Observability, Observers, and Observer Based Control

Control Engineering (Reguleringsteknik)

Christoffer Sloth

chsl@mmmi.sdu.dk

SDU Robotics
The Maersk Mc-Kinney Moller Institute
University of Southern Denmark

SDU

Agenda

Introduction

Observability

Full Order Observer

Observer Design

Observer Based Control

Introduction

Curriculum for Reguleringsteknik (REG)

Matematiske og grafiske metoder til syntese af lineære tidsinvariante systemer:[1]

- diskret og kontinuert tilstandsbeskrivelse
- analyse i tid og frekvens
- stabilitet, reguleringshastighed, følsomhed og fejl
- digitale PI, PID, LEAD og LAG regulatorer (serieregulatorer)
- tilstandsregulering, pole-placement og tilstands-estimering (observer)
- optimal regulering (least squares) og optimal tilstands-estimation (Kalman-filter)

Færdigheder:
Efter gennemførelse af kurset kan den succesfulde studerende:

- kunne analysere, dimensionere og implementere såvel kontinuert som tidsdiskret regulering af lineære tidsinvariante og stokastiske systemer

Kompetencer:
Efter gennemførelse af kurset kan den succesfulde studerende:

- anvende og implementere klassiske og moderne reguleringsteknikker for at kunne styre og regulere en robot hurtig og præcist

[1] Based on https://fagbesk.sam.sdu.dk/?fag_id=39673

The twelve lectures of the course are

- **Lecture 1**: Introduction to Linear Time-Invariant Systems
- **Lecture 2**: Stability and Performance Analysis
- **Lecture 3**: Introduction to Control
- **Lecture 4**: Design of PID Controllers
- **Lecture 5**: Root Locus
- **Lecture 6**: The Nyquist Plot
- **Lecture 7**: Dynamic Compensators and Stability Margins
- **Lecture 8**: Implementation
- **Lecture 9**: State Feedback
- **Lecture 10**: Observer Design
- **Lecture 11**: Optimal Control (Linear Quadratic Control)
- **Lecture 12**: The Kalman Filter

Observability

A continuous time system

$$\dot{x}(t) = Ax(t)\,, \quad y(t) = Cx(t)$$

is said to be *observable* iff $y(t) \equiv 0 \Rightarrow x(t) \equiv 0$.

A discrete time system

$$x_{k+1} = \Phi x_k\,, \quad y_k = Cx_k$$

is said to be *observable* iff $y_k \equiv 0 \Rightarrow x_k \equiv 0$.

We consider the discrete time system

$$x_{k+1} = \Phi x_k \,, \quad y_k = C x_k \,, \quad x_0 = x_0$$

and iterate:

$$x_0 = x_0 \qquad y_0 = C x_0$$

We consider the discrete time system

$$x_{k+1} = \Phi x_k \,, \quad y_k = C x_k \,, \quad x_0 = x_0$$

and iterate:

$$\begin{aligned} x_0 &= x_0 & \qquad y_0 &= C x_0 \\ x_1 &= \Phi x_0 & & \end{aligned}$$

We consider the discrete time system

$$x_{k+1} = \Phi x_k \,, \quad y_k = C x_k \,, \quad x_0 = x_0$$

and iterate:

$$\begin{aligned} x_0 &= x_0 & \qquad y_0 &= C x_0 \\ x_1 &= \Phi x_0 \end{aligned}$$

Observability

Condition for Observability (1)

We consider the discrete time system

$$x_{k+1} = \Phi x_k \,, \quad y_k = C x_k \,, \quad x_0 = x_0$$

and iterate:

$$\begin{aligned} x_0 &= x_0 & y_0 &= C x_0 \\ x_1 &= \Phi x_0 & y_1 &= C\Phi x_0 \end{aligned}$$

We consider the discrete time system

$$x_{k+1} = \Phi x_k\,, \quad y_k = C x_k\,, \quad x_0 = x_0$$

and iterate:

$$\begin{aligned} x_0 &= x_0 & \qquad y_0 &= C x_0 \\ x_1 &= \Phi x_0 & \qquad y_1 &= C \Phi x_0 \\ x_2 &= \Phi x_1 & & \end{aligned}$$

We consider the discrete time system

$$x_{k+1} = \Phi x_k\,, \quad y_k = C x_k\,, \quad x_0 = x_0$$

and iterate:

$$\begin{aligned} x_0 &= x_0 & y_0 &= C x_0 \\ x_1 &= \Phi x_0 & y_1 &= C\Phi x_0 \\ x_2 &= \Phi x_1 & & \end{aligned}$$

We consider the discrete time system

$$x_{k+1} = \Phi x_k\,, \quad y_k = C x_k\,, \quad x_0 = x_0$$

and iterate:

$$\begin{aligned} x_0 &= x_0 & y_0 &= Cx_0 \\ x_1 &= \Phi x_0 & y_1 &= C\Phi x_0 \\ x_2 &= \Phi^2 x_0 & y_2 &= C\Phi^2 x_0 \end{aligned}$$

We consider the discrete time system

$$x_{k+1} = \Phi x_k \,, \quad y_k = C x_k \,, \quad x_0 = x_0$$

and iterate:

$$\begin{aligned} x_0 &= x_0 & y_0 &= C x_0 \\ x_1 &= \Phi x_0 & y_1 &= C\Phi x_0 \\ x_2 &= \Phi^2 x_0 & y_2 &= C\Phi^2 x_0 \\ &\vdots \\ x_{n-1} &= \Phi x_{n-2} \end{aligned}$$

We consider the discrete time system

$$x_{k+1} = \Phi x_k\,, \quad y_k = C x_k\,, \quad x_0 = x_0$$

and iterate:

$$\begin{aligned}
x_0 &= x_0 & y_0 &= C x_0 \\
x_1 &= \Phi x_0 & y_1 &= C\Phi x_0 \\
x_2 &= \Phi^2 x_0 & y_2 &= C\Phi^2 x_0 \\
&\vdots \\
x_{n-1} &= \Phi x_{n-2}
\end{aligned}$$

We consider the discrete time system

$$x_{k+1} = \Phi x_k \,, \quad y_k = C x_k \,, \quad x_0 = x_0$$

and iterate:

$$\begin{aligned}
x_0 &= x_0 & y_0 &= C x_0 \\
x_1 &= \Phi x_0 & y_1 &= C \Phi x_0 \\
x_2 &= \Phi^2 x_0 & y_2 &= C \Phi^2 x_0 \\
&\vdots \\
x_{n-1} &= \Phi^{n-1} x_0 & y_{n-1} &= C \Phi^{n-1} x_0
\end{aligned}$$

Writing the equations

$$y_k = C\Phi^k x_0 \, , \, k = 0, \ldots, n-1$$

in matrix form we obtain:

Writing the equations

$$y_k = C\Phi^k x_0\,,\ k = 0, \ldots, n-1$$

in matrix form we obtain:

$$\underbrace{\begin{bmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{n-1} \end{bmatrix}}_{\text{Observability matrix}} x_0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Writing the equations

$$y_k = C\Phi^k x_0\,,\ k = 0, \ldots, n-1$$

in matrix form we obtain:

$$\begin{bmatrix} C \\ C\Phi \\ \vdots \\ C\Phi^{n-1} \end{bmatrix} x_0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

When is this equation solvable for some $x_0 \neq 0$?

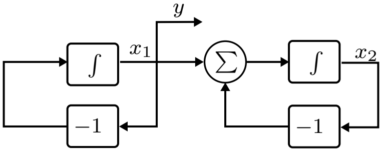
THEOREM. A system

continuous time	discrete time
$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) \\ y(t) = Cx(t) \end{cases}$	$\Sigma : \begin{cases} x_{k+1} = \Phi x_k \\ y_k = Cx_k \end{cases}$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, is observable if and only if

$$\operatorname{rank} \mathcal{O} = \operatorname{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

Example: Series Connection (1)



State and output equations:

$$\left\{\begin{array}{ccc} \dot{x}_1 & = & -x_1 \\ \dot{x}_2 & = & -x_2 + x_1 \\ y & = & x_1 \end{array}\right\}$$

State space model:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Observability
Example: Series Connection (2)

For the state space matrices:

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

the observability matrix $\mathcal{O}$ becomes:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

For the state space matrices:

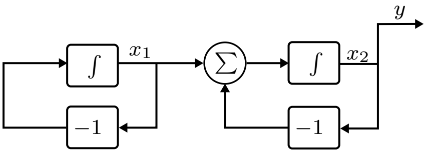
$$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

the observability matrix $\mathcal{O}$ becomes:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

$\det \mathcal{O} = 0 \implies$ system is unobservable.

Observability
Example: Series Connection (3)



State and output equations:

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & -x_1 \\ \dot{x}_2 & = & -x_2 + x_1 \\ y & = & x_2 \end{array} \right\}$$

State space model:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For the state space matrices:

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

the observability matrix $\mathcal{O}$ becomes:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

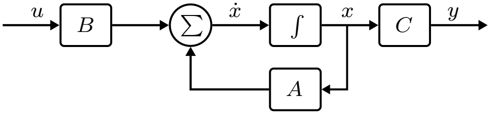
For the state space matrices:

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

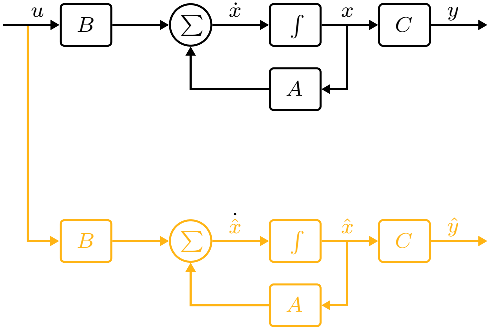
the observability matrix $\mathcal{O}$ becomes:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$\det \mathcal{O} = -1 \neq 0 \implies$ system is observable.



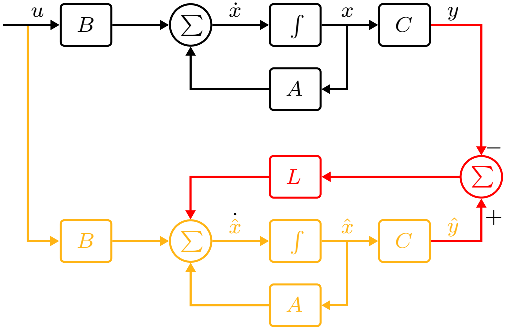
u
B
$\sum$
$\dot{x}$
$\int$
x
C
y
A



u
B
Σ
ẋ
∫
x
C
y
A
B
Σ
x̂̇
∫
x̂
C
ŷ
A

Full Order Observer

The Full Order Observer (1)



Full Order Observer

The Full Order Observer (2)

System:
$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

Observer:
$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\ \hat{y} &= C\hat{x} \end{aligned}$$

System:
$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

Observer:
$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\ \hat{y} &= C\hat{x} \end{aligned}$$

Error, $e = \hat{x} - x$:

$$\dot{e} = \dot{\hat{x}} - \dot{x} = A\hat{x} + Bu + L(C\hat{x} - y) - (Ax + Bu)$$

Full Order Observer

The Full Order Observer (2)

System:
$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

Observer:
$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\ \hat{y} &= C\hat{x} \end{aligned}$$

Error, $e = \hat{x} - x$:

$$\begin{aligned} \dot{e} = \dot{\hat{x}} - \dot{x} &= A\hat{x} + Bu + L(C\hat{x} - y) - (Ax + Bu) \\ &= A(\hat{x} - x) + L(C\hat{x} - Cx) \end{aligned}$$

Full Order Observer

The Full Order Observer (2)

System:
$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

Observer:
$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\ \hat{y} &= C\hat{x} \end{aligned}$$

Error, $e = \hat{x} - x$:

$$\begin{aligned} \dot{e} = \dot{\hat{x}} - \dot{x} &= A\hat{x} + Bu + L(C\hat{x} - y) - (Ax + Bu) \\ &= A(\hat{x} - x) + L(C\hat{x} - Cx) \\ &= (A + LC)(\hat{x} - x) = (A + LC)e \end{aligned}$$

THEOREM. A full order observer for the system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

with observer gain L is stable, if and only if the eigenvalues of the matrix $A + LC$ all have negative real part.

Moreover, such an L always exists, if (A, C) is observable.

Full Order Observer

Observable Canonical Form (1)

Any observable *single output* system can be written in the form:

$$\dot{x}_o = A_o x_o\,, \quad y = C_o x_o\,, \quad x_o \in \mathbb{R}^n\,, y \in \mathbb{R}$$

where

$$A_o = \left[\begin{array}{c|c} a & \begin{array}{c} I_{n-1} \\ \hline 0_{1\times(n-1)} \end{array} \end{array}\right], \quad C_o = \left[\begin{array}{c|c} 1 & 0_{1\times(n-1)} \end{array}\right]$$

and where $a \in \mathbb{R}^{n\times 1}$, $a^T = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$. It can be shown that

$$\det(\lambda I - A_o) = \lambda^n - a_1\lambda^{n-1} - \ldots - a_n$$

For $n = 3$ the observable canonical form becomes:

$$A_o = \left[\begin{array}{c|cc} a_1 & 1 & 0 \\ a_2 & 0 & 1 \\ \hline a_3 & 0 & 0 \end{array}\right], \; C_o = \left[\begin{array}{c|cc} 1 & 0 & 0 \end{array}\right]$$

which is indeed observable:

$$\mathcal{O}_o = \begin{bmatrix} C_o \\ C_o A_o \\ C_o A_o^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ a_1^2 + a_2 & a_1 & 1 \end{bmatrix}$$

$\det(\mathcal{O}) = 1 \neq 0 \implies$ system is observable.

Consider a system:

$$\dot{x} = Ax\,, \quad y = Cx\,, \quad x \in \mathbb{R}^n\,, \quad y \in \mathbb{R}$$

For $n = 3$, the observable canonical form for this system can be found through the following procedure:

1. Compute $t_3 = \mathcal{O}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ where $\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$

Full Order Observer

Observable Canonical Form (3)

1. Compute $t_3 = \mathcal{O}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ where $\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$
2. Compute $t_2 = At_3$, $t_1 = At_2$.

Full Order Observer

Observable Canonical Form (3)

1. Compute $t_3 = \mathcal{O}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ where $\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$
2. Compute $t_2 = At_3$, $t_1 = At_2$.
3. Define $T = \begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix}$

Full Order Observer

Observable Canonical Form (3)

1. Compute $t_3 = \mathcal{O}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ where $\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$
2. Compute $t_2 = At_3$, $t_1 = At_2$.
3. Define $T = \begin{bmatrix} t_1 & t_2 & t_3 \end{bmatrix}$
4. The state space matrices for the observable canonical form are now given by $A_o = T^{-1}AT$, and $C_o = CT$.

Full Order Observer

Example: Observable Canonical Form (1)

We consider the system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} x + \begin{bmatrix} 2 \\ 3 \end{bmatrix} u \\ y &= \begin{bmatrix} -3 & 2 \end{bmatrix} x \end{aligned}$$

having the observability matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}, \quad \det(\mathcal{O}) = -1 \neq 0$$

We compute the columns of T by

$$t_2 = \mathcal{O}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$t_1 = At_2 = \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \end{bmatrix}$$

Thus,

$$T = \begin{bmatrix} -5 & 2 \\ -7 & 3 \end{bmatrix} \quad \Longrightarrow \quad T^{-1} = \begin{bmatrix} -3 & 2 \\ -7 & 5 \end{bmatrix}$$

Eventually, we have

$$A_o = T^{-1}AT = \begin{bmatrix} -3 & 2 \\ -7 & 5 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ -7 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}$$

and

Eventually, we have

$$
\begin{aligned}
A_o = T^{-1}AT &= \begin{bmatrix} -3 & 2 \\ -7 & 5 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ -7 & 3 \end{bmatrix} \\
&= \left[\begin{array}{c|c} -3 & 1 \\ \hline -2 & 0 \end{array}\right]
\end{aligned}
$$

and

Full Order Observer

Example: Observable Canonical Form (3)

Eventually, we have

$$A_o = T^{-1}AT = \begin{bmatrix} -3 & 2 \\ -7 & 5 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ -7 & 3 \end{bmatrix}$$

$$= \left[\begin{array}{c|c} -3 & 1 \\ \hline -2 & 0 \end{array}\right]$$

and

$$C_o = CT = \begin{bmatrix} -3 & 2 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Eventually, we have

$$A_o = T^{-1}AT = \begin{bmatrix} -3 & 2 \\ -7 & 5 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ -7 & 3 \end{bmatrix}$$

$$= \left[\begin{array}{c|c} -3 & 1 \\ \hline -2 & 0 \end{array}\right]$$

and

$$C_o = CT = \begin{bmatrix} -3 & 2 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ -7 & 3 \end{bmatrix} = \left[\begin{array}{c|c} 1 & 0 \end{array}\right]$$

Eventually, we have

$$
\begin{aligned}
A_o = T^{-1}AT &= \begin{bmatrix} -3 & 2 \\ -7 & 5 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ -7 & 3 \end{bmatrix} \\
&= \left[\begin{array}{c|c} -3 & 1 \\ \hline -2 & 0 \end{array}\right] \Rightarrow \det\left(\lambda I - A\right) = \lambda^2 + 3\lambda + 2
\end{aligned}
$$

and

$$
C_o = CT = \begin{bmatrix} -3 & 2 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ -7 & 3 \end{bmatrix} = \left[\begin{array}{c|c} 1 & 0 \end{array}\right]
$$

Eventually, we have

$$A_o = T^{-1}AT = \begin{bmatrix} -3 & 2 \\ -7 & 5 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ -7 & 3 \end{bmatrix}$$

$$= \left[\begin{array}{c|c} -3 & 1 \\ \hline -2 & 0 \end{array}\right] \Rightarrow \det(\lambda I - A) = (\lambda+1)(\lambda+2)$$

and

$$C_o = CT = \begin{bmatrix} -3 & 2 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ -7 & 3 \end{bmatrix} = \left[\begin{array}{c|c} 1 & 0 \end{array}\right]$$

For a single output system in observable canonical form, an observer state matrix takes a particular simple form:

$$A_o = \left[\begin{array}{c|cc} a_1 & 1 & 0 \\ a_2 & 0 & 1 \\ \hline a_3 & 0 & 0 \end{array}\right], C_o = \left[\begin{array}{c|cc} 1 & 0 & 0 \end{array}\right]$$

Applying the observer gain

$$L_o = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix}$$

we obtain:

$$
\begin{aligned}
A_o + L_o C_o &= \left[\begin{array}{c|cc} a_1 & 1 & 0 \\ a_2 & 0 & 1 \\ \hline a_3 & 0 & 0 \end{array}\right] + \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix} \left[\begin{array}{c|cc} 1 & 0 & 0 \end{array}\right] \\
&= \left[\begin{array}{c|cc} a_1 + \ell_1 & 1 & 0 \\ a_2 + \ell_2 & 0 & 1 \\ \hline a_3 + \ell_3 & 0 & 0 \end{array}\right]
\end{aligned}
$$

Thus, the characteristic polynomial has been changed from

$$\det(\lambda I - A_o) = \lambda^n - a_1\lambda^{n-1} - \ldots - a_n$$

to

$$\det(\lambda I - (A_o + L_o C_o)) = \lambda^n - (a_1 + \ell_1)\lambda^{n-1} - \ldots - (a_n + \ell_n)$$

By choosing $\ell_1, \ldots, \ell_n$ appropriately, *any* observer pole configuration can be obtained. This is known as *observer pole assignment*.

Let $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{1\times n}$ be given.

1. Choose desired observer polynomial
   $\det(\lambda I - (A + LC)) = \lambda^n - a_{\text{obs},1}\lambda^{n-1} - \ldots - a_{\text{obs},n}$.

Let $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{1\times n}$ be given.

1. Choose desired observer polynomial
   $\det(\lambda I - (A + LC)) = \lambda^n - a_{\text{obs},1}\lambda^{n-1} - \ldots - a_{\text{obs},n}$.
2. Determine T, such that $A_o = T^{-1}AT$ and $C_o = CT$ are in observable canonical form.

Observer Design
Observer Pole Assignment

Let $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{1\times n}$ be given.

1. Choose desired observer polynomial
   $\det(\lambda I - (A + LC)) = \lambda^n - a_{\text{obs},1}\lambda^{n-1} - \ldots - a_{\text{obs},n}$.
2. Determine T, such that $A_o = T^{-1}AT$ and $C_o = CT$ are in observable canonical form.
3. Determine open loop polynomial $\det(\lambda I - A) = \lambda^n - a_1\lambda^{n-1} - \ldots - a_n$

Let $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{1\times n}$ be given.

1. Choose desired observer polynomial
   $\det(\lambda I - (A + LC)) = \lambda^n - a_{\text{obs},1}\lambda^{n-1} - \ldots - a_{\text{obs},n}$.
2. Determine T, such that $A_o = T^{-1}AT$ and $C_o = CT$ are in observable canonical form.
3. Determine open loop polynomial $\det(\lambda I - A) = \lambda^n - a_1\lambda^{n-1} - \ldots - a_n$
4. Define $L_o = \begin{bmatrix} a_{\text{obs},1} - a_1 \\ \vdots \\ a_{\text{obs},n} - a_n \end{bmatrix}$.

Let $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{1\times n}$ be given.

1. Choose desired observer polynomial
   $\det(\lambda I - (A + LC)) = \lambda^n - a_{\text{obs},1}\lambda^{n-1} - \ldots - a_{\text{obs},n}$.
2. Determine T, such that $A_o = T^{-1}AT$ and $C_o = CT$ are in observable canonical form.
3. Determine open loop polynomial $\det(\lambda I - A) = \lambda^n - a_1\lambda^{n-1} - \ldots - a_n$
4. Define $L_o = \begin{bmatrix} a_{\text{obs},1} - a_1 \\ \vdots \\ a_{\text{obs},n} - a_n \end{bmatrix}$.
5. Compute resulting observer gain $L = TL_o$.

We consider again the system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} x + \begin{bmatrix} 2 \\ 3 \end{bmatrix} u \\ y &= \begin{bmatrix} -3 & 2 \end{bmatrix} x \end{aligned}$$

for which we would like to assign observer poles to $\{-4, -5\}$, i.e. to design L such that $A + LC$ has eigenvalues in $\{-4, -5\}$.

1. Desired closed loop polynomial: $\lambda^2 + 9\lambda + 20$

1. Desired closed loop polynomial: $\lambda^2 + 9\lambda + 20$
2. $T = \begin{bmatrix} -5 & 2 \\ -7 & 3 \end{bmatrix} \Rightarrow A_o = \left[\begin{array}{c|c} -3 & 1 \\ \hline -2 & 0 \end{array}\right], C_o = [1 \mid 0]$

Observer Design

Example: Observer Pole Assignment (2)

1. Desired closed loop polynomial: $\lambda^2 + 9\lambda + 20$
2. $T = \begin{bmatrix} -5 & 2 \\ -7 & 3 \end{bmatrix} \Rightarrow A_o = \left[\begin{array}{c|c} -3 & 1 \\ \hline -2 & 0 \end{array}\right], C_o = \left[\begin{array}{c|c} 1 & 0 \end{array}\right]$
3. Open loop polynomial: $\lambda^2 + 3\lambda + 2$

Observer Design

Example: Observer Pole Assignment (2)

1. Desired closed loop polynomial: $\lambda^2 + 9\lambda + 20$
2. $T = \begin{bmatrix} -5 & 2 \\ -7 & 3 \end{bmatrix} \Rightarrow A_o = \left[\begin{array}{c|c} -3 & 1 \\ \hline -2 & 0 \end{array}\right], C_o = \left[\begin{array}{c|c} 1 & 0 \end{array}\right]$
3. Open loop polynomial: $\lambda^2 + 3\lambda + 2$
4. $L_o = \begin{bmatrix} -9-(-3) \\ -20-(-2) \end{bmatrix} = \begin{bmatrix} -6 \\ -18 \end{bmatrix}$

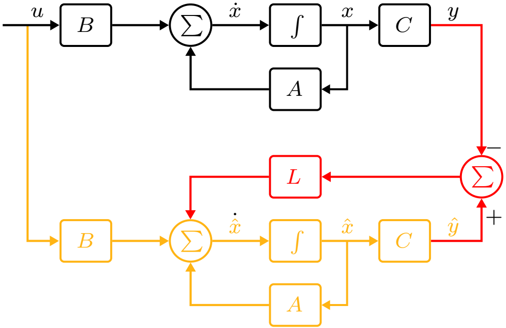
Observer Design

Example: Observer Pole Assignment (2)

1. Desired closed loop polynomial: $\lambda^2 + 9\lambda + 20$
2. $T = \begin{bmatrix} -5 & 2 \\ -7 & 3 \end{bmatrix} \Rightarrow A_o = \left[\begin{array}{c|c} -3 & 1 \\ \hline -2 & 0 \end{array}\right], C_o = [1 \mid 0]$
3. Open loop polynomial: $\lambda^2 + 3\lambda + 2$
4. $L_o = \begin{bmatrix} -9-(-3) \\ -20-(-2) \end{bmatrix} = \begin{bmatrix} -6 \\ -18 \end{bmatrix}$
5. $L = TL_o = \begin{bmatrix} -5 & 2 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} -6 \\ -18 \end{bmatrix} = \begin{bmatrix} -6 \\ -12 \end{bmatrix}$

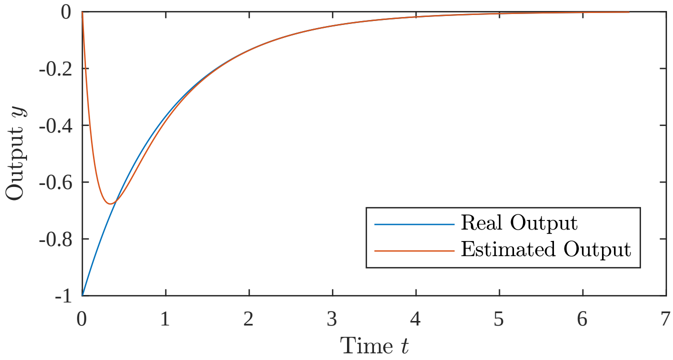
Observer Design

The Full Order Observer



Observer Design

Example: Observer Pole Assignment



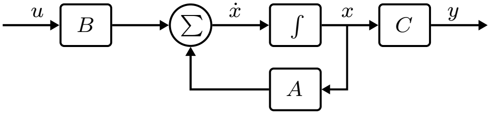
Introduction

Observability

Full Order Observer

Observer Design

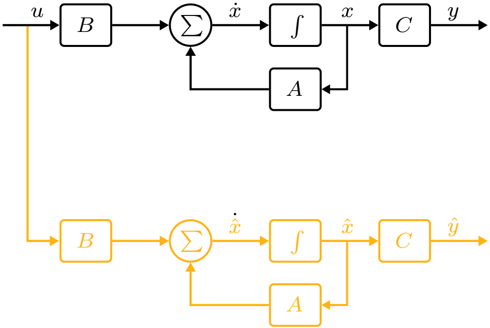
Observer Based Control



u
B
$\sum$
$\dot{x}$
$\int$
x
C
y
A

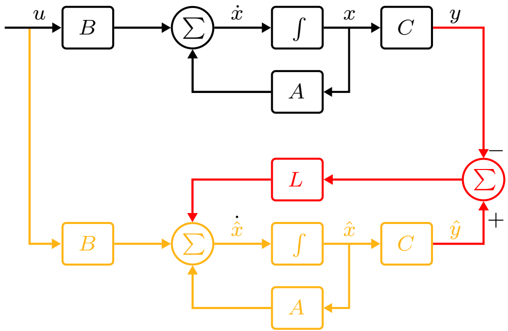
Observer Based Control

Observer Based Control (1)



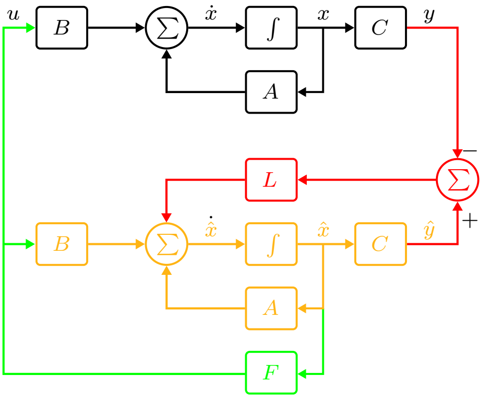
Observer Based Control

Observer Based Control (1)



Observer Based Control

Observer Based Control (1)



System:
$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

Observer:
$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\ \hat{y} &= C\hat{x} \end{aligned}$$

Feedback:
$$u = F\hat{x}$$

System: $$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

Observer: $$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\ \hat{y} &= C\hat{x} \end{aligned}$$

Feedback: $$u = F\hat{x}$$

Error, $e = \hat{x} - x$:

System:
$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx\end{aligned}$$

Observer:
$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y)\\ \hat{y} &= C\hat{x}\end{aligned}$$

Feedback: $u = F\hat{x}$

Error, $e = \hat{x} - x$:

$$\begin{aligned}\dot{e} &= \dot{\hat{x}} - \dot{x}\\ &= A\hat{x} + BF\hat{x} + L(C\hat{x} - y) - (Ax + BF\hat{x})\end{aligned}$$

Observer Based Control

Observer Based Control (2)

System:
$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

Observer:
$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\ \hat{y} &= C\hat{x} \end{aligned}$$

Feedback: $u = F\hat{x}$

Error, $e = \hat{x} - x$:

$$\begin{aligned} \dot{e} &= \dot{\hat{x}} - \dot{x} \\ &= A\hat{x} + BF\hat{x} + L(C\hat{x} - y) - (Ax + BF\hat{x}) \\ &= A(\hat{x} - x) + L(C\hat{x} - Cx) \end{aligned}$$

Observer Based Control

Observer Based Control (2)

System: $\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$

Observer: $\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(C\hat{x} - y) \\ \hat{y} &= C\hat{x} \end{aligned}$

Feedback: $u = F\hat{x}$

Error, $e = \hat{x} - x$:

$$\begin{aligned} \dot{e} &= \dot{\hat{x}} - \dot{x} \\ &= A\hat{x} + BF\hat{x} + L(C\hat{x} - y) - (Ax + BF\hat{x}) \\ &= A(\hat{x} - x) + L(C\hat{x} - Cx) \\ &= (A + LC)(\hat{x} - x) = (A + LC)e \end{aligned}$$

Combining the two equations:

$$\begin{aligned}\dot{x} &= Ax + Bu = Ax + BF\hat{x} = Ax + BF(e + x)\\ &= (A + BF)x + BFe\end{aligned}$$

and

$$\dot{e} = (A + LC)e$$

gives:

$$\begin{bmatrix}\dot{x}\\ \dot{e}\end{bmatrix} = \begin{bmatrix}A + BF & BF\\ 0 & A + LC\end{bmatrix}\begin{bmatrix}x\\ e\end{bmatrix}$$

THEOREM. An observer based controller for the system

$$\begin{array}{rcllll} \dot{x} & = & Ax & + & Bu & , \quad x \in \mathbb{R}^n \\ y & = & Cx & & & \end{array}$$

with observer gain L and feedback gain F results in $2n$ closed loop poles, coinciding with the eigenvalues of the two matrices:

$$A + BF \quad \text{and} \quad A + LC$$

We consider again the system

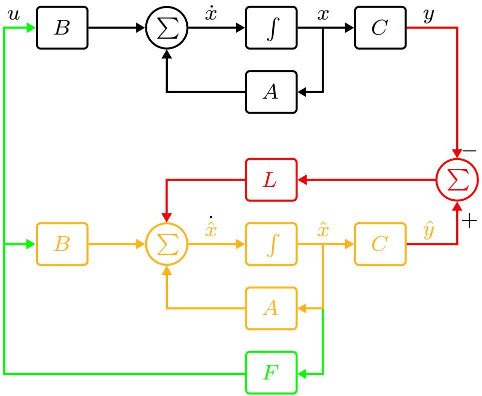
$$\begin{aligned} \dot{x} &= \begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix} x + \begin{bmatrix} 2 \\ 3 \end{bmatrix} u \\ y &= \begin{bmatrix} -3 & 2 \end{bmatrix} x \end{aligned}$$

for which we apply an observer based controller with

$$L = \begin{bmatrix} -6 \\ -12 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 42 & -30 \end{bmatrix}$$

Observer Based Control

Observer Based Control



The transfer function of the controller becomes:

$$\begin{aligned} K(s) &= -F\,(sI - A - BF - LC)^{-1}\,L \\ &= -108\frac{s+\frac{7}{3}}{s^2+15s+74} \end{aligned}$$

The closed loop transfer function becomes:

$$G(s)\,(I - K(s)G(s))^{-1} = \frac{s^2+15s+74}{(s+5)^2(s+4)^2}$$

Observer Based Control

Example: Observer Based Control (3)