PW/TP 11-12 Partial Differential Equations

First-order Quasilinear Equation

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

Solving the Homogeneous Equation of two independent variables

$$a(x,y)u_x + b(x,y)u_y = 0$$

the characteristic curve: $a\mathrm{d}y - b\mathrm{d}x = 0$

Calculate solution $f(x,y) = k$ with k an arbitrary constant.

$$u = F(f(x,y))$$

Exercise 1. Find the general solution of

1. $3u_x - 7u_y = 0$
2. $y^2 u_x + \frac{1}{x}u_y = 0$

the characteristic curve is $y^2\mathrm{d}y - \frac{1}{x}\mathrm{d}x = 0$, which gives as solution $y^3 - 3\ln(x) = k$.

The general solution of the equation is then $u = F(y^3 - 3\ln(x))$.

```
syms u(x,y)
u = y^3-3*log(x);
Eq = y^2*diff(u,x)+1/x*diff(u,y)
```

Eq = 0

But $u = F(y^3 - 3\ln(x))$, so also the following are possible solutions:

```
u1(x,y) = cos(u);
u2(x,y) = 5*u^3-2*u^2;
u3(x,y) = exp(u);
subs(Eq,u,u1), subs(Eq,u,u2), subs(Eq,u,u3)
```

ans = 0

ans = 0

ans = 0

3. $2xyu_x + (x^2 + y^2)u_y = 0$

Solving the Homogeneous Equation of more independent variables

$$a(x,y,z)u_x + b(x,y,z)u_y + c(x,y,z)u_z = 0$$

Define the system $\frac{\mathrm{d}x}{a} = \frac{\mathrm{d}y}{b} = \frac{\mathrm{d}z}{c}$

Calculate solution $f_1(x, y, z) = k_1$ and $f_2(x, y, z) = k_2$ with k_1 and k_2 arbitrary constants.

$$u = F(f_1(x, y, z), f_2(x, y, z))$$

Exercise 2. Find the general solution of

1. $3u_x + 7u_y + 5u_z = 0$
2. $xu_x + zu_y + z^2u_z = 0$

```
syms u(x,y,z)
u1(x,y,z) =z/exp(y);
u2(x,y,z) = -exp(-y)*(z*log(x)+1);
Eq = x*diff(u,x)+z*diff(u,y)+z^2*diff(u,z);
subs(Eq,u,u1), simplify(subs(Eq,u,u2))
```

`ans(x, y, z) =` 0

`ans(x, y, z) =` 0

The general solution of two independent variables

We search $u = f(x, y)$ of $a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$.

Write $F(x, y, u) = 0$ of $a(x, y, u)F_x + b(x, y, u)F_y + c(x, y, u)F_u = 0$

Define the system $\frac{\mathrm{d}x}{a} = \frac{\mathrm{d}y}{b} = \frac{\mathrm{d}u}{c}$

Calculate solution $f_1(x, y, u) = k_1$ and $f_2(x, y, u) = k_2$ with k_1 and k_2 arbitrary constants.

$$F(x, y, u) = G(f_1(x, y, u), f_2(x, y, u)) = 0$$

Exercise 3. Find the general solution of

1. $xu_x + yu_y = u + 1$

```
syms u(x,y)
Eq = x*diff(u,x)+y*diff(u,y)==u+1;
clear
syms f(x,y,u)
Eq2 = x*diff(f,x)+y*diff(f,y)+(u+1)*diff(f,u);
f1(x,y,u) = log(u+1)-log(y);
f2(x,y,u) = log(y)-log(x);
subs(Eq2,f,f1), subs(Eq2,f,f2)
```

`ans(x, y, u) =` 0

`ans(x, y, u) =` 0

2. $u_x + axu_y = bxu^2$, with $a, b \in \mathbb{R}$

```
syms f(x,y,u) a b
Eq2 = diff(f,x)+a*x*diff(f,y)+b*x*u^2*diff(f,u);
f1(x,y,u) = 2*y-a*x^2;
f2(x,y,u) = -b*x^2/2-1/u;
subs(Eq2,f,f1), subs(Eq2,f,f2)
```

ans(x, y, u) = 0

ans(x, y, u) = 0

3. $u_x + yu^2u_y + au = 0$

The general solution of two independent variables with boundary conditions

using boundary conditions $\begin{cases} \gamma_1(x,y,u) = 0 \\ \gamma_2(x,y,u) = 0 \end{cases}$, we get $\begin{cases} f_1(x,y,u) = k_1 \\ f_2(x,y,u) = k_2 \\ \gamma_1(x,y,u) = 0 \\ \gamma_2(x,y,u) = 0 \end{cases}$

Find the relation between k_1 and k_2: $F_1(k_1, k_2) = 0$

Giving the solution: $F_1(f_1(x,y,u), f_2(x,y,u)) = 0$

Exercise 4. Find the general solution of

1. $xu_x + yu_y = u + 1$ with $u(x,y) = x^2$ on $y = x^2$

```
syms u(x,y)
u(x,y) = (x^2+y^2-y)/y;
Eq = x*diff(u,x)+y*diff(u,y)==u+1;
simplify(Eq)
```

ans(x, y) = symtrue

2. $u_x + uu_y = 0$, with $u = \varphi(y)$ at $x = 0$

```
syms u(x,y) phi(z)
phi(z) = z;
u(x,y) = y/(1+x);
Eq = diff(u,x)+u*diff(u,y)==0
```

Eq(x, y) = $0 = 0$

3. $u_x + u_y = u$, with $u(x,0) = \cos(x)$

```
syms u(x,y)
u(x,y) = cos(x-y)*exp(y);
Eq = diff(u,x)+diff(u,y) == u, simplify(Eq)
```

```
Eq(x, y) = 
```
$\cos(x-y)\,e^y = \cos(x-y)\,e^y$

```
ans(x, y) = 
```
symtrue

4. $xu_x + yu_y = cu,$ with $u(x,1) = f(x)$