Exercise 3

$$y'' + 2y' - 3y = xe^{-x}$$

$$m^2 + 2m - 3 = 0 \qquad D = 4 + 12 = 16$$

$$m = \frac{-2 \pm 4}{2} = 1 \vee -3$$

$$y_H(x) = C_1 e^x + C_2 e^{-3x}$$

general solution $y(x) = K_1 e^x + K_2 e^{-3x}$

$$y'(x) = K_1 e^x - 3K_2 e^{-3x} + \underbrace{K_1' e^x + K_2' e^{-3x}}_{\text{Set } = 0}$$

$$y''(x) = K_1 e^x + 9K_2 e^{-3x} + K_1' e^x - 3K_2' e^{-3x}$$

$$y''(x) + 2y' - 3y =$$
$$K_1 e^x + 9K_2 e^{-3x} + K_1' e^x - 3K_2' e^{-3x}$$
$$+ 2K_1 e^x - 6K_2 e^{-3x}$$
$$-3K_1 e^x - 3K_2 e^{-3x} = xe^{-x}$$

$$\begin{cases} K_1' e^x + K_2' e^{-3x} = 0 \\ K_1' e^x - 3K_2' e^{-3x} = xe^{-x} \end{cases} \Leftrightarrow \begin{cases} K_1' = \frac{x}{4} e^{-2x} \\ K_2' = -\frac{x}{4} e^{2x} \end{cases}$$

$$K_1 = e^{-2x}\left(-\frac{x}{8} - \frac{1}{16}\right)$$

$$K_2 = e^{2x}\left(-\frac{x}{8} + \frac{1}{16}\right)$$

$$e^{-2x}\left(-\frac{x}{8} - \frac{1}{16}\right)e^x + e^{2x}\left(-\frac{x}{8} + \frac{1}{16}\right)e^{-3x}$$

$$\left(-\frac{x}{8} - \cancel{\frac{1}{16}}\right)e^{-x} + \left(-\frac{x}{8} + \cancel{\frac{1}{16}}\right)e^{-x} = -\frac{1}{4}xe^{-x}$$

Exercise 4

(1) $$\frac{1}{D^2+D-12}\left(9e^{5x}-4e^{-x}\right)=\frac{1}{(D-3)(D+4)}\left(9e^{5x}-4e^{-x}\right)$$

$$= 9\cdot\frac{1}{(D-3)(D+4)}e^{5x}-4\cdot\frac{1}{(D-3)(D+4)}e^{-x}$$

(A)
$m=-4$

$$= \cancel{9}\cdot\frac{1}{D-3}\left(e^{-4x}\underbrace{\int e^{5x}\cdot e^{4x}dx}_{\frac{1}{\cancel{9}}e^{9x}}\right)-4\cdot\frac{1}{D-3}\left(e^{-4x}\underbrace{\int e^{-x}e^{4x}dx}_{\frac{1}{3}e^{3x}}\right)$$

(A)
$m=3$

$$= e^{3x}\underbrace{\int e^{-3x}\cdot e^{5x}dx}_{\frac{1}{2}e^{2x}}-\frac{4}{3}e^{3x}\underbrace{\int e^{-3x}e^{-x}dx}_{-\frac{1}{4}e^{-4x}}$$

$$= \frac{1}{2}e^{5x}+\frac{1}{3}e^{-x}$$

OR non-operator technique:

$y_H = c_1e^{3x}+c_2e^{-4x}$

trial $y_p = ae^{5x}+be^{-x}$

$(D^2+D-12)(y_p) = 9e^{5x}-4e^{-x}$

$Dy_p = 5\cdot a\cdot e^{5x}-be^{-x}$

$D^2y_p = 25\cdot a\cdot e^{5x}+b\cdot e^{-x}$

$$\Rightarrow \begin{cases} 25\cdot a+5\cdot a-12\cdot a = 9 \\ b-b-12b=-4 \end{cases} \Rightarrow \begin{cases} a = 1/2 \\ b = 1/3 \end{cases}$$

$y_p = \frac{1}{2}e^{5x}+\frac{1}{3}e^{-x}$

(2) $\frac{1}{(D+1)^2}(4\sin 2x + 3\cos 2x)$

$$= 4\cdot\frac{1}{(D+1)^2}\sin 2x + 3\cdot\frac{1}{(D+1)^2}\cos 2x$$

$(D+1)^2 = D^2+2D+1$

(D): D^2 can be replaced by $-p^2 = -4$

$$= 4\cdot\frac{1}{2D-3}\sin 2x + 3\cdot\frac{1}{2D-3}\cos 2x$$

$\frac{T}{N}\cdot\frac{2D+3}{2D+3}$

$$= 4\,\frac{2D+3}{4D^2-9}\sin 2x + 3\cdot\frac{2D+3}{4D^2-9}\cos 2x$$

$D^2=-4 \to 4D^2-9=-25$

$$= -\frac{4}{25}(2D+3)\sin 2x - \frac{3}{25}(2D+3)\cos 2x$$

$$= -\frac{8}{25}\cdot(2\cos 2x) - \frac{12}{25}\sin 2x - \frac{6}{25}(-2\sin 2x) - \frac{9}{25}\cos 2x$$

$$= -\cos 2x$$

(3) $\frac{1}{(D-4)^5}(xe^{4x}) \underset{(G)}{=} e^{4x}\cdot\frac{1}{(D+4-4)^5}(x) = e^{4x}\cdot\frac{1}{D^5}x$

$p=4$

$$= e^{4x}\frac{1}{D^4}\left(\frac{1}{D}x\right) \underset{\substack{m=0\\(A)}}{=} e^{4x}\cdot\frac{1}{D^4}\underbrace{\int x\,dx}_{\frac{x^2}{2}} = e^{4x}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}\cdot\frac{1}{5}\cdot\frac{1}{6}\cdot x^6$$

$$= \frac{e^{4x}x^6}{6!}$$

(4) $\frac{1}{D^2-4}(16x^3) = 16 \cdot \frac{1}{D^2-4} x^3$

(f) $\frac{1}{x^2-4} = f(0) + f'(0)\,x + \frac{f''(0)}{2} x^2$

$= -\frac{1}{4} - \frac{1}{16} x^2$

$= 16\left(-\frac{1}{4} - \frac{1}{16} D^2\right) x^3$

$= -4x^3 - 6x$

<u>Exercise 5</u>

(1) $(D^3+8)\,y = 0$

$m^3 + 8 = 0 \qquad m = -2$

$(m+2)(m^2 - 2m + 4) = 0$

$D = 4 - 16 = -12$

$m = \frac{2 \pm \sqrt{-12}}{2} = 1 \pm \sqrt{3}\,i$

$y(x) = c_1 e^{-2x} + e^{x}\left(c_1 \cos\sqrt{3}\,x + c_2 \sin\sqrt{3}\,x\right)$

(2) $(D+6)^4 (D-3)^2\, y = 0$

$(D-3)\,y = 0 \Leftrightarrow y(x) = C e^{3x}$

$(D+6)\,y = 0 \Leftrightarrow y(x) = C e^{-6x}$

$y(x) = e^{3x}(C_1 + C_2 x)$

$\quad + e^{-6x}(C_3 + C_4 x + C_5 x^2 + C_6 x^3)$

(3) $D^4(D+1)^2(D^2+4D+5)^2(D^2+4)y=0$

$D^4y=0 \iff y_1 = C_1 + C_2x + C_3x^2 + C_4x^3$

$(D+1)^2y=0 \iff y_2 = (C_5 + C_6x)e^{-x}$

$(D^2+4D+5)^2y=0 \iff m^2+4m+5=0$

$D = 16-20 = -4$

$m = \frac{-4 \pm \sqrt{-4}}{2} = -2 \pm i$

$y_3(x) = e^{-2x}(C_7\cos x + C_8\sin x)$

$+ xe^{-2x}(C_9\cos x + C_{10}\sin x)$

$(D^2+4)y=0 \iff m^2+4=0 \quad m = \pm 2i$

$y_4(x) = C_{11}\cos 2x + C_{12}\sin 2x$

$y(x) = y_1(x) + y_2(x) + y_3(x) + y_4(x)$

Exercise 6

$(D^6 - 2D^5 + D^4)y = 120x + 8e^x$

$m^6 - 2m^5 + m^4 = 0 = m^4(m^2 - 2m + 1) = m^4(m-1)^2$

$y_H(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + (c_5 + c_6 x)e^x$

$y_p(x) = (ax+b)x^4 + cx^2e^x$

$y_p'(x) = 5ax^4 + 4bx^3 + 2cxe^x + cx^2e^x$

$y_p''(x) = 20ax^3 + 12bx^2 + 2ce^x + \cancel{2}_{4}cxe^x + \cancel{2cxe^x} + cx^2e^x$

$y_p'''(x) = 60ax^2 + 24bx + \cancel{2}_{6}ce^x + \cancel{4ce^x} + \cancel{4}_{6}cxe^x + \cancel{2cxe^x} + cx^2e^x$

$y_p^{(4)}(x) = 120ax + 24b + \cancel{6}_{12}ce^x + \cancel{6ce^x} + \cancel{6}_{8}cxe^x + \cancel{2cxe^x} + cx^2e^x$

$y_p^{(5)}(x) = 120a \quad + \cancel{12}_{20}ce^x + \cancel{8ce^x} + \cancel{8}_{10}cxe^x + \cancel{2cxe^x} + cx^2e^x$

$y_p^{(6)}(x) = \quad \cancel{20}_{30}ce^x + \cancel{10ce^x} + \cancel{10}_{12}cxe^x + \cancel{2cxe^x} + cx^2e^x$

$+30ce^x + 12cxe^x + cx^2e^x - 240a + 120ax$

$-40ce^x - 20cxe^x - 2cx^2e^x + 24b$

$+12ce^x \quad \underbrace{+8cxe^x}_{=0} \quad \underbrace{+cx^2e^x}_{=0}$

$= 8e^x \qquad + 120x$

$$\begin{cases} a = 1 \\ b = 10 \\ c = 4 \end{cases}$$

$$\boxed{y(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + (c_5 + c_6 x)e^x + (x+10)x^4 + 4x^2e^x}$$

Exercise 7

(1) $y'' + 4y = \csc 2x$

$m^2 + 4 = 0 \quad m = \pm 2i$

$y_H(x) = C_1 \cos 2x + C_2 \sin 2x$

general solution $y(x) = K_1 \cos 2x + K_2 \sin 2x$

$$y'(x) = -2K_1 \sin 2x + 2K_2 \cos 2x + \underbrace{K_1' \cos 2x + K_2' \sin 2x}_{SET\ =0}$$

$$y''(x) = -4K_1 \cos 2x - 4K_2 \sin 2x - 2K_1' \sin 2x + 2K_2' \cos 2x$$

$$\begin{cases} K_1' \cos 2x + K_2' \sin 2x = 0 \\ -2K_1' \sin 2x + 2K_2' \cos 2x = \csc 2x = \dfrac{1}{\sin 2x} \end{cases}$$

$$\Leftrightarrow \begin{cases} K_1' = -1/2 \\ K_2' = 1/2 \cot 2x \end{cases} \Leftrightarrow \begin{cases} K_1 = -1/2\, x + C_1 \\ K_2 = \dfrac{1}{4} \ln(\sin 2x) + C_2 \end{cases}$$

$$y(x) = \left(-\frac{1}{2}x + C_1\right) \cos 2x + \left(\frac{1}{4} \ln(\sin 2x) + C_2\right) \sin 2x$$

$$= C_1 \cos 2x + C_2 \sin 2x - \frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x \cdot \ln(\sin 2x)$$

(2) $(D^3+D)y = 4\tan x$

$$m^3+m = 0 = m(m^2+1) \begin{cases} \nearrow m=0 \\ \searrow m=\pm i \end{cases}$$

$$y_H = C_1 + C_2\cos x + C_3 \sin x$$

general solution $y(x) = K_1 + K_2\cos x + K_3 \sin x$

$$y'(x) = -K_2\sin x + K_3\cos x + \overbrace{K_1' + K_2'\cos x + K_3'\sin x}^{=0}$$

$$y''(x) = -K_2\cos x - K_3 \sin x \underbrace{- K_2'\sin x + K_3'\cos x}_{=0}$$

$$y'''(x) = K_2\sin x - K_3\cos x - K_2'\cos x - K_3'\sin x$$

$$\begin{cases} K_1' + K_2'\cos x + K_3'\sin x = 0 \\ -K_2'\sin x + K_3'\cos x = 0 \\ -K_2'\cos x - K_3'\sin x = 4\tan x \end{cases} \iff \begin{cases} K_1' = 4\tan x \\ K_2' = -4\sin x \\ K_3' = -4\sin^2 x/\cos x \end{cases}$$

$$\Rightarrow \begin{cases} K_1 = -4\ln(\cos x) + C_1 \\ K_2 = 4\cos x + C_2 \\ K_3 = 4\sin x + 4\ln(\cos x) - 4\ln(1+\sin x) + C_3 \\ \quad = 4\sin x - 4\ln\left(\frac{1}{\cos x} + \tan x\right) + C_3 \end{cases}$$

$$y(x) = -4\ln(\cos x) + C_1 + 4\cos^2 x + C_2\cos x + 4\sin^2 x$$
$$-4\sin x \ln\left(\frac{1}{\cos x} + \tan x\right) + C_3 \sin x$$

$$= C_1 + C_2\cos x + C_3 \sin x - 4\ln(\cos x) + 4 - 4\sin x \ln\left(\frac{1}{\cos x} + \tan x\right)$$

Exercise 8

(1) $\frac{1}{D+3}\left(e^{-2x}\right)$

$$y = \frac{1}{D+3}\left(e^{-2x}\right) \iff (D+3)y = e^{-2x}$$

① linear equation: $\frac{dy}{dx} + 3y = e^{-2x}$ $\qquad \mu(x) = e^{\int 3dx} = e^{3x}$

$$e^{3x}y = \int e^{3x}\cdot e^{-2x}dx + c = e^{x} + c$$

$$y = \underbrace{e^{-2x}}_{y_P} + \underbrace{ce^{-3x}}_{y_H}$$

② non-operator technique

$m+3=0 \iff m=-3 \Rightarrow y_H = ce^{-3x}$

Trial solution $= a\cdot e^{-2x}$

$$(D+3)(a e^{-2x}) = -2ae^{-2x} + 3ae^{-2x} == e^{-2x}$$

$$\iff a = 1$$

$$\Rightarrow y_P = e^{-2x}$$

③ operator technique (only gives y_P)

$$\frac{1}{\underset{m=-3}{D+3}}e^{-2x} = e^{-3x}\int e^{3x}e^{-2x}dx = e^{-2x} = y_P$$

(2) $\frac{D-1}{D^4+D^2+1}(8\cos x) = \frac{D-1}{1-1+1}(8\cos x) = (D-1)(8\cos x)$

$(D)\ D^2 = -1^2 = -1$ $\qquad = -8\sin x - 8\cos x$

$$(3)\quad \frac{1}{D^2+D-2}\left(x^2e^{2x}\right) \underset{(G)}{=} e^{2x}\cdot\frac{1}{(D+2)^2+D+2-2}x^2 = e^{2x}\cdot\frac{1}{D^2+5D+4}x^2$$

$$(F)\quad \frac{1}{x^2+5x+4} = f(0)+f'(0)x+\frac{f''(0)}{2}x^2$$

$$= \frac{1}{4}-\frac{5}{16}x+\frac{21}{64}x^2$$

$$= e^{2x}\left(\frac{1}{4}-\frac{5}{16}D+\frac{21}{64}D^2\right)x^2$$

$$= \frac{1}{4}x^2e^{2x}-\frac{5}{8}xe^{2x}+\frac{21}{32}e^{2x}$$

$$(4)\quad \frac{1}{D^2-1}\left(e^x(\sin x+\cos x)\right) \underset{(G)}{=} e^x\cdot\frac{1}{(D+1)^2-1}(\sin x+\cos x)$$

$$= e^x\cdot\frac{1}{D^2+2D}(\sin x+\cos x) \underset{\substack{(D)\\ D^2=-1^2}}{=} e^x\cdot\frac{1}{2D-1}(\sin x+\cos x)$$

$$\underset{\cdot\frac{2D+1}{2D+1}}{=} e^x\cdot\frac{2D+1}{4D^2-1}(\sin x+\cos x) \underset{(D)}{=} e^x\cdot\left(-\frac{1}{5}\right)\cdot(2D+1)(\sin x+\cos x)$$

$$= -\frac{1}{5}e^x\left(2\cos x-2\sin x+\sin x+\cos x\right)$$

$$= -\frac{1}{5}e^x\left(3\cos x-\sin x\right)$$

(5) $\frac{1}{(D-4)(D+3)(D+1)}\left(e^{-2x}\cos 2x\right) \underset{(G)}{=} e^{-2x}\cdot\frac{1}{(D-6)(D+1)(D-1)}\cos 2x$

$$= e^{-2x}\frac{D+6}{(D^2-36)(D^2-1)}\cos 2x \underset{\substack{(D)\\ D^2=-4}}{=} e^{-2x}\frac{D+6}{(-40)(-5)}\cos 2x$$

$\cdot\frac{D+6}{D+6}$

$$= \frac{1}{200}\cdot e^{-2x}\left(-2\sin 2x + 6\cos 2x\right)$$

$$= \frac{1}{100}e^{-2x}\left(3\cos 2x - \sin 2x\right)$$

<u>Exercise 9</u>

$$(D^2+4D+4)y = 18e^x - 8\sin 2x$$

$0 = m^2+4m+4 = (m+2)^2 \qquad m=-2$

$y_H = (C_1 + C_2 x)e^{-2x}$

$$\frac{1}{(D+2)^2}\left(18e^x - 8\sin 2x\right) = 18\frac{1}{(D+2)^2}e^x - 8\cdot\frac{1}{D^2+4D+4}\sin 2x$$

$$\underset{(A)}{=} 18\cdot\frac{1}{D+2}\, e^{-2x}\underbrace{\int e^{3x}dx}_{\frac{1}{3}e^{3x}} - 8\cdot\frac{1}{-4+4D+4}\sin 2x$$

(D) $D^2=-4$

$$= 6\cdot\frac{1}{D+2}e^x - 2\cdot\frac{1}{D}\sin 2x$$

$$= 6e^{-2x}\int e^{3x}dx - 2\frac{D}{\underbrace{D^2}_{=-4}}\sin 2x$$

$$= 2e^x + \frac{1}{2}\cdot 2\cos 2x = \cos 2x + 2e^x$$

$y_p = \cos 2x + 2e^x$