

Exercise 7 (2.56)

$$(a) x^2 + cy^2 = 1$$

$$2x + 2cy y' = 0$$

$$c = -\frac{x}{y y'}$$

$$x^2 - \frac{x y^2}{y y'} = 1 \quad \Leftrightarrow \quad x^2 y' - x y = y'$$

$$\Leftrightarrow \boxed{0 = (1 - x^2) y' + x y}$$

$$(b) y^2 = ax + b$$

$$2yy' = a \quad \Leftrightarrow \quad y^2 = 2yy'x + b$$

$$2yy' = 2y'^2 x + 2yy''x + 2yy'$$

$$0 = y'^2 x + yy''x$$

$$\boxed{0 = y'^2 + yy''}$$

$$1) (2.59 b) x^3 y''' = 1 + x^4; \quad y(1) = y'(1) = y''(1) = 0$$

\Rightarrow separable

$$\int dy'' = \int \frac{1+x^4}{x^3} dx$$

$$y'' = -\frac{1}{2x^2} + \frac{x^2}{2} + C_1 \quad y''(1) = 0$$

$$0 = -\frac{1}{2} + \frac{1}{2} + C_1 \Rightarrow C_1 = 0$$

$$\int dy' = \int \left(-\frac{1}{2x^2} + \frac{x^2}{2} \right) dx$$

$$y' = -\frac{1}{2} \left(-\frac{1}{x} - \frac{x^3}{3} + C_2 \right) \quad y'(1) = 0$$

$$0 = -\frac{1}{2} \left(-1 - \frac{1}{3} + C_2 \right)$$

$$\int dy = \int \left(\frac{1}{2x} + \frac{x^3}{6} - \frac{2}{3} \right) dx \quad C_2 = \frac{4}{3}$$

$$y = \frac{1}{2} \ln x + \frac{x^4}{24} - \frac{2}{3} x + C_3 \quad y(1) = 0$$

$$0 = 0 + \frac{1}{24} - \frac{2}{3} + C_3$$

$$C_3 = \frac{15}{24} = \frac{5}{8}$$

$$y = \frac{1}{2} \ln x + \frac{x^4}{24} - \frac{2}{3} x + \frac{5}{8}$$

$$3) (2.67 \text{ b}) \quad 2x^2 y' = xy + y^3$$

$$\frac{dy}{dx} - \frac{1}{2x} y = \frac{1}{2x^2} y^3$$

$$n=3 \\ v = y^{-2}$$

$$\frac{dv}{dx} = -2y^{-3} \frac{dy}{dx}$$

$$-\frac{1}{2} y^3 \frac{dv}{dx} - \frac{1}{2x} y = \frac{1}{2x^2} y^3$$

$$\frac{dy}{dx} \hat{=} -\frac{1}{2} y^3 \frac{dv}{dx}$$

$$\frac{dv}{dx} + \frac{y^{-2}}{x} = -\frac{1}{x^2}$$

$$\frac{dv}{dx} + \frac{1}{x} v = -\frac{1}{x^2}$$

$$\mu = e^{\int \frac{1}{x} dx} = x$$

$$x \cdot v = \int x \cdot \left(-\frac{1}{x^2}\right) dx + C$$

$$x \cdot v = -\ln x + C \quad (\text{with } v = y^{-2})$$

$$\frac{x}{y^2} = -\ln x + C$$

$$\boxed{e^{x/y^2} = C x^{-1}}$$

$$4) (2.72 a) \quad xy'' - 3y' = x^2$$

$$y \text{ is missing : } y' = p \quad y'' = \frac{dp}{dx}$$

$$x \frac{dp}{dx} - 3p = x^2$$

$$\frac{dp}{dx} - \frac{3}{x} p = x \quad (= \text{linear first order equation})$$

$$\mu = e^{-\int \frac{3}{x} dx} = e^{-3 \ln x} = x^{-3}$$

$$x^{-3} p = \int x^{-3} \cdot x \, dx + c_1 = -\frac{1}{x} + c_1$$

$$\frac{dy}{dx} = p = -x^2 + c_1 x^3$$

$$\boxed{y = -\frac{1}{3} x^3 + \frac{c_1}{4} x^4 + c_2}$$

$$5) (2.61 b) \quad \frac{dy}{dx} = \frac{3 - 4xy^2}{4x^2y + 6y^2} ; y(1) = -1$$

$$\underbrace{(4x^2y + 6y^2)}_M dy + \underbrace{(4xy^2 - 3)}_N dx = 0$$

$$\frac{\partial M}{\partial x} = 8xy \quad \frac{\partial N}{\partial y} = 8xy$$

$$\int M dy = \int (4x^2y + 6y^2) dy = 2x^2y^2 + 2y^3$$

$$\frac{\partial}{\partial x} (2x^2y^2 + 2y^3) = 4xy^2$$

$$\int (N - 4xy^2) dx = \int (-3) dx = -3x$$

$$2x^2y^2 + 2y^3 - 3x = C \quad \text{with } y(1) = -1 \Leftrightarrow C = -3$$

$$\boxed{2x^2y^2 + 2y^3 - 3x = -3}$$

$$6) (2.70 \text{ a}) \quad y'^2 + (y-1)y' - y = 0$$

$$y'^2 + yy' - y' - y = 0$$

$$y(y'-1) = y' - y'^2$$

$$y = \frac{y'(1-y')}{y'-1} = -y' \quad \text{if } y'-1 \neq 0$$

$$y = -y'$$

$$\int dx = - \int \frac{1}{y} dy$$

$$x + c_1 = -\ln y$$

$$y = c_2 e^{-x}$$

$$y' = 1$$

$$y = x + c_1$$

$$\Leftrightarrow (y - x + c_1)(y - c_2 e^{-x}) = 0$$

$$7) (2.65 \text{ b}) \quad xy' - 4y = x$$

$$y' - \frac{4}{x}y = 1$$

$$\mu = e^{\int -\frac{4}{x} dx} = x^{-4}$$

$$x^{-4}y = \int x^{-4} dx + c$$

$$y = -\frac{1}{3}x^{-3} \cdot x^4 + cx^4$$

$$3y = -x + cx^4$$

$$8) (2.63 \text{ a}) \quad (3y - 2xy^3) dx + (4x - 3x^2y^2) dy = 0$$

$$\frac{\partial}{\partial y} = 3 - 6xy^2 \neq \frac{\partial}{\partial x} = 4 - 6xy^2$$

$$\cdot x^p y^{-q} \quad \underbrace{\frac{3y - 2xy^3}{x^p y^q}}_N dx + \underbrace{\frac{4x - 3x^2y^2}{x^p y^q}}_M dy = 0$$

$$\frac{\partial N}{\partial y} = 3x^{-p}y^{-q}(1-q) - 2x^{1-p}y^{2-q}(3-q)$$

$$\frac{\partial M}{\partial x} = 4(1-p)x^{-p}y^{-q} - 3(2-p)x^{1-p}y^{2-q}$$

$$\begin{cases} 3(1-q) = 4(1-p) \\ -2(3-q) = -3(2-p) \end{cases} \Leftrightarrow \begin{cases} p = -2 \\ q = -3 \end{cases}$$

$$(3x^3y^4 - 2x^3y^6) dx + (4x^3y^3 - 3x^4y^5) dy = 0$$

$$\frac{\partial}{\partial y} = 12x^3y^3 - 12x^3y^5 \quad \frac{\partial}{\partial x} = 12x^2y^3 - 12x^3y^5$$

$$U = \int 3x^3y^4 - 2x^3y^6 dx = x^3y^4 - \frac{1}{2}x^4y^6 + F(y)$$

$$\frac{\partial U}{\partial y} = 4x^3y^3 - 3x^4y^5 + F'(y) = 4x^3y^3 - 3x^4y^5 \Leftrightarrow F'(y) = 0$$

$$\boxed{C = x^3y^4 - \frac{1}{2}x^4y^6}$$

9) (2.71 a) $y = px + 2p^2$, with $p = y'$

$$\frac{dy}{dx} = \frac{dp}{dx} \cdot x + p + 4p \frac{dp}{dx}$$

"
p

$$x \frac{dp}{dx} + 4p \frac{dp}{dx} = 0$$

$$(x + 4p) \frac{dp}{dx} = 0$$

Case 1, $x + 4p \neq 0$. $\frac{dp}{dx} = 0$

$$y' = p = c$$

$$\boxed{y = cx + 2c^2} \quad (\text{general})$$

Case 2, $x + 4p = 0$. $p = -\frac{x}{4}$

$$\boxed{y = -\frac{1}{8}x^2}$$

$$10) (2.62 d) (2y \sin x + 3y^4 \sin x \cos x) dx - (4y^3 \cos^2 x + \cos x) dy = 0$$

$$\frac{\partial}{\partial y} = 2 \sin x + 12y^3 \sin x \cos x \quad \frac{\partial}{\partial x} = 2 \cos x \sin x \cdot 4y^3 + \sin x$$

$$R(x) = \frac{-1}{4y^3 \cos^2 x + \cos x} \left(\sin x + 4y^3 \sin x \cos x \right)$$

$$= -\frac{1}{\cos x} \cdot \sin x \left(\frac{1 + \cancel{4y^3 \cos x}}{4y^3 \cos x + 1} \right) = -\tan x$$

$$F(x) = e^{-\int \tan x dx} \quad -\int \frac{\sin x}{\cos x} dx \quad \begin{array}{l} \cos x = t \\ dt = -\sin x dx \end{array}$$

$$= \int \frac{1}{t} dt = \ln(t)$$

$$= \cos x$$

$$\Rightarrow (2y \sin x \cos x + 3y^4 \sin x \cos^2 x) dx - (4y^3 \cos^3 x + \cos^2 x) dy = 0$$

$$\frac{\partial}{\partial y} = 2 \sin x \cos x + 12y^3 \sin x \cos^2 x$$

$$\frac{\partial}{\partial x} = 12y^3 \cos^2 x \cdot \sin x + 2 \cos x \sin x \quad \checkmark$$

$$\int 2y \sin x \cos x + 3y^4 \sin x \cos^2 x dx = -\int (2y t + 3y^4 t^2) dt$$

$$\cos x = t \rightarrow dt = -\sin x dx$$

$$= -y t^2 - y^4 t^3 + F(y)$$

$$\frac{\partial}{\partial y} (-y \cos^2 x - y^4 \cos^3 x + F(y)) = -\cos^2 x - 4y^3 \cos^3 x + F'(y)$$

$$= -4y^3 \cos^3 x - \cos^2 x$$

$$\Rightarrow F'(y) = 0 \Leftrightarrow F(y) = C$$

$$\boxed{C = -y \cos^2 x - y^4 \cos^3 x}$$

$$11) (2.66 a) \quad \frac{dy}{dx} = \frac{2y}{x} - \frac{y^2}{x^2}$$

$$\left(v = \frac{y}{x}\right) \quad v + x \frac{dv}{dx} = 2v - v^2$$

$$\int \frac{1}{v-v^2} dv = \int \frac{1}{x} dx$$

$$\int \left(\frac{1}{v} + \frac{1}{1-v}\right) dv = \ln(v) - \ln(1-v) = \ln x + C$$

$$\frac{v}{1-v} = Cx$$

$$\frac{y/x}{1-y/x} = \frac{y}{x-y} = Cx$$

$$y = Cx^2 - Cxy$$

$$\boxed{Cy = x^2 - xy}$$

$$12) (2.66 d) \quad (x-y)y' + 3y - 5x = 0$$

$$\frac{dy}{dx} = \frac{5x-3y}{x-y} \quad (y=vx) \rightarrow v + x \frac{dv}{dx} = \frac{5-3v}{1-v}$$

$$x \frac{dv}{dx} = \frac{5-4v+v^2}{1-v}$$

$$\int \frac{1-v}{5-4v+v^2} dv = \int \frac{1}{x} dx$$

$$\int \frac{2v-4}{5-4v+v^2} dv + \int \frac{2}{\underbrace{5-4v+v^2}_{(v-2)^2+1}} dv = -2 \int \frac{1}{x} dx$$

$$\ln(5-4v+v^2) + 2 \tan^{-1}(v-2) = -2 \ln x + C$$

$$\ln\left(5-4\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2\right) + \ln(x^2) + 2 \tan^{-1}\left(\frac{y}{x}-2\right) = C$$

$$\boxed{\ln(y^2 - 4xy + 5x^2) + 2 \tan^{-1}\left(\frac{y-2x}{x}\right) - C}$$

$$13) (2.72c) \quad y'' + 4y = 0$$

missing x : $y' = p \quad y'' = \frac{dp}{dx} = \frac{dp}{dy} \frac{dy}{dx} = p \frac{dp}{dy}$

$$p \frac{dp}{dy} + 4y = 0$$

$$p \frac{dp}{dy} = -4y$$

$$\int p dp = \int -4y dy$$

$$\frac{p^2}{2} = -2y^2 + C_1$$

$$\frac{dy}{dx} = p = \sqrt{2C_1 - 4y^2} = 2\sqrt{\frac{C_1}{2} - y^2}$$

$$\int \frac{1}{\sqrt{\frac{C_1}{2} - y^2}} dy = \int 2 dx$$

$$\sin^{-1}\left(\frac{y}{C_1/2}\right) = 2x + C_2$$

$$y = \frac{2}{C_1} \sin(2x + C_2)$$

$$y = C_1 \sin(2x + C_2)$$

$$14) (2.69) \quad \frac{dy}{dx} = x^2 + 2xy + y^2 + 2x + 2y; \quad y(0) = 0$$

$$\frac{dy}{dx} = (x+y)^2 + 2(x+y)$$

$$\frac{du}{dx} - 1 = u^2 + 2u$$

$$u = x + y \quad \frac{du}{dx} = 1 + \frac{dy}{dx}$$

$$\int \frac{1}{(u+1)^2} du = \int dx$$

$$-\frac{1}{u+1} = x + C$$

$$-\frac{1}{x+y+1} = x + C \quad \text{with } y(0) = 0 \Leftrightarrow C = -1$$

$$1 = (1-x)(x+y+1)$$

$$15) (2.71 d) \quad x^2 y = x^3 p - y p^2, \text{ with } p = y'$$

$$x^2 y + p^2 y = x^3 p$$

$$y = \frac{x^3 p}{x^2 + p^2}$$

$$p = \frac{dy}{dx} = \frac{(x^2 + p^2) \frac{d(x^3 p)}{dx} - x^3 p \frac{d(x^2 + p^2)}{dx}}{(x^2 + p^2)^2}$$

$$p(x^2 + p^2)^2 = (x^2 + p^2) \left(3x^2 p + x^3 \frac{dp}{dx} \right) - x^3 p (2x + 2p \frac{dp}{dx})$$

$$p(x^2 + p^2)^2 - (x^2 + p^2) 3x^2 p + 2x^4 p = ((x^2 + p^2) x^3 - 2x^3 p^2) \frac{dp}{dx}$$

$$p \left((x^2 + p^2) \underbrace{(x^2 + p^2 - 3x^2)}_{p^2 - 2x^2} + 2x^4 \right) = (x^5 - x^3 p^2) \frac{dp}{dx}$$

$$p \left(\underbrace{p^2 x^2 + p^4 - 2x^4 - 2p^2 x^2 + 2x^4}_{p^4 - p^2 x^2} \right) - x^3 (x^2 - p^2) \frac{dp}{dx} = 0$$

$$p^3 (p^2 - x^2) + x^3 (p^2 - x^2) \frac{dp}{dx} = 0$$

$$(p^2 - x^2) \left(p^3 + x^3 \frac{dp}{dx} \right) = 0$$

$$\text{Case 1, } p^2 - x^2 = 0. \quad p = \pm x$$

$$x^2 y = x^3 (\pm x) - y x^2 \quad \Rightarrow \quad \boxed{y = \pm \frac{x^2}{2}} \quad (\text{singular})$$

$$2x^2 y = \pm x^4$$

$$\text{Case 2, } p^3 + x^3 \frac{dp}{dx} = 0$$

$$dp^2 = \frac{2x^2}{2cx^2 - 1} \quad p = \frac{\pm x}{\sqrt{2cx^2 - 1}}$$

$$\frac{1}{x^3} dx = - \frac{1}{p^3} dp$$

$$-\frac{1}{2x^2} + c = \frac{1}{2p^2}$$

$$p = \frac{\pm x}{\sqrt{2cx^2-1}} \quad x^2 y = x^3 p - y p^2$$

$$x^2 y = \frac{\pm x^4}{\sqrt{2cx^2-1}} - \frac{y x^2}{2cx^2-1}$$

$$x^2 y + \frac{x^2 y}{2cx^2-1} = \frac{\pm x^4}{\sqrt{2cx^2-1}}$$

$$\frac{x^2 y}{2cx^2-1} (2cx^2) = \frac{\pm x^4}{\sqrt{2cx^2-1}}$$

$$\frac{2cy}{2cx^2-1} = \frac{\pm 1}{\sqrt{2cx^2-1}}$$

$$\frac{4c^2 y^2}{(2cx^2-1)^2} = \frac{1}{2cx^2-1}$$

$$4c^2 y^2 = 2cx^2 - 1$$

$$y^2 = \frac{x^2}{2c} - \frac{1}{4c^2} \quad (c = \frac{1}{2c})$$

$$\boxed{y^2 = cx^2 - c^2} \quad (\text{general})$$

$$16) (2.65 d) \quad \frac{dy}{dx} + 2y \cot x = \csc x$$

$$\mu = e^{\int 2 \cot x \, dx} = \sin^2 x$$

$$\sin^2 x \cdot y = \int \sin^2 x \cdot \frac{1}{\sin x} \, dx + C$$

$$= -\cos x + C$$

$$\boxed{y = -\cos x \cdot \sin^{-2} x + C \cdot \sin^{-2} x}$$

$$17) (2.60 d) \frac{dy}{dx} = \frac{x\sqrt{1-y^2}}{y\sqrt{1-x^2}}$$

$$\int \frac{y}{\sqrt{1-y^2}} dy = \int \frac{x}{\sqrt{1-x^2}} dx$$

$$1-y^2 = t$$

$$-2y dy = dt$$

$$-\frac{1}{2} \int \frac{1}{\sqrt{t}} dt = -\frac{1}{2} \int \frac{1}{\sqrt{u}} du$$

$$\sqrt{1-y^2} = \sqrt{1-x^2} + C$$

$$\boxed{\sqrt{1-x^2} - \sqrt{1-y^2} = C}$$

$$18) (2.61 c) (ye^x - e^{-y})dx + (xe^{-y} + e^x)dy = 0$$

$$\frac{\partial (ye^x - e^{-y})}{\partial y} = e^x + e^{-y}$$

$$\frac{\partial (xe^{-y} + e^x)}{\partial x} = e^{-y} + e^x$$

$$\int ye^x - e^{-y} dx = ye^x$$

$$\frac{\partial (ye^x)}{\partial y} = e^x$$

$$\int (xe^{-y} + e^x - e^x) dy = -xe^{-y}$$

$$\boxed{ye^x - xe^{-y} = C}$$