PW/TP 7-8: Linear Differential Equations - Variable coefficients (CH3)

Cauchy or Euler equation

This equation has the form

$$\left(b_n x^n \mathrm{D}^n + b_{n-1} x^{n-1} \mathrm{D}^{n-1} + \cdots + b_1 x \mathrm{D} + b_0\right) y = R(x)$$

where $b_0,\ b_1,\ \ldots,\ b_n$ are constants. It can be solved by letting $x = \mathrm{e}^t$ and using the results

$$x\mathrm{D}_x = \mathrm{D}_t, \quad x^2\mathrm{D}_x^2 = \mathrm{D}_t(\mathrm{D}_t - 1), \quad x^3\mathrm{D}_x^3 = \mathrm{D}_t(\mathrm{D}_t - 1)(\mathrm{D}_t - 2), \quad \ldots$$

Exercise 1. Prove the equalities $x\mathrm{D}_x = \mathrm{D}_t, \quad x^2\mathrm{D}_x^2 = \mathrm{D}_t(\mathrm{D}_t - 1).$

Exercise 2. Solve

1. $x^2y'' + xy' - \lambda^2 y = 0$
2. $x^3y'' + 3x^2y'' + xy' + 8y = 7x^{\frac{-1}{2}}$

Exercise 3. Solve

1. $r^2R'' + 2rR' - n(n+1)R = 0$
2. $(2x^2D^2 + 5xD + 1)y = \ln x$
3. $x^2y'' - 2y = 0$
4. $4x^2y'' + 5y = 0$
5. $x^2y'' + xy' + 9y = 0$ with $y(1) = 0, y'(1) = 5/2$
6. $x^2y'' + 2xy' - 6y = 0$ with $y(1) = 1/2, y'(1) = 3/2$
7. $x^2y'' - 3xy' + 4y = 0$ with $y(1) = -\pi, y'(1) = 2\pi$

System of First-Order Equations

Homogeneous linear system ODE

We study systems of n equation of the form $Y' = AY$. We restrict ourselves to systems with constant coefficients. Depending on the properties of the matrix A, the solution of the homogeneous system will look slightly different.

n independent eigenvectors

If A is a matrix of constants having n real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with the associated linear independent eigenvectors $X_1, X_2, \ldots, X_n$, then the functions $y_1 = X_1 e^{\lambda_1 t}, y_2 = X_2 e^{\lambda_2 t}, \ldots, y_n = X_n e^{\lambda_n t}$ are a fundamentel set of solutions of $Y' = AY$. The solution of such a system is given by:

$$Y_h = c_1 X_1 e^{\lambda_1 t} + \ldots + c_n X_n e^{\lambda_n t}$$

Exercise 4. Solve the following system

$$Y' = \underbrace{\begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix}}_{A} Y$$

n independent eigenvectors (number of eigenvalues with multiplicity ≥ 2)

An eigenvalue with a multiplicity greater than 1, can lead to a number of different cases.

Multiplicity ≥ 2, associated eigenspace of dimension 1

In this case, all the eigenvectors associated with the eigenvalue with multiplicity ≥ 2 are a multiple of an eigenvector x. So there are an infinite number of vectors u for which holds that

$$(A - \lambda I)u = x$$

For such a vector u, the linear independent solutions of $Y' = AY$ are given by

$$y_1 = xe^{\lambda t}$$

$$y_2 = ue^{\lambda t} + xte^{\lambda t}$$

Exercise 5. Check that the above proposition for y_2 is indeed a solution of $Y' = AY$ and that y_1 and y_2 are linearly independent.

Exercise 6. Solve $Y' = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} Y$

Multiplicity ≥ 3, associated eigenspace of dimension 1

In this case, all the eigenvectors associated with the eigenvalue with multiplicity ≥ 3 are again multiples of an eigenvector x. So there are an infinite number of vectors u for which it holds that

$$(A - \lambda I)u = x$$

Moreover, there are an infinite number of vectors v for which it holds that

$$(A - \lambda I)v = u$$

For such vectors u and v it holds that the linear independen solutions of $Y' = AY$ are given by

$$y_1 = xe^{\lambda t}$$

$$y_2 = ue^{\lambda t} + xte^{\lambda t}$$

$$y_3 = ve^{\lambda t} + ute^{\lambda t} + x\frac{t^2}{2}e^{\lambda t}$$

Multiplicity ≥ 3, associated eigenspace of dimension 2

Inthis case, all the eigenvectors associated with the eigenvalue are a linear combination of the two linearly independent eigenvectors x_1 and x_2. There exist constants α and β (not together 0) so that

$$x_3 = \alpha x_1 + \beta x_2$$

there exist an infinite number of vectors u such that

$$(A - \lambda I)u = x_3$$

For such a u, the linear independent solutions of $Y' = AY$ are given by

$$y_1 = x_1 e^{\lambda t}$$

$$y_2 = x_2 e^{\lambda t}$$

$$y_3 = u e^{\lambda t} + x_3 t e^{\lambda t}$$

n independent, complex eigenvalues

For a complex eigenvalue $\lambda = \alpha + i\beta$ (with $\beta \neq 0$) of A, with associated eigenvector $x = u + iv$ $(u, v \neq 0)$ then

$$y_1 = e^{\alpha t}(u\cos(\beta t) - v\sin(\beta t))$$

$$y_2 = e^{\alpha t}(u\sin(\beta t) + v\cos(\beta t))$$

which are equal to the real and imaginary part of

$$e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))(u + iv)$$

the linear independent solutions of $Y' = AY$.

Exercise 7. Solve $Y' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} Y$.

Higher order equation to system of first order ODE

By making a change of variables a higher order ODE can be converted to a system of first order order equations.

Exercise 8. Solve the following differential equation by first reducing it to a system of first order ODE's:

$$y'' + y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 2$$

Exercises

Exercise 9. Solve

1. $Y' = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} Y$ with $Y(0) = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$

2. $Y' = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} Y$

3. $Y' = \begin{bmatrix} 3 & 4 & -10 \\ 2 & 1 & -2 \\ 2 & 2 & -6 \end{bmatrix} Y$

4. $Y' = \begin{bmatrix} -14 & 39 \\ -6 & 16 \end{bmatrix} Y$

5. $Y' = \begin{bmatrix} -5 & 5 & 4 \\ -8 & 7 & 6 \\ 1 & 0 & 0 \end{bmatrix} Y$

6. $Y' = \begin{bmatrix} 21 & -12 \\ 24 & -15 \end{bmatrix} Y$ with $Y(0) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

7. $Y' = \begin{bmatrix} 15 & -9 \\ 16 & -9 \end{bmatrix} Y$

8. $Y' = \begin{bmatrix} -7 & 24 \\ -6 & 17 \end{bmatrix} Y$

9. $Y' = \begin{bmatrix} 4 & -5 \\ 5 & -2 \end{bmatrix} Y$

Exercise 10. Solve the following differential equation by first reducing it to a system of first order ODE's:

1. $y'' - 5y' + 4y = 0; \quad y(0) = 5; \quad y'(0) = -1$

Exercise 11. Solve the homogeneous part of the exercises from PW/TP 5-6 by first reducing it to a system of first order ODE's.