Integral Control and Optimal Control

Control Engineering (Reguleringsteknik)

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SDU

Agenda

Matematiske og grafiske metoder til syntese af lineære tidsinvariante systemer:[1]

- diskret og kontinuert tilstandsbeskrivelse
- analyse i tid og frekvens
- stabilitet, reguleringshastighed, følsomhed og fejl
- digitale PI, PID, LEAD og LAG regulatorer (serieregulatorer)
- tilstandsregulering, pole-placement og tilstands-estimering (observer)
- optimal regulering (least squares) og optimal tilstands-estimation (Kalman-filter)

Færdigheder:
Efter gennemførelse af kurset kan den succesfulde studerende:

- kunne analysere, dimensionere og implementere såvel kontinuert som tidsdiskret regulering af lineære tidsinvariante og stokastiske systemer

Kompetencer:
Efter gennemførelse af kurset kan den succesfulde studerende:

- anvende og implementere klassiske og moderne reguleringsteknikker for at kunne styre og regulere en robot hurtig og præcist

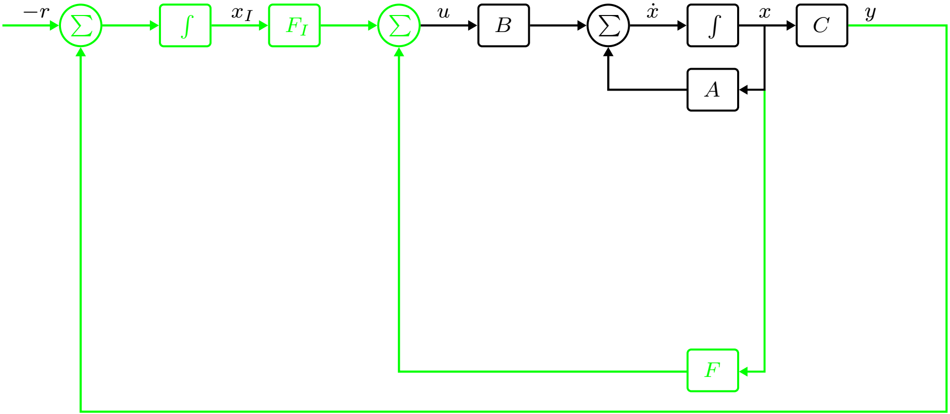
[1] Based on https://fagbesk.sam.sdu.dk/?fag_id=39673

The twelve lectures of the course are

- **Lecture 1**: Introduction to Linear Time-Invariant Systems
- **Lecture 2**: Stability and Performance Analysis
- **Lecture 3**: Introduction to Control
- **Lecture 4**: Design of PID Controllers
- **Lecture 5**: Root Locus
- **Lecture 6**: The Nyquist Plot
- **Lecture 7**: Dynamic Compensators and Stability Margins
- **Lecture 8**: Implementation
- **Lecture 9**: State Feedback
- **Lecture 10**: Observer Design
- **Lecture 11**: Optimal Control (Linear Quadratic Control)
- **Lecture 12**: The Kalman Filter

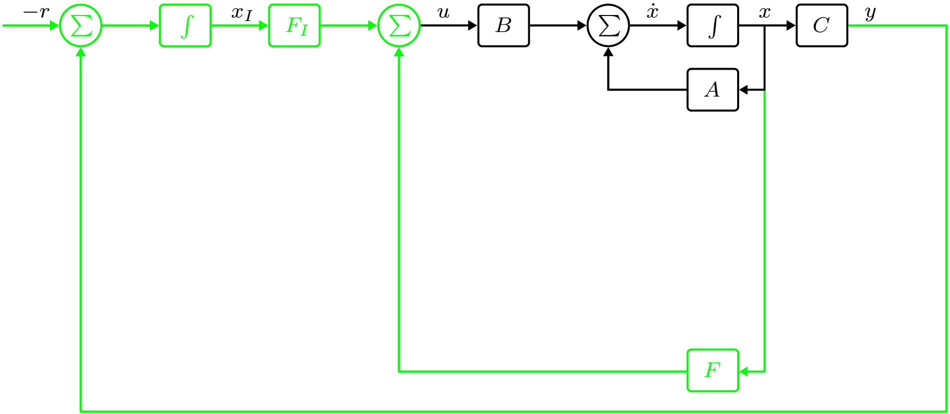
Integral Control

Block Diagram



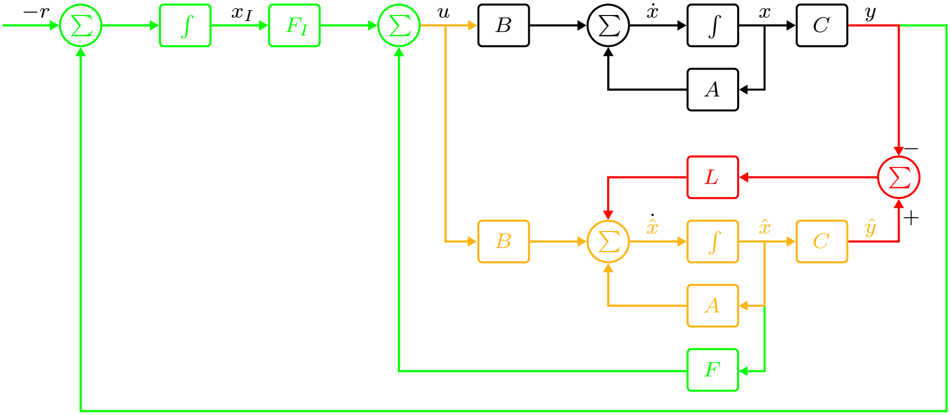
Integral Control

Block Diagram of Observer-Based Integral Control

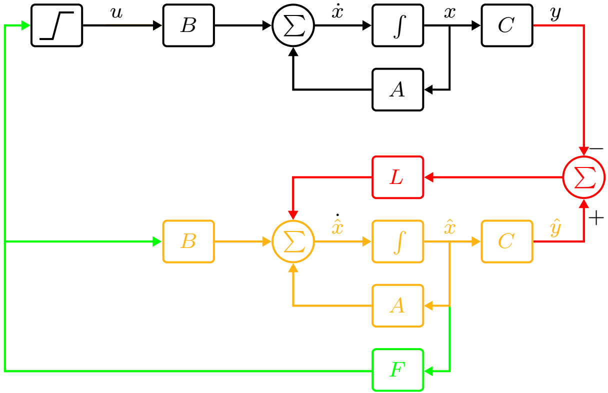


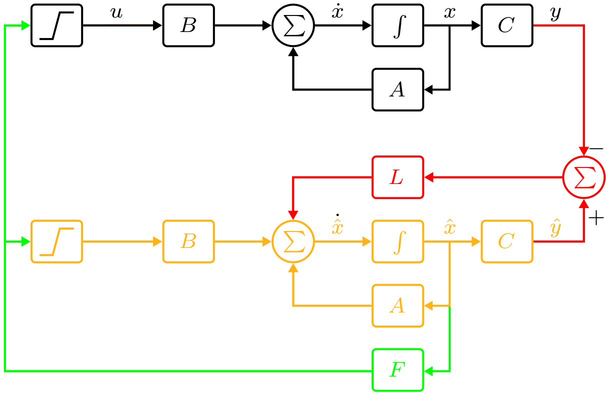
Integral Control

Block Diagram of Observer-Based Integral Control



Anti-Windup
Architecture





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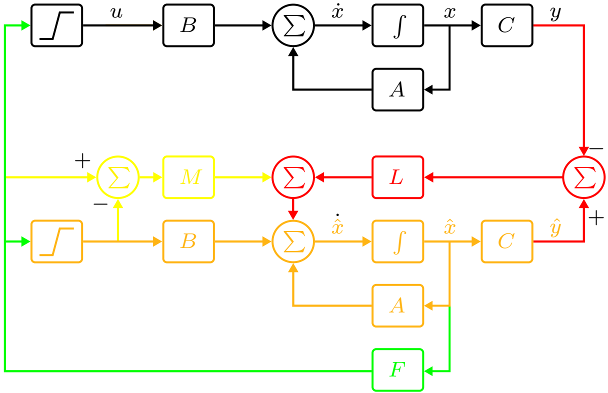
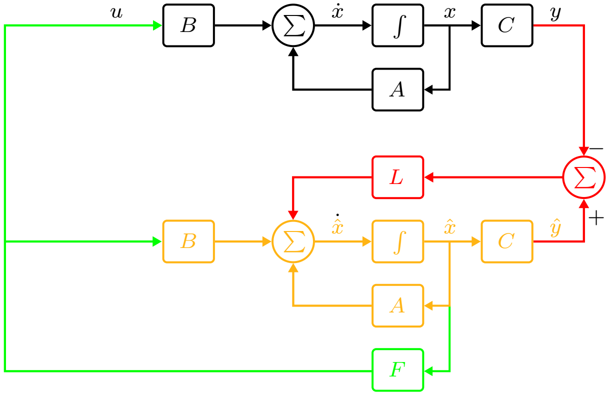
Anti-Windup
Architecture

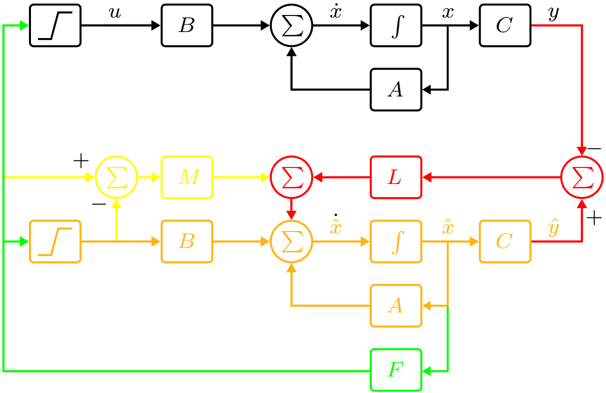


Anti-Windup
Nominal

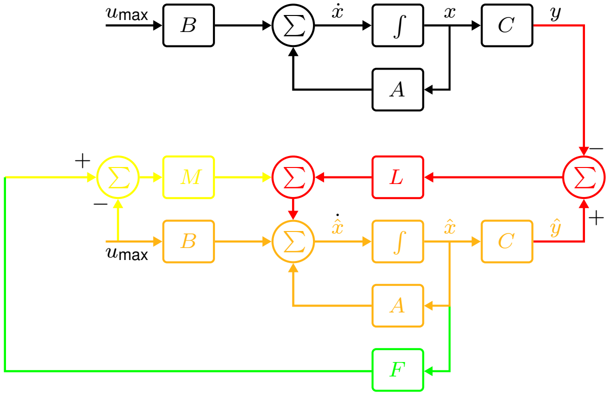


Anti-Windup

Saturated



Anti-Windup
Saturated



Dynamics of controller during saturation:

$$\dot{\hat{x}} = A\hat{x} + LC\hat{x} + MF\hat{x}$$

or

$$\dot{\hat{x}} = (A + LC + MF)\,\hat{x}$$

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or

$$\dot{\hat{x}} = (A + LC + MF)\,\hat{x}$$

Determining M can be recognized as an observer gain design problem:

$$\dot{\hat{x}} = \left(\tilde{A} + \tilde{L}\tilde{C}\right)\hat{x}$$

with $\tilde{A} = A + LC$, $\tilde{L} = M$, and $\tilde{C} = F$, from which the unknown $\tilde{L} = M$ can be chosen to assign any desired poles to the saturated controller.

We consider a linear control system of the form:

$$\begin{aligned} \dot{x} &= Ax + Bu\,, \quad x(0) = x_0 \\ y &= Cx \end{aligned}$$

A control law for such a system is said to be *optimal*, if it minimizes the cost functional:

$$\mathcal{J} = \int_0^\infty x^T Q x + u^T R u \; dt$$

where $Q = Q^T$ is a positive semi-definite matrix and $R = R^T$ is a positive definite matrix.

An *Algebraic Riccati Equation* is a second order matrix equation in an indeterminate $P = P^T \in \mathbb{R}^{n \times n}$ of the form:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ are matrices, $R = R^T \in \mathbb{R}^{m \times m}$ is a positive definite matrix, and $Q = Q^T \in \mathbb{R}^{n \times n}$ is a positive semidefinite matrix.

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P is called a *stabilizing solution* to the ARE, if it satisfies the equation, and further satisfies that the eigenvalues of $A - BR^{-1}B^T P$ are in the open left half plane.

THEOREM. Consider a linear system of the form:

$$\begin{aligned} \dot{x} &= Ax + Bu\,, \quad x(0) = x_0 \\ y &= Cx \end{aligned}$$

Let P be a stabilizing solution to the ARE:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

Then the optimal state feedback law is given by:

$$u = Fx \quad \text{where} \quad F = -R^{-1}B^T P$$

Introducing $y = Cx$ into a cost functional of the type

$$\mathcal{J} = \int_0^\infty \rho y^T y + u^T u \, dt \,, \quad \rho \in \mathbb{R}$$

this can be written as an optimal control problem

$$\begin{aligned}
\mathcal{J} &= \int_0^\infty \rho y^T y + u^T u \, dt \\
&= \int_0^\infty \rho x^T C^T C x + u^T u \, dt \\
&= \int_0^\infty x^T Q x + u^T R u \, dt \,, \quad Q = \rho C^T C \,, R = I
\end{aligned}$$

Alternatively, use a cost functional of the type

$$\mathcal{J} = \int_0^\infty x^T Q x + u^T R u \, dt$$

where Q and R are diagonal matrices with this can be written as an optimal control problem

$$\begin{aligned} Q_{ii} &= \frac{1}{\text{maximum acceptable value of } x_i^2} \\ R_{jj} &= \frac{1}{\text{maximum acceptable value of } u_j^2}. \end{aligned}$$

Given the system

$$
\begin{array}{ccccccc}
\dot{x} & = & Ax & + & Bu & + & Gw \\
y & = & Cx & + & Du & + & v
\end{array}
$$

with unbiased process noise w and measurement noise v with covariances

$$
\mathcal{E}\{ww^T\} = Q\,, \quad \mathcal{E}\{vv^T\} = R
$$

Then an optimal state estimator is given by:

$$
\dot{\hat{x}} = A\hat{x} + Bu + L(C\hat{x} - y)
$$

with unbiased process noise w and measurement noise v with covariances

$$\mathcal{E}\{ww^T\} = Q\,, \quad \mathcal{E}\{vv^T\} = R$$

Then an optimal state estimator is given by:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(C\hat{x} - y)$$

where

$$L = -PC^T R^{-1}$$

P is a stabilizing solution to the ARE:

$$AP + PA^T - PC^T R^{-1} CP + Q = 0$$

We consider once again the system

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} x + \begin{bmatrix} 2 \\ 3 \end{bmatrix} u \\
y &= \begin{bmatrix} -3 & 2 \end{bmatrix} x
\end{aligned}
$$

Computing an optimal state feedback for the cost functional:

$$
\mathcal{J} = \int_0^{\infty} 800\, y^T y + u^T u \, dt
$$

can be done with the MATLAB command

```
Fopt = -lqr(A,B,800*C'*C,1)
```

This yields the result:

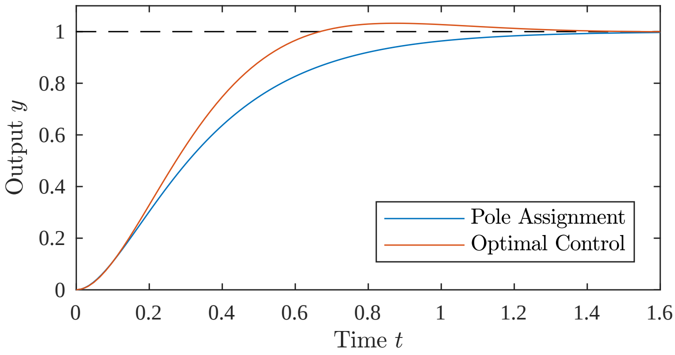
$$F_{\text{opt}} = \begin{bmatrix} 69.3536 & -47.8542 \end{bmatrix}$$

In comparison, a pole assignment with the poles $\{-4, -8\}$ leads to the gain:

$$F = \begin{bmatrix} 72 & -51 \end{bmatrix}$$

A first glance would suggest that the pole assignment with its larger gains would have faster dynamics. However, the optimal feedback assigns complex poles, giving a better rise-time.

Optimal Control
Example (3)



Introducing Reference Signals

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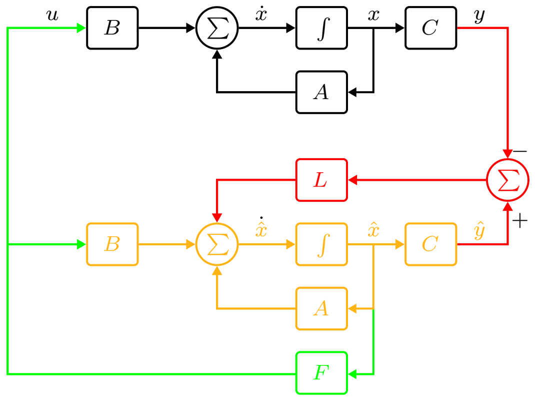
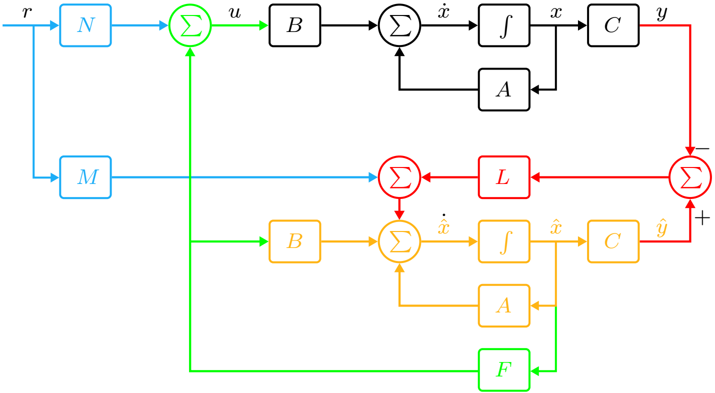
Block Diagram



Introducing Reference Signals

Block Diagram



Introducing Reference Signals

System Description

System:

$$\begin{aligned} \dot{x} &= Ax + B(F\hat{x} + Nr) \\ y &= Cx \end{aligned}$$

Observer:

$$\dot{\hat{x}} = A\hat{x} + BF\hat{x} + L(C\hat{x} - y) + Mr$$

System:

$$
\begin{aligned}
\dot{x} &= Ax + B(F\hat{x} + Nr) \\
y &= Cx
\end{aligned}
$$

Observer:

$$
\dot{\hat{x}} = A\hat{x} + BF\hat{x} + L(C\hat{x} - y) + Mr
$$

$$
\begin{aligned}
\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} &= \begin{bmatrix} A & BF \\ -LC & A + BF + LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} BN \\ M \end{bmatrix} r \\
y &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}
\end{aligned}
$$

Recall from Lecture 2 how to find the zeros of a state space model.

LEMMA. A square (#inputs=#outputs) system with a state space model of the form

$$\begin{array}{rcrcr} \dot{x} & = & Ax & + & Bu \\ y & = & Cx & + & Du \end{array}$$

has a zero with value $z \in \mathbb{C}$ only if

$$\det \begin{bmatrix} A - zI & B \\ C & D \end{bmatrix} = 0$$

$$\det\left(\begin{bmatrix} A_{\mathsf{cl}} - zI & B_{\mathsf{cl}} \\ C_{\mathsf{cl}} & D_{\mathsf{cl}} \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} A_{\text{cl}} - zI & B_{\text{cl}} \\ C_{\text{cl}} & D_{\text{cl}} \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} A - zI & BF & BN \\ -LC & A + BF + LC - zI & M \\ C & 0 & 0 \end{bmatrix}\right) = 0$$

Introducing Reference Signals
Zeros of Closed-Loop System

$$\det\left(\begin{bmatrix} A_{\text{cl}} - zI & B_{\text{cl}} \\ C_{\text{cl}} & D_{\text{cl}} \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} A - zI & BF & BN \\ -LC & A + BF + LC - zI & M \\ C & 0 & 0 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} A - zI & BF - BNN^{-1}F & BN \\ -LC & A + BF + LC - zI - MN^{-1}F & M \\ C & 0 & 0 \end{bmatrix}\right) = 0$$

Introducing Reference Signals

Zeros of Closed-Loop System

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Introducing Reference Signals

Zeros of Closed-Loop System

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$$\det\left(\begin{bmatrix} A - zI & 0 & BN \\ -LC & A + BF + LC - MN^{-1}F - zI & M \\ C & 0 & 0 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} A - zI & 0 & B \\ -LC & A + BF + LC - \tilde{M}F - zI & \tilde{M} \\ C & 0 & 0 \end{bmatrix}\right) = 0$$

Introducing Reference Signals

Zeros of Closed-Loop System

$$\det\left(\begin{bmatrix} A-zI & BF-BNN^{-1}F & BN \\ -LC & A+BF+LC-zI-MN^{-1}F & M \\ C & 0 & 0 \end{bmatrix}\right)=0$$

$$\det\left(\begin{bmatrix} A-zI & 0 & BN \\ -LC & A+BF+LC-MN^{-1}F-zI & M \\ C & 0 & 0 \end{bmatrix}\right)=0$$

$$\det\left(\begin{bmatrix} A-zI & 0 & B \\ -LC & A+BF+LC-\tilde{M}F-zI & \tilde{M} \\ C & 0 & 0 \end{bmatrix}\right)=0$$

$$\begin{cases} \det\left(\begin{bmatrix} A-zI & B \\ C & 0 \end{bmatrix}\right)=0 \qquad \text{or} \\ \det\left(A+BF+LC-\tilde{M}F-zI\right)=0 \end{cases}$$

LEMMA. If $\tilde{M}$ is an 'observer gain' such that the characteristic polynomial of the matrix $A_{\text{za}} + \tilde{M} C_{\text{za}}$ has the characteristic polynomial

$$\det\left(sI - \left(A_{\text{za}} + \tilde{M} C_{\text{za}}\right)\right) = (s - z_1) \cdots (s - z_n)$$

with $A_{\text{za}} = A + BF + LC$ and $C_{\text{za}} = -F$, then the numbers $z_1, \ldots, z_n$ are all zeros of the closed loop transfer function from r to y.

1. Design $\tilde{M}$ assigning zeros close to the cut-off frequency of the Bode plot, such that the 'horizontal' part is extended.

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2. Compute N such that the DC-value of the transfer function from r to y is unity:

$$N = -\left(C_{\text{cl}} A_{\text{cl}}^{-1} \tilde{B}_{\text{cl}}\right)^{-1}$$

where

$$A_{\text{cl}} = \begin{bmatrix} A & BF \\ -LC & A + BF + LC \end{bmatrix}, \quad \tilde{B}_{\text{cl}} = \begin{bmatrix} B \\ \tilde{M} \end{bmatrix}$$

$$C_{\text{cl}} = \begin{bmatrix} C & 0 \end{bmatrix}$$

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$$C_{\text{cl}} = \begin{bmatrix} C & 0 \end{bmatrix}$$

3. Compute $M = MN^{-1}N = \tilde{M}N$.

Zero Assignment

Example (1)

We consider again the system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} x + \begin{bmatrix} 2 \\ 3 \end{bmatrix} u \\ y &= \begin{bmatrix} -3 & 2 \end{bmatrix} x \end{aligned}$$

A state feedback F that assign poles in $\{-3, -4\}$ and an observer gain L that assigns poles in $\{-9, -12\}$ are given by:

$$F = \begin{bmatrix} 22 & -16 \end{bmatrix}, \quad L = \begin{bmatrix} -122 \\ -192 \end{bmatrix}$$

We would like to assign zeros from r to y in $\{-3, -4\}$ to cancel the poles from F.

Zero Assignment

Example (2)

With these values of F and L we obtain:

$$A_{\text{za}} = A + BF + LC = \begin{bmatrix} 412 & -279 \\ 646 & -437 \end{bmatrix}$$

$$C_{\text{za}} = -F = \begin{bmatrix} -22 & 16 \end{bmatrix}$$

An 'observer gain' that assigns poles in $\{-3, -4\}$ for $A_{\text{za}} + \tilde{M} C_{\text{za}}$ is

$$\tilde{M} = \begin{bmatrix} 7.0460 \\ 10.8133 \end{bmatrix}$$

N can be computed as:

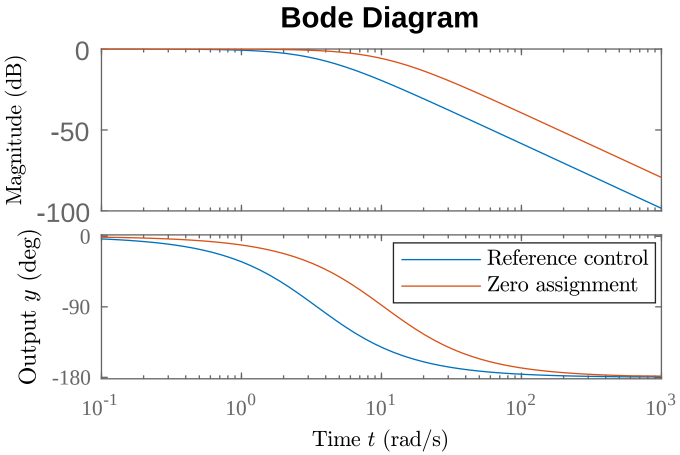
$$N = -\left(\begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} A & BF \\ -LC & A + BF + LC \end{pmatrix}^{-1} \begin{pmatrix} B \\ \tilde{M} \end{pmatrix}\right)^{-1}$$
$$= 108$$

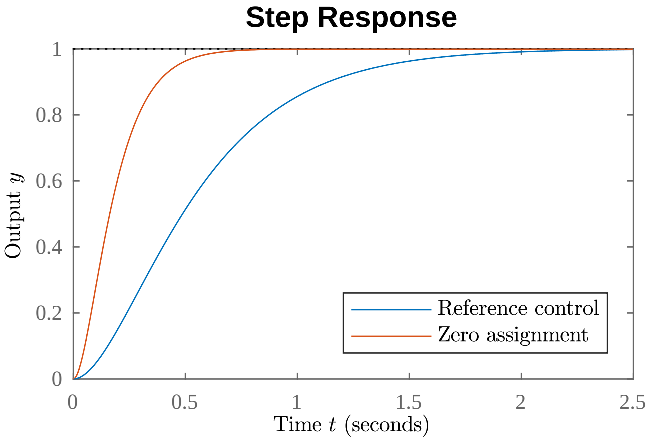
M is obtained from:

$$M = \tilde{M} N = \begin{bmatrix} 7.0460 \\ 10.8133 \end{bmatrix} \cdot 108 = \begin{bmatrix} 760.97 \\ 1167.84 \end{bmatrix}$$

Zero Assignment

Example: Bode Plot





Step Response
1
0.8
0.6
0.4
0.2
0
Output y
0
0.5
1
1.5
2
2.5
Time t (seconds)
Reference control
Zero assignment