Exercise 1.

(a) $x \dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial z}{\partial y} = 0$

$$\frac{\partial}{\partial y}\left(x \frac{\partial z}{\partial x} + z\right) = 0 \longrightarrow x \frac{\partial z}{\partial x} + z = F(x)$$

$$\frac{\partial z}{\partial x} + \frac{1}{x} z = \frac{1}{x} F(x) \quad \text{(linear)}$$

$$\mu = e^{\int \frac{1}{x} dx} = x$$

$$xz = \int F(x)\, dx + G(y) \iff z = \frac{1}{x}\int F(x)\, dx + \frac{1}{x} G(y)$$

$$z(x,y) = \frac{1}{x} H(x) + \frac{1}{x} G(y)$$

(b) $\begin{cases} z(x,0) = x^5 + x \\ z(2,y) = 3y^4 \end{cases} \iff \begin{cases} z(x,0) = \frac{1}{x} H(x) + \frac{1}{x} G(0) = x^5 + x \\ z(2,y) = \frac{1}{2} H(2) + \frac{1}{2} G(y) = 3y^4 \end{cases}$

$$\iff \begin{cases} H(x) = x^6 + x^2 - G(0), \quad H(2) = 68 - G(0) \\ 34 - \frac{1}{2} G(0) + \frac{1}{2} G(y) = 3y^4 \end{cases}$$

$$\iff G(y) = 6y^4 - 68 + G(0)$$

$$z(x,y) = \frac{1}{x} H(x) + \frac{1}{x} G(y) = x^5 + x - \frac{1}{x} G(0) + 6\frac{y^4}{x} - 68 \cdot \frac{1}{x} + \frac{1}{x} G(0)$$

$$= x^5 + x + 6\frac{y^4}{x} - 68\frac{1}{x}$$

Exercise 2

(1) $u_{yy} = 0$

$$\frac{\partial}{\partial y}(u_y) = 0 \iff \frac{\partial u}{\partial y} = f(x)$$

$$\iff u(x,y) = f(x)\,y + g(x)$$

(2) $u_{xx} + 16\pi^2 u = 0$

$$\overset{u(x)}{\iff} u'' + 16\pi^2 u = 0$$

$$m^2 + 16\pi^2 = 0 \iff m = \pm 4\pi i$$

$$u(x) = C_1 \cos 4\pi x + C_2 \sin 4\pi x$$

$$\iff u(x,y) = f(y)\cos 4\pi x + g(y)\sin 4\pi x$$

(3) $25 u_{yy} - 4u = 0$

$$\overset{u(y)}{\iff} 25u'' - 4u = 0 \iff 25m^2 - 4 = 0 \iff m = \pm\frac{2}{5}$$

$$u(y) = C_1 e^{2/5 y} + C_2 e^{-2/5 y}$$

$$\iff u(x,y) = f(x)e^{2/5 y} + g(x)e^{-2/5 y}$$

(4) $u_y + y^2 u = 0$

$$\overset{u(y)}{\iff} u' + y^2 u = 0 \iff \frac{du}{dy} = -y^2 u$$

$$\frac{1}{u}\,du = -y^2\,dy$$

$$\ln u = -\frac{y^3}{3} + C_1$$

$$u(y) = C_1 e^{-y^3/3}$$

$$\iff u(x,y) = f(x)e^{-y^3/3}$$

(5) $2u_{xx} + 9u_x + 4u = -3\cos x - 29\sin x$

$\xleftrightarrow{u(x)}$ $2u_H'' + 9u_H' + 4u_H = 0 \iff 2m^2 + 9m + 4 = 0$

$D = 81 - 32 = 49$

$m = \dfrac{-9 \pm 7}{4} = -\dfrac{1}{2} \vee -4$

$u_H(x) = C_1 e^{-1/2x} + C_2 e^{-4x}$

$$\left.\begin{aligned} u_p(x) &= A\cos x + B\sin x \\ u_p'(x) &= -A\sin x + B\cos x \\ u_p''(x) &= -A\cos x - B\sin x \end{aligned}\right\} -2A\cos x - 2B\sin x - 9A\sin x + 9B\cos x + 4A\cos x + 4B\sin x = -3\cos x - 29\sin x$$

$$\iff \begin{cases} -2A + 9B + 4A = -3 \\ -2B - 9A + 4B = -29 \end{cases} \iff \begin{cases} A = 3 \\ B = -1 \end{cases}$$

$u_p(x) = 3\cos x - \sin x$

$\iff u(x,y) = f_1(y) e^{-1/2x} + f_2(y) e^{-4x} + 3\cos x - \sin x$

(6) $u_{yy} + 6u_y + 13u = 4e^{3y}$

$\xleftrightarrow{u(y)}$ $u_H'' + 6u_H' + 13u_H = 0 \iff m^2 + 6m + 13 = 0$

$D = 36 - 52 = -16$

$m = \dfrac{-6 \pm 4i}{2} = -3 \pm 2i$

$u_H(y) = e^{-3y}(C_1\cos 2y + C_2 \sin 2y)$

$$\left.\begin{aligned} u_p(y) &= Ae^{3y} \\ u_p'(y) &= 3Ae^{3y} \\ u_p''(y) &= 9Ae^{3y} \end{aligned}\right\} \Rightarrow 9Ae^{3y} + 18Ae^{3y} + 13Ae^{3y} = 4e^{3y}$$

$\iff \{ 40A = 4 \iff A = 1/10$

$u_p(y) = \dfrac{1}{10} e^{3y}$

$\iff u(x,y) = e^{-3y}\left(f(x)\cos 2y + g(x)\sin 2y\right) + \dfrac{1}{10} e^{3y}$

(7) $u_{xy} = u_x$

$U_x = p \Leftrightarrow p_y = p \Leftrightarrow \frac{dp}{dy} = p \Leftrightarrow \ln p = y + c(x)$

$\Leftrightarrow p = c(x)e^y$

$$u(x,y) = \int c(x)e^y\,dx = f(x)e^y + g(y)$$

with $f(x) = \int c(x)\,dx$

OR

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} - u\right) = 0 \Leftrightarrow \frac{\partial u}{\partial y} - u = G(y) \quad \text{(linear)}$$

$$\mu = e^{-\int dy} = e^{-y}$$

$$e^{-y}u = \int e^{-y}G(y)\,dy + f(x)$$

$$u(x,y) = e^y\int e^{-y}G(y)\,dy + f(x)e^y$$

$$= g(y) + f(x)e^y$$

(8) $x^2u_{xx} + 2xU_x - 2u = 0$

$\Leftrightarrow_{u(x)} \quad x^2u'' + 2xu' - 2u = 0 \quad \overset{x=e^t}{\Leftrightarrow} \quad D_t(D_t - 1) + 2D_t - 2 = 0$

$D_t^2 + D_t - 2 = 0$

$m^2 + m - 2 = 0 \qquad D = 1 + 8 = 9$

$m = \frac{-1 \pm 3}{2} = 1 \vee -2$

$u(t) = c_1e^t + c_2e^{-2t}$

$u(x) = c_1x + c_2x^{-2}$

$$u(x,y) = f(y)x + g(y)x^{-2}$$

Exercise 3.

(1) $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} \rightsquigarrow u = e^{ax+by} \Rightarrow (a^2-b^2)e^{ax+by} = 0$

$a = \pm b$

$\rightsquigarrow$ solutions: $e^{b(x+y)}$ and $e^{b(x-y)}$ $\forall b$

$u(x,y) = F(x+y) + G(x-y)$

(2) $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$

$\rightsquigarrow u = e^{ax+by} \Rightarrow (a^2 - 2ab + b^2)e^{ax+by} = 0$

$(a-b)^2 = 0$

$a = b \quad (\times 2)$

$\rightsquigarrow$ Solutions: $e^{b(x+y)}$, $xe^{b(x+y)}$ OR $ye^{b(x+y)}$ $\forall b$

$u(x,y) = F(x+y) + xG(x+y)$ OR $F(x+y) + yG(x+y)$

(3) $\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = x$

$\rightsquigarrow u_H = e^{ax+by} \Rightarrow (a+2b)e^{ax+by} = 0 \Leftrightarrow a = -2b$

$\rightsquigarrow$ Solutions: $e^{b(y-2x)}$ $\forall b$

$u_H(x,y) = F(y-2x)$

$u_p(x,y) = \alpha x^2$

$\rightsquigarrow \frac{\partial u_p}{\partial x} + 2\frac{\partial u_p}{\partial y} = 2\alpha x = x \Leftrightarrow \alpha = \frac{1}{2}$

$u(x,y) = F(y-2x) + \frac{x^2}{2}$

(4) $\dfrac{\partial^4 u}{\partial x^4} + 2\dfrac{\partial^4 u}{\partial x^3 \partial y} = 4$

$u = e^{ax+by} \Rightarrow (a^4 + 2a^3 b)\, e^{ax+by} = 0$

$$a^3(a+2b) = 0 \iff \begin{cases} a = 0 \quad (\times 3) \\ a = -2b \end{cases}$$

$u_H(x,y) = F(y) + x\,G(y) + x^2 H(y) + I(y-2x)$

$u_p(x,y) = \alpha x^3 y$

$2 \cdot \alpha \cdot 3 \cdot 2 = 4 \iff \alpha = \frac{1}{3}$

$u(x,y) = F(y) + x\,G(y) + x^2 H(y) + I(y-2x) + \frac{1}{3}x^3 y$

(5) $\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = 3u$

$\rightsquigarrow u = e^{ax+by} \Rightarrow (a + 2b - 3)\, e^{ax+by} = 0$

$$a = 3 - 2b$$

$\rightsquigarrow$ Solutions: $e^{3x} e^{b(y-2x)} \quad \forall b$

$u(x,y) = e^{3x} F(y-2x)$

(6) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

$\rightsquigarrow u = e^{ax+by} \Rightarrow (a^2 + b^2)\, e^{ax+by} = 0$

$$a = \pm bi$$

$\rightsquigarrow$ Solutions: $e^{b(y+ix)} \quad e^{b(y-ix)} \quad \forall b$

$u(x,y) = F(y+ix) + G(y-ix)$

(7) $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x \cdot \sin y$

$z = e^{ax+by} \Rightarrow (a^2 - 3ab + 2b^2)e^{ax+by} = 0$

$(a-2b)(a-b) = 0 \quad a = 2b \vee a = b$

$z_H(x,y) = F(y+2x) + G(x+y)$

$z_p(x,y) = \alpha_1 \sin y + \alpha_2 x \sin y + \beta_1 \cos y + \beta_2 x \cos y$

$\frac{\partial z_p}{\partial x} = \alpha_2 \sin y + \beta_2 \cos y \qquad \frac{\partial^2 z}{\partial x \partial y} = \alpha_2 \cos y - \beta_2 \sin y$

$\frac{\partial^2 z_p}{\partial x^2} = 0$

$\frac{\partial z}{\partial y} = \alpha_1 \cos y + \alpha_2 x \cos y - \beta_1 \sin y - \beta_2 x \sin y$

$\frac{\partial^2 z}{\partial y^2} = -(\alpha_1 \sin y + \alpha_2 x \sin y + \beta_1 \cos y + \beta_2 x \cos y)$

$\Leftrightarrow \quad -3\alpha_2 \cos y + 3\beta_2 \sin y$
$-2\beta_1 \cos y - 2\alpha_1 \sin y - \underline{2\alpha_2 x \sin y} - 2\beta_2 x \cos y = \underline{x \sin y}$

$\Leftrightarrow \quad -2\alpha_2 = 1 \Leftrightarrow \alpha_2 = -1/2 \quad \& \quad 2\beta_2 = 0 \Leftrightarrow \beta_2 = 0$
$-2\beta_1 - 3\alpha_2 = 0 \Leftrightarrow \beta_1 = 3/4 \qquad 3\beta_2 - 2\alpha_1 = 0 \Leftrightarrow \alpha_1 = 0$

$z_p = -\frac{1}{2} x \sin y + \frac{3}{4} \cos y$

$z(x,y) = F(y+2x) + G(x+y) - \frac{1}{2} x \sin y + \frac{3}{4} \cos y$

Exercise 4.

(1) $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$, $u(x,0) = 4e^{-x}$

Let $u = X(x)\,Y(y)$

$$3X'Y = -2XY' \iff \frac{X'}{-2X} = \frac{Y'}{3Y} = c$$

$X' + 2cX = 0$ & $Y' - 3cY = 0$

$$\frac{d(X)}{dx} = -2cX \iff \frac{1}{X}dX = -2c\,dx$$

$$\ln(X) = -2cx + A$$

$$X = Ae^{-2cx}$$

and $Y = Be^{3cy}$

$$u = ABe^{c(3y-2x)} = Ke^{c(3y-2x)} \text{ with } u(x,0) = 4e^{-x} = Ke^{c(-2x)}$$

$\iff K = 4$

$c = 1/2$

$$u = 4e^{\frac{3}{2}y - x}$$

(2) $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ $\quad u(0,t) = 0$

$u(\pi,t) = 0$

$u(x,0) = 2\sin 3x - 4\sin 5x$

need complex solution !

$u = X(x)\,T(t)$

$XT' = 4X''T \iff \frac{T'}{4T} = \frac{X''}{X} = -\lambda^2$

$T' = -4\lambda^2 T$

$T = Ae^{-4\lambda^2 t}$

$X'' + \lambda^2 X = 0$

$m^2 + \lambda^2 = 0 \iff m = \pm \lambda i$

$X = B_1 \cos\lambda x + B_2 \sin\lambda x$

$$u = XT = e^{-4\lambda^2 t}\left(\underbrace{AB_1}_{K_1}\cos\lambda x + \underbrace{AB_2}_{K_2}\sin\lambda x\right)$$

$u(0,t) = 0 = e^{-4\lambda^2 t}\cdot K_1 \iff K_1 = 0$

$u(\pi,t) = 0 = K_2 \sin(\pi\lambda)\cdot e^{-4\lambda^2 t} = 0 \iff \sin(\pi\lambda) = 0$

$\iff \pi\lambda = k\pi$ with $k \in \mathbb{Z}$

$\lambda = k$ with $k \in \mathbb{Z}$

$$u(x,t) = K_2 e^{-4k^2 t}\sin(kx) \text{ with } k \in \mathbb{Z}$$

Principle of superposition:

$$u(x,t) = K_1 e^{-4k_1^2 t}\sin(k_1 x) + K_2 e^{-4k_2^2 t}\sin(k_2 x) + \dots$$

$$u(x,0) = 2\sin 3x - 4\sin 5x = \sum_i K_i \sin(k_i x)$$

$$\iff u(x,t) = 2e^{-36t}\sin(3x) - 4e^{-100t}\sin(5x)$$

(3) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $\quad u_x(0,t) = 0$

$u(2,t) = 0$

$u(x,0) = 8\cos\frac{3\pi x}{4} - 6\cos\frac{9\pi x}{4}$

$u = XT$

$XT' = X''T \Leftrightarrow \frac{X''}{X} = \frac{T'}{T} = -\lambda^2$

$T' = -\lambda^2 T$

$T = Ae^{-\lambda^2 t}$

$X'' = -\lambda^2 X$

$m^2 + \lambda^2 = 0 \Leftrightarrow m = \pm\lambda i$

$X = B_1\cos\lambda x + B_2\sin\lambda x$

$$u = e^{-\lambda^2 t}(\underbrace{AB_1}_{K_1}\cos\lambda x + \underbrace{AB_2}_{K_2}\sin\lambda x)$$

$$u_x = e^{-\lambda^2 t}(-\lambda K_1\sin\lambda x + \lambda K_2\cos\lambda x)$$

$u_x(0,t) = 0 = \lambda K_2 e^{-\lambda^2 t} \Leftrightarrow K_2 = 0$

$$u = e^{-\lambda^2 t}(K_1\cos\lambda x)$$

$u(2,t) = 0 = K_1\cos(2\lambda)e^{-\lambda^2 t} = 0 \Leftrightarrow \cos 2\lambda = 0$

$2\lambda = \frac{2k+1}{2}\pi, \ k\in\mathbb{Z}$

$\lambda = \frac{2k+1}{4}\pi, \ k\in\mathbb{Z}$

$$u = e^{-\left(\frac{2k+1}{4}\pi\right)^2 t}\cdot K_1\cos\left(\frac{2k+1}{4}\pi x\right), \ k\in\mathbb{Z}$$

$$= \sum_i K_i e^{-\left(\frac{2k_i+1}{4}\pi\right)^2}\cos\left(\frac{2k_i+1}{4}\pi x\right)$$

$u(x,0) = 8\cos\frac{3\pi}{4}x - 6\cos\frac{9\pi x}{4}$

$$u = 8e^{-\frac{9\pi^2}{16}t}\cos\frac{3\pi}{4}x - 6e^{-\frac{81\pi^2}{16}t}\cos\frac{9\pi}{4}x$$

(4) $\frac{\partial u}{\partial t} = 4 \frac{\partial^2 u}{\partial x^2}$ $\quad u(0,t)=0$

$u(\pi,t)=0$

$u(x,0) = 2\sin 3x - 4\sin 5x$

need complex solution!

$u = XT$

$XT' = 4X''T \Leftrightarrow \frac{T'}{4T} = \frac{X''}{X} = -\lambda^2$

$T' = -4\lambda^2 T$

$T = Ae^{-4\lambda^2 t}$

$X'' + \lambda^2 X = 0$

$m^2 + \lambda^2 = 0 \Leftrightarrow m = \pm \lambda i$

$X = B_1 \cos \lambda x + B_2 \sin \lambda x$

$u = XT = e^{-4\lambda^2 t}\left(\underbrace{AB_1}_{K_1} \cos \lambda x + \underbrace{AB_2}_{K_2} \sin \lambda x\right)$

$u(0,t) = 0 = e^{-4\lambda^2 t} \cdot K_1 \Leftrightarrow K_1 = 0$

$u(\pi,t) = 0 = K_2 \sin(\pi\lambda) \cdot e^{-4\lambda^2 t} = 0 \Leftrightarrow \sin(\pi\lambda) = 0$

$\Leftrightarrow \pi\lambda = k\pi$ with $k \in \mathbb{Z}$

$\lambda = k$ with $k \in \mathbb{Z}$

$u(x,t) = K_2 e^{-4k^2 t} \sin(kx)$ with $k \in \mathbb{Z}$

Principle of superposition:

$u(x,t) = K_1 e^{-4k_1^2 t} \sin(k_1 x) + K_2 e^{-4k_2^2 t} \sin(k_2 x) + \dots$

$u(x,0) = 2\sin 3x - 4\sin 5x = \sum_i K_i \sin(k_i x)$

$\Leftrightarrow u(x,t) = 2e^{-36t} \sin(3x) - 4e^{-100t} \sin(5x)$

(5) $\frac{\partial^2 y}{\partial t^2} = 4\frac{\partial^2 y}{\partial x^2}$, $y(0,t) = y(5,t) = 0$

$y(x,0) = 0$, $y_t(x,0) = 5\sin\pi x$

$y = XT$ $\quad XT'' = 4X''T \iff \frac{T''}{4T} = \frac{X''}{X} = -\lambda^2$

$\frac{T''}{4T} = -\lambda^2$

$T'' + 4\lambda^2 T = 0$

$m^2 + 4\lambda^2 = 0$

$m = \pm 2\lambda i$

$T = C_1\cos 2\lambda t + C_2 \sin 2\lambda t$

$X'' = -\lambda^2 X$

$X'' + \lambda^2 X = 0$

$m^2 + \lambda^2 = 0$

$m = \pm\lambda i$

$X = C_3\cos\lambda x + C_4\sin\lambda x$

$y(x,t) = (C_1\cos 2\lambda t + C_2\sin 2\lambda t)(C_3\cos\lambda x + C_4\sin\lambda x)$

$y(0,t) = (C_1\cos 2\lambda t + C_2\sin 2\lambda t)C_3 = 0 \quad \Rightarrow C_3 = 0$

$y(x,0) = C_1(C_3\cos\lambda x + C_4\sin\lambda x) = 0 \quad \Rightarrow C_1 = 0$

$y(x,t) = C_2\sin 2\lambda t \, C_4\sin\lambda x = \underbrace{C_2C_4}_{K}\sin 2\lambda t\sin\lambda x$

$y(5,t) = \sin 2\lambda t\sin 5\lambda = 0$

$\iff 5\lambda = k\pi \quad k \in \mathbb{Z}$

$\iff \lambda = \frac{k\pi}{5}$

$y(x,t) = K\sin\left(\frac{2}{5}k\pi t\right)\sin\left(\frac{1}{5}k\pi x\right)$

$y_t(x,t) = K\sin\left(\frac{1}{5}k\pi x\right)\cdot\left(\frac{2}{5}k\pi\right)\cos\left(\frac{2}{5}k\pi t\right)$

$y_t(x,0) = K\sin\left(\frac{1}{5}k\pi x\right)\left(\frac{2}{5}k\pi\right) = 5\sin\pi x \iff \begin{cases} k = 5 \\ 2K\pi = 5 \Rightarrow K = \frac{5}{2\pi}\end{cases}$

$y(x,t) = \frac{5}{2\pi}\sin(2\pi t)\sin(\pi x)$

(6) $\dfrac{\partial^2 y}{\partial t^2} = 4\dfrac{\partial^2 y}{\partial x^2}$, $y(0,t) = y(5,t) = 0$

$y(x,0) = 0$, $y_t(x,0) = 3\sin 2\pi x - 2\sin 5\pi x$

$y(x,t) = K\sin\left(\frac{2}{5}k\pi t\right)\sin\left(\frac{1}{5}k\pi x\right)$ (see ex 4.5)

$y_t(x,t) = K\sin\left(\frac{1}{5}k\pi x\right)\cdot\left(\frac{2}{5}k\pi\right)\cos\left(\frac{2}{5}k\pi t\right)$

$y_t(x,0) = K\sin\left(\frac{1}{5}k\pi x\right)\left(\frac{2}{5}k\pi\right) = 3\sin 2\pi x - 2\sin 5\pi x$

$\parallel$

$\frac{2}{5}K_1 k_1\pi\sin\left(\frac{1}{5}k_1\pi x\right) + \frac{2}{5}K_2 k_2\pi\sin\left(\frac{1}{5}k_2\pi x\right)$

$\Leftrightarrow \begin{cases} k_1 = 10\,, & K_1 = \dfrac{3}{4\pi} \\ k_2 = 25\,, & K_2 = -\dfrac{1}{5\pi} \end{cases}$

$y(x,t) = \dfrac{3}{4\pi}\sin(4\pi t)\sin(2\pi x) - \dfrac{1}{5\pi}\sin(10\pi t)\sin(5\pi x)$

(7) $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$ $\quad u(0,t) = u(4,t) = 0 \quad 0 < x < 4,\ t > 0$

$u(x,0) = 25x$

$u = XT \Rightarrow XT' = 2X''T \Leftrightarrow \frac{T'}{2T} = \frac{X''}{X} = -\lambda^2$

$T' = -2\lambda^2 T$

$T = Ae^{-2\lambda^2 t}$

$X'' + \lambda^2 X = 0$

$m^2 + \lambda^2 = 0$

$m = \pm \lambda i$

$X = B_1 \cos \lambda x + B_2 \sin \lambda x$

$$u = \underbrace{AB_1}_{K_1} e^{-2\lambda^2 t} \cos \lambda x + \underbrace{AB_2}_{K_2} e^{-2\lambda^2 t} \sin \lambda x$$

$u(0,t) = K_1 e^{-2\lambda^2 t} = 0 \Rightarrow K_1 = 0$

$u(4,t) = K_2 e^{-2\lambda^2 t} \sin(4\lambda) = 0 \Rightarrow 4\lambda = k\pi \quad k \in \mathbb{Z}$

$\lambda = \frac{k\pi}{4}$

$$u(x,t) = K_2 e^{-2\lambda^2 t} \sin\left(\frac{k\pi}{4}x\right) = \sum_{m=1}^{\infty} B_m e^{-\frac{m^2\pi^2}{8}t} \sin\left(\frac{m\pi}{4}x\right)$$

$$u(x,0) = 25x = \sum_{m=1}^{\infty} B_m \sin\left(\frac{m\pi}{4}x\right)$$

$L = 4$: $\quad B_m = \frac{2}{L}\int_0^L f(x) \sin \frac{m\pi x}{L}\, dx$

$$= \frac{1}{2}\int_0^4 25x \sin \frac{m\pi x}{4}\, dx = \frac{25}{2}\left(-\frac{4}{m\pi} x \cos \frac{m\pi}{4}x\right)_0^4 + \frac{25}{2}\int_0^4 \frac{4}{m\pi} \cos \frac{m\pi x}{4}\, dx$$

$u = x \qquad du = dx$

$dv = \sin \frac{m\pi x}{4} dx \rightarrow v = -\frac{4}{m\pi} \cos \frac{m\pi}{4} x$

$$= -\frac{200}{m\pi} \cos m\pi + \frac{50}{m\pi} \underbrace{\left(\frac{4}{m\pi} \sin \frac{m\pi x}{4}\right)_0^4}_{=0}$$

$$u(x,t) = \sum_{m=1}^{\infty} -\frac{200}{m\pi} \cos(m\pi)\, e^{-\frac{m^2\pi^2}{8}t} \sin\left(\frac{m\pi}{4}x\right)$$

(8) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $\quad u_x(0,t) = u_x(\pi,t) = 0$ $\quad 0<x<\pi,\ t>0$

$u(x,0) = f(x)$

$u = XT \Rightarrow XT' = X''T \Leftrightarrow \frac{T'}{T} = \frac{X''}{X} = -\lambda^2$

$T' + \lambda^2 T = 0$

$T = Ae^{-\lambda^2 t}$

$X'' + \lambda^2 X = 0$

$m^2 + \lambda^2 = 0 \Rightarrow m = \pm\lambda i$

$X = B_1 \cos\lambda x + B_2 \sin\lambda x$

$u(x,t) = \underbrace{AB_1}_{K_1} e^{-\lambda^2 t}\cos\lambda x + \underbrace{AB_2}_{K_2} e^{-\lambda^2 t}\sin\lambda x$

$u_x(x,t) = -K_1\lambda e^{-\lambda^2 t}\sin\lambda x + K_2\lambda e^{-\lambda^2 t}\cos\lambda x$

$u_x(0,t) = K_2\lambda e^{-\lambda^2 t} = 0 \Rightarrow K_2 = 0$

$u_x(\pi,t) = -K_1\lambda e^{-\lambda^2 t}\sin(\lambda\pi) = 0 \Leftrightarrow \sin\lambda\pi = 0 \rightsquigarrow \lambda\pi = k\pi,\ k\in\mathbb{Z}$

$\lambda = k$

$u(x,t) = K_2 e^{-k^2 t}\cos kx$

$= \sum_{m=1}^{\infty} A_m e^{-m^2 t}\cos mx + \frac{A_0}{2}$

$u(x,0) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos mx = f(x)$

$(L=\pi)$ $\quad A_m = \frac{2}{L}\int_0^L f(x)\cos\frac{m\pi x}{L}\,dx = \frac{2}{\pi}\int_0^\pi f(x)\cos mx\,dx$

$A_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx$

$u(x,t) = \frac{1}{\pi}\int_0^\pi f(x)\,dx + \sum_{m=1}^{\infty} e^{-m^2 t}\cdot\frac{2}{\pi}\int_0^\pi f(x)\cos mx\,dx\cdot\cos mx$