

Problem 3:

Determine whether the given vector field is conservative, and find a potential if it is

$$\mathbf{F}(x, y, z) = x\mathbf{i} - 2y\mathbf{j} + 3z\mathbf{k}$$

Solution:

$$\phi(x, y, z) = \frac{x^2}{2} - y^2 + \frac{3z^2}{2}$$

Check if conservative

$$f_1 = x, f_2 = -2y, f_3 = 3z$$

$$\left. \begin{aligned} \frac{\partial f_1}{\partial y} &= \frac{\partial f_2}{\partial x} \Rightarrow 0 = 0 \quad \checkmark \\ \frac{\partial f_1}{\partial z} &= \frac{\partial f_3}{\partial x} \Rightarrow 0 = 0 \quad \checkmark \\ \frac{\partial f_2}{\partial z} &= \frac{\partial f_3}{\partial y} \Rightarrow 0 = 0 \quad \checkmark \end{aligned} \right\} \text{Conservative!}$$

$$\nabla f = \vec{F}(x, y, z) \Rightarrow \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} x \\ -2y \\ 3z \end{bmatrix} \Rightarrow \begin{bmatrix} f \\ f \\ f \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x^2 + A(y, z) \\ -y^2 + B(x, z) \\ \frac{3}{2}z^2 + C(x, y) \end{bmatrix}$$

$$\Rightarrow f = \frac{1}{2}x^2 + A(y, z) = -y^2 + B(x, z) = \frac{3}{2}z^2 + C(x, y)$$

$$\Rightarrow A(y, z) = -y^2 + \frac{3}{2}z^2, B(x, z) = \frac{1}{2}x^2 + \frac{3}{2}z^2, C(x, y) = \frac{1}{2}x^2 - y^2$$

$$\Rightarrow \underline{\underline{f = \frac{1}{2}x^2 - y^2 + \frac{3}{2}z^2}}$$

Problem 4:

Determine whether the given vector field is conservative, and find a potential if it is

$$\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$$

Solution:

$$\phi(x, y, z) = xy + \frac{z^3}{3}$$

$$f_1 = y, \quad f_2 = x, \quad f_3 = z^2$$

$$\left. \begin{aligned} \frac{\partial f_1}{\partial y} &= \frac{\partial f_2}{\partial x} \Rightarrow 1 = 1 \quad \checkmark \\ \frac{\partial f_1}{\partial z} &= \frac{\partial f_3}{\partial x} \Rightarrow 0 = 0 \quad \checkmark \\ \frac{\partial f_2}{\partial z} &= \frac{\partial f_3}{\partial y} \Rightarrow 0 = 0 \quad \checkmark \end{aligned} \right\} \text{ CONSERVATIVE! }$$

$$\Rightarrow \nabla f = \vec{F}(x, y, z) \Rightarrow \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \begin{bmatrix} y \\ x \\ z^2 \end{bmatrix} \Rightarrow \begin{cases} f = xy + A(y, z) \\ f = xy + B(x, z) \\ f = \frac{1}{3}z^3 + C(x, y) \end{cases}$$

All three equations are true if: $A(y, z) = B(x, z) = \frac{1}{3}z^3$, $C(x, y) = xy$

$$\Rightarrow \underline{\underline{f = xy + \frac{1}{3}z^3}}$$

Problem 6:

Evaluate the given line integral over the specified curve \mathcal{C}

$$\int_{\mathcal{C}} (x + y) \, ds, \quad \mathbf{r} = at\mathbf{i} + bt\mathbf{j} + ct\mathbf{k}, \quad 0 \leq t \leq m$$

Solution:

$$\int_{\mathcal{C}} (x + y) \, ds = \frac{(a + b)\sqrt{a^2 + b^2 + c^2}}{2} m^2$$

$$f(x, y, z) = x + y$$

$$x(t) = at, \quad y(t) = bt, \quad z(t) = ct$$

$$f(x(t), y(t), z(t)) = at + bt$$

$$\int_{t=0}^{t=m} f(x(t), y(t), z(t)) \, dS = \int_{t=0}^{t=m} (at + bt) \, dS$$

$$dS = \sqrt{x_t^2 + y_t^2 + z_t^2} \, dt = \sqrt{a^2 + b^2 + c^2} \, dt$$

$$\Rightarrow \int_{t=0}^{t=m} (at + bt) \sqrt{a^2 + b^2 + c^2} \, dt = \sqrt{a^2 + b^2 + c^2} \cdot \int_0^m (at + bt) \, dt$$

$$= \sqrt{a^2 + b^2 + c^2} \cdot \left[\frac{a}{2} t^2 + \frac{b}{2} t^2 \right]_0^m = \sqrt{a^2 + b^2 + c^2} \cdot \left(\frac{am^2}{2} + \frac{bm^2}{2} \right)$$

$$= \sqrt{a^2 + b^2 + c^2} \cdot \frac{1}{2} \cdot (a + b) \cdot m^2 = \underline{\underline{\frac{\sqrt{a^2 + b^2 + c^2} \cdot (a + b) m^2}{2}}}$$

$$\int_{t=a}^{t=b} f(x(t), y(t)) \, dS$$

$$dS = \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2} \, dt$$

Problem 7:

Find the mass of a wire along the curve

$$\mathbf{r} = 3t\mathbf{i} + 3t^2\mathbf{j} + 2t^3\mathbf{k}, \quad (0 \leq t \leq 1)$$

if the density at $\mathbf{r}(t)$ is $1 + t$ g/unit length.

Solution:

$$m = 8 \text{ g}$$

$$\int_0^1 f(\vec{r}(t)) \, dS = \int_0^1 1+t \, dS$$

$$dS = \sqrt{x_t^2 + y_t^2 + z_t^2} = \sqrt{3^2 + (6t)^2 + (6t^2)^2} = \sqrt{9 + 36t^2 + 36t^4} = \sqrt{9 \cdot (2t^2+1)^2} = \sqrt{9} \sqrt{(2t^2+1)^2} = 3 \cdot (2t^2+1) \, dt$$
$$= 9 \cdot (4t^4 + 4t^2 + 1) = 9 \cdot (2t^2+1)^2$$

$$\Rightarrow \int_0^1 (t+1)(6t^2+3) \, dt = \int_0^1 6t^3 + 3t + 6t^2 + 3 \, dt = \int_0^1 6t^3 + 6t^2 + 3t + 3 \, dt = \left[\frac{6}{4} t^4 + 2t + \frac{3}{2} t + 3t \right]_0^1$$
$$= \frac{3}{2} + 2 + \frac{3}{2} + 3 = \underline{\underline{8}}$$

Problem 8:

Evaluate the line integral of the tangential component of the given vector field along the given curve

$$\mathbf{F}(x, y) = xy\mathbf{i} - x^2\mathbf{j} \text{ along } y = x^2 \text{ from } (0, 0) \text{ to } (1, 1)$$

Solution:

$$-\frac{1}{4}$$

$$x(t) = t, \quad y(t) = t^2 \Rightarrow \vec{r}(t) = t\hat{i} + t^2\hat{j}$$

$$\Rightarrow F(\vec{r}(t)) = t^3\hat{i} - t^2\hat{j}$$