$$\partial_x \Big( \underbrace{(1-x^2)\,\partial_x P}_{=\,-Q} \Big) + \Big( \lambda - \frac{M^2}{1-x^2} \Big) P = 0 \qquad (1)$$

$\Rightarrow$ coupled ODE

$$(1-x^2)\,\partial_x P = -Q \qquad (2)$$

$$(1-x^2)\,\partial_x Q = \big( \lambda (1-x^2) - M^2 \big) P \qquad (3)$$

use

$$P = (1-x^2)^{\alpha} f \qquad (4)$$

$$Q = (1-x^2)^{\alpha} g \qquad (5)$$

$\Rightarrow$

$$(1-x^2)^{\alpha+1}\,\partial_x f - 2\alpha x (1-x^2)^{\alpha} f = -(1-x^2)^{\alpha} g \qquad (6)$$

$$(1-x^2)^{\alpha+1}\,\partial_x g - 2\alpha x (1-x^2)^{\alpha} g = \big[ \lambda (1-x^2) - M^2 \big] (1-x^2)^{\alpha} f \qquad (7)$$

$\Rightarrow$

$$(1-x^2)\,\partial_x f = 2\alpha x f - g \qquad (8)$$

$$(1-x^2)\,\partial_x g = \big[ \lambda (1-x^2) - M^2 \big] f + 2\alpha x g \qquad (9)$$

inspect boundary $\pm 1$, require consistency in boundary condition on $f, g$.
cannot appeal to $P, Q$ since $P, Q \to 0$ at $\pm 1$ satisfied by transform

$x \to \pm 1$ eqn:

$$0 = 2\alpha f - g \qquad (10)$$

$$0 = -M^2 f + 2\alpha g \qquad (11)$$

$\Rightarrow$ two conditions:

$$g = 2\alpha f \qquad (12)$$

$$g = (M^2/2\alpha)\, f \qquad (13)$$

$\Rightarrow$ require $2\alpha = \dfrac{M^2}{2\alpha}$ $\Rightarrow$ $\alpha = \pm \dfrac{|M|}{2}$ (14)

for consistency
(always pick +ve $\alpha$, though)*

<u>Method of sol$^n$:</u>

- start at $\pm 1$ with initial condition satisfying eqn (12, 13). Since normalisation is arbitrary, choose $f = 1$, and so $g = |M| f$
- shoot for solution using guess of $\lambda$
- notice: $P = f$, $Q = g$ at $x = 0$
  $\Rightarrow$ boundary condition at $x=0$ is the same
- perform root finding on this function to find $\lambda$ (we already know that $\lambda = l(l+1)$, see Orthogonal_functions.pdf)
- to find eigenfunction $P$ use eqn (4, 5)

* use +ve $\alpha$ since in eqn (4,5) −ve $\alpha$ would result in worse singularity