PW/TP 15-16 Characteristics of second-order Linear PDE's

The general *linear partial differential equation* of order two in two independent variables has the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + 2B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + D(x,y)\frac{\partial u}{\partial x} + E(x,y)\frac{\partial u}{\partial y} + F(x,y)u = G(x,y)$$

where $A, B, \ldots, G$ may depend on x and y but not on u.

Characteristics and normal form of second-order linear PDE's

Calculate $B^2 - \mathrm{AC}$.

1. $B^2 - \mathrm{AC} > 0$**: hyperbolic curve**

The PDE can be reduced to $\frac{\partial^2 u}{\partial \xi \partial \eta} + (\text{lower order terms}) = 0$.

The characteristics $\xi(x,y) = \text{constant}$, and $\eta(x,y) = \text{constant}$ satisfy $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a}$.

2. $B^2 - \mathrm{AC} = 0$**: parabolic curve**

The PDE can be reduced to $\frac{\partial^2 u}{\partial \eta^2} + (\text{lower order terms}) = 0$.

The characteristics $\xi(x,y) = \text{constant}$ satisfy $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b}{a}$.

$\eta(x,y)$ can be choosen arbitrarily, provided that ξ and η are independent (thus $\xi_x \eta_y - \xi_y \eta_x \neq 0$).

3. $B^2 - \mathrm{AC} < 0$**: elliptic curve**

The PDE can be reduced to $\frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2} + (\text{lower order terms}) = 0$.

We known $\xi(x,y) = \text{constant}$, and $\eta(x,y) = \text{constant}$ satisfy $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b \pm \sqrt{b^2 - ac}}{a}$.

Apply a further change of variables $(\xi, \eta) \rightarrow (\alpha = \xi + \eta, \beta = i(\xi - \eta))$ to find the canonical form

We have now real characteristics.

Exercise 1.

Investigate the following PDE's:

1. $y^2 \frac{\partial^2 u}{\partial x^2} - 2xy \frac{\partial^2 u}{\partial x \partial y} + x^2 \frac{\partial^2 u}{\partial y^2} = \frac{1}{xy}(y^3 \frac{\partial u}{\partial x} + x^3 \frac{\partial u}{\partial y})$
2. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - 2\frac{\partial^2 u}{\partial y^2} + 1 = 0 \text{ in } 0 \leq x \leq 1, y > 0 \text{ with } u = \frac{\partial u}{\partial y} = x \text{ on } y = 0$
3. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$

4. $u_{xx} + 4u_{xy} + u_x = 0$
5. $x^2u_{xx} - 2xyu_xy + y^2u_{yy} + xu_x + yu_y = 0$, $\forall x > 0$
6. $u_{xx} + xu_{yy} = 0$, $\forall x > 0$