PW/TP 15-16 Characteristics of second-order Linear PDE's

The general *linear partial differential equation* of order two in two independent variables has the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2}+2B(x,y)\frac{\partial^2 u}{\partial x\partial y}+C(x,y)\frac{\partial^2 u}{\partial y^2}+D(x,y)\frac{\partial u}{\partial x}+E(x,y)\frac{\partial u}{\partial y}+F(x,y)u=G(x,y)$$

where $A, B, \ldots, G$ may depend on x and y but not on u.

Characteristics and normal form of second-order linear PDE's

Calculate $B^2-\mathrm{AC}$.

1. $B^2-\mathrm{AC}>0$**: hyperbolic curve**

The PDE can be reduced to $\frac{\partial^2 u}{\partial\xi\partial\eta}+(\text{lower order terms})=0$.

The characteristics $\xi(x,y)=\text{constant}$, and $\eta(x,y)=\text{constant}$ satisfy $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{b\pm\sqrt{b^2-ac}}{a}$.

2. $B^2-\mathrm{AC}=0$**: parabolic curve**

The PDE can be reduced to $\frac{\partial^2 u}{\partial\eta^2}+(\text{lower order terms})=0$.

The characteristics $\xi(x,y)=\text{constant}$ satisfy $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{b}{a}$.

$\eta(x,y)$ can be choosen arbitrarily, provided that ξ and η are independent (thus $\xi_x\eta_y-\xi_y\eta_x\neq 0$).

3. $B^2-\mathrm{AC}<0$**: elliptic curve**

The PDE can be reduced to $\frac{\partial^2 u}{\partial\alpha^2}+\frac{\partial^2 u}{\partial\beta^2}+(\text{lower order terms})=0$.

We known $\xi(x,y)=\text{constant}$, and $\eta(x,y)=\text{constant}$ satisfy $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{b\pm\sqrt{b^2-ac}}{a}$.

Apply a further change of variables $(\xi,\eta)\rightarrow(\alpha=\xi+\eta,\beta=i(\xi-\eta))$ to find the canonical form

We have now real characteristics.

Exercise 1.

Investigate the following PDE's:

1. $y^2\frac{\partial^2 u}{\partial x^2}-2xy\frac{\partial^2 u}{\partial x\partial y}+x^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{xy}(y^3\frac{\partial u}{\partial x}+x^3\frac{\partial u}{\partial y})$
2. $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial y}-2\frac{\partial^2 u}{\partial y^2}+1=0$ in $0\le x\le 1, y>0$ with $u=\frac{\partial u}{\partial y}=x$ on $y=0$

3. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$
4. $u_{xx} + 4u_{xy} + u_x = 0$
5. $x^2 u_{xx} - 2xyu_x y + y^2 u_{yy} + xu_x + yu_y = 0$, $\forall x > 0$
6. $u_{xx} + xu_{yy} = 0$, $\forall x > 0$