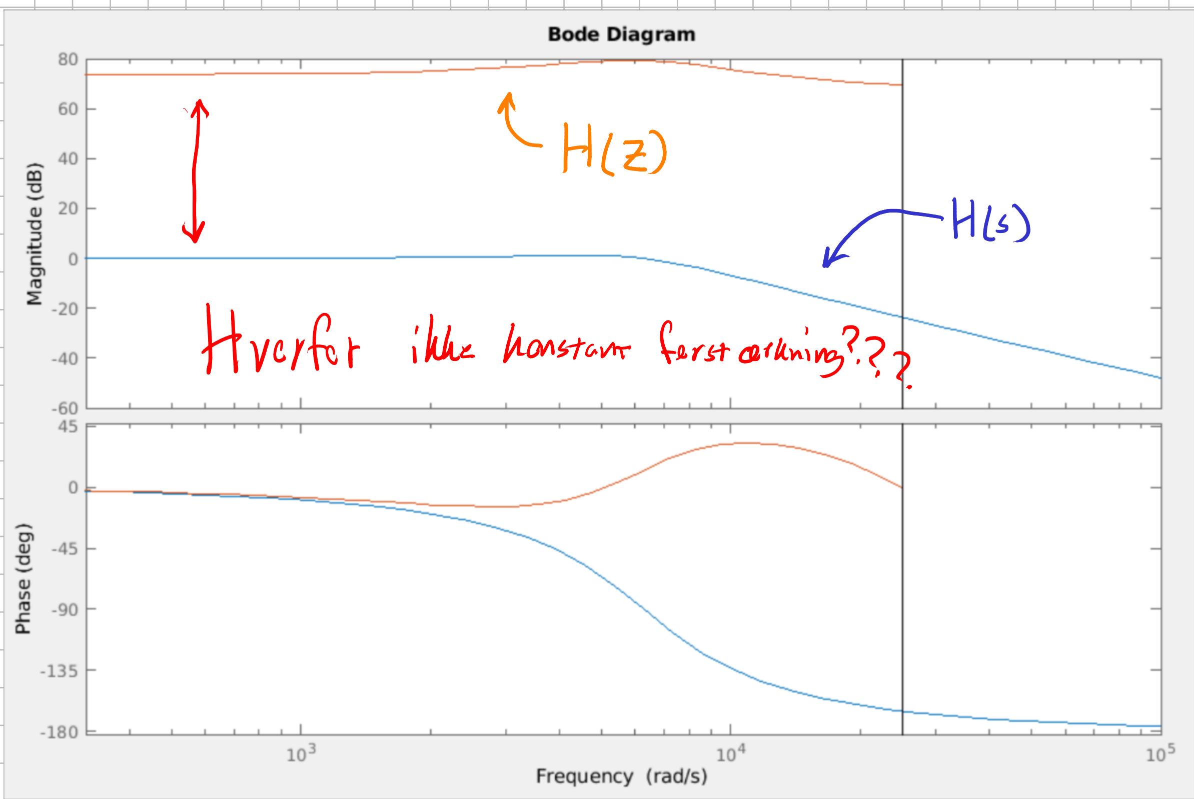
$$\omega_2 T = -1000\pi\sqrt{3} \cdot \frac{1}{8000} = \frac{-\pi\sqrt{3}}{8}$$

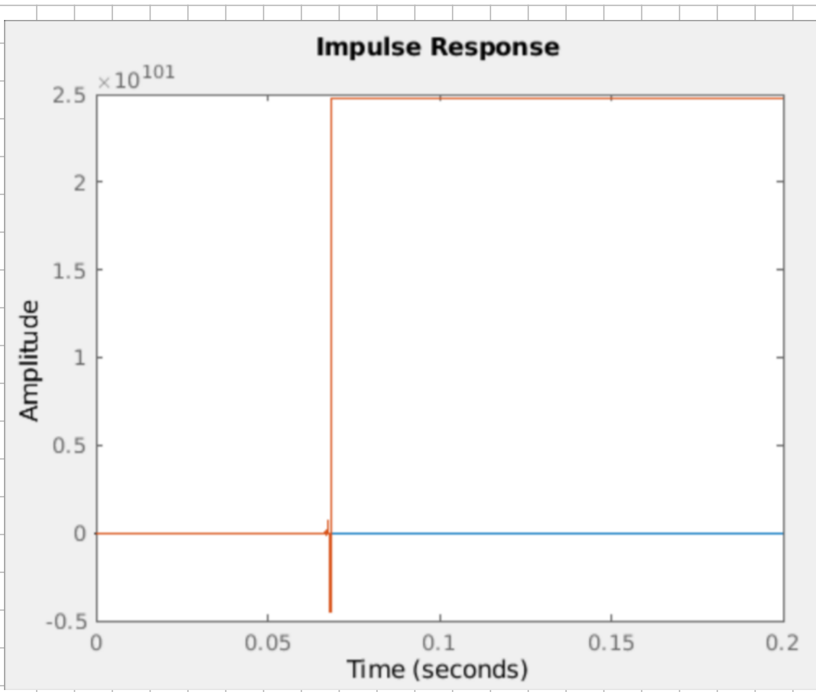
$$\Rightarrow \begin{cases} a_0 = 2\alpha_2 \\ a_1 = -2e^{\sigma_2 T}\left(\alpha_2 \cos(\omega_2 T) - \beta \cdot \sin(\omega T)\right) \\ b_1 = -\left(2e^{\sigma_2 T} \cdot \cos(\omega_2 T)\right) \\ b_2 = e^{2\sigma_2 T} \end{cases} \Rightarrow \begin{cases} a0 \to -2000\,\pi \\ a1 \to -2\, e^{\pi/8}\left(-1000\,\pi \operatorname{Cos}\left[\frac{\sqrt{3}\,\pi}{8}\right] - \frac{1000\,\pi \operatorname{Sin}\left[\frac{\sqrt{3}\,\pi}{8}\right]}{\sqrt{3}}\right) \\ b1 \to -2\, e^{\pi/8} \operatorname{Cos}\left[\frac{\sqrt{3}\,\pi}{8}\right] \\ b2 \to e^{\pi/4} \end{cases}$$

$$\approx \begin{cases} a0 \to -6283.19 \\ a1 \to 10613.3 \\ b1 \to -2.3028 \\ b2 \to 2.19328 \end{cases} \Rightarrow H_2(z) \approx \frac{-6283 + 10613 z^{-1}}{1 - 2{,}3028\, z^{-1} + 2{,}1933\, z^{-2}}$$

Den samplede overførinsfunktion i z-domænet!

$$\Rightarrow H(z) = H_1(z) + H_2(z) = \frac{\pi}{4} \cdot \frac{1}{1 - e^{\frac{-\pi}{4}} \cdot z^{-1}} + \frac{-6283 + 10613 z^{-1}}{1 - 2{,}3028\, z^{-1} + 2{,}1933\, z^{-2}}$$




Impulse Response
×10^101
2.5
2
1.5
1
0.5
0
-0.5
Amplitude
0
0.05
0.1
0.15
0.2
Time (seconds)