PW/TP 19-20: Calculus of Variations

Find the curve $Y = y(x)$ with $y(x_1) = y_1,$ and $y(x_2) = y_2$ such that for some given function $F(x, y, y')$, $\int_{x_1}^{x_2} F(x, y, y')\mathrm{d}x$ an extremum is (1)

Euler's Equation: $y = y(x)$ satisfies $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$. if $F(x, y, y') = F(y, y'): \leftrightarrow F - y'\frac{\partial F}{\partial y'} = c$

Generalizations for other functionals

1) Integrand $F(t, x_1, x_2, \dot{x}_1, \dot{x}_2)$

The Euler's equation becomes the system $\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial \dot{x}_1}\right) - \frac{\partial F}{\partial x_1} = 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial \dot{x}_2}\right) - \frac{\partial F}{\partial x_2} = 0 \end{cases}$

This can be generalized for $F(t, x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n)$, to $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial \dot{x}_k}\right) - \frac{\partial F}{\partial x_k} = 0, \quad \forall k = 1, \ldots, n$

2) Integrand $F(x, y, y', y'')$

The Euler's equation becomes: $\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial F}{\partial y''}\right)$

This can be generalized for $F(x, y, y', y'', \ldots, y^{(n)})$, to

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial F}{\partial y''}\right) - \frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(\frac{\partial F}{\partial y'''}\right) + \ldots + (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{\partial F}{\partial y^{(n)}}\right)$$

Exercise 1. Find the extremal for the functional $I = \int_0^1 (1 + (y'')^2)\mathrm{d}x$ satisfying the end conditions $y(0) = 0, y'(0) = 1, y(1) = 1, y'(1) = 0$.

Exercise 2. Find the function which creates a stationary value for the functional $I = \int_0^{\pi/2} (y'^2 + 2xyy')\mathrm{d}x$ satisfying the end conditions $y(0) = y_0$ and $y\left(\frac{\pi}{2}\right) = y_1$.

Exercise 3. Find the curve joining the points $(0,0)$ and $(1,0)$ for which the functional $I = \int_0^1 (y'')^2\mathrm{d}x$ is a minimum and y is subject to the conditions $y'(0) = a$ and $y'(1) = b$, where a, b are given constants.

Exercise 4. Find a stationary value for the functional $I = \int_0^{\pi/2} [(\dot{x}_1)^2 + (\dot{x}_2)^2 + 2x_1x_2]\mathrm{d}t$ subject to the end point conditions $x_1(0) = 0,\ x_1\left(\frac{\pi}{2}\right) = 1, x_2(0) = 1, x_2\left(\frac{\pi}{2}\right) = -1$.

Exercise 5.

Exercise 6.

Exercise 7.

Lagrange multipliers

If we want to keep at te same time $\int_{x_1}^{x_2} G(x, y, y')\mathrm{d}x$ equal to some constant, (1) can be rewritten as:

$\int_{x_1}^{x_2}(F + \lambda G)\ \mathrm{d}x$

Exercise 8. Find $w = w(x)$ such that the functional $I = \int_0^{\pi/2} w^2(x)\mathrm{d}x$ is to be minimum, where w is subject to the constraints $\frac{\mathrm{d}w}{\mathrm{d}x} + y - (y - z)^2 y = 0, \quad \frac{\mathrm{d}y}{\mathrm{d}x} - w = 0$ with w, y, z subject to the boundary conditions

$y(0) = 0, z(0) = 0, w(0) = 1, y(\pi/2) = 1, z(\pi/2) = 1, w(\pi/2) = 0$.

Exercise 9. Find a function $y(x)$ for which $\int_0^{\pi} (y'^2 - y^2)\mathrm{d}x$ if $\int_0^{\pi} y\mathrm{d}x = 1$ and $y(0) = 0$, $y(\pi) = 1$.

Exercise 10.

Exercise 11.

Exercise 12.

Natural boundary conditions

If one or both of the end point conditions $y(x_1) = y_1$ and $y(x_2) = y_2$ are not prescribed (when one or both ends can vary), we use the natural boundary conditions or transversality conditions:

$$\frac{\partial F}{\partial y'}\Big|_{x=x_1} \text{ and } \frac{\partial F}{\partial y'}\Big|_{x=x_2}$$

Exercise 13. Find the curve $y = y(x)$ producing the shortest distance between the points x_0 and x_1 subject to natural boundary conditions.

Exercise 14. Find the extremum for the functional $I = \int_1^2 [y' + x^2(y')^2]\mathrm{d}x$ with boundary conditions $y(1) = 1, y(2) = 2$. Define also the natural boundary conditions for this problem.

Exercise 15. Find the extremum for the functional $I = \int_0^{\pi} [y'^2 + 2y\sin x]\mathrm{d}x$ with boundary conditions $y(0) = 0, y(\pi\ ) = 0$. Define also the natural boundary conditions for this problem.

Exercise 16. Determine the stationary functions associated with the integral $I = \int_0^1 [(y')^2 - 2\alpha y y' - 2\beta y']\mathrm{d}x$ where α and β are constants, in each of the following situations:

- The end conditions $y(0) = 0$ and $y(1) = 1$ are preassigned.
- Only the end condition $y(0) = 0$ is preassigned.
- Only the end condition $y(1) = 1$ is preassigned.
- No end conditions are preassigned.

Exercise 17.

Exercise 18.