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Introduction to Intelligent Systems

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INFERENCE IN DYNAMIC MODELS

Inference Problems in Dynamic Models

- **Filtering**: the task of computing the belief state (the posterior distribution over the most recent state) given all evidence to date, i.e., computing $P(\mathbf{X}_t|\mathbf{e}_{1:t})$.
- **Prediction**: the task of computing the posterior distribution over the future state, given all evidence to date, i.e., $P(\mathbf{X}_{t+k}|\mathbf{e}_{1:t})$ for some $k \geq 1$.
- **Smoothing**: the task of computing the posterior distribution over a past state, given all evidence up to the present, i.e., $P(\mathbf{X}_k|\mathbf{e}_{1:t})$ for some $0 \leq k \leq t$. Note that smoothing provides a better estimate of the state than filtering because it incorporates more evidence.
- **Most likely explanation**: Given a sequence of observations, find the sequence of states that is most likely to have generated those observations, i.e., $\arg\max_{\mathbf{x}_{1:t}} P(\mathbf{x}_{1:t}|\mathbf{e}_{1:t})$.
- **Learning**: Learning the transition and sensor models from observations. Note that learning requires smoothing for better estimates.

Filtering

- **Recursive estimation**: given the result of filtering up to time t, the agent needs to compute the result for $t+1$ from the new evidence $\mathbf{e}_{t+1}$.

$$\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) = f(\mathbf{e}_{t+1}, \mathbf{P}(\mathbf{X}_t|\mathbf{e}_{1:t}))$$

$$\begin{aligned}
\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1}) &= \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}, \mathbf{e}_{t+1}) \quad \text{(dividing up the evidence)} \\
&= \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}, \mathbf{e}_{1:t})\, \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}) \quad \text{(using Bayes' rule)} \\
&= \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1})\, \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t}) \quad \text{(by the sensor Markov assumption).}
\end{aligned}$$

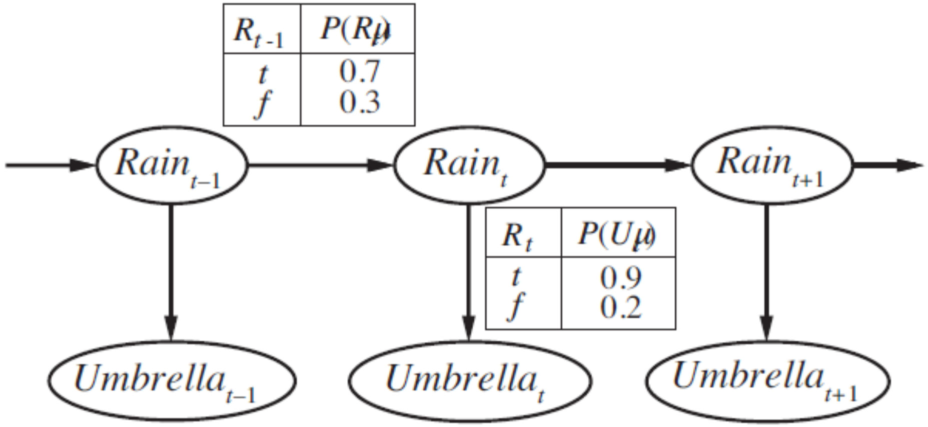
$$\begin{aligned}
\mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{e}_{1:t+1}) &= \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{x}_t, \mathbf{e}_{1:t}) P(\mathbf{x}_t \mid \mathbf{e}_{1:t}) \\
&= \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{x}_t) P(\mathbf{x}_t \mid \mathbf{e}_{1:t}) \quad \text{(Markov assumption)}
\end{aligned}$$

Requires two steps:

- *Prediction*: $\mathbf{P}(\mathbf{X}_t|\mathbf{e}_{1:t}) \to \mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t})$
- *Measurement Update*: $\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t}) \to \mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1})$

Filtering Example

$$\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) = \alpha\,\mathbf{P}(\mathbf{e}_{t+1}\,|\,\mathbf{X}_{t+1})\sum_{\mathbf{x}_t}\mathbf{P}(\mathbf{X}_{t+1}\,|\,\mathbf{x}_t)P(\mathbf{x}_t\,|\,\mathbf{e}_{1:t})$$



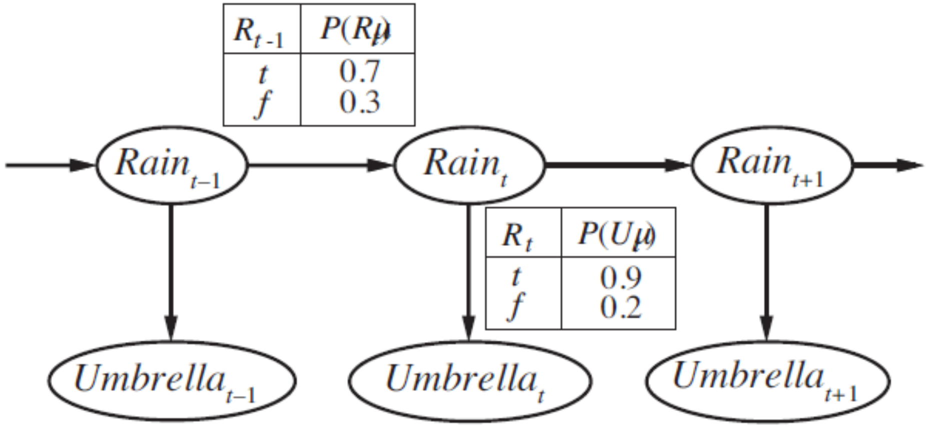
- On day 0, we have no observations, only the security guard's prior beliefs; let's assume that consists of $\mathbf{P}(R_0) = \langle 0.5, 0.5\rangle$.
- On day 1, the umbrella appears, so $U_1 = true$. The prediction from $t=0$ to $t=1$ is

$$\begin{aligned}\mathbf{P}(R_1) &= \sum_{r_0}\mathbf{P}(R_1\,|\,r_0)P(r_0)\\ &= \langle 0.7, 0.3\rangle \times 0.5 + \langle 0.3, 0.7\rangle \times 0.5 = \langle 0.5, 0.5\rangle\ .\end{aligned}$$

Then the update step simply multiplies by the probability of the evidence for $t=1$ and normalizes, as shown in Equation (15.4):

$$\begin{aligned}\mathbf{P}(R_1\,|\,u_1) &= \alpha\,\mathbf{P}(u_1\,|\,R_1)\mathbf{P}(R_1) = \alpha\,\langle 0.9, 0.2\rangle\langle 0.5, 0.5\rangle\\ &= \alpha\,\langle 0.45, 0.1\rangle \approx \langle 0.818, 0.182\rangle\ .\end{aligned}$$

Filtering Example



$$\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) = \alpha\, \mathbf{P}(\mathbf{e}_{t+1} \mid \mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1} \mid \mathbf{x}_t) P(\mathbf{x}_t \mid \mathbf{e}_{1:t})$$

$$\mathbf{P}(R_1|u_1) = \langle 0.818, 0.182 \rangle$$

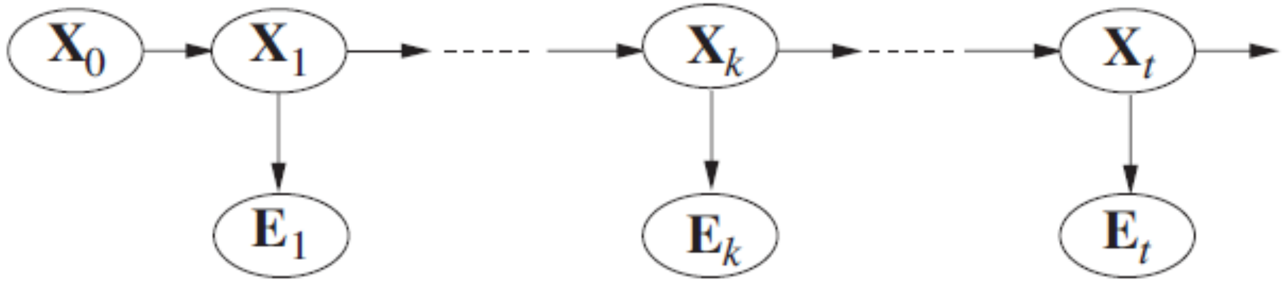
- On day 2, the umbrella appears, so $U_2 = true$. The prediction from $t = 1$ to $t = 2$ is

$$\begin{aligned} \mathbf{P}(R_2 \mid u_1) &= \sum_{r_1} \mathbf{P}(R_2 \mid r_1) P(r_1 \mid u_1) \\ &= \langle 0.7, 0.3 \rangle \times 0.818 + \langle 0.3, 0.7 \rangle \times 0.182 \approx \langle 0.627, 0.373 \rangle \ , \end{aligned}$$

and updating it with the evidence for $t = 2$ gives

$$\begin{aligned} \mathbf{P}(R_2 \mid u_1, u_2) &= \alpha\, \mathbf{P}(u_2 \mid R_2) \mathbf{P}(R_2 \mid u_1) = \alpha \langle 0.9, 0.2 \rangle \langle 0.627, 0.373 \rangle \\ &= \alpha \langle 0.565, 0.075 \rangle \approx \langle 0.883, 0.117 \rangle \ . \end{aligned}$$

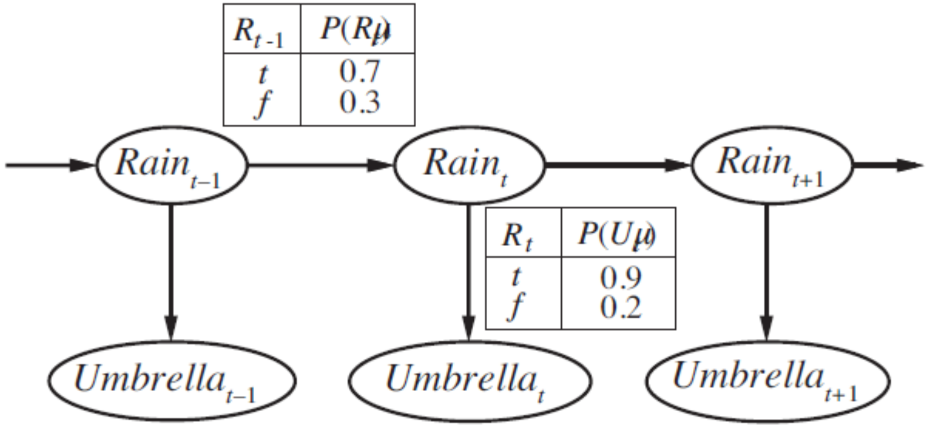
Smoothing



$$
\begin{aligned}
\mathbf{P}(\mathbf{X}_k \mid \mathbf{e}_{1:t}) &= \mathbf{P}(\mathbf{X}_k \mid \mathbf{e}_{1:k}, \mathbf{e}_{k+1:t}) \\
&= \alpha\, \mathbf{P}(\mathbf{X}_k \mid \mathbf{e}_{1:k})\mathbf{P}(\mathbf{e}_{k+1:t} \mid \mathbf{X}_k, \mathbf{e}_{1:k}) \quad \text{(using Bayes' rule)} \\
&= \alpha\, \mathbf{P}(\mathbf{X}_k \mid \mathbf{e}_{1:k})\mathbf{P}(\mathbf{e}_{k+1:t} \mid \mathbf{X}_k) \quad \text{(using conditional independence)} \\
&= \alpha\, \mathbf{f}_{1:k} \times \mathbf{b}_{k+1:t}\,.
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{P}(\mathbf{e}_{k+1:t} \mid \mathbf{X}_k) &= \sum_{\mathbf{x}_{k+1}} \mathbf{P}(\mathbf{e}_{k+1:t} \mid \mathbf{X}_k, \mathbf{x}_{k+1})\mathbf{P}(\mathbf{x}_{k+1} \mid \mathbf{X}_k) \quad \text{(conditioning on } \mathbf{X}_{k+1}\text{)} \\
&= \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1:t} \mid \mathbf{x}_{k+1})\mathbf{P}(\mathbf{x}_{k+1} \mid \mathbf{X}_k) \quad \text{(by conditional independence)} \\
&= \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1}, \mathbf{e}_{k+2:t} \mid \mathbf{x}_{k+1})\mathbf{P}(\mathbf{x}_{k+1} \mid \mathbf{X}_k) \\
&= \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1} \mid \mathbf{x}_{k+1})P(\mathbf{e}_{k+2:t} \mid \mathbf{x}_{k+1})\mathbf{P}(\mathbf{x}_{k+1} \mid \mathbf{X}_k)
\end{aligned}
$$

Smoothing Example



$$\mathbf{P}(R_1 \mid u_1, u_2) = \alpha\, \mathbf{P}(R_1 \mid u_1)\, \mathbf{P}(u_2 \mid R_1)$$

$$\mathbf{P}(R_1|u_1) = \langle 0.818, 0.182 \rangle$$

$$\sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1} \mid \mathbf{x}_{k+1}) P(\mathbf{e}_{k+2:t} \mid \mathbf{x}_{k+1}) \mathbf{P}(\mathbf{x}_{k+1} \mid \mathbf{X}_k)$$

$$\begin{aligned} \mathbf{P}(u_2 \mid R_1) &= \sum_{r_2} P(u_2 \mid r_2) P(\ \mid r_2) \mathbf{P}(r_2 \mid R_1) \\ &= (0.9 \times 1 \times \langle 0.7, 0.3 \rangle) + (0.2 \times 1 \times \langle 0.3, 0.7 \rangle) = \langle 0.69, 0.41 \rangle \end{aligned}$$

$$\mathbf{P}(R_1 \mid u_1, u_2) = \alpha\, \langle 0.818, 0.182 \rangle \times \langle 0.69, 0.41 \rangle \approx \langle 0.883, 0.117 \rangle$$

Forward and Backward Recursions

- **Forward recursion**

$$
\begin{aligned}
\mathbf{f}_{1:k+1} &:= P(\mathbf{X}_{k+1}|\mathbf{e}_{1:k+1}) \\
&= \alpha P(\mathbf{e}_{k+1}|\mathbf{X}_{k+1}) \sum_{\mathbf{x}_k} P(\mathbf{X}_{k+1}|\mathbf{x}_k) P(\mathbf{x}_k|\mathbf{e}_{1:k}) \\
&= \alpha \cdot Forward(\mathbf{f}_{1:k}, \mathbf{e}_{k+1})
\end{aligned}
$$

Initialized with $\mathbf{f}_{1:0} = P(\mathbf{X}_0)$.

- **Backward recursion**

$$
\begin{aligned}
\mathbf{b}_{k+1:t} &:= P(\mathbf{e}_{k+1:t}|\mathbf{X}_k) \\
&= \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1}|\mathbf{x}_{k+1}) P(\mathbf{e}_{k+2:t}|\mathbf{x}_{k+1}) P(\mathbf{x}_{k+1}|\mathbf{X}_k) \\
&= Backward(\mathbf{b}_{k+2:t}, \mathbf{e}_{k+1})
\end{aligned}
$$

Initialized with $\mathbf{b}_{t+1:t} = \mathbf{1}$.

Forward-Backward Algorithm

```
function FORWARD-BACKWARD(ev, prior) returns a vector of probability distributions
  inputs: ev, a vector of evidence values for steps 1, ..., t
          prior, the prior distribution on the initial state, P(X_0)
  local variables: fv, a vector of forward messages for steps 0, ..., t
                   b, a representation of the backward message, initially all 1s
                   sv, a vector of smoothed estimates for steps 1, ..., t

  fv[0] ← prior
  for i = 1 to t do
     fv[i] ← FORWARD(fv[i − 1], ev[i])
  for i = t downto 1 do
     sv[i] ← NORMALIZE(fv[i] × b)
     b ← BACKWARD(b, ev[i])
  return sv
```

Finding the Most Likely Sequence

- Consider the umbrella problem and we have observations for 5 days
  - Umbrella_1 = true
  - Umbrella_2 = true
  - Umbrella_3 = false
  - Umbrella_4 = true
  - Umbrella_5 = true
- What is the most likely weather sequence?
  - [Rain, Rain, Rain, Rain, Rain]?
  - [Rain, Rain, No Rain, Rain, Rain]?
  - [Rain, Rain, No Rain, No Rain, Rain]?
  - There are $\mathbf{2^5}$ possible sequences.

Viterbi Algorithm

- Viterbi algorithm is used to find the most likely sequence.
- There is a recursive relationship between most likely paths to each state $\mathbf{x}_{t+1}$ and most likely paths to each state $\mathbf{x}_t$. (**Principle of Optimality**)

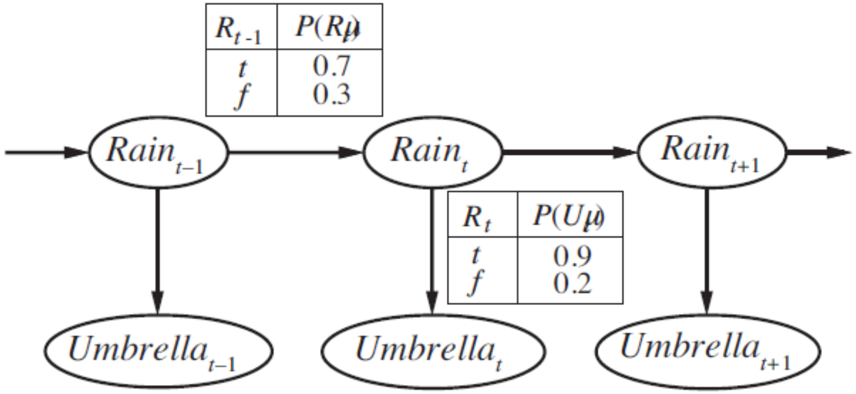
$$\max_{\mathbf{x}_1 \dots \mathbf{x}_t} \mathbf{P}(\mathbf{x}_1, \dots, \mathbf{x}_t, \mathbf{X}_{t+1} \,|\, \mathbf{e}_{1:t+1})$$
$$= \alpha \, \mathbf{P}(\mathbf{e}_{t+1} \,|\, \mathbf{X}_{t+1}) \max_{\mathbf{x}_t} \left(\mathbf{P}(\mathbf{X}_{t+1} \,|\, \mathbf{x}_t) \max_{\mathbf{x}_1 \dots \mathbf{x}_{t-1}} P(\mathbf{x}_1, \dots, \mathbf{x}_{t-1}, \mathbf{x}_t \,|\, \mathbf{e}_{1:t}) \right)$$

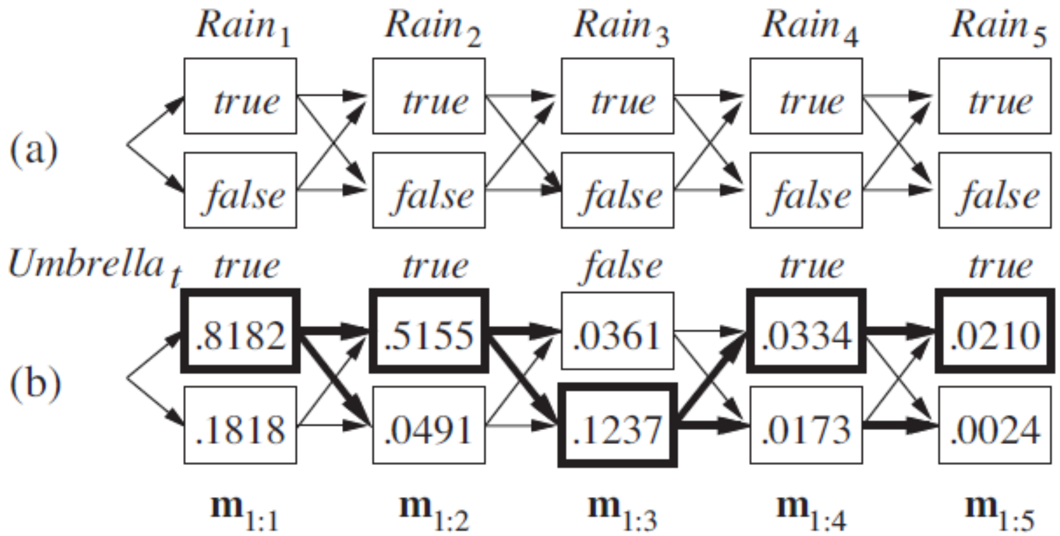
$$\mathbf{m}_{1:t} = \max_{\mathbf{x}_1 \dots \mathbf{x}_{t-1}} \mathbf{P}(\mathbf{x}_1, \dots, \mathbf{x}_{t-1}, \mathbf{X}_t \,|\, \mathbf{e}_{1:t})$$

cf. Forward recursion

$$\begin{aligned} \mathbf{f}_{1:k+1} &:= P(\mathbf{X}_{k+1} | \mathbf{e}_{1:k+1}) \\ &= \alpha P(\mathbf{e}_{k+1} | \mathbf{X}_{k+1}) \sum_{\mathbf{x}_k} P(\mathbf{X}_{k+1} | \mathbf{x}_k) P(\mathbf{x}_k | \mathbf{e}_{1:k}) \\ &= \alpha \cdot Forward(\mathbf{f}_{1:k}, \mathbf{e}_{k+1}) \end{aligned}$$

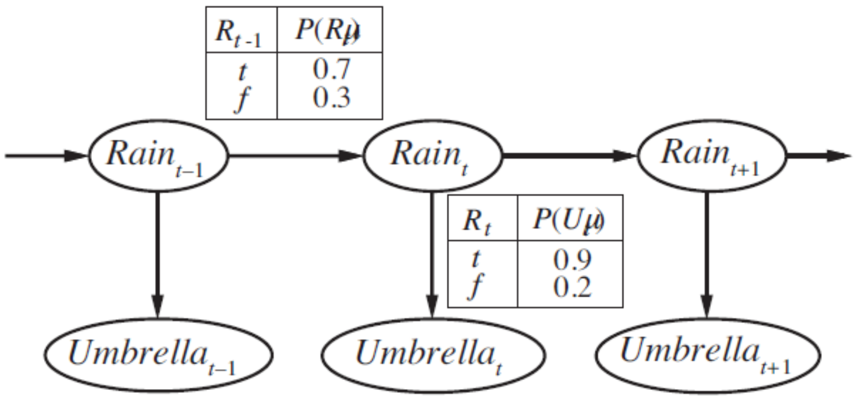
Viterbi Algorithm: Example

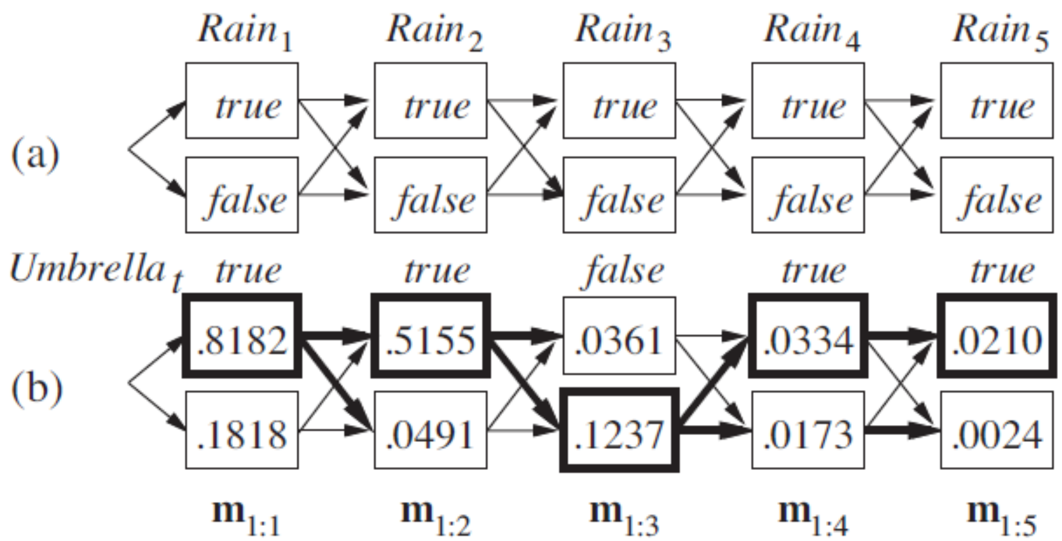




$$
\begin{aligned}
\mathbf{m}_{1:1} &= P(\mathbf{X}_1|\mathbf{e}_{1:1}) = P(R_1|u_1 = t) = \langle 0.8182, 0.1818\rangle \\
\mathbf{m}_{1:2} &= \alpha P(u_2 = t|R_2) \max_{r_1} \left(P(R_2|r_1)\mathbf{m}_{1:1}(r_1)\right) \\
&= \alpha\langle 0.9, 0.2\rangle \max\left(\langle 0.7, 0.3\rangle 0.8182, \langle 0.3, 0.7\rangle 0.1818\right) \\
&= \alpha\langle 0.9, 0.2\rangle \max\left(\langle 0.5727, 0.2455\rangle, \langle 0.0546, 0.1274\rangle\right) \\
&= \alpha\langle 0.9, 0.2\rangle \langle 0.5727, 0.2455\rangle \qquad \boxed{\arg\max(\cdot) = \langle 1, 1\rangle} \\
&= \alpha\langle 0.5155, 0.0491\rangle
\end{aligned}
$$

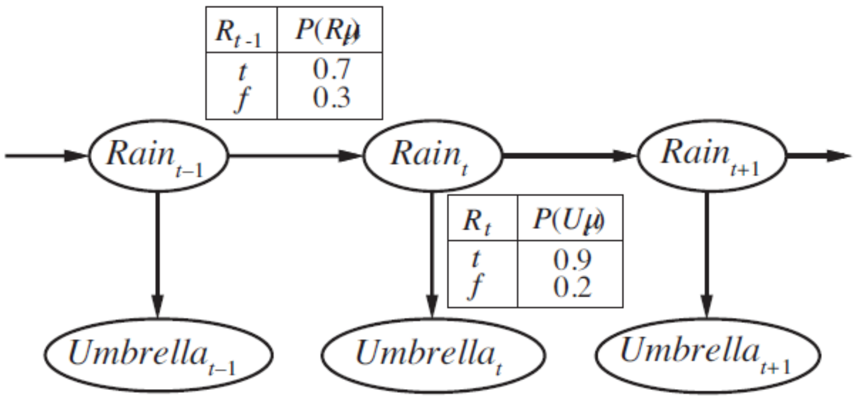
Viterbi Algorithm: Example

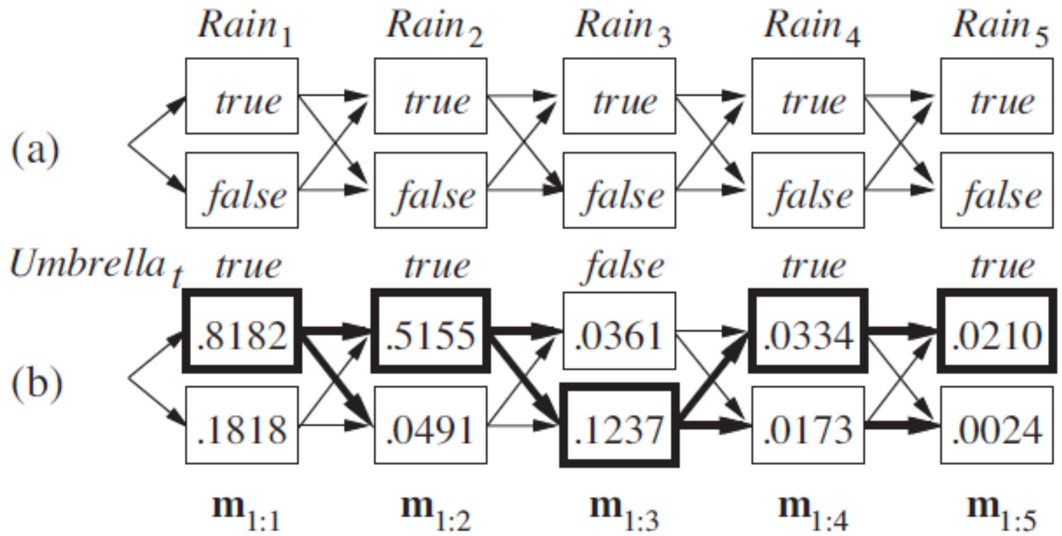




$$
\begin{aligned}
\mathbf{m}_{1:2} &= \alpha\langle 0.5155, 0.0491\rangle \\
\mathbf{m}_{1:3} &= \alpha P(u_3 = f|R_3) \max_{r_2} \left(P(R_3|r_2)\mathbf{m}_{1:2}(r_2)\right) \\
&= \alpha\langle 0.1, 0.8\rangle \max\left(\langle 0.7, 0.3\rangle 0.5155, \langle 0.3, 0.7\rangle 0.0491\right) \\
&= \alpha\langle 0.1, 0.8\rangle \max\left(\langle 0.3608, 0.1546\rangle, \langle 0.0147, 0.0344\rangle\right) \\
&= \alpha\langle 0.1, 0.8\rangle\langle 0.3608, 0.1546\rangle \qquad \boxed{\arg\max(\cdot) = \langle 1, 1\rangle} \\
&= \alpha\langle 0.0361, 0.1237\rangle
\end{aligned}
$$

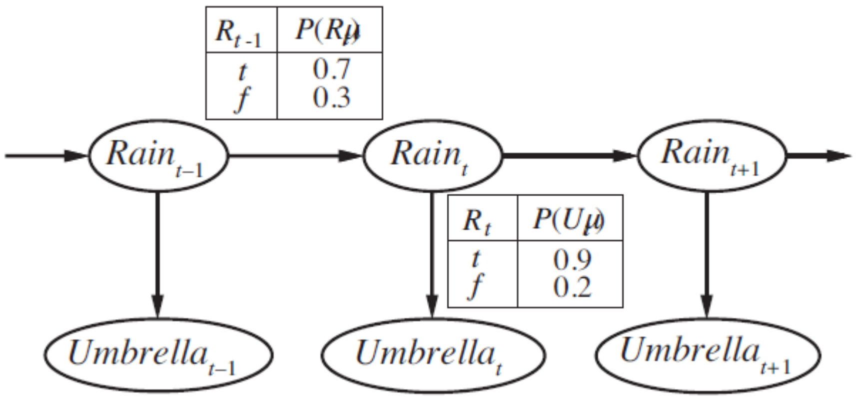
Viterbi Algorithm: Example

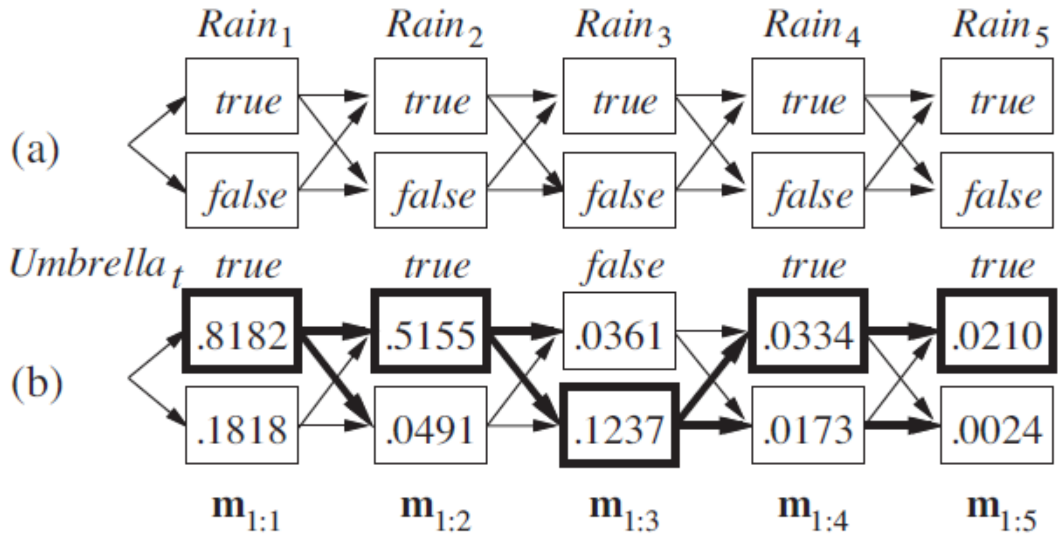




$$
\begin{aligned}
\mathbf{m}_{1:3} &= \alpha\langle 0.0361, 0.1237\rangle \\
\mathbf{m}_{1:4} &= \alpha P(u_4 = t | R_4) \max_{r_3} \left(P(R_4|r_3)\mathbf{m}_{1:3}(r_3)\right) \\
&= \alpha\langle 0.9, 0.2\rangle \max\left(\langle 0.7, 0.3\rangle 0.0361, \langle 0.3, 0.7\rangle 0.1237\right) \\
&= \alpha\langle 0.9, 0.2\rangle \max\left(\langle 0.0252, 0.0108\rangle, \langle 0.0371, 0.0866\rangle\right) \\
&= \alpha\langle 0.9, 0.2\rangle \langle 0.0371, 0.0866\rangle \qquad \boxed{\arg\max(\cdot) = \langle 2, 2\rangle} \\
&= \alpha\langle 0.0334, 0.0173\rangle
\end{aligned}
$$

Viterbi Algorithm: Example





$$
\begin{aligned}
\mathbf{m}_{1:4} &= \alpha\langle 0.0334, 0.0173\rangle \\
\mathbf{m}_{1:5} &= \alpha P(u_5 = t|R_5)\max_{r_4}\left(P(R_5|r_4)\mathbf{m}_{1:4}(r_4)\right) \\
&= \alpha\langle 0.9, 0.2\rangle \max\left(\langle 0.7, 0.3\rangle 0.0334, \langle 0.3, 0.7\rangle 0.0173\right) \\
&= \alpha\langle 0.9, 0.2\rangle \max\left(\langle 0.0234, 0.0100\rangle, \langle 0.0052, 0.0121\rangle\right) \\
&= \alpha\langle 0.9, 0.2\rangle\langle 0.0234, 0.0121\rangle \qquad \boxed{\arg\max(\cdot) = \langle 1, 2\rangle} \\
&= \alpha\langle 0.0210, 0.0024\rangle
\end{aligned}
$$