Problem 1: Find the type, transform to normal form, and solve. (Show the details of your work)

$$u_{xy} - u_{yy} = 0$$

$$Au_{xx} + Bu_{xy} + C_{yy} + Du_x + Eu_y + Fu = G$$

$$\Rightarrow B = 1, \ C = -1, \ A = D = E = F = G = 0$$

Calculate discriminant

$$d = B^2 - 4AC = 1^2 - 4 \cdot 0 \cdot (-1) = 1$$

$$d > 0 \Rightarrow \text{Hyperbolic!!}$$

Get xi and eta

$$A\lambda^2 - B\lambda + C = 0 \Rightarrow -\lambda - 1 = 0 \Rightarrow \lambda = 1$$

$$\frac{dy}{dx} = \lambda \Rightarrow \int dy = \int \lambda \, dx = y = x$$

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Calculate discriminant

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Characteristic Equations

$$Ay'^2 + By' + C = 0 \Rightarrow 0 \cdot y'^2 + y' - 1 = 0 \Rightarrow y' = 1$$

$$\Rightarrow y = x + C_1 \Rightarrow C_1 = y - x$$

New variable w

$$w = y - x \Rightarrow w_x = -1, \quad w_y = 1$$

Describe u in terms of w

$$u_x = u_w \overset{-1}{w_x} = -u_w$$

$$u_{xy} = (-u_w)_w \overset{1}{w_y} = -u_{ww} \cdot 1 = -u_{ww}$$

$$u_y = u_w \overset{1}{w_y} = u_w$$

$$u_{yy} = (u_w)_w \cdot \overset{1}{w_y} = u_{ww}$$

Substitute variables

$$u_{xy} - u_{yy} = 0 \Rightarrow -u_{ww} - u_{ww} = 0 \Rightarrow -2u_{ww} = 0$$

$$\Rightarrow u_{ww} = 0 \quad \textcolor{red}{\text{Normal form !!}}$$

Solution

$$u_{ww} = f_1(w) \Rightarrow \iint f_1(w) \, dw \, dw = f_2(w)$$

$$\underline{u(x,y) = f_2(y-x)}$$

Problem 1: Find the type, transform to normal form, and solve. (Show the details of your work)

$$u_{xy} - u_{yy} = 0$$

This is a linear PDE as nothing is a function of u

$$u(x,y) = F(x)G(y)$$

We find the double derivatives

$$u_{yy} = F(x)G''(y)$$

$$u_{xy} = F'(x)G'(y)$$

Substitute in the original equation

$$F'(x)G'(y) - F(x)G''(y) = 0 \implies F'(x)G'(y) = F(x)G''(y)$$

$$\implies \frac{F'(x)}{F(x)} = \frac{G''(y)}{G'(y)} = \lambda$$

$$\textcircled{1} \begin{cases} F'(x) - \lambda \cdot F(x) = 0 \\ \textcircled{2} \quad G''(x) - \lambda \cdot G'(y) = 0 \end{cases}$$

Solving the first ODE:

$$f'(x) - \lambda \cdot f(x) = 0 \implies a = 0, \; b = 1, \; c = -\lambda$$

$$(a \cdot r^2 + b \cdot r + c) \cdot e^{rt} = 0 \implies a \cdot r^2 + b \cdot r + c = 0$$

$$\implies r - \lambda = 0 \implies r = \lambda$$

We have one root. Therefore we use this solution:

$$f(x) = A \cdot e^{rx} + B \cdot x \cdot e^{rx} = A \cdot e^{\lambda x} + B \cdot x \cdot e^{rx}$$

Solving the second ODE:

$$f''(x) - \lambda \cdot f'(x) = 0 \implies a = 1, \ b = -\lambda, \ c = 0$$

$$\implies r^2 - \lambda b = 0$$

Solving with quadratic equation

$$d = b^2 - 4ac = \lambda^2$$

$$r = \frac{-b \pm \sqrt{d}}{2a} = \frac{\lambda \mp \sqrt{d}}{2} = \frac{\lambda \pm \lambda}{2} \implies \begin{cases} 0 \\ \lambda \end{cases}$$

Here we have two solutions

$$f(x) = A \cdot e^{r_1 x} + B \cdot e^{r_2 x} = A \cdot e^0 + B \cdot e^{\lambda x} = A + B \cdot e^{\lambda x}$$

We now have solutions to the ODEs

$$F(x) = A \cdot e^{\lambda x} + B \cdot x \cdot e^{rx} \qquad G(y) = C + D \cdot e^{\lambda y}$$

Combining them to get PDE solution

$$u(x,y) = F(x) G(y) = \left(A \cdot e^{\lambda x} + B \cdot x \cdot e^{rx} \right) \left(C + D \cdot e^{\lambda y} \right)$$

Problem 2: Find the type, transform to normal form, and solve. (Show the details of your work)

$$u_{xx} + 2u_{xy} + u_{yy} = 0$$

$$Au_{xx} + Bu_{xy} + C_{yy} + Du_x + Eu_y + Fu = G$$

$\Rightarrow A=1, B=2, C=1, E=F=G=0$

$B^2 - 4AC = 4 - 4 \cdot 1 \cdot 1 = 0 \Rightarrow$ Parabolic!

Characteristic Equation

$$\frac{dy}{dx} = \frac{B}{2A} = \frac{2}{2 \cdot 1} = 1 \Rightarrow y = \int 1 dx \Rightarrow y = x + C_1$$

$C_1 = y - x \Rightarrow \xi = y - x$

I choose $\eta = x$

Check jacobian

$$\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0 \checkmark$$

Get needed derivatives (from chain rule)

$\xi_x = -1, \xi_y = 1, \xi_{xx} = 0, \xi_{yy} = 0, \xi_{xy} = 0$

$\eta_x = 1, \eta_y = 0, \eta_{xx} = 0, \eta_{yy} = 0, \xi_{xy} = 0$

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$
$$u_y = u_\xi \xi_y + u_\eta \eta_y$$
$$u_{xx} = u_{\xi\xi}(\xi_x)^2 + 2u_{\xi\eta}(\xi_x)(\eta_x) + u_{\eta\eta}(\eta_x)^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$$
$$u_{yy} = u_{\xi\xi}(\xi_y)^2 + 2u_{\xi\eta}(\xi_y)(\eta_y) + u_{\eta\eta}(\eta_y)^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}$$
$$u_{xy} = u_{\xi\xi}(\xi_x)(\xi_y) + u_{\eta\eta}(\eta_x)(\eta_y) + u_\xi \xi_{xy} + u_\eta \eta_{xy} + u_{\xi\eta}(\xi_x \eta_y) + (\xi_y \eta_x)$$

$u_{xx} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$

$u_{yy} = u_{\xi\xi}$

$u_{xy} = -u_{\xi\xi} + u_{\xi\eta}$

Replace in original PDE

$u_{xx} + 2u_{xy} + u_{yy} = 0$

$\Rightarrow u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} - 2u_{\xi\xi} + 2u_{\xi\eta} + u_{\xi\xi} = 0$

$\Rightarrow u_{\eta\eta} = 0$

Solve dat shit!

$$u_\eta = \int 0 \, d\eta = f_1(\varepsilon)$$

$$u = \int f_1(\varepsilon) \, d\eta = f_1(\varepsilon) \cdot \int 1 \, d\eta = f_1(\varepsilon) \cdot \eta + f_2(\varepsilon)$$

Remeber when we were young and naive

$$\varepsilon = y - x, \quad \eta = x$$

$$\underline{\underline{u = f_1(y-x) \cdot x + f_2(y-x)}}$$

Problem 3: Find the type, transform to normal form, and solve. (Show the details of your work)

$$u_{xx} - 4u_{xy} + 3u_{yy} = 0$$

$$Au_{xx} + Bu_{xy} + C_{yy} + Du_x + Eu_y + Fu = G$$

$A = 1, \quad B = -4, \quad C = 3$

$d = B^2 - 4ac = (-4)^2 - 4(1)(3) = 16 - 12 = 4$

$d > 0 \Rightarrow$ Hypobolic!

Find new variables

$A\lambda^2 - B\lambda + C = 0 \Rightarrow \underset{a}{\lambda^2} + \underset{b}{4\lambda} + \underset{c}{3} = 0$

$\lambda = \dfrac{-4 \pm \sqrt{4}}{2} = \dfrac{-4 \pm 2}{2} = -2 \pm 1 \Rightarrow \begin{cases} -1 \\ -3 \end{cases}$

$\dfrac{dy}{dx} = -1 \Rightarrow \int dy = \int -1 \, dx \Rightarrow y = -x + C_1 \Rightarrow C_1 = x + y$

$\dfrac{dy}{dx} = -3 \Rightarrow \int dy = \int -3 \, dx \Rightarrow y = -3x + C_2 \Rightarrow C_2 = 3x + y$

$\Rightarrow \begin{cases} \xi = x + y \\ \eta = 3x + y \end{cases}$

Find the needed derivatives

$\xi_x = 1, \quad \xi_y = 1, \quad \xi_{xx} = \xi_{yy} = \xi_{xy} = 0$

$\eta_x = 3, \quad \eta_y = 1, \quad \eta_{xx} = \eta_{yy} = \eta_{xy} = 0$

$$\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x \\
u_y &= u_\xi \xi_y + u_\eta \eta_y \\
u_{xx} &= u_{\xi\xi}(\xi_x)^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}(\eta_x)^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx} \\
u_{yy} &= u_{\xi\xi}(\xi_y)^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}(\eta_y)^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy} \\
u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\eta\eta}\eta_x\eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy} + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x)
\end{aligned}$$

$u_{xx} = u_{\xi\xi} + 6u_{\xi\eta} + 9u_{\eta\eta}$

$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

$u_{xy} = u_{\xi\xi} + 3u_{\eta\eta} + 4u_{\xi\eta}$

Substitute in original equation

$u_{xx} - 4u_{xy} + 3u_{yy} = 0$

$\Rightarrow u_{\xi\xi} + 6u_{\xi\eta} + 9u_{\eta\eta} - 4u_{\xi\xi} - 12u_{\eta\eta} - 16u_{\xi\eta} + 3u_{\xi\xi} + 6u_{\xi\eta} + 3\eta\eta = 0$

$\Rightarrow -4u_{\xi\eta} = 0 \Rightarrow u_{\xi\eta} = 0$

A laterally insulated bar of length 10cm and constant cross-sectional area $1cm^2$, of density $10.6\ gm/cm^3$, thermal conductivity $1.04\ cal/(cm\sec{}^\circ C)$, and specific heat $0.056\ cal/(gm\ {}^\circ C)$ (this corresponds to silver, a good heat conductor) has initial temperature $f(x)$ and is kept at $0^\circ C$ at the ends $x = 0$ and $x = 10$. Find the temperature $u(x,t)$ at later times. Here $f(x)$ equals:

Problem 4: $f(x) = \sin 0.4\,\pi x$

$c = 0,056$

$k = 1,04$

$\rho = 10.6$

$$\frac{\partial u}{\partial t} = \alpha \cdot \nabla^2 u, \qquad \alpha = \frac{k}{c\rho}$$

t: Time

u: Temperature as a function of position and time.

k: Thermal conductivity

c: Specific Heat Capacity

ρ: Density

Calculate alpha

$$\alpha = \frac{k}{c\rho} \approx 1,752 \qquad\qquad u_t = \alpha \cdot u_{xx}$$

Initial contitions

$$u(0,t) = 0 \qquad u(x,0) = f(x)$$
$$u(10,t) = 0$$

Assume that we can solve with seperation of variables

$$u(x,t) = F_{(x)}\, G_{(t)}$$

$$\Rightarrow u_t = F_{(x)}\, G'_{(t)}, \quad u_{xx} = F''_{(x)}\, G_{(t)}$$

Rewrite equation

$$F_{(x)}\, G'_{(t)} = \alpha \cdot F''_{(x)}\, G_{(t)} \;\Rightarrow\; \frac{G'_{(t)}}{G_{(t)}} = \alpha \cdot \frac{F''_{(x)}}{F_{(x)}} = \lambda$$

Convert to ODEs

$$\begin{cases} \alpha \cdot F''(x) - \lambda \cdot F_{(x)} = 0 \\ G'(t) - \lambda \cdot G_{(t)} = 0 \end{cases}$$

Solving the first ODE

$$\alpha \cdot F''(x) - \lambda \cdot F(x) = 0 \implies a = \alpha, \ b = -\lambda, \ c = 0$$

$$\implies e^{rt} \cdot (a \cdot r^2 + b \cdot r + c) = 0 \implies a \cdot r^2 + b \cdot r + c = 0$$

$$\implies d = b^2 - 4ac = \lambda^2$$

$$\implies r = \frac{-b \pm \sqrt{d}}{2a} = \frac{\lambda \pm \sqrt{\lambda^2}}{2\alpha} = \frac{\lambda \pm \lambda}{2\alpha} = \begin{cases} r_1 = 0 \\ r_2 = \frac{\lambda}{\alpha} \end{cases}$$

Because we find two root we use the following solution

$$F(x) = A \cdot e^{r_1 x} + B \cdot e^{r_2 x} = A \cdot e^0 + B \cdot e^{\frac{\lambda}{\alpha} \cdot x} = A + B \cdot e^{\frac{\lambda}{\alpha} \cdot x}$$

Solving the second one

$$G'(t) - \lambda \cdot G(t) = 0 \implies a = 0, \ b = 1, \ c = -\lambda$$

$$e^{rt} \cdot (a \cdot r^2 + b \cdot r + c) = 0 \implies a \cdot r^2 + b \cdot r + c = 0 \implies r - \lambda = 0$$

$$\implies r = \lambda$$

Here we only get one solution

$$G(t) = C \cdot e^{\lambda t} + D \cdot x \cdot e^{\lambda t}$$

We combine the functions to get a general solution

$$u(x,t) = F(x) G(t) = \left(A + B \cdot e^{\frac{\lambda}{\alpha} \cdot x} \right) \left(C \cdot e^{\lambda t} + D \cdot x \cdot e^{\lambda t} \right)$$

Plugging in initial conditions

$$u(0,t) = F(0) G(t) = 0 \implies F(0) = 0$$

G(t) = 0 is also a solution, but it is uninteresting.

$$u(10,t) = F(10) G(t) = 0 \implies F(10) = 0$$

We use this to find A and B

$$F(0) = 0 \implies A + B \cdot e^{\frac{\lambda}{\alpha} \cdot 0} = 0 \implies A + B = 0 \implies B = -A$$

$$F(10) = 0 \implies A - A \cdot e^{\frac{\lambda}{\alpha} \cdot 10} = 0 \implies A \cdot (1 - e^{\frac{\lambda}{\alpha} \cdot 10}) = 0$$

This is not great. Here either A or λ must be 0 leading to u = 0

$$\lambda = 0 \implies u(x, t) = 0$$
$$A = 0 \implies u(x, t) = 0$$

Instead we now assume that $\lambda < 0$

And that we can express i like this: $\lambda = -p^2$

We now solve the first ODE again

$$\alpha \cdot F''(x) - \lambda \cdot F(x) = 0 \implies \alpha \cdot F''(x) + p^2 \cdot F(x) = 0$$

$$\implies a = \alpha, \quad b = 0, \quad c = p^2, \quad d = b^2 - 4ac = -4\alpha p^2$$

$$\implies a \cdot r^2 + b \cdot r + c = 0 \implies r = \frac{-b \pm \sqrt{d}}{2a} = \frac{\pm \sqrt{-4\alpha p^2}}{2\alpha} = \frac{\pm \sqrt{-1}\sqrt{4}\sqrt{\alpha}\sqrt{p^2}}{2\alpha}$$

$$= \frac{\pm i \cdot 2 \cdot \sqrt{\alpha} \cdot p}{2\alpha} \cdot \frac{\sqrt{\alpha}}{\sqrt{\alpha}} = \frac{\pm i \cdot \alpha \cdot p}{\alpha \cdot \sqrt{\alpha}} = \pm i \cdot \frac{p}{\sqrt{\alpha}}$$

$$r = k \pm w i \implies k = 0, \quad u = \frac{p}{\sqrt{\alpha}}$$

The solution can now be expressed like this

$$F(x) = A \cdot e^{kx} \cdot \cos(wx) + B \cdot e^{kx} \cdot \sin(wx) = A \cdot e^0 \cdot \cos\left(\frac{p}{\sqrt{\alpha}} x\right) + B \cdot e^0 \cdot \sin\left(\frac{p}{\sqrt{\alpha}} x\right)$$

$$= A \cdot \cos\left(\frac{p}{\sqrt{\alpha}} x\right) + B \cdot \sin\left(\frac{p}{\sqrt{\alpha}}\right)$$

Plugging in initial conditions

$$u(0, t) = 0 \implies F(0) = 0 \implies A \cdot \cos(0) + B \cdot \sin(0) = 0 \implies A = 0 \implies F(x) = B \cdot \sin\left(\frac{p}{\sqrt{\alpha}} x\right)$$

We assume that B != 0 because otherwise u(x, t) = 0

$$u(10,t) = 0 \Rightarrow F(10) = 0 \Rightarrow B \cdot \sin\left(\frac{p}{\sqrt{\alpha}} \cdot 10\right) = 0 \Rightarrow \sin\left(p \cdot \frac{10}{\sqrt{\alpha}}\right) = 0$$

$$\Rightarrow p = \frac{n\pi\sqrt{\alpha}}{10}, \quad n = \{1, 2, 3 \ldots\}$$

Inserting p

$$F(x) = B \cdot \sin\left(\frac{n\pi\sqrt{\alpha}}{10} \cdot \frac{x}{\sqrt{\alpha}}\right) = B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right), \quad n = \{1, 2, 3 \ldots\}$$

Let's now take a look at G(y)

$$G'(t) - \lambda \cdot G(t) = 0 \Rightarrow G'(t) + p^2 \cdot G(t) = 0$$

$$\Rightarrow a = 0, \; b = 1, \; c = p^2 \Rightarrow a \cdot r^2 + b \cdot r + c = 0 \Rightarrow r + p^2 = 0 \Rightarrow r = -p^2$$

$$\Rightarrow r = -\left(\frac{n\pi\sqrt{\alpha}}{10}\right)^2 = -\frac{n^2\pi^2\alpha}{100}$$

We still get one double root

$$G(t) = C \cdot e^{r_1 t} + D \cdot x \cdot e^{r_2 t} = C \cdot e^{-\frac{n^2\pi^2\alpha}{100} \cdot t} + D \cdot t \cdot e^{-\frac{n^2\pi^2\alpha}{100} \cdot t}$$

Kombining solutions

$$u(x, t) = F(x) G(t) = B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right) \cdot \left(C \cdot e^{-\frac{n^2\pi^2\alpha}{100} \cdot t} + D \cdot t \cdot e^{-\frac{n^2\pi^2\alpha}{100} \cdot t}\right)$$

Plugging in initial condition

$$u(x, 0) = f(x) \Rightarrow \sin(0, 4\pi x) = B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right) \cdot \left(C \cdot e^0 + D \cdot 0 \cdot e^0\right)$$

$$\Rightarrow \sin(0, 4\pi x) = B \cdot \sin\left(\frac{n\pi}{10}\right) \cdot (C) = C \cdot B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right)$$

$$\Rightarrow \sin(0, 4\pi x) = C \cdot B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right) \Rightarrow \begin{cases} C = 1 \\ B = 1 \end{cases}$$
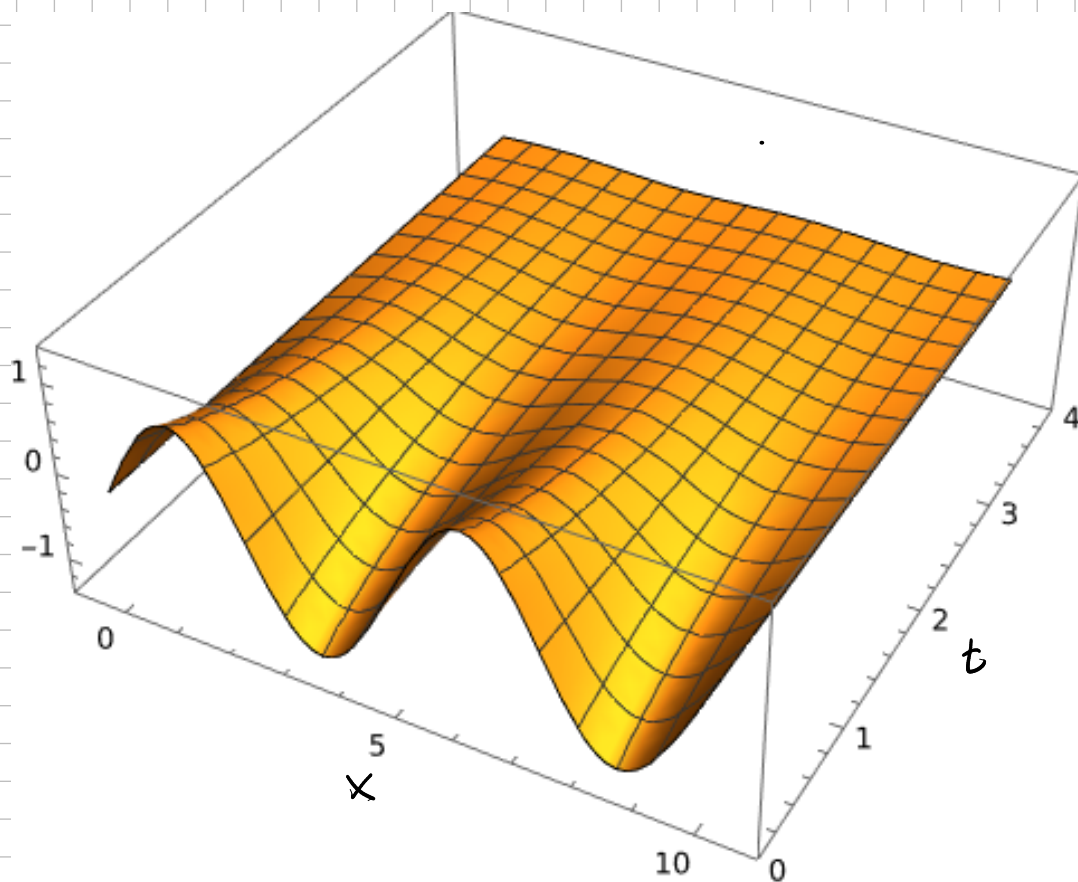
$$\Rightarrow \sin(0, 4\pi x) = \sin\left(\frac{n}{10}\pi x\right) \Rightarrow n = 4$$

We now have all our constants and can write the specific solution

$$u(x,t) = B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right) \cdot \left(C \cdot e^{-\frac{n^2\pi^2\alpha}{100} \cdot t} + D \cdot t \cdot e^{\frac{n^2\pi^2\alpha}{100} \cdot t}\right)$$

$$= 1 \cdot \sin\left(\frac{4}{10}\pi x\right) \cdot \left(1 \cdot e^{-\frac{n^2\pi^2\alpha}{100} \cdot t} + 0 \cdot t \cdot e^{-\frac{n^2\pi^2\alpha}{100} \cdot t}\right)$$

$$\Rightarrow \underline{\underline{u(x,t) = \sin\left(\frac{4}{10}\pi x\right) \cdot e^{-\frac{16\cdot\pi^2\alpha}{100} \cdot t}}}$$

Problem 5: $f(x) = \sin 0.1\,\pi x + \frac{1}{2}\sin 0.2\,\pi x$

We start with our general solution:

$$u(x,t) = B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right) \cdot \left(C \cdot e^{-\frac{n^2\pi^2\alpha}{100} \cdot t} + D \cdot t \cdot e^{-\frac{n^2\pi^2\alpha}{100} \cdot t}\right)$$

Plugging in initial condition

$$u(x,0) = f(x) = B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right) \cdot \left(C \cdot e^{0} + D \cdot 0 \cdot e^{0}\right)$$

$$= B \cdot \sin\left(\frac{n\pi}{10}x\right) \cdot C$$

Because this is a linear PDE the sum of two solutions will also be a solution. We can therefore split this up into two solutions and add them togeather afterwards.

$$u(x,0) = f_1(x) + f_2(x) = u_1(x,t) + u_2(x,t)$$

$$f_1(x) = \sin(0,1\pi x), \quad f_2(x) = \frac{1}{2} \cdot \sin(0,2\pi x)$$

Get first solution

$$\sin(0,1\pi x) = B \cdot C \cdot \sin\left(\frac{n\pi}{10}x\right) \Rightarrow \begin{cases} n = 1 \\ B \cdot C = 1 \Rightarrow \underline{B=1, C=1} \end{cases}$$

I could also have chosen values like B=2 and C=0.5 but this is a more elegant solution

$$u_1(x,t) = 1 \cdot \sin\left(\frac{\pi}{10} \cdot x\right) \cdot \left(1 \cdot e^{-\frac{1 \cdot \pi^2\alpha}{100} \cdot t} + D \cdot t \cdot e^{-\frac{1 \cdot \pi^2\alpha}{100} \cdot t}\right)$$

I also set D=0 to simplify

$$u_1(x,t) = \sin\left(\frac{\pi}{10} \cdot x\right) \cdot e^{-\frac{\pi^2\alpha}{100}}$$

Second solution

$$\frac{1}{2} \cdot \sin(0,2\pi x) = B \cdot C \cdot \sin\left(\frac{n\pi}{10}x\right) \Rightarrow \begin{cases} n = 2 \\ B = \frac{1}{2} \\ C = 1 \end{cases}$$
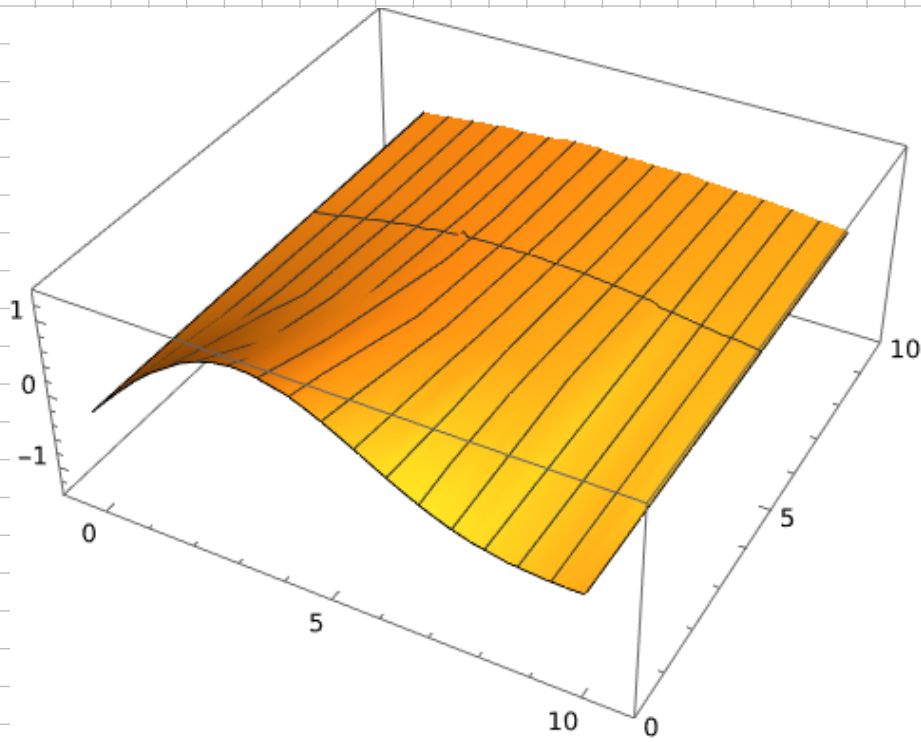
ag sin i set D=0

$$u_2(x,t) = \frac{1}{2}\sin\left(\frac{2\pi}{10} \cdot x\right) \cdot \left(1 \cdot e^{-\frac{4 \cdot \pi^2\alpha}{100} \cdot t} + 0 \cdot t \cdot e^{-\frac{4 \cdot \pi^2\alpha}{100} \cdot t}\right)$$

$$= \frac{1}{2} \cdot \sin\left(\frac{2\pi}{10} \cdot x\right) \cdot e^{-\frac{4\pi^2\alpha}{100} \cdot t}$$

We can now add these solutions togeather to get the final solution

$$u(x,t) = u_1(x,t) + u_2(x,t) = \sin\left(\frac{\pi}{10} \cdot x\right) \cdot e^{\frac{-\pi^2 \alpha}{100} \cdot t} + \frac{1}{2} \cdot \sin\left(\frac{2\pi}{10} \cdot x\right) \cdot e^{\frac{-4\pi^2 \alpha}{100} \cdot t}$$

Problem 6: $f(x) = 1 - 0.2|x - 5|$

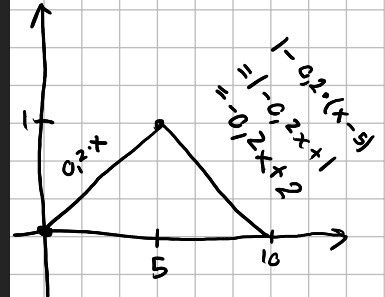We start by approximating the function as a fourier-series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

n: A how many times of the base frequency.

L: The half period.

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 0$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n > 0$$
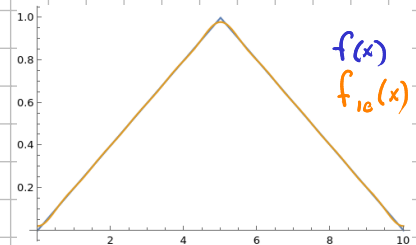
We know that we can only use

$$f(x) = 1 - 0.2 \cdot |x - 5| \implies \int_0^{10} f(x)\, dx = \int_0^5 0.2x\, dx + \int_5^{10} -0.2x + 2\, dx$$

$$2L = 10 \implies L = 5$$

$$N = 10$$

Calculated with mathematica:

$$f_a(x) = \underbrace{-0.405285 \cos\left(\frac{\pi x}{5}\right)}_{f_1} \underbrace{- 0.0450316 \cos\left(\frac{3\pi x}{5}\right)}_{f_2} \underbrace{- 0.0162114 \cos(\pi x)}_{f_3} -$$

$$\underbrace{0.00827112 \cos\left(\frac{7\pi x}{5}\right)}_{f_4} \underbrace{- 0.00500352 \cos\left(\frac{9\pi x}{5}\right)}_{f_5} + 0.5, \quad \text{offset?}$$

Find general solution to subfunctions on the following form:

$$f(x) = a \cdot \cos\left(\frac{b\,\pi\,x}{5}\right)$$

We start with the general solution

$$u(x,t) = B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right) \cdot \left(C \cdot e^{\frac{-n^2\pi^2\alpha}{100} \cdot t} + D \cdot t \cdot e^{\frac{-n^2\pi^2\alpha}{100} \cdot t} \right)$$

Insert the initial condition

$$u(x,0) = f(x) = a \cdot \cos\left(\frac{b\,\pi\,x}{5}\right) = B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right) \cdot \left(C \cdot e^0 + D \cdot 0 \cdot e^0 \right)$$

$$\Rightarrow \quad a \cdot \cos\left(\frac{b\pi x}{5}\right) = B \cdot C \cdot \sin\left(\frac{n\pi}{10} \cdot x\right)$$

$$\textcolor{red}{\cos(x) = \sin\left(x + \frac{\pi}{2}\right)}$$

I will set D=0 and C=1 for simplicity

$$\Rightarrow a \cdot \cos\left(\frac{b\pi x}{5}\right) = B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right) \Rightarrow a \cdot \sin\left(\frac{2b\pi}{10} \cdot x + \frac{\pi}{2}\right) = B \cdot \sin\left(\frac{n\pi}{10} \cdot x\right)$$

$$a = B \Rightarrow \sin\left(\frac{2b\pi}{10} \cdot x + \frac{\pi}{2}\right) = \sin\left(\frac{n\pi}{10} \cdot x\right)$$

Idk what to do here, i will assume that n = 2b

$$u(x,t) = a \cdot \sin\left(\frac{2b\pi}{10} \cdot x\right) \cdot \left(e^{-\frac{4b^2\pi^2\alpha}{100} \cdot t} + 0 \cdot t \cdot e^{-\frac{4b^2\pi^2\alpha}{100} \cdot t}\right)$$

$$= a \cdot \sin\left(\frac{2b\pi}{10} \cdot x\right) \cdot e^{-\frac{4b^2\pi^2\alpha}{100} \cdot t}$$

Problem 7: Arbitrary temperatures at ends. If the ends $x = 0$ and $x = L$ of the bar in the text are kept at constant temperatures U_1 and U_2 respectively, what is the temperature $u_1(x)$ in the bar after a long time (theoretically, as $t \to \infty$)? First guess, then calculate.