Exercise 1.

(1) $$\underbrace{y^2}_{A}\frac{\partial^2 u}{\partial x^2}\underbrace{-2xy}_{2B}\frac{\partial^2 u}{\partial x\partial y}+\underbrace{x^2}_{C}\frac{\partial^2 u}{\partial y^2}=\frac{1}{xy}\left(y^3\frac{\partial u}{\partial x}+x^3\frac{\partial u}{\partial y}\right)$$

$B^2-AC = x^2y^2-x^2y^2=0 \Rightarrow$ Parabolic

On $\xi =$ constant, $$\frac{dy}{dx}=\frac{B}{A}=\frac{-xy}{y^2}=-\frac{x}{y} \Leftrightarrow x^2+y^2 = \text{constant} = \xi$$

Choose $\eta = y$ (independent of ξ)

We have $\frac{\partial \xi}{\partial x}=2x$, $\frac{\partial \xi}{\partial y}=2y$ $\quad \frac{\partial \eta}{\partial x}=0$, $\frac{\partial \eta}{\partial y}=1$

$$\frac{\partial u}{\partial x}=2x\frac{\partial u}{\partial \xi}+0 \qquad \frac{\partial u}{\partial y}=2y\frac{\partial u}{\partial \xi}+\frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2}=2\frac{\partial u}{\partial \xi}+2x\left(u_{\xi\xi}\cdot 2x+u_{\xi\eta}\cdot 0\right) \qquad \frac{\partial^2 u}{\partial y^2}=2u_\xi+2y\left(u_{\xi\xi}\cdot 2y+u_{\xi\eta}\right)+u_{\eta\xi}\,2y+u_{\eta\eta}$$

$$\frac{\partial^2 u}{\partial x\partial y}=2x\left(u_{\xi\xi}\cdot 2y+u_{\xi\eta}\right)$$

Replacing this in the original equation, gives us

$$x^2\frac{\partial^2 u}{\partial \eta^2}=\frac{x^2}{y}\frac{\partial u}{\partial \eta} \Leftrightarrow \frac{\partial^2 u}{\partial \eta^2}-\frac{1}{\eta}\frac{\partial u}{\partial \eta}=0 \quad \text{(canonical form)}$$

$$\Leftrightarrow \frac{\partial}{\partial \eta}\left(\frac{1}{\eta}\frac{\partial u}{\partial \eta}\right)=0$$

$$\Leftrightarrow \frac{1}{\eta}\frac{\partial u}{\partial \eta}=f(\xi) \Leftrightarrow \frac{\partial u}{\partial \eta}=\eta f(\xi)$$

$$\Leftrightarrow u=\frac{\eta^2}{2}f(\xi)+g(\xi)$$ with f, g arbitrary functions

$$u = g(x^2+y^2)+\frac{y^2}{2}f(x^2+y^2)$$