PW/TP 3-4: Ordinary Differential Equations - Applications (CH2)

Solutions

Exercise 1. (2.74) An object moves along the x axis, acted upon by a constant force. If its initial velocity in the positive direction is 40 meters/sec while 5 seconds later it is 20 meters/sec, find (a) the velocity at any time, (b) the position at any time assuming the object starts from the origin $x = 0$.

Net force: $F = m \cdot a$

$$\frac{d^2x}{dt^2} = \frac{F}{m} = constant$$

```
syms v(t) t Fm
cond1 = v(0) == 40;
cond2 = v(5) == 20;
eq = diff(v,t) == Fm;
dsolve(eq,cond1)
```

ans = $\mathrm{Fm}\, t + 40$

$v(t) = \frac{F}{m} \cdot t + 40$, using the second condition to find the value of the constant $\frac{F}{m}$.

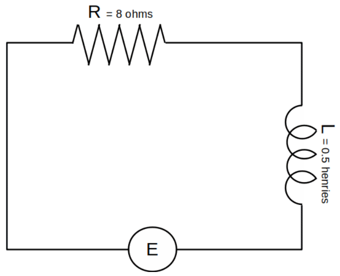
$v(t) = -4 \cdot t + 40$

```
syms x(t)
cond1 = x(0) == 0;
eq = diff(x,t) == -4*t+40;
dsolve(eq,cond1)
```

ans = $-2\,t\,(t - 20)$

$x(t) = -2t^2 + 40$.

Exercise 2. (2.77) An electric circuit contains an 9 ohm resistor in series with an inductor of 0.5 henries and a battery of E volts. At $t = 0$ the current is zero. Find the current at any time $t > 0$ and the maximum current if (a) $E = 64$, (c) $E = 32e^{-8t}$.



- Potential drop across R = $8I$
- Potential drop across L = $0.5\frac{dI}{dt}$

- Potential drop across $E = -E$

Then by Kirchhoff's law,

$$8I + 0.5\frac{dI}{dt} - E = 0$$

(a) $E = 64$

```
syms I(t) E
cond = I(0) == 0;
eq = 8*I+0.5*diff(I,t)-E == 0;
EN = 64;
dsolve(subs(eq,E,EN), cond)
```

ans = $8 - 8\,e^{-16\,t}$

(b) $E = 8te^{-16t}$

```
EN = 8*t*exp(-16*t);
dsolve(subs(eq,E,EN),cond)
```

ans = $8\,t^2\,e^{-16\,t}$

OR $\frac{dI}{dt} + 16I = 16te^{-16t}$ is a linear equation, thus can be solved as follows:

```
mu = exp(int(16,t));
syms c
eq = mu*I == int(mu*16*t*exp(-16*t),t) + c
```

eq(t) = $e^{16\,t}\,I(t) = 8\,t^2 + c$

```
cond = I(0) == 0;
subs(subs(eq,t,0),I,0)
```

ans(t) = $0 = c$

```
eq = subs(eq,c,0);
eq = lhs(eq)*exp(-16*t) == rhs(eq)*exp(-16*t)
```

eq(t) = $I(t) = 8\,t^2\,e^{-16\,t}$

Exercise 3. (2.81 c) Find the orthogonal trajectories of the family of curves $y^2 = cx^2 - 2y$.

1) The differential equation of the family is: $2y\frac{dy}{dx} = 2cx - 2\frac{dy}{dx}$, substituting in this $c = \frac{y^2 + 2y}{x^2}$

```
syms y(x) x c
eq = y(x)^2 == c*x^2-2*y(x);
eq_diff = diff(eq,x);
eq = simplify(subs(eq_diff,c,solve(eq,c)));
```

```
syms df
S = solve(subs(eq,diff(y,x),df),df)
```

S =

$$\frac{y(x)\ (y(x)+2)}{x\ (y(x)+1)}$$

2) Since the slope of each member of the orthogonal family must be the negative reciprocal of this slope, we see that the slope of the orthogonal family is:

```
OF = -S^(-1)
```

OF =

$$-\frac{x\ (y(x)+1)}{y(x)\ (y(x)+2)}$$

Solving this gives us the equation of the orthogonal trajectories:

```
syms y x c
eq = int(y*(y+2)/(y+1),y) == int(-x,x) + c;
eq = log(y+1) == c+x^2/2 + y^2/2 + y;
eq = exp(lhs(eq)) == exp(simplify(rhs(eq)))
```

eq =

$$y+1=\mathrm{e}^{\frac{x^2}{2}+\frac{y^2}{2}+y+c}$$

Exercise 4. (2.82) Find the equation of that curve passing through $(0,1)$ which is orthogonal to each member of the family $x^2+y^2=ce^x$.

1) The differential equation of the family $x^2+y^2=ce^x$

```
syms x y(x) c
eq = x^2+y^2 == c*exp(x);
eq = subs(diff(eq,x),c,solve(eq,c));
syms df;
S = solve(subs(eq,diff(y,x),df),df)
```

S =

$$\frac{x^2-2\,x+y(x)^2}{2\,y(x)}$$

2) Since the slope of each member of the orthogonal family must be the negative reciprocal of this slope, we see that the slope of the orthogonal family is:

```
OF = -S^(-1)
```

OF =

$$-\frac{2\,y(x)}{x^2-2\,x+y(x)^2}$$

We find the differential equation: $(x^2-2x+y^2)\mathrm{d}y+2y\mathrm{d}x=0$. Dividing by x^2+y^2, gives us $\mathrm{d}y+2\frac{y\mathrm{d}x-x\mathrm{d}y}{x^2+y^2}=0$.

```
syms y(x) x
eq = atan(x/y);
simplify(diff(eq,x))
```

ans(x) =

$$\frac{y(x)-x\frac{\partial}{\partial x}\ y(x)}{x^2+y(x)^2}$$

We know that $\frac{y\mathrm{d}x-x\mathrm{d}y}{x^2+y^2}=\mathrm{d}(\tan^{-1}\frac{x}{y})$, giving us the differential equation $\mathrm{d}y+\mathrm{d}(2\tan^{-1}\frac{x}{y})=0$ which gives as solution: $y+2\tan^{-1}(\frac{x}{y})=c$.

Using the extra constraint they need to pass through $(0,1)$, to eliminate c.

```
syms y x c
eq = y+2*atan(x/y) == c;
subs(subs(eq,x,0),y,1)
```

ans = $1=c$

```
eq = subs(eq,c,1)
```

eq =

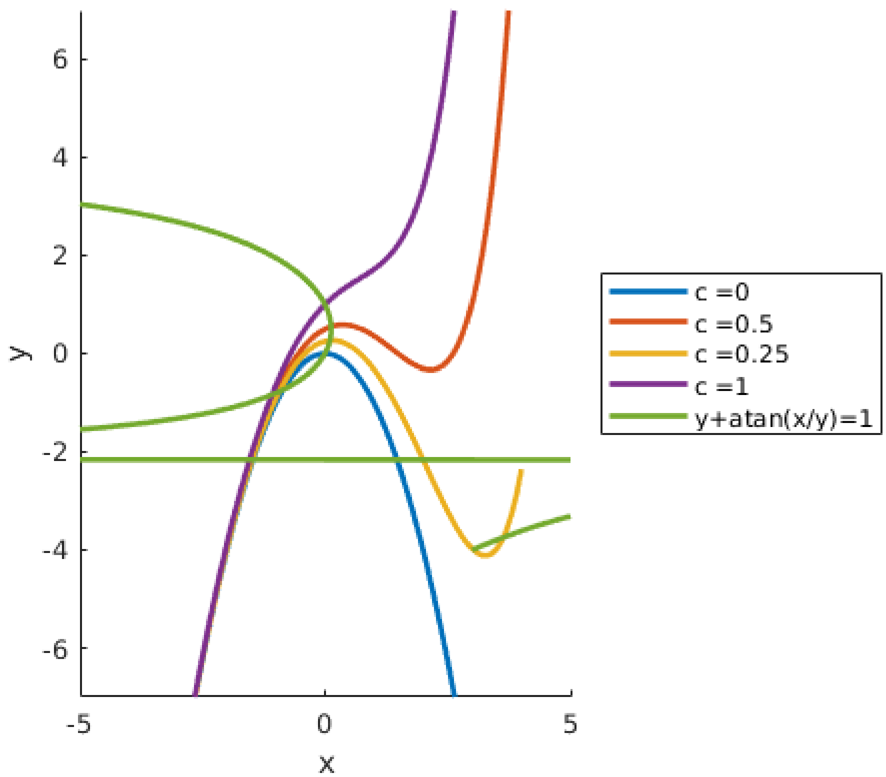
$$y+2\,\mathrm{atan}\left(\frac{x}{y}\right)=1$$

And so we find: $y+2\tan^{-1}(\frac{x}{y})=1$

The family $x^2+y^2=ce^x$ and the member of the orthogonal family $y+2\tan^{-1}(\frac{x}{y})=1$ going through $(0,1)$ are shown below:

```
clear
figure
hold on
x = -4:0.1:4;
legends = {};
for c = [0, 1/2, 1/4, 1]
    y = c*exp(x)-x.^2;
    plot(x, y, "LineWidth", 2)
    legends{end+1} = strcat('c = ', num2str(c));
end
y = -4:0.1:4;
x = y.*tan((1-y)./2);
plot(x,y, "LineWidth", 2)
legends{end+1} = 'y+atan(x/y)=1';
```

```
axis equal
axis([-5 5 -7 7])
legend(legends, 'Location', 'eastoutside')
xlabel('x'); ylabel('y')
```



Exercise 5. (2.85) A tank contains 100 gallons of water. A salt solution containing 2 lb of salt per gallon flows in at the rate of 3 gallons per minute and the well-stirred mixture flows out at the same rate. (a) How much salt is in the tank at any time? (b) When will the tank have 100 lb of salt?

Let A= lb salt in the tank at time t minutes.

Rate of change of amount of salt = Rate of entrance - Rate of exit

$$\frac{dA}{dt}\frac{\text{lb}}{\text{min}} = 2\frac{\text{lb}}{\text{gal}}\cdot 3\frac{\text{gal}}{\text{min}} - \frac{A}{100}\frac{\text{lb}}{\text{gal}}\cdot 3\frac{\text{gal}}{\text{min}}$$

$$\frac{dA}{dt} = 6 - \frac{3A}{100}$$

```
syms A(t) t
eq = diff(A,t) == 6 - (3*A)/100;
cond = A(0) == 0;
S = dsolve(eq,cond)
```

S =

$$200 - 200\,e^{-\frac{3\,t}{100}}$$

$A(t) = 200(1 - e^{-0.03t})$

When will $A = 100$?

```
double(solve(S==100,t))
```

```
ans = 23.1049
```

$t = 23,1$ minutes.

Exercise 6. (2.89) The rate at which bacteria multiply is proportional to the instantaneous number present. If the original number doubles in 2 hours, in how many hours will it triple?

B = the number of bacteria present after t hours.

Time rate of change of B is proportional to B: $\frac{dB}{dt} \sim B \Rightarrow \frac{dB}{dt} = k \cdot B$ with $B = B_0$ at $t = 0$.

```
syms B(t) t B0 k
eq = diff(B,t) == k*B;
cond = B(0) == B0;
S = dsolve(eq,cond)
```

S = $B_0 \, e^{k\,t}$

```
Sk = solve(subs(S,t,2)==2*B0,k)
```

Sk =

$$\frac{\log(2)}{2}$$

```
double(solve(subs(S,k,Sk)==3*B0,t))
```

```
ans = 3.1699
ans = 3.1699
```

Exercise 7. (2.93) It takes 15 minutes for an object to warm up from $10\,°\mathrm{C}$ to $20\,°\mathrm{C}$ in a room whose temperature is $30\,°\mathrm{C}$. Assuming Newton's law of cooling, how long would it take to warm up from $20\,°\mathrm{C}$ to $25\,°\mathrm{C}$?

U=temperature of object after t minutes.

time rate of chage in temperature of an object $\sim$ difference in temperature between object and surroundings.

$$\frac{dU}{dt} = k \cdot (U - 30)$$

```
syms U(t) k
eq = diff(U,t) == k*(U-30);
cond1 = U(0) == 10;
cond2 = U(15) == 20;
S = dsolve(eq,cond1)
```

```
S = 
```
$30 - 20\,e^{k\,t}$

```
Sk = solve(subs(S,t,15)==20,k)
```

```
Sk =
```
$$-\frac{\log(2)}{15}$$

```
solve(subs(S,k,Sk)==25,t)
```

```
ans = 
```
30

We reach the temperature of $25\,°\mathrm{C}$ after 15 minutes.

Exercise 8. (2.75 a,b) A 64lb object falls from rest. The limiting velocity is 4 ft/sec. Find the velocity after t seconds assuming a force of resistance proportional to (a) v (b) v^2.

Exercise 9. (2.94) At $1:00$ P.M. the temperature of a tank of water is $200\,°\mathrm{F}$. At $1:30$ P.M. its temperature is $160\,°\mathrm{F}$. Assuming the surrounding temperature is maintained at $80\,°\mathrm{F}$, (a) what is the temperature at $2:00$ P.M. and (b) at what time will the temperature be $100\,°\mathrm{F}$? Assume Newton's law of cooling.

Exercise 10. (2.90) After 2 days, 10 grams of a radioactive chemical is preent. Three days later 5 grams is present. How much of the chemical was present initially assuming the rate of disintegration is proportional to the instantaneous amount which is present?

Exercise 11. (2.78) An electric circuit contains an 9 ohm resistor in series with an inductor of 0.5 henries and a battery of E volts. At $t = 0$ the current is zero. Find the current at any time $t > 0$ if $E = 64\sin 8t$. What is the transient current and steady-stade current?

Exercise 12. (2.81 b) Find the orthogonal trajectories of the family of curves $x^2 + y^2 = cx$.

Exercise 13. (2.79 b) An electric circuit contains a 20 ohm resistor in series with a capacitor of 0.05 farads and a battery of E volts. At $t = 0$ there is no charge on the capacitor. Find the charge and current at any time $t > 0$ if $E = 100\mathrm{te}^{-2t}$.

Exercise 14. (2.86) A tank contains 100 gallons of water. A salt solution containing 2 lb of salt per gallon flows in at the rate of 3 gallons per minute. How much salt is in the tank at any time if the well-stirred mixture flows out at the rate of (a) 2 gal/min (b) 4 gal/min?