Exercise 8 (a) Resistance $\sim v$

$W = 64\,lb$ $\quad v(t=0) = 0$ $\qquad \lim_{t\to\infty} v(t) = 4\ ft/s$

$\downarrow \quad \uparrow F \quad$ a) v $\quad$ b) v^2 $\quad$ c) $\sqrt{v}$

Net force = Weight - Resistance $\qquad g = 32\ ft/s^2$

$$\frac{64}{g}\frac{dv}{dt} = 64 - kv$$

$$\frac{dv}{dt} = 32 - \frac{k}{2}v$$

$$\int \frac{2}{64 - kv}\,dv = \int dt$$

$$-\frac{2}{k}\ln(64 - kv) = t + c$$

$$64 - kv = c \cdot e^{-\frac{k}{2}t}$$

$$v = \frac{64}{k} - c e^{-\frac{k}{2}t} \qquad v(t=0) = 0$$

$$0 = \frac{64}{k} - c \iff c = \frac{64}{k}$$

$$v = \frac{64}{k} - \frac{64}{k}e^{-\frac{k}{2}t} \qquad \lim_{t\to\infty} v(t) = 4\ ft/s$$

$$\lim v(t) = \frac{64}{k} - 0 = 4 \iff k = 16$$

$$\boxed{v(t) = 4 - 4e^{-8t} = 4\left(1 - e^{-8t}\right)}$$

(b) Resistance $\sim v^2$

Net force = Weight - Resistance $\qquad g = 32\ \text{ft/s}^2$

$$\frac{64}{g}\frac{dv}{dt} = 64 - kv^2$$

$$\frac{dv}{dt} = 32 - \frac{k}{2}v^2$$

$$\int \frac{2}{64-kv^2}\,dv = \int dt$$

$$\frac{2}{64-kv^2} = \frac{1/8}{8-\sqrt{k}\,v} + \frac{1/8}{8+\sqrt{k}\,v}$$

$$a(8+\sqrt{k}\,V) + b(8-\sqrt{k}\,v)$$

$$a+b = 1/16$$

$$a-b = 0$$

$$-\frac{1}{8\sqrt{k}}\ln(8-\sqrt{k}\,v) + \frac{1}{8\sqrt{k}}\ln(8+\sqrt{k}\,v) = t + c$$

$$\ln\left(\frac{8-\sqrt{k}\,v}{8+\sqrt{k}\,v}\right) = -8\sqrt{k}\,t + c \qquad v(t=0)=0 \iff c=0$$

$$\frac{8-\sqrt{k}\,v}{8+\sqrt{k}\,v} = e^{-8\sqrt{k}\,t}$$

$$8-\sqrt{k}\,v = (8+\sqrt{k}\,v)e^{-8\sqrt{k}\,t}$$

$$8(1-e^{-8\sqrt{k}\,t}) = \sqrt{k}\,v\,(1+e^{-8\sqrt{k}\,t})$$

$$v = \frac{8}{\sqrt{k}}\left(\frac{1-e^{-8\sqrt{k}\,t}}{1+e^{-8\sqrt{k}\,t}}\right)$$

$$\lim_{t\to\infty}(v(t)) = \frac{8}{\sqrt{k}} = 4 \iff k=4$$

$$v(t) = 2\left(\frac{1-e^{-16t}}{1+e^{-16t}}\right) = 2\left(\frac{e^{8t}-e^{-8t}}{e^{8t}+e^{-8t}}\right)$$

$$\boxed{v(t) = 4\tanh(8t)}$$

Exercise 9

$t=0 \leftarrow$ 1:00 PM $\longrightarrow 200°F$

$t=30 \leftarrow$ 1:30 PM $\longrightarrow 160°F$

Surrounding $= 80°F$

(a) $t =$ 2:00 PM

$$\frac{du}{dt} = k(U-80) \Leftrightarrow U = 80 + ce^{kt}$$

$$t=0,\ U=200 \Leftrightarrow U = 80 + 120\, e^{kt}$$

$$t=30,\ U=160 \Leftrightarrow e^{k} = \left(\tfrac{2}{3}\right)^{1/30}$$

$$t=60,\quad U = 80 + 120\left(\tfrac{2}{3}\right)^{60/30}$$

$$= 80 + 120 \cdot \frac{4}{9} = 80 + \frac{160}{3} = \frac{400}{3}$$

$$= 133{,}33\ °F$$

(b) $U = 100°F$

$$100 = 80 + 120\left(\tfrac{2}{3}\right)^{t/30}$$

$$\left(\tfrac{2}{3}\right)^{t/30} = \frac{1}{6}$$

$$t \cdot \ln\left(\tfrac{2}{3}\right) = 30 \ln\left(\tfrac{1}{6}\right)$$

$$t = 30\,\frac{\ln(1/6)}{\ln(2/3)} = 132{,}57 \text{ minutes}$$

approximatly 3:12 PM

Exercise 10

A gr present after t days

$$\frac{dA}{dt} = -kA$$

$$A = c \cdot e^{-kt} \qquad \text{with } A = A_0 \text{ at } t=0$$

$$A = A_0 \cdot e^{-kt}$$

with $A = 10$ at $t = 2$
$A = 5$ at $t = 5$

$$\begin{cases} 10 = A_0 \cdot e^{-2k} \\ 5 = A_0 \cdot e^{-5k} \end{cases}$$

$$\frac{1}{2} \cdot \ln\left(\frac{A_0}{10}\right) = k = \frac{1}{5} \ln\left(\frac{A_0}{5}\right)$$

$$\left(\frac{A_0}{10}\right)^{1/2} = \left(\frac{A_0}{5}\right)^{1/5}$$

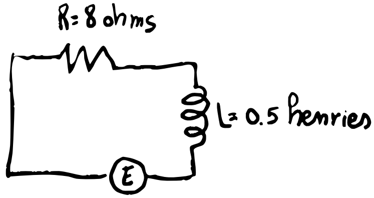
$$A_0^{1/2 - 1/5} = \frac{10^{1/2}}{5^{1/5}}$$

$$A_0^{3/10} = \sqrt{2} \cdot 5^{3/10}$$

$$A_0 = 2^{5/3} \cdot 5 = 10\sqrt[3]{4}$$

$$\boxed{A_0 = 15,87 \text{ gram}}$$

Exercise 11



$I=0$, for $t=0$

$I(t>0)$? and the maximum I?

potential drop across $R = 8I$

potential drop across $L = 0.5\dfrac{dI}{dt}$

potential drop across $E = -E$

By Kirchhoff's laws,

$$8I + 0{,}5\frac{dI}{dt} = E$$

a) $E = 64\sin 8t$

$$\frac{dI}{dt} + 16I = 128\sin 8t \qquad \mu = e^{16t}$$

$$e^{16t} I = \int 128 e^{16t}\sin 8t\, dt + c$$

$$\int e^{16t}\sin 8t\,dt = -\frac{1}{8}e^{16t}\cos 8t + 2\int e^{16t}\cos 8t\,dt$$

$$\left[\begin{array}{ll|ll} u = e^{16t} & du = 16e^{16t}dt & u = e^{16t} & du = 16e^{16t}dt \\ dv = \sin 8t\,dt \rightarrow & v = -\frac{1}{8}\cos 8t & dv = \cos 8t\,dt \rightarrow & v = \frac{1}{8}\sin 8t \end{array}\right]$$

$$= -\frac{1}{8}e^{16t}\cos 8t + \frac{2}{8}e^{16t}\sin 8t - 4\int e^{16t}\sin 8t\,dt$$

$$\int e^{16t}\sin 8t\,dt = \frac{1}{40}e^{16t}(2\sin 8t - \cos 8t)$$

$$e^{16t} I = \frac{128}{40}\cdot e^{16t}(2\sin 8t - \cos 8t) + c$$

$$I = \frac{16}{5}(2\sin 8t - \cos 8t) + ce^{-16t} \qquad . \quad I=0,\ t=0$$

$$0 = \frac{16}{5}(-1) + c$$

$$\boxed{I = \frac{16}{5}\left(2\sin 8t - \cos 8t + e^{-16t}\right)}$$

b) Transient current? Steady-state current?

$$I = \frac{16}{5}\left(2\sin 8t - \cos 8t + e^{-16t}\right)$$

$t \to \infty$

$\frac{16}{5} e^{-16t}$ approches zero as t increases $\Rightarrow$ transient Current

$\frac{16}{5}(2\sin 8t - \cos 8t) \Rightarrow$ Steady-state current.

Exercise 12

$$x^2+y^2=cx$$

$$2x+2y\frac{dy}{dx}=c=x+y^2/x$$

$$2y\frac{dy}{dx}=\frac{y^2}{x}-x$$

$$\frac{dy}{dx}=\frac{1}{2}\frac{y}{x}-\frac{1}{2}\frac{x}{y}=\frac{1}{2}\left(\frac{y^2-x^2}{xy}\right)$$

(slope) $\frac{dy}{dx}=2\left(\frac{xy}{x^2-y^2}\right)$ $\qquad v=\frac{y}{x} \qquad y=vx$

$$\frac{dy}{dx}=v+x\frac{dv}{dx}=2\left(\frac{vx^2}{x^2-v^2x^2}\right)=2\left(\frac{v}{1-v^2}\right)$$

$$x\frac{dv}{dx}=\frac{2v}{1-v^2}-\frac{v(1-v^2)}{1-v^2}=\frac{v+v^3}{1-v^2}$$

$$\int\frac{1-v^2}{v+v^3}dv=\int\frac{1}{x}dx=\ln x+c$$

$$\int\frac{1}{v}-\frac{2v}{1+v^2}dv=\ln v-\ln(1+v^2)$$

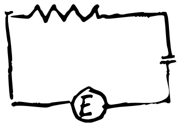
$$\frac{v}{1+v^2}=cx$$

$$\frac{v}{1+v^2}=cx \qquad y=vx \Rightarrow \frac{y/x}{1+(y/x)^2}=cx$$

$$\Leftrightarrow \boxed{x^2+y^2=cy}$$

Exercise 13

R = 20 ohm



C = 0.05 farads

$t=0,\ Q=0$

Potential drop over $R = 20I = 20\dfrac{dQ}{dt}$

over $C = \dfrac{Q}{0.05} = 20Q$

over $E = -E$

Kirchoff's law, $20\dfrac{dQ}{dt} + 20Q = E$

$E = 100te^{-2t}$

$\dfrac{dQ}{dt} + Q = 5te^{-2t} \qquad \mu = e^{t}$

$e^{t}Q = \int 5te^{-t}\,dt + c$

$u = t \quad du = dt$

$dv = e^{-t}dt \quad v = -e^{-t}$

$= 5\left(-te^{-t} + \int e^{-t}dt\right) + c$

$Q = -5te^{-2t} - 5e^{-2t} + ce^{-t}$

with $c = 5$

$$\boxed{Q = 5\left(e^{-t} - e^{-2t} - te^{-2t}\right)}$$

$$\boxed{I = \frac{dQ}{dt} = 5e^{-2t} + 10te^{-2t} - 5e^{-t}}$$