Exercise 1

(1) $\underbrace{3}_{a}u_x \underbrace{-7}_{b}u_y = 0$

$$a\,dy - b\,dx = 0 \Leftrightarrow 3dy + 7dx = 0$$

$$3y + 7x = k$$

$$u = F(3y + 7x)$$

(2) $y^2 u_x + \frac{1}{x} u_y = 0$

$$a\,dy - b\,dx = 0 \Leftrightarrow y^2 dy - \frac{1}{x} dx = 0$$

$$\frac{y^3}{3} - \ln(x) = k$$

$$y^3 - 3\ln(x) = k$$

$$u = F(y^3 - 3\ln(x))$$

(3) $\underbrace{2xy}_{a}u_x + \underbrace{(x^2+y^2)}_{b}u_y = 0$

$$a\,dy - b\,dx = 0 \Leftrightarrow 2xy\,dy - (x^2+y^2)\,dx = 0$$

$$\frac{dy}{dx} = \frac{x^2+y^2}{2xy} = \frac{1}{2}\left(\frac{x}{y} + \frac{y}{x}\right)$$

$$\left[\begin{array}{l} v = \frac{y}{x} \rightarrow xv = y \\ \quad\Updownarrow \\ v + \frac{dv}{dx}x = \frac{dy}{dx} \end{array}\right]$$

$$v + x\frac{dv}{dx} = \frac{1}{2}\left(\frac{1}{v} + v\right) = \frac{1+v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{1+v^2-2v^2}{2v} = \frac{1-v^2}{2v}$$

$$\int \frac{2v}{1-v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\begin{pmatrix} u = 1-v^2 \\ du = -2v\,dv \end{pmatrix} \quad -\ln(1-v^2) = \ln x + c$$

$$1 - v^2 = c\,x^{-1}$$

$$1 - \left(\frac{y}{x}\right)^2 = c\,x^{-1} \Leftrightarrow c = x - \frac{y^2}{x}$$

$$u = F\left(x - \frac{y^2}{x}\right)$$

Exercise 2.

(1) $3u_x + 7u_y + 5u_z = 0$

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c} \iff \frac{dx}{3} = \frac{dy}{7} = \frac{dz}{5}$$

$$\iff \begin{cases} 7dx = 3dy \\ 5dy = 7dz \end{cases}$$

$$\iff \begin{cases} 7x - 3y = k_1 \\ 5y - 7z = k_2 \end{cases}$$

$$u = F(7x-3y, 5y-7z)$$

(2) $x u_x + z u_y + z^2 u_z = 0$

$$\frac{dx}{x} = \frac{dy}{z} = \frac{dz}{z^2} \iff \begin{cases} z^2 dy = z\,dz \\ z\,dx = x\,dy \end{cases}$$

$$z^2 dy = z\,dz \iff z\,dy = dz \iff dy = \frac{1}{z}dz$$

$$y + k_1 = \ln(z)$$

$$z = k_1 \cdot e^y$$

$$z\,dx = x\,dy \iff k_1 e^y dx = x\,dy$$

$$\frac{k_1}{x}dx = e^{-y}dy$$

$$k_1 \ln(x) + k_2 = -e^{-y}$$

$$\Rightarrow k_1 = ze^{-y}$$

$$k_2 = -e^{-y} - ze^{-y}\ln x = -e^{-y}(z\ln x + 1)$$

$$u = F(ze^{-y}, -e^{-y}(z\ln x + 1))$$

Exercise 3.

(1) $x u_x + y u_y = u + 1$

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c} \Leftrightarrow \frac{dx}{x} = \frac{dy}{y} = \frac{du}{u+1} \Leftrightarrow \begin{cases} (u+1)\,dy = y\,du \\ y\,dx = x\,dy \end{cases}$$

$$\Leftrightarrow \begin{cases} \ln y + k_1 = \ln(u+1) \\ \ln x + k_2 = \ln y \end{cases}$$

$$F(\ln(u+1) - \ln(y), \ln(y) - \ln(x)) = 0$$

(2) $u_x + a x u_y = b x u^2$, with $a, b \in \mathbb{R}$

$$dx = \frac{dy}{ax} = \frac{du}{bxu^2} \Leftrightarrow \begin{cases} ax\,dx = dy \\ bx\,dx = \frac{1}{u^2}\,du \end{cases}$$

$$\Leftrightarrow \begin{cases} a\frac{x^2}{2} + k_1 = y \\ b\frac{x^2}{2} + k_2 = -\frac{1}{u} \end{cases} \Leftrightarrow \begin{cases} k_1 = 2y - ax^2 \\ k_2 = -\frac{b}{2}x^2 - \frac{1}{u} \end{cases}$$

$$F\left(2y - ax^2, -\frac{b}{2}x^2 - \frac{1}{u}\right) = 0$$

(3) $u_x + y u^2 u_y + au = 0$

$$dx = \frac{dy}{yu^2} = \frac{du}{-au} \Leftrightarrow \begin{cases} -a\,dx = \frac{1}{u}\,du \\ -\frac{a}{y}\,dy = u\,du \end{cases}$$

$$\Leftrightarrow \begin{cases} -ax + k_1 = \ln u \\ -a\ln y + k_2 = \frac{u^2}{2} \end{cases} \Leftrightarrow \begin{cases} k_1 = ue^{ax} \\ k_2 = u^2 + 2a\ln y \end{cases}$$

$$F(ue^{ax}, u^2 + 2a\ln y) = 0$$

Exercise 4

(1) $x u_x + y u_y = u+1$ with $u(x,y) = x^2$ on $y = x^2$

$$\begin{cases} k_1 = \ln(u+1) - \ln(y) \\ k_2 = \ln(y) - \ln(x) \\ u(x, x^2) = x^2 \end{cases} \Leftrightarrow \begin{cases} k_1 = \ln(x^2+1) - \ln(x^2) \\ k_2 = \ln(x^2) - \ln(x) \end{cases}$$

$$\Leftrightarrow \begin{cases} k_1 = \ln\left(1 + \frac{1}{x^2}\right) \\ k_2 = \ln(x) \end{cases}$$

$$\Leftrightarrow \frac{1}{e^{k_1} - 1} = e^{2k_2}$$

$$\Rightarrow \frac{1}{e^{\ln\left(\frac{u+1}{y}\right)} - 1} = e^{2\ln(y/x)} \Leftrightarrow u = \frac{x^2 + y^2 - y}{y}$$

(2) $u_x + u u_y = 0$ with $u = \varphi(y)$ at $x = 0$

$$\frac{dx}{1} = \frac{dy}{u} = \frac{du}{0} \Leftrightarrow \begin{cases} u\,du = 0 \\ u\,dx = dy \end{cases} \Leftrightarrow \begin{cases} \frac{u^2}{2} = k_1 \\ u\,dx = dy \end{cases}$$

$$\Leftrightarrow \begin{cases} k_1 = u \\ k_1\,dx = dy \end{cases} \Leftrightarrow \begin{cases} k_1 = u \\ k_1 x + k_2 = y \end{cases} \Leftrightarrow \begin{cases} k_1 = u \\ k_2 = y - ux \end{cases}$$

using $u = \varphi(y)$ at $x = 0$

$\Rightarrow$ Search $F_1(k_1, k_2) = 0$

$$\begin{cases} k_1 = \varphi(y) \\ k_2 = y \end{cases} \Leftrightarrow k_1 - \varphi(k_2) = 0$$

Replace k_1, k_2 with original formula's:

$$u - \varphi(y - ux) = 0$$

(3) $u_x + u_y = u$, with $u(x,0) = \cos x$

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{u} \Leftrightarrow \begin{cases} dx = dy \\ u\,dy = du \end{cases} \Leftrightarrow \begin{cases} x + k_1 = y \\ y + k_2 = \ln(u) \end{cases}$$

$$\Leftrightarrow \left.\begin{matrix} \begin{cases} k_1 = y - x \\ k_2 = u e^{-y} \end{cases} \\ \text{with } u(x,0) = \cos x \end{matrix}\right\} \Leftrightarrow \begin{cases} k_1 = -x \\ k_2 = \cos x \end{cases} \Leftrightarrow k_2 - \cos(-k_1) = 0$$

$$\Rightarrow u e^{-y} - \cos(x - y) = 0$$

$$\Leftrightarrow u = \cos(x-y)e^{y}$$

(4) $x u_x + y u_y = cu$, with $u(x,1) = f(x)$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{cu} \Leftrightarrow \begin{cases} \frac{1}{x}dx = \frac{1}{y}dy \\ \frac{1}{x}dx = \frac{1}{cu}du \end{cases}$$

$$\Leftrightarrow \begin{cases} \ln x + k_1 = \ln y \\ \ln x + k_2 = \frac{1}{c}\ln u \end{cases} \Leftrightarrow \begin{cases} y = k_1 x \\ u = k_2 x^{c} \end{cases}$$

$$c \ln x + k_2 = \ln u$$
$$k_2 \cdot x^c = u$$

$$\Leftrightarrow \left.\begin{matrix} \boxed{\begin{cases} k_1 = y/x \\ k_2 = u x^{-c} \end{cases}} \\ \text{with } u(x,1) = f(x) \end{matrix}\right\} \Leftrightarrow \begin{cases} k_1 = 1/x \\ k_2 = f(x)\,x^{-c} \end{cases} \Longleftrightarrow \boxed{k_2 = f\left(\frac{1}{k_1}\right) \cdot \left(\frac{1}{k_1}\right)^{-c}}$$

$$u x^{-c} = f\left(\frac{x}{y}\right) \cdot \left(\frac{x}{y}\right)^{-c} \Leftrightarrow u = y^c f\left(\frac{x}{y}\right)$$