PW/TP 19-20: Calculus of Variations

Find the curve $Y=y(x)$ with $y(x_1)=y_1,$ and $y(x_2)=y_2$ such that for some given function $F(x,y,y')$, $\int_{x_1}^{x_2} F(x,y,y')\mathrm{d}x$ an extremum is (1)

Euler's Equation: $y=y(x)$ satisfies $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial y}=0.$ if $F(x,y,y')=F(y,y'):\leftrightarrow F-y'\frac{\partial F}{\partial y'}=c$

Generalizations for other functionals

1) Integrand $F(t,x_1,x_2,\dot{x}_1,\dot{x}_2)$

The Euler's equation becomes the system $\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{x}_1}\right)-\frac{\partial F}{\partial x_1}=0\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{x}_2}\right)-\frac{\partial F}{\partial x_2}=0\end{cases}$

This can be generalized for $F(t,x_1,x_2,\ldots,x_n,\dot{x}_1,\dot{x}_2,\ldots,\dot{x}_n)$, to $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{x}_k}\right)-\frac{\partial F}{\partial x_k}=0,\quad \forall k=1,\ldots,n$

2) Integrand $F(x,y,y',y'')$

The Euler's equation becomes: $\frac{\partial F}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)+\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial F}{\partial y''}\right)$

This can be generalized for $F(x,y,y',y'',\ldots,y^{(n)})$, to

$$\frac{\partial F}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)+\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial F}{\partial y''}\right)-\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(\frac{\partial F}{\partial y'''}\right)+\ldots+(-1)^n\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{\partial F}{\partial y^{(n)}}\right)$$

Exercise 1. Find the extremal for the functional $I=\int_0^1(1+(y'')^2)\mathrm{d}x$ satisfying the end conditions $y(0)=0, y'(0)=1, y(1)=1, y'(1)=0$.

Exercise 2. Find the function which creates a stationary value for the functional $I=\int_0^{\pi/2}(y'^2+2xyy')\mathrm{d}x$ satisfying the end conditions $y(0)=y_0$ and $y\left(\frac{\pi}{2}\right)=y_1$.

Exercise 3. Find the curve joining the points $(0,0)$ and $(1,0)$ for which the functional $I=\int_0^1(y'')^2\mathrm{d}x$ is a minimum and y is subject to the conditions $y'(0)=a$ and $y'(1)=b$, where a,b are given constants.

Exercise 4. Find a stationary value for the functional $I=\int_0^{\pi/2}[(\dot{x}_1)^2+(\dot{x}_2)^2+2x_1x_2]\mathrm{d}t$ subject to the end point conditions $x_1(0)=0,\ x_1\left(\frac{\pi}{2}\right)=1, x_2(0)=0, x_2\left(\frac{\pi}{2}\right)=-1$.

Exercise 5. Find the extremal for the functional $I=\int_0^1(y'(x)z'(x)+y(x)^2)\mathrm{d}x$ subject to the conditions $y(0)=z(0)=0$ and $y(1)=z(1)=1$.

Exercise 6. Find the extremal for the functional $I = \int_0^{\frac{\pi}{4}} (4y_1^2 + y_2^2 + y_1{}'y_2{}')\mathrm{d}x$ subject to the conditions $y_1(0) = 1, y_1\left(\frac{\pi}{4}\right) = 0$ and $y_2(0) = 0, y_2\left(\frac{\pi}{4}\right) = 1$.

Exercise 7. Find the extremal for the functional $I = \int_{x_0}^{x_1} (y^2 - (y'')^2)\mathrm{d}x$.

Lagrange multipliers

If we want to keep at te same time $\int_{x_1}^{x_2} G(x, y, y')\mathrm{d}x$ equal to some constant, (1) can be rewritten as:

$\int_{x_1}^{x_2} (F + \lambda G)\ \mathrm{d}x$

Exercise 8. Find $w = w(x)$ such that the functional $I = \int_0^{\pi/2} w^2(x)\mathrm{d}x$ is to be minimum, where w is subject to the constraints $\frac{\mathrm{d}w}{\mathrm{d}x} + y - (y-z)^2 y = 0, \quad \frac{\mathrm{d}y}{\mathrm{d}x} - w = 0$ with w, y, z subject to the boundary conditions

$y(0) = 0, z(0) = 0, w(0) = 1, y(\pi/2) = 1, z(\pi/2) = 1, w(\pi/2) = 0$.

Exercise 9. Find a function $y(x)$ for which $\int_0^{\pi} (y'^2 - y^2)\mathrm{d}x$ if $\int_0^{\pi} y\mathrm{d}x = 1$ and $y(0) = 0$, $y(\pi) = 1$.

Exercise 10. Find the extremal curves $y = y(x)$ of $\int_0^1 y'^2\mathrm{d}x$, such that $y(0) = y(\pi) = 0$, and $\int_0^1 y^2\mathrm{d}x = 2$.

Natural boundary conditions

If one or both of the end point conditions $y(x_1) = y_1$ and $y(x_2) = y_2$ are not prescribed (when one or both ends can vary), we use the natural boundary conditions or transversality conditions:

$$\frac{\partial F}{\partial y'}\Big|_{x=x_1} \text{ and } \frac{\partial F}{\partial y'}\Big|_{x=x_2}$$

Exercise 11. Find the curve $y = y(x)$ producing the shortest distance between the points x_0 and x_1 subject to natural boundary conditions.

Exercise 12. Find the extremum for the functional $I = \int_1^2 [y' + x^2(y')^2]\mathrm{d}x$ with boundary conditions $y(1) = 1, y(2) = 2$. Define also the natural boundary conditions for this problem.

Exercise 13. Find the extremum for the functional $I = \int_0^{\pi} [y'^2 + 2y\sin x]\mathrm{d}x$ with boundary conditions $y(0) = 0, y(\pi) = 0$. Define also the natural boundary conditions for this problem.

Exercise 14. Determine the stationary functions associated with the integral $I = \int_0^1 [(y')^2 - 2\alpha yy' - 2\beta y']\mathrm{d}x$ where α and β are constants, in each of the following situations:

- The end conditions $y(0) = 0$ and $y(1) = 1$ are preassigned.
- Only the end condition $y(0) = 0$ is preassigned.

- Only the end condition $y(1) = 1$ is preassigned.
- No end conditions are preassigned.