Ex 1 $\quad I = \int_0^1 (1 + y''^2)\, dx \qquad y(0)=0,\ y'(0)=1$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad y(1)=1,\ y'(1)=0$

$$F(x,y,y',y'') = 1 + y''^2$$

$$\frac{\partial F}{\partial y} = 0 \qquad \frac{\partial F}{\partial y'} = 0 \qquad \frac{\partial F}{\partial y''} = 2y''$$

$$\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0 \iff \frac{\partial F}{\partial y''} = C_1 x + C_2$$

$$2y'' = C_1 x + C_2$$

$$2y' = C_1 \frac{x^2}{2} + C_2 x + C_3$$

$$y = \frac{C_1}{2\cdot 2\cdot 3} x^3 + \frac{C_2}{2} x^2 + C_3 x + C_4$$

$$y = ax^3 + bx^2 + cx + d$$

$y(0) = d = 0$

$y(1) = a + b + c = 1$

$y'(x) = 3ax^2 + 2bx + c \rightarrow y'(0) = c = 1$

$\qquad\qquad\qquad\qquad\qquad\quad y'(1) = 3a + 2b + c = 0$

$$\left.\begin{cases} a + b = 0 \\ 3a + 2b = -1 \end{cases} \iff \begin{cases} b = 1 \\ a = -1 \end{cases}\right.$$

$$y = -x^3 + x^2 + x$$

Ex2 $\quad I = \int_0^{\pi/2} (y'^2 + 2xyy')\,dx \qquad y(0) = y_0 \,,\; y(\frac{\pi}{2}) = y_1$

$F(x, y, y') = y'^2 + 2xyy'$

$\frac{\partial F}{\partial y} = 2xy' \qquad \frac{\partial F}{\partial y'} = 2y' + 2xy$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 2y'' + 2y + 2xy'$$

$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \Leftrightarrow 2y'' + 2y = 0 \Leftrightarrow y'' + y = 0$

$$m^2 + 1 = 0 \rightarrow m = \pm i$$

$$y = A\cos x + B\sin x$$

$y(0) = A = y_0$

$y(\frac{\pi}{2}) = B = y_1 \qquad \Rightarrow \quad y = y_0 \cos x + y_1 \sin x$

Ex3 $\quad I = \int_0^1 (y'')^2 dx \qquad y'(0) = a \qquad y(0) = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad y'(1) = b \qquad y(1) = 0$

$$\frac{\partial F}{\partial y} = 0 \quad \frac{\partial F}{\partial y'} = 0 \quad \frac{\partial F}{\partial y''} = 2y''$$

$$\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) = 0 \iff 2y'' = c_1 x + c_2$$

$$2y' = \frac{c_1}{2}x^2 + c_2 x + c_3$$

$$y = \underbrace{\frac{c_1}{12}}_{c_1} x^3 + \underbrace{\frac{c_2}{4}}_{c_2} x^2 + \underbrace{\frac{c_3}{2}}_{c_3} x + \underbrace{c_4}_{c_4}$$

$y(0) = c_4 = 0$, $y(1) = c_1 + c_2 + c_3 = 0$

$$\left.\begin{aligned} y'(x) &= 3c_1 x^2 + 2c_2 x + c_3 \\ y'(0) &= c_3 = a \\ y'(1) &= 3c_1 + 2c_2 + a = b \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= b + a \\ c_2 &= -2a - b \end{aligned}$$

$$y = (b+a)x^3 - (2a+b)x^2 + ax$$

EX4 $\quad I = \int_0^{\pi/2} \dot{x}_1^2 + \dot{x}_2^2 + 2x_1x_2 \, dt \qquad \begin{array}{ll} x_1(0) = 0 & x_1(\frac{\pi}{2}) = 1 \\ x_2(0) = 0 & x_2(\frac{\pi}{2}) = -1 \end{array}$

$$F(t, x_1, x_2, \dot{x}_1, \dot{x}_2) = \dot{x}_1^2 + \dot{x}_2^2 + 2x_1x_2$$

$$\frac{\partial F}{\partial x_1} = 2x_2 \qquad \frac{\partial F}{\partial \dot{x}_1} = 2\dot{x}_1 \qquad \frac{d}{dx}\left(\frac{\partial F}{\partial \dot{x}_1}\right) = 2\ddot{x}_1$$

$$\frac{\partial F}{\partial x_2} = 2x_1 \qquad \frac{\partial F}{\partial \dot{x}_2} = 2\dot{x}_2 \qquad \frac{d}{dx}\left(\frac{\partial F}{\partial \dot{x}_2}\right) = 2\ddot{x}_2$$

$$\Leftrightarrow \begin{cases} x_2 - \ddot{x}_1 = 0 \\ x_1 - \ddot{x}_2 = 0 \end{cases} \Leftrightarrow \begin{cases} \ddot{x}_2 = x_1^{(4)} \\ x_1 = \ddot{x}_2 \end{cases} \Leftrightarrow x_1^{(4)} - x_1 = 0$$

$$m^4 - 1 = 0$$
$$(m^2-1)(m^2+1) = 0$$
$$m = \pm 1 \qquad m = \pm i$$

$$x_1 = ae^t + be^{-t} + c\cos t + d\sin t$$
$$\dot{x}_1 = ae^t - be^{-t} - c\sin t + d\cos t$$
$$x_2 = \ddot{x}_1 = ae^t + be^{-t} - c\cos t - d\sin t$$

with boundary conditions : $\begin{cases} x_1 = \sin t \\ x_2 = -\sin t \end{cases}$

Original exercise in class had a mistake in the boundary conditions : $\begin{array}{ll} x_1(0) = 0 & x_1(\frac{\pi}{2}) = 1 \\ x_2(0) = 1 & x_2(\frac{\pi}{2}) = -1 \end{array}$

These boundary conditions would give the following final solution :

$$\begin{cases} x_1 = \frac{1}{2} \cdot \frac{1}{1-e^{\pi}} e^t - \frac{1}{2} \frac{1}{e^{\pi}-1} \cdot e^{-t} - \frac{1}{2}\cos t + \sin t \\ x_2 = \frac{1}{2} \cdot \frac{1}{1-e^{\pi}} e^t - \frac{1}{2} \frac{1}{e^{\pi}-1} e^{-t} + \frac{1}{2}\cos t - \sin t \end{cases}$$

Ex 5 $I = \int_0^1 (y'z' + y^2)\,dx$ with $y(0) = z(0) = 0$ and $y(1) = z(1) = 1$

$F(x, y(x), z(x), y'(x), z'(x))$

$$\begin{cases} \dfrac{\partial F}{\partial y} = 2y \quad \dfrac{\partial F}{\partial y'} = z' \Rightarrow 2y - z'' = 0 \\ \dfrac{\partial F}{\partial z} = 0 \quad \dfrac{\partial F}{\partial z'} = y' \Rightarrow y' = c \end{cases}$$

$\Leftrightarrow y' = c \Leftrightarrow y = c_1 x + c_2$ with $y(0) = 0$, $y(1) = 1$

$\Leftrightarrow y = x$

$2y - z'' = 0 \underset{y=x}{\Leftrightarrow} z'' = 2x \Leftrightarrow z = \frac{1}{3}x^3 + c_3 x + c_4$

with $z(0) = 0$, $z(1) = 1$

$\Leftrightarrow z = \frac{1}{3}x^3 + \frac{2}{3}x$

Ex 6 $I = \int_0^{\pi/4} (4y_1^2 + y_2^2 + y_1' y_2')\,dx$ with $y_1(0) = 1$, $y_1(\frac{\pi}{4}) = 0$, $y_2(0) = 0$, $y_2(\frac{\pi}{4}) = 1$

$F(x, y_1(x), y_2(x), y_1'(x), y_2'(x))$

$$\left.\begin{cases} \dfrac{\partial F}{\partial y_1} = 8y_1 \quad \dfrac{\partial F}{\partial y_1'} = y_2' \Leftrightarrow 8y_1 - y_2'' = 0 \\ \dfrac{\partial F}{\partial y_2} = 2y_2 \quad \dfrac{\partial F}{\partial y_2'} = y_1' \Leftrightarrow 2y_2 - y_1'' = 0 \end{cases}\right\} \Rightarrow \begin{array}{l} y_2 = \frac{1}{2}y_1'' \\ 8y_1 - \frac{1}{2}y_1^{(4)} = 0 \end{array}$$

$\Leftrightarrow y_1^{(4)} - 16y_1 = 0$ $\quad m^4 - 16 = 0$, $m^2 = \pm 4$, $m = \pm 2$, $m = \pm 2i$

$y_1 = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$

$y_2 = 2c_1 e^{2x} + 2c_2 e^{-2x} - 2c_3 \cos 2x - 2c_4 \sin 2x$

with boundary conditions: $C_1 = 0{,}0317$ $\quad C_3 = 1/2$

$C_2 = 0{,}4683$ $\quad C_4 = -1/4$

Ex 7 $\quad I=\int_{x_0}^{x_1}(y^2-(y'')^2)\,dx$

$F(x,y,y',y'')$

$\frac{\partial F}{\partial y}=2y \qquad \frac{\partial F}{\partial y'}=0 \qquad \frac{\partial F}{\partial y''}=-2y''$

$$\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)=-2y^{(4)}$$

$\Leftrightarrow \quad 2y-2y^{(4)}=0 \Leftrightarrow \quad y^{(4)}-y=0$

$m^4-1=0=(m^2-1)(m^2+1)$

$m=\pm 1 \quad \vee \quad m=\pm i$

$y=C_1e^{x}+C_2e^{-x}+C_3\cos x+C_4\sin x$

Ex8 $\quad \min I = \int_0^{\pi/2} w^2(x)\,dx \quad$ constraints $\frac{dw}{dx} + y - (y-z)^2 y = 0$

$$\frac{dy}{dx} - w = 0$$

$$H(x, w(x), y(x), z(x), \dot{w}, \dot{y}, \dot{z}) = w^2 + \lambda_1(\dot{w} + y - (y-z)^2 y) + \lambda_2(\dot{y} - w)$$

$$\frac{\partial H}{\partial w} = 2w - \lambda_2 \qquad \frac{\partial H}{\partial \dot{w}} = \lambda_1$$

$$\frac{\partial H}{\partial y} = \lambda_1(1 - 2(y-z)y - (y-z)^2) \qquad \frac{\partial H}{\partial \dot{y}} = \lambda_2$$

$$\frac{\partial H}{\partial z} = -2\lambda_1(y-z) \qquad \frac{\partial H}{\partial \dot{z}} = 0$$

$$\begin{cases} 2w - \lambda_2 - \frac{d}{dx}(\lambda_1) = 0 \\ \lambda_1(1 - 2(y-z)y - (y-z)^2) - \frac{d}{dx}(\lambda_2) = 0 \\ -2\lambda_1(y-z) = 0 \end{cases} \Leftrightarrow \begin{cases} y = z \\ \lambda_1 - \frac{d}{dx}(\lambda_2) = 0 \\ 2w - \lambda_2 - \frac{d}{dx}(\lambda_1) = 0 \end{cases}$$

$$2w - \lambda_2 - \frac{d^2}{dx^2}(\lambda_2) = 0 \Leftrightarrow w = \frac{1}{2}\lambda_2 + \frac{1}{2}\frac{d^2}{dx^2}(\lambda_2)$$

$$\begin{matrix} w(\pi/2) = 0 = \lambda_2 + \lambda_2'' \\ w(0) = 1 = \lambda_2 + \lambda_2'' \end{matrix} \Rightarrow \exists \lambda_2 ?\ \text{yes, ex. } \lambda_2 = A\cos x + B\sin x - \frac{2}{\pi}\left(x - \frac{\pi}{2}\right)$$

We search a w, $y = z$ so that the initial conditions are true and $\frac{dy}{dx} - w = 0$ (= given constraint)

using $y(0) = 0$, $y(\pi/2) = 1$ gives a possible $y(x) = \sin x$.

And we find $\begin{cases} y = z = \sin x \\ w = \cos x \end{cases}$

Ex9 $\int_0^{\pi} (y'^2 - y^2)\, dx$ with $\int_0^{\pi} y\, dx = 1$ and $y(0) = 0$, $y(\pi) = 1$

$H(x, y, y') = y'^2 - y^2 + \lambda y$

$\frac{\partial H}{\partial y} = -2y + \lambda \qquad \frac{\partial H}{\partial y'} = 2y'$

$-2y + \lambda - 2y'' = 0 \iff y'' + y = \frac{\lambda}{2}$

$m^2 + 1 = 0 \iff m = \pm i$

$y_H = A\cos x + B\sin x$

$y_P = \frac{\lambda}{2}$

$\left.\begin{array}{l} y(0) = 0 \text{ and } y(\pi) = 1 \\ \text{and using } \int_0^{\pi} y\, dx = 1 \end{array}\right\}$ Gives $A = -\frac{1}{2}$, $\lambda = 1$, $B = \frac{1}{2} - \frac{\pi}{4}$

$\Rightarrow y = -\frac{1}{2}\cos x + \left(\frac{1}{2} - \frac{\pi}{4}\right)\sin x + \frac{1}{2}$

Ex10 $\int_0^{1} y'^2\, dx$ with $y(0) = y(\pi) = 0$ and $\int_0^{1} y^2\, dx = 2$

$H(x, y, y') = y'^2 - \lambda^2 y^2 \iff \frac{\partial H}{\partial y} = -2\lambda^2 y \qquad \frac{\partial H}{\partial y'} = 2y'$

$-2\lambda^2 y - 2y'' = 0 \iff y'' + \lambda^2 y = 0 \qquad m^2 + \lambda^2 = 0$

$m = \pm \lambda i$

$y = C_1 \cos\lambda x + C_2 \sin\lambda x$

$y(0) = C_1 = 0 \qquad y(\pi) = C_2 \sin\lambda\pi = 0$

$\lambda = k \in \mathbb{Z}$

$y = C_2 \sin kx \quad \forall k \in \mathbb{Z}$

Ex 11 $y = y(x)$ Shortest distance : $L = \int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx$

$$\frac{\partial F}{\partial y} = 0 \qquad \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \Leftrightarrow \frac{y'}{\sqrt{1+y'^2}} = C \Leftrightarrow y = Ax + B$$

$$\left.\frac{\partial F}{\partial y'}\right|_{x=x_1} = 0 \Rightarrow \frac{y'(x_1)}{\sqrt{1+y'^2(x_1)}} = 0 \Rightarrow y'(x_1) = 0$$

$$y(x_1) = \text{constant}$$

$$\left.\frac{\partial F}{\partial y'}\right|_{x=x_2} = 0 \Rightarrow y(x_2) = \text{constant}$$

$$\Leftrightarrow y = \text{constant}$$

Ex 12

$$I = \int_1^2 \left(y' + x^2 (y')^2\right) dx$$

$$\frac{\partial F}{\partial y} = 0 \qquad \frac{\partial F}{\partial y'} = 1 + 2x^2 y' \Rightarrow 1 + 2x^2 y' = C$$

$$y' = \frac{C-1}{2x^2}$$

$$\Leftrightarrow y(x) = \frac{C-1}{2} \cdot \left(-\frac{1}{x}\right) + C_2$$

$$\left.\frac{\partial F}{\partial y'}\right|_{x=1} = 0 \Rightarrow 1 + 2 \cdot 1^2 y'(1) = 0 \Leftrightarrow y'(1) = -1/2$$

$$\left.\frac{\partial F}{\partial y'}\right|_{x=2} = 0 \Rightarrow y'(2) = -1/8$$

$$\left.\begin{aligned} y(1) &= \frac{1-C}{2} + C_2 = 1 \\ y(2) &= \frac{1-C}{4} + C_2 = 2 \end{aligned}\right\} \frac{2 - 2C - 1 + C}{4} = -1 \Leftrightarrow \begin{cases} C = 5 \\ C_2 = 3 \end{cases} \Leftrightarrow y(x) = -\frac{2}{x} + 3$$

<u>Ex 13</u> $\quad I = \int_0^{\pi} (y'^2 + 2y\sin x)\,dx$

$$\frac{\partial F}{\partial y} = 2\sin x \qquad \frac{\partial F}{\partial y'} = 2y' \Rightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 2y''$$

$$\sin x - y'' = 0 \iff y'' = \sin x$$

$$y' = -\cos x + C_1$$

$$y = -\sin x + C_1 x + C_2$$

$$\left.\frac{\partial F}{\partial y'}\right|_{x=0} = 0 \iff 2y'(0) = 0 \rightarrow y(0) = \text{Constant}$$

$$\left.\frac{\partial F}{\partial y'}\right|_{x=\pi} = 0 \iff y(\pi) = \text{constant}$$

$$y(0) = -\sin 0 + C_1 + C_2 = 0$$

$$y(\pi) = -\sin\pi + 2C_1 + C_2 = 0 \qquad \iff C_1 = C_2 = 0$$

Ex 1.4 $\quad I = \int_0^1 (y'^2 - 2\alpha y' y - 2\beta y')\, dx$

$$\frac{\partial F}{\partial y} = -2\alpha y' \qquad \frac{\partial F}{\partial y'} = 2y' - 2\alpha y - 2\beta$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 2y'' - 2\alpha y'$$

$$\Leftrightarrow 2y'' = 0 \Leftrightarrow y'' = 0$$

$$y = C_1 x + C_2$$

(a) $y(0) = 0 \to C_2 = 0$

$y(1) = 1 \to C_1 = 1 \quad \Rightarrow y(x) = x$

(b) $y(0) = 0 \to C_2 = 0$

$$\left.\frac{\partial F}{\partial y'}\right|_{x=1} = 0 = 2y'(1) - 2\alpha y(1) - 2\beta$$

$$= y'(1) - \alpha y(1) - \beta \qquad y'(1) = C_1$$

$$\dots \qquad y(1) = C_1$$

$$0 = C_1 - \alpha C_1 - \beta$$

$$C_1 = \frac{\beta}{1-\alpha} \qquad \Rightarrow y(x) = \frac{\beta}{1-\alpha} x$$

(c) $y(1) = 1 \to C_1 + C_2 = 1, \quad C_1 = 1 - C_2$

$$\left.\frac{\partial F}{\partial y'}\right|_{x=0} = 0 = 2y'(0) - 2\alpha y(0) - 2\beta \qquad y'(0) = C_1$$

$$= \cancel{2}C_1 - \cancel{2}\alpha C_2 - \cancel{2}\beta \qquad y(0) = C_2$$

$$0 = 1 - C_2 - \alpha C_2 - \beta$$

$$1 - \beta = (1+\alpha) C_2 \Rightarrow C_2 = \frac{1-\beta}{1+\alpha}, \quad C_1 = 1 - \frac{1-\beta}{1+\alpha} = \frac{\alpha+\beta}{1+\alpha}$$

$$\Rightarrow y(x) = \frac{\alpha+\beta}{1+\alpha} x + \frac{1-\beta}{1+\alpha}$$

(d) $\left.\dfrac{\partial F}{\partial y'}\right|_{x=1} = 0 = y'(1) - \alpha y(1) - \beta$ $\qquad$ $y'(1) = C_1$, $\quad y(1) = C_1 + C_2$

$\left.\dfrac{\partial F}{\partial y'}\right|_{x=0} = 0 = y'(0)\ \alpha y(0) - \beta$ $\qquad$ $y'(0) = C_1$, $\quad y(0) = C_2$

$$0 = C_1 - \alpha(C_1 + C_2) - \beta$$

$$0 = C_1 - \alpha C_2 - \beta \qquad -$$

$$0 = 0 - \alpha C_1 \iff C_1 = 0$$

$$C_2 = \frac{-\beta}{\alpha}$$

$$\Rightarrow y(x) = -\frac{\beta}{\alpha}$$