PW/TP 7-8: Linear Differential Equations - Variable coefficients (CH3)

Cauchy or Euler equation

This equation has the form

$$\left(b_n x^n \mathrm{D}^n + b_{n-1}x^{n-1}\mathrm{D}^{n-1} + \cdots + b_1 x\mathrm{D} + b_0\right) y = R(x)$$

where $b_0,\ b_1,\ \ldots,\ b_n$ are constants. It can be solved by letting $x = \mathrm{e}^t$ and using the results

$$x\mathrm{D}_x = \mathrm{D}_t, \quad x^2\mathrm{D}_x^2 = \mathrm{D}_t(\mathrm{D}_t - 1), \quad x^3\mathrm{D}_x^3 = \mathrm{D}_t(\mathrm{D}_t - 1)(\mathrm{D}_t - 2), \quad \ldots$$

Exercise 1. (3.23) Prove the equalities $x\mathrm{D}_x = \mathrm{D}_t, \quad x^2\mathrm{D}_x^2 = \mathrm{D}_t(\mathrm{D}_t - 1)$ for $x = e^t$.

(a) $xD_x = D_t$

$$D_x y = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$

$$= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{e^t} \text{ because } x = e^t\text{, thus } \frac{\mathrm{d}x}{\mathrm{d}t} = e^t$$

$$= e^{-t}\frac{\mathrm{d}y}{\mathrm{d}t} = e^{-t}D_t y$$

(b) $x^2D_x^2 = D_t(D_t - 1)$

$$D_x^2 = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-t}\frac{\mathrm{d}y}{\mathrm{d}t}\right) = e^{-t}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-t}\frac{\mathrm{d}y}{\mathrm{d}t}\right)$$

$$= e^{-t}\left(\frac{\mathrm{d}}{\mathrm{d}t}(e^{-t})\cdot\frac{\mathrm{d}y}{\mathrm{d}t} + e^{-t}\cdot\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)$$

$$= e^{-t}\left(-e^{-t}\cdot\frac{\mathrm{d}y}{\mathrm{d}t} + e^{-t}\cdot\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)$$

$$= e^{-2t}(D_t^2 - D_t)y$$

Exercise 2. Solve

1. (3.70 a) $x^2y'' + xy' - \lambda^2 y = 0$

```
syms y(x) x l
eq = x^2 * diff(y,x,2) + x*diff(y,x)- l^2*y == 0;
```

```
S = dsolve(eq);
simplify(S)
```

ans =

$$\frac{C_1}{x^l} + C_2\, x^l$$

2. (3.70 d) $x^3y''' + 3x^2y'' + xy' + 8y = 7x^{-\frac{1}{2}}$

```
syms y(x) x
eq = x^3*diff(y,x,3)+3*x^2*diff(y,x,2)+x*diff(y,x)+8*y == 0;
S = dsolve(eq)
```

S =

$$\frac{C_3}{x^2} + C_2\, x\cos(\sqrt{3}\ \log(x)) - C_1\, x\sin(\sqrt{3}\ \log(x))$$

```
syms y(t) t
Dt = diff(y,t);
eq = Dt*(Dt-1)*(Dt-2)+3*Dt*(Dt-1)+Dt+8*y == 0;
simplify(eq)
```

ans(t) =

$$\left(\frac{\partial}{\partial t}\ y(t)\right)^3 + 8\ y(t) = 0$$

```
eq = diff(y,t,3)+8*y == 0;
dsolve(eq)
```

ans = $C_1\,\mathrm{e}^{-2\,t} + C_2\,\mathrm{e}^{t}\cos(\sqrt{3}\ t) - C_3\,\mathrm{e}^{t}\sin(\sqrt{3}\ t)$

Giving us the homogeneous solution:

$$y_H = c_1e^{-2t} + c_2e^t\cos(\sqrt{3}\,t) - c_3e^t\sin(\sqrt{3}\,t) = c_1x^{-2} + c_2x\cos(\sqrt{3}\ln x) - c_3x\sin(\sqrt{3}\ln x)$$

```
syms a
yp = a*exp(-t/2);
eq = diff(yp,t,3)+ 8*yp == 7*exp(-t/2)
```

eq =

$$\frac{63\,a\,\mathrm{e}^{-\frac{t}{2}}}{8} = 7\,\mathrm{e}^{-\frac{t}{2}}$$

```
solve(eq,a)
```

ans =

$$\frac{8}{9}$$

And the partial solution: $y_P = \frac{8}{9}e^{\frac{-t}{2}} = \frac{8}{9}x^{\frac{-1}{2}}$

Exercise 3. Solve

1. (3.70 e) $r^2R'' + 2rR' - n(n+1)R = 0$
2. (3.70 c) $(2x^2D^2 + 5xD + 1)y = \ln x$
3. $x^2y'' - 2y = 0$
4. $4x^2y'' + 5y = 0$
5. $x^2y'' + xy' + 9y = 0$ with $y(1) = 0, y'(1) = 5/2$
6. $x^2y'' + 2xy' - 6y = 0$ with $y(1) = 1/2, y'(1) = 3/2$
7. $x^2y'' - 3xy' + 4y = 0$ with $y(1) = -\pi, y'(1) = 2\pi$

System of First-Order Equations

Homogeneous linear system ODE

We study systems of n equation of the form $Y' = AY$. We restrict ourselves to systems with constant coefficients. Depending on the properties of the matrix A, the solution of the homogeneous system will look slightly different.

n independent eigenvectors

If A is a matrix of constants having n real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with the associated linear independent eigenvectors $X_1, X_2, \ldots, X_n$, then the functions $y_1 = X_1e^{\lambda_1 t}, y_2 = X_2e^{\lambda_2 t}, \ldots, y_n = X_ne^{\lambda_n t}$ are a fundamentel set of solutions of $Y' = AY$. The solution of such a system is given by:

$$Y_h = c_1X_1e^{\lambda_1 t} + \ldots + c_nX_ne^{\lambda_n t}$$

Exercise 4. Solve the following system

$$Y' = \underbrace{\begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix}}_{A} Y$$

```
A = [-4, -3 ; 6, 5];
[V,D] = eig(A)
```

```
V = 2×2
   -0.7071    0.4472
    0.7071   -0.8944
D = 2×2
    -1     0
     0     2
```

We find that the eigenvalues of A are 2 and -1. And the corresponding eigenvectors are respectivily $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

The solution of this system is:

$$Y = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$

```
syms y1(t) y2(t)
eq = [diff(y1,t); diff(y2,t)] == A*[y1; y2]
```

```
eq(t) =
```

$$\begin{pmatrix} \frac{\partial}{\partial t}\, y_1(t) = -4\, y_1(t) - 3\, y_2(t) \\ \frac{\partial}{\partial t}\, y_2(t) = 6\, y_1(t) + 5\, y_2(t) \end{pmatrix}$$

```
Y = dsolve(eq);
Y.y1, Y.y2
```

```
ans =
```

$$-C_1\, \mathrm{e}^{-t} - \frac{C_2\, \mathrm{e}^{2\,t}}{2}$$

`ans =` $C_1\, \mathrm{e}^{-t} + C_2\, \mathrm{e}^{2\,t}$

n independent eigenvectors (number of eigenvalues with multiplicity ≥ 2)

An eigenvalue with a multiplicity greater than 1, can lead to a number of different cases.

Multiplicity ≥ 2, associated eigenspace of dimension 1

In this case, all the eigenvectors associated with the eigenvalue with multiplicity ≥ 2 are a multiple of an eigenvector x. So there are an infinite number of vectors u for which holds that

$$(A - \lambda I)u = x$$

For such a vector u, the linear independent solutions of $Y' = AY$ are given by

$$y_1 = xe^{\lambda t}$$

$$y_2 = ue^{\lambda t} + xte^{\lambda t}$$

Exercise 5. Check that the above proposition for y_2 is indeed a solution of $Y' = AY$ and that y_1 and y_2 are linearly independent.

1) Check that y_1 and y_2 are solutions of $Y' = AY$:

$y_1 = xe^{\lambda t} \leftrightarrow y_1{}' = \lambda xe^{\lambda t} = Axe^{\lambda t} = Ay_1$ using that λ is an eigenvalue of A with corresponding eigenvector x so that $\lambda x = Ax$.

$y_2 = ue^{\lambda t} + xte^{\lambda t} \leftrightarrow y_2{}' = \lambda ue^{\lambda t} + xe^{\lambda t} + \lambda xte^{\lambda t} = (\lambda u + x)e^{\lambda t} + \lambda xte^{\lambda t} = Aue^{\lambda t} + Axte^{\lambda t} = A(ue^{\lambda t} + xte^{\lambda t} = Ay_2$ using the previous and $(A - \lambda I)u = x \leftrightarrow Au = x + \lambda u$.

2) Check that they are linearly independent:

y_1 and y_2 are linearly independent if and only if: $c_1y_1 + c_2y_2 = 0 \Leftrightarrow c_1 = c_2 = 0$.

$$c_1y_1 + c_2y_2 = 0 = c_1xe^{\lambda t} + c_2(ue^{\lambda t} + xte^{\lambda t}) \leftrightarrow \begin{cases} c_2x = 0 \\ c_1x + c_2u = 0 \end{cases} \leftrightarrow \begin{cases} c_2 = 0 \\ c_1 = 0 \end{cases}.$$

Or using the Wronskian:

```
clear
syms t lambda x u
y1(t) = x*exp(lambda*t);
y2(t) = u*exp(lambda*t)+x*t*exp(lambda*t);
W = [y1 y2; diff(y1,t) diff(y2,t)];
```

W(t) =

$$\begin{pmatrix} x\,e^{\lambda t} & u\,e^{\lambda t} + t\,x\,e^{\lambda t} \\ \lambda\,x\,e^{\lambda t} & x\,e^{\lambda t} + \lambda\,u\,e^{\lambda t} + \lambda\,t\,x\,e^{\lambda t} \end{pmatrix}$$

```
det(W)
```

ans(t) = $x^2\,e^{2\lambda t}$

Since we always choose a nontrivial vector x as eigenvector, we have that $\det(W) \neq 0 \Rightarrow y_1$ and y_2 are linearly independent.

Exercise 6. Solve $Y' = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} Y$.

```
A = [11, -25; 4, -9];
[V,D] = eig(A)
```

```
V = 2×2
    0.9285   -0.9285
    0.3714   -0.3714
D = 2×2
    1.0000         0
         0    1.0000
```

```
V(1,1)/V(2,1)
```

```
ans = 2.5000
```

We find the eigenvalue $\lambda = 1$ with multiplicity 2 and eigenvector $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

Calculating: $(A - \lambda I)u = x$ to find u.

```
syms u1 u2
lambda = 1
```

```
lambda = 1
```

```
x = [5;2];
eq = (A-lambda*[1,0;0,1])*[u1;u2] == x;
S = solve(eq,[u1,u2]); S.u1, S.u2
```

ans =

$\frac{1}{2}$

ans = 0

We find $u = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$.

And the solution to the system is $Y = c_1 \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 5 \\ 2 \end{bmatrix} t\, e^t \right)$.

Multiplicity ≥ 3, associated eigenspace of dimension 1

In this case, all the eigenvectors associated with the eigenvalue with multiplicity ≥ 3 are again multiples of an eigenvector x. So there are an infinite number of vectors u for which it holds that

$$(A - \lambda I)u = x$$

Moreover, there are an infinite number of vectors v for which it holds that

$$(A - \lambda I)v = u$$

For such vectors u and v it holds that the linear independen solutions of $Y' = AY$ are given by

$$y_1 = xe^{\lambda t}$$

$$y_2 = ue^{\lambda t} + xte^{\lambda t}$$

$$y_3 = ve^{\lambda t} + ute^{\lambda t} + x\frac{t^2}{2}e^{\lambda t}$$

Multiplicity ≥ 3, associated eigenspace of dimension 2

Inthis case, all the eigenvectors associated with the eigenvalue are a linear combination of the two linearly independent eigenvectors x_1 and x_2. There exist constants α and β (not together 0) so that

$$x_3 = \alpha x_1 + \beta x_2$$

there exist an infinite number of vectors u such that

$$(A - \lambda I)u = x_3$$

For such a u, the linear independent solutions of $Y' = AY$ are given by

$$y_1 = x_1 e^{\lambda t}$$

$$y_2 = x_2 e^{\lambda t}$$

$$y_3 = u e^{\lambda t} + x_3 t e^{\lambda t}$$

n independent, complex eigenvalues

For a complex eigenvalue $\lambda = \alpha + i\beta$ (with $\beta \neq 0$) of A, with associated eigenvector $x = u + iv$ ($u, v \neq 0$) then

$$y_1 = e^{\alpha t}(u\cos(\beta t) - v\sin(\beta t))$$

$$y_2 = e^{\alpha t}(u\sin(\beta t) + v\cos(\beta t))$$

which are equal to the real and imaginary part of

$$e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))(u + iv)$$

the linear independent solutions of $Y' = AY$.

Exercise 7. Solve $Y' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} Y$.

```
A = [1,1;-1,1];
[D,V] = eig(A)
```

```
D = 2×2 complex
   0.7071 + 0.0000i   0.7071 + 0.0000i
   0.0000 + 0.7071i   0.0000 - 0.7071i
V = 2×2 complex
   1.0000 + 1.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   1.0000 - 1.0000i
```

We find the eigenvalues $1 \pm i$ with eigenvectors $\begin{bmatrix} 1 \\ \pm i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pm i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

We find that the solutions are: $y_1 = e^t(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin t)$ and $y_2 = e^t(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin t + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos t)$.

$$\Rightarrow y = c_1 e^t(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin t) + c_2 e^t(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin t + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos t)$$

Higher order equation to system of first order ODE

By making a change of variables a higher order ODE can be converted to a system of first order order equations.

Exercise 8. Solve the following differential equation by first reducing it to a system of first order ODE's:

$$y'' + y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 2$$

Define new variables $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ with $\begin{matrix} y_1 = y \\ y_2 = y' \end{matrix}$, we find that $Y' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix}$.

We can rewrite the original equation as: $Y' = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

```
A = [0,1;2,-1];
[D,V] = eig(A)
```

```
D = 2×2
    0.7071   -0.4472
    0.7071    0.8944
V = 2×2
     1     0
     0    -2
```

We find the solution: $Y = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-2t}$.

Using the initial conditions, we can find values for c_1 and c_2.

```
syms c1 c2
y1(t) = c1*exp(t) + c2*(-1*exp(-2*t));
y2(t) = c1*exp(t)+c2*2*exp(-2*t);
S = solve([y1(0)==1;y2(0)==2],[c1;c2]); S.c1, S.c2
```

ans =

$\frac{4}{3}$

ans =

$\frac{1}{3}$

We find $Y = \frac{4}{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + \frac{1}{3} \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-2t}$, and thus is $y(= y_1) = \frac{4}{3} e^t - \frac{1}{3} e^{-2t}$.

```
syms y(t) t
eq = diff(y,t,2)+diff(y,t)-2*y == 0;
cond1 = y(0) == 1;
Dy(t) = diff(y,t);
cond2 = Dy(0) == 2;
dsolve(eq,cond1,cond2)
```

ans =

$$\frac{4\,e^t}{3} - \frac{e^{-2t}}{3}$$

Exercises

Exercise 9. Solve

1. $Y' = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} Y$ with $Y(0) = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$
2. $Y' = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} Y$
3. $Y' = \begin{bmatrix} 3 & 4 & -10 \\ 2 & 1 & -2 \\ 2 & 2 & -6 \end{bmatrix} Y$
4. $Y' = \begin{bmatrix} -14 & 39 \\ -6 & 16 \end{bmatrix} Y$
5. $Y' = \begin{bmatrix} -5 & 5 & 4 \\ -8 & 7 & 6 \\ 1 & 0 & 0 \end{bmatrix} Y$
6. $Y' = \begin{bmatrix} 21 & -12 \\ 24 & -15 \end{bmatrix} Y$ with $Y(0) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$
7. $Y' = \begin{bmatrix} 15 & -9 \\ 16 & -9 \end{bmatrix} Y$
8. $Y' = \begin{bmatrix} -7 & 24 \\ -6 & 17 \end{bmatrix} Y$
9. $Y' = \begin{bmatrix} 4 & -5 \\ 5 & -2 \end{bmatrix} Y$

Exercise 10. Solve the following differential equation by first reducing it to a system of first order ODE's:

1. $y'' - 5y' + 4y = 0; \quad y(0) = 5; \quad y'(0) = -1$

Exercise 11. Solve the homogeneous part of the exercises from PW/TP 5-6 by first reducing it to a system of first order ODE's.