430.457

Introduction to Intelligent Systems

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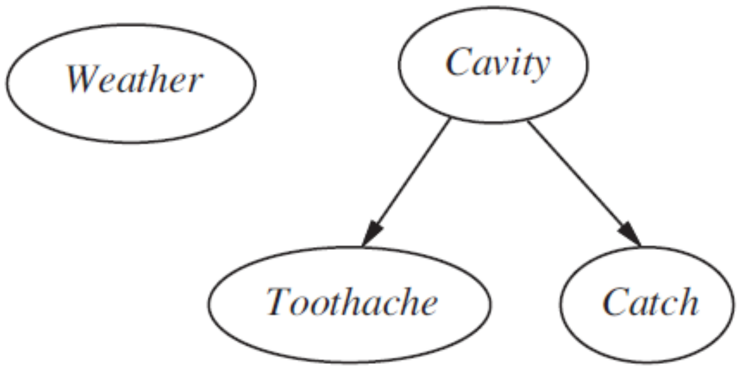
Chapter 13. Probabilistic Reasoning

BAYESIAN NETWORKS

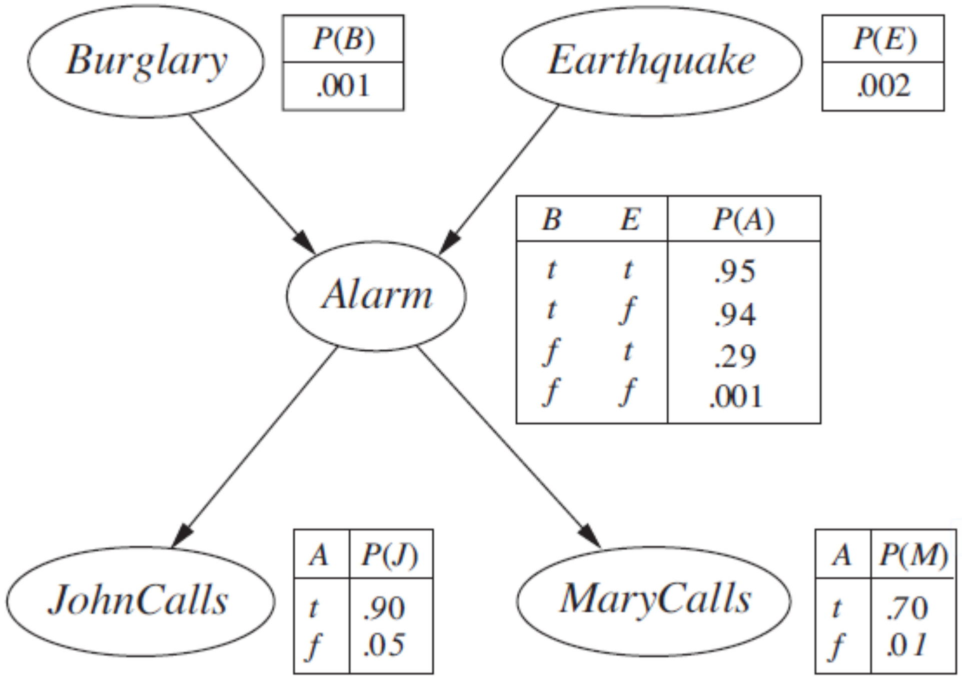
Bayesian Networks

- Probability theory + Graph theory
- Compact representation for a complex probability distribution (cf. full joint distribution)
  - By taking advantage of independence and conditional independence
- A Bayesian network is a DAG (directed acyclic graph)
  - Node = random variable
  - Node X_i has a conditional probability distribution $P(X_i|\text{Parents}(X_i))$



Example: Burglar Alarm

- A new burglar alarm at home
  - Fairly reliable at detecting a burglary, but also responds to minor earthquake
- Two neighbors, John and Mary, who calls you at work when they hear the alarm
  - John nearly always calls when he hears the alarm but sometimes confuses the telephone ringing with the alarm
  - Mary likes loud music and often misses the alarm
- Given the evidence of who has or has not called, what is the probability of a burglary?



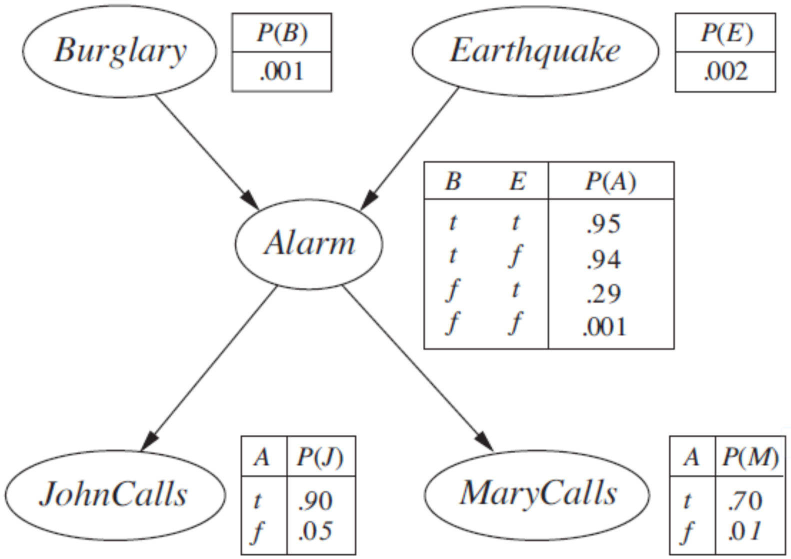
Semantics of Bayesian Networks

- Two views on the semantics of Bayesian networks
  - A representation of the joint distribution
  - An encoding of a collection of conditional independence statements

  * Two views are equivalent
- Full joint distribution for a Bayesian network:

$$\boxed{P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))}$$

$$\begin{aligned} P(B, E, A, J, M) &= P(B)P(E)P(A|B, E) \\ &\times P(J|A)P(M|A) \end{aligned}$$



Constructing a Bayesian Network

- Chain rule:

$$\begin{aligned} P(x_1, \ldots, x_n) &= P(x_n|x_{n-1}, \ldots, x_1)P(x_{n-1}|x_{n-2}, \ldots, x_1) \cdots P(x_2|x_1)P(x_1) \\ &= \prod_{i=1}^{n} P(x_i|x_{i-1}, \ldots, x_1) \end{aligned}$$

- Bayesian networks: $P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i|parents(X_i))$.
- Hence, we must have $P(X_i|X_{i-1}, \ldots, X_1) = P(X_i|Parents(X_i))$, provided that $Parents(X_i) \subseteq \{X_{i-1}, \ldots, X_1\}$.

Algorithm for constructing a Bayesian network

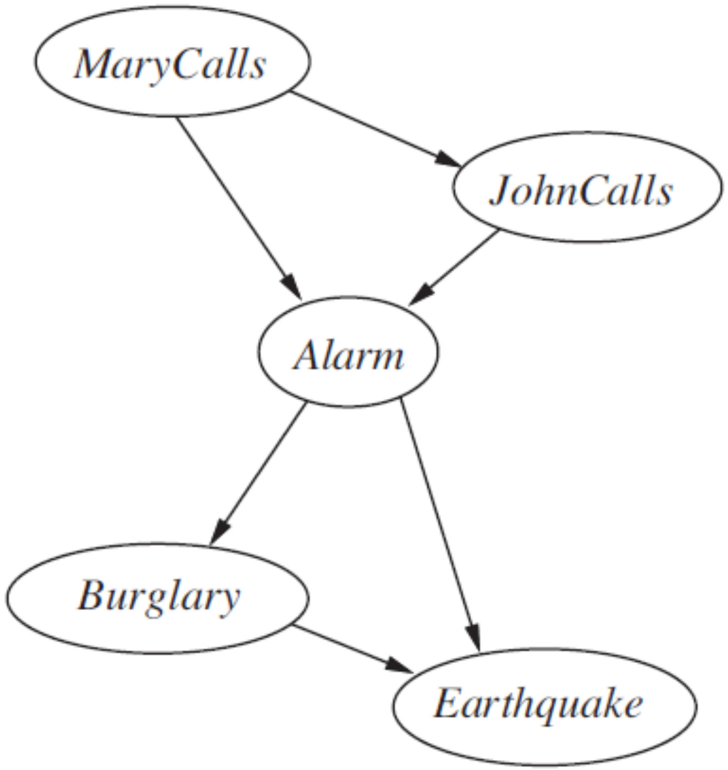
- Nodes: Determine the set of variables to model the domain and order them, $\{X_1, \ldots, X_n\}$.
- Links: For $i = 1$ to n do:
  - Choose a minimal set of parents for X_i from $\{X_1, \ldots, X_{i-1}\}$
  - For each parent insert a link from the parent to X_i
  - Associate the conditional probability tables, $P(X_i|Parents(X_i))$ to node X_i.

Compactness of Bayesian Networks

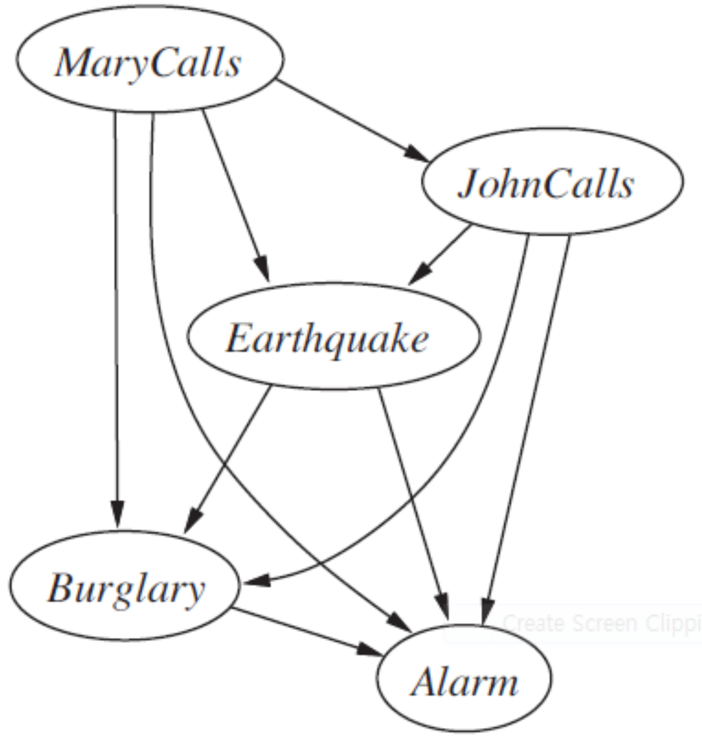
- Suppose we have n Boolean variables.
- Joint distribution requires 2^n numbers.
- A Bayesian network with at most degree k requires only $n2^k$ numbers.
- E.g., if $n = 30$ and each node has five parents ($k = 5$), a Bayesian network requires 960 numbers while the full joint distribution requires $2^{30} \approx 10^9 = 1,000,000,000$ numbers.
- The ordering of nodes at the construction of a Bayesian network determines the complexity of the resulting network.

Effects of Node Ordering

{ MaryCalls, JohnCalls,
Alarm, Burglary, Earthquake}



{ MaryCalls, JohnCalls,
Earthquake, Burglary, Alarm}

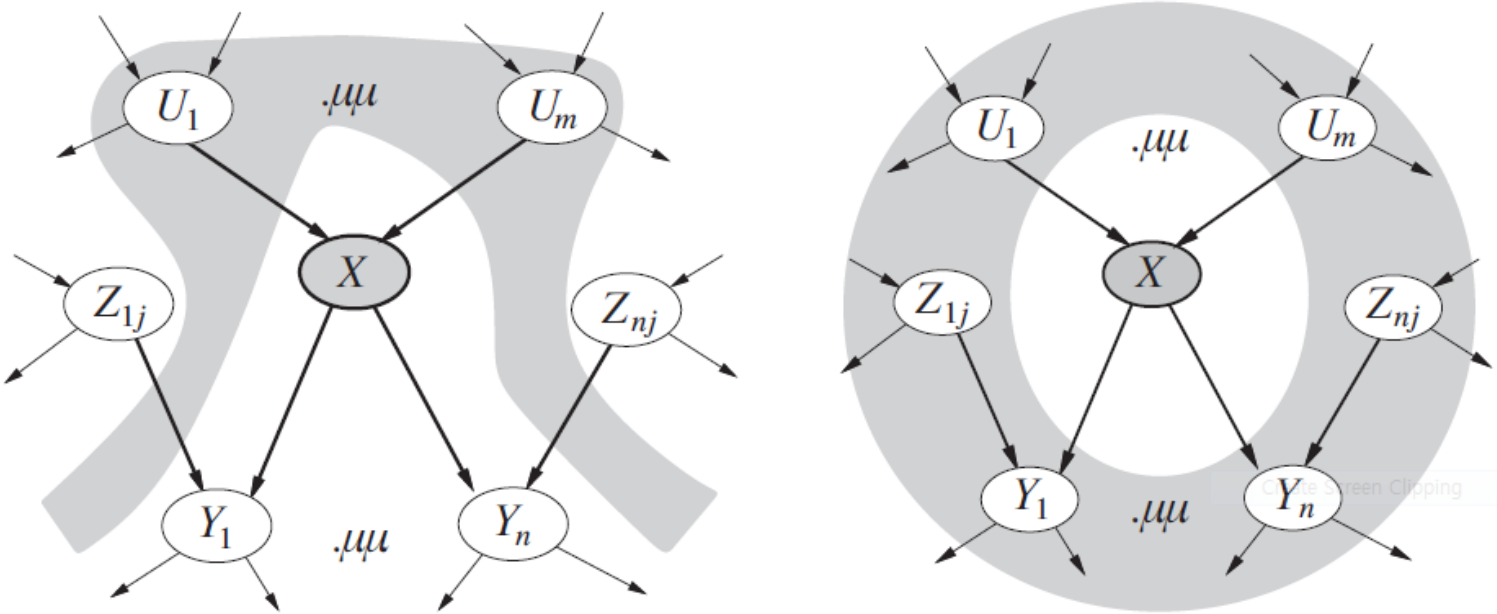


Compact representation if causes precede effects (causal model)

Conditional Independence

a. Each variable is conditionally independent of its non-descendants given its parents (recall $P(X_i \mid Parents(X_i))$)

b. A node is conditionally independent of all other nodes in the networks, given its parents, children, and co-parents (children's parents) - **Markov blanket**



(a) (b)

Efficient Representation of Conditional Distributions

Noisy-OR Model

- Generalization of the logical OR. (E.g., Fever is true iff Cold, Flu, or Malaria is true.)
- Why noisy? There is an uncertainty about the ability of each parent to cause the child to be true.
- Assumptions: (1) all possible causes are listed, (2) inhibition of each parent is independent of inhibition of any other parents.
- Inhibition probabilities:

$$
\begin{aligned}
q_{cold} &= P(\neg fever | cold, \neg flu, \neg malaria) = 0.6 \\
q_{flu} &= P(\neg fever | \neg cold, flu, \neg malaria) = 0.2 \\
q_{malaria} &= P(\neg fever | \neg cold, \neg flu, malaria) = 0.1
\end{aligned}
$$

  $Fever$ is $false$ iff all its $true$ parents are inhibited.

- General rule:

$$P(x_i | parents(X_i)) = 1 - \prod_{\{j : X_j = true\}} q_j$$

Noisy-OR Model

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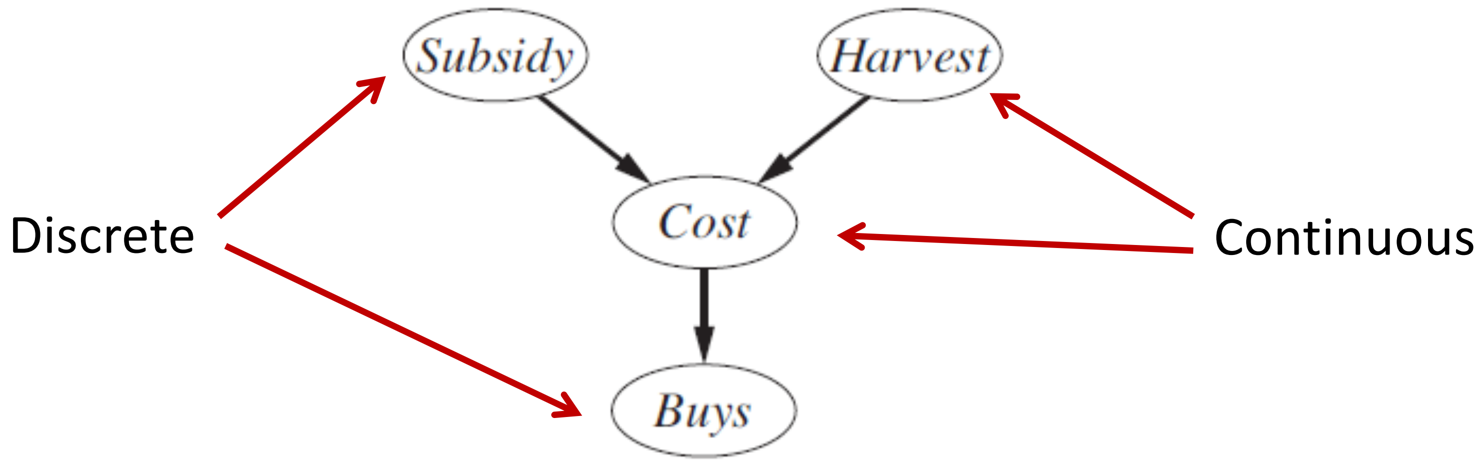
$$P(x_i | parents(X_i)) = 1 - \prod_{\{j : X_j = true\}} q_j$$

Cold	*Flu*	*Malaria*	$P(Fever)$	$P(\neg Fever)$
F	F	F	0.0	1.0
F	F	T	0.9	**0.1**
F	T	F	0.8	**0.2**
F	T	T	0.98	$0.02 = 0.2 \times 0.1$
T	F	F	0.4	**0.6**
T	F	T	0.94	$0.06 = 0.6 \times 0.1$
T	T	F	0.88	$0.12 = 0.6 \times 0.2$
T	T	T	0.988	$0.012 = 0.6 \times 0.2 \times 0.1$

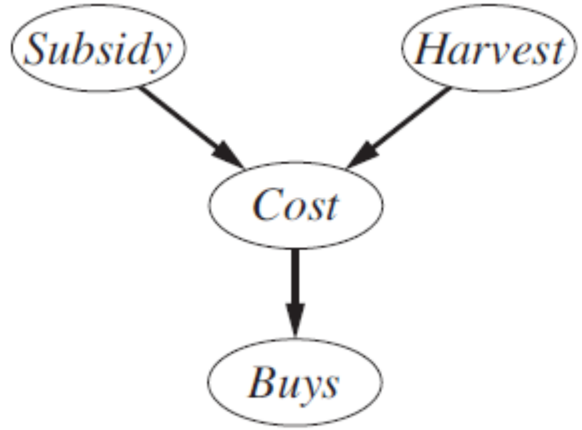
- Requires only $O(k)$ parameters instead of $O(2^k)$ for the full conditional probability table, where k is the number of parents.

Hybrid Bayesian Networks

- Representing continuous variables in a Bayesian network
  - Discretization (may requires large CPTs)
  - Parametric model (Gaussian, Gamma, Beta, etc.)
  - Nonparametric model
- Hybrid Bayesian network
  - A network with both discrete and continuous variables



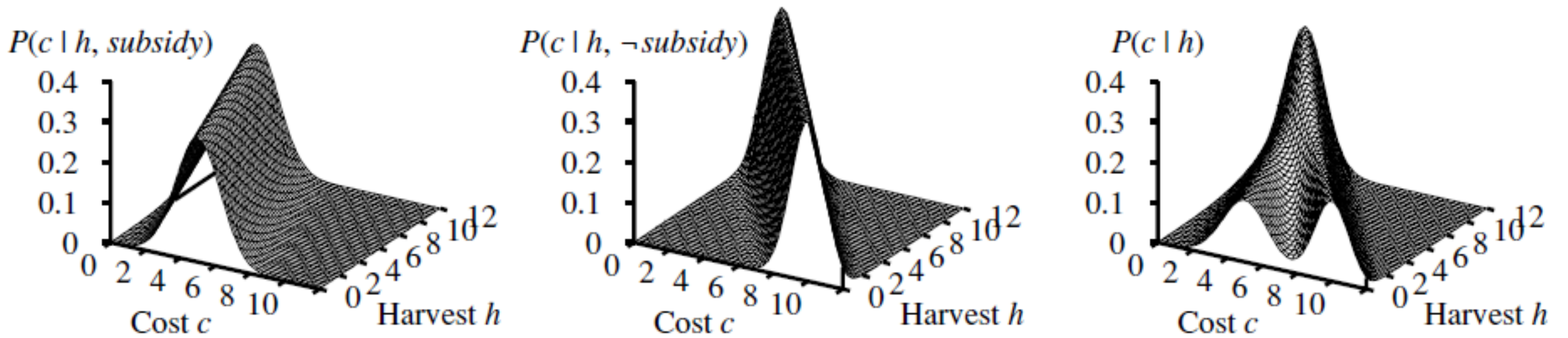
From Discrete to Continuous



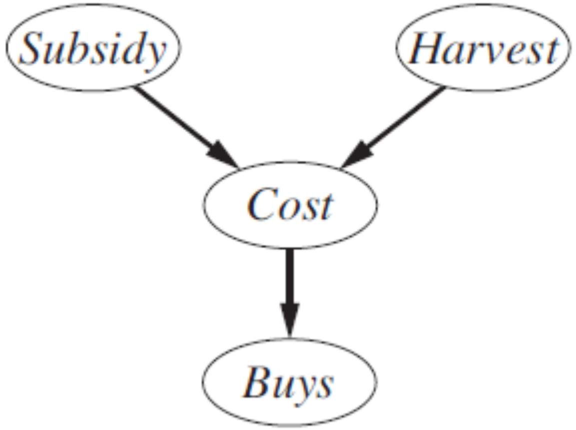
- $P(Cost|Harvest, Subsidy)$ can be specified using
  - $P(Cost|Harvest, Subsidy)$ and
  - $P(Cost|Harvest, \neg Subsidy)$.
- **Linear Gaussian model**

$$P(c|h, subsidy) = \mathcal{N}(c|a_t h + b_t, \sigma_t^2) = \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left(-\frac{1}{2\sigma_t^2}\left(c - (a_t h + b_t)\right)^2\right)$$

$$P(c|h, \neg subsidy) = \mathcal{N}(c|a_f h + b_f, \sigma_f^2) = \frac{1}{\sqrt{2\pi}\sigma_f} \exp\left(-\frac{1}{2\sigma_f^2}\left(c - (a_f h + b_f)\right)^2\right)$$



From Continuous to Discrete



Specifying $Buys$ given $Cost$:

- **Probit distribution**:

$$P(buys|Cost = c) = \Phi\left(\frac{-c+\mu}{\sigma}\right),$$

where $\Phi(x) = \int_{-\infty}^{x} \mathcal{N}(x|0,1)dx$.

- **Logit distribution**:

$$P(buys|Cost = c) = \frac{1}{1+\exp\left(-2\frac{-c+\mu}{\sigma}\right)},$$

where $1/(1+e^{-x})$ is called a logistic function.