Exercise 1.

(1) $$\underbrace{y^2}_{A} \frac{\partial^2 u}{\partial x^2} \underbrace{- 2xy}_{2B} \frac{\partial^2 u}{\partial x \partial y} + \underbrace{x^2}_{C} \frac{\partial^2 u}{\partial y^2} = \frac{1}{xy}\left(y^3 \frac{\partial u}{\partial x} + x^3 \frac{\partial u}{\partial y}\right)$$

$B^2 - AC = x^2y^2 - x^2y^2 = 0 \Rightarrow$ Parabolic

On ξ = constant, $$\frac{dy}{dx} = \frac{B}{A} = \frac{-xy}{y^2} = -\frac{x}{y} \Leftrightarrow x^2 + y^2 = \text{constant} = \xi$$

Choose $\eta = y$ (independent of ξ)

We have $\frac{\partial \xi}{\partial x} = 2x$, $\frac{\partial \xi}{\partial y} = 2y$ $\quad \frac{\partial \eta}{\partial x} = 0$, $\frac{\partial \eta}{\partial y} = 1$

$$\frac{\partial u}{\partial x} = 2x \frac{\partial u}{\partial \xi} + 0 \qquad \frac{\partial u}{\partial y} = 2y \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \frac{\partial u}{\partial \xi} + 2x\left(u_{\xi\xi} \cdot 2x + u_{\xi\eta} \cdot 0\right) \qquad \frac{\partial^2 u}{\partial y^2} = 2u_\xi + 2y\left(u_{\xi\xi} \cdot 2y + u_{\xi\eta}\right) + u_{\eta\xi}\, 2y + u_{\eta\eta}$$

$$\frac{\partial^2 u}{\partial x \partial y} = 2x\left(u_{\xi\xi} \cdot 2y + u_{\xi\eta}\right)$$

Replacing this in the original equation, gives us

$$x^2 \frac{\partial^2 u}{\partial \eta^2} = \frac{x^2}{y}\frac{\partial u}{\partial \eta} \Leftrightarrow \frac{\partial^2 u}{\partial \eta^2} - \frac{1}{\eta}\frac{\partial u}{\partial \eta} = 0 \quad \text{(canonical form)}$$

$$\Leftrightarrow \frac{\partial}{\partial \eta}\left(\frac{1}{\eta}\frac{\partial u}{\partial \eta}\right) = 0$$

$$\Leftrightarrow \frac{1}{\eta}\frac{\partial u}{\partial \eta} = f(\xi) \Leftrightarrow \frac{\partial u}{\partial \eta} = \eta f(\xi)$$

$$\Leftrightarrow u = \frac{\eta^2}{2} f(\xi) + g(\xi)$$ with f, g arbitrary functions

$$u = g(x^2+y^2) + \frac{y^2}{2} f(x^2+y^2)$$

(2) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - 2\frac{\partial^2 u}{\partial y^2} + 1 = 0$ in $0 \le x \le 1,\ y > 0$ with $u = \frac{\partial u}{\partial y}$ on $y = 0$

$B^2 - AC = \frac{1}{4} + 2 > 0 \Rightarrow$ hyperbolic

$$\frac{dy}{dx} = \frac{\frac{1}{2} \pm \sqrt{9/4}}{1} = \frac{1}{2} \pm \frac{3}{2} = -1 \vee 2$$

$\frac{dy}{dx} = -1 \Leftrightarrow y + x = k_1$

$\frac{dy}{dx} = 2 \Leftrightarrow y - 2x = k_2$
$\Leftrightarrow x - \frac{1}{2}y = k_2$

$$\Rightarrow \begin{cases} \xi = x + y \\ \eta = x - \frac{1}{2}y \end{cases}$$

we do this because of the initial condition for $y = 0$.

Replacing this in the original equation, gives us
$\frac{9}{2} u_{\eta\xi} + 1 = 0$ (canonical form)

$$\frac{\partial^2 u}{\partial \eta \partial \xi} = -\frac{2}{9} \Leftrightarrow \frac{\partial u}{\partial \eta} = -\frac{2}{9}\xi + f(\eta)$$

$\Leftrightarrow u = -\frac{2}{9}\xi\eta + f(\eta) + g(\xi)$ with f, g arbitrary functions

when $y = 0$, then $u = \frac{\partial u}{\partial y} = x$ our transformation is $\begin{cases} \xi = x \\ \eta = x \end{cases}$

$u(\xi = x, \eta = x) = x \Leftrightarrow f(x) + g(x) = x + \frac{2}{9}x^2$ (1)

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi} - \frac{1}{2}\frac{\partial u}{\partial \eta} = -\frac{2}{9}\eta + g'(\xi) - \frac{1}{2}\cdot\left(-\frac{2}{9}\xi\right) - \frac{1}{2}f'(\eta)$$

$\frac{\partial u}{\partial y}(\xi = x, \eta = x) = x \Leftrightarrow g'(x) - \frac{1}{2}f'(x) = \frac{10}{9}x$

$\Leftrightarrow g(x) - \frac{1}{2}f(x) = \frac{5}{9}x^2 + k$ (2)

(1) & (2) $\Leftrightarrow f(x) = -\frac{2}{9}x^2 + \frac{2}{3}x - \frac{2}{3}k$ & $g(x) = \frac{1}{3}x + \frac{2}{3}k + \frac{4}{9}x^2$

In terms of ξ and $\eta \Rightarrow f(\eta) = -\frac{2}{9}\eta^2 + \frac{2}{3}\eta - \frac{2}{3}k$ and $g(\xi) = \frac{1}{3}\xi + \frac{2}{3}k + \frac{4}{9}\xi^2$

So $u(\xi, \eta) = -\frac{2}{9}\xi\eta - \frac{2}{9}\eta^2 + \frac{2}{3}\eta + \frac{1}{3}\xi + \frac{4}{9}\xi^2$

$$\Rightarrow u(x, y) = x + xy + \frac{y^2}{2}$$

(3) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$

$A = 1 \quad B = \frac{1}{2} \quad C = 1 \qquad B^2 - AC = -\frac{3}{4} < 0 \Rightarrow$ elliptic curve

find ξ and η by

$$\left.\begin{array}{l} \xi = \text{constant on } \dfrac{dy}{dx} = \dfrac{1+\sqrt{3}\,i}{2} \\ \eta = \text{constant on } \dfrac{dy}{dx} = \dfrac{1-\sqrt{3}\,i}{2} \end{array}\right\} \Rightarrow \begin{array}{l} \xi = y - \frac{1}{2}(1+\sqrt{3}\,i)x \\ \eta = y - \frac{1}{2}(1-\sqrt{3}\,i)x \end{array}$$

We apply the further change of variables:

$$\begin{cases} \alpha = \xi + \eta = 2y - x \\ \beta = i(\xi - \eta) = \sqrt{3}\,x \end{cases} \quad \text{with} \quad \begin{array}{l} \dfrac{\partial \alpha}{\partial x} = -1, \ \dfrac{\partial \alpha}{\partial y} = 2 \\ \dfrac{\partial \beta}{\partial x} = \sqrt{3}, \ \dfrac{\partial \beta}{\partial y} = 0 \end{array}$$

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial \alpha} + \sqrt{3}\frac{\partial u}{\partial \beta} \qquad \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial \alpha}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \alpha^2} - \sqrt{3}\frac{\partial^2 u}{\partial \alpha \partial \beta} - \sqrt{3}\frac{\partial^2 u}{\partial \beta \partial \alpha} + 3\frac{\partial^2 u}{\partial \beta^2} \qquad \frac{\partial^2 u}{\partial y^2} = 4\frac{\partial^2 u}{\partial \alpha^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = -2\frac{\partial^2 u}{\partial \alpha^2} + 2\sqrt{3}\frac{\partial^2 u}{\partial \alpha \partial \beta}$$

We get: $3\dfrac{\partial^2 u}{\partial \alpha^2} + 3\dfrac{\partial^2 u}{\partial \beta^2} = 0 \Leftrightarrow \dfrac{\partial^2 u}{\partial \alpha^2} + \dfrac{\partial^2 u}{\partial \beta^2} = 0$ (canonical form)

$u = e^{A\alpha + B\beta} \qquad A^2 u + B^2 u = 0 \Leftrightarrow A = \pm Bi$

Solutions $e^{B(\alpha+\beta i)}, e^{B(\alpha-\beta i)} \quad \forall B$

$$\Rightarrow u(\alpha,\beta) = F(\alpha + \beta i) + G(\alpha - \beta i)$$

$$u(x,y) = F(2y + (\sqrt{3}i - 1)x) + G(2y - (\sqrt{3}i + 1)x)$$

(4) $u_{xx} + 4u_{xy} + u_x = 0$

$A = 1 \quad B = 2 \quad C = 0 \quad \Rightarrow \quad B^2 - AC = 4 > 0$ hyperbolic

ξ = constant, η = constant satisfy $\dfrac{dy}{dx} = \dfrac{b \pm \sqrt{b^2 - ac}}{a}$

$\dfrac{dy}{dx} = 2 \pm \sqrt{4} = 4 \vee 0 \qquad \dfrac{dy}{dx} = 4 \Leftrightarrow y = 4x + k_1 \qquad \dfrac{dy}{dx} = 0 \Leftrightarrow y = k_2 \qquad \begin{cases} \xi = k_1 = y - 4x \\ \eta = k_2 = y \end{cases}$

Replacing this in the original equation, gives us

$\dfrac{\partial \xi}{\partial x} = -4 \qquad \dfrac{\partial \xi}{\partial y} = 1 \qquad \dfrac{\partial \eta}{\partial x} = 0 \qquad \dfrac{\partial \eta}{\partial y} = 1$

$\dfrac{\partial u}{\partial x} = -4 \dfrac{\partial u}{\partial \xi} \qquad \dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial \xi} + \dfrac{\partial u}{\partial \eta}$

$\dfrac{\partial^2 u}{\partial x^2} = 16 \dfrac{\partial^2 u}{\partial \xi^2} \qquad \dfrac{\partial^2 u}{\partial y \partial x} = -4 \dfrac{\partial^2 u}{\partial \xi^2} - 4 \dfrac{\partial^2 u}{\partial \eta \partial \xi}$

$u_{xx} + 4u_{xy} + u_x = \cancel{16 \dfrac{\partial^2 u}{\partial \xi^2}} - \cancel{16 \dfrac{\partial^2 u}{\partial \xi^2}} - 16 \dfrac{\partial^2 u}{\partial \eta \partial \xi} - 4 \dfrac{\partial u}{\partial \xi} = 0$

$\Leftrightarrow \quad \dfrac{\partial^2 u}{\partial \eta \partial \xi} + \dfrac{1}{4} \dfrac{\partial u}{\partial \xi} = 0$ (canonical form)

$\left(\dfrac{\partial u}{\partial \xi} = p\right) \quad \dfrac{\partial p}{\partial \eta} + \dfrac{1}{4} p = 0 \overset{p(\eta)}{\Leftrightarrow} \dfrac{1}{p} dp = -4 \, d\eta \overset{p(\eta,\xi)}{\Leftrightarrow} \ln p = -4\eta + f(\xi)$

$\Leftrightarrow \quad \dfrac{\partial u}{\partial \xi} = p = f(\xi) e^{-4\eta}$

$u = e^{-4\eta} \int f(\xi) d\xi + g(\eta)$

$u(\xi, \eta) = F(\xi) e^{-4\eta} + G(\eta)$

$u(x, y) = F(y - 4x) e^{-4y} + G(y)$

(5) $x^2 u_{xx} - 2xy u_{xy} + y^2 u_{yy} + x u_x + y u_y = 0 \quad \forall x > 0$

$A = x^2 \qquad B = -xy \qquad C = y^2$

$B^2 - AC = x^2y^2 - x^2y^2 = 0 \rightarrow$ parabolic

ξ = constant on $\frac{dy}{dx} = \frac{-xy}{x^2} = -\frac{y}{x} \Leftrightarrow \frac{1}{y}dy = -\frac{1}{x}dx \Leftrightarrow \ln y = -\ln x + k_1$

$\Leftrightarrow y = k_1 x^{-1} \Leftrightarrow k_1 = xy$

$\begin{cases} \xi = xy \\ \eta = x \end{cases}$ (η is choosen independent of ξ)

$\frac{\partial \xi}{\partial x} = y \quad \frac{\partial \xi}{\partial y} = x \quad \frac{\partial \eta}{\partial x} = 1 \quad \frac{\partial \eta}{\partial y} = 0$

$\frac{\partial u}{\partial x} = y\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \qquad \frac{\partial u}{\partial y} = x\frac{\partial u}{\partial \xi}$

$\frac{\partial^2 u}{\partial x^2} = y^2 \frac{\partial^2 u}{\partial \xi^2} + 2y\frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \eta^2} \qquad \frac{\partial^2 u}{\partial y^2} = x^2 \frac{\partial^2 u}{\partial \xi^2}$

$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial \xi} + xy\frac{\partial^2 u}{\partial \xi^2} + x\frac{\partial^2 u}{\partial \eta \partial \xi}$

$x^2 u_{xx} - 2xy u_{xy} + y^2 u_{yy} + x u_x + y u_y = 0$

$\Leftrightarrow 0 = x^2y^2 u_{\xi\xi} + 2x^2y u_{\eta\xi} + x^2 u_{\eta\eta} - 2xy u_\xi$
$\quad -2x^2y^2 u_{\xi\xi} - 2x^2y u_{\eta\xi}$
$\quad + x^2y^2 u_{\xi\xi} \qquad + xy u_\xi + x u_\eta$
$\qquad + xy u_\xi$

$\Leftrightarrow x^2 u_{\eta\eta} + x u_\eta = 0$

$\Leftrightarrow u_{\eta\eta} + \frac{1}{\eta} u_\eta = 0$ (canonical form)

$\eta u_{\eta\eta} + u_\eta = \frac{\partial}{\partial \eta}(\eta \cdot u_\eta) = 0$

$\Leftrightarrow \eta u_\eta = f(\xi) \Leftrightarrow du = \frac{f(\xi)}{\eta} d\eta \Leftrightarrow u(\eta, \xi) = f(\xi)\ln(\eta) + g(\xi)$

$u(x,y) = f(xy)\ln(x) + g(xy)$

(6) $u_{xx} + x u_{yy} = 0 \quad \forall x > 0$

$A = 1 \quad B = 0 \quad C = x \qquad B^2 - AC = -x < 0 \Rightarrow$ elliptic

$$\frac{dy}{dx} = \pm\sqrt{-x} \qquad \int dy = \int \sqrt{-x}\, dx \Leftrightarrow y = -\frac{2}{3}\sqrt{-x^3} + k_1$$

$$\begin{cases} \xi = y + \frac{2}{3}\sqrt{x^3}\, i \\ \eta = y - \frac{2}{3}\sqrt{x^3}\, i \end{cases}$$

We apply the further change of variables:

$$\begin{cases} \alpha = \xi + \eta = 2y \\ \beta = i(\xi - \eta) = -\frac{4}{3}\sqrt{x^3} \end{cases} \qquad \frac{\partial \alpha}{\partial x} = 0 \quad \frac{\partial \alpha}{\partial y} = 2 \qquad \frac{\partial \beta}{\partial x} = -2\sqrt{x} \quad \frac{\partial \beta}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} = -2\sqrt{x}\frac{\partial u}{\partial \beta} \qquad \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial \alpha}$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{\sqrt{x}}\frac{\partial u}{\partial \beta} + 4x\frac{\partial^2 u}{\partial \beta^2} \qquad \frac{\partial^2 u}{\partial y} = 4\frac{\partial^2 u}{\partial \alpha^2}$$

$$u_{xx} + x u_{yy} = 0 = -\frac{1}{\sqrt{x}}\frac{\partial u}{\partial \beta} + 4x\frac{\partial^2 u}{\partial \beta^2} + 4x\frac{\partial^2 u}{\partial \alpha^2}$$

$$\Leftrightarrow \frac{\partial^2 u}{\partial \beta^2} + \frac{\partial^2 u}{\partial \alpha^2} - \frac{1}{4x\sqrt{x}}\frac{\partial u}{\partial \beta} = 0$$

$$\Leftrightarrow \frac{\partial^2 u}{\partial \beta^2} + \frac{\partial^2 u}{\partial \alpha^2} + \frac{1}{3\beta}\frac{\partial u}{\partial \beta} = 0 \quad \text{(canonical form)}$$