Exercise 7 (2.56)

(a) $x^2 + cy^2 = 1$

$\cancel{2}x + \cancel{2}cyy' = 0$

$c = -\dfrac{x}{yy'}$

$x^2 - \dfrac{xy^{\cancel{2}}}{\cancel{y}y'} = 1 \quad \Leftrightarrow \quad x^2y' - xy = y'$

$\Leftrightarrow \quad \boxed{0 = (1-x^2)y' + xy}$

(b) $y^2 = ax + b$

$2yy' = a \quad \Leftrightarrow \quad y^2 = 2yy'x + b$

$\cancel{2yy'} = \cancel{2}y'^2x + \cancel{2}yy''x + \cancel{2yy'}$

$0 = y'^2\cancel{x} + yy''\cancel{x}$

$\boxed{0 = y'^2 + yy''}$

1) $\underline{(2.59\ b)\ x^3 y''' = 1 + x^4\ ;\ y(1) = y'(1) = y''(1) = 0}$

$\Rightarrow$ separable

$$\int dy'' = \int \frac{1+x^4}{x^3}\, dx$$

$$y'' = -\frac{1}{2x^2} + \frac{x^2}{2} + c_1 \qquad y''(1) = 0$$

$$\Updownarrow$$

$$0 = -\frac{1}{2} + \frac{1}{2} + c_1 \Leftrightarrow c_1 = 0$$

$$\int dy' = \int \left(-\frac{1}{2x^2} + \frac{x^2}{2}\right) dx$$

$$y' = -\frac{1}{2}\left(-\frac{1}{x} - \frac{x^3}{3} + c_2\right) \qquad y'(1) = 0$$

$$\Updownarrow$$

$$0 = -\frac{1}{2}\left(-1 - \frac{1}{3} + c_2\right)$$

$$\Updownarrow$$

$$c_2 = \frac{4}{3}$$

$$\int dy = \int \left(\frac{1}{2x} + \frac{x^3}{6} - \frac{2}{3}\right) dx$$

$$y = \frac{1}{2}\ln x + \frac{x^4}{24} - \frac{2}{3}x + c_3 \qquad y(1) = 0$$

$$\Updownarrow$$

$$0 = 0 + \frac{1}{24} - \frac{2}{3} + c_3$$

$$\Updownarrow$$

$$c_3 = \frac{15}{24} = \frac{5}{8}$$

$$\boxed{y = \frac{1}{2}\ln x + \frac{x^4}{24} - \frac{2}{3}x + \frac{5}{8}}$$

2) (2.64 b) $\underline{(\underbrace{x^2+x-y^2}_{M})dx \underbrace{-y}_{N}dy = 0}$

$$\frac{\partial M}{\partial y} = -2y \neq \frac{\partial N}{\partial x} = 0$$

$\Rightarrow$ integrating factor?

$$R(x) = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{1}{y}(-2y - 0) = 2$$

$$F(x) = e^{\int 2dx} = e^{2x}$$

$$\underbrace{e^{2x}(x^2+x-y^2)}dx \underbrace{- e^{2x}y}dy = 0$$

$$\frac{\partial}{\partial y} = -2e^{2x}\cdot y \qquad \frac{\partial}{\partial x} = -2e^{2x}y$$

$$U(x,y) = \int(-e^{2x}y)dy + F(x)$$

$$= -e^{2x}\cdot\frac{y^2}{2} + F(x)$$

$$\frac{\partial}{\partial x}U(x,y) = -e^{2x}y^2 + F'(x) = M = e^{2x}(x^2+x-y^2)$$

$$\int F'(x) = \int e^{2x}(x^2+x)\,dx$$

$$\boxed{\begin{array}{l} \int x^2 e^{2x} \quad u = x^2 \quad du = 2x\,dx \\ \quad\quad\quad\; dv = e^{2x}dx \rightarrow v = \frac{1}{2}e^{2x} \\ \quad \parallel \\ \frac{1}{2}x^2e^{2x} - \int x e^{2x} dx \end{array}}$$

$$F(x) = \frac{1}{2}x^2e^{2x} + c$$

$$U(x,y) = \frac{1}{2}x^2e^{2x} - \frac{1}{2}e^{2x}y^2 + c$$

$$\boxed{ce^{-2x} = x^2 - y^2}$$

3) $\underline{(2.67\ b)\quad 2x^2y' = xy + y^3}$

$$\frac{dy}{dx} - \frac{1}{2x}y = \frac{1}{2x^2}y^3$$

$n = 3$

$v = y^{-2}$

$$\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$$

$$\Updownarrow$$

$$\frac{dy}{dx} = -\frac{1}{2}y^3\frac{dv}{dx}$$

$$-\frac{1}{2}y^3\frac{dv}{dx} - \frac{1}{2x}y = \frac{1}{2x^2}y^3$$

$$\frac{dv}{dx} + \frac{y^{-2}}{x} = -\frac{1}{x^2}$$

$$\frac{dv}{dx} + \frac{1}{x}v = -\frac{1}{x^2}$$

$$\mu = e^{\int 1/x\, dx} = x$$

$$x \cdot v = \int x \cdot \left(-\frac{1}{x^2}\right) dx + c$$

$$x \cdot v = -\ln x + c \qquad (\text{with } v = y^{-2})$$

$$\frac{x}{y^2} = -\ln x + c$$

$$\boxed{e^{x/y^2} = c\,x^{-1}}$$

4) (2.72 a) $xy'' - 3y' = x^2$

y is missing : $y' = p \qquad y'' = \frac{dp}{dx}$

$$x\frac{dp}{dx} - 3p = x^2$$

$$\frac{dp}{dx} - \frac{3}{x}p = x$$ (= linear first order equation)

$$\mu = e^{-\int 3/x \, dx} = e^{-3\ln x} = x^{-3}$$

$$x^{-3}p = \int x^{-3} \cdot x \, dx + c_1 = -\frac{1}{x} + c_1$$

$$\frac{dy}{dx} = p = -x^2 + c_1 x^3$$

$$\boxed{y = -\frac{1}{3}x^3 + \frac{c_1}{4}x^4 + c_2}$$

5) (2.61 b) $\frac{dy}{dx} = \frac{3 - 4xy^2}{4x^2y + 6y^2}$; $y(1) = -1$

$$\underbrace{(4x^2y + 6y^2)}_{M} dy + \underbrace{(4xy^2 - 3)}_{N} dx = 0$$

$$\frac{\partial M}{\partial x} = 8xy \qquad \frac{\partial N}{\partial y} = 8xy$$

$$\int M dy = \int (4x^2y + 6y^2) dy = 2x^2y^2 + 2y^3$$

$$\frac{\partial}{\partial x}(2x^2y^2 + 2y^3) = 4xy^2$$

$$\int (N - 4xy^2) dx = \int (-3) dx = -3x$$

$2x^2y^2 + 2y^3 - 3x = c$ with $y(1) = -1 \iff c = -3$

$$\boxed{2x^2y^2 + 2y^3 - 3x = -3}$$

6) (2.70 a) $y'^2 + (y-1)y' - y = 0$

$$y'^2 + yy' - y' - y = 0$$

$$y(y'-1) = y' - y'^2$$

$$y = \frac{y'(1-y')}{y'-1} = -y' \quad \text{if} \quad y'-1 \neq 0$$

$$y' = 1$$

$$y = x + c_1$$

$$y = -y'$$

$$\int dx = -\int \frac{1}{y} dy$$

$$x + c_2 = -\ln y$$

$$y = c_2 e^{-x}$$

$$\Leftrightarrow \quad \boxed{(y - x + c_1)(y - c_2 e^{-x}) = 0}$$

7) (2.65 b) $xy' - 4y = x$

$$y' - \frac{4}{x} y = 1$$

$$\mu = e^{\int -4/x \, dx} = x^{-4}$$

$$x^{-4} y = \int x^{-4} dx + c$$

$$y = -\frac{1}{3} x^{-3} \cdot x^4 + c x^4$$

$$\boxed{3y = -x + c x^4}$$

8) (2.63 a) $(3y - 2xy^3)dx + (4x - 3x^2y^2)dy = 0$

$$\frac{\partial}{\partial y} = 3 - 6xy^2 \neq \frac{\partial}{\partial x} = 4 - 6xy^2$$

$\cdot x^{-p}y^{-q}$ $$\underbrace{\frac{3y - 2xy^3}{x^p y^q}}_{N} dx + \underbrace{\frac{4x - 3x^2y^2}{x^p y^q}}_{M} dy = 0$$

$$\frac{\partial N}{\partial y} = 3\underline{x^{-p}y^{-q}}(1-q) - 2\underline{x^{1-p}y^{2-q}}(3-q)$$

$$\frac{\partial M}{\partial x} = 4(1-p)\underline{x^{-p}x^{-q}} - 3(2-p)\underline{x^{1-p}y^{2-q}}$$

$$\begin{cases} 3(1-q) = 4(1-p) \\ -2(3-q) = -3(2-p) \end{cases} \Leftrightarrow \begin{cases} p = -2 \\ q = -3 \end{cases}$$

$$(3x^2y^4 - 2x^3y^6)dx + (4x^3y^3 - 3x^4y^5)cy = 0$$

$$\frac{\partial}{\partial y} = 12x^2y^3 - 12x^3y^5 \quad \frac{\partial}{\partial x} = 12x^2y^3 - 12x^3y^5$$

$$U = \int 3x^2y^4 - 2x^3y^6\,dx = x^3y^4 - \frac{1}{2}x^4y^6 + F(y)$$

$$\frac{\partial U}{\partial y} = 4x^3y^3 - 3x^4y^5 + F'(y) = 4x^3y^3 - 3x^4y^5 \Leftrightarrow F'(y) = 0$$

$$\boxed{C = x^3y^4 - \frac{1}{2}x^4y^6}$$

9) (2.71 a) $y = px + 2p^2$, with $p = y'$

$$\underbrace{\frac{dy}{dx}}_{p} = \frac{dp}{dx}\cdot x + p + 4p\frac{dp}{dx}$$

$$x\frac{dp}{dx} + 4p\frac{dp}{dx} = 0$$

$$(x+4p)\frac{dp}{dx} = 0$$

case 1, $x+4p \neq 0$. $\frac{dp}{dx} = 0$

$y' = p = c$

$\boxed{y = cx + 2c^2}$ (general)

case 2, $x + 4p = 0$. $p = -\frac{x}{4}$

$\boxed{y = -\frac{1}{8}x^2}$

10) $(2.62\ d)\ (2y\sin x + 3y^4 \sin x \cos x)dx - (4y^3 \cos^2 x + \cos x)dy = 0$

$\frac{\partial}{\partial y} = 2\sin x + 12y^3 \sin x \cos x \qquad \frac{\partial}{\partial x} = 2\cos x \sin x\ 4y^3 + \sin x$

$R(x) = \frac{-1}{4y^3\cos^2 x + \cos x}\left(\sin x + 4y^3 \sin x \cos x\right)$

$= -\frac{1}{\cos x} \cdot \sin x \left(\frac{1 + \cancel{4y^3\cos x}}{\cancel{4y^3\cos x} + 1}\right) = -\tan x$

$\cos x = t$

$F(x) = e^{-\int \tan x\, dx} \qquad -\int \frac{\sin x}{\cos x} dx \qquad dt = -\sin x\ dx$

$= \int \frac{1}{t} dt = \ln(t)$

$= \cos x$

$\Rightarrow (2y\sin x \cos x + 3y^4 \sin x \cos^2 x)\,dx - (4y^3 \cos^3 x + \cos^2 x)\,dy = 0$

$\frac{\partial}{\partial y} = 2\sin x \cos x + 12y^3 \sin x \cos^2 x$

$\frac{\partial}{\partial x} = +12y^3 \cos^2 x \cdot \sin x + 2\cos x \sin x$ ✓

$\int 2y\sin x \cos x + 3y^4 \sin x \cos^2 x\ dx = -\int (2yt + 3y^4 t^2)\, dt$

$\cos x = t \rightarrow dt = -\sin x\ dx$

$= -\cancel{2}yt^2 - y^4 t^3 + F(y)$

$\frac{\partial}{\partial y}\left(-y\cos^2 x - y^4 \cos^3 x + F(y)\right) = \cancel{-\cos^2 x} \cancel{- 4y^3 \cos^3 x} + F'(y)$

$= \cancel{-4y^3 \cos^3 x} \cancel{- \cos^2 x}$

$\Rightarrow F'(y) = 0 \Leftrightarrow F(y) = c$

$\boxed{c = -y\cos^2 x - y^4 \cos^3 x}$

11) (2.66 a) $\dfrac{dy}{dx} = \dfrac{2y}{x} - \dfrac{y^2}{x^2}$

$\left(v = \dfrac{y}{x}\right) \qquad v + x\dfrac{dv}{dx} = 2v - v^2$

$$\int \frac{1}{v - v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\int \left(\frac{1}{v} + \frac{1}{1-v}\right)dv = \ln(v) - \ln(1-v) = \ln x + c$$

$$\frac{v}{1-v} = cx$$

$$\frac{y/x}{1 - y/x} = \frac{y}{x-y} = cx$$

$$y = cx^2 - cxy$$

$$\boxed{cy = x^2 - xy}$$

12) (2.66 d) $(x-y)y' + 3y - 5x = 0$

$\dfrac{dy}{dx} = \dfrac{5x - 3y}{x - y} \xrightarrow{(y = vx)} v + x\dfrac{dv}{dx} = \dfrac{5 - 3v}{1 - v}$

$$x\frac{dv}{dx} = \frac{5 - 4v + v^2}{1 - v}$$

$$\int \frac{1 - v}{5 - 4v + v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\int \frac{2v - 4}{5 - 4v + v^2}\,dv + \int \frac{2}{\underbrace{5 - 4v + v^2}_{(v-2)^2 + 1}}\,dv = -2\int \frac{1}{x}\,dx$$

$$\ln(5 - 4v + v^2) + 2\tan^{-1}(v - 2) = -2\ln x + c$$

$$\ln\left(5 - 4\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2\right) + \ln(x^2) + 2\tan^{-1}\left(\frac{y}{x} - 2\right) = c$$

$$\boxed{\ln(y^2 - 4xy + 5x^2) + 2\tan^{-1}\left(\frac{y - 2x}{x}\right) = c}$$

13) $\underline{(2.72\,c)\ y'' + 4y = 0}$

missing x : $\quad y' = p \quad y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$

$$p\frac{dp}{dy} + 4y = 0$$

$$p\frac{dp}{dy} = -4y$$

$$\int p\,dp = \int -4y\,dy$$

$$\frac{p^2}{2} = -2y^2 + C_1$$

$$\frac{dy}{dx} = p = \sqrt{2C_1 - 4y^2} = 2\sqrt{\frac{C_1}{2} - y^2}$$

$$\int \frac{1}{\sqrt{\frac{C_1}{2} - y^2}}\,dy = \int 2\,dx$$

$$\sin^{-1}\left(\frac{y}{C_1/\sqrt{2}}\right) = 2x + C_2$$

$$y = \frac{\sqrt{2}}{C_1}\sin(2x + C_2)$$

$$\boxed{y = C_1 \sin(2x + C_2)}$$

14) $\underline{(2.69)\ \frac{dy}{dx} = x^2 + 2xy + y^2 + 2x + 2y\ ;\ y(0) = 0}$

$$\frac{dy}{dx} = (x+y)^2 + 2(x+y) \qquad u = x + y$$

$$\frac{du}{dx} - 1 = u^2 + 2u \qquad \frac{du}{dx} = 1 + \frac{dy}{dx}$$

$$\int \frac{1}{(u+1)^2}\,du = \int dx$$

$$-\frac{1}{u+1} = x + C$$

$-\frac{1}{x+y+1} = x + C$ with $y(0) = 0 \Leftrightarrow C = -1$

$$\boxed{1 = (1-x)(x+y+1)}$$

15) $\underline{(2.71\ d)\ x^2y = x^3p - yp^2, \text{with } p = y'}$

$$x^2y + p^2y = x^3p$$

$$y = \frac{x^3p}{x^2+p^2}$$

$$p = \frac{dy}{dx} = \frac{(x^2+p^2)\frac{d(x^3p)}{dx} - x^3p\frac{d(x^2+p^2)}{dx}}{(x^2+p^2)^2}$$

$$p(x^2+p^2)^2 = (x^2+p^2)\left(3x^2p + x^3\frac{dp}{dx}\right) - x^3p\left(2x + 2p\frac{dp}{dx}\right)$$

$$p(x^2+p^2)^2 - (x^2+p^2)3x^2p + 2x^4p = \left((x^2+p^2)x^3 - 2x^3p^2\right)\frac{dp}{dx}$$

$$p\left((x^2+p^2)\underbrace{(x^2+p^2-3x^2)}_{p^2-2x^2} + 2x^4\right) = (x^5 - x^3p^2)\frac{dp}{dx}$$

$$p\underbrace{(p^2x^2 + p^4 \cancel{- 2x^4} - 2p^2x^2 \cancel{+ 2x^4})}_{p^4 - p^2x^2} - x^3(x^2-p^2)\frac{dp}{dx} = 0$$

$$p^3(p^2 - x^2) + x^3(p^2 - x^2)\frac{dp}{dx} = 0$$

$$(p^2 - x^2)\left(p^3 + x^3\frac{dp}{dx}\right) = 0$$

Case 1, $p^2 - x^2 = 0$. $p = \pm x$

$$\left.\begin{aligned} x^2y &= x^3(\pm x) - yx^2 \\ 2x^2y &= \pm x^4 \end{aligned}\right\} \Rightarrow \boxed{y = \pm\frac{x^2}{2}} \quad \text{(singular)}$$

Case 2, $p^3 + x^3\dfrac{dp}{dx} = 0$

$$\frac{1}{x^3}dx = -\frac{1}{p^3}dp$$

$$-\frac{1}{2x^2} + c = \frac{1}{2p^2}$$

$$\cancel{2}p^2 = \frac{\cancel{2}x^2}{2cx^2 - 1} \qquad p = \frac{\pm x}{\sqrt{2cx^2-1}}$$

$$x^2y = x^3p - yp^2$$

$$p = \frac{\pm x}{\sqrt{2cx^2-1}}$$

$$x^2y = \frac{\pm x^4}{\sqrt{2cx^2-1}} - \frac{yx^2}{2cx^2-1}$$

$$x^2y + \frac{x^2y}{2cx^2-1} = \frac{\pm x^4}{\sqrt{2cx^2-1}}$$

$$\frac{x^2y}{2cx^2-1}\left(2cx^2\right) = \frac{\pm x^4}{\sqrt{2cx^2-1}}$$

$$\frac{2cy}{2cx^2-1} = \frac{\pm 1}{\sqrt{2cx^2-1}}$$

$$\frac{4c^2y^2}{(2cx^2-1)^2} = \frac{1}{2cx^2-1}$$

$$4c^2y^2 = 2cx^2 - 1$$

$$y^2 = \frac{x^2}{2c} - \frac{1}{4c^2} \qquad \left(c = \frac{1}{2c}\right)$$

$$\boxed{y^2 = cx^2 - c^2} \quad \text{(general)}$$

16) (2.65 d) $\frac{dy}{dx} + 2y\cot x = \csc x$

$$\mu = e^{\int 2\cot x\,dx} = \sin^2 x$$

$$\sin^2 x \cdot y = \int \sin^2 x \cdot \frac{1}{\sin x}\,dx + c$$

$$= -\cos x + c$$

$$\boxed{y = -\cos x \cdot \sin^{-2} x + c \cdot \sin^{-2} x}$$

17) (2.60 d) $\frac{dy}{dx} = \frac{x\sqrt{1-y^2}}{y\sqrt{1-x^2}}$

$$\int \frac{y}{\sqrt{1-y^2}}\,dy = \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$1-y^2 = t$

$-2y\,dy = dt$

$$-\frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt = -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$$

$$\sqrt{1-y^2} = \sqrt{1-x^2} + c$$

$$\boxed{\sqrt{1-x^2} - \sqrt{1-y^2} = c}$$

18) (2.61 c) $(ye^x - e^{-y})\,dx + (xe^{-y} + e^x)\,dy = 0$

$$\frac{\partial(ye^x - e^{-y})}{\partial y} = e^x + e^{-y} \qquad \frac{\partial(xe^{-y} + e^x)}{\partial x} = e^{-y} + e^x$$

$$\int ye^x - e^{-y}\,dx = ye^x \qquad \frac{\partial(ye^x)}{\partial y} = e^x$$

$$\int (xe^{-y} + \cancel{e^x} - \cancel{e^x})\,dy = -xe^{-y}$$

$$\boxed{ye^x - xe^{-y} = c}$$