# zero-to-hero

Ed Southwood

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# Welcome

This course is designed to refresh your knowledge of maths to get you ready to use calculus in your course. There is no right or wrong way to use it. Each section includes written notes, a video (with the same content as the notes) and practice questions. It's chunked into bitesized sections to allow you make progress in 10 min windows. You may like to try the questions first and then just go back to the notes if you get stuck. Feel free to start anywhere you like.

This is a work in progress, the videos are appearing and things may change! If you find a mistake please email edrs20@bath.ac.uk and good luck!

# 1 Negative numbers

On a number line negative numbers are typically written to the left of zero and have values smaller than zero. Negative numbers are tricky. Often when an error creeps into a calculation it's due to a misplaced minus sign, they are a source of problems for everyone - don't worry if they seem tricky, they have only relatively recently lost their mysteriousness. The evidence of humans counting dates from $35,000$BCE yet as recently as 1758 British mathematician Francis Maseres said that negative numbers...

*"... darken the very whole doctrines of the equations and make dark of the things which are in their nature excessively obvious and simple".*

## 1.1 Multiplication and Division

When multiplying and dividing using negative numbers the answer will be the same as the equivalent calculation with positive numbers only, but, you may have to change the sign - to either positive or negative. The rules for deciding if the answer is positive or negative are below:

Note

- positive $\times$ positive $=$ positive
- negative $\times$ positive $=$ negative
- positive $\times$ negative $=$ negative
- negative $\times$ negative $=$ positive

Notice that the order is not important. Here are some examples:

$$-2 \times 3 = -6$$

$$10 \times -5 = -50$$

$$-4 \times -6 = 24$$

If you have more that two numbers to multiplying you can just count the number of negative numbers and apply the following rule:

> **Note**
>
> - If the total number of negative numbers is **even** the answer is **positive**.
> - If the total number of negative numbers is **odd** the answer is **negative**.

Here's a longer example:

$$-2 \times -2 \times -2 \times -2 = 16$$

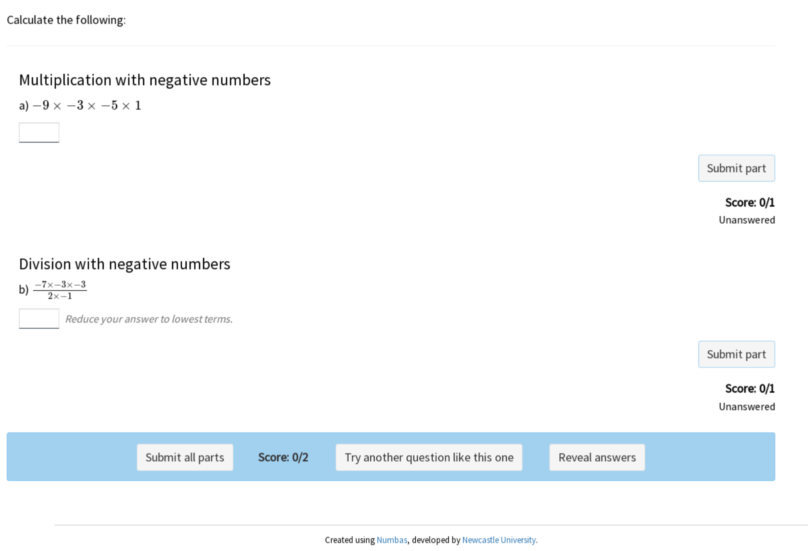
since there are even number of negatives in the question the answer will be positive.

Since division and multiplication are so closely related, division works in exactly the same way. For example:

$$\frac{-3 \times -6}{-9} = -2$$

.

You can practice these techniques with the following questions. You can refresh the question to change the numbers. Try them as much as you like.



### 1.1.1 But why?!!?

Building a physical idea of a negative number is tricky. For example thinking of $2 \times 3$ as two lots of 3 things is fine, but what does $-2 \times -3$ even mean? Hopefully but looking at the pattern below it will be become clear that our definition of what happens with two negative numbers is the only one that makes sense. Consider extending the two times table into negative numbers.

$$
\begin{array}{ccccc}
3 & \times & 2 & = & 6 \\
2 & \times & 2 & = & 4 \\
1 & \times & 2 & = & 2 \\
0 & \times & 2 & = & 0 \\
-1 & \times & 2 & = & -2 \\
-2 & \times & 2 & = & -4 \\
-3 & \times & 2 & = & -6
\end{array}
$$

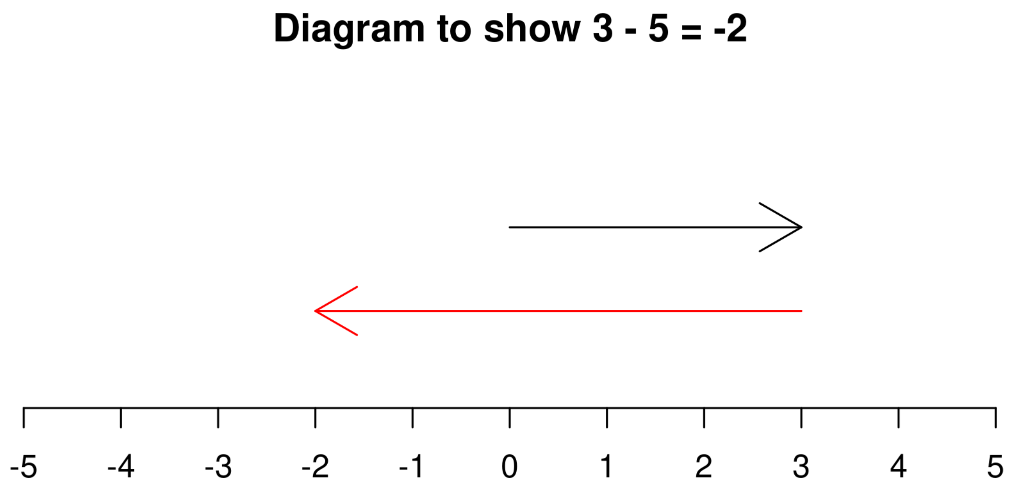
Now with the negative two times table.

$$
\begin{array}{ccccc}
3 & \times & -2 & = & -6 \\
2 & \times & -2 & = & -4 \\
1 & \times & -2 & = & -2 \\
0 & \times & -2 & = & 0 \\
-1 & \times & -2 & = & 2 \\
-2 & \times & -2 & = & 4 \\
-3 & \times & -2 & = & 6
\end{array}
$$

Our definition fits the pattern. Horrah!

## 1.2 Addition and subtraction

It helps to think about addition and subtraction of negative numbers on a number line. We can think about positive numbers as arrows pointing *forwards*, shifts to the right from zero, and negative numbers as arrows *backwards*, shifts to the left. Add to this the idea that addition and subtraction is then combining these arrows. When you add two numbers you place them one after another, the end of the second arrow on the tip of the first. With subtraction you reverse the direction of the second arrow and then place them together just like addition.




Consider the following examples:

- $3 - 5 = -2$ can be thought of as: start at three then move five back to the left.
- $-4 + 1 = -3$, start at $-4$ then move one to the right.
- $5 + -2 = 3$, start at 5 then add on a shift of 2 to the left.
- $1 - -3 = 4$, start at 1 then reverse a shift of 3 to the left (I know it seams bonkers!). The double negative cancels out to give a calculation equivalent to $1 + 3 = 4$.

> **Warning**
>
> It's tempting to cling on to the idea that *two negatives make a positive* when it comes to addition and subtraction. But consider the following statements, they are all correct, but imagine how easy it is to be confused if you just apply the *two negatives make a positive* rule.
>
> - $-3 - 5 = -8$
> - $-10 - -4 = -6$

You can practice these techniques with the following questions. The numbers change each time to try them as much as you like.

Calculate:

a)

$-7 + -2 =$

Submit part

**Score: 0/1**
Unanswered

b)

$-6 - -6$

Submit part

**Score: 0/1**
Unanswered

Submit all parts | **Score: 0/2** | Try another question like this one | Reveal answers

# 2 Algebraic expressions

Algebraic expressions are just statements about numbers. However, letters are used as place holders for some of the numbers. There are many reasons this is useful, it could be because we would like to uncover the structure of something, or, because we don't know the specific numbers to use yet.

## 2.1 Substitution

In order to evaluate an algebraic expression we have to substitute the letters for numbers. After the numbers are written in place of the letters we must take care to evaluate the statement in the correct order. BIDMAS is often used to remember the order:

- **Brackets** Work out anything in brackets first.
- **Indices** Powers are next, something like $3^2$.
- **Division and Multiplication** these two have equal priority. If there is a 'tie' work left to right. However if you see a large division they have implicit brackets in them. For example $\frac{2+10}{2\times 3}$ should be thought of as $\frac{(2+10)}{(2\times 3)}$.
- **Addition and Subtraction** like multiplication and division these are equal priority. If there is a tie work left to right.

One more thing to know before we start making substitutions is that the multiplication symbol $\times$ is often not used in algebraic expressions. Letters and numbers that are next to each other are multiplied together. For example $3a$ means $3 \times a$. You can show two numbers multiplied together like this $2 \times 3 = (2)(3) = 6$.

Here are some examples:

If $a = 2$ and $b = -3$ then we can evaluate $5a + 4b$ like this:

$$5(2) + 4(-3)$$

When things are written next to each other this means multiplication.

$$5 \times 2 + 4 \times -3$$

Using BIDMAS to do the multiplication first and remembering that a positive number multiplied by a negative gives a negative number.

$$10 + -12 = -2$$

Substituting $n = 3$ and $x = 2$ into $5x^n$. By replacing the letters with numbers we have:

$$5(2)^3$$

Remembering that when things are next to each other it means multiplication, which gives:

$$5 \times 2^3$$

Following BIDMAS we must deal with the powers first. Since $2^3 = 2 \times 2 \times 2 = 8$ we have:

$$5 \times 8 = 40$$

Finally consider $\frac{2p+q}{r}$ where $p = 6$, $q = 3$ and $r = 5$. Replacing the letters with numbers we have:

$$\frac{2(6) + 3}{5} = \frac{2 \times 6 + 3}{5}$$

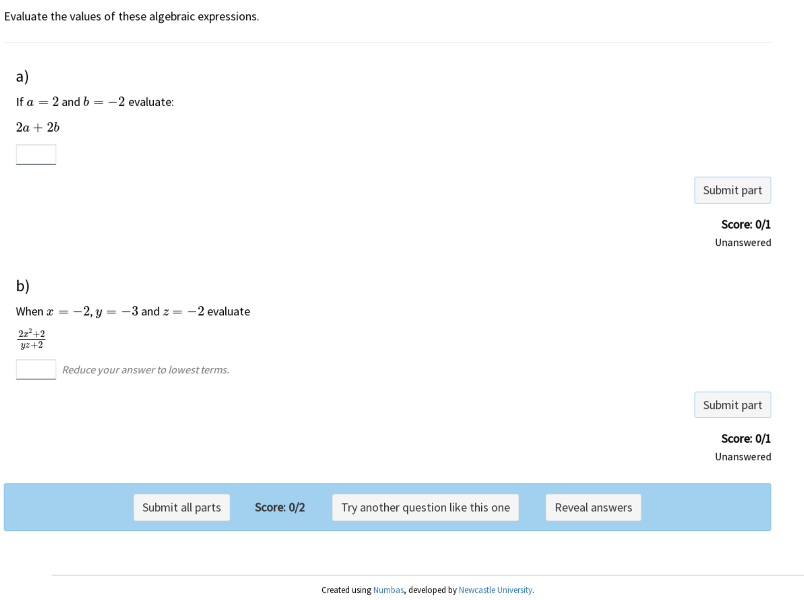
Remembering that there are implicit brackets in fractions, the numerator needs to be evaluated first.

$$\frac{(2 \times 6 + 3)}{5} = \frac{(12 + 3)}{5} = \frac{15}{5}$$

Now the fraction can be evaluated.

$$\frac{15}{5} = 3$$

You can practice these techniques with the following questions. The numbers change each time to try them as much as you like.




## 2.2 Simplification

Algebraic expressions are made up of terms. Similar terms can be combined to create a simplified expression, this processes is called *collecting like terms*. For example $2a + 3a$ can be simplified to $5a$ by collecting the $a$ terms. Here's another example with a bit more going on:

$$5x + 7y - 3x + 3y = \overbrace{5x - 3x}^{x\text{ terms}} + \overbrace{7y + 3y}^{y\text{ terms}} = 2x + 10y$$

Notice that the like terms were grouped first to make it easier to simplify. Also, each term *owns* the positive of negative symbol ahead of it.

Terms can be more complex too. Although it's tempting to find something to simplify there are no like terms in this expression: $3xy + 6x^2 + 2x - 5y$. Only the exact same multiples can be simplified. For example:

$$6x^2 + 2x - 5x^2 - 8x = \overbrace{6x^2 - 5x^2}^{x^2\text{ terms}} + \overbrace{2x - 8x}^{x\text{ terms}} = x^2 - 6x$$

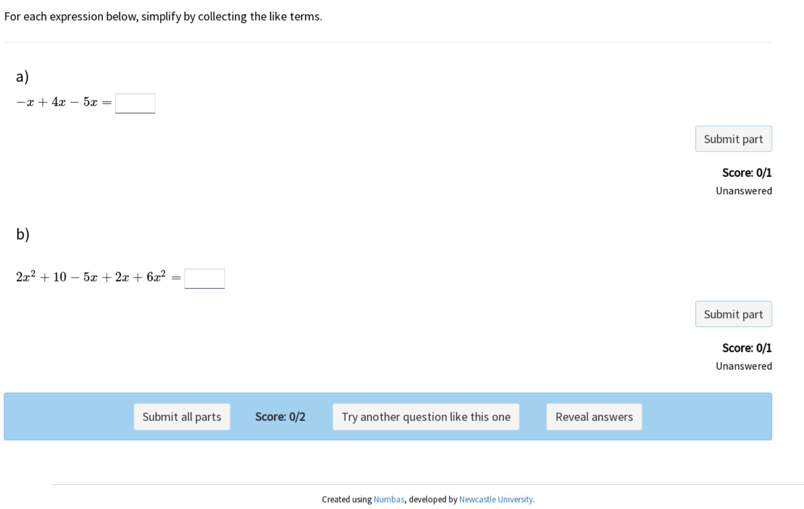
Notice that the two different types of term are $x$ and $x^2$. Also, I could have written $1x^2$ but we normally don't bother with the 1. It's also important to note that capitalisation matters; $x$ is different from $X$.

Take care when simplifying multiples of different letters $3xy+5yx$ can be simplified. This is because the order of multiplication doesn't matter so $3xy + 5yx = 3xy + 5xy = 8xy$. Terms are normally written in alphabetical order with the highest powers first.

**Key point:**

- $x \times x = x^2$
- $x + x = 2x$
- $x$ is different from $X$
- $1x$ is written as $x$

Have a go at simplifying with these questions.



# 3 Expressions with brackets

Dealing with algebraic expressions containing brackets is a useful skill. This section looks at removing brackets by *expanding* and adding brackets back in by *factorising*.

## 3.1 Expanding

### 3.1.1 Single brackets

Expanding a bracket in an algebraic expression is an example of the distributive law. You probably are already familiar with that law. Here is an example of how the law could be used to work out $6 \times 15$ using a mental method.

$$\begin{aligned}6 \times 15 &= 6 \times (10 + 5)\\&= 6 \times 10 + 6 \times 5\\&= 60 + 30\\&= 90\end{aligned}$$

The same procedure is followed with an algebraic expression.

$$\begin{aligned}6(2x + 5) &= 6 \times (2x + 5)\\&= 6 \times 2x + 6 \times 5\\&= 12x + 30\end{aligned}$$

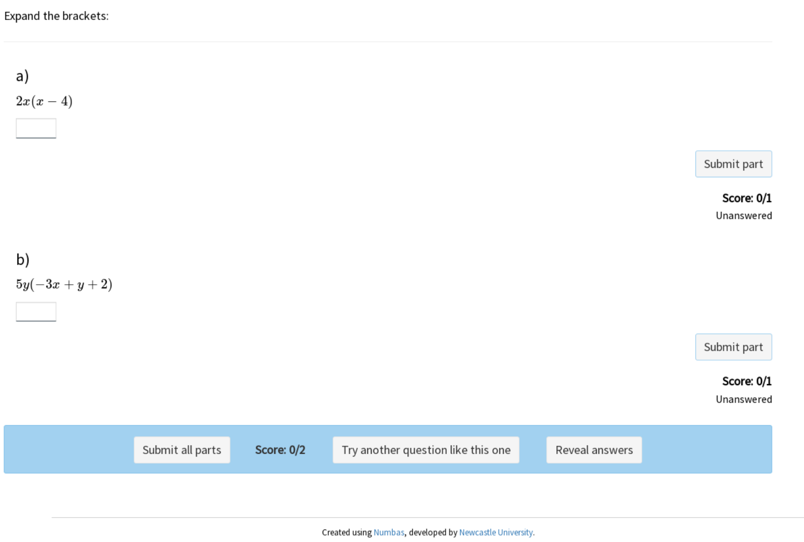
The number of terms within the bracket isn't limited to two. For example:

$$\begin{aligned}x(y + 3x - 5) &= x \times (y + 3x - 5)\\&= x \times y + x \times 3x + x \times -5\\&= xy + 3x^2 - 5x\end{aligned}$$

Finally, another common pattern is to have a negative sign before a bracket. This just means everything inside the bracket is multiplied by $-1$. It just *flips* the sign of everything in the brackets.

$$
\begin{aligned}
-(3-x) &= -1 \times (3-x) \\
&= -1 \times 3 + -1 \times -x \\
&= -3 + x
\end{aligned}
$$

Here are some practice questions.




### 3.1.2 Expanding pairs of brackets

This will be covered in Quadratics.

## 3.2 Factorising

The reverse of expanding brackets is called factorising. We look for a common factor in each term to take outside of the bracket.

### 3.2.1 Factorising - single bracket

For each term in the expression look for a common factor. We can then write this in front of the bracket so when you expand the bracket the original expression is returned. For example:

$$
\begin{aligned}
12x^2 - 15x &= 3x \times 4x + 3x \times -5 \\
&= 3x(4x - 5)
\end{aligned}
$$

Notice that $3x$ is a factor of both $12x^2$ and $-15x$. Also, if we expand our answer we should get back to where we started from.

Here are some practice questions.


## 3.2.2 Factorising - pairs of brackets

This will be covered in the Quadratics section.

# 4 Fractions

Fractions can be written in two ways:

- as decimals fractions, for example 0.5, 0.25 and $0.\dot{3}$.
- as vulgar fractions, the following fractions have the same values as the examples above, $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{3}$. Vulgar fractions consist of two parts. The top, or **numerator**, and the bottom, the **denominator**.

Vulgar fractions are useful in algebra. The next section looks at some techniques for dealing with them.

## 4.1 Simplifying

Fractions can be *cancelled down* or simplified by dividing the numerator and denominator by the same thing. For example:

$$\begin{aligned}\frac{18}{24} &= \frac{3 \times 6}{4 \times 6} \\ &= \frac{3 \times \cancel{6}}{4 \times \cancel{6}} \\ &= \frac{3}{4}\end{aligned}$$

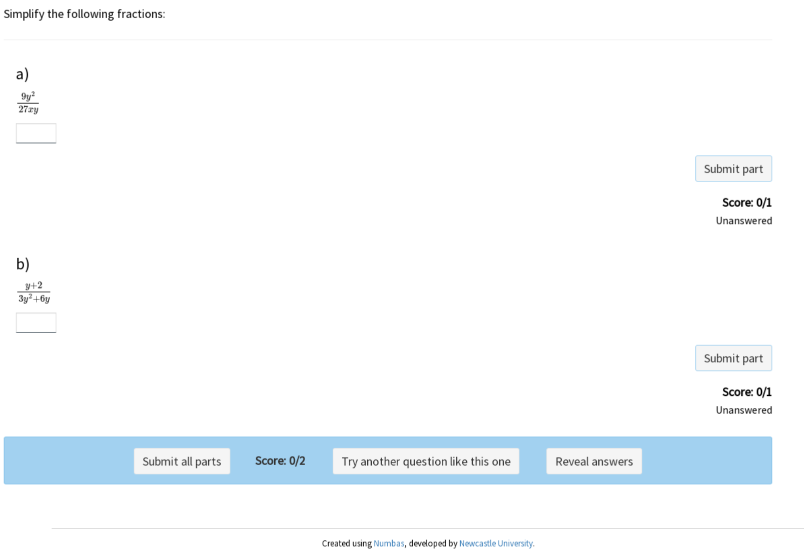
The same can be done with algebraic fractions.

$$\begin{aligned}\frac{4xy}{6x} &= \frac{2y \times 2x}{3 \times 2x} \\ &= \frac{2y \times \cancel{2x}}{3 \times \cancel{2x}} \\ &= \frac{2y}{3}\end{aligned}$$

Sometimes you'll need to factorise expressions in the fraction in order to cancel it down.

$$\begin{aligned}\frac{10x^2+5x}{4x+2} &= \frac{5x\times 2x+5x\times 1}{2\times 2x+2\times 1}\\ &= \frac{5x(2x+1)}{2(2x+1)}\\ &= \frac{5x\cancel{(2x+1)}}{2\cancel{(2x+1)}}\\ &= \frac{5x}{2}\end{aligned}$$

Here are some practice questions.




**! Warning!**

It is tempting to want to make cancellations like this:

$$\begin{aligned}\frac{2x^2}{3x+7} &= \frac{2x\cancel{x}}{3\cancel{x}+1}\\ &= \frac{2x}{3+7}\\ &= \frac{2x}{10}\\ &= \frac{x}{5}\end{aligned}$$

However, please don't do it, as it's just plain wrong! Lets let $x = 3$ and substitute it into the original $\frac{2x}{3x+7}$ and into incorrectly simplified version $\frac{x}{5}$. If the algebra is correct it should give the same answer.
We claim:

$$\frac{2x^2}{3x+7} = \frac{x}{5}$$

but if we substitute $x = 2$ into both sides we get:

$$\begin{aligned}\frac{2(3)^2}{3(3)+7} &= \frac{(3)}{5}\\ \frac{2\times 9}{9+7} &= \frac{3}{5}\\ \frac{18}{16} &= \frac{3}{5}\\ \frac{9}{8} &= \frac{3}{5}\end{aligned}$$

Which is nonsense!

## 4.2 Multiplication and division

Multiplication and division of fractions is, thankfully, really easy!

### 4.2.1 Multiplicaiton

For multiplication you simply multiply the numerators and denominators together. After the multiplication you may be able to cancel down the fraction. Just like this:

$$\begin{aligned}\frac{2}{5}\times\frac{3}{4} &= \frac{2\times 3}{5\times 4}\\ &= \frac{6}{20}\\ &= \frac{3\times 2}{10\times 2}\\ &= \frac{3\times \cancel{2}}{10\times \cancel{2}}\\ &= \frac{3}{10}\end{aligned}$$

**Pro-tip**

It is possible to cancel before multiplying. Here is the same example revisited:

$$\begin{aligned}\frac{2}{5}\times\frac{3}{4} &= \frac{2\times 3}{5\times 4}\\ &= \frac{2\times 3}{5\times 2\times 2}\\ &= \frac{\cancel{2}\times 3}{5\times 2\times \cancel{2}}\\ &= \frac{3}{10}\end{aligned}$$

This can be super useful when dealing with large numbers or complex algebraic fractions.

### 4.2.2 Division

We can change a division into a multiplication by remembering **keep, change, flip**. We keep the first fraction as it is. Change the division, $\div$, symbol to a multiplication, $\times$, and flip the last fraction - swap the places of the numerator and denominator. This is called taking the reciprocal of the fraction. For example:

$$\begin{aligned}\frac{3}{7}\div\frac{5}{2} &= \frac{3}{7}\times\frac{2}{5}\\ &= \frac{3\times 2}{7\times 5}\\ &= \frac{6}{35}\end{aligned}$$