Problem 1:

calculate **div F** and **curl F** for the vector field $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$

Solution: div F $= 0$, **curl F** $= -\mathbf{i} - \mathbf{j} - \mathbf{k}$

$$\textbf{div}\, \vec{v}(x,y) = \nabla \bullet \vec{v}(x,y) = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y}$$

$$\textbf{curl}\, \vec{v}(x,y,z) = \nabla \times \vec{v}(x,y,z) = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{bmatrix} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix}$$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \begin{bmatrix} \frac{d}{dx} \\ \frac{d}{dy} \\ \frac{d}{dz} \end{bmatrix} \cdot \begin{bmatrix} y \\ z \\ x \end{bmatrix} = \frac{d}{dx} y i + \frac{d}{dy} z j + \frac{d}{dz} x k = 0 + 0 + 0 = 0$$

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$det(M) = |M| = a \cdot \left|\begin{pmatrix} e & f \\ h & i \end{pmatrix}\right| - b \cdot \left|\begin{pmatrix} d & f \\ g & i \end{pmatrix}\right| + c \cdot \left|\begin{pmatrix} d & e \\ g & h \end{pmatrix}\right|$$

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F}(x,y,z) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ y & z & x \end{vmatrix}$$

$$= i \cdot \left(\frac{d}{dy} \cdot x - \frac{d}{dz} \cdot z\right) - j \cdot \left(\frac{d}{dx} \cdot x - \frac{d}{dz} \cdot y\right) + k \cdot \left(\frac{d}{dx} z - \frac{d}{dy} \cdot y\right)$$

$$= i \cdot (0 - 1) - j(1 - 0) + k \cdot (0 - 1) = \underline{-i - j - k}$$

Problem 2:

calculate **div F** and **curl F** for the vector field $\mathbf{F} = x\mathbf{i} + x\mathbf{k}$

Solution: div F $= 1$, **curl F** $= -\mathbf{j}$

$$\textbf{curl}\, \vec{v}(x,y) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}, \text{ where } \vec{v}(x,y) = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$$

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$det(M) = |M| = a \cdot \left|\begin{pmatrix} e & f \\ h & i \end{pmatrix}\right| - b \cdot \left|\begin{pmatrix} d & f \\ g & i \end{pmatrix}\right| + c \cdot \left|\begin{pmatrix} d & e \\ g & h \end{pmatrix}\right|$$

$$\operatorname{div} \vec{F} = \nabla \vec{F} = \frac{d}{dx} x + \frac{d}{dy} 0 + \frac{d}{dz} x = 1$$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ x & 0 & x \end{vmatrix} = i \cdot \left(\frac{d}{dy} \cdot x - \frac{d}{dz} \cdot 0\right) - j\left(\frac{d}{dx} \cdot x - \frac{d}{dz} \cdot x\right) + k\left(\frac{d}{dx} \cdot 0 - \frac{d}{dy} x\right)$$

$$= i \cdot (0 - 0) - j(1 - 0) + k(0 - 0) = -j$$

Problem 4:

Evaluate $\oint_c (\sin x + 3y^2)\, dx + (2x - e^{-y^2})dy$, where c is the boundary of the half-disk $x^2 + y^2 \leq a^2$, $y \geq 0$, oriented counterclockwise.

Solution: $\pi a^2 - 4a^3$

$$\oint_C \vec{F}(x,y)\, \mathrm{d}\vec{r} = \oint_C P(x,y)\mathrm{dx} + Q(x,y)\mathrm{dy} = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, \mathrm{dA}$$

$P(x,y) = \sin(x) + 3y^2$

$\frac{dP}{dy} = 6y$

$Q(x,y) = 2x - e^{-y^2}$

$\frac{dQ}{dx} = 2$

$$\Rightarrow \iint_R 2 - 6y \; dA$$

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$$\iint_R (2 - 6 \cdot r \cdot \sin(\theta)) \cdot r \; dr \; d\theta$$

$r = 0..\pi$

$\theta = 0..a$

$$= \int_0^{\pi}\int_0^{a} 2r - 6 \cdot r^2 \sin(\theta) \; dr \; d\theta$$

$$= 2 \cdot \int_0^{\pi}\int_0^{a} r \; dr\, d\theta - 6 \int_0^{\pi}\int_0^{a} r^2 \cdot \sin(\theta) \; dr \; d\theta$$

$$= 2 \cdot \int_0^{\pi} \left[\frac{r^2}{2}\right]_0^a d\theta - 6 \cdot \int_0^{\pi} \sin(\theta) \cdot \left[\frac{r^3}{3}\right]_0^a \; d\theta = \cancel{2} \cdot \int_0^{\pi} \frac{a^2}{\cancel{2}} d\theta - 6 \cdot \int_0^{\pi} \sin(\theta) \cdot \frac{1}{3} \cdot a^3 \; d\theta$$

$$= a^2 \cdot \left[\theta\right]_0^{\pi} - 2a^3 \left[-\cos(\theta)\right]_0^{\pi} = \pi a^2 - 2a^3 \left(-0 - (-1)\right) = \underline{\underline{\pi a^2 - 2a^3}}$$

Problem 5:

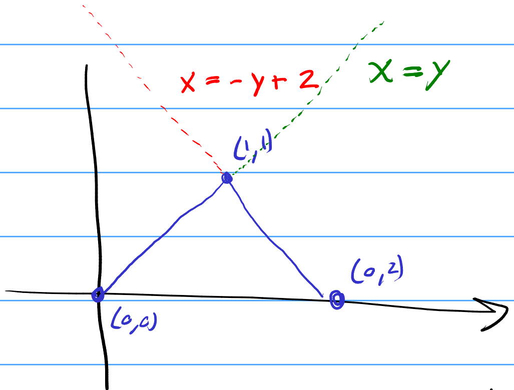
Evaluate $\oint_e (x^2 - xy)\,dx + (xy - y^2)dy$, clockwise around the triangle with vertices $(0,0)$, $(1,1)$, and $(2,0)$.

Solution: $-\frac{4}{3}$

$$\oint_C \vec{F}(x,y)\,\mathrm{d}\vec{r} = \oint_C P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{dA}$$

$$= -\iint_R \frac{d}{dx}(xy - y^2) - \frac{d}{dy}(x^2 - xy)\,dA$$

$$= -\iint_R y - (-x)\,dA = \iint y + x\,dA$$



$$-\int_0^1 \int_y^{-y+2} y + x\,dx\,dy = -\int_0^1 \int_y^{-y+2} y\,dxdy - \int_0^1 \int_y^{-y+2} x\,dxdy$$

$$= -\int_0^1 \left[xy\right]_y^{-y+2} dy - \int_0^1 \left[\frac{x^2}{2}\right]_y^{-y+2} dy$$

$$= -\int_0^1 ((-y+2)y) - (y \cdot y)\,dy - \frac{1}{2}\int_0^1 (-y+2)^2 - y^2\,dy$$

$$= -\int_0^1 -y^2 + 2y - y^2\,dy - \frac{1}{2}\int_0^1 \cancel{y^2} - 4y + 4 - \cancel{y^2}\,dy$$

$$= -\int_0^1 2y - 2y^2\,dy - \frac{1}{2}\int_0^1 -4y + 4\,dy$$

$$= 2\int_0^1 y^2 + 2y\,dy + 2\int_0^1 y - 1\,dy$$

$$= 2 \cdot \left[\frac{1}{3} \cdot y^3 + y^2\right]_0^1 - 2\left[\frac{y^2}{2} - y\right]_0^1 = 2 \cdot \left(\frac{1}{3} + 1\right) - 2 \cdot \left(\frac{1}{2} - 1\right) = \frac{2}{3} + 2 - 1 + 2$$

$$= \frac{6}{3}$$

Problem 6:

Use a line integral to find the plane area enclosed by the curve $r = a\cos^3 t\mathbf{i} + b\sin^3 t\mathbf{j}$, $(0 \le t \le 2\pi)$.

Solution: $\frac{3\pi ab}{8}$

How?

$r = a \cdot \cos(t)^3 i + b \cdot \sin(t)^3 j$

Problem 7:

Use the Divergence Theorem to calculate the flux of the given vector field out of the sphere $\mathcal{S}$ with equation $x^2 + y^2 + z^2 = a^2$, where $a > 0$

$$\mathbf{F} = x\mathbf{i} - 2y\mathbf{j} + 4z\mathbf{k}$$

$$\iint_S \vec{F} \bullet \vec{n}\, \mathrm{dS} = \iiint_V \mathbf{div}\, \vec{F}\, \mathrm{dV}$$

$$\iiint_V div\, \vec{F}\, dV = \iiint_V \nabla \cdot \vec{F}\, dV = \iiint_V \frac{d}{dx} x + \frac{d}{dy} \cdot (-2y) + \frac{d}{dz} 4z\, dV$$

$$= \iiint_V 1 - 2 + 4\, dV = \iiint 3\, dV$$

Convert to spherical coordinates

$$x = \rho \cdot \sin\phi \cdot \cos\theta$$
$$y = \rho \cdot \sin\phi \cdot \sin\theta$$
$$z = \rho \cdot \cos\phi$$
$$\mathrm{dV} = \rho \cdot \sin\phi\, \mathrm{d}\rho\, \mathrm{d}\phi\, \mathrm{d}\theta$$

$$x^2 + y^2 + z^2 = a^2 \Rightarrow \rho = 0 \ldots a$$

$$\iiint_V 3\, dV \Rightarrow 3 \cdot \iiint \rho^2 \cdot \sin(\phi)\, d\theta\, d\rho\, d\phi$$

$$= 3 \cdot \iiint \rho^2 \cdot \sin(\phi)\, d\theta\, d\phi\, d\rho$$

We can now assign limits

$$= 3 \cdot \int_0^a \rho^2 \int_0^\pi \sin(\phi) \int_0^{2\pi} 1\, d\theta\, d\phi\, d\rho$$

$$= 3 \cdot \int_0^a \rho^2 \int_0^\pi \sin(\phi) \cdot 2\pi\, d\phi\, d\rho$$

$$= 6\pi \cdot \int_0^a \rho^2 \cdot \Big[-\cos(\phi)\Big]_0^\pi\, d\rho = 6\pi \cdot \int_0^a \rho \cdot \big(\overset{2}{-(-1) - (-1)}\big)\, d\rho$$

$$= 12\pi \int_0^a \rho^2\, d\rho = 12\pi \cdot \left[\frac{\rho^3}{3}\right]_0^a = 12\pi \cdot \frac{a^3}{3} = \underline{\underline{4\pi a^3}}$$

Problem 8:

Use the Divergence Theorem to calculate the flux of the given vector field out of the sphere $\mathcal{S}$ with equation $x^2 + y^2 + z^2 = a^2$, where $a > 0$

$$\mathbf{F} = x^3\mathbf{i} + 3yz^2\mathbf{j} + (3y^2z + x^2)\mathbf{k}$$

Solution: $\frac{12}{5}\pi a^5$

$$\iint_S \vec{F} \bullet \vec{n}\, \mathrm{dS} = \iiint_V \mathbf{div}\, \vec{F}\, \mathrm{dV}$$

$$\operatorname{div} \vec{F} = \nabla \vec{F} = \frac{d}{dx}(x^3) + \frac{d}{dy}(3yz^2) + \frac{d}{dz}(3y^2z + x^2)$$

$$= 3x^2 + 3z^2 + 3y^2 = 3(x^2+y^2+z^2)$$

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_V 3(x^2+y^2+z^2)\, dV = 3 \cdot \iiint_V x^2+y^2+z^2\, dV$$

Convert to spherical coordinates

$$x^2+y^2+z^2 = r^2$$

$$x^2+y^2+z^2 \leq a^2 \Rightarrow 0 \leq r \leq a$$

$$dV = r^2 \cdot \sin(\phi)\, dr\, d\phi\, d\theta$$

Re-writing the integral

$$3 \cdot \iiint_V x^2+y^2+z^2\, dV \Rightarrow 3 \cdot \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2 \cdot r^2 \cdot \sin(\phi)\, dr\, d\phi\, d\theta$$

$$= 3 \cdot \int_0^{2\pi}\int_0^{\pi} \left[\frac{r^5}{5} \cdot \sin(\phi)\right]_0^a d\phi\, d\theta = \frac{3}{5} \cdot \int_0^{2\pi}\int_0^{\pi} a^5 \cdot \sin(\phi)\, d\phi\, d\theta = \frac{3}{5} \cdot \int_0^{2\pi} \left[a^5 \cdot \cos(\phi)\right]_0^{\pi} d\theta$$

$$= \frac{3}{5} \cdot \int_0^{2\pi} a^5 \cdot \overbrace{\cos(0)}^{1} - a^5 \cdot \overbrace{\cos(\pi)}^{-1}\, d\theta = \frac{3}{5} \cdot \int_0^{2\pi} 2a^5\, d\theta = \frac{3}{5}\left[2a^5\theta\right]_0^{2\pi} = \frac{3}{5} \cdot 2a^5 \cdot 2\pi = \underline{\underline{\frac{12\pi a^5}{5}}}$$

Problem 9:

Evaluate the flux of $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ outward across the boundary of the given solid region.

The solid ellipsoid $x^2 + y^2 + 4(z-1)^2 \leq 4$

Solution: $32\pi/3$

$$\operatorname{div} \vec{F} = \nabla \vec{F} = \frac{d}{dx}x^2 + \frac{d}{dy}y^2 + \frac{d}{dz}z^2 = 2x + 2y + 2z$$

$$x^2+y^2+4(z-1)^2 \leq 4 \Rightarrow 4(z-1)^2 \leq 4-x^2-y^2 \Rightarrow 2(z-1) \leq \sqrt{4-x^2-y^2}$$

$$z \leq \frac{1}{2}\sqrt{4-x^2-y^2}+1$$

The elipsoid is widest at z=0, finding limits from there

$$x^2+y^2+4(0-1)^2 \leq 4 \Rightarrow x^2+y^2+4 \leq 4$$

Convert to spherical coordinates

$$x \rightarrow \rho \cdot \sin\phi \cdot \cos\theta$$
$$y \rightarrow \rho \cdot \sin\phi \cdot \sin\theta$$
$$z \rightarrow \rho \cdot \cos\phi$$
$$\mathrm{dV} \rightarrow \rho^2 \cdot \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta$$

$$\Rightarrow \operatorname{div}\vec{F} = 2\rho\cdot\sin(\phi)\cdot\cos(\theta) + 2\rho\cdot\sin(\phi)\cdot\sin(\theta) + 2\rho\cdot\cos(\phi)$$
$$= 2\rho\left(\sin(\phi)\cdot\cos(\theta) + \sin(\phi)\cdot\sin(\theta) + \cos(\phi)\right)$$

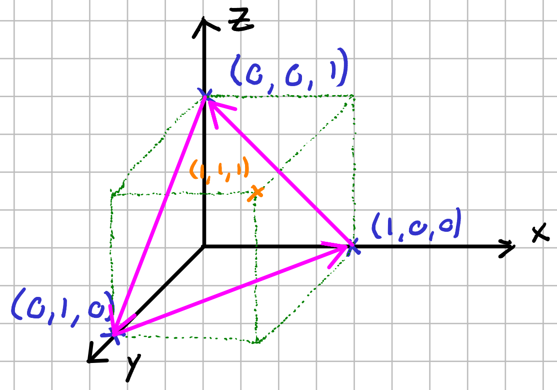
$$x^2+y^2+4(z-1)^2 \leq 4 \Rightarrow \rho^2\cdot\sin^2(\phi)\cdot\cos^2(\theta) + \rho^2\cdot\sin^2(\phi)\sin^2(\theta) + 4(\rho\cdot\cos(\phi)-1)^2 \leq 4$$
$$= \rho^2\sin^2(\phi)\cdot\left(\overbrace{\cos^2(\theta)+\sin^2(\theta)}^{1}\right) + 4(\rho\cdot\cos(\phi)-1)^2 \leq 4$$
$$= \rho^2\sin^2(\phi) + 4\left(\rho^2\cos^2(\phi) - 2\rho\cos(\phi)+1\right) \leq 4$$

Problem 10:

Evaluate $\oint_e \overset{f_1}{xy}\, dx + \overset{f_2}{yz}\, dy + \overset{f_3}{zx}\, dz$ around the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, oriented clockwise as seen from the point $(1, 1, 1)$.

Solution: $\frac{1}{2}$

$$\vec{F} = xy\,\hat{i} + yz\,\hat{j} + zx\,\hat{k}$$



$$\Rightarrow \iint_R \text{curl}\,\vec{F} \cdot \vec{n}\, dA$$

$$= \iint_R (-y\hat{i} - z\hat{j} - x\hat{k}) \cdot \vec{n}\, dA$$

Write z in terms of x and y, so the resulting plane contains the three points.

$$z = 1 - x - y \quad , \quad \vec{n} = \hat{i} + \hat{j} + \hat{k}$$

Writing the integral

$$\int_0^1 \int_0^{-y+1} (-y\hat{i} - (1-x-y)\hat{j} - x\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k})\, dx\, dy$$

$$\Rightarrow \int_0^1 \int_0^{-y+1} \cancel{-y} - 1 \cancel{+x} \cancel{+y} \cancel{-x}\; dx\, dy = \int_0^1 [x]_0^{-y+1}\, dy = \int_0^1 -y + 1\, dy = \left[\frac{-1}{2}y^2 + y\right]_0^1 = \frac{-1}{2} + 1 = \underline{\underline{\frac{1}{2}}}$$

Problem 11:

Evaluate $\iint_{\mathcal{S}} \mathbf{curl\ F} \cdot \widehat{\mathbf{N}}\ dS$, where $\mathcal{S}$ is the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$, with outward normal, and $\mathbf{F} = 3y\mathbf{i} - 2xz\mathbf{j} + (x^2 - y^2)\mathbf{k}$.

Solution: $-3\pi a^2$

Finding the normal vector

$$g(x,y,z) = x^2+y^2+z^2 \Rightarrow \nabla g(x,y,z) = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\Rightarrow \vec{n} = x\hat{i} + y\hat{j} + z\hat{k}$$

Curl the field

$$\vec{F} = \overset{f_1}{3y}\hat{i} - \overset{f_2}{2xz}\hat{j} + \overset{f_3}{(x^2-y^2)}\hat{k}$$

$$\text{curl } \vec{F} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\hat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{k}$$

$$= (-2y+2x)\hat{i} - 2x\hat{j} + (-2z-3)\hat{k}$$

Write the integral

$$\iint_R \text{curl } \vec{F} \cdot \vec{n}\, ds \Rightarrow \iint_R \left((-2y+2x)\hat{i} - 2x\hat{j} + (-2z-3)\hat{k}\right) \cdot \left(x\hat{i} + y\hat{j} + z\hat{k}\right)\, ds$$

$$\Rightarrow \iint_R \underline{-2yx} + 2x^2 \underline{-2xy} - 2z^2 - 3z\, ds$$

$$\Rightarrow \iint_R -4yx + 2x^2 - 2z^2 - 3z\, ds$$

Write z in terms of x and y

$$x^2 + y^2 + z^2 = a^2 \Rightarrow z = \left|\sqrt{a^2 - x^2 - y^2}\right|$$

$$\Rightarrow \iint_R -4yx + 2x^2 - 2(a^2-x^2-y^2) - 3\sqrt{a^2-x^2-y}\, ds$$

Get z limits

$$z = 0 \Rightarrow x^2 + y^2 = a^2 \Rightarrow y = \pm\sqrt{a^2-x^2}$$

$$\Rightarrow \begin{cases} -1 \leq x \leq 1 \\ \sqrt{a^2-x^2} \leq y \leq +\sqrt{a^2-x^2} \end{cases}$$

Write integral

$$\Rightarrow \int_{-1}^{1} \int_{\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} -4yx + 2x^2 - 2(a^2-x^2-y^2) - 3\sqrt{a^2-x^2-y}\, dy\, dx$$

Problem 12:

Evaluate $\iint_{\mathcal{S}} \mathbf{curl}\ \mathbf{F} \cdot \widehat{\mathbf{N}}\ dS$, where $\mathcal{S}$ is the surface $x^2 + y^2 + 2(z-1)^2 = 6$, $z \geq 0$, $\widehat{\mathbf{N}}$ is the outward (away from the origin) normal on $\mathcal{S}$, and $\mathbf{F} = (xz - y^3 \cos z)\mathbf{i} + x^3 e^z \mathbf{j} + xyze^{x^2+y^2+z^2}\mathbf{k}$.

Solution: 24π