

Problem 1:calculate **div F** and **curl F** for the vector field $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$ **Solution:** $\text{div } \mathbf{F} = 0$, $\text{curl } \mathbf{F} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$

$$\text{div } \vec{v}(x, y) = \nabla \cdot \vec{v}(x, y) = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y}$$

$$\text{curl } \vec{v}(x, y, z) = \nabla \times \vec{v}(x, y, z) = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} P(x, y, z) \\ Q(x, y, z) \\ R(x, y, z) \end{bmatrix} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix}$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} y \\ z \\ x \end{bmatrix} = \frac{\partial}{\partial x} y + \frac{\partial}{\partial y} z + \frac{\partial}{\partial z} x = 0 + 0 + 0 = 0$$

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\det(M) = |M| = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\text{curl } \vec{F} = \nabla \times \vec{F}(x, y, z) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= \hat{i} \cdot \left(\frac{\partial}{\partial y} x - \frac{\partial}{\partial z} z \right) - \hat{j} \cdot \left(\frac{\partial}{\partial x} x - \frac{\partial}{\partial z} y \right) + \hat{k} \cdot \left(\frac{\partial}{\partial x} z - \frac{\partial}{\partial y} y \right)$$

$$= \hat{i} \cdot (0 - 1) - \hat{j} \cdot (1 - 0) + \hat{k} \cdot (0 - 1) = \underline{\underline{-\hat{i} - \hat{j} - \hat{k}}}$$

Problem 2:calculate **div F** and **curl F** for the vector field $\mathbf{F} = x\mathbf{i} + x\mathbf{k}$ **Solution:** $\text{div } \mathbf{F} = 1$, $\text{curl } \mathbf{F} = -\mathbf{j}$

$$\text{curl } \vec{v}(x, y) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}, \text{ where } \vec{v}(x, y) = \begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix}$$

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\det(M) = |M| = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} x = 1$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 0 & x \end{vmatrix} = \hat{i} \cdot \left(\frac{\partial}{\partial y} x - \frac{\partial}{\partial z} 0 \right) - \hat{j} \cdot \left(\frac{\partial}{\partial x} x - \frac{\partial}{\partial z} x \right) + \hat{k} \cdot \left(\frac{\partial}{\partial x} 0 - \frac{\partial}{\partial y} x \right)$$

$$= \hat{i} \cdot (0 - 0) - \hat{j} \cdot (1 - 0) + \hat{k} \cdot (0 - 0) = -\hat{j}$$

Evaluate $\oint_c (\sin x + 3y^2) dx + (2x - e^{-y^2}) dy$, where c is the boundary of the half-disk $x^2 + y^2 \leq a^2$, $y \geq 0$, oriented counterclockwise.

$$\oint_C \vec{F}(x, y) \, d\vec{r} = \oint_C P(x, y) dx + Q(x, y) dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA$$

$$Q(x, y) = 2x - e^{-y^2}$$

$$\frac{dQ}{dx} = 2$$

$$\Rightarrow \iint_R 2-6y \, dA$$

$$\iint_R (z - b \cdot r \cdot \sin(\varphi)) \cdot r \, dr \, d\varphi$$

$$0 = 0..a$$

$$= \int_0^{\pi} \int_0^a 2r \cdot b \cdot r^2 \sin(\theta) \, dr \, d\theta$$

$$= 2 \int_0^{\pi} \int_0^a r \, dr \, d\theta - 6 \int_0^{\pi} \int_0^a r^3 \cdot \sin(\theta) \, dr \, d\theta$$

$$= 2 \cdot \int_0^{\pi} \left[\frac{r^2}{2} \right]_0^a d\theta - b \cdot \int_0^{\pi} \sin(\theta) \cdot \left[\frac{r^3}{3} \right]_0^a d\theta = 2 \cdot \int_0^{\pi} \frac{a^2}{2} d\theta - b \cdot \int_0^{\pi} \sin(\theta) \cdot \frac{1}{3} \cdot a^3 d\theta$$

$$= a^2 \left[\theta \right]_0^\pi - 2a^3 \left[-\cos(\theta) \right]_0^\pi = \pi a^2 - 2a^3 (-0 - (-1)) = \pi a^2 - 2a^3$$

Problem 5:

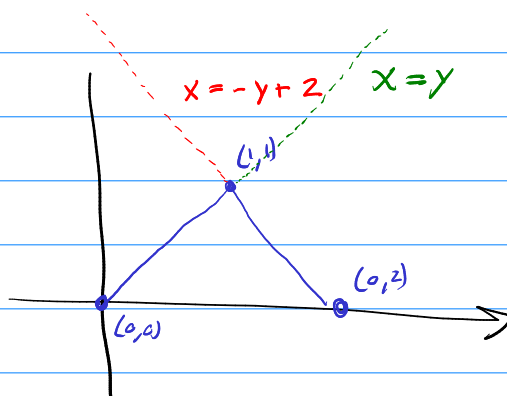
Evaluate $\oint_C (x^2 - xy) dx + (xy - y^2) dy$, clockwise around the triangle with vertices $(0,0)$, $(1,1)$, and $(2,0)$.

Solution: $-\frac{4}{3}$

$$\oint_C \vec{F}(x,y) d\vec{r} = \oint_C P(x,y) dx + Q(x,y) dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$$

$$= - \iint_R \frac{\partial}{\partial x} (xy - y^2) - \frac{\partial}{\partial y} (x^2 - xy) dA$$

$$= - \iint_R y - (-x) dA = \iint_R y + x dA$$



$$- \int_0^1 \int_y^{-y+2} y + x dx dy = - \int_0^1 \int_y^{-y+2} y dx dy - \int_0^1 \int_y^{-y+2} x dx dy$$

$$= - \int_0^1 [xy]_y^{-y+2} dy - \int_0^1 \left[\frac{x^2}{2} \right]_y^{-y+2} dy$$

$$= - \int_0^1 ((-y+2)y) - (y \cdot y) dy - \frac{1}{2} \int_0^1 (-y+2)^2 - y^2 dy$$

$$= - \int_0^1 -y^2 + 2y - y^2 dy - \frac{1}{2} \int_0^1 \cancel{y^2} - 4y + 4 - \cancel{y^2} dy$$

$$= - \int_0^1 2y - 2y^2 dy - \frac{1}{2} \int_0^1 -4y + 4 dy$$

$$= 2 \int_0^1 y^2 + 2y dy + 2 \int_0^1 y - 1 dy$$

$$= 2 \cdot \left[\frac{1}{3} y^3 + y^2 \right]_0^1 - 2 \left[\frac{y^2}{2} - y \right]_0^1 = 2 \cdot \left(\frac{1}{3} + 1 \right) - 2 \cdot \left(\frac{1}{2} - 1 \right) = \frac{2}{3} + 2 - 1 + 2$$

$$= \frac{8}{3}$$

Problem 6:

Use a line integral to find the plane area enclosed by the curve $r = a \cos^3 t \mathbf{i} + b \sin^3 t \mathbf{j}$, $(0 \leq t \leq 2\pi)$.

Solution: $\frac{3\pi ab}{8}$

How?

$$r = a \cdot \cos(t)^3 i + b \cdot \sin(t)^3 j$$

Problem 7:

Use the Divergence Theorem to calculate the flux of the given vector field out of the sphere S with equation $x^2 + y^2 + z^2 = a^2$, where $a > 0$

$$\mathbf{F} = x\mathbf{i} - 2y\mathbf{j} + 4z\mathbf{k}$$

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iiint_V \operatorname{div} \vec{F} \, dV$$

$$\iiint_V \operatorname{div} \vec{F} \, dV = \iiint_V \nabla \cdot \vec{F} \, dV = \iiint_V \frac{d}{dx} x + \frac{d}{dy} (-2y) + \frac{d}{dz} 4z \, dV$$

$$= \iiint_V 1 - 2 + 4 \, dV = \iiint_V 3 \, dV$$

Convert to spherical coordinates

$$x^2 + y^2 + z^2 = a^2 \Rightarrow \rho = 0 \dots a$$

$$\begin{aligned} x &= \rho \cdot \sin \phi \cdot \cos \theta \\ y &= \rho \cdot \sin \phi \cdot \sin \theta \\ z &= \rho \cdot \cos \phi \\ dV &= \rho \cdot \sin \phi \, d\rho \, d\phi \, d\theta \end{aligned}$$

$$\iiint_V 3 \, dV \Rightarrow 3 \cdot \iiint \rho^2 \sin(\phi) \, d\theta \, d\phi \, d\rho$$

$$= 3 \cdot \iiint \rho^2 \sin(\phi) \, d\theta \, d\phi \, d\rho$$

We can now assign limits

$$= 3 \cdot \int_0^a \rho^2 \int_0^\pi \sin(\phi) \int_0^{2\pi} 1 \, d\theta \, d\phi \, d\rho$$

$$= 3 \cdot \int_0^a \rho^2 \int_0^\pi \sin(\phi) \cdot 2\pi \, d\phi \, d\rho$$

$$= 6\pi \cdot \int_0^a \rho^2 \left[-\cos(\phi) \right]_0^\pi \, d\rho = 6\pi \cdot \int_0^a \rho \cdot (-(-1) - (-1)) \, d\rho$$

$$= 12\pi \int_0^a \rho^2 \, d\rho = 12\pi \cdot \left[\frac{\rho^3}{3} \right]_0^a = 12\pi \cdot \frac{a^3}{3} = \underline{\underline{4\pi a^3}}$$

Problem 8:

Use the Divergence Theorem to calculate the flux of the given vector field out of the sphere S with equation $x^2 + y^2 + z^2 = a^2$, where $a > 0$

$$\mathbf{F} = x^3 \mathbf{i} + 3yz^2 \mathbf{j} + (3y^2z + x^2) \mathbf{k}$$

Solution: $\frac{12}{5} \pi a^5$

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iiint_V \operatorname{div} \vec{F} \, dV$$

$$\begin{aligned} \operatorname{div} \vec{F} &= \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(3yz^2) + \frac{\partial}{\partial z}(3y^2z + x^2) \\ &= 3x^2 + 3z^2 + 3y^2 = 3(x^2 + y^2 + z^2) \end{aligned}$$

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iiint_V 3(x^2 + y^2 + z^2) \, dV = 3 \cdot \iiint_V x^2 + y^2 + z^2 \, dV$$

Convert to spherical coordinates

$$x^2 + y^2 + z^2 = r^2$$

$$x^2 + y^2 + z^2 \leq a^2 \Rightarrow 0 \leq r \leq a$$

$$dV = r^2 \cdot \sin(\phi) \, dr \, d\phi \, d\theta$$

Re-writing the integral

$$\begin{aligned} 3 \cdot \iiint_V x^2 + y^2 + z^2 \, dV &\Rightarrow 3 \cdot \int_0^{2\pi} \int_0^\pi \int_0^a r^2 \cdot r^2 \cdot \sin(\phi) \, dr \, d\phi \, d\theta \\ &= 3 \cdot \int_0^{2\pi} \int_0^\pi \left[\frac{r^5}{5} \cdot \sin(\phi) \right]_0^a \, d\phi \, d\theta = \frac{3}{5} \cdot \int_0^{2\pi} \int_0^\pi a^5 \cdot \sin(\phi) \, d\phi \, d\theta = \frac{3}{5} \cdot \int_0^{2\pi} \left[a^5 \cdot \cos(\phi) \right]_0^\pi \, d\theta \\ &= \frac{3}{5} \cdot \int_0^{2\pi} a^5 \cdot \overbrace{\cos(0)}^1 - a^5 \cdot \overbrace{\cos(\pi)}^{-1} \, d\theta = \frac{3}{5} \cdot \int_0^{2\pi} 2a^5 \, d\theta = \frac{3}{5} \left[2a^5 \theta \right]_0^{2\pi} = \frac{3}{5} \cdot 2a^5 \cdot 2\pi = \underline{\underline{\frac{12\pi a^5}{5}}} \end{aligned}$$

Problem 9:

Evaluate the flux of $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ outward across the boundary of the given solid region.

The solid ellipsoid $x^2 + y^2 + 4(z - 1)^2 \leq 4$

Solution: $32\pi/3$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x} x^2 + \frac{\partial}{\partial y} y^2 + \frac{\partial}{\partial z} z^2 = 2x + 2y + 2z$$

$$x^2 + y^2 + 4(z-1)^2 \leq 4 \Rightarrow 4(z-1)^2 \leq 4 - x^2 - y^2 \Rightarrow 2(z-1) \leq \sqrt{4 - x^2 - y^2}$$

$$z \leq \frac{1}{2}\sqrt{4 - x^2 - y^2} + 1$$

The ellipsoid is widest at $z=0$, finding limits from there

$$x^2 + y^2 + 4(0-1)^2 \leq 4 \Rightarrow x^2 + y^2 + 4 \leq 4$$

Convert to spherical coordinates

$$x \rightarrow \rho \cdot \sin \phi \cdot \cos \theta$$

$$y \rightarrow \rho \cdot \sin \phi \cdot \sin \theta$$

$$z \rightarrow \rho \cdot \cos \phi$$

$$dV \rightarrow \rho^2 \cdot \sin \phi \, d\rho \, d\phi \, d\theta$$

$$\Rightarrow \operatorname{div} \vec{F} = 2\rho \cdot \sin(\theta) \cdot \cos(\theta) + 2\rho \cdot \sin(\theta) \cdot \sin(\theta) + 2\rho \cdot \cos(\theta)$$

$$= 2\rho (\sin(\phi) \cdot \cos(\theta) + \sin(\phi) \cdot \sin(\theta) + \cos(\phi))$$

$$x^2 + y^2 + 4(z-1)^2 \leq 4 \Rightarrow \rho^2 \cdot \sin^2(\phi) \cdot \cos^2(\theta) + \rho^2 \cdot \sin^2(\phi) \cdot \sin^2(\theta) + 4(\rho \cdot \cos(\phi) - 1)^2 \leq 4$$

$$= \rho^2 \sin^2(\phi) \cdot (\cos^2(\theta) + \sin^2(\theta)) + 4(\rho \cdot \cos(\phi) - 1)^2 \leq 4$$

$$= \rho^2 \sin^2(\phi) + 4(\rho^2 \cos^2(\phi) - 2\rho \cos(\phi) + 1) \leq 4$$

Problem 10:

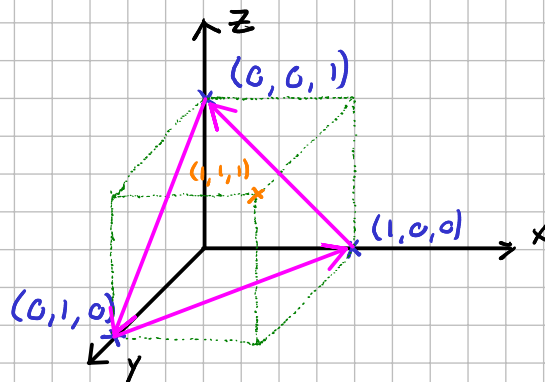
Evaluate $\oint_C \overset{f_1}{xy} dx + \overset{f_2}{yz} dy + \overset{f_3}{zx} dz$ around the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, oriented clockwise as seen from the point $(1, 1, 1)$.

Solution: $\frac{1}{2}$

$$\vec{F} = xy\hat{i} + yz\hat{j} + zx\hat{k}$$

$$\Rightarrow \iint_R \text{curl } \vec{F} \cdot \vec{n} \, dA$$

$$= \iint_R (-y\hat{i} - z\hat{j} - x\hat{k}) \cdot \vec{n} \, dA$$



Write z in terms of x and y , so the resulting plane contains the three points.

$$z = 1 - x - y, \quad \vec{n} = \hat{i} + \hat{j} + \hat{k}$$

Writing the integral

$$\int_0^1 \int_0^{-y+1} (-y\hat{i} - (1-x-y)\hat{j} - x\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) \, dx \, dy$$

$$\Rightarrow \int_0^1 \int_0^{-y+1} -y - 1 + x + y - x \, dx \, dy = \int_0^1 [x]_0^{-y+1} \, dy = \int_0^1 -y + 1 \, dy = \left[-\frac{1}{2}y^2 + y \right]_0^1 = -\frac{1}{2} + 1 = \underline{\underline{\frac{1}{2}}}$$

Problem 11:

Evaluate $\iint_S \mathbf{curl} \mathbf{F} \cdot \hat{\mathbf{N}} \, dS$, where S is the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$, with outward normal, and $\mathbf{F} = 3y\mathbf{i} - 2xz\mathbf{j} + (x^2 - y^2)\mathbf{k}$.

Solution: $-3\pi a^2$

Finding the normal vector

$$g(x, y, z) = x^2 + y^2 + z^2 \Rightarrow \nabla g(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\Rightarrow \vec{n} = x\hat{i} + y\hat{j} + z\hat{k}$$

Curl the field

$$\vec{F} = 3y\hat{i} - 2xz\hat{j} + (x^2 - y^2)\hat{k}$$

$$\begin{aligned} \mathbf{curl} \vec{F} &= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \hat{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \hat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \hat{k} \\ &= (-2y + 2x)\hat{i} - 2x\hat{j} + (-2z - 3)\hat{k} \end{aligned}$$

Write the integral

$$\iint_R \mathbf{curl} \vec{F} \cdot \vec{n} \, dS \Rightarrow \iint_R ((-2y + 2x)\hat{i} - 2x\hat{j} + (-2z - 3)\hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) \, dS$$

$$\Rightarrow \iint_R -2yx + 2x^2 - 2xy - 2z^2 - 3z \, dS$$

$$\Rightarrow \iint_R -4yx + 2x^2 - 2z^2 - 3z \, dS$$

Write z in terms of x and y

$$x^2 + y^2 + z^2 = a^2 \Rightarrow z = \sqrt{a^2 - x^2 - y^2}$$

$$\Rightarrow \iint_R -4yx + 2x^2 - 2(a^2 - x^2 - y^2) - 3\sqrt{a^2 - x^2 - y^2} \, dS$$

Get z limits

$$z = 0 \Rightarrow x^2 + y^2 = a^2 \Rightarrow y = \pm \sqrt{a^2 - x^2}$$

$$\Rightarrow \begin{cases} -1 \leq x \leq 1 \\ \sqrt{a^2 - x^2} \leq y \leq +\sqrt{a^2 - x^2} \end{cases}$$

Write integral

$$\Rightarrow \int_{-1}^1 \int_{\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} -4yx + 2x^2 - 2(a^2 - x^2 - y^2) - 3\sqrt{a^2 - x^2 - y^2} \, dy \, dx$$

Problem 12:

Evaluate $\iint_{\mathcal{S}} \mathbf{curl} \mathbf{F} \cdot \hat{\mathbf{N}} \, dS$, where \mathcal{S} is the surface $x^2 + y^2 + 2(z - 1)^2 = 6$, $z \geq 0$, $\hat{\mathbf{N}}$ is the outward (away from the origin) normal on \mathcal{S} , and $\mathbf{F} = (xz - y^3 \cos z)\mathbf{i} + x^3 e^z \mathbf{j} + xyze^{x^2+y^2+z^2} \mathbf{k}$.

Solution: 24π