Design of PID Controllers

Control Engineering (Reguleringsteknik)

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The Maersk Mc-Kinney Moller Institute
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Agenda

Introduction
Curriculum for Reguleringsteknik (REG)

Matematiske og grafiske metoder til syntese af lineære tidsinvariante systemer:[1]

- diskret og kontinuert tilstandsbeskrivelse
- analyse i tid og frekvens
- stabilitet, reguleringshastighed, følsomhed og fejl
- digitale PI, PID, LEAD og LAG regulatorer (serieregulatorer)
- tilstandsregulering, pole-placement og tilstands-estimering (observer)
- optimal regulering (least squares) og optimal tilstands-estimation (Kalman-filter)

Færdigheder:
Efter gennemførelse af kurset kan den succesfulde studerende:

- kunne analysere, dimensionere og implementere såvel kontinuert som tidsdiskret regulering af lineære tidsinvariante og stokastiske systemer

Kompetencer:
Efter gennemførelse af kurset kan den succesfulde studerende:

- anvende og implementere klassiske og moderne reguleringsteknikker for at kunne styre og regulere en robot hurtig og præcist

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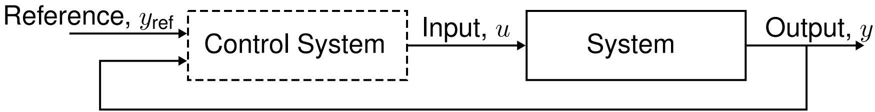
[1] Based on https://fagbesk.sam.sdu.dk/?fag_id=39673

The twelve lectures of the course are

- **Lecture 1**: Introduction to Linear Time-Invariant Systems
- **Lecture 2**: Stability and Performance Analysis
- **Lecture 3**: Introduction to Control
- **Lecture 4**: Design of PID Controllers
- **Lecture 5**: Root Locus
- **Lecture 6**: The Nyquist Plot
- **Lecture 7**: Dynamic Compensators and Stability Margins
- **Lecture 8**: Implementation
- **Lecture 9**: State Feedback
- **Lecture 10**: Observer Design
- **Lecture 11**: Optimal Control (Linear Quadratic Control)
- **Lecture 12**: The Kalman Filter

Task: Design a cruise control for a car.



- **Control Input**: Throttle position u
- **Measured Output**: Velocity of the car y
- **Reference Input**: Desired velocity of the car y_{ref}

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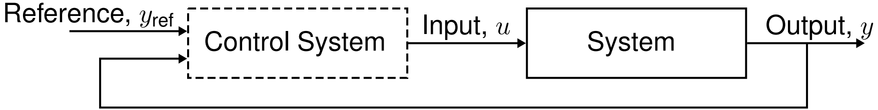

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1. How should the controller structure be chosen to ensure that a desired velocity y_{ref} is reached?

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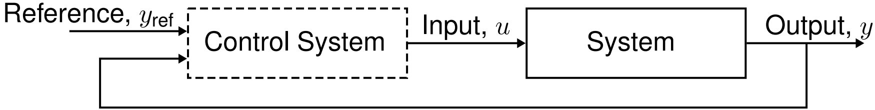

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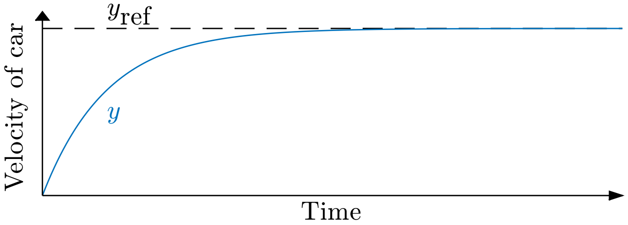

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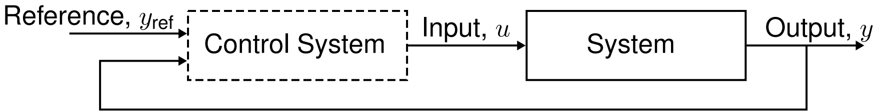
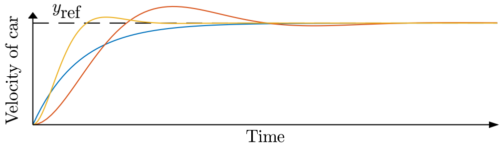
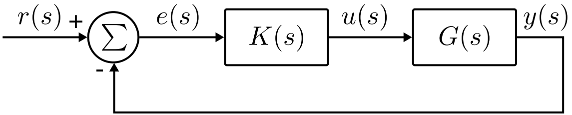


Today, answers to the following questions are provided:

1. How should the controller structure be chosen to ensure that a desired velocity y_{ref} is reached?
2. How do PID gains affect the performance of the system?



Recall that a feedback control is defined as shown in the block diagram.



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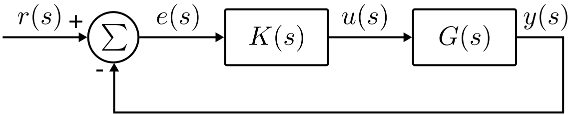

The closed-loop transfer function is

$$T_{\mathsf{cl}}(s) = \frac{y(s)}{r(s)} = \frac{G(s)K(s)}{1 + G(s)K(s)}$$

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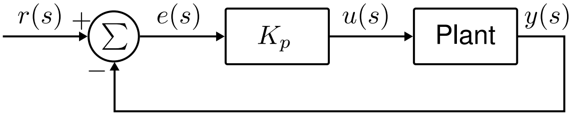

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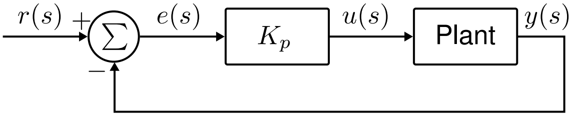

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It is seen that K_p should be large to make the steady state error small.

Consider a plant with transfer function

$$G(s) = \frac{y(s)}{u(s)} = \frac{1}{(s+1)^3}$$

and proportional control $u = K_p e$.

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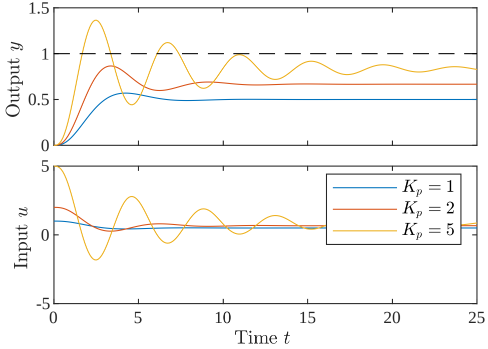
The steady-state error of the system is $G(0) = 1$

$$e_{ss} = \frac{1}{1+K_p} r$$

Unit step responses are shown below for $K_p = 1, 2, 5$.



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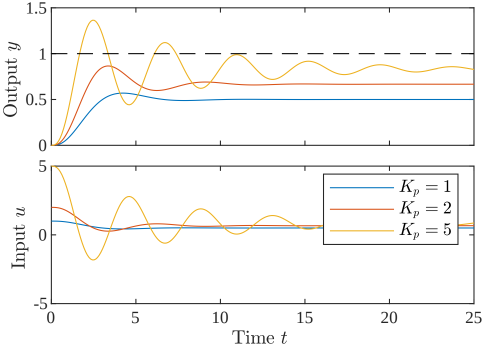
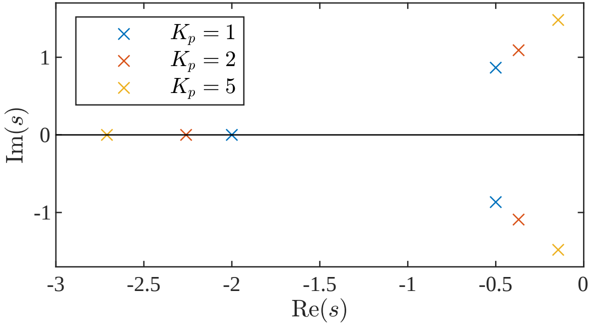


There are also disadvantages of choosing a high controller gain.

Map of closed-loop poles.



To eliminate the steady state error, feedforward can be added to the P-Controller

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$$u(t) = K_p e(t) + \frac{1}{G(0)} r(t)$$

The feedforward eliminates the steady state error if the dynamics of the system is perfectly known.

The closed-loop transfer function of the system with feedforward is

$$T(s) = \frac{G(s)/G(0) + G(s)K_p}{1 + G(s)K_p}$$

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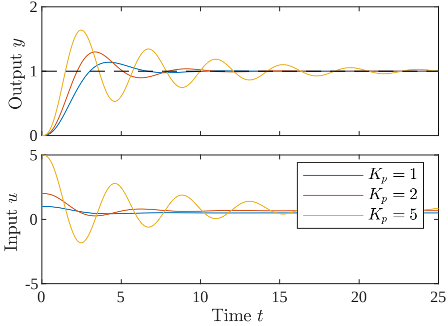

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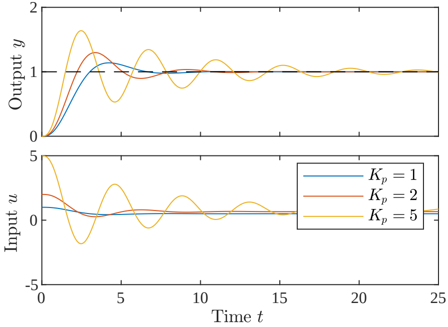

Note that the feedforward does not affect the poles of the closed loop system.

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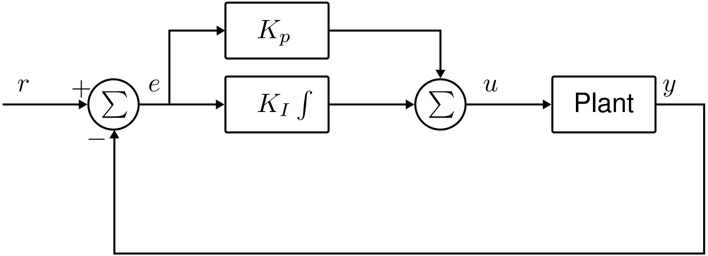
The controller adds a term, which is proportional to the integral of the error e to the P-control term.



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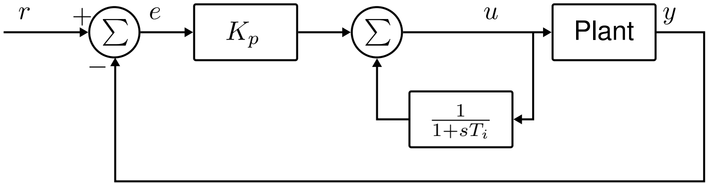
Proof. Assume that there is a steady state error with $u = u_0$ and e_0. Then

$$u_0 = K_p e_0 + k_I e_0 t$$

which is a contradiction except if $e_0 = 0$ or $K_I = 0$.

The following implementation of a PI-controller is called *automatic reset*.



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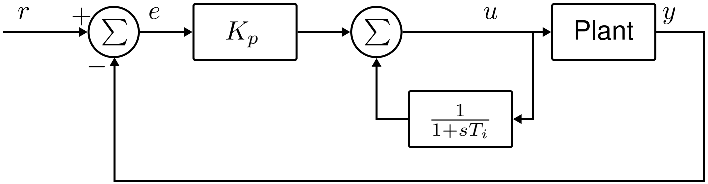
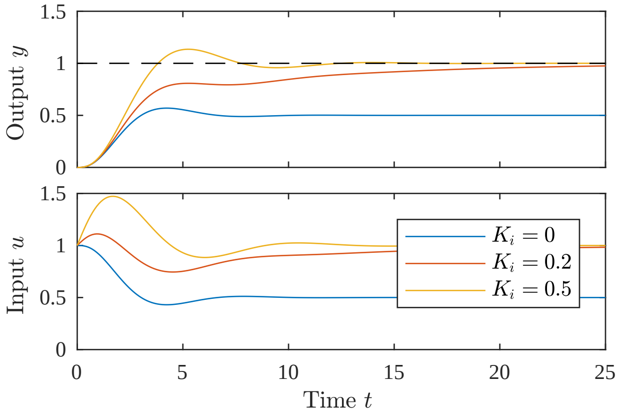


The transfer function from e to u is

$$T_{ue} = K_p \frac{1+sT_i}{sT_i} = K_p + \frac{K_p}{sT_i}$$

Unit step responses are shown below for $K_i = 0, 0.2, 0.5$ and $K_p = 1$.



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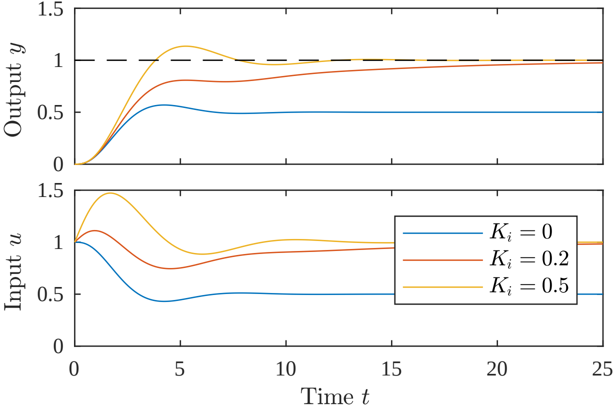
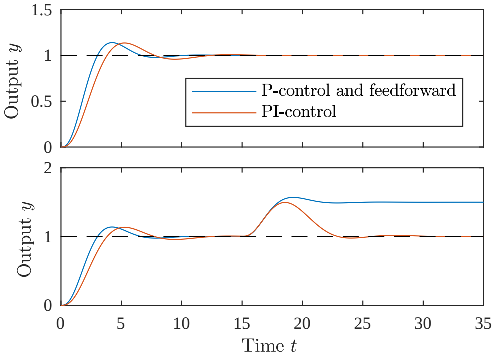


It is seen that the integral action removes the steady state error.

The feedforward control can only eliminate the steady state error if the model and disturbances are known.



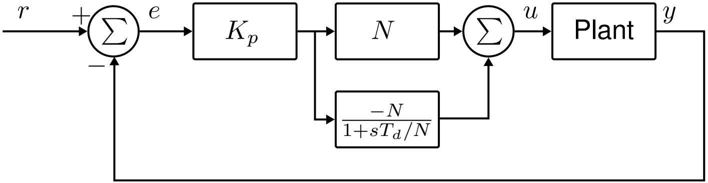
A step disturbance is added at time 15 s resulting in responses shown in the lower subplot.

The derivative term provides an anticipatory action to the control, by doing feedback based on the trend of the error, i.e.

$$u(t) = K_p e(t) + K_d \frac{de(t)}{dt} = K_p \underbrace{\left(e + T_d \frac{de(t)}{dt}\right)}_{=e_p}$$

where K_d is the derivative gain, T_d is the derivative time constant, and e_p is a prediction of the error (T_d forwards in time).

Noise is an issue for controllers that include a derivative term. Therefore, they can be implemented in a low-pass filtered version.



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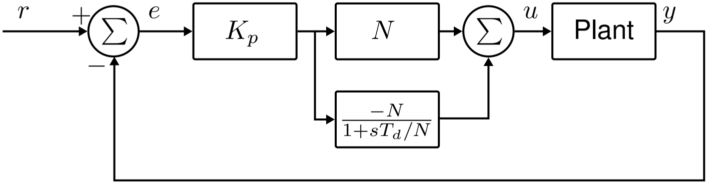
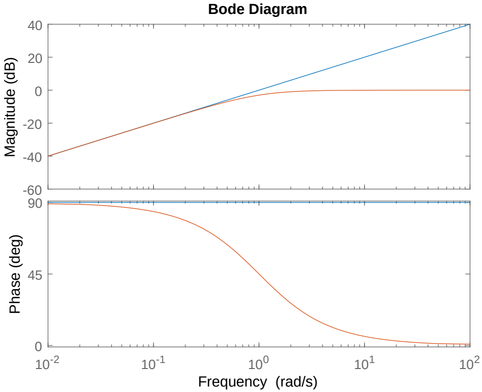


The transfer function of the controller is

$$T_{ue}(s) = K_p\left(N - \frac{N}{1+sT_d/N}\right) = K_p\frac{sT_d}{1+sT_d/N}$$

where N is a filter constant (typical values of N are 2 to 20)

The bode plot of an ideal PD-controller and a filtered PD-controller are similar for low frequencies (frequencies below $1/T_d$).



The control law of a proportional-integral-derivative feedback controller (PID-Controller) is

$$u(t) = K_p e(t) + k_I \int_{t_0}^{t} e(\tau) d\tau + k_D \frac{de(t)}{dt}$$

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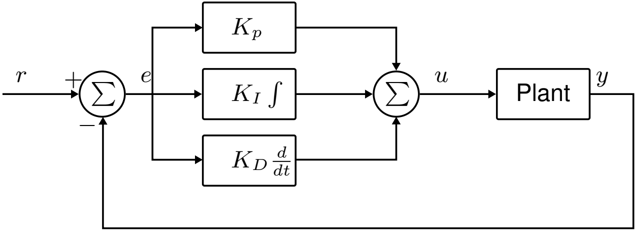
A block diagram of the controller is given below.



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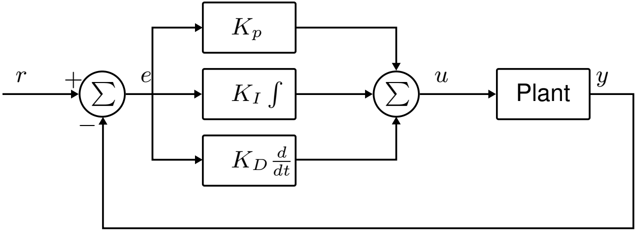


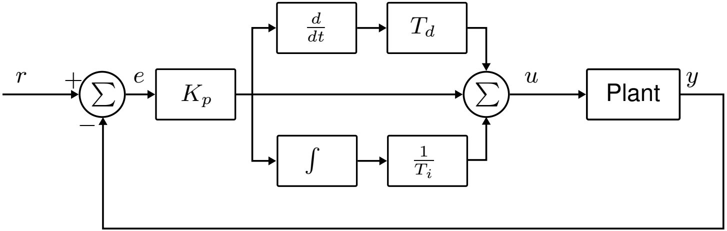
Alternatively, the PID controller with filter on the D-term is

$$K(s) = K_p \left(1 + \frac{1}{sT_i} + \frac{sT_d}{1 + sT_d/N}\right)$$

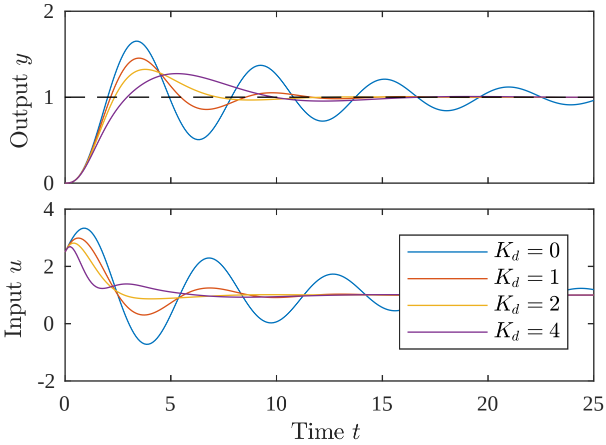
The PID controller

$$K(s) = K_p \left(1 + \frac{1}{sT_i} + sT_d\right)$$

has the following diagram.



Unit step responses are shown below for $K_d = 0, 1, 2, 4$ and $K_p = 2.5, K_i = 1.5$.



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In the next lecture, the root locus method is presented to give an in-depth explanation of how the change of a gain affects the closed-loop poles.

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Zeigler and Nichols has proposed two methods for tuning PID controllers without explicit use of a plant model

- Zeigler-Nichols tuning based on step response
- Zeigler-Nichols tuning - The ultimate sensitivity method

The considered system is assumed to have the transfer function

$$\frac{y(s)}{u(s)} = \frac{A}{\tau s + 1} e^{-st_d}$$

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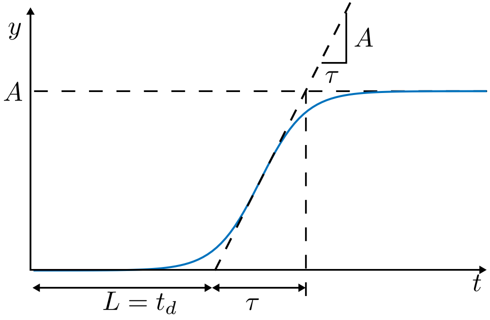
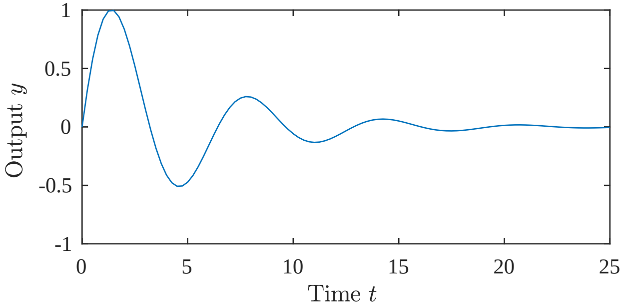


By using the Ziegler-Nichols tuning method, a closed-loop system is obtained that has a decay ratio of ~ 0.25 (damping ration $\zeta \approx 0.21$).



The parameters for the PID controller

$$u(s) = K_p \left(1 + \frac{1}{T_i s} + T_d s\right) e(s)$$

are given in the table[2].

Type of Controller	PID Gains
P	$K_p = \frac{1}{RL}$
PI	$\begin{cases} K_p = \frac{0.9}{RL} \\ T_i = \frac{L}{0.3} \end{cases}$
PID	$\begin{cases} K_p = \frac{1.2}{RL} \\ T_i = 2L \\ T_d = 0.5L \end{cases}$

[2] $R = A/\tau$

Consider the system

$$T(s) = \frac{y(s)}{u(s)} = \frac{e^{-5s}}{(10s+1)(60s+1)}$$

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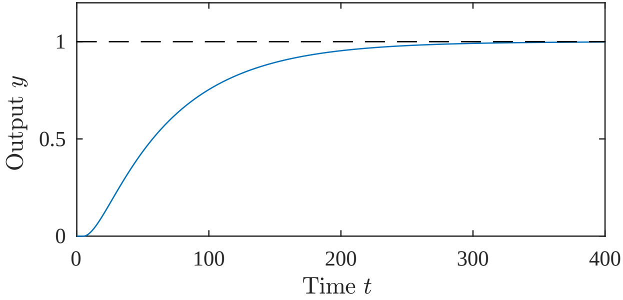
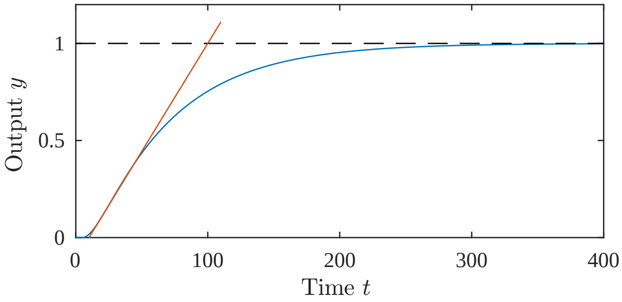
A step response for the system is given below.



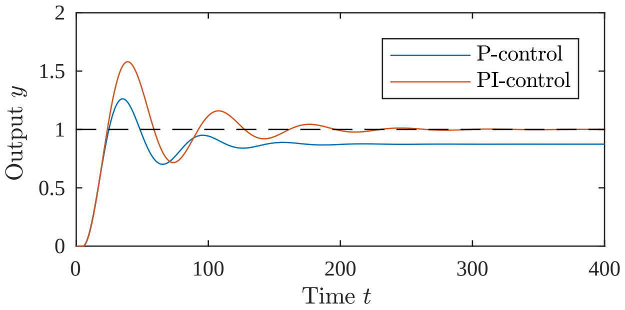
The lag is approximated to be 13 s ($L = 13$), the amplitude is one ($A = 1$), and the time-constant is 90 s $\tau = 90$ s ($R = A/\tau = 1/90$).



Tuning a PID Controller

Ziegler-Nichols Tuning: Example (3)

Step responses of the resulting closed-loop systems are shown below.



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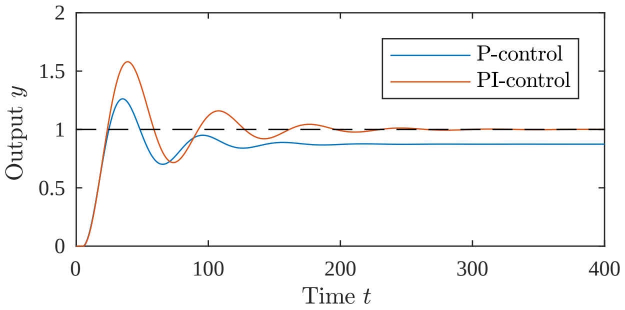
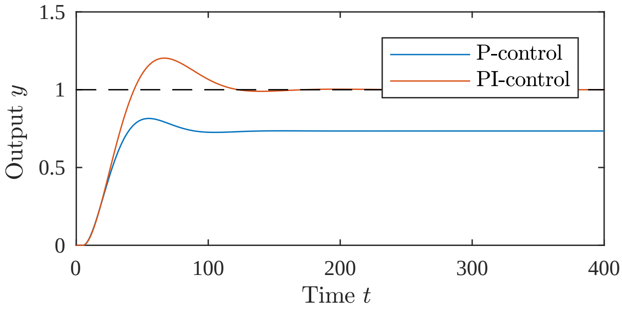
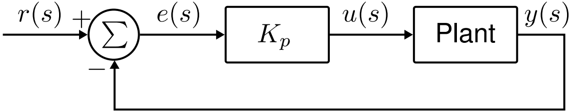


Remark: The method only provides an initial guess for the controller gains. The controller most often needs manual tuning as well.

By reducing the proportional gain to 40% of the original gain, the following response is obtained.



An alternative to studying the step response is to connect a P-controller to the system, and increase the gain K_p until the output oscillates.



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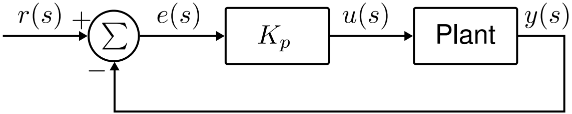
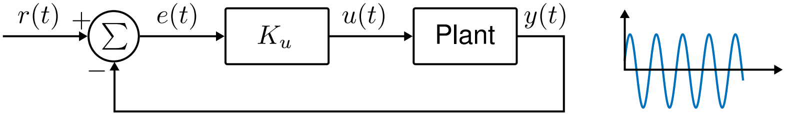


The value of K_p when the output oscillates with a constant amplitude is called the ***ultimate gain*** K_u.



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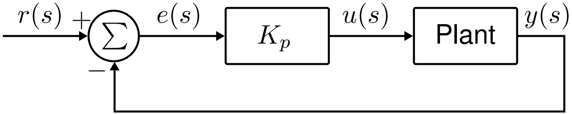

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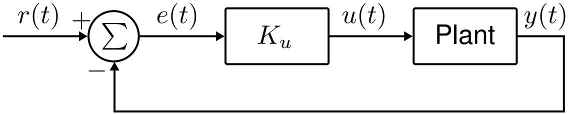

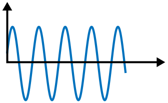

The period of the oscillation P_u is called the ***ultimate period***.

The parameters for the PID controller

$$u = K_p \left(1 + \frac{1}{T_i s} + T_d s \right) e$$

are given in the table.

Type of Controller	PID Gains
P	$K_p = 0.5 K_u$
PI	$\begin{cases} K_p = 0.45 K_u \\ T_i \ = \frac{P_u}{1.2} \end{cases}$
PID	$\begin{cases} K_p = 0.6 K_u \\ T_i = 0.5 P_u \\ T_d = \frac{1}{8} P_u \end{cases}$

Consider the system

$$T(s) = \frac{y(s)}{u(s)} = \frac{e^{-5s}}{(10s+1)(60s+1)}$$

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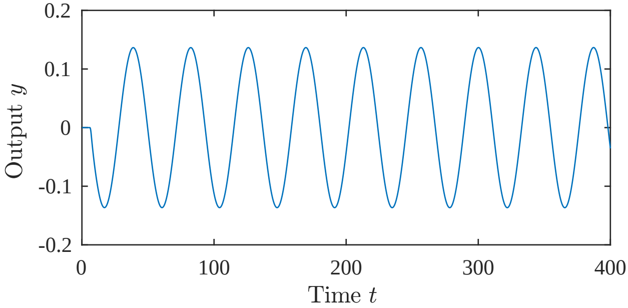
By increasing the gain of a P-controller to $K_u = 15.293$ the following response is observed.



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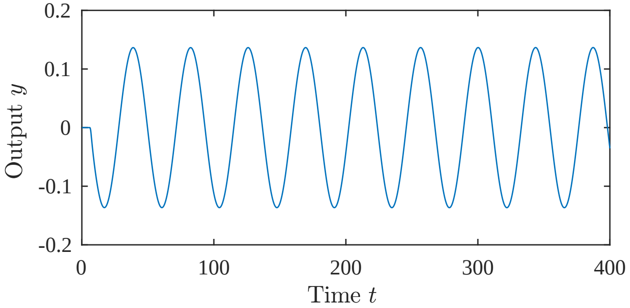
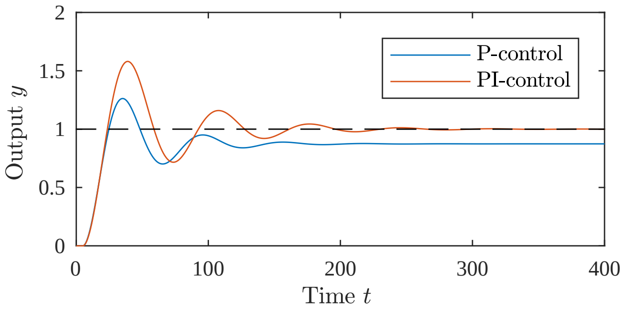


It is seen that the ultimate period P_u is about $P_u = 43.5$.

Tuning a PID Controller

Ziegler-Nichols Tuning (USM): Example (2)

Step responses of the closed-loop systems are shown below.



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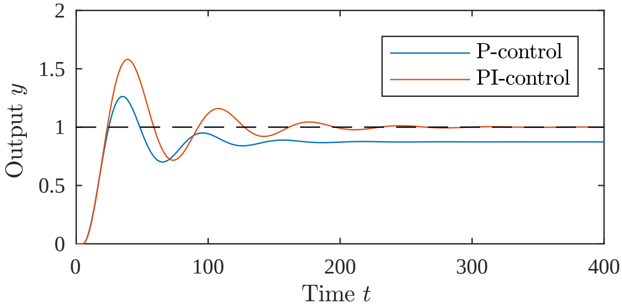
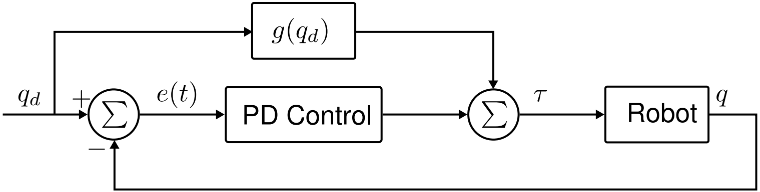


Remark: The method only provides an initial guess for the controller gains. The controller most often needs manual tuning as well.

PID Control of Robotic Manipulator

Setup

Most robots are controlled by PID controllers. One control scheme is PD control with desired gravity compensation ($g(q)$ gives the gravitational force, and q denotes the angles of revolute joints).



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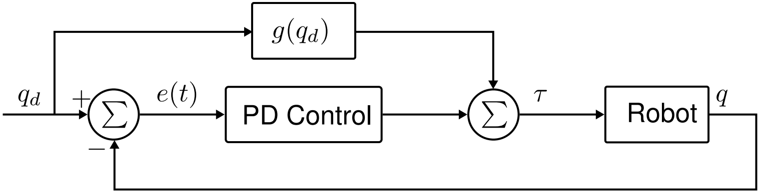

Global asymptotic stability is obtained if the proportional gain is chosen high enough.

A step response of the closed-loop system is shown below.