

Betragt følgende Butterworth 3. ordens lavpasfilter (frekvensnormeret filter)

$$\tilde{H}(s) = \frac{1}{s+1} \frac{1}{s^2+s+1}$$

og design et tilsvarende digitalt lavpasfilter med afskæringsfrekvens på 1 kHz og samplefrekvens på 8 kHz. Design filtret som følger

1. Design et digitalt lavpasfilter $H_1(z)$ ved brug af matched z -transformation.
2. Design et digitalt lavpasfilter $H_2(z)$ ved brug af impuls invariant z -transformation.
3. Benyt MATLAB til at sammenligne Bode plot for denormerede lavpasfilter $H(s)$, $H_1(z)$ og $H_2(z)$.
4. Benyt MATLAB til at sammenligne impulsresponsen for $H(s)$, $H_1(z)$ og $H_2(z)$.

Denormerer

$$\omega_a = 2\pi \cdot f_a = 2000\pi$$

$$H(s) = \tilde{H}(s) \Big|_s = \frac{s}{\omega_a} = \frac{1}{\frac{s}{2000\pi} + 1} \cdot \frac{1}{\left(\frac{s}{2000\pi}\right)^2 + \frac{s}{2000\pi} + 1} = \frac{2000\pi}{s + 2000\pi} \cdot \frac{(2000\pi)^2}{s^2 + 2000\pi s + (2000\pi)^2}$$

Finder poler og nulpunkter

$$s + 2000\pi = 0 \Rightarrow p_1 = -2000\pi$$

$$s^2 + 2000\pi s + (2000\pi)^2 = 0 \Rightarrow \begin{cases} p_2 = -3141 - 5441i \\ p_3 = -3141 + 5441i \end{cases}$$

Overfør poler til z -domæne

$$\boxed{z = e^{sT}} \quad T = 8000^{-1}$$

$$p'_1 = e^{-2000\pi \cdot 8000^{-1}} = e^{-\frac{\pi}{4}} \approx 0,4559$$

$$\boxed{\text{Define: } a = 0,5249, b = 0,4247}$$

$$p'_2 = e^{(-3141 - 5441i) \cdot 8000^{-1}} \approx 0,5249 - 0,4247i = a - bi$$

$$p'_3 = e^{(-3141 + 5441i) \cdot 8000^{-1}} \approx 0,5249 + 0,4247i = a + bi$$

Opskriv overføringsfunktion i z -domæne

$$H(z) = \frac{k}{(z - p'_1)(z - p'_2)(z - p'_3)} = \frac{k}{(z - 0,4559) \underbrace{(z - (a - bi))(z - (a + bi))}_{(z^2 - 2az + a^2 + b^2)}}$$

$$\begin{aligned} (z - (a - bi))(z - (a + bi)) &= z^2 - (a - bi)z - (a + bi)z + (a - bi)(a + bi) \\ &= z^2 - az + bzi - az - bzi + a^2 + b^2 = z^2 - 2az + a^2 + b^2 \end{aligned}$$

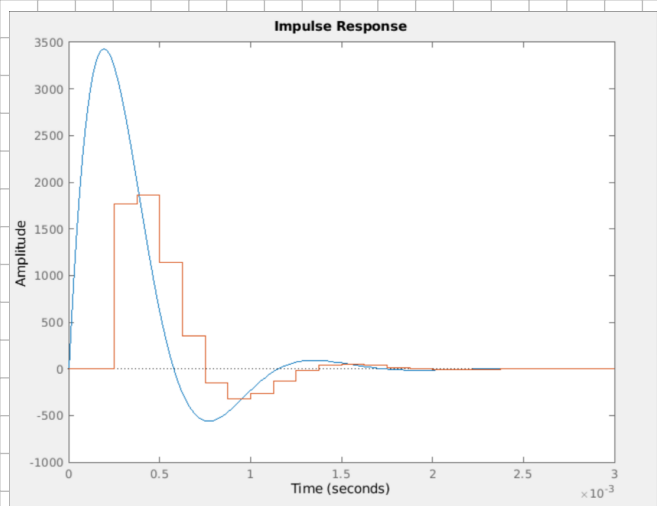
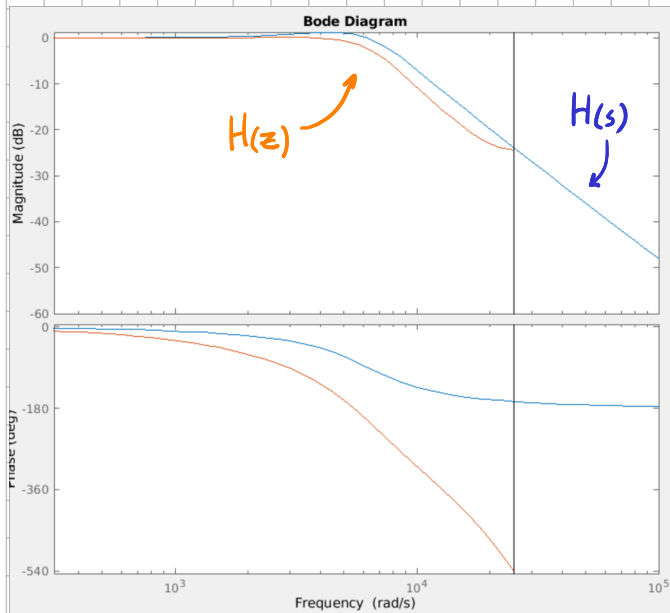
$$= \frac{k}{(z - 0,4559)(z^2 - 2az + a^2 + b^2)} = \frac{k}{(z - 0,4559)(z^2 - 1,049z + 0,4559)}$$

Lav konstant forstærkning

$$H(s)|_{s=0} = H(z)|_{z=1} \Rightarrow \frac{\cancel{2000\pi}}{0 + \cancel{2000\pi}} \cdot \frac{(\cancel{2000\pi})^2}{0 + 0 + (\cancel{2000\pi})^2} = \frac{k}{(1-0,4559)(1-1,049+0,4559)}$$

$$\Rightarrow 1 = \frac{k}{(1-0,4559)(1-1,049+0,4559)} = \frac{k}{0,2214} \Rightarrow k = 0,2214$$

$$H(z) = \frac{0,2214}{(z-0,4559)(z^2-1,049z+0,4559)}$$



(2)

$$\tilde{H}(s) = \frac{1}{s+1} \cdot \frac{1}{s^2+s+1}$$

Partialbrøkopløs

$$\tilde{H}(s) = \left(\frac{1}{s+1} \cdot \frac{1}{s^2+s+1} = \frac{A}{s+1} + \frac{Bs+C}{s^2+s+1} \right) \cdot (s+1)(s^2+s+1)$$

$$\Rightarrow 1 = A(s^2+s+1) + (Bs+C)(s+1)$$

$$\Rightarrow 1 = As^2 + As + A + Bs^2 + Bs + Cs + C$$

Samle konstanter

$$s^2: 0 = A+B$$

$$s: 0 = A+B+C$$

$$1: 1 = A+C$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{array} \right] \xrightarrow{\begin{smallmatrix} R_2 - R_1 \\ R_3 - R_1 \end{smallmatrix}} \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{array} \right] \xrightarrow{R_3 \leftrightarrow R_2} \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{R_2 \cdot (-1)} \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{R_2 + R_3} \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{R_1 - R_2} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array} \right] \Rightarrow \begin{cases} A = 1 \\ B = -1 \\ C = 0 \end{cases}$$

$$\Rightarrow \tilde{H}(s) = \frac{A}{s+1} + \frac{Bs+C}{s^2+s+1} = \frac{1}{s+1} + \frac{-s}{s^2+s+1}$$

Denormer overføringsfunktionen

$$H(s) = \tilde{H}(s) \Big|_{s = \frac{s}{\omega_n}} = \frac{1}{\frac{s}{2000\pi} + 1} + \frac{\frac{-s}{2000\pi}}{\left(\frac{s}{2000\pi}\right)^2 + \frac{s}{2000\pi} + 1} = \underbrace{\frac{2000\pi}{s + 2000\pi}}_{H_1(s)} + \underbrace{\frac{-2000\pi \cdot s}{s^2 + 2000\pi \cdot s + (2000\pi)^2}}_{H_2(s)}$$

Find konstanter og z-transformer

$$H_1(s) = \frac{2000\pi}{s + 2000\pi} \Rightarrow \begin{cases} k_1 = 2000\pi \\ s_1 = -2000\pi \end{cases} \Rightarrow H_1(z) = \frac{1}{8000} \cdot 2000\pi \cdot \frac{1}{1 - e^{-2000\pi \cdot \frac{1}{8000}} \cdot z^{-1}} = \frac{\pi}{4} \cdot \frac{1}{1 - e^{-\frac{\pi}{4}} \cdot z^{-1}}$$

$$H_2(s) = \frac{-2000\pi \cdot s}{s^2 + 2000\pi \cdot s + (2000\pi)^2} = \frac{-2000\pi \cdot s}{(s - 1000\pi + 1000\pi\sqrt{3}i) \cdot (s - 1000\pi - 1000\pi\sqrt{3}i)}$$

Definition: $s_2 = 1000\pi - 1000\pi\sqrt{3}i$

Partialbrøkopløs

$$H_2(s) = \frac{-2000\pi \cdot s}{(s - s_1) \cdot (s - s_2^*)} = \frac{k_2}{s - s_2} + \frac{k_2^*}{s - s_1^*}$$

$$\Rightarrow k_2 = (s - s_1) \cdot H_2(s) \Big|_{s=s_2} = \frac{-2000\pi s}{s - s_2^*} \Big|_{s=s_2} = \frac{-2000\pi \cdot (1000\pi - 1000\pi\sqrt{3}i)}{(1000\pi - 1000\pi\sqrt{3}i) - (1000\pi + 1000\pi\sqrt{3}i)}$$

$$= \frac{-2000\pi \cdot (1000\pi - 1000\pi\sqrt{3}i)}{-2000\pi\sqrt{3}i} = \frac{1000\pi - 1000\pi\sqrt{3}i}{\sqrt{3}i} = \frac{1000\pi i + 1000\pi\sqrt{3}}{-\sqrt{3}} = -1000\pi - 1000\pi \frac{1}{\sqrt{3}}i$$

$$\begin{cases} k_2 = -1000\pi - 1000\pi \frac{1}{\sqrt{3}} i \\ s_2 = 1000\pi - 1000\pi \frac{1}{\sqrt{3}} i \end{cases} \Rightarrow \begin{cases} \alpha_2 = -1000\pi \\ \beta_2 = -1000\pi \frac{1}{\sqrt{3}} \\ \sigma_2 = 1000\pi \\ \omega_2 = -1000\pi \frac{1}{\sqrt{3}} \end{cases}$$

$$f_s = 8000 \Rightarrow T = \frac{1}{8000}$$

$$\sigma_2 T = 1000\pi \cdot \frac{1}{8000} = \frac{\pi}{8}$$

$$\omega_2 T = -1000\pi \frac{1}{\sqrt{3}} \cdot \frac{1}{8000} = -\frac{\pi\sqrt{3}}{8}$$

$$\Rightarrow \begin{cases} \alpha_0 = 2\alpha_2 \\ a_1 = -2e^{\sigma_2 T} (\alpha_2 \cos(\omega_2 T) - \beta_2 \sin(\omega_2 T)) \\ b_1 = -(2e^{\sigma_2 T} \cos(\omega_2 T)) \\ b_2 = e^{2\sigma_2 T} \end{cases} \Rightarrow \begin{cases} a_0 \rightarrow -2000\pi \\ a_1 \rightarrow -2e^{\pi/8} \left(-1000\pi \cos\left[\frac{\sqrt{3}\pi}{8}\right] - \frac{1000\pi \sin\left[\frac{\sqrt{3}\pi}{8}\right]}{\sqrt{3}} \right) \\ b_1 \rightarrow -2e^{\pi/8} \cos\left[\frac{\sqrt{3}\pi}{8}\right] \\ b_2 \rightarrow e^{\pi/4} \end{cases}$$

$$\Rightarrow \begin{cases} a_0 \rightarrow -6283.19 \\ a_1 \rightarrow 10613.3 \\ b_1 \rightarrow -2.3028 \\ b_2 \rightarrow 2.19328 \end{cases} \Rightarrow H_2(z) \approx \frac{-6283 + 10613z^{-1}}{1 - 2.3028z^{-1} + 2.1933z^{-2}}$$

Den samlede overføringsfunktion i z-domænet!

$$\Rightarrow H(z) = H_1(z) + H_2(z) = \frac{\pi}{4} \cdot \frac{1}{1 - e^{\frac{\pi}{4}} z^{-1}} + \frac{-6283 + 10613z^{-1}}{1 - 2.3028z^{-1} + 2.1933z^{-2}}$$

