

Problem 1:

$$\iiint xy^2 \cos(z) \, dydzdx \quad R: \left\{ x, y, z \mid \begin{array}{l} 0 \leq x \leq 2 \\ 0 \leq y \leq 3 \text{ and } 0 \leq z \leq \pi/2 \end{array} \right.$$

Answer: 18

$$\begin{aligned} \int_0^2 \int_0^{\frac{\pi}{2}} \int_0^3 xy^2 \cdot \cos(z) \, dydzdx &= \int_0^2 \int_0^{\frac{\pi}{2}} \left[x \cdot \cos(z) \cdot \frac{y^3}{3} \right]_0^3 dzdx \\ &= \int_0^2 \int_0^{\frac{\pi}{2}} x \cdot \cos(z) \cdot \frac{3^3}{3} dzdx = 9 \cdot \int_0^2 \int_0^{\frac{\pi}{2}} x \cdot \cos(z) dzdx \\ &= 9 \cdot \int_0^2 \left[x \cdot \sin(z) \right]_0^{\frac{\pi}{2}} dx = 9 \cdot \int_0^2 x \, dx = 9 \cdot \left[\frac{x^2}{2} \right]_0^2 = 9 \cdot \frac{4}{2} = \underline{\underline{18}} \end{aligned}$$

Problem 2:

$$\iiint 2\sqrt{y}e^{-x^2} dz dx dy \quad R: \left\{ x, y, z \mid \begin{array}{l} 0 \leq x \leq 1 \\ 0 \leq y \leq 4 \text{ and } 0 \leq z \leq x \end{array} \right.$$

Answer: $\frac{16}{3}(1 - e^{-1})$

$$\begin{aligned} \int_0^4 \int_0^1 \int_0^x 2\sqrt{y}e^{-x^2} dz dx dy &= 2 \cdot \int_0^4 \sqrt{y} \int_0^1 e^{-x^2} \int_0^x 1 dz dx dy \\ &= 2 \cdot \int_0^4 \sqrt{y} \int_0^1 e^{-x^2} \left[z \right]_0^x dx dy = 2 \cdot \int_0^4 \sqrt{y} \int_0^1 e^{-x^2} \cdot x dx dy \end{aligned}$$

$$u = -x^2 \Rightarrow \frac{du}{dx} = -2x \Rightarrow dx = \frac{-1}{2x} du$$

$$= 2 \cdot \int_0^4 \sqrt{y} \int_{x=0}^{x=1} e^u \cdot \cancel{x} \cdot \frac{-1}{\cancel{2x}} du dy = - \int_0^4 \sqrt{y} \int_{x=0}^{x=1} e^u du dy$$

$$= - \int_0^4 \sqrt{y} \cdot \left[e^u \right]_{x=0}^{x=1} dy = - \int_0^4 \sqrt{y} \cdot \left[e^{-x^2} \right]_0^1 dy = - \int_0^4 \sqrt{y} \cdot (e^{-1} - e^{-0}) dy$$

$$= -(e^{-1} - 1) \cdot \int_0^4 \sqrt{y} dy = (1 - e^{-1}) \cdot \int_0^4 y^{\frac{1}{2}} dy = (1 - e^{-1}) \cdot \int_0^4 y^{\frac{1}{2}} dy$$

$$= (1 - e^{-1}) \cdot \left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^4 = (1 - e^{-1}) \cdot \left[\frac{2}{3} \cdot y^{\frac{3}{2}} \right]_0^4 = (1 - e^{-1}) \cdot \frac{2}{3} \left[(\sqrt{y})^3 \right]_0^4$$

$$= (1 - e^{-1}) \cdot \frac{2}{3} \cdot \left((\sqrt{4})^3 - (\sqrt{0})^3 \right) = (1 - e^{-1}) \cdot \frac{2}{3} \cdot 2^3 = \underline{\underline{(1 - e^{-1}) \cdot \frac{16}{3}}}$$

Problem 3:

$$\iiint 1 + 2x - 3y \, dx \, dy \, dz$$

$$R: \left\{ x, y, z \mid \begin{array}{l} -a \leq x \leq a \\ -b \leq y \leq b \text{ and } -c \leq z \leq c \end{array} \right.$$

Answer: 8abc

$$\int_{-c}^c \int_{-b}^b \int_{-a}^a 1 + 2x - 3y \, dx \, dy \, dz = \int_{-c}^c \int_{-b}^b \left[x + x^2 - 3yx \right]_{-a}^a dy \, dz$$

$$= \int_{-c}^c \int_{-b}^b a + a^2 - 3ya - (-a + (-a)^2 - 3y(-a)) \, dy \, dz$$

$$= \int_{-c}^c \int_{-b}^b \cancel{a} + \cancel{a^2} - 3ya + \cancel{a} - \cancel{a^2} - 3ay \, dy \, dz = \int_{-c}^c \int_{-b}^b 2a - 6ay \, dy \, dz$$

$$= \int_{-c}^c \left[2ay - 3ay^2 \right]_{-b}^b dz = \int_{-c}^c 2ab - 3ab^2 - (2a(-b) - 3a(-b)^2) \, dz$$

$$= \int_{-c}^c \cancel{2ab} - \cancel{3ab^2} + \cancel{2ab} + \cancel{3ab^2} \, dz = \int_{-c}^c 4ab \, dz = \left[4abz \right]_{-c}^c$$

$$= 4abc - 4ab(-c) = \underline{\underline{8abc}}$$

Problem 4:

Find triple integral of function $\frac{1}{(x+y+z)^3} dx dy dz$ when R is bounded by 6 planes $z = 1, z = 2, y = 0, y = z, x = 0$, and $x = y + z$.

Answer: $\frac{3}{16} \ln 2$.

$$\int_1^2 \int_0^z \int_0^{y+z} \frac{1}{(x+y+z)^3} dx dy dz = \int_1^2 \int_0^z \int_0^{y+z} u \cdot \frac{1}{(x+y+z)^2} \cdot (-1) du dy dz$$

$$u = \frac{1}{x+y+z} \Rightarrow \frac{du}{dx} = \frac{-1}{(x+y+z)^2} \cdot 1 \Rightarrow dx = -(x+y+z)^2 du$$

$$= \int_1^2 \int_0^z \int_0^{y+z} u \cdot (-1) du dy dz = - \int_1^2 \int_0^z \int_0^{y+z} u du dy dz = - \int_1^2 \int_0^z \left[\frac{u^2}{2} \right]_0^{y+z} dy dz$$

$$= -\frac{1}{2} \int_1^2 \int_0^z \left[u^2 \right]_0^{y+z} dy dz = -\frac{1}{2} \int_1^2 \int_0^z \left[\left(\frac{1}{x+y+z} \right)^2 \right]_0^{y+z} dy dz = -\frac{1}{2} \int_1^2 \int_0^z \left[\frac{1}{(x+y+z)^2} \right]_0^{y+z} dy dz$$

$$= -\frac{1}{2} \int_1^2 \int_0^z \left(\frac{1}{(y+z+y+z)^2} - \frac{1}{(y+z)^2} \right) dy dz = -\frac{1}{2} \int_1^2 \int_0^z \left(\frac{1}{(2y+2z)^2} - \frac{1}{(y+z)^2} \right) dy dz$$

$$\frac{1}{(2y+2z)^2} = \frac{1}{(2 \cdot (y+z))^2} = \frac{1}{4 \cdot (y+z)^2} = \frac{1}{4} \cdot \frac{1}{(y+z)^2}$$

$$= -\frac{1}{2} \int_1^2 \int_0^z \left(\frac{1}{4} \cdot \frac{1}{(y+z)^2} - \frac{1}{(y+z)^2} \right) dy dz$$

$$= -\frac{1}{2} \int_1^2 \int_0^z \left(-\frac{3}{4} \cdot \frac{1}{(y+z)^2} \right) dy dz = \frac{3}{8} \int_1^2 \int_0^z \frac{1}{(y+z)^2} dy dz = \frac{3}{8} \int_1^2 \int_{y=0}^{y=z} u^{-2} du dz$$

$$u = y+z \Rightarrow du = dy$$

$$= \frac{3}{8} \int_1^2 \left[-u^{-1} \right]_{y=0}^{y=z} dz = -\frac{3}{8} \int_1^2 \left[(y+z)^{-1} \right]_0^z dz = -\frac{3}{8} \int_1^2 \left((z+z)^{-1} - (0+z)^{-1} \right) dz = -\frac{3}{8} \int_1^2 \left(\frac{1}{2z} - \frac{1}{z} \right) dz$$

$$= -\frac{3}{8} \int_1^2 \left(\frac{1}{2z} - \frac{1}{z} \right) dz = \frac{3}{8} \int_1^2 \left(\frac{1}{z} - \frac{1}{2z} \right) dz = \frac{3}{16} \int_1^2 \frac{1}{z} dz = \frac{3}{16} \cdot [\ln(z)]_1^2 = \frac{3}{16} \cdot (\ln(2) - \ln(1)) = \underline{\underline{\frac{3}{16} \cdot \ln(2)}}$$

Problem 5:

determinant of the

Find Jacobian for $x=2u+w$ $y=-v^2$ and $z = u + v^2 - 2w^2$

Answer: $16vw+2v$

$$f(u, v, w) = (f_1, f_2, f_3) = (2u+w, -v^2, u+v^2-2w^2)$$

$$J = \begin{bmatrix} \frac{f_1}{\partial u} & \frac{f_1}{\partial v} & \frac{f_1}{\partial w} \\ \frac{f_2}{\partial u} & \frac{f_2}{\partial v} & \frac{f_2}{\partial w} \\ \frac{f_3}{\partial u} & \frac{f_3}{\partial v} & \frac{f_3}{\partial w} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -2v & 0 \\ 1 & 2v & -4w \end{bmatrix}$$

$$|J| = 2(8vw-0) - 0 + 1 \cdot (0-(-2v)) = \underline{\underline{16vw+2v}}$$

I choose to believe that this is the correct answer!

Problem 6:

Find the volume of solid bounded by $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 = 8z$

Answer: $\frac{40}{3}\pi$

Convert to cylindrical coordinates

$$\begin{aligned} x &\rightarrow r \cdot \cos \theta \\ y &\rightarrow r \sin \theta \\ z &\rightarrow z \\ dx dy dz &\rightarrow r \cdot d\theta dr dz \\ x^2 + y^2 = a^2 &\rightarrow r = a \end{aligned}$$

$$x^2 + y^2 = 8z \Rightarrow r^2 \cdot \cos^2(\theta) + r^2 \cdot \sin^2(\theta) = 8z \Rightarrow r^2 = 8z \Rightarrow z = \frac{r^2}{8}$$

$$x^2 + y^2 + z^2 = 9 \Rightarrow r^2 + z^2 = 9 \Rightarrow z^2 = 9 - r^2 \Rightarrow z = \sqrt{9 - r^2}$$

$$\Rightarrow \int_0^{2\pi} \int_{\frac{r^2}{8}}^{\sqrt{9-r^2}} r \, dz \, dr \, d\theta$$

Try to find the limits for r

$$\begin{cases} z = \frac{r^2}{8} \\ z = \sqrt{9 - r^2} \end{cases} \Rightarrow \left(\frac{r^2}{8} = \sqrt{9 - r^2} \right)^2 \Rightarrow \frac{1}{64} r^4 = 9 - r^2 \Rightarrow \frac{1}{64} r^4 + r^2 = 9 \Rightarrow r^4 + 64r^2 - 9 \cdot 64 = 0$$

$$\Rightarrow r^4 + 64r^2 - 9 \cdot 64 = 0 \Rightarrow (r^2 - 8)(r^2 + 72) = 0$$

$$\Rightarrow \begin{cases} r^2 - 8 = 0 \Rightarrow r^2 = 8 \Rightarrow r_1 = \sqrt{8} \\ r^2 + 72 = 0 \Rightarrow r^2 = -72 \Rightarrow r_2 = \sqrt{-72} \end{cases} \Rightarrow r = \sqrt{8}$$

Imaginary part is ignored This is the boundary for r.

Rewriting the integral

$$\int_0^{2\pi} \int_{\frac{r^2}{8}}^{\sqrt{9-r^2}} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^{\sqrt{8}} \left[rz \right]_{\frac{r^2}{8}}^{\sqrt{9-r^2}} dr \, d\theta = \int_0^{2\pi} \int_0^{\sqrt{8}} r \sqrt{9-r^2} - \frac{1}{8} r^3 dr \, d\theta$$

From table: $\int x \sqrt{-x^2 \pm a^2} dx = -\frac{1}{3} (x^2 \pm a^2)^{3/2}$

$$\begin{aligned} \int_0^{2\pi} \left[-\frac{1}{3} (9-r^2)^{\frac{3}{2}} - \frac{1}{32} r^4 \right]_0^{\sqrt{8}} d\theta &= \int_0^{2\pi} \left[-\frac{1}{3} (9-(\sqrt{8})^2)^{\frac{3}{2}} - \frac{1}{32} (\sqrt{8})^4 + \frac{1}{3} (9-0)^{\frac{3}{2}} + \frac{1}{32} 0^4 \right] d\theta \\ &= \int_0^{2\pi} \left[-\frac{1}{3} (9-8)^{\frac{3}{2}} - \frac{1}{32} \cdot 8^2 + \frac{1}{3} \cdot 9^{\frac{3}{2}} \right] d\theta = \int_0^{2\pi} \left[-\frac{1}{3} \cdot (1)^{\frac{3}{2}} - \frac{64}{32} + \frac{1}{3} \cdot (\sqrt{9})^3 \right] d\theta \\ &= \int_0^{2\pi} \left[-\frac{1}{3} - 2 + (\sqrt{9})^2 \right] d\theta = \int_0^{2\pi} \left[-\frac{1}{3} - 2 + 9 \right] d\theta = \int_0^{2\pi} \left[-\frac{1}{3} + 7 \right] d\theta \\ &= \int_0^{2\pi} \frac{20}{3} d\theta = \left[\frac{20\theta}{3} \right]_0^{2\pi} = \underline{\underline{\frac{40\pi}{3}}} \end{aligned}$$