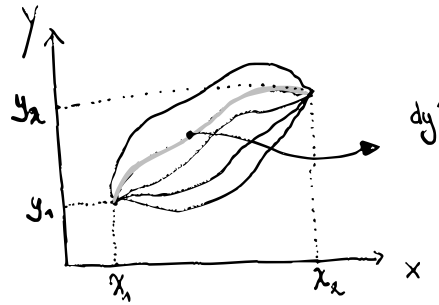
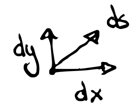
1)





$$ds = \sqrt{dx^2 + dy^2}$$
with $y(x)$

$$\Leftrightarrow \quad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$
$$= \sqrt{1 + y'^2}\, dx$$
$=:$ Arc length

$\Rightarrow$ distance of curve $y(x)$ from x_1 to x_2 is $\int_{x_1}^{x_2} ds$

Thus we search $y(x)$ for which $\int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx$ is minimal

Using Euler's equation with $F(x, y, y') = \sqrt{1 + y'^2}$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0, \text{ becomes } \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Leftrightarrow \quad \frac{\partial F}{\partial y'} = c$$

$$\Leftrightarrow \quad \frac{1}{\cancel{2}\sqrt{1+y'^2}} \cdot \cancel{2} y' = c \quad \Leftrightarrow \quad y' = c\sqrt{1 + y'^2}$$

$$\Leftrightarrow \quad y'^2 = c^2(1 + y'^2) \quad \Leftrightarrow \quad (1-c^2)\, y'^2 = c^2$$

$$\Leftrightarrow \quad \frac{dy}{dx} = y' = k \quad \left(\text{with } k = \sqrt{\frac{c^2}{1-c^2}}\right)$$

$$\Leftrightarrow \quad y = kx + m$$
with constants k and m to be determined from points $(x_1, y_1), (x_2, y_2)$.

2.1) $F(x,y,y') = (y')^2 + 2y$

$$F - y'\frac{\partial F}{\partial y'} = c \qquad \frac{\partial F}{\partial y'} = 2y'$$

$$\Leftrightarrow \quad y'^2 + 2y - 2y'^2 = c$$

$$\Leftrightarrow \quad y'^2 - 2y + c = 0$$

$$\Leftrightarrow \frac{dy}{dx} = y' = \sqrt{2y - c} \qquad \Leftrightarrow \int \frac{1}{\sqrt{2y-c}}\,dy = \int dx$$

$$\Leftrightarrow \quad \sqrt{2y-c} = x + k \quad \Leftrightarrow \quad y = \frac{1}{2}(x+k)^2 + \frac{c}{2}$$

2.2) $F = y'^2 + 4xy'$

$$\frac{\partial F}{\partial y} = 0 \qquad \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \Leftrightarrow \frac{\partial F}{\partial y'} = c$$

$$\frac{\partial F}{\partial y'} = 2y' + 4x = c \quad \Leftrightarrow \quad 2y' = c - 4x$$

$$\Leftrightarrow \int 2\,dy = \int (c - 4x)\,dx$$

$$\Leftrightarrow \quad 2y = cx - 2x^2 + k$$

$$\Leftrightarrow y = -x^2 + \frac{c}{2}x + \frac{k}{2}$$

2.3) $F = y'^2 + yy' + y^2$

$$\frac{\partial F}{\partial y'} = 2y' + y \quad : \quad F - y'\frac{\partial F}{\partial y'} = c \Leftrightarrow y'^2 + yy' + y^2 - 2y'^2 - yy' = c$$

$$\Leftrightarrow \quad y^2 - y'^2 = c \quad \Leftrightarrow \quad \frac{dy}{dx} = y' = \sqrt{y^2 - c}$$

$$\Leftrightarrow \int \frac{1}{\sqrt{y^2 - c}}\,dy = \int dx \Leftrightarrow \ln\left|y + \sqrt{y^2 - c}\right| = x + k$$

$$\Leftrightarrow \quad y + \sqrt{y^2 - c} = ke^x$$

2.4) $F = xy'^2 - yy' + y$

$$\frac{\partial F}{\partial y} = -y' + 1 \qquad \frac{\partial F}{\partial y'} = 2xy' - y$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 2y' + 2xy'' - y'$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0 \iff y' + 2xy'' + y' - 1 = 0$$

$$\iff 2xy'' + 2y' - 1 = 0$$

$$\iff (2xy')' = 1 \qquad \xrightarrow{2xy' = p} \quad \frac{dp}{dx} = 1$$

$$\iff 2xy' = p = x + C_1$$

$$\iff y' = \frac{1}{2} + \frac{C_1}{2} \cdot \frac{1}{x}$$

$$\iff y = \frac{1}{2}x + \frac{C_1}{2}\ln x + C_2$$

2.5) $I = \int_0^b (y'^2 - y^2)\, dx$

$$\frac{\partial F}{\partial y'} = 2y' \quad \Rightarrow \quad F - y'\frac{\partial F}{\partial y'} = c$$

$$\iff y'^2 - y^2 - 2y'^2 = c$$

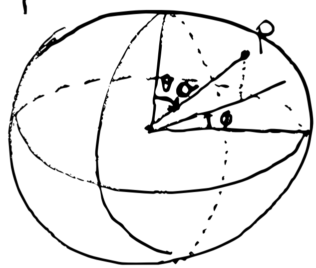
$$\iff y' = \sqrt{-y^2 - c}$$

$$\iff \int \frac{1}{\sqrt{-c - y^2}}\, dy = \int dx$$

$$\iff \sin^{-1}\left(\frac{y}{\sqrt{-c}}\right) = x + C_1$$

$$\iff y = \sqrt{-c}\, \sin(x + C_1)$$

3) Spherical coordinates



$$\begin{cases} x = a\sin\theta\cos\phi \\ y = a\sin\theta\sin\phi \\ z = a\cos\theta \end{cases}$$

Take $\phi(t), \theta(t)$

$$\frac{dx}{dt} = a\cos\theta\cos\phi\,\frac{d\theta}{dt} - a\sin\theta\sin\phi\,\frac{d\phi}{dt}$$

$$\frac{dy}{dt} = a\cos\theta\sin\phi\,\frac{d\theta}{dt} + a\sin\theta\cos\phi\,\frac{d\phi}{dt}$$

$$\frac{dz}{dt} = -a\sin\theta\,\frac{d\theta}{dt}$$

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = a^2\left(\frac{d\theta}{dt}\right)^2 + a^2\sin^2\theta\left(\frac{d\phi}{dt}\right)^2$$

$$L = \int_{t_1}^{t_2}\sqrt{a^2(d\theta^2 + \sin^2\theta\,d\phi^2)} = a\int_{\theta_1}^{\theta_2}\sqrt{1 + \sin^2\theta\,\phi'^2}\,d\theta \qquad F(\theta, \phi, \phi')$$

with the path described as $\phi(\theta)$

$$\frac{\partial F}{\partial \phi} = 0 \Rightarrow \frac{\partial F}{\partial \phi'} = c = \frac{\sin^2\theta\,\phi'}{\sqrt{1 + \sin^2\theta\,\phi'^2}} \Leftrightarrow c^2 + c^2\sin^2\theta\,\phi'^2 = \sin^4\theta\,\phi'^2$$

$$\Leftrightarrow \frac{d\phi}{d\theta} = \phi' = \frac{c}{\sqrt{\sin^4\theta - c^2\sin^2\theta}} = \frac{c}{\sin\theta\sqrt{\sin^2\theta - c^2}}$$

$$\Leftrightarrow \int d\phi = \int \frac{c}{\sin\theta\sqrt{\sin^2\theta - c^2}}\,d\theta \qquad \text{Let } u = \cot\theta,\ du = -\frac{1}{\sin^2\theta}\,d\theta$$

(with $1 + \cot^2\theta = \frac{1}{\sin^2\theta}$)

$$\Leftrightarrow \phi = \int \frac{-c\sin^2\theta\,du}{\sin\theta\sqrt{\sin^2 - c^2}} = \int \frac{-c\sin^2\theta}{\sin^2\theta\sqrt{1 - \frac{c^2}{\sin^2\theta}}}\,du = \int \frac{-c}{\sqrt{1 - c^2(1+u^2)}}\,du = \int \frac{1}{\sqrt{a^2 - u^2}} \qquad \left(a^2 = \frac{1-c^2}{c^2}\right)$$

$$\Leftrightarrow \phi = -\sin^{-1}\left(\frac{u}{a}\right) + \phi_0 \quad \Rightarrow a\sin(\phi_0 - \phi) = \cot\theta$$

4.1)

$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \\ z = z \end{cases}$$

$\phi(t)$, $z(t)$

$$\frac{dx}{dt} = -r\sin\phi\,\frac{d\phi}{dt}$$

$$\frac{dy}{dt} = r\cos\phi\,\frac{d\phi}{dt}$$

$$\frac{dz}{dt} = \frac{dz}{dt}$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = r^2\left(\frac{d\phi}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

$$L = \int_a^b \sqrt{r^2(d\phi)^2 + (dz)^2} = \int_a^b \sqrt{r^2 + (z')^2}\,d\phi$$ with path $z(\phi)$

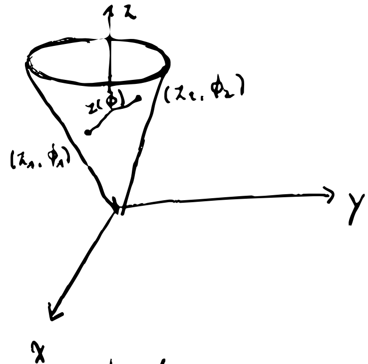
F does not contain z : $$\frac{\partial F}{\partial z'} = \frac{1}{2\sqrt{r^2 + z'^2}} \cdot 2z' = c$$

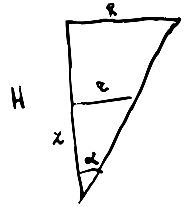
$$z'^2 = c^2(r^2 + z'^2) \Leftrightarrow z'^2(1-c^2) = c^2 \cdot r^2$$

$$\Leftrightarrow z' = \frac{c\cdot r}{\sqrt{1-c^2}} \Leftrightarrow \int dz = \int \frac{c\cdot r}{\sqrt{1-c^2}}\,d\phi$$

$$\Leftrightarrow z = \frac{c\cdot r}{\sqrt{1-c^2}}\,\phi + z_0$$

4.2)





$\Rightarrow k = \frac{R}{H}$ or $k = \tan\alpha$

Cartesian coordinates or cylindrical coordinates

$$\begin{cases} x = kz\cos\phi \\ y = kz\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \rho = kz \\ \phi = \phi \\ z = z \end{cases}$$

$$ds^2 = dx^2 + dy^2 + dx^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$$
$$= k^2 dz^2 + k^2 z^2 d\phi^2 + dz^2$$

$$ds = \underbrace{\sqrt{(1+k^2)z'^2 + k^2 z^2}}_{F(\phi, z, z')}\, d\phi$$

[divide F by $(1+k^2)^{1/2}$, using $\frac{k^2}{1+k^2} = \sin^2\alpha$: Let $a = \sin\alpha$]

$$I = \int_{\phi_1}^{\phi_2} \sqrt{(1+k^2)z'^2 + k^2 z^2}\, d\phi = \sqrt{1+k^2} \int_{\phi_1}^{\phi_2} \underbrace{\sqrt{z'^2 + a^2 z^2}}_{\text{new } F}\, d\phi$$

$$\frac{\partial F}{\partial z'} = \frac{z'}{\sqrt{z'^2 + a^2 z^2}} \underset{\text{Euler}}{\Rightarrow} \frac{z'^2 + a^2 z^2 - z'^2}{\sqrt{z'^2 + a^2 z^2}} = c \iff a^4 z^4 = c^2 (z'^2 + a^2 z^2)$$

$$\iff z' = \sqrt{\frac{a^4 z^4 - c^2 a^2 z^2}{c^2}} = az\sqrt{\frac{a^2 z^2}{c^2} - 1} \iff \int \frac{dz}{z\sqrt{\frac{a^2 z^2}{c^2} - 1}} = a\phi + C_1$$

$$\underset{\left(\begin{array}{c} z = \sec u \\ dz = \frac{\sin u}{\cos^2 u} du \end{array}\right)}{\iff} \sec^{-1}\left(\frac{a}{c} z\right) = a\phi + c_1 \iff z(\phi) = \frac{c}{a\cos(a\phi + c_1)}$$

5.1) $\int_0^1 y' dx$ $\quad \frac{\partial F}{\partial y} = 0 \Rightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$\Leftrightarrow \frac{\partial F}{\partial y'} = c \Leftrightarrow 1 = c$

which is satisfied for all y
Clearly y should also satisfy the boundary conditions, i.e $y = x$

5.2) $\int_0^1 yy' dx$ $\quad \frac{\partial F}{\partial y'} = y$

$F - y'\frac{\partial F}{\partial y'} = c \Leftrightarrow yy' - y'y = c \Leftrightarrow 0 = c$

$\Rightarrow$ Again looking at the boundary conditions gives us $y = x$

5.3) $\int_0^1 (1+x) y'^2 dx$ $\quad \frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial F}{\partial y'} = c$

$\frac{\partial F}{\partial y'} = 2(1+x)y' = c \Leftrightarrow y' = \frac{c}{2(1+x)}$

$\Leftrightarrow y = \frac{c}{2}\ln(1+x) + C_2 = C_1 \ln(1+x) + C_2$

$y(0) = 0 : \quad 0 = C_1 \ln(1) + C_2 \Leftrightarrow C_2 = 0$

$y(1) = 1 : \quad 1 = C_1 \ln(2) \Leftrightarrow C_1 = \frac{1}{\ln 2}$

$\Leftrightarrow y = \frac{1}{\ln 2}\ln(1+x)$