Problem 1: $u_{yy} + 16u = 0$ $\Rightarrow$ $u = f(x)$

Problem 2: $u_{yy} = 0$ $\Rightarrow$ $u = f(x)$

Problem 3: $u_y + 2yu = 0$ $\Rightarrow$ $u = f(x)$

Problem 4: $u_y + u = e^{xy}$

Problem 5: $u_{xx} = 4y^2 u$

Problem 6: $u_y = 2xyu$

④ $u_y + u = e^{xy}$ $\Rightarrow$ $u_y + u - e^{xy} = 0$

$\Rightarrow f(x),$

Verify (by substitution) that the given function is a solution of the indicated PDE (wave equation).

Problem 7: $u = \sin kx \cos kct$

Wave equation: $\quad \boxed{u_{tt} = c^2 \nabla^2 u}$

$u = \sin(kx) \cdot \cos(kct)$

$u_t = \sin(kx) \cdot \left(-\sin(kct) \cdot kc \right) = -kc \cdot \sin(kx) \cdot \sin(kct)$

$u_{tt} = -kc \cdot \sin(kx) \cdot \cos(kct) \cdot kc = -k^2 c^2 \cdot \sin(kx) \cdot \cos(kct)$

$c^2 \nabla^2 u = c^2 \dfrac{d^2 u}{dx^2} = c^2 \dfrac{d}{dx} \left(\cos(kx) \cdot k \cdot \cos(kct) \right)$

$= c^2 \cdot \left(-k^2 \cdot \sin(kx) \cdot \cos(kct) \right)$

$= -k^2 c^2 \cdot \sin(kx) \cdot \cos(kct)$

Equal!

Verify (by substitution) that the given function is a solution of the indicated PDE (heat equation).

Problem 8: $u = e^{-\omega^2 c^2 t} \cos \omega x$

Heat equation: $$\frac{\partial u}{\partial t} = \alpha \cdot \nabla^2 u$$

$$u_t = \frac{d}{dt}\left(e^{-\omega^2 c^2 t} \cdot \cos(\omega x) \right)$$

$$= -\omega^2 c^2 \cdot e^{-\omega^2 c^2 t} \cdot \cos(\omega x)$$

$$\alpha \cdot \nabla^2 u = \alpha \cdot \frac{d^2}{dx^2}\left(e^{-\omega^2 c^2 t} \cdot \cos(\omega x) \right)$$

$$= \alpha \cdot \frac{d}{dx}\left(e^{-\omega^2 c^2 t} \cdot \left(-\sin(\omega x) \cdot \omega \right) \right)$$

$$= -\alpha \cdot e^{-\omega^2 c^2 t} \cdot \cos(\omega x) \cdot \omega^2$$

$$= -\omega^2 \cdot \alpha \cdot e^{-\omega^2 c^2 t} \cdot \cos(\omega x)$$

Putting them together

$$-\omega^2 c^2 \cdot e^{-\omega^2 c^2 t} \cdot \cos(\omega x) = -\omega^2 \cdot \alpha \cdot e^{-\omega^2 c^2 t} \cdot \cos(\omega x)$$

$$\Rightarrow u \text{ is a solution if } \alpha = c^2$$

Problem 10: Boundary value problem: Verify that the function $u(x, y) = a \ln(x^2 + y^2) + b$ satisfies Laplace's equation and determine a and b so that u satisfies the boundary conditions $u = 110$ on the circle $x^2 + y^2 = 1$ and $u = 0$ on the circle $x^2 + y^2 = 100$.

$$u(x, y) = a \cdot \ln(x^2 + y^2) + b$$

Check: $\nabla^2 u = 0$

$$\frac{d^2}{dx^2}\left(a \cdot \ln(x^2 + y^2) + b\right) + \frac{d^2}{dy^2}\left(a \cdot \ln(x^2 + y^2) + b\right) = 0$$

$$\Rightarrow \frac{d}{dx}\left(a \cdot \frac{1}{x^2 + y^2} \cdot 2x\right) + \frac{d}{dy}\left(a \cdot \frac{1}{x^2 + y^2} \cdot 2y\right) = 0$$

$$a \cdot \left(\frac{d}{dx}\left(\frac{1}{x^2 + y^2}\right) \cdot 2x + \frac{d}{dx}(2x) \cdot \frac{1}{x^2 + y^2} \right)$$

$$= a \cdot \left(\frac{-1}{(x^2 + y^2)^2} \cdot 2x \cdot 2x + 2 \cdot \frac{1}{x^2 + y^2} \right)$$

$$= \frac{-4ax^2}{(x^2 + y^2)^2} + \frac{2a}{x^2 + y^2} = \frac{-4ax^2}{(x^2 + y^2)^2} + \frac{2a(x^2 + y^2)}{(x^2 + y^2)^2}$$

$$= \frac{-4ax^2 + 2a(x^2 + y^2)}{(x^2 + y^2)^2}$$

$$\Rightarrow \frac{-4ax^2 + 2a(x^2+y^2)}{(x^2+y^2)^2} + \frac{-4ay^2 + 2a(x^2+y^2)}{(x^2+y^2)^2} = 0$$

$$\Rightarrow \frac{-4ax^2 - 4ay^2 + 4a(x^2+y^2)}{(x^2+y^2)^2} = 0$$

$$\Rightarrow \frac{-4ax^2 - 4ay^2 + 4ax^2 + 4ay^2}{(x^2+y^2)^2} = 0$$

$$\Rightarrow \frac{0}{(x^2+y^2)^2} = 0 \qquad \text{True!}$$

Problem 10: Boundary value problem: Verify that the function $u(x, y) = a \ln(x^2 + y^2) + b$ satisfies Laplace's equation and determine a and b so that u satisfies the boundary conditions $u = 110$ on the circle $x^2 + y^2 = 1$ and $u = 0$ on the circle $x^2 + y^2 = 100$.

$$u(x, y) = a \cdot \ln(x^2 + y^2) + b$$

$x^2 + y^2 = 1 \Rightarrow y^2 = 1 - x^2$	$x^2 + y^2 = 100 \Rightarrow y^2 = 100 - x^2$
$\Rightarrow a \cdot \ln(x + 1 - x^2) + b = 110$	$\Rightarrow a \cdot \ln(x + 100 - x^2) = 0$
$\Rightarrow a \cdot \ln(-x^2 + x + 1) + b = 110$	$\Rightarrow a \cdot \ln(-x^2 + x + 100) = 0$

$$\Rightarrow \begin{cases} a \cdot \ln(-x^2 + x + 1) + b = 110 \\ a \cdot \ln(-x^2 + x + 100) + b = 0 \end{cases}$$

$$\Rightarrow \quad a \cdot \ln(-x^2+x+1) + \cancel{b} - a \cdot \ln(-x^2+x+100) - \cancel{b} = 110 - 0$$

$$\Rightarrow \quad a \cdot \left(\ln(-x^2+x+1) - \ln(-x^2+x+100) \right) = 110$$

$$\Rightarrow \quad a = \frac{110}{\ln\left(\dfrac{-x^2+x+1}{-x^2+x+100}\right)}$$