zero-to-hero

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09/09/2022

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Welcome

This course is designed to refresh your knowledge of maths to get you ready to use calculus in your course. There is no right or wrong way to use it. Each section includes written notes, ~~a video (with the same content as the notes)~~ and practice questions. It's chunked into bitesized sections to allow you make progress in 10 min windows. You may like to try the questions first and then just go back to the notes if you get stuck. Feel free to start anywhere you like.

This is a work in progress, things may change! If you find a mistake please email edrs20@bath.ac.uk and good luck!

1 Negative numbers

On a number line negative numbers are typically written to the left of zero and have values smaller than zero. Negative numbers are tricky. Often when an error creeps into a calculation it's due to a misplaced minus sign, they are a source of problems for everyone - don't worry if they seem tricky, they have only relatively recently lost their mysteriousness. The evidence of humans counting dates from $35,000$BCE yet as recently as 1758 British mathematician Francis Maseres said that negative numbers...

> *"... darken the very whole doctrines of the equations and make dark of the things which are in their nature excessively obvious and simple".*

1.1 Multiplication and Division

When multiplying and dividing using negative numbers the answer will be the same as the equivalent calculation with positive numbers only, but, you may have to change the sign - to either positive or negative. The rules for deciding if the answer is positive or negative are below:

positive $\times$ positive $=$ positive
negative $\times$ positive $=$ negative
positive $\times$ negative $=$ negative
negative $\times$ negative $=$ positive

Notice that the order is not important. Here are some examples:

$$-2 \times 3 = -6$$

$$10 \times -5 = -50$$

$$-4 \times -6 = 24$$

If you have more that two numbers to multiplying you can just count the number of negative numbers and apply the following rule:

> If the total number of negative numbers is **even** the answer is **positive**.
> If the total number of negative numbers is **odd** the answer is **negative**.

Here's a longer example:

$$-2 \times -2 \times -2 \times -2 = 16$$

since there are even number of negatives in the question the answer will be positive.

Since division and multiplication are so closely related, division works in exactly the same way. For example:

$$\frac{-3 \times -6}{-9} = -2$$

.

You can practice these techniques with the following questions. You can refresh the question to change the numbers. Try them as much as you like.

1.1.1 But why?!!?

Building a physical idea of a negative number is tricky. For example thinking of 2×3 as two lots of 3 things is fine, but what does -2×-3 even mean? Hopefully but looking at the pattern below it will be become clear that our definition of what happens with two negative numbers is the only one that makes sense. Consider extending the two times table into negative numbers.

$$\begin{array}{ccccc} 3 & \times & 2 & = & 6 \\ 2 & \times & 2 & = & 4 \\ 1 & \times & 2 & = & 2 \\ 0 & \times & 2 & = & 0 \\ -1 & \times & 2 & = & -2 \\ -2 & \times & 2 & = & -4 \\ -3 & \times & 2 & = & -6 \end{array}$$

Now with the negative two times table.

$$\begin{array}{ccccc} 3 & \times & -2 & = & -6 \\ 2 & \times & -2 & = & -4 \\ 1 & \times & -2 & = & -2 \\ 0 & \times & -2 & = & 0 \\ -1 & \times & -2 & = & 2 \\ -2 & \times & -2 & = & 4 \\ -3 & \times & -2 & = & 6 \end{array}$$

Question 1

Calculate the following:

Multiplication with negative numbers

a) $-9 \times 4 \times 2 \times -2$

______

Score: 0/1

Division with negative numbers

b) $\frac{-3 \times 1 \times -5}{1 \times 2}$

______ *Reduce your answer to lowest terms.*

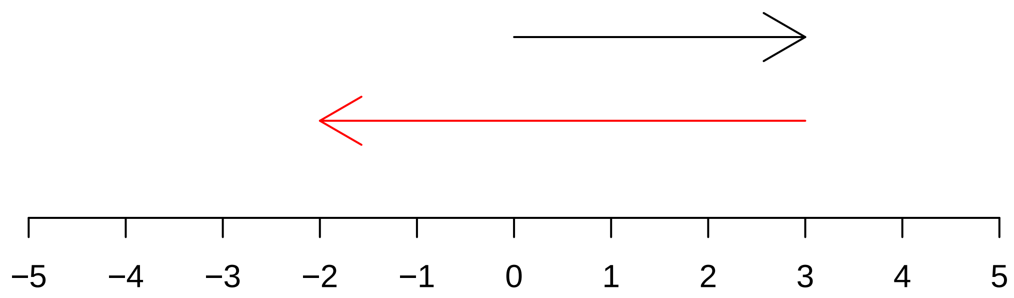
Score: 0/1

Our definition fits the pattern. Horrah!

1.2 Addition and subtraction

It helps to think about addition and subtraction of negative numbers on a number line. We can think about positive numbers as arrows pointing *forwards*, shifts to the right from zero, and negative numbers as arrows *backwards*, shifts to the left. Add to this the idea that addition and subtraction is then combining these arrows. When you add two numbers you place them one after another, the end of the second arrow on the tip of the first. With subtraction you reverse the direction of the second arrow and then place them together just like addition.

Diagram to show 3 − 5 = −2



Consider the following examples:

- $3 - 5 = -2$ can be thought of as: start at three then move five back to the left.
- $-4 + 1 = -3$, start at -4 then move one to the right.
- $5 + -2 = 3$, start at 5 then add on a shift of 2 to the left.
- $1 - -3 = 4$, start at 1 then reverse a shift of 3 to the left. The moving left backwards cancels out to give our answer of 4.

Warning: It's tempting to cling on to the idea that *two negatives make a positive* when it comes to addition and subtraction. But consider the following statements, they are all correct, but imagine how easy it is to be confused if you just apply the *two negatives make a positive* rule.

- $-3 - 5 = -8$
- $-10 - -4 = -6$

You can practice these techniques with the following questions. The numbers change each time to try them as much as you like.

Question 1

Calculate:

---

a)

$9 + -6 =$

______

Score: 0/1

b)

$6 - -7$

______

Score: 0/1

Created using Numbas (https://www.numbas.org.uk), developed by Newcastle University (http://www.newcastle.ac.uk).

2 Algebraic expressions

Algebraic expressions are just statements about numbers. However, letters are used as place holders for some of the numbers. There are many reasons this is useful, it could be because we would like to uncover the structure of something, or, because we don't know the specific numbers to use yet.

2.1 Substitution

In order to evaluate an algebraic expression we have to substitute the letters for numbers. After the numbers are written in place of the letters we must take care to evaluate the statement in the correct order. BIDMAS is often used to remember the order:

- **Brackets** Work out anything in brackets first.
- **Indices** Powers are next, something like 3^2.
- **Division and Multiplication** these two have equal priority. If there is a 'tie' work left to right. However if you see a large division they have implicit brackets in them. For example $\frac{2+10}{2\times 3}$ should be thought of as $\frac{(2+10)}{(2\times 3)}$.
- **Addition and Subtraction** like multiplication and division these are equal priority. If there is a tie work left to right.

One more thing to know before we start making substitutions is that the multiplication symbol $\times$ is often not used in algebraic expressions. Letters and numbers that are next to each other are multiplied together. For example $3a$ means $3 \times a$. You can show two numbers multiplied together like this $2 \times 3 = (2)(3) = 6$.

Here are some examples:

If $a = 2$ and $b = -3$ then we can evaluate $5a + 4b$ like this:

$$5(2) + 4(-3)$$

When things are written next to each other this means multiplication.

$$5 \times 2 + 4 \times -3$$

Using BDMAS to do the multiplication first and remembering that a positive number multiplied by a negative gives a negative number.

$$10 + -12 = -2$$

Substituting $n = 3$ and $x = 2$ into $5x^n$. By replacing the letters with numbers we have:

$$5(2)^3$$

Remembering that when things are next to each other it means multiplication, which gives:

$$5 \times 2^3$$

Following BIDMAS we must deal with the powers first. Since $2^3 = 2 \times 2 \times 2 = 8$ we have:

$$5 \times 8 = 40$$

Finally consider $\frac{2p+q}{r}$ where $p = 6$, $q = 3$ and $r = 5$. Replacing the letters with numbers we have:

$$\frac{2(6)+3}{5} = \frac{2 \times 6 + 3}{5}$$

Remembering that there are implicit brackets in fractions, the numerator needs to be evaluated first.

$$\frac{(2 \times 6 + 3)}{5} = \frac{(12 + 3)}{5} = \frac{15}{5}$$

Now the fraction can be evaluated.

$$\frac{15}{5} = 3$$

You can practice these techniques with the following questions. The numbers change each time to try them as much as you like.

Question 1

Evaluate the values of these algebraic expressions.

a)

If $p = 2$ and $q = -3$ evaluate:

$5p + 3q$

______

Score: 0/1

b)

When $a = -2, b = 3$ and $c = 3$ evaluate

$\frac{3a^3+3}{bc+3}$

______ *Reduce your answer to lowest terms.*

Score: 0/1

2.2 Simplification

Algebraic expressions are made up of terms. Similar terms can be combined to create a simplified expression, this processes is called *collecting like terms*. For example $2a + 3a$ can be simplified to $5a$ by collecting the a terms. Here's another example with a bit more going on:

$$5x + 7y - 3x + 3y = \overbrace{5x - 3x}^{x \text{ terms}} + \overbrace{7y + 3y}^{y \text{ terms}} = 2x + 10y$$

Notice that the like terms were grouped first to make it easier to simplify. Also, each term *owns* the positive of negative symbol ahead of it.

Terms can be more complex too. Although it's tempting to find something to simplify there are no like terms in this expression: $3xy + 6x^2 + 2x - 5y$. Only the exact same multiples can be simplified. For example:

$$6x^2 + 2x - 5x^2 - 8x = \overbrace{6x^2 - 5x^2}^{x^2 \text{ terms}} + \overbrace{2x - 8x}^{x \text{ terms}} = x^2 - 6x$$

Notice that the two different types of term are x and x^2. Also, I could have written $1x^2$ but we normally don't bother with the 1. It's also important to note that capitalisation matters; x is different from X.

Take care when simplifying multiples of different letters $3xy+5yx$ can be simplified. This is because the order of multiplication doesn't matter so $3xy + 5yx = 3xy + 5xy = 8xy$. Terms are normally written in alphabetical order with the highest powers first.

Key point:

> $x \times x = x^2$
> $x + x = 2x$
> x is different from X
> $1x$ is written as x

Have a go at simplifying with these questions.

Question 1

For each expression below, simplify by collecting the like terms.

---

a)

$-5x - 3x + 5x =$ _____

Score: 0/1

b)

$10x^2 + 2 + 7x + 2x - 2x^2 =$ _____

Score: 0/1

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Typesetting math: 100%

3 Expressions with brackets

Dealing with algebraic expressions containing brackets is a useful skill. This section looks at removing brackets by *expanding* and adding brackets back in by *factorising*.

3.1 Expanding

3.1.1 Single brackets

Expanding a bracket in an algebraic expression is an example of the distributive law. You probably are already familiar with that law. Here is an example of how the law could be used to work out 6×15 using a mental method.

$$\begin{aligned} 6 \times 15 &= 6 \times (10 + 5) \\ &= 6 \times 10 + 6 \times 5 \\ &= 60 + 30 \\ &= 90 \end{aligned}$$

The same procedure is followed with an algebraic expression.

$$\begin{aligned} 6(2x + 5) &= 6 \times (2x + 5) \\ &= 6 \times 2x + 6 \times 5 \\ &= 12x + 30 \end{aligned}$$

The number of terms within the bracket isn't limited to two. For example:

$$\begin{aligned} x(y + 3x - 5) &= x \times (y + 3x - 5) \\ &= x \times y + x \times 3x + x \times -5 \\ &= xy + 3x^2 - 5x \end{aligned}$$

Finally, another common pattern is to have a negative sign before a bracket. This just means everything inside the bracket is multiplied by -1. It just *flips* the sign of everything in the brackets.

$$\begin{aligned} -(3-x) &= -1 \times (3-x) \\ &= -1 \times 3 + -1 \times -x \\ &= -3 + x \end{aligned}$$

Here are some practice questions.

3.1.2 Expanding pairs of brackets

This will be covered in Quadratics.

3.2 Factorising

The reverse of expanding brackets is called factorising. We look for a common factor in each term to take outside of the bracket.

3.2.1 Factorising - single bracket

For each term in the expression look for a common factor. We can then write this in front of the bracket so when you expand the bracket the original expression is returned. For example:

$$\begin{aligned} 12x^2 - 15x &= 3x \times 4x + 3x \times -5 \\ &= 3x(4x-5) \end{aligned}$$

Notice that $3x$ is a factor of both $12x^2$ and $-15x$. Also, if we expand our answer we should get back to where we started from.

Here are some practice questions.

3.2.2 Factorising - pairs of brackets

This will be covered in the Quadratics section.

Question 1

Expand the brackets:

a)

$3x(y-1)$

Score: 0/1

b)

$-2y(5x+5y+5)$

Score: 0/1

Question 1

Factorise:

Note: you must enter a multiplication sign between the factor and the bracket type `x*(x+1)` to get $x \times (x+1)$.

a)

$3x^2 + 3x$

______

Score: 0/1

b)

$-5xy - 3x^2 - 5x$

______

Score: 0/1

Typesetting math: 100%

4 Fractions

Fractions can be written in two ways:

- as decimals fractions, for example 0.5, 0.25 and $0.\dot{3}$.
- as vulgar fractions, the following fractions have the same values as the examples above, $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{3}$. Vulgar fractions consist of two parts. The top, or **numerator**, and the bottom, the **denominator**.

Vulgar fractions are useful in algebra. The next section looks at some techniques for dealing with them.

4.1 Simplifying

Fractions can be *cancelled down* or simplified by dividing the numerator and denominator by the same thing. For example:

$$\begin{aligned}\frac{18}{24} &= \frac{3 \times 6}{4 \times 6} \\ &= \frac{3 \times \cancel{6}}{4 \times \cancel{6}} \\ &= \frac{3}{4}\end{aligned}$$

The same can be done with algebraic fractions.

$$\begin{aligned}\frac{4xy}{6x} &= \frac{2y \times 2x}{3 \times 2x} \\ &= \frac{2y \times \cancel{2x}}{3 \times \cancel{2x}} \\ &= \frac{2y}{3}\end{aligned}$$

Sometimes you'll need to factorise expressions in the fraction in order to cancel it down.

$$\begin{aligned}\frac{10x^2+5x}{4x+2} &= \frac{5x\times 2x+5x\times 1}{2\times 2x+2\times 1}\\ &= \frac{5x(2x+1)}{2(2x+1)}\\ &= \frac{5x\cancel{(2x+1)}}{2\cancel{(2x+1)}}\\ &= \frac{5x}{2}\end{aligned}$$

Here are some practice questions.

4.1.0.1 Warning!

It is tempting to want to make cancellations like this:

$$\begin{aligned}\frac{2x^2}{3x+7} &= \frac{2x\cancel{x}}{3\cancel{x}+1}\\ &= \frac{2x}{3+7}\\ &= \frac{2x}{10}\\ &= \frac{x}{5}\end{aligned}$$

However, please don't do it, as it's just plain wrong! Lets let $x=3$ and substitute it into the original $\frac{2x}{3x+7}$ and into incorrectly simplified version $\frac{x}{5}$. If the algebra is correct it should give the same answer.

We claim:

$$\frac{2x^2}{3x+7} = \frac{x}{5}$$

but if we substitute $x=2$ into both sides we get:

$$\begin{aligned}\frac{2(3)^2}{3(3)+7} &= \frac{(3)}{5}\\ \frac{2\times 9}{9+7} &= \frac{3}{5}\\ \frac{18}{16} &= \frac{3}{5}\\ \frac{9}{8} &= \frac{3}{5}\end{aligned}$$

Question 1

Simplify the following fractions:

---

a)

$\frac{6xy}{9y}$

______

Score: 0/1

b)

$\frac{2x-6}{3x-9}$

______

Score: 0/1

Which is nonsense!

4.2 Multiplication and division

Multiplication and division of fractions is, thankfully, really easy!

4.2.1 Multiplicaiton

For multiplication you simply multiply the numerators and denominators together. After the multiplication you may be able to cancel down the fraction. Just like this:

$$\begin{aligned}\frac{2}{5} \times \frac{3}{4} &= \frac{2 \times 3}{5 \times 4} \\ &= \frac{6}{20} \\ &= \frac{3 \times 2}{10 \times 2} \\ &= \frac{3 \times \cancel{2}}{10 \times \cancel{2}} \\ &= \frac{3}{10}\end{aligned}$$

Pro-tip It is possible to cancel before multiplying. Here is the same example revisited:

$$\begin{aligned}\frac{2}{5} \times \frac{3}{4} &= \frac{2 \times 3}{5 \times 4} \\ &= \frac{2 \times 3}{5 \times 2 \times 2} \\ &= \frac{\cancel{2} \times 3}{5 \times 2 \times \cancel{2}} \\ &= \frac{3}{10}\end{aligned}$$

This can be super useful when dealing with large numbers or complex algebraic fractions.

4.2.2 Division

We can change a division into a multiplication by remembering **keep, change, flip**. We keep the first fraction as it is. Change the division, $\div$, symbol to a multiplication, $\times$, and flip the last fraction - swap the places of the numerator and denominator. This is called taking the reciprocal of the fraction. For example:

$$\begin{aligned}\frac{3}{7} \div \frac{5}{2} &= \frac{3}{7} \times \frac{2}{5} \\ &= \frac{3 \times 2}{7 \times 5} \\ &= \frac{6}{35}\end{aligned}$$

4.3 Addition and subtraction

Addition and subtraction is easy if the denominators are the same. We just add the numerators together and the demoninator stays the same.

$$\begin{aligned}\frac{2}{5} + \frac{1}{5} &= \frac{2+1}{5} \\ &= \frac{3}{5}\end{aligned}$$

If the denominators are different we must make equivalent fractions with a common denominators first. It's like simplification, or cancelling down, in reverse.

$$\begin{aligned}\frac{2}{3} + \frac{2}{9} &= \frac{2 \times 3}{3 \times 3} + \frac{2}{9} \\ &= \frac{6}{9} + \frac{2}{9} \\ &= \frac{6+2}{9} \\ &= \frac{8}{9}\end{aligned}$$

5 Solving equations

When we workout the value of an unknown, say x, in an equation we say that we *are solving for* x. To workout the value we are free to apply any mathematical operation we like to the equation so long as we *do the same to both sides*.

Note: We can't quite do any operation. Division by zero, $\div 0$, is not allowed as it is undefined.

5.1 Linear equations

5.1.1 Single unknown

Keeping the idea of doing the same thing to both sides in mind lets solve the following equation by *undoing* each operation with it's inverse.

$$3x + 8 = 10$$

First subtract 8 from each side.

$$\begin{aligned} 3x + 8 - 8 &= 10 - 8 \\ 3x &= 2 \end{aligned}$$

Now divide both sides by 3 to find the value of one x.

$$\begin{aligned} \frac{3x}{3} &= \frac{2}{3} \\ x &= \frac{2}{3} \end{aligned}$$

The nice thing here is that we can leave the answer as $\frac{2}{3}$. No need to find a decimal fraction if we don't need to.

Solve the following equations by applying the same operation to both sides. Remember the questions come with full solutions, so, if you get stuck have a look at the answers and then try a different one.

Question 1

a)

Solve

$3x + 11 = 18$

______

Score: 0/1

b)

Solve

$\frac{-2x - 5}{3} = 18$

______

Score: 0/1

c)

Solve

$11\left(18x + 3\right) = -2$

______

Score: 0/1

5.1.2 Unknown on both sides

If the unknown appears twice in an equation collect the unknown like terms first and then solve as before.

Given $\frac{4y}{y-9} = -2$, we can multiply both sides by $(y-9)$ to get rid of the fraction, then get all the ys on one side, then finally solve as before.

$$\begin{aligned}
\frac{4y}{y-9} &= -2 \\
\frac{4y}{y-9} \times (y-9) &= -2 \times (y-9) \\
\frac{4y}{\cancel{y-9}} \times \cancel{(y-9)} &= -2 \times y - 2 \times -9 \\
4y &= -2y + 18 \\
4y + 2y &= -2y + 18 + 2y \\
6y &= 18 \\
\frac{6y}{6} &= \frac{18}{6} \\
y &= 3
\end{aligned}$$

Have a go at some questions. You'll need a pen and paper to work these out.

5.2 Inequalities

Solving inequalities works just like solving a normal equation except when you divide or multiply by a negative number the inequality symbol changes direction. Here are some examples.

Addition and subtraction work.

$$\begin{aligned}
1 &< 2 \\
1 + 5 &< 2 + 5 \\
6 &< 7 \\
&\checkmark
\end{aligned}$$

$$\begin{aligned}
1 &< \quad 2 \\
1 - 4 &< 2 - 4 \\
-3 &< -2 \\
&\checkmark
\end{aligned}$$

Question 1

a)

Solve $3\,(7w-11)=2w+8$

$w=$ ______ *Round your answer to 1 decimal place.*

Score: 0/1

b)

Given $\dfrac{12y}{y-10}=-1$,

$y=$ ______ *Round your answer to 1 decimal place.* .

Score: 0/1

c)

Solve $\dfrac{x+5}{3}+\dfrac{x}{2}=5$.

$x=$ ______ *Round your answer to 1 decimal place.*

Score: 0/1

Remember -3 is less than -2 since it is further to the left on a numberline. In other words -3 is more negative than -2.

Multiplication and division work as expected with positive numbers.

$$\begin{gathered} 4 < 6 \\ 4 \times 2 < 6 \times 2 \\ 8 < 12 \\ \checkmark \end{gathered}$$

$$\begin{gathered} 4 < 6 \\ 4 \div 2 < 6 \div 2 \\ 2 < 3 \\ \checkmark \end{gathered}$$

We need to be careful when multiplying and dividing by negative numbers.

$$\begin{gathered} 4 < \qquad 6 \\ 4 \times -2 < 6 \times -2 \\ -8 \not< -12 \\ -8 > -12 \end{gathered}$$

Remember the following key point when using inequalities:

> When multiplying or dividing by a negative number change the direction of the inequality.

5.3 Simultaneous equations

Sometimes equations have more than one unknown. Take $x + y = 4$ for example. There are infinitely many pairs of numbers, x and y, that work for this. Take the following pairs for example: $x = 1$ and $y = 3$, $x = -100$ and $y = 104$, and $x = 0.1$ and $y = 3.9$. **Pro-tip** These pairs of solutions are often given as co-ordinate pairs like $(1, 3)$, $(-100, 104)$ and $(0.1, 3.9)$. We'll do more about co-ordinates later.

However, if I give you some more information, say $x = y$, now there is only one solution, namely $x = 2$ and $y = 2$. We can use the information in two equations together to find the values that satisfy both equations.

5.3.1 Elimination method

The idea with this method is to combine the two equations to create a new equation with only one variable in it.

$$\begin{aligned} 4x + 2y &= -6 \quad (1) \\ -2x + 3y &= 7 \qquad (2) \end{aligned}$$

To get a solution for y, if we multiply equation (2) by 2 we will have two equations with equal and opposite x-coefficients:

$$\begin{aligned} 4x + 2y &= -6 \\ -4x + 6y &= 14 \quad (3) \end{aligned}$$

If we add equation (1) to equation (3) this eliminates the x-terms, leaving us with one equation in terms of y:

$$\begin{aligned} (2+6)y &= -6 + 14 \\ 8y &= 8 \\ y &= 1 \end{aligned}$$

To obtain a solution for x we can substitute this y-value into either of our initial equations. Using equation (1), we obtain:

$$\begin{aligned} 4x + 2 \times 1 &= -6 \\ 4x + 2 &= -6 \\ 4x &= -8 \\ x &= -2 \end{aligned}$$

We can check our values for x and y by substituting them into equation 2.

$$\begin{aligned} -2x + 3y &= 7 \\ -2 \times -2 + 3 \times 1 &= 7 \\ 4 + 3 &= 7 \end{aligned}$$

Which works out!

You can try other examples in the exercise below. Sometime you may have to multiply both of your starting equations in order to get the same amount of one variable. Also, don't worry if you have eliminated the other variable - it doesn't matter which you get rid of first, you should get the same answer in the end.

Question 1

Solve the simultaneous equations for x and y.

$$\begin{aligned} -2x + 3y &= 6 \\ -4x + 9y &= 6 \end{aligned}$$

x= ______

y= ______

5.3.2 Substitution method

It is also possible to re-arrange one equation and substitute it into the other. This method will be covered in the Quadratics section.

6 Straight line graphs

It is often useful to plot graphs of functions to gain an understanding of what they mean. Straight line graphs are produced by linear equations. Linear equations like $y = 2x + 4$ only have x to the power of one only. Note: this doesn't just apply to x, it could be whatever variable you are using.

6.1 Coordinates

To build a picture of a function we work out pairs of values that satisfy the function. Take for example $y = \frac{1}{2}x + 1$. If we choose values of x we can work out the corresponding y values.

x	y
0	$\frac{1}{2}(0) + 1 = 1$
1	$\frac{1}{2}(1) + 1 = 1.5$
2	$\frac{1}{2}(2) + 1 = 2$

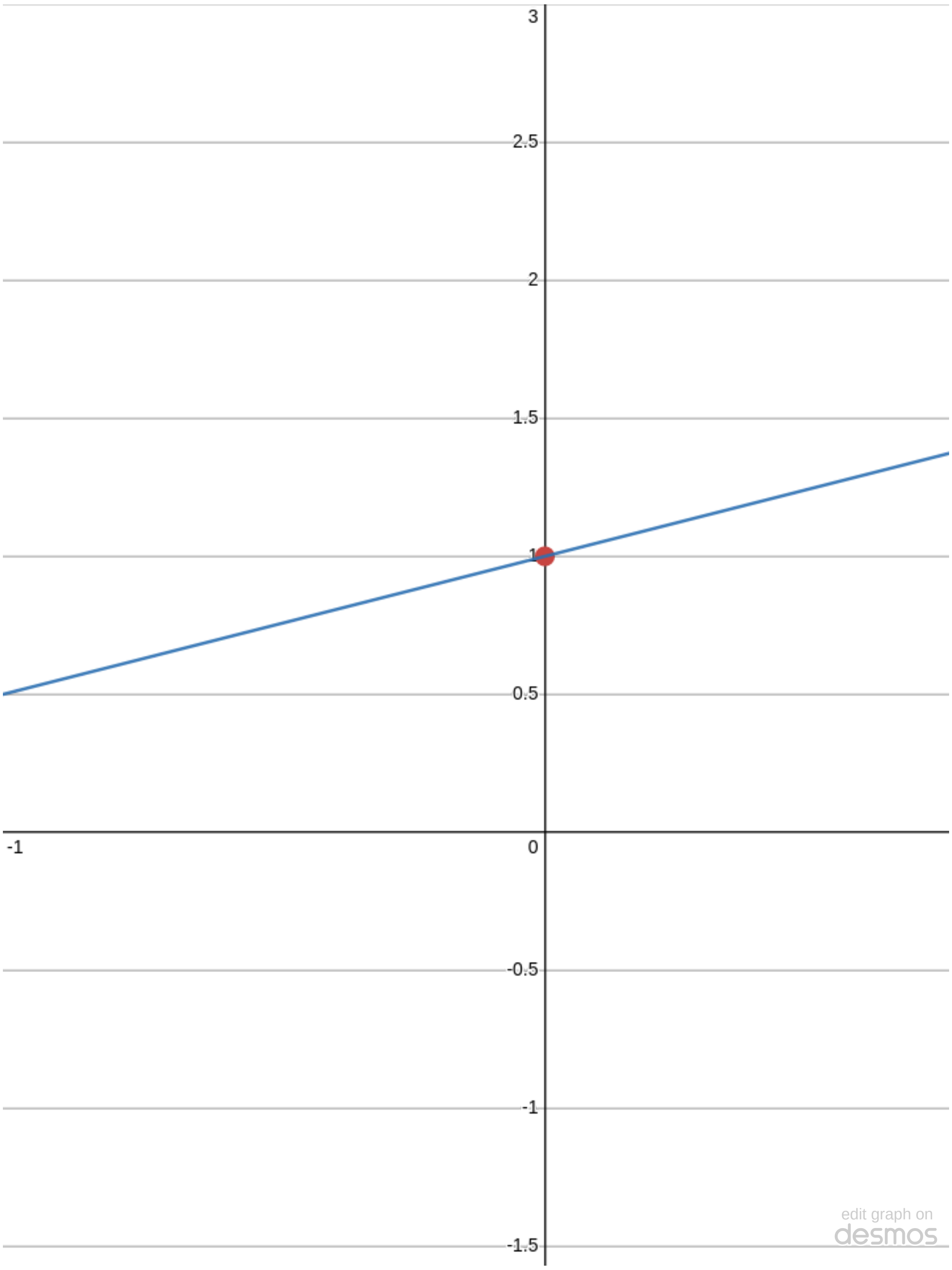
Once we have these values they can be plotted on graph.

The red dots show the points and the blue line shows the equation.

By working out some co-ordinates in the following question try to generate the correct line.

6.2 The formula for a straight line graph: $y = mx + c$

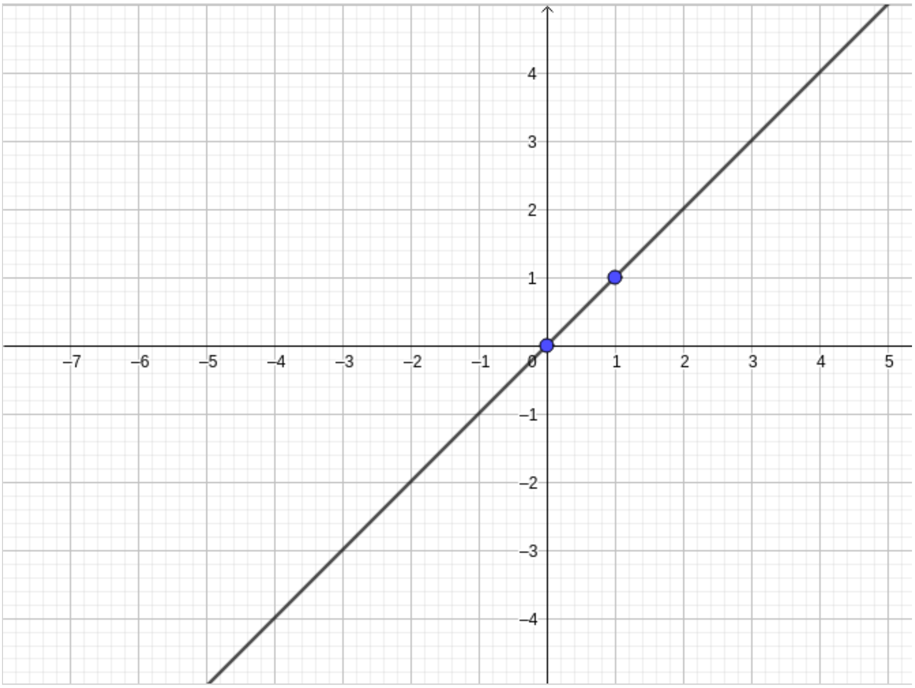
Straight line graphs can be defined by two quantities. The gradient, m, a measure of how steep the line is, and the y intercept, c, where the line crosses the y axis.


3
2.5
2
1.5
1
0.5
-1
0
-0.5
-1
-1.5
edit graph on
desmos

Question 1

Move the points A and B to make line $y = 4x + 1$

You can zoom and pan this image.



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6.2.1 The y intercept: c

The y intercept is where line crosses the y axis. We can quickly work out the co-ordinate by substituting $x = 0$ into the equation of a line, or, by noticing the constant term in equation where $y = mx + c$. Here are two examples:

For the line $y = 3x + 4$, the y intercept is at $(0, 4)$ i.e. it crosses the y axis at 4. We can check this by substituting $x = 0$ into the equation.

$$\begin{aligned} y &= 3x + 4 \\ &= 3(0) + 4 \\ &= 3 \times 0 + 4 \\ y &= 4 \end{aligned}$$

We need to be careful with the next example: $y + 2 = 5x$. It's tempting to say that the y intercept is 2 but it's not. First we must re-arrange the equation into the form of $y = mx + c$. We'll use the idea of doing the same thing to both sides again.

$$\begin{aligned} y + 2 &= 5x \\ y + 2 - 2 &= 5x - 2 \\ y &= 5x - 2 \end{aligned}$$

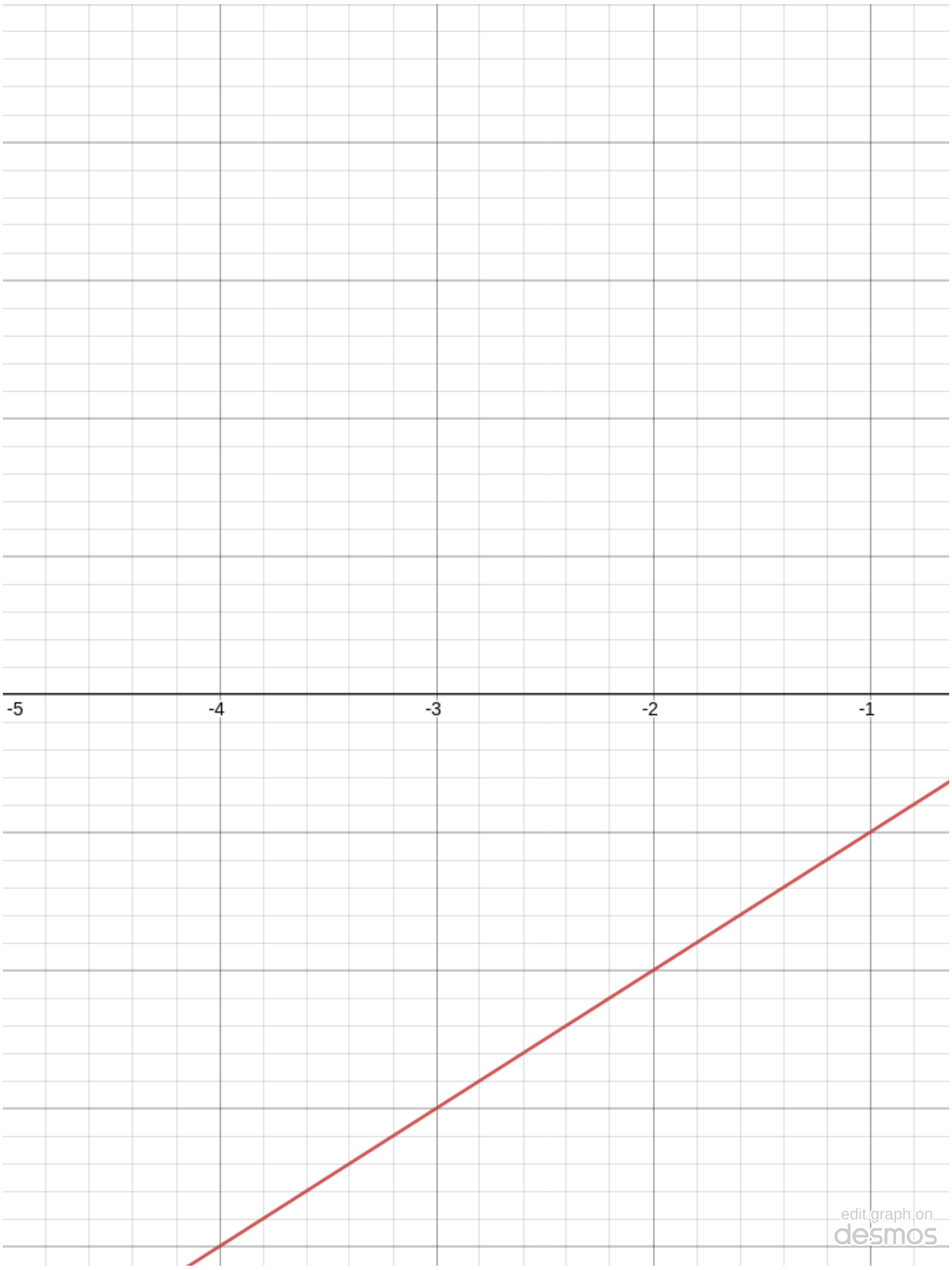
Once we've done this we can see that the intercept is when $y = -2$. Notice if we substituted $x = 0$ in the original equaiton we would get this answer too.

$$\begin{aligned} y + 2 &= 5x \\ y + 2 &= 5(0) \\ y + 2 &= 0 \\ y &= -2 \end{aligned}$$

Click on the graph below and play with the slider for c. Notice how the graph moves up and down.

6.2.2 The gradient: m

The gradient of a graph is a measure of how much steep the line is. The value of m is the change in the y axis for each increase of 1 in the x axis. So a gradient of $m = 2$ would mean the y values increase by 2 for each increase of 1 in the x direction. This is a positive gradient. Contrast this to a value of m such as -0.5. This means for each increase of 1 in the x direction, the corresponding y value decreases by 0.5 or a half. This is a negative gradient.

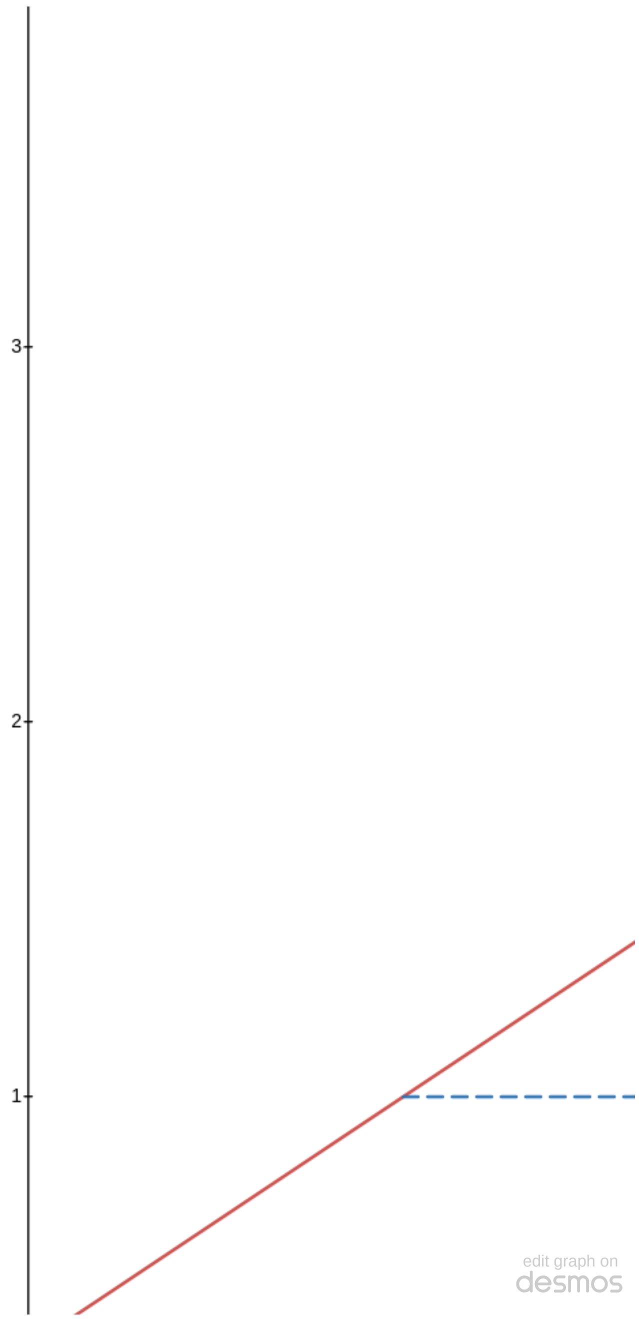

-5
-4
-3
-2
-1
edit graph on
desmos

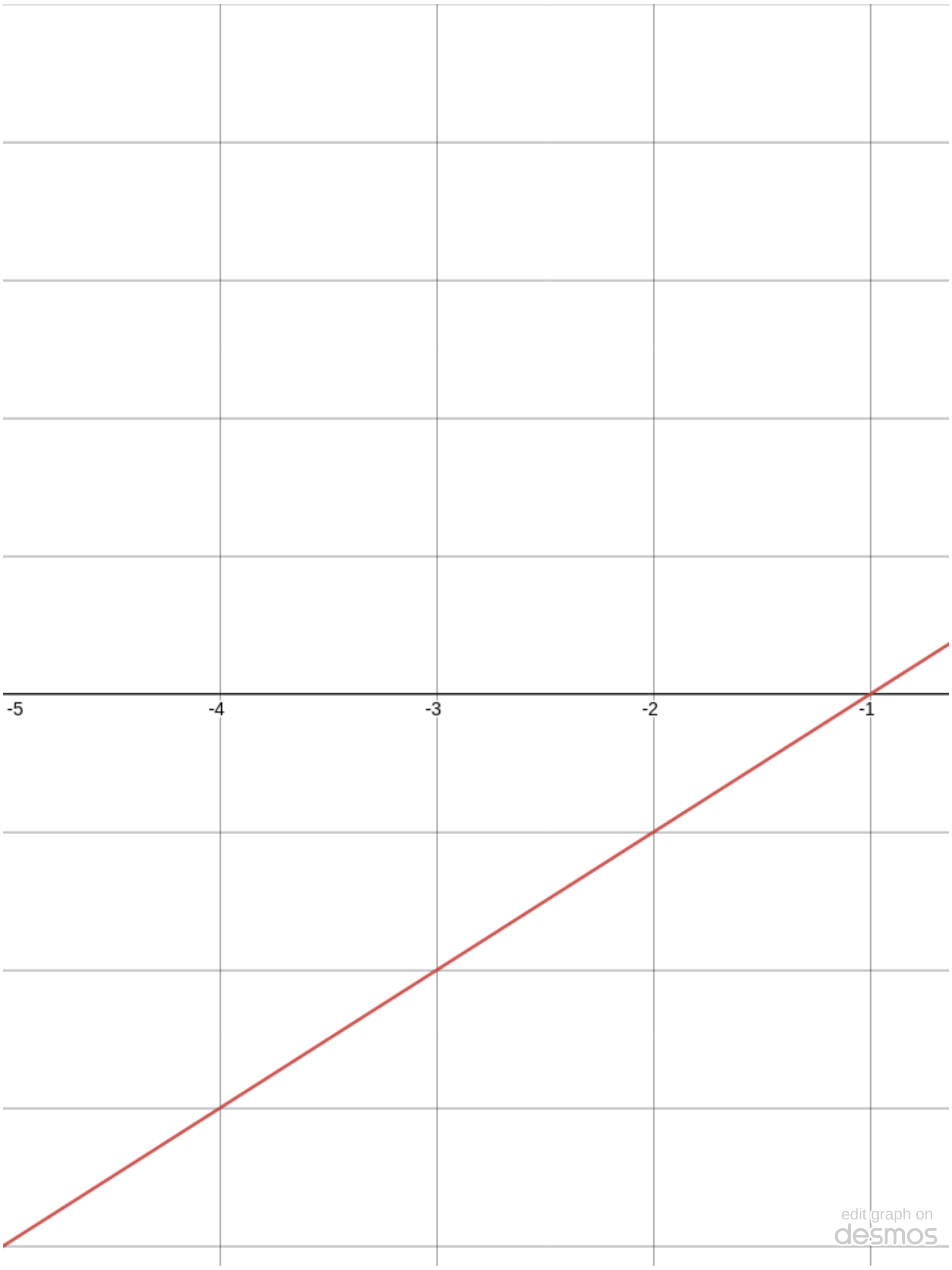
The gradient can also be found by calculating the change in the y direction divided by the change in the x direction. The graph below shows how you could calculate the gradient of the line. The line shown has a gradient of $\frac{2}{3}$.

Pro tip A change in a quantity is often represented by the Greek letter delta, Δ, so we can rewrite m as: $m = \frac{\Delta y}{\Delta x}$

Click on the graph below and then change the value of m with the slider. Notice how the gradient changes but the y intercept stays the same.

Using you knowledge of $y = mx + c$ try the following questions. Don't be afraid to look at the answers and then try a fresh set of questions if it seems tricky at first.


3
2
1
edit graph on
desmos


-5
-4
-3
-2
-1
edit graph on
desmos

Question 1

$y = 2x$

a)

Does this line have a positive or negative gradient?

○ Positive ○ Negative

Score: 0/1

b)

What is the gradient of this line?

______

Score: 0/1

c)

What is the y-intercept of this line?

This is the point where the function intersects the y-axis.

(0, ______)

Score: 0/1

7 Quadratics

Quadratics often appear in mathematics, they occur when you have something squared, like x^2. They produce ‘U’ shaped graphs that can be either way up (depending on the sign of the x^2 term), and, a powerful formula is know that we can use to solve them.

A plot of $y = x^2$ is below:

Quadratics can occur when we expand pairs of brackets, so I’ve included in this section.

7.1 Expanding paris of brackets

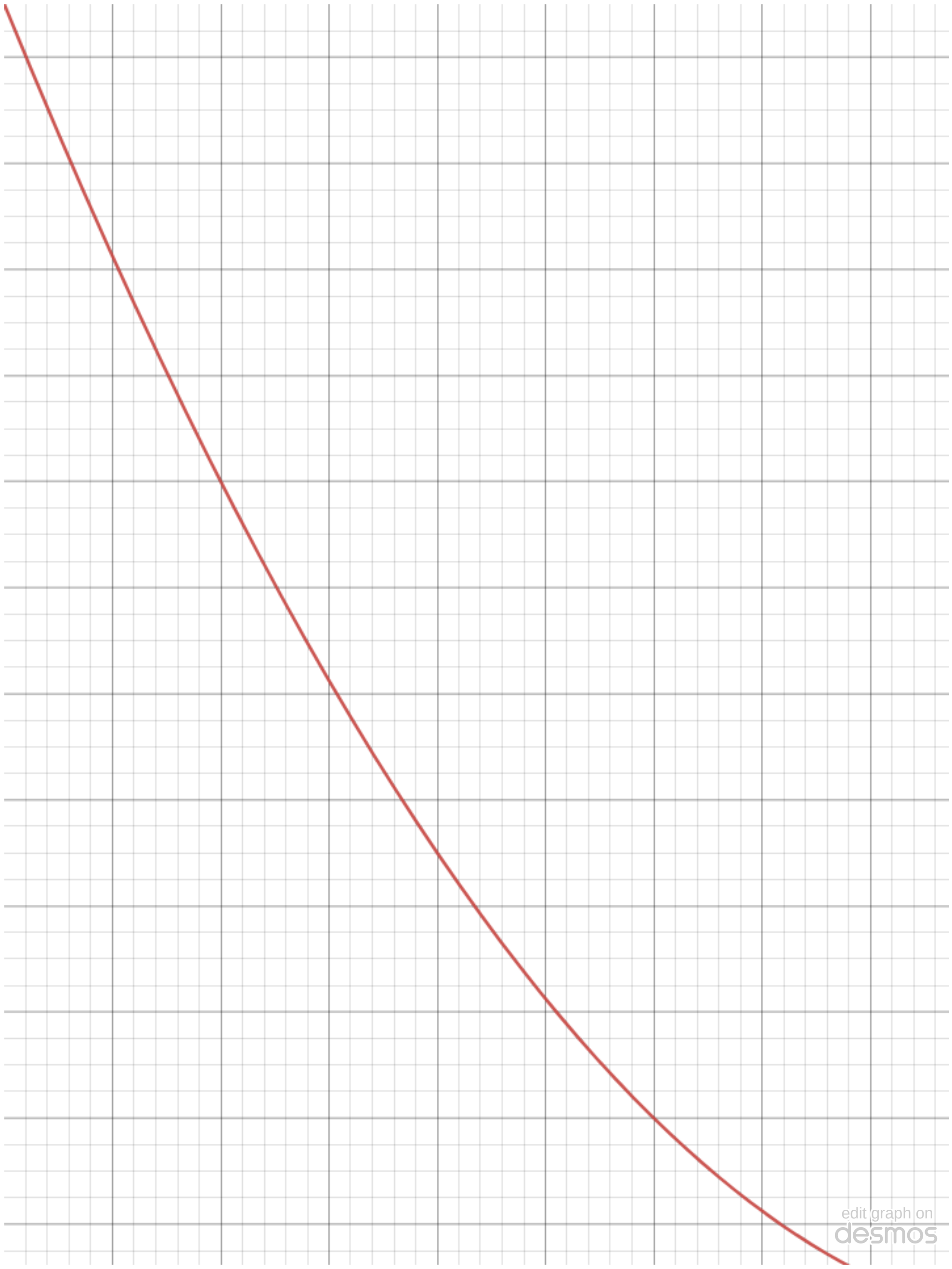
Expanding a pair of brackets is much the same as a single bracket. However there is a little more going on. Consider this example of a mental method to calculate 25×16.

$$\begin{aligned}
25 \times 16 &= (20 + 5) \times (10 + 6) \\
&= \overbrace{20 \times 10 + 20 \times 6}^{20\times(10+6)} + \overbrace{5 \times 10 + 5 \times 6}^{5\times(10+6)} \\
&= 200 + 120 + 50 + 30 \\
&= 400
\end{aligned}$$

With algebra it works in the same way:

$$\begin{aligned}
(a + b)(c + d) &= (a + b) \times (c + d) \\
&= \overbrace{a \times c + a \times d}^{a\times(c+d)} + \overbrace{b \times c + b \times d}^{b\times(c+d)} \\
&= ac + ad + bc + bd
\end{aligned}$$

Here are some practice questions. **NEED to create on NUMBAS placeholder only**


edit graph on
desmos

Question 1

Evaluate the values of these algebraic expressions.

---

a)

If $p = 3$ and $q = -1$ evaluate:

$5p + 3q$

______

Score: 0/1

b)

When $x = 2, y = 2$ and $z = -2$ evaluate

$\frac{-2x^2+5}{yz+5}$

______ *Reduce your answer to lowest terms.*

Score: 0/1

7.2 Factorising pairs of brackets

To factorise a quadratic in the form x^2+bx+c into a pair of brackets like $(x+p)(x+q)$. We look to see if there are a pair of numbers p and q that add to get b, $p+q=b$, and multiply to get c, $pq=c$. If we can find this pair of numbers we can factorise the quadratic. For example for the quadratic $x^2+8x+12$ we can look at the factors of 12 to help us.

$$\begin{aligned} 12 &= 1\times 12, & 1+12 &= 13 \\ 12 &= 2\times 6, & 2+6 &= 8 \\ 12 &= 3\times 4, & 3+4 &= 7 \end{aligned}$$

Notice how 2 and 6 multiply to get 12 but add to get 8. This means we have the correct pair. So we can now factorise the quadratic:

$$x^2+8x+12=(x+2)(x+6)$$

Here are some practice questions.

7.3 Solving Quadratics

Interestingly three things can happen when we solve a quadratic. There can be:

- two different values that satisfy the equation
- one *repeated* value
- no real values (only imaginary ones - and yes that is a thing!)

7.3.1 Factorisation

We can solve some quadratics by factorisation. Take for example the following equation $x^2+8x=-12$. To solve via factorisation we must first make it equal to zero and then factorise. So we have:

$$\begin{aligned} x^2+8x &= -12 \\ x^2+8x+12 &= -12+12 \\ x^2+8x+12 &= 0 \end{aligned}$$

Now, with a little sense of deja vu (see the example in the previous section) we can factorise our quadratic to get $(x+2)(x+6)=0$. Notice that this is one bracket multiplied by another to get the answer zero. When this happens, i.e. when you multiply

Question 1

Factorise the following quadratic expression:

$$x^2 + 7x + 10$$

two numbers and the answer is zero, either the first number is zero or the second one is. This means either $x + 2 = 0$ or $x + 6 = 0$. Solving these two mini-equations gives the two solutions: either $x = -2$ or $x = -6$.

Pro tip We can quickly get from the factorised quadratic to the solutions by *flipping* the signs in the bracket.

Try some questions.

7.3.2 Quadratic Formula

For a quadratic equation of the form $ax^2 + bx + c = 0$ we can use the quardrtic formula to find solutions for x.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We can use the formula on the equation $x^2 - 4x + 2 = 0$. In this example the values of a, b and c are:

$a = 1$ since x^2 means $1 \times x^2$ $b = -4$ notice how the negative sign is *owned* by the x coefficient $c = 2$ finially we just have 2

Substituting into the quadratic formula we have:

$$\begin{aligned} x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)} \\ &= \frac{4 \pm \sqrt{16 - 8}}{2} \\ &= \frac{4 \pm \sqrt{8}}{2} \end{aligned}$$

It is possible to simplify the square roots in this answer to give $2 \pm \sqrt{2}$. So don't be surprised if your calculator gives you that answer.

Finally, we must deal with the $\pm$ symbol. This means do the calculation once using $+$ and another time using $-$. This will give two possible answers for x, given to 2 decimal places.

$$\begin{aligned} x_1 &= \frac{4 + \sqrt{8}}{2} \\ &= 3.41 \end{aligned}$$

and

Question 1

Solve the following quadratic equation by factorisation or otherwise:

$$x^2 + 9x + 8 = 0$$

$x =$ ______ *Enter a list of numbers separated by , .*

$$x_2 = \frac{4 - \sqrt{8}}{2}$$
$$= 0.59$$

Pro tip Notice the use of x_1 and x_2. It is common in maths to use subscript numbers to show different particular values of the same variable. That's all it's doing x_1 is just a value for x named x_1 and x_2 is just a value for x named x_2.

7.4 Simultaneous equations

We are going to solve this type of equation by substitution i.e. substituting one equation into another.

To solve a pair of simultaneous equations of this type we want to rearrange the linear equation such that it is in terms of x or y, which we can then substitute into the equation with the quadratic terms. This will result in a quadratic equation in terms of one variable only.

For the equations:

$$\begin{aligned} 2x + y &= 1 \quad (1) \\ 3x^2 + 3y^2 &= 4 \quad (2) \end{aligned}$$

we can rearrange equation (1) to make y the subject:

$$y = 1 - 2x \qquad (3)$$

Substituting equation (3) into equation (2) we have:

$$\begin{aligned} 3x^2 + 3y^2 &= 4 \\ 3x^2 + 3(1 - 2x)^2 &= 4 \\ 3x^2 + 3(1 - 4x + 4x^2) &= 4 \\ 3x^2 + 3 - 12x + 12x^2 &= 4 \\ 15x^2 - 12x - 1 &= 0 \end{aligned}$$

There are a few things to be careful of here:

- $(1-2x)^2$ was expanded as a pair of brackets, $(1-2x)(1-2x)$ before being multiplied by 3.
- The finial stage was to make the equation equal zero so we can use the quadratic formula.

Question 1

Using the quadratic formula, solve the following quadratic equation:

$$3x^2 + 2x - 4 = 0$$

$x =$ ______ *Enter a list of numbers separated by , .*

Give your answers to 3 decimal places where necessary.

Now we have an equation we can solve we can use the quadratic formula. To find values of x. This gives two solutions $x_1 = -0.08$ to 2 decimal places, and, $x_2 = -0.88$ again to 2 decimal places.

Finally, since our equations for x and y we need to find corresponding y values for each x. The easiest way to do this is to use equation (3). This gives, $ y_1 =1.15 $ and $y_2 = -0.75$. Note, to maintain accuracy you'll need to put your *full* values for x_1 and x_2 into equation (3) and then round to 2 decimal places afterwards.

This gives two pairs of numbers for our answer. $(x_1, y_1) = (-0.08, 1.15)$ and $(x_2, y_2) = (0.88, -0.75)$.

Pro tip notice our answers look a lot like co-ordinates on a graph. That's because they are. If you plot the lines $2x + y = 1$ and $3x^2 + 3y^2 = 4$ on the same graph (don't do this by hand! Use something like desmos) the places where the two lines cross will correspond with our answers.

Here are some practice questions. Don't forget you can graph them if it helps.

Question 1

Solve the following simultaneous equations:

$$\begin{aligned} 4x + y &= 3 \\ 4x^2 + 3y^2 &= 36 \end{aligned}$$

Give your answers to 2 decimal places where necessary.

$(x_1, y_1) =$ ______ ______

$(x_2, y_2) =$ ______ ______

Created using Numbas (https://www.numbas.org.uk), developed by Newcastle University (http://www.newcastle.ac.uk).

Typesetting math: 100%

8 Indices

Indices is another word for powers. In this section we move beyond the idea that powers are just repeated multiplications.

8.1 Index notation

Being comfortable moving between different ways to write powers helps when rearranging algebra.

$x^0 = 1$ except when $x = 0$ then it's undefined
$x^-n = \frac{1}{x^n}$
$x^{\frac{1}{n}} = \sqrt[n]{x}$

Here are some examples:

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

More generally.

$$x^{-3} = \frac{1}{x^3}$$

Anything to the power of zero is 1:

$$\pi^0 = 1$$

Remember good old π? From working stuff out about circles $\pi = 3.14159...$

We can write roots too:

$$16^{\frac{1}{2}} = \sqrt{16} = \pm 4$$

Pro tip When taking square roots remember there are two possible solutions. Since in the above example $4 \times 4 = 16$ and $-4 \times -4 = 16$. So either answer is just fine.

8.1.1 But why?

Just like we did with negative numbers we can extend the idea of what a power means by following a pattern. Here's a pattern to justify $x^0 = 1$ and $x^{-n} = \frac{1}{x^n}$.

$$
\begin{aligned}
10^3 &= 10 \times 10 \times 10 &= 1000 \\
10^2 &= 10 \times 10 &= 100 \\
10^1 &= 10 &= 10 \\
10^0 &= 1 &= 1 \\
10^{-1} &= \frac{1}{10} &= 0.1 \\
10^{-2} &= \frac{1}{10 \times 10} &= 0.01 \\
10^{-3} &= \frac{1}{10 \times 10 \times 10} &= 0.001
\end{aligned}
$$

I'll come back to the justification about square roots after the next section.

8.2 Rules of indices

There is a neat set of rules we can use when combining numbers with indices:

$$
\begin{aligned}
&x^n \times x^m = x^{n+m} \\
&x^n \div x^m = x^{n-m} \\
&(x^n)^m = x^{n \times m}
\end{aligned}
$$

When you multiply terms you add the powers.

$$
\begin{aligned}
3x^4 \times 5x^6 &= 3 \times 5 \times x^4 \times x^5 \\
&= 15 \times x^{4+5} \\
&= 15x^9
\end{aligned}
$$

Lets put it all together with a complicated example:

To rewrite $\dfrac{\sqrt[4]{x^5 x^3}}{\sqrt[3]{x}\sqrt[6]{x^3}}$ in the form x^n, we need to use the following rules:

1. $a^n a^m = a^{n+m}$;
2. $\sqrt[n]{a} = a^{1/n}$;
3. $\left(a^n\right)^m = a^{n \times m}$;
4. $\frac{a^n}{a^m} = a^{n-m}$.

We will simplify the numerator and denominator separately to make the steps clearer. Firstly, applying rule 1, then rule 2, and then rule 3 to the numerator:

$$\begin{aligned}\frac{\sqrt[4]{x^5x^3}}{\sqrt[3]{x}\sqrt[6]{x^3}} &= \frac{\sqrt[4]{x^8}}{\sqrt[3]{x}\sqrt[6]{x^3}} \\ &= \frac{(x^8)^{1/4}}{\sqrt[3]{x}\sqrt[6]{x^3}} \\ &= \frac{x^2}{\sqrt[3]{x}\sqrt[6]{x^3}}\end{aligned}$$

To simplify the denominator, we want to apply rule 2, then rule 3, and then rule 1:

$$\begin{aligned}\frac{x^2}{\sqrt[3]{x}\sqrt[6]{x^3}} &= \frac{x^2}{x^{1/3}(x^3)^{1/6}} \\ &= \frac{x^2}{x^{1/3}x^{1/2}} \\ &= \frac{x^2}{x^{5/6}}\end{aligned}$$

Remember that we'll need to get common denominators when adding the fractions at the end:

$$\begin{aligned}\frac{1}{3} + \frac{1}{2} &= \frac{1 \times 2}{3 \times 2} + \frac{1 \times 3}{2 \times 3} \\ &= \frac{2}{6} + \frac{3}{6} \\ &= \frac{5}{6}\end{aligned}$$

Finally, applying rule 4 and simplifying,

$$\begin{aligned}\frac{x^2}{x^{5/6}} &= x^2 \times x^{-5/6} \\ &= x^{2-5/6} \\ &= x^{12/6-5/6} \\ &= x^{7/6}\end{aligned}$$

Lots of work with fractions here!

Now try these questions. Don't worry if it takes a while to just solve one!

Question 1

Rewrite the following expression as a single term, in the form x^n, where n is a fraction.

$$\frac{\sqrt[4]{x^5 x^3}}{\sqrt[3]{x}\,\sqrt[6]{x^3}}$$

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8.2.1 But why? Square roots

As promised here is an explanation of why $x^{\frac{1}{n}} = \sqrt[n]{x}$.

When we take a square root we look for the a number that when it is multiplied by it's self we get the answer i.e. $? \times ? = x$. Since one x is the same as x^1 we can rewrite out statement again:

$$\begin{aligned} ? \times ? &= x^1 \\ x^? \times x^? &= x^1 \\ x^{?+?} &= x^1 \end{aligned}$$

This means $? + ? = 1$ so $? = \frac{1}{2}$ so $x^{\frac{1}{2}} = \sqrt{x}$.

9 Exponetial function

10 Logarithms

10.1 Reverse of indices

10.2 Rules of logarithms

11 Differentiation