PW/TP 1-2: Ordinary Differential Equations (CH2)

Solutions

Exercise 1. (2.60 e) Solve $(1-x^2)y' = 4y;\ y(0) = 1$.

```
clear
syms y(x) x
eqn = (1-x^2)*diff(y,x) == 4*y;
S = dsolve(eqn)
```

S = $C_1\, e^{4\,\mathrm{atanh}(x)}$

For real values x in the domain $-1 < x < 1$, the inverse hyperbolic tangent satisfies $\tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$.

```
S = simplify(S)
```

S =

$$\frac{C_1\,(x+1)^2}{(x-1)^2}$$

```
cond = y(0) == 1;
S = dsolve(eqn,cond);
S = simplify(S)
```

S =

$$\frac{(x+1)^2}{(x-1)^2}$$

We get the final solution $y = \frac{(1+x)^2}{(x-1)^2}$.

Exercise 2. (2.62 c) Solve $(2y\sin x + 3y^4 \sin x \cos x)\mathrm{d}x - (4y^3\cos^2 x + \cos x)\mathrm{d}y = 0$.

```
clear
syms x y
M = 2*y*sin(x) + 3*y^4*sin(x)*cos(x);
N = -(4*y^3*cos(x)^2+cos(x));
```

We can check if it is an exact equation:

```
My = diff(M,y);
Nx = diff(N,x);
isequal(My,Nx)
```

```
ans = logical
   0
```

Since it is not, we search an integrating factor:

```
f(x) = (My - Nx)/N;
```

```
f(x) = simplify(f(x));
mu(x) = exp(int(f(x),x))
```

mu(x) = $\cos(x)$

We find an integrating factor, only depending on x, so we can multiply M en N with this integrating factor to find an exact equation:

```
M = simplify(mu(x)*M); My = simplify(diff(M,y));
N = simplify(mu(x)*N); Nx = simplify(diff(N,x));
isequal(My,Nx)
```

```
ans = logical
   1
```

Now we can solve the exact equation:

```
U = simplify(int(M,x))
```

U = $-y\cos(x)^2\ (\cos(x)\ y^3+1)$

```
syms F(y)
eq = diff(U+F(y),y) == N
```

eq =

$$\frac{\partial}{\partial y}\ F(y) - 3\ y^3\cos(x)^3 - \cos(x)^2\ (\cos(x)\ y^3+1) = -\cos(x)^2\ (4\cos(x)\ y^3+1)$$

```
syms dF
dF = -cos(x)^2*(4*cos(x)*y^3+1)+3*y^3*cos(x)^3+cos(x)^2*(cos(x)*y^3+1);
simplify(dF)
```

ans = 0

```
syms c1
F(y) = c1;
U = U + F(y)
```

U = $c_1 - y\cos(x)^2\ (\cos(x)\ y^3+1)$

$U(x, y) = c_1 - y \cdot cos^2(x) \cdot (\cos(x) \cdot y^3 + 1)$

Thus $M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y = \mathrm{d}U(x, y) = 0 = \mathrm{d}(c_1 - y \cdot cos^2(x) \cdot (\cos(x) \cdot y^3 + 1)) = 0$ from which we must have

$$c = y \cdot cos^2(x) \cdot (\cos(x) \cdot y^3 + 1)$$

Exercise 3. (2.67 a) Solve $x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^3y^2$.

We see that it takes the form of a Bernouilli's equation: $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}y = x^2y^2$, with $n = 2$.

We let $v = y^{1-2} = y^{-1}$ and using $\frac{\mathrm{d}v}{\mathrm{d}x} = -y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x}$ we get:

$-y^2\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{1}{x}y = x^2y^2 \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} + \frac{1}{x}v = -x^2$ giving us an linear equation.

```
syms x v(x)
eq = diff(v,x)-v/x == -x^2;
dsolve(eq)
```

ans =

$$C_1\,x - \frac{x^3}{2}$$

```
syms c1
eq = 1/y == c1*x - x^3/2;
simplify(eq)
```

ans = $y\,x^3 + 2 = 2\,c_1\,x\,y$

Giving us the final solution: $2 = cxy - yx^3$.

Exercise 4. (2.66 b) Solve $x^2\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + 3xy + y^2$.

Moving x^2 to the other side, gives us an homogeneous equation: $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + 3\frac{y}{x} + \left(\frac{y}{x}\right)^2$.

Let $v = \frac{y}{x}$ or $y = \mathrm{v}x$, giving us $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$. So the equation will be $v + x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + 3v + v^2$, which is a seperable equation $\frac{1}{1+2v+v^2}\mathrm{d}v = \frac{1}{x}\mathrm{d}x$.

```
syms v x c1
eq = int(1/(1+2*v+v^2),v) - int(1/x,x) == c1
```

eq =

$$-\log(x) - \frac{1}{v+1} = c_1$$

Giving us $-\frac{1}{v+1} = c + \ln x$

```
eq = exp(-1/(v+1)) == exp (c1 + log(x));
eq = subs(eq,v,y/x);
simplify(eq)
```

ans =

$$x\,\mathrm{e}^{c_1} = \mathrm{e}^{-\frac{x}{x+y}}$$

Giving us the final solution: $cx = e^{-\frac{x}{x+y}}$.

Exercise 5. (2.71 c) Solve $(xp - y)^2 = p^2 - 1$ with $p = y'$.

We can write this as $y = xp - \sqrt{p^2 - 1}$ which is a clairaut's equation. Differentiate both sides of the equation with respect to x.

$\frac{\mathrm{d}y}{\mathrm{d}x} = p = p + x\frac{\mathrm{d}p}{\mathrm{d}x} - \frac{2p}{2\sqrt{p^2 - 1}}\frac{\mathrm{d}p}{\mathrm{d}x}$ from which $\frac{\mathrm{d}p}{\mathrm{d}x}\left(x - \frac{p}{\sqrt{p^2 - 1}}\right) = 0$.

Case 1, $\frac{\mathrm{d}p}{\mathrm{d}x} = 0$. In this case $p = c$ and so the general solution is $y = cx - \sqrt{c^2 - 1}$.

Case 2, $x - \frac{p}{\sqrt{p^2 - 1}} = 0$. In this case $x = \frac{p}{\sqrt{p^2 - 1}}$, and $y = \frac{p^2}{\sqrt{p^2 - 1}} - \sqrt{p^2 - 1} = \frac{1}{\sqrt{p^2 - 1}}$.

We find $x^2 - y^2 = 1$ which is a solution of the differential equation, as van be checked.

However, it cannot be obtained from the general solution by any choice of c. Thus $x^2 - y^2 = 1$ is a singular solution.

Exercise 6. (2.57) Find the differential equation for (a) the family of straight lines which intersect at the point $(2, 1)$ and (b) the family of circles tangent to the x axis and having unit radius.

(a) We take the general equation for a straight line: $y = ax + b$. Since they intersect at the point $(2, 1)$, we have $1 = 2a + b$.

We have $\frac{\mathrm{d}y}{\mathrm{d}x} = a$. Substituting all the constants, gives us the required differential equation of the family:

$y = y'x + (1 - 2y') \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y - 1}{x - 2}$.

(b) We take the general equation for a circle with unit radius: $(x - a)^2 + (y - b)^2 = 1$. Since they have to be tangent to the x axis, the point $(a, 0) \in C \Rightarrow b = \pm 1$.

We have $2(x - a) + 2(y - b)y' = 0$. Substituting the constant b, gives us $2(x - a) + 2(y \pm 1)y' = 0 \Rightarrow a = (y \pm 1)y' + x$.

Substituting all the constants, gives us the required differential equation of the family: $((y \pm 1)y')^2 + (y \pm 1)^2 = 1 \Rightarrow (y \pm 1)^2(y'^2 + 1) = 1$.

Exercise 7. (2.56) Find differential equations for the following families of curves: (a) $x^2 + cy^2 = 1$, (b) $y^2 = ax + b$.

See Exercises_01_Solutions.pdf

Exercise 8. Solve the following differential equations:

1. (2.59 b) $x^3 y''' = 1 + x^4; \quad y(1) = y'(1) = y''(1) = 0$

2. (2.64 b) $(x^2+x-y^2)\mathrm{d}x-y\mathrm{d}y=0$
3. (2.67 b) $2x^2y'=xy+y^3$
4. (2.72 a) $xy''-3y'=x^2$
5. (2.61 b) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{3-4xy^2}{4x^2y+6y^2}; \quad y(1)=-1$
6. (2.70 a) $y'^2+(y-1)y'-y=0$
7. (2.65 b) $xy'-4y=x$
8. (2.63 a) $(3y-2xy^3)\mathrm{d}x+(4x-3x^2y^2)\mathrm{d}y=0$
9. (2.71 a) $y=px+2p^2$, with $p=y'$
10. (2.62 d) $(2y\sin x+3y^4\sin x\cos x)\mathrm{d}x-(4y^3\cos^2 x+\cos x)\mathrm{d}y=0$
11. (2.66 a) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2y}{x}-\dfrac{y^2}{x^2}$
12. (2.66 d) $(x-y)y'+3y-5x=0$
13. (2.72 c) $y''+4y=0$
14. (2.69) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=x^2+2xy+y^2+2x+2y; \quad y(0)=0$
15. (2.71 d) $x^2y=x^3p-yp^2$, with $p=y'$
16. (2.65 d) $\dfrac{\mathrm{d}y}{\mathrm{d}x}+2y\cot x=\csc x$
17. (2.60 d) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{x\sqrt{1-y^2}}{y\sqrt{1-x^2}}$
18. (2.61 c) $(ye^x-e^{-y})\mathrm{d}x+(xe^{-y}+e^x)\mathrm{d}y=0$

See Exercises_01_Solutions.pdf