PW/TP 11-12 Partial Differential Equations

First-order Quasilinear Equation

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$$

Solving the Homogeneous Equation of two independent variables

$$a(x, y)u_x + b(x, y)u_y = 0$$

the characteristic curve: $a\mathrm{d}y - b\mathrm{d}x = 0$

Calculate solution $f(x, y) = k$ with k an arbitrary constant.

$$u = F(f(x, y))$$

Exercise 1. Find the general solution of

1. $3u_x - 7u_y = 0$
2. $y^2u_x + \frac{1}{x}u_y = 0$
3. $2xyu_x + (x^2 + y^2)u_y = 0$

Solving the Homogeneous Equation of more independent variables

$$a(x, y, z)u_x + b(x, y, z)u_y + c(x, y, z)u_z = 0$$

Define the system $\frac{\mathrm{d}x}{a} = \frac{\mathrm{d}y}{b} = \frac{\mathrm{d}z}{c}$

Calculate solution $f_1(x, y, z) = k_1$ and $f_2(x, y, z) = k_2$ with k_1 and k_2 arbitrary constants.

$$u = F(f_1(x, y, z), f_2(x, y, z))$$

Exercise 2. Find the general solution of

1. $3u_x + 7u_y + 5u_z = 0$
2. $xu_x + zu_y + z^2u_z = 0$

The general solution of two independent variables

We search $u = f(x, y)$ of $a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$.

Write $F(x, y, u) = 0$ of $a(x, y, u)F_x + b(x, y, u)F_y + c(x, y, u)F_u = 0$

Define the system $\frac{\mathrm{d}x}{a} = \frac{\mathrm{d}y}{b} = \frac{\mathrm{d}u}{c}$

Calculate solution $f_1(x, y, u) = k_1$ and $f_2(x, y, u) = k_2$ with k_1 and k_2 arbitrary constants.

$$F(x, y, u) = G(f_1(x, y, u), f_2(x, y, u)) = 0$$

Exercise 3. Find the general solution of

1. $xu_x + yu_y = u + 1$
2. $u_x + axu_y = bxu^2,$ with $a, b \in \mathbb{R}$
3. $u_x + yu^2u_y + au = 0$

The general solution of two independent variables with boundary conditions

using boundary conditions $\begin{cases} \gamma_1(x, y, u) = 0 \\ \gamma_2(x, y, u) = 0 \end{cases}$, we get $\begin{cases} f_1(x, y, u) = k_1 \\ f_2(x, y, u) = k_2 \\ \gamma_1(x, y, u) = 0 \\ \gamma_2(x, y, u) = 0 \end{cases}$

Find the relation between k_1 and k_2: $F_1(k_1, k_2) = 0$

Giving the solution: $F_1(f_1(x, y, u), f_2(x, y, u)) = 0$

Exercise 4. Find the general solution of

1. $xu_x + yu_y = u + 1$ with $u(x, y) = x^2$ on $y = x^2$
2. $u_x + uu_y = 0,$ with $u = \varphi(y)$ at $x = 0$
3. $u_x + u_y = u,$ with $u(x, 0) = \cos(x)$
4. $xu_x + yu_y = cu,$ with $u(x, 1) = f(x)$