PW/TP 17-18: Calculus of Variations

Find the curve $Y = y(x)$ with $y(x_1) = y_1,$ and $y(x_2) = y_2$ such that for some given function $F(x, y, y')$,

$\int_{x_1}^{x_2} F(x, y, y')\mathrm{d}x$ an extremum is (1)

Euler's Equation: $y = y(x)$ satisfies $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$.

if $F(x, y, y') = F(y, y') : \leftrightarrow F - y' \frac{\partial F}{\partial y'} = c$

If we want to keep at te same time $\int_{x_1}^{x_2} G(x, y, y')\mathrm{d}x$ equal to some constant, (1) can be rewritten as:

$\int_{x_1}^{x_2} (F + \lambda G)\ \mathrm{d}x$

Exercise 1. Show that the shortest distance between two points in a plane is a straight line.

Exercise 2. Find the extremals of $I = \int_{x_1}^{x_2} F(x, y, y')\mathrm{d}x$ for each case

1. $F = (y')^2 + 2y$
2. $F = (y')^2 + 4xy'$
3. $F = (y')^2 + yy' + y^2$
4. $F = x(y')^2 - yy' + y$
5. $F = (y')^2 - y^2$

Exercise 3. The shortest distance between two points on any surface is called a *geodesic* of the surface. Show that the geodesics on the surface of a sphere of radius a are the arcs of great circles.

Exercise 4. Find the geodesics for

1. a right circular cylinder
2. a right circular cone

Exercise 5. Find the extremals

1. $$\int_0^1 y'\mathrm{d}x, \qquad y(0) = 0, y(1) = 1$$
2. $$\int_0^1 yy'\mathrm{d}x, \qquad y(0) = 0, y(1) = 1$$

3. $\displaystyle\int_0^1 (1+x)(y')^2\mathrm{d}x, \qquad y(0)=0,\, y(1)=1$