PW/TP 13-14 Second-order Linear PDE's

Using methods of ODE's

Exercise 1.

(a) Solve $x\dfrac{\partial^2 z}{\partial x \partial y}+\dfrac{\partial z}{\partial y}=0$

(b) Find the particular solution for which $z(x,0)=x^5+x, z(2,y)=3y^4$

Exercise 2.

Solve for $u=u(x,y)$:

1. $u_{yy}=0$
2. $u_{xx}+16\pi^2 u=0$
3. $25u_{yy}-4u=0$
4. $u_y+y^2u=0$
5. $2u_{xx}+9u_x+4u=-3\cos x-29\sin x$
6. $u_{yy}+6u_y+13u=4e^{3y}$
7. $u_{xy}=u_x$
8. $x^2u_{xx}+2xu_x-2u=0$

Second-order Linear PDE's with constant coefficients

Method 1

Solve the homogeneous equation by assuming $u=e^{ax+by}$, where a and b are constants to be determined.

Solve the non-homogeneous equation by using the method of undetermined coefficients.

Exercise 3.

1. $\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial^2 u}{\partial y^2}$
2. $\dfrac{\partial^2 z}{\partial x^2}-2\dfrac{\partial^2 z}{\partial x \partial y}+\dfrac{\partial^2 z}{\partial y^2}=0$
3. $\dfrac{\partial u}{\partial x}+2\dfrac{\partial u}{\partial y}=x$
4. $\dfrac{\partial^4 u}{\partial x^4}+2\dfrac{\partial^4 u}{\partial x^3 \partial y}=4$
5. $\dfrac{\partial u}{\partial x}+2\dfrac{\partial u}{\partial y}=3u$
6. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$

7. $\dfrac{\partial^2 z}{\partial x^2} - 3\dfrac{\partial^2 z}{\partial x \partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = x \sin y$

Method 2. Seperation of variables

Assume that a solution can be expressed as a product of unknown functions each of which depends on only one of the independent variables.

$$u(x, y) = X(x)Y(y)$$

Rewrite the differential equation as $f(X, X', X'', \ldots) = g(Y, Y', Y'', \ldots) = \text{ constant}$, with a well chosen constant depending on the given initial conditions.

For general function (as initial condition) use a Half Range Fourier Series: $f(x) = \dfrac{A_0}{2}\sum_{m=1}^{\infty}\left(A_m \cos(\dfrac{m\pi}{L}x) + B_m \sin(\dfrac{m\pi}{L}x)\right)$ with

$$\begin{cases} A_m = 0, \quad B_m = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin(\dfrac{m\pi}{L}x)\mathrm{d}x & \text{for } f(x) \text{ odd function with period } 2L \\ B_m = 0, \quad A_m = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos(\dfrac{m\pi}{L}x)\mathrm{d}x & \text{for } f(x) \text{ even function with period } 2L \end{cases}$$

Exercise 4.

1. $3\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = 0, \quad u(x, 0) = 4e^{-x}$
2. $\dfrac{\partial u}{\partial t} = 4\dfrac{\partial^2 u}{\partial x^2}, \; u(0, t) = 0, u(\pi, t) = 0, \quad u(x, 0) = 2\sin 3x - 4\sin 5x$
3. $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, \; u_x(0, t) = 0, u(2, t) = 0, \quad u(x, 0) = 8\cos\dfrac{3\pi x}{4} - 6\cos\dfrac{9\pi x}{4}$
4. $\dfrac{\partial u}{\partial x} = 2\dfrac{\partial u}{\partial y} + u, \quad u(x, 0) = 3e^{-5x} + 2e^{-3x}$
5. $\dfrac{\partial^2 y}{\partial t^2} = 4\dfrac{\partial^2 y}{\partial x^2}, \quad y(0, t) = y(5, t) = 0, \quad y(x, 0) = 0, \quad y_t(x, 0) = 5\sin \pi x$
6. $\dfrac{\partial^2 y}{\partial t^2} = 4\dfrac{\partial^2 y}{\partial x^2}, \quad y(0, t) = y(5, t) = 0, \quad y(x, 0) = 0, \quad y_t(x, 0) = 3\sin 2\pi x - 2\sin 5\pi x$
7. $\dfrac{\partial u}{\partial t} = 2\dfrac{\partial^2 u}{\partial x^2}, \quad u(0, t) = u(4, t) = 0, \quad u(x, 0) = 25x \quad \text{where } 0 < x < 4, t > 0$
8. $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, \quad u_x(0, t) = u_x(\pi, t) = 0, \quad u(x, 0) = f(x) \quad \text{where } 0 < x < 4, t > 0$