PW/TP 13-14 Second-order Linear PDE's

Using methods of ODE's

Exercise 1. (12.41)

(a) Solve $x\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial z}{\partial y} = 0$

(b) Find the particular solution for which $z(x, 0) = x^5 + x, z(2, y) = 3y^4$

Exercise 2.

Solve for $u = u(x, y)$:

1. $u_{yy} = 0$

2. $u_{xx} + 16\pi^2 u = 0$

3. $25u_{yy} - 4u = 0$

4. $u_y + y^2 u = 0$

5. $2u_{xx} + 9u_x + 4u = -3\cos x - 29\sin x$

6. $u_{yy} + 6u_y + 13u = 4e^{3y}$

7. $u_{xy} = u_x$

8. $x^2 u_x x + 2xu_x - 2u = 0$

Second-order Linear PDE's with constant coefficients

Method 1

Solve the homogeneous equation by assuming $u = e^{ax+by}$, where a and b are constants to be determined.

Solve the non-homogeneous equation by using the method of undetermined coefficients.

Exercise 3.

1. (12.42 a) $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial y^2}$

Let $u(x, y) = e^{ax+by}$:

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syms a b u(x,y) z
u(x,y)=exp(a*x+b*y);
eq = diff(u,x,2)-diff(u,y,2)==0;
simplify(eq)
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ans(x, y) = $a^2 = b^2$

We find $a = \pm b$, and thus are the solutions $e^{b(x+y)}$ and $e^{b(y-x)} \quad \forall b$.

The final solution is $u(x, y) = F(x + y) + G(y - x)$. In which F, G are arbitrary functions.

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syms u(x,y) z
F1(z) = z; F2(z) = z^2; F3(z) = cos(z);
G1(z) = z; G2(z) = 3*z^3-2; G3(z) = sin(z);
u1(x,y) = F1(x+y)+G1(y-x);
u2(x,y) = F2(x+y)+G2(y-x);
u3(x,y) = F3(x+y)+G3(y-x);
eq = diff(u,x,2)-diff(u,y,2)==0;
subs(eq,u,u1), subs(eq,u,u2), subs(eq,u,u3)
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ans(x, y) = $0 = 0$

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2. (12.42 e) $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$

The solution is $u(x, y) = F(x + y) + xG(x + y)$ or $F(x + y) + yG(x + y)$.

```
syms u(x,y) z
F1(z) = z; F2(z) = z^2; F3(z) = cos(z);
G1(z) = z; G2(z) = 3*z^3-2; G3(z) = sin(z);
u1(x,y) = F1(x+y)+x*G1(x+y);
u2(x,y) = F2(x+y)+y*G2(x+y);
u3(x,y) = F3(x+y)+x*G2(x+y)+y*G3(x+y);
eq = diff(u,x,2)-2*diff(diff(u,x),y)+diff(u,y,2)==0;
subs(eq,u,u1), subs(eq,u,u2), subs(eq,u,u3)
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ans(x, y) = $0 = 0$

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3. (12.43 a) $\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = x$

4. (12.43 c) $\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^3 \partial y} = 4$

5. (12.42 b) $\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 3u$

6. (12.42 c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

7. (12.43 d) $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x \sin y$

Method 2. Seperation of variables

Assume that a solution can be expressed as a product of unknown functions each of which depends on only one of the independent variables. $u(x, y) = X(x)Y(y)$.

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$$u(x, y) = X(x)Y(y)$$

Rewrite the differential equation as $f(X, X', X'', \ldots) = g(Y, Y', Y'', \ldots) = \text{ constant}$, with a well chosen constant depending on the given initial conditions.

For general function (as initial condition) use a Half Range Fourier Series: $f(x) = \frac{A_0}{2} \sum_{m=1}^{\infty} \left(A_m \cos(\frac{m\pi}{L} x) + B_m \sin(\frac{m\pi}{L} x) \right)$ with

$$\begin{cases} A_m = 0, \quad B_m = \frac{2}{L} \int_0^L f(x) \sin(\frac{m\pi}{L} x) \mathrm{d}x & \text{for } f(x) \text{ odd function with period } 2L \\ B_m = 0, \quad A_m = \frac{2}{L} \int_0^L f(x) \cos(\frac{m\pi}{L} x) \mathrm{d}x & \text{for } f(x) \text{ even function with period } 2L \end{cases}$$

Exercise 4.

1. (12.46 a) $3 \frac{\partial u}{\partial x} + 2 \frac{\partial u}{\partial y} = 0$, $u(x, 0) = 4e^{-x}$
2. (12.46 c) $\frac{\partial u}{\partial t} = 4 \frac{\partial^2 u}{\partial x^2}$, $u(0, t) = 0, u(\pi, t) = 0, u(x, 0) = 2 \sin 3x - 4 \sin 5x$
3. (12.46 d) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $u_x(0, t) = 0, u(2, t) = 0, u(x, 0) = 8 \cos \frac{3\pi x}{4} - 6 \cos \frac{9\pi x}{4}$
4. $\frac{\partial u}{\partial x} = 2 \frac{\partial u}{\partial y} + u, \quad u(x, 0) = 3e^{-5x} + 2e^{-3x}$
5. $\frac{\partial^2 y}{\partial t^2} = 4 \frac{\partial^2 y}{\partial x^2}, \quad y(0, t) = y(5, t) = 0, \quad y(x, 0) = 0, \quad y_t(x, 0) = 5 \sin \pi x$
6. $\frac{\partial^2 y}{\partial t^2} = 4 \frac{\partial^2 y}{\partial x^2}, \quad y(0, t) = y(5, t) = 0, \quad y(x, 0) = 0, \quad y_t(x, 0) = 3 \sin 2\pi x - 2 \sin 5\pi x$
7. $\frac{\partial u}{\partial t} = 2 \frac{\partial^2 u}{\partial x^2}, \quad u(0, t) = u(4, t) = 0, \quad u(x, 0) = 25x \quad \text{where } 0 < x < 4, t > 0$
8. $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u_x(0, t) = u_x(\pi, t) = 0, \quad u(x, 0) = f(x) \quad \text{where } 0 < x < 4, t > 0$